

CS231A

Computer Vision: From 3D Reconstruction to Recognition



Optimal Estimation

Perception as a Continuous Process



Perception as a Multi-Modal Experience



Perception as Inference

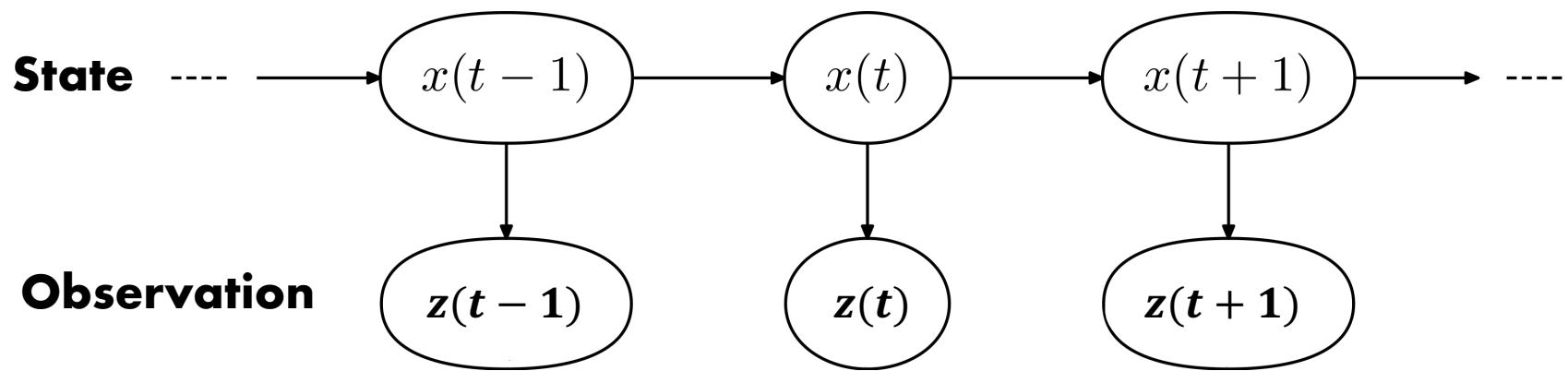


Recursive State Estimation

Mathematical Formalism to:

- continuously integrate measurements
- from different sensor sources
- to infer the state of a latent variable

What is a state? What is a representation?



Hidden Markov Model

Representations for Autonomous Driving

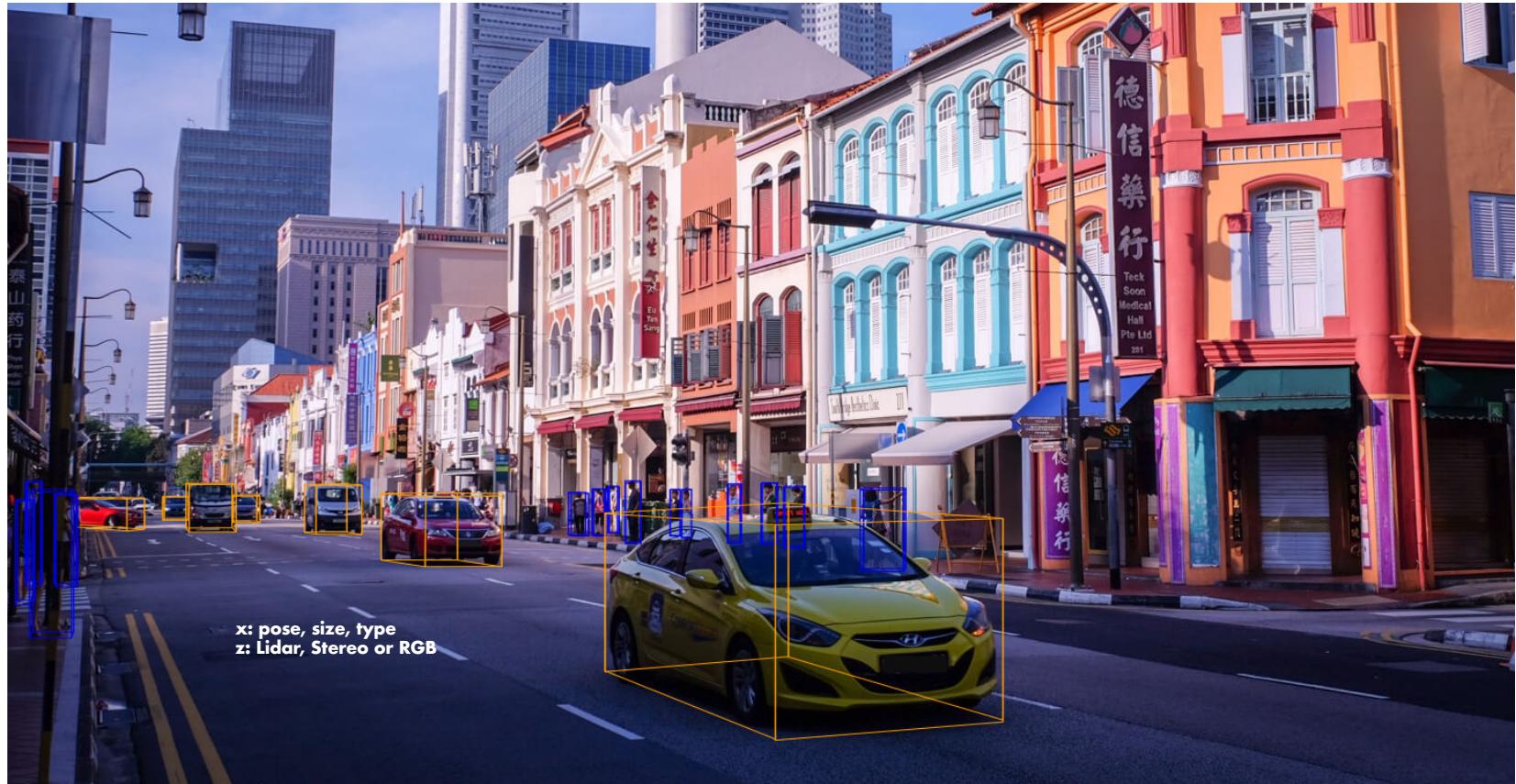


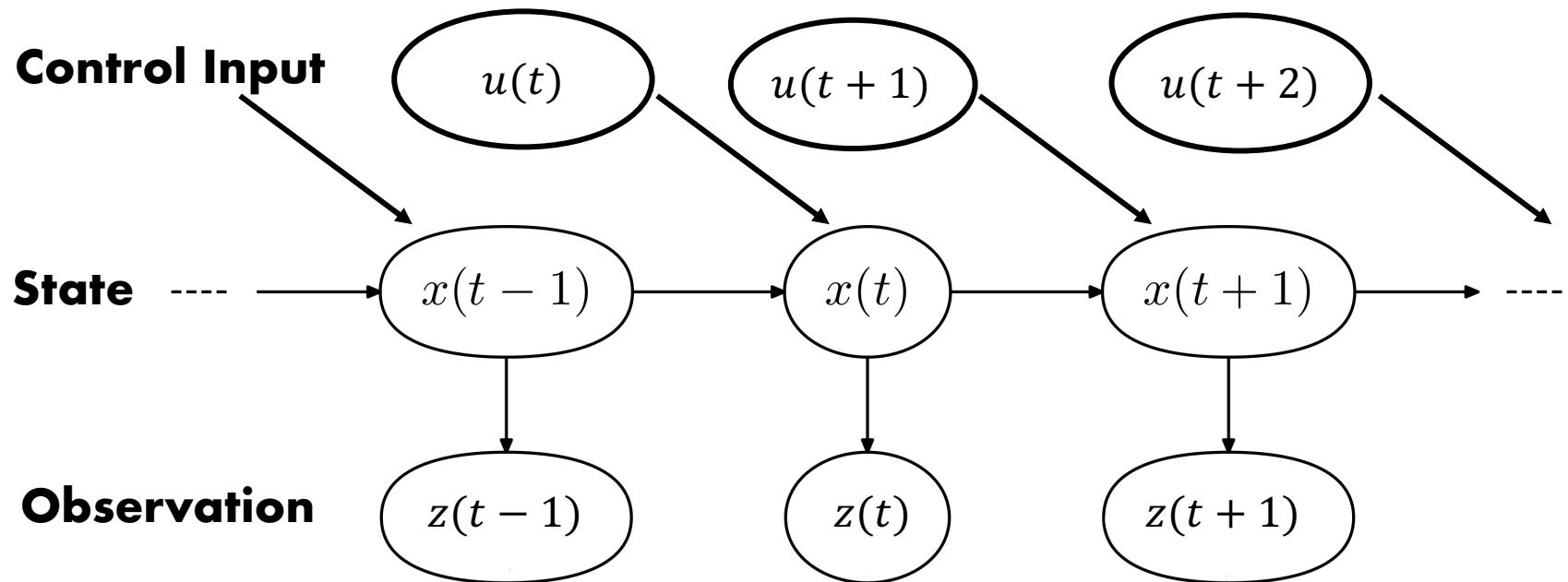
Image adapted from NuScenes by Motional. nuscenes.org

Representations for Manipulation



Manuel Wüthrich et al. "Probabilistic Object Tracking using a Depth Camera", IROS 2013

Why do we care about state estimation in Robotics?

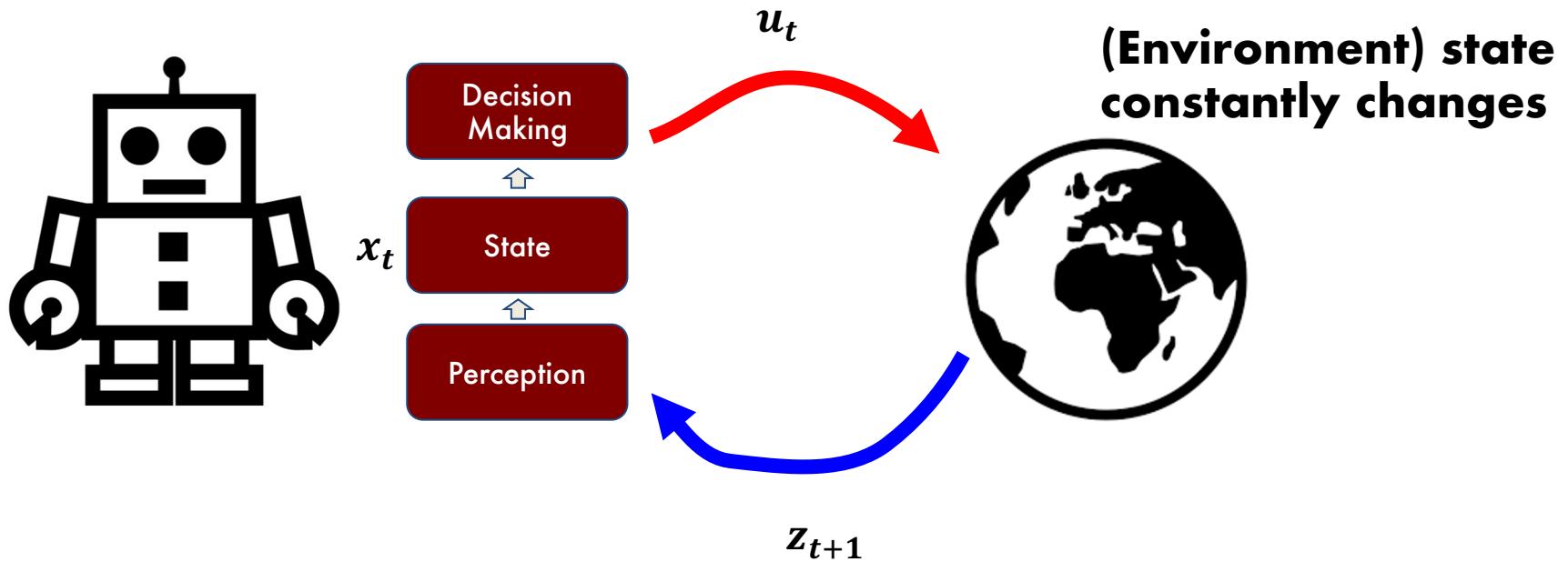


Partially Observable Markov Decision Process

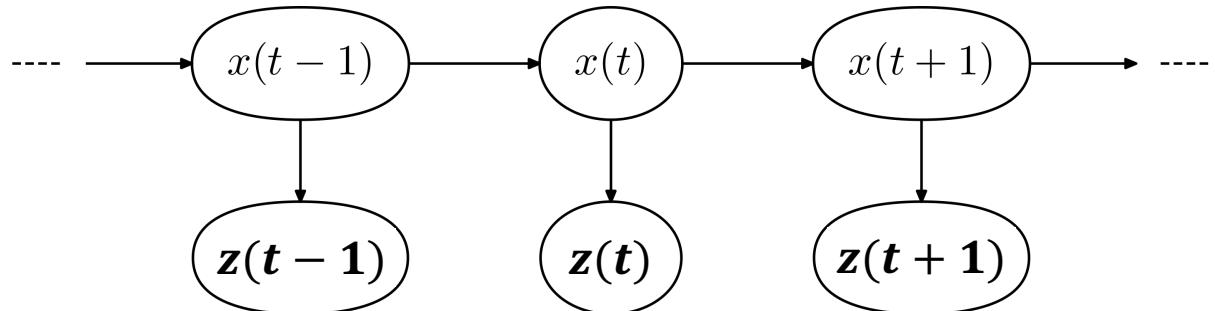
Today

- Intro: Why state estimation?
- Bayes Filter
- Kalman Filter
- Extended Kalman Filter
- For more depth:
 - **AA 273: State Estimation and Filtering for Robotic Perception – Mac Schwager**

The Agent and the Environment



Notation



- x State of dynamical system, dim n
- x_t Instantiation of system state at time t
- z Sensor Observation Vector, dim k
- z_t Specific Observation at time t
- u Robot action / control input, dim m
- u_t Robot action / control input at time t
- $p(x_t | z_{0:t}, u_{0:t})$ Probability distribution

Markov Assumption

State is complete

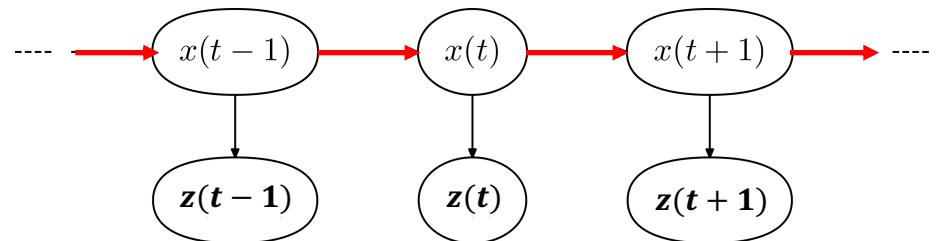
Probabilistic Generative Laws

- Evolution of state and measurement governed by probabilistic laws
- x_t generated stochastically

State Transition Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t})$$



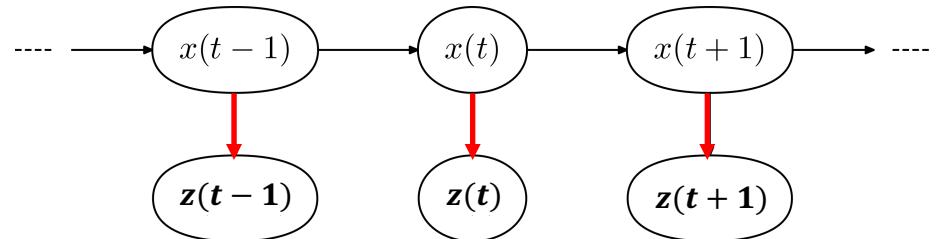
- Assumption: State complete

$$p(x_t | x_{0:t-1}, z_{1:t-1}, u_{1:t}) = p(x_t | x_{t-1}, u_t)$$

Measurement Model

- Probability distribution conditioned on all previous states, measurements and controls

$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t})$$



- Assumption: State complete

$$p(z_t | x_{0:t}, z_{1:t-1}, u_{1:t}) = p(z_t | x_t)$$

Belief Distribution

- Assigns probability to each possible hypothesis about what the true state may be
- Posterior distributions over state conditioned on all the data

$$bel(x_t) = p(x_t | z_{1:t}, u_{1:t})$$

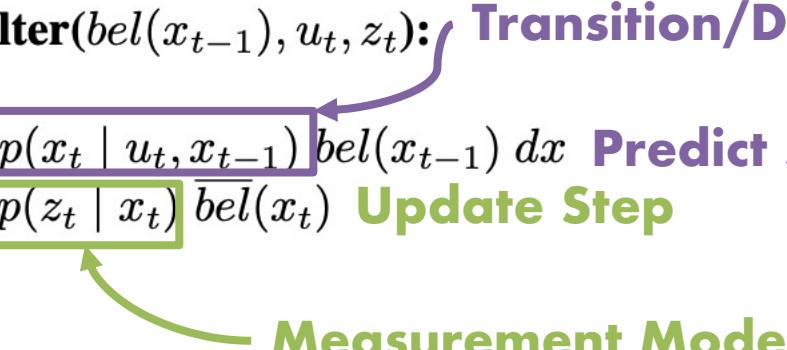
- Before incorporating measurement z_t = prediction

$$\overline{bel}(x_t) = p(x_t | z_{1:t-1}, u_{1:t})$$

The Bayes Filter

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_t and not from the ever-growing history $z_{1:t}, u_{1:t}$

```
1:   Algorithm Bayes_filter( $bel(x_{t-1})$ ,  $u_t$ ,  $z_t$ ): Transition/Dynamics model
2:     for all  $x_t$  do
3:        $\bar{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx$  Predict Step
4:        $bel(x_t) = \eta p(z_t | x_t) \bar{bel}(x_t)$  Update Step
5:     endfor
6:     return  $bel(x_t)$ 
```



Simple example – Belief & Measurement Model

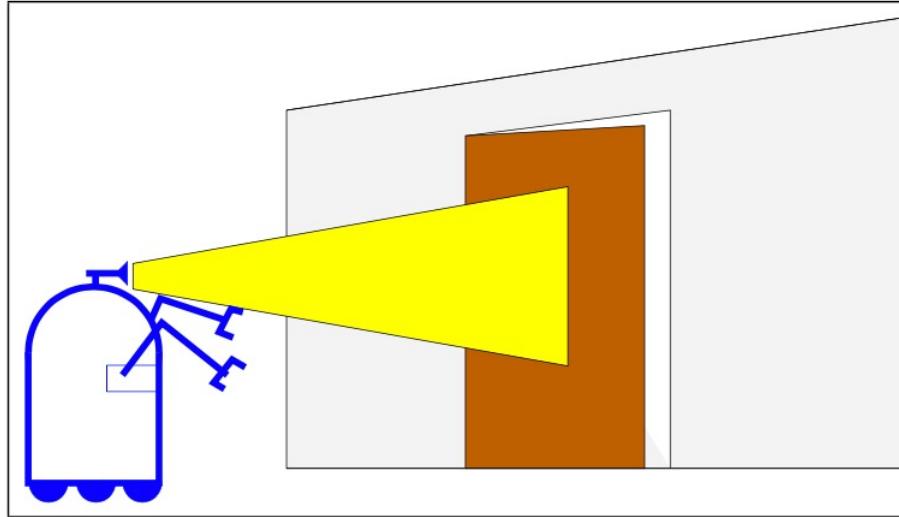


Figure 2.2 A mobile robot estimating the state of a door.

$$bel(X_0 = \text{open}) = 0.5$$

$$bel(X_0 = \text{closed}) = 0.5$$

$$p(Z_t = \text{sense_open} | X_t = \text{is_open}) = 0.6$$

$$p(Z_t = \text{sense_closed} | X_t = \text{is_open}) = 0.4$$

$$p(Z_t = \text{sense_open} | X_t = \text{is_closed}) = 0.2$$

$$p(Z_t = \text{sense_closed} | X_t = \text{is_closed}) = 0.8$$

Simple example – Transition Model

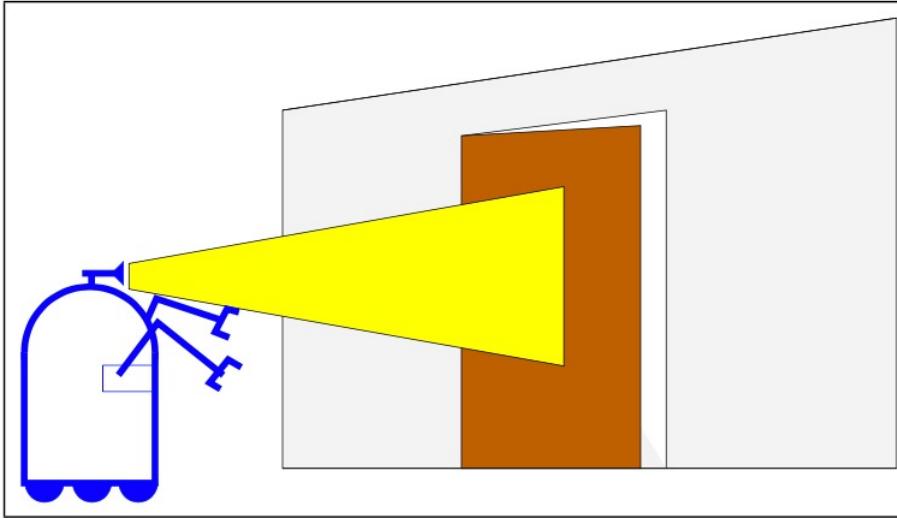


Figure 2.2 A mobile robot estimating the state of a door.

$$\begin{array}{lll} p(X_t = \text{is_open} | U_t = \text{push}, X_{t-1} = \text{is_open}) & = & 1 \\ p(X_t = \text{is_closed} | U_t = \text{push}, X_{t-1} = \text{is_open}) & = & 0 \\ p(X_t = \text{is_open} | U_t = \text{push}, X_{t-1} = \text{is_closed}) & = & 0.8 \\ p(X_t = \text{is_closed} | U_t = \text{push}, X_{t-1} = \text{is_closed}) & = & 0.2 \end{array} \quad \begin{array}{lll} p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) & = & 1 \\ p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) & = & 0 \\ p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) & = & 0 \\ p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) & = & 1 \end{array}$$

The Bayes Filter - Derivation

- Bayes Rule

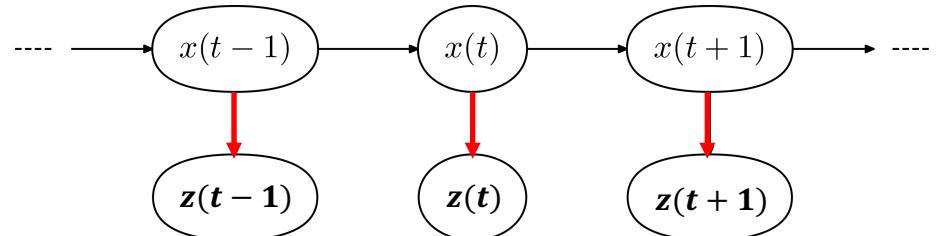
$$p(a, b) = p(a|b)p(b) = p(b|a)p(a)$$

$$p(a|b) = \frac{p(b|a)p(a)}{p(b)}$$

$$p(x_t \mid z_{1:t}, u_{1:t}) = \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \text{ Normalization}$$

The Bayes Filter - Derivation

- State is complete



$$p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) = p(z_t \mid x_t)$$

- Simplify

$$\begin{aligned} p(x_t \mid z_{1:t}, u_{1:t}) &= \frac{p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t})}{p(z_t \mid z_{1:t-1}, u_{1:t})} \\ &= \eta p(z_t \mid x_t, z_{1:t-1}, u_{1:t}) p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \eta \boxed{p(z_t \mid x_t)} p(x_t \mid z_{1:t-1}, u_{1:t}) \end{aligned}$$

new

The Bayes Filter - Derivation

$$p(x_t \mid z_{1:t}, u_{1:t}) = \eta p(z_t \mid x_t) p(x_t \mid z_{1:t-1}, u_{1:t})$$

$$bel(x_t) = \eta p(z_t \mid x_t) \overline{bel}(x_t)$$

This still depends on entire history

```
1: Algorithm Bayes_filter( $bel(x_{t-1})$ ,  $u_t$ ,  $z_t$ ):  
2:   for all  $x_t$  do  
3:      $\overline{bel}(x_t) = \int p(x_t \mid u_t, x_{t-1}) bel(x_{t-1}) dx$   
4:      $bel(x_t) = \eta p(z_t \mid x_t) \overline{bel}(x_t)$  Update Step  
5:   endfor  
6:   return  $bel(x_t)$ 
```



Measurement Model

The Bayes Filter - Derivation

$$bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$$

- Total probability $p(a) = \int p(a|b)p(b)db$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t | z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} | z_{1:t-1}, u_{1:t}) dx_{t-1}\end{aligned}$$

- State is complete

Previous Belief over x

$$\underbrace{p(x_t | x_{t-1}, z_{1:t-1}, u_{1:t})}_{=} = p(x_t | x_{t-1}, u_t)$$

The Bayes Filter - Derivation

$$p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) = p(x_t \mid x_{t-1}, u_t)$$

$$\begin{aligned}\overline{bel}(x_t) &= p(x_t \mid z_{1:t-1}, u_{1:t}) \\ &= \int p(x_t \mid x_{t-1}, z_{1:t-1}, u_{1:t}) p(x_{t-1} \mid z_{1:t-1}, u_{1:t}) dx_{t-1}\end{aligned}$$

$$\overline{bel}(x_t) = \int p(x_t \mid x_{t-1}, u_t) p(x_{t-1} \mid z_{1:t-1}, u_{1:t-1}) dx_{t-1}$$

```
1: Algorithm Bayes filter( $bel(x_{t-1})$ ,  $u_t$ ,  $z_t$ ): Transition/Dynamics model
2:   for all  $x_t$  do
3:      $\overline{bel}(x_t) = \int [p(x_t \mid u_t, x_{t-1})] bel(x_{t-1}) dx$  Predict Step
4:      $bel(x_t) = \eta p(z_t \mid x_t) bel(x_t)$ 
5:   endfor
6:   return  $bel(x_t)$ 
```

Limitations

1. $p(x)$ is defined $\forall x$ – intractable
 - Discrete and small spaces
 - Continuous and/or large spaces – Moments,
Finite # of samples
2. The integral term -> costly to compute

Re-Iterate Example

- Is door open or not?

$$bel(X_0 = \text{open}) = 0.5$$

$$bel(X_0 = \text{closed}) = 0.5$$

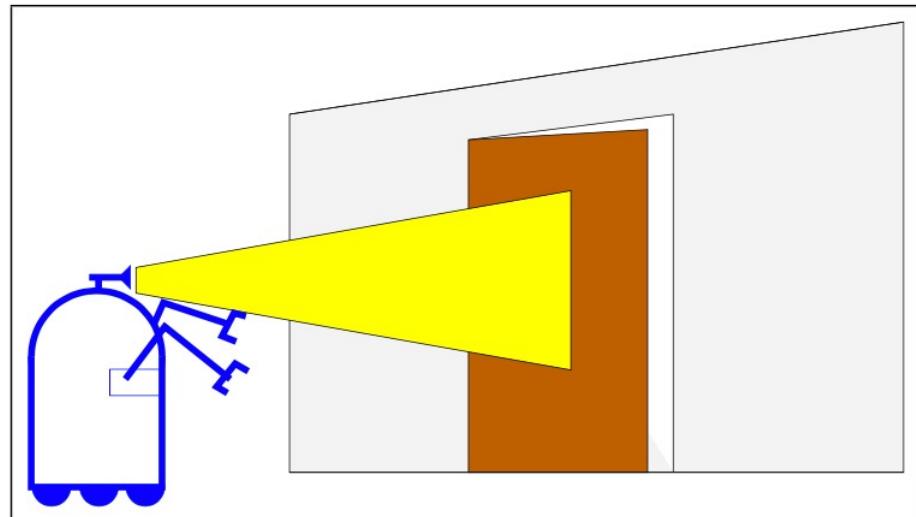


Figure 2.2 A mobile robot estimating the state of a door.

Measurement Model

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$$p(Z_t = \text{sense_closed} | X_t = \text{is_closed}) = 0.8$$

Transition Model for do_nothing

$$p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) = 1$$

$$p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) = 0$$

$$p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) = 0$$

$$p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) = 1$$

Re-Iterate Example

- Is door open or not?

$$bel(X_0 = \text{open}) = 0.5$$

$$bel(X_0 = \text{closed}) = 0.5$$

Received Sensor Measurement:

$$bel(x_1) = \eta p(Z_1 = \text{sense_open} | x_1) \overline{bel}(x_1)$$

Measurement Model

$$p(Z_t = \text{sense_open} | X_t = \text{is_open}) = 0.6$$

$$p(Z_t = \text{sense_closed} | X_t = \text{is_open}) = 0.4$$

$$p(Z_t = \text{sense_open} | X_t = \text{is_closed}) = 0.2$$

$$p(Z_t = \text{sense_closed} | X_t = \text{is_closed}) = 0.8$$

$$bel(x_t) = (0.75, 0.25) \quad \text{Don't forget normalization!}$$

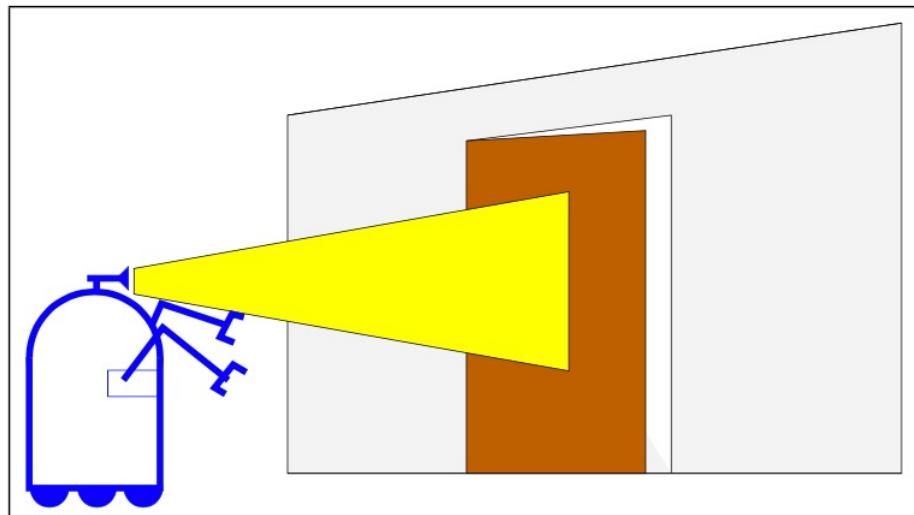


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Transition Model for do_nothing

$$p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) = 1$$

$$p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_open}) = 0$$

$$p(X_t = \text{is_open} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) = 0$$

$$p(X_t = \text{is_closed} | U_t = \text{do_nothing}, X_{t-1} = \text{is_closed}) = 1$$

The Bayes Filter

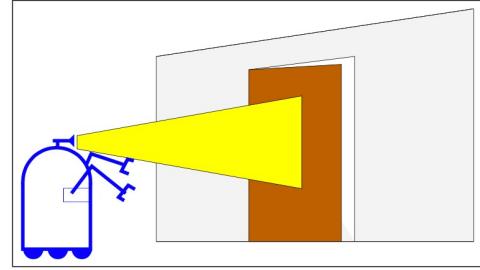


Figure 2.2 A mobile robot estimating the state of a door.

- Recursive filter for estimating x_t only from x_{t-1}, z_t and u_t and not from the ever-growing history $z_{1:t}, u_{1:t}$

```
1: Algorithm Bayes_filter( $bel(x_{t-1})$ ,  $u_t$ ,  $z_t$ ): Transition/Dynamics model
2:   for all  $x_t$  do
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4:      $bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$  Update Step
5:   endfor
6:   return  $bel(x_t)$ 
```

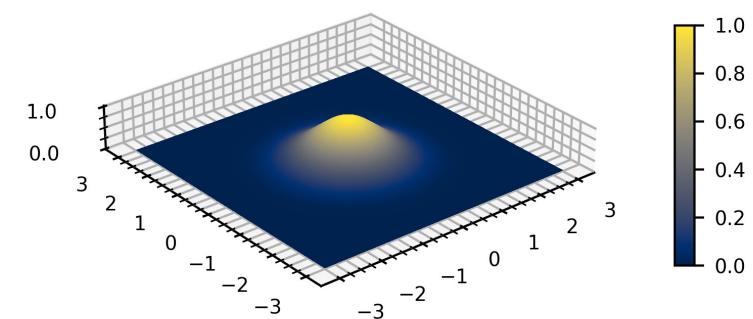
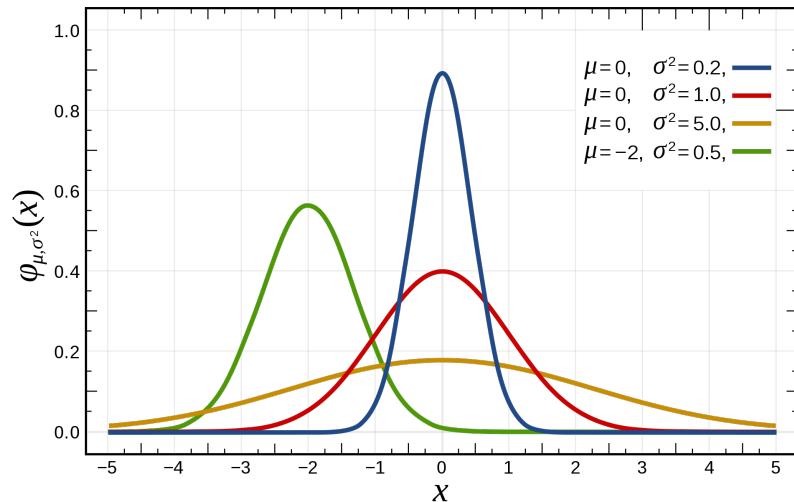
Annotations:

- A purple box highlights the term $p(x_t | u_t, x_{t-1})$ in step 3, with a purple arrow pointing to it from the text "Transition/Dynamics model".
- A green box highlights the term $p(z_t | x_t)$ in step 4, with a green arrow pointing to it from the text "Measurement Model".

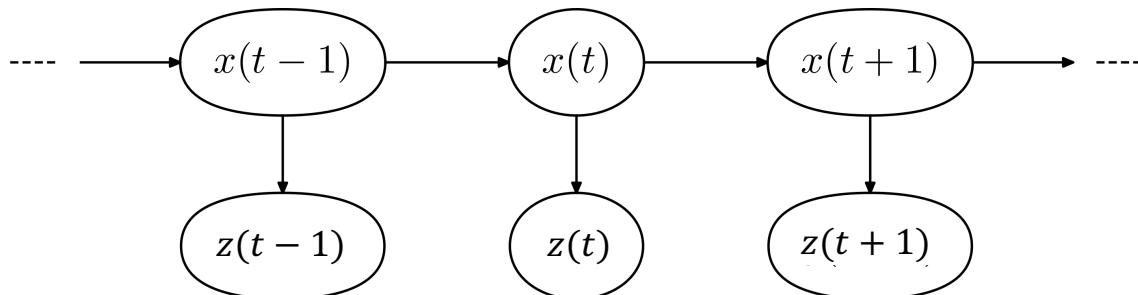
Gaussian Filters - Kalman Filter

$$\mathbf{x} \sim N(\mu, \Sigma)$$

$$p(x) = \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x - \mu)^T \Sigma^{-1} (x - \mu)\right\}$$



Kalman Filter



- Gaussian Belief
- Linear Transition Model

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

Process Noise $\varepsilon \sim N(0, R)$

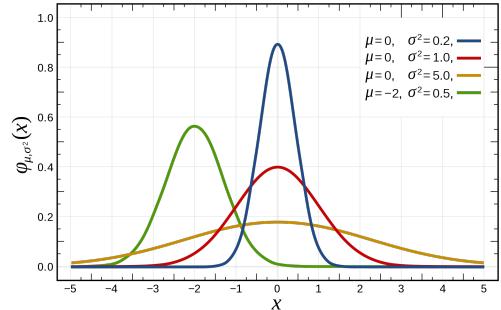
- Linear Measurement Model

$$z_t = C_t x_t + \delta_t$$

Measurement Noise $\delta \sim N(0, Q)$

$$x_t = \begin{pmatrix} x_{1,t} \\ x_{2,t} \\ \vdots \\ x_{n,t} \end{pmatrix} \quad u_t = \begin{pmatrix} u_{1,t} \\ u_{2,t} \\ \vdots \\ u_{m,t} \end{pmatrix}$$

Kalman Filter



- Initial Belief $\mathbf{x}_0 \sim N(\mu_0, \Sigma_0)$

$$bel(x_0) = p(x_0) = \det(2\pi\Sigma_0)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_0 - \mu_0)^T \Sigma_0^{-1} (x_0 - \mu_0)\right\}$$

- Distribution over next state

$$p(x_t | u_t, x_{t-1}) = \det(2\pi R_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(x_t - \underbrace{A_t x_{t-1} - B_t u_t}_{\text{Transition Model}})^T R_t^{-1} (x_t - \underbrace{A_t x_{t-1} - B_t u_t}_{\text{Transition Model}})\right\}$$

Process Noise

- Likelihood of Measurement

$$p(z_t | x_t) = \det(2\pi Q_t)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(z_t - \underbrace{C_t x_t}_{\text{Measurement Model}})^T Q_t^{-1} (z_t - \underbrace{C_t x_t}_{\text{Measurement Model}})\right\}$$

Measurement Noise

The Kalman Filter Algorithm

```
1: Algorithm Kalman filter( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):  
2:    $\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$   
3:    $\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$   
4:    $K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$   
5:    $\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$   
6:    $\Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$   
7:   return  $\mu_t, \Sigma_t$ 
```

Uncertainty increases
 $K = \text{Kalman Gain}$ $K \approx \frac{R}{Q}$

Uncertainty decreases

```
1: Algorithm Bayes filter( $bel(x_{t-1}), u_t, z_t$ ):  
2:   for all  $x_t$  do  
3:      $\overline{bel}(x_t) = \int p(x_t | u_t, x_{t-1}) bel(x_{t-1}) dx$    Predict Step  
4:      $bel(x_t) = \eta p(z_t | x_t) \overline{bel}(x_t)$    Update Step  
5:   endfor  
6:   return  $bel(x_t)$ 
```

If R large, then K is large.
Update dominated by innovation.

If Q large, then K is small.
Update dominated by prediction.

Example

$p(x_0)$

Measurement

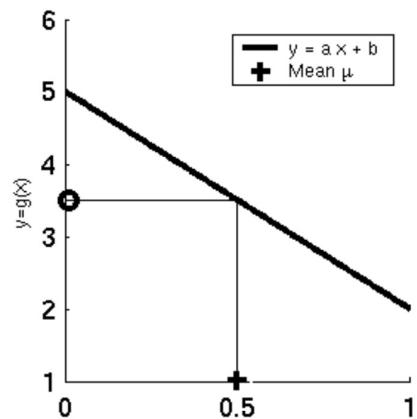
After Update

After Prediction

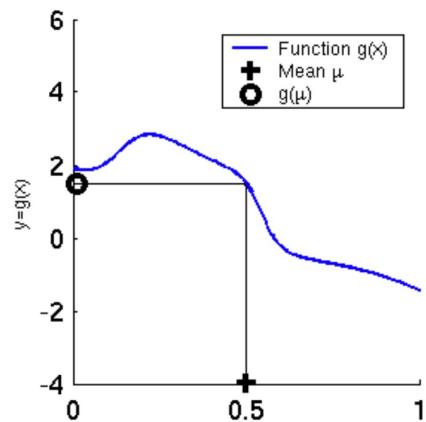
Measurement

After Update

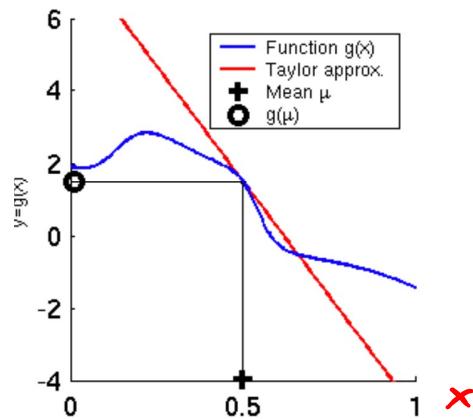
Propagating a Gaussian through a Linear Model



Propagating a Gaussian through a Non-Linear Model



Linearizing the Non-Linear Model



Extended Kalman filter - Process Model

$$x_t = g(u_t, x_{t-1}) + \varepsilon_t \quad \textbf{Process Model}$$

$$z_t = h(x_t) + \delta_t . \quad \textbf{Measurement Model}$$

First order Taylor Expansion – linear approximation around value and slope

$$g'(u_t, x_{t-1}) := \frac{\partial g(u_t, x_{t-1})}{\partial x_{t-1}} \quad \textbf{Gradient of Nonlinear function
around } x_{t-1}$$

$$\begin{aligned} g(u_t, x_{t-1}) &\stackrel{\text{Taylor Expansion}}{\approx} g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=: G_t} (x_{t-1} - \mu_{t-1}) \\ &= g(u_t, \mu_{t-1}) + \underline{G_t (x_{t-1} - \mu_{t-1})} \end{aligned}$$

Jacobian

Extended Kalman filter - Process Model

$$\begin{aligned} g(u_t, x_{t-1}) &\approx g(u_t, \mu_{t-1}) + \underbrace{g'(u_t, \mu_{t-1})}_{=: G_t} (x_{t-1} - \mu_{t-1}) \\ &= g(u_t, \mu_{t-1}) + G_t (x_{t-1} - \mu_{t-1}) \end{aligned}$$

Same equations as in previous slide

Written as Gaussian:

$$\begin{aligned} p(x_t | u_t, x_{t-1}) & \\ \approx \det(2\pi R_t)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} [x_t - g(u_t, \mu_{t-1}) - G_t (x_{t-1} - \mu_{t-1})]^T \right. & \\ \left. R_t^{-1} [x_t - g(u_t, \mu_{t-1}) - G_t (x_{t-1} - \mu_{t-1})] \right\} & \end{aligned}$$

Extended Kalman Filter – Measurement Model

$$x_t = g(u_t, x_{t-1}) + \varepsilon_t \quad \textbf{Process Model}$$

$$z_t = h(x_t) + \delta_t . \quad \textbf{Measurement Model}$$

First order Taylor Expansion – linear approximation around value and slope

$$\begin{aligned} h(x_t) &\approx h(\bar{\mu}_t) + \underbrace{h'(\bar{\mu}_t)}_{=: H_t} (x_t - \bar{\mu}_t) \\ &= h(\bar{\mu}_t) + \underline{H_t (x_t - \bar{\mu}_t)} \end{aligned}$$

Jacobian

Written as Gaussian:

$$\begin{aligned} p(z_t | x_t) &= \det(2\pi Q_t)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} [z_t - h(\bar{\mu}_t) - H_t (x_t - \bar{\mu}_t)]^T \right. \\ &\quad \left. Q_t^{-1} [z_t - h(\bar{\mu}_t) - H_t (x_t - \bar{\mu}_t)] \right\} \end{aligned}$$

The Extended Kalman Filter Algorithm

```
1: Algorithm Extended Kalman filter( $\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$ ):  
2:    $\bar{\mu}_t = g(u_t, \mu_{t-1})$                                 Predict  
3:    $\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t$   
4:    $K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1}$   
5:    $\mu_t = \bar{\mu}_t + K_t(z_t - h(\bar{\mu}_t))$                 Update  
6:    $\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t$   
7:   return  $\mu_t, \Sigma_t$ 
```

	Kalman filter	EKF
state prediction (Line 2)	$A_t \mu_{t-1} + B_t u_t$	$g(u_t, \mu_{t-1})$
measurement prediction (Line 5)	$C_t \bar{\mu}_t$	$h(\bar{\mu}_t)$

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Introduction to Computer Vision



Next lecture:

Optimal Estimation cont'