Power Method for Dominant Eigenvalue

Let λ_1, λ_2, λ_3, and λ_n be the eigenvalues of an **n X n** matrix **A**. λ_1 is called the dominant eigenvalue of **A** if

$|\lambda_1| > |\lambda_i|, \quad i = 2, 3, \ldots, n$

The eigenvectors corresponding to λ_1 are called dominant eigenvectors of **A**.

Procedure

1. Choose an **n X n** matrix
   Otherwise-Pop up error – the number of rows and columns should be the same (or matrix dimension mismatched)

2. Like the Jacobi and Gauss-Seidel methods, the power method for approximating eigenvalues is iterative. First, we assume that matrix **A** has a dominant eigenvalue with corresponding dominant eigenvectors. Then we choose an initial approximation $\mathbf{x_0}$ of one of the dominant eigenvectors of **A**. This initial approximation must be a nonzero vector in R^n

   Finally, we form the sequence given by
   $\mathbf{x_1 = Ax_0}$
   $\mathbf{x_2 = Ax_1 = A(Ax_0) = A^2x_0}$
   $\mathbf{x_3 = Ax_2 = A(A^2x_0) = A^3x_0}$
   …
   $\mathbf{x_n = Ax_{n-1} = A(A^{n-1}x_0) = A^nx_0}$
   *(In the above, $\mathbf{x_1}$ denotes the value of vector **x** at the first iteration and so on)*

   Compare the updated value of **x** with its previous value (obtained from the previous iteration)

   For large powers of k, and by properly scaling this sequence, we will see that we obtain a good approximation of the dominant eigenvector of **A**.

3. Repeat the iteration process until convergence

4. **The formula for finding the corresponding eigenvalue from eigenvector x.**
   If **x** is an eigenvector of **A**, then its corresponding eigenvalue is given by

   λ = (**Ax.x / x.x**)

5. If they do not converge even after many iterations (maybe after 1000 iterations), then show pop-up message
   Otherwise-Pop up error – Entered matrix has no dominant eigenvalue

Example

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

We begin with an initial nonzero approximation of

$$\mathbf{x_0} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We then obtain the following approximations

$$\mathbf{x_1} = \mathbf{Ax_0} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10 \\ -4 \end{bmatrix} = -4\begin{bmatrix} 2.50 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x_2} = \mathbf{Ax_1} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -10 \\ -4 \end{bmatrix} = \begin{bmatrix} 28 \\ 10 \end{bmatrix} = 10\begin{bmatrix} 2.80 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x_3} = \mathbf{Ax_2} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 28 \\ 10 \end{bmatrix} = \begin{bmatrix} -64 \\ -22 \end{bmatrix} = -22\begin{bmatrix} 2.91 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x_4} = \mathbf{Ax_3} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -64 \\ -22 \end{bmatrix} = \begin{bmatrix} 136 \\ 46 \end{bmatrix} = 46\begin{bmatrix} 2.96 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x_5} = \mathbf{Ax_4} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 136 \\ 46 \end{bmatrix} = \begin{bmatrix} -280 \\ -94 \end{bmatrix} = -94\begin{bmatrix} 2.98 \\ 1.00 \end{bmatrix}$$

$$\mathbf{x_6} = \mathbf{Ax_5} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}\begin{bmatrix} -280 \\ -94 \end{bmatrix} = \begin{bmatrix} 568 \\ 190 \end{bmatrix} = 190\begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix}$$

Note that the approximations in Example appear to be approaching scalar multiples of $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$

So, the obtained dominant eigenvector from the above iterations is

$$\mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Now, we'll find the corresponding eigenvalue from the obtained eigenvector

Formula

If **x** is an eigenvector of **A**, then its corresponding eigenvalue is given by

$\lambda = (\mathbf{Ax.x} / \mathbf{x.x})$

$\mathbf{Ax} = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix}$

Then, $\mathbf{Ax.x} = \begin{bmatrix} -6.02 \\ -2.01 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = -20.0$ (approx.)

And $\mathbf{x.x} = \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = \begin{bmatrix} 2.99 \\ 1.00 \end{bmatrix} = 9.94$ (approx.)

So, the corresponding eigenvalue, $\lambda = (-20.0 / 9.94) = -2$ (approx.)

Some more discussion about eigenvalue & eigenvector

Let's assume a square matrix **A**

The characteristic equation,

$| \mathbf{A} - \lambda * \mathbf{I} | = 0$

*(where **I** is an identity matrix)*

After calculating the values of λs we attempt to find eigenvectors for corresponding eigenvalues like this

For eigenvalue, $\lambda = \lambda_1$

$\mathbf{A} * \mathbf{x} = \lambda_1 * \mathbf{I} * \mathbf{x}$ *(where, **x** is an unknown vector)*

Or, $(\mathbf{A} - \lambda_1 * \mathbf{I}) * \mathbf{x} = 0$

The value of **x** is the corresponding eigenvector of λ_1

We've already discussed the topic of 'eigenvalue and eigenvector' with the proper example in 'Jordan Decomposition'. Please go thru it if you want.