Cholesky Decomposition

Procedures

1. Choose an **n X n** matrix
   Otherwise-Pop up error – select number of rows and Columns should be same (or matrix dimension mismatched)

2. The chosen matrix should be symmetric

   Let's assume, **A** = $\begin{bmatrix} a1 & a2 & a3 \\ a2 & a4 & a5 \\ a3 & a5 & a6 \end{bmatrix}$
   a matrix **A** is symmetric if
   (a) **A** is a square matrix
   (b) and **A = A^T^**

   Otherwise-Pop up error – entered matrix should be symmetric

3. The chosen matrix should be positive definite as well.
   a matrix **A** is positive definite if **determinants of all upper-left sub-matrices are positive** or,

   $|a1| > 0$

   $\begin{vmatrix} a1 & a2 \\ a2 & a4 \end{vmatrix} > 0$

   $\begin{vmatrix} a1 & a2 & a3 \\ a2 & a4 & a5 \\ a3 & a5 & a6 \end{vmatrix} > 0$

   Otherwise-Pop up error – entered matrix should be positive definite

4. Every symmetric positive definite matrix **A** can be decomposed into a product of a unique lower triangular matrix **L** and its transpose.

$$\mathbf{A = L*L^T}$$

5. There is a clear-cut formula to find the elements of matrix **L**

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}}{l_{ii}}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

Where, for example, l_{ki} denotes the matrix **L**'s entry/element in k^{th} row and i^{th} column.

l_{kk} denotes the main diagonal elements of matrix **L**

Example:

Given matrix, **A** = $\begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$

The matrix **A** is symmetric as $\mathbf{A = A^T}$

Now, we'll check given matrix **A** is positive definite or not.

$|4| > 0$

$\begin{vmatrix} 4 & 2 \\ 2 & 17 \end{vmatrix} = 68 - 4 = 64 > 0$

$\begin{vmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{vmatrix} = 1600 > 0$

So, the matrix is positive definite as well.

Any symmetric positive definite matrix **A** can be decomposed into a product of a unique lower triangular matrix **L** and its transpose or,

$$\mathbf{A = L*L^T}$$

Formula

$$l_{ki} = \frac{a_{ki} - \sum_{j=1}^{i-1} l_{ij} \cdot l_{kj}}{l_{ii}}$$

$$l_{kk} = \sqrt{a_{kk} - \sum_{j=1}^{k-1} l_{kj}^2}$$

$l_{11} = \sqrt{4} = 2$

$l_{21} = a_{21} / l_{11} = 2/2 = 1$

$l_{22} = \sqrt{(a_{22} - (l_{21})^2)} = \sqrt{(17 - 1)} = 4$

$l_{31} = a_{31} / l_{11} = 14/2 = 7$

$l_{32} = (a_{32} - l_{31} * l_{21}) / l_{22} = (-5 - (7)*(1)) / 4 = -3$

$l_{33} = \sqrt{(a_{33} - (l_{31})^2 - (l_{32})^2)} = \sqrt{(83 - (7)^2 - (-3)^2)} = 5$

Now, from the solutions of the aforementioned equations we obtain the matrix **L**.

$$\mathbf{L} = \begin{bmatrix} l11 & 0 & 0 \\ l21 & l22 & 0 \\ l31 & l32 & l33 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix}$$

And $\mathbf{L}*\mathbf{L}^{\mathbf{T}} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{bmatrix} * \begin{bmatrix} 2 & 1 & 7 \\ 0 & 4 & -3 \\ 0 & 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix} = \mathbf{A}$

Some more discussion about Cholesky decomposition (Extra)

a square matrix **A** can be factorized as

A = L*U

Where **L** is a lower triangular matrix and **U** is an upper triangular matrix. We've already discussed in detail about '**LU decomposition**'

For a symmetric and positive definite matrix

U = L^{T}

So, in this case, **A = L*U = L * L^{T}**

For a system of linear equations

Ax = b

Or, **L* L^{T} * x = b**

Let, **L^{T} * x = y**

Then, **L * y = b**

Now, one need to find the value of **L** by calculating

L * L^{T} = A

Or, $\begin{bmatrix} l11 & 0 & 0 \\ l21 & l22 & 0 \\ l31 & l32 & l33 \end{bmatrix} * \begin{bmatrix} l11 & l21 & l31 \\ 0 & l22 & l23 \\ 0 & 0 & l33 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{bmatrix}$

See above to find the solution of matrix **L**. This is already discussed as a formula.

Once, one has solved **y** from **L*y = b**

Then solve **x** from '**L^{T} * x = y**'

In the above, we've discussed the whole procedure of finding matrix **L** from a given matrix.