Jordan Decomposition

The goal of a Jordan decomposition is to diagonalize a given square matrix. If there is an invertible n×n matrix C and a diagonal matrix D such that A=CDC-1, then an n×n matrix A is diagonalizable.

Procedure-

1. Choose a square matrix (**m X m)** (e.g., 3 X 3, 4 X 4, 5 X 5, etc.,)
   Otherwise-Pop up error – select number of rows and Columns should be same (or matrix dimension mismatched)

2. For a given matrix,

   $$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

   The aim of Jordan decomposition is to diagonalize a given square matrix A, if $A=PDP^{-1}$ is possible, where P is an invertible matrix and D is diagonal matrix. We'll go into the specifics of how matrix P and matrix D are formed later. Matrix P and D are derived from matrix A.

3. Firstly, we'll find the eigen values of the matrix A

   $| A - \lambda{*}I | = 0$ *(I = identity matrix)*

   $$\text{Or, } \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \lambda * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

Or, $\begin{bmatrix} (\mathbf{2}-\lambda) & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & (\mathbf{2}-\lambda) & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & (\mathbf{2}-\lambda) \end{bmatrix} = 0$

Or, $(2-\lambda)\ ((2-\lambda) \times (2-\lambda)-1\times1)-1(1\times (2-\lambda)-1\times0) +0(1\times1-(2-\lambda) \times0) = 0$

Or,$(2-\lambda)((4-4\lambda+\lambda2)-1)-1((2-\lambda)-0)+0(1-0)=0$

Or,$(2-\lambda)(3-4\lambda+\lambda2)-1(2-\lambda)+0(1)=0$

Or,$(6-11\lambda+6\lambda2-\lambda3)-(2-\lambda)+0=0$

Or,$(-\lambda3+6\lambda2-10\lambda+4)=0$

Or,$-(\lambda-2)(\lambda-0.5858)(\lambda-3.4142)=0$

Or,$(\lambda-2)=0$or$(\lambda-0.5858)=0$or$(\lambda-3.4142)=0$

So, The eigenvalues of the matrix A are given by λ=0.5858,2,3.4142
You can apply Newton Raphson method to find a good approximation for the **root** of a real-valued function. You can use this method here to find the eigen values (or, λ's)

Please read through the matrix's minor and co-factor in to understand the finding of the determinant value in step 3. I've already written an article regarding minors of a matrix.

4. Now, calculate the eigen vectors from the corresponding eigen values.

In our case, eigen values are 0.5858, 2, 3.4142

For, $\lambda = 0.5858$

A - $\lambda * \mathrm{I}$

= A - $0.5858 * \mathrm{I}$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} - 0.5858 * \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

=

$$\begin{bmatrix} 1.4142 & 1 & 0 \\ 1 & 1.4142 & 1 \\ 0 & 1 & 1.4142 \end{bmatrix}$$

Now, do row operations to reduce the matrix

Now, reduce this matrix

R1←R1÷1.4142

$$= \begin{bmatrix} 1 & 0.7071 & 0 \\ 1 & 1.4142 & 1 \\ 0 & 1 & 1.4142 \end{bmatrix}$$

R2←R2-R1

$$= \begin{bmatrix} 1 & 0.7071 & 0 \\ 0 & 0.7071 & 1 \\ 0 & 1 & 1.4142 \end{bmatrix}$$

Interchanging rows R2↔R3

$$= \begin{bmatrix} 1 & 0.7071 & 0 \\ 0 & 1 & 1.4142 \\ 0 & 0.7071 & 1 \end{bmatrix}$$

R1←R1-0.7071×R2

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.4142 \\ 0 & 0.7071 & 1 \end{bmatrix}$$

R3←R3 - 0.7071×R2

$$= \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.4142 \\ 0 & 0 & 0 \end{bmatrix}$$

One can calculate row echelon form to reduce a matrix

Now, compute

A*x - λI*x = 0

Or, (A - λI)x = 0

Or, (A - 0.5858 * I)x = 0

Or, $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1.4142 \\ 0 & 0 & 0 \end{bmatrix} * \begin{bmatrix} x1 \\ x2 \\ x3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Or, $x1$-$x3$=0, $x2$+1.4142$x3$=0

Or, $x1$=$x3$, $x2$=-1.4142$x3$

Now, for eigen value, $\lambda = 0.5858$, corresponding eigen vector is

$$v_1 = \begin{bmatrix} x3 \\ -1.4142x3 \\ x3 \end{bmatrix}$$

let x3 = 1

$$v_1 = \begin{bmatrix} 1 \\ -1.4142 \\ 1 \end{bmatrix}$$

We found the eigen vector for the eigen value, =0.5858, only in step 4 above. The same method may be used to calculate the eigen vectors for λ=2 and 3.4142.

Corresponding eigen vectors for eigen values 2 & 3.4142 are

$$v_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$v_3 = \begin{bmatrix} 1 \\ 1.4142 \\ 1 \end{bmatrix}$$ respectively.

5. To allow diagonalization, the number of eigenvectors must be equal the given square matrix's dimension.

   If the number of eigenvalues is less than the dimension of the given square matrix, a matrix cannot be diagonalized and show pop-up error.

Pop up error – 'not diagonalizable!'

6. Now, initialize the P matrix. P matrix columns are formed from the eigen vectors derived from the eigen values 0.5858, 2, and 3.4142 or they are

[$\mathbf{v_1}$ $\mathbf{v_2}$ $\mathbf{v_3}$]

$$\text{or, } P = \begin{bmatrix} 1 & -1 & 1 \\ -1.4142 & 0 & 1.4142 \\ 1 & 1 & 1 \end{bmatrix}$$

7. The diagonal matrix (D) of the above matrix A contains the eigen values of matrix A as the following diagonal elements:

$$D = \begin{bmatrix} 0.5858 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3.4142 \end{bmatrix}$$

8. Now the final step is to check whether the matrix P is invertible or not. If matrix P is not invertible then display the pop-up notification
Pop up error – 'not diagonalizable!'

The values of matrices A, P, and P^{-1} will only be displayed if matrix P is invertible.