LU Decomposition using Doolittle Factorization

We can write an **m X n** matrix **A** as a product of two matrices, **L** and **U**. And **A = L*U**

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l21 & 1 & 0 & 0 \\ l31 & l32 & 1 & 0 \\ l41 & l42 & l43 & 1 \end{bmatrix} ; \quad \mathbf{U} = \begin{bmatrix} u11 & u12 & u13 & u14 \\ 0 & u22 & u23 & u24 \\ 0 & 0 & u33 & u34 \\ 0 & 0 & 0 & u44 \end{bmatrix}$$

L= lower triangular matrix;   U= upper triangular matrix

Procedure-

1. Choose a matrix (**m X n)** (e.g., 3X 3, 3 X 4, 4 X 4, etc.,)

2. Initialize the **L** and **U** matrices. For L matrix, take a matrix with all diagonal elements assigned to 1, and the matrix elements above the diagonal are zeroes. L matrix size will be (**m X m**). The values of matrix elements below the main diagonal can be assigned to l_{21}, l_{31}, etc., and so on.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ l21 & 1 & 0 & 0 \\ l31 & l32 & 1 & 0 \\ l41 & l42 & l43 & 1 \end{bmatrix}$$

l_{21}, l_{31}, etc. are unknown

3. Size of matrix **U** will be as same as matrix **A (or, m X n)**. Matrix **U** can be written as

$$\begin{bmatrix} u11 & u12 & u13 & u14 \\ 0 & u22 & u23 & u24 \\ 0 & 0 & u33 & u34 \\ 0 & 0 & 0 & u44 \end{bmatrix}$$

u_{11}, u_{12}, u_{13}, u_{14}, u_{22}, etc. – all are unknown.

4. For a given example,

$$\mathbf{A} = \begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{-4} & \mathbf{6} & \mathbf{3} \\ \mathbf{-4} & \mathbf{-2} & \mathbf{8} \end{bmatrix}$$

Now, $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{-2} \\ \mathbf{-4} & \mathbf{6} & \mathbf{3} \\ \mathbf{-4} & \mathbf{-2} & \mathbf{8} \end{bmatrix}$ = **L * U** = $\begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \boldsymbol{l21} & \mathbf{1} & \mathbf{0} \\ \boldsymbol{l31} & \boldsymbol{l32} & \mathbf{1} \end{bmatrix} * \begin{bmatrix} \boldsymbol{u11} & \boldsymbol{u12} & \boldsymbol{u13} \\ \mathbf{0} & \boldsymbol{u22} & \boldsymbol{u23} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{u33} \end{bmatrix}$

Next, resolve the basic equations, $u_{11} = 2$, $u_{12} = -1$, $u_{13} = -2$

To get the above equations, take the sum of the multiplications of the elements in row 1 of matrix **L** and the elements in column 1 of matrix **U** to obtain the equations above. Then compare the sum with corresponding element of matrix **A**.

Then follow the same process for matrix **L's** row 1 and matrix **U's** column 2, then matrix **L's** row 1 and matrix **U's** column 3.

Then follow the same procedure for matrix **L's** row 2 and 3

Apply the same procedure to the first row of matrix **L** and the second column of matrix **U**, then to the first row of matrix **L** and third column of matrix **U**.

Then repeat these steps for rows 2 and 3 of matrix **L**.

$l_{21}*u_{11} = -4$

$l_{21}*u_{12} + u_{22} = 6$

$l_{21}*u_{13} + l_{23} =3$

$l_{31} * u_{11} = -4$

$l_{31} * u_{12} + l_{32} * u_{22} = -2$

$l_{31} * u_{13} + l_{32} * u_{23} + u_{33} = 8$

We obtain matrices **L** and **U** by solving the aforementioned equations.