Singular Value Decomposition (SVD)

Theory:

Singular Value Decomposition (SVD) is a matrix factorization technique that decomposes any m×n matrix A into three matrices: $A=U\Sigma V^T$

Where:

- U is an m×m orthogonal matrix (or unitary if complex).
- Σ is m×n diagonal matrix with non-negative real numbers on the diagonal (singular values).
- V^T is an n×n orthogonal matrix (or unitary if complex), and V^T is the transpose of V.

Example

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Compute A^TA and AA^T

$$A^TA = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 10 & 14 \\ 14 & 20 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 11 \\ 11 & 25 \end{bmatrix}$$

Find Eigenvalues and Eigenvectors:

For A^TA:

- Eigenvalues are $\lambda 1= 29.8661$ and $\lambda 2 = 0.1339$
- Corresponding eigenvectors (normalized) are $v1 = \begin{bmatrix} -0.5760 \\ -0.8174 \end{bmatrix}$ and $v2 = \begin{bmatrix} 0.8174 \\ -0.5760 \end{bmatrix}$

For AA^T:

- Eigenvalues are $\lambda 1= 29.8661$ and $\lambda 2 = 0.1339$

- Corresponding eigenvectors (normalized) are u1 = $\begin{bmatrix} -0.4046 \\ -0.9145 \end{bmatrix}$ and u2 = $\begin{bmatrix} 0.9145 \\ -0.4046 \end{bmatrix}$

Compute Singular Values:

Singular values are σ1 = √ 29.8661= 5.4650

And σ1 = √ 0.1339 = 0.3660

Final SVD:

$A=U\Sigma V^T$

Where:

U = $\begin{bmatrix} -0.4046 & 0.9145 \\ -0.9145 & -0.4046 \end{bmatrix}$; Σ = $\begin{bmatrix} 5.4650 & 0 \\ 0 & 0.3660 \end{bmatrix}$;

V^T = $\begin{bmatrix} -0.5760 & -0.8174 \\ 0.8174 & -0.5760 \end{bmatrix}$