Properties of Matrix Operations

Properties of Addition

The basic properties of addition for real numbers also hold true for matrices.

Let A, B and C be m x n matrices

1. $A + B = B + A$ commutative
2. $A + (B + C) = (A + B) + C$ associative
3. There is a unique m x n matrix O with

   $A + O = A$ additive identity

4. For any m x n matrix A there is an m x n matrix B (called -A) with

   $A + B = O$ additive inverse

Properties of Matrix Multiplication

Unlike matrix addition, the properties of multiplication of real numbers do not all generalize to matrices. Matrices rarely commute even if AB and BA are both defined. There often is no multiplicative inverse of a matrix, even if the matrix is a square matrix. There are a few properties of multiplication of real numbers that generalize to matrices. We state them now.

Let A, B and C be matrices of dimensions such that the following are defined. Then

1. $A(BC) = (AB)C$ associative
2. $A(B + C) = AB + AC$ distributive
3. $(A + B)C = AC + BC$ distributive
4. There are unique matrices I_m and I_n with

   $I_m A = A I_n = A$ multiplicative identity

Properties of Scalar Multiplication

Since we can multiply a matrix by a scalar, we can investigate the properties that this multiplication has. All of the properties of multiplication of real numbers generalize. In particular, we have

Let r and s be real numbers and A and B be matrices. Then

1. $r(sA) = (rs)A$
2. $(r + s)A = rA + sA$
3. $r(A + B) = rA + rB$
4. $A(rB) = r(AB) = (rA)B$

Properties of the Transpose of a Matrix

Recall that the transpose of a matrix is the operation of switching rows and columns. We state the following properties. We proved the first property in the last section.

Let r be a real number and A and B be matrices. Then

1. $(A^T)^T = A$
2. $(A + B)^T = A^T + B^T$
3. $(AB)^T = B^TA^T$
4. $(rA)^T = rA^T$

Properties of Determinants

1. **$\det(A) = \det(A^T)$**
2. If any row or column of a determinant, is multiplied by any scalar value, that is, a non-zero constant, the entire determinant gets multiplied by the same scalar, that is, if any row or column is multiplied by constant k, the determinant value gets multiplied by k.

$$\mathbf{det(\Delta') = k\, det(\Delta)}$$

3. If all elements of any column or row are zero, then the determinant is zero
4. If all the elements in the determinant above or below the diagonal are zero, then the determinant is a product of diagonal elements

Minor, Co-factor, Adjoint, Inverse

$[A]_{(i \times k)} \cdot [B]_{(k \times j)}$

$$= \begin{bmatrix} a11 & \cdots & a1n \\ a21 & \dots & a2n \\ \vdots & \ddots & \vdots \\ an1 & \cdots & ann \end{bmatrix} \begin{bmatrix} b11 & \cdots & b1n \\ b21 & \dots & b2n \\ \vdots & \ddots & \vdots \\ bn1 & \cdots & bnn \end{bmatrix}$$

$$= \begin{bmatrix} c11 & \cdots & c1n \\ c21 & \dots & c2n \\ \vdots & \ddots & \vdots \\ cn1 & \cdots & cnn \end{bmatrix} = [C]_{ij}$$

Where, $c_{ij} = \sum_1^k a_{ik}b_{kj}$

$$C^T = \begin{bmatrix} c11 & \cdots & cn1 \\ c12 & \dots & cn2 \\ \vdots & \ddots & \vdots \\ c1n & \cdots & cnn \end{bmatrix} = [C]_{ji}$$

Where, C^T is transpose of matrix C

1. a.)

$\mathbf{(A.B)^T = B^T.A^T}$

$(A.B)^T_{ij}$

$= (A.B)_{ji}$

$= a_{jk}b_{ki}$

$= b_{ki}a_{jk}$

$= (b^T)_{ik}\,(a^T)_{kj}$

$= (B^TA^T)_{ij}$

(1. b.)

$B^T.A^T = (A.B)^T$

$(B^TA^T)_{ij}$

$= (b^T)_{ik} \,.\, (a^T)_{kj}$

$= (b)_{ki} \,.\, (a)_{jk}$

$= (a)_{jk} \,.\, (b)_{ki}$

$= (AB)_{ji}$

$= (AB)^T_{ij}$

A **minor** of a matrix is a determinant of a smaller square matrix, which is formed by deleting one or more rows and/or columns from the original matrix. The concept of a minor is crucial in various areas of linear algebra, including in calculating determinants, cofactors, and in solving systems of linear equations.

Cofactor Matrix of a matrix A

$$[CO_{ij}] = (-1)^{(i+j)} * \begin{bmatrix} M11 & \cdots & M1n \\ M21 & \ldots & M2n \\ \vdots & \ddots & \vdots \\ Mn1 & \cdots & Mnn \end{bmatrix}$$

Where, $\begin{bmatrix} M11 & \cdots & M1n \\ M21 & \ldots & M2n \\ \vdots & \ddots & \vdots \\ Mn1 & \cdots & Mnn \end{bmatrix}$ *is minor matrix of A*

M_{ij} is the entry in the i^{th} row and j^{th} column

Adjoint of a matrix, A

$Adj(A) = [CO_{ij}]^T$

Now, the inverse of the matrix A

$A^{-1} = Adj(A) / det(A)$

Where 'det' denotes the determinant

2. $(AB)^{-1}=B^{-1}A^{-1}$

If A and B are invertible matrices or Both A and B are **n X n** square matrices and determinants are not zeroes then

$(AB)(AB)^{-1} = I$

Where I is the identity matrix of size **n X n**

Pre-multiply by A^{-1}

$(A^{-1}). (AB)(AB)^{-1} = A^{-1}I$

Or, $I.(B).(AB)^{-1} = A^{-1}$

Or, $(B).(AB)^{-1} = A^{-1}$

Pre-multiply by B^{-1}

$(B^{-1}). (B).(AB)^{-1} = B^{-1}A^{-1}$

Or, $I(AB)^{-1} = B^{-1}A^{-1}$

Or, $(AB)^{-1} = B^{-1}A^{-1}$

3. Adj(A . B) = Adj(B) . Adj(A)

$(AB)^{-1}=\text{adj}(AB)/\det(AB)$

Or, $\text{adj}(AB) = (AB)^{-1}\cdot\det(AB)$ . . . (1)

It is also known that, $(AB)^{-1}=B^{-1}A^{-1}$

And, $\det(AB)=\det(A)\cdot\det(B)$ . . . (2)

Also

$A^{-1}=\text{adj}(A)/\det(A)$

$B^{-1}=\text{adj}(B)/\det(B)$

Or, $\text{adj}(A)=A^{-1}\det(A)$

Or, $\text{adj}(B)=B^{-1}\det(B)$

$$\mathrm{adj}(B)\cdot\mathrm{adj}(A)=\det A\cdot\det B\cdot B^{-1}\cdot A^{-1} \qquad \ldots (3)$$

Putting (2) in equation (1)

$$\mathrm{adj}(AB)=\det(A)\cdot\det(B)\cdot B^{-1}\cdot A^{-1} \qquad \ldots (4)$$

From (3) and (4)

$$\mathrm{adj}(AB)=\mathrm{adj}(B)\cdot\mathrm{adj}(A)$$