# Chapter 2: Matrix Algebra*

## Matrix Addition

If $A$ and $B$ are matrices of the same size, their sum $A + B$ is the matrix formed by adding corresponding entries.

If $A = [a_{ij}]$ and $B = [b_{ij}]$, this takes the form

$$A + B = [a_{ij} + b_{ij}]$$

Note that addition is not defined for matrices of different sizes.

## Scalar Multiplication

If $A$ is any matrix and $k$ is any number, the scalar multiple $kA$ is the matrix obtained from $A$ by multiplying each entry of $A$ by $k$.

If $A = [a_{ij}]$, this is

$$kA = [ka_{ij}]$$

*Revised on 24 Feb 2020, the materials on matrix transformation are moved to Chapter 5 and the topics on Composition and LU-factorization are removed.

**Theorem 1**

Let $A$, $B$, and $C$ denote arbitrary $m \times n$ matrices where $m$ and $n$ are fixed. Let $k$ and $p$ denote arbitrary real numbers. Then

1. $A + B = B + A$.
2. $A + (B + C) = (A + B) + C$.
3. There is an $m \times n$ matrix 0, such that $0 + A = A$ for each $A$.
4. For each $A$ there is an $m \times n$ matrix, $-A$, such that $A + (-A) = 0$.
5. $k(A + B) = kA + kB$.
6. $(k + p)A = kA + pA$.
7. $(kp)A = k(pA)$.
8. $1A = A$. ■

## Transpose

If $A$ is an $m \times n$ matrix, the transpose of $A$, written $A^T$, is the $n \times m$ matrix whose rows are just the columns of $A$ in the same order.

$$\text{If } A = [a_{ij}], \text{ then } A^T = [a_{ji}].$$

**Theorem 2**

Let $A$ and $B$ denote matrices of the same size, and let $k$ denote a scalar.

1. If $A$ is an $m \times n$ matrix, then $A^T$ is an $n \times m$ matrix.
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A + B)^T = A^T + B^T$. ■

## Matrix Multiplication

**Definition** The product $AB$ of an $m \times n$ matrix $A$ and an $n \times k$ matrix $B$ is defined to be the $m \times k$ matrix $AB$ whose $(i, j)$-entry is the dot product of row $i$ of $A$ and column $j$ of $B$.

Define $I$ or $I_n$ as the identity matrix.

$$AI = A \quad \text{and} \quad IB = B$$

**Theorem 1**
Assume that $k$ is an arbitrary scalar and that $A$, $B$, and $C$ are matrices of sizes such that the indicated operations can be performed.

1. $IA = A, BI = B$.
2. $A(BC) = (AB)C$.
3. $A(B + C) = AB + AC; \quad A(B - C) = AB - AC$.
4. $(B + C)A = BA + CA; \quad (B - C)A = BA - CA$.
5. $k(AB) = (kA)B = A(kB)$.
6. $(AB)^T = B^T A^T$. ■

$$\boxed{\text{In general, } AB \neq BA}$$

Recall that

$$\begin{aligned} p \Leftrightarrow q &\equiv p \Rightarrow q \,\wedge\, q \Rightarrow p \\ &\equiv p \Rightarrow q \wedge \sim p \Rightarrow \sim q \end{aligned}$$

**Example 7** Show that $AB = BA$ if and only if $(A-B)(A+B) = A^2 - B^2$.
**Solution** Theorem 1 shows that the following always holds:

$$(A-B)(A+B) = A(A+B) - B(A+B) = A^2 + AB - BA - B^2.$$

Hence if $AB = BA$, then $(A-B)(A+B) = A^2 - B^2$ follows.

Conversely, if this last equation holds, then equation becomes

$$A^2 - B^2 = A^2 + AB - BA - B^2$$

This gives $0 = AB - BA$, and $AB = BA$ follows.

## Matrices and Linear Equations

**Theorem 2**
Suppose $X_1$ is a particular solution to $AX = B$. Then, every solution $X_2$ to $AX = B$ has the form

$$X_2 = X_0 + X_1$$

where $X_0$ is the solution of the associated homogeneous system $AX = 0$. ■

**Proof** Suppose $AX_1 = B$ and $AX_0 = 0$. Then

$$AX_1 + AX_0 = B + 0$$

$$A(X_1 + X_0) = B$$

Thus, from the above, any solution $X_2 = X_1 + X_0$. ■

The importance of Theorem 2 lies in the fact that sometimes a particular solution $X_1$ is easily found, and so the problem of finding all solutions is reduced to solving the associated homogeneous system.

**Example 9** Solve the homogeneous system $AX = 0$ where

$$A = \begin{bmatrix} 1 & -2 & 3 & -2 \\ -3 & 6 & 1 & 0 \\ -2 & 4 & 4 & -2 \end{bmatrix}$$

**Solution** The reduction of the augmented matrix to reduced form is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 3 & -2 & 0 \\ -3 & 6 & 1 & 0 & 0 \\ -2 & 4 & 4 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrrr|r} 1 & -2 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 & -\frac{3}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

so the solutions are $x_1 = 2s + \frac{1}{5}t$, $x_2 = s$, $x_3 = \frac{3}{5}t$, and $x_4 = t$ by gaussian elimination.

Hence we can write the general solution $X$ in the matrix form

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2s + \frac{1}{5}t \\ s \\ \frac{3}{5}t \\ t \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} \frac{1}{5} \\ 0 \\ \frac{3}{5} \\ 1 \end{bmatrix}$$

$$X = sX_1 + tX_2$$

where $X_1 = [2 \quad 1 \quad 0 \quad 0]^T$ and $X_2 = [\frac{1}{5} \quad 0 \quad \frac{3}{5} \quad 1]^T$ are particular solutions determined by the gaussian algorithm.

The solutions $X_1$ and $X_2$ in Example 9 are called **basic solutions** to the homegeneous system, and a solution of the form $sX_1+tX_2$ is called a **linear combination** of $X_1$ and $X_2$.

**Definition**
Any non zero scalar multiple of a basic solution will be called a basic solution.

**Theorem 3**
Let $A$ be an $m\times n$ matrix of rank $r$, and consider the homogeneous system $AX = 0$ in $n$ variables. Then:

1. The system has exactly $n - r$ basic solutions, one for each parameter.

2. Every solution is a linear combination of these basic solutions.

■

**Example 10** Find basic solutions of the system $AX = 0$ and express every solution as a linear combination of the basic solutions, where

$$A = \begin{bmatrix} -1 & -3 & 0 & 2 & 2 \\ -2 & 6 & 1 & 2 & -5 \\ 3 & -9 & -1 & 0 & 7 \\ -3 & 9 & 2 & 6 & -8 \end{bmatrix}$$

**Solution** The reduction of the augmented matrix to reduced row-echelon form is

$$\left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ -2 & 6 & 1 & 2 & -5 & 0 \\ 3 & -9 & -1 & 0 & 7 & 0 \\ -3 & 9 & 2 & 6 & -8 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{rrrrr|r} 1 & -3 & 0 & 2 & 2 & 0 \\ 0 & 0 & 1 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

So the general solution is $x_1 = 3r - 2s - 2t$, $x_2 = r$, $x_3 = -6s + t$, $x_4 = s$, and $x_5 = t$ where $r$, $s$, and $t$ are parameters.

In matrix form, this is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3r - 2s - 2t \\ r \\ -6s + t \\ s \\ t \end{bmatrix}.$$

$$X = r \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

## Block Multiplication

### Theorem 4

Let $A = \left[ \begin{array}{cccc} C_1 & C_2 & ... & C_n \end{array} \right]$ be an $m \times n$ matrix with column $C_1$, $C_2$, ..., $C_n$. If $X = \left[ \begin{array}{cccc} x_1 & x_2 & ... & x_n \end{array} \right]$ is any column, then

$$AX = \left[ \begin{array}{cccc} C_1 & C_2 & ... & C_n \end{array} \right] \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} = x_1C_1 + x_2C_2 + ... + x_nC_n$$

■

## Matrix Inverses

If $A$ is a square matrix, a matrix $B$ is called an **inverse** of $A$ if and only if

$$AB = I \quad \text{and} \quad BA = I$$

A matrix $A$ that has an inverse is called an **invertible matrix**

### Theorem 1

If $B$ and $C$ are both inverses of $A$, then $B = C$. ■

## Inverses and Linear Systems

Matrix inverses can be used to solve certain systems of linear equations.

Recall that a system of linear equations can be written as a single matrix equation

$$AX = B$$

If $A$ is invertible, we multiply each side of the equation on the left by $A^{-1}$ to get

$$X = A^{-1}B$$

This gives the solution to the system of equations.

**Theorem 2**
Suppose a system of $n$ equations in $n$ variables is written in matrix form as

$$AX = B$$

If the $n \times n$ coefficient matrix $A$ is invertible, the system has the unique solution

$$X = A^{-1}B$$

■

## An Inversion Method

Given a particular $n \times n$ matrix $A$, it is desirable to have an effcient technique to determine whether $A$ has an inverse. If so, to find it.

## Matrix Inversion Algorithm

If $A$ is a (square) invertible matrix, there exists a sequence of elementary row operations that carry $A$ to the identity matrix $I$ of the same size, written $A \to I$.

This same series of row operations carries $I$ to $A^{-1}$.

This algorithm can be summarized as follows:

$$\left[\begin{array}{cc} A & I \end{array}\right] \to \left[\begin{array}{cc} I & A^{-1} \end{array}\right]$$

where the row operations on $A$ and $I$ are carried out simultaneously.

**Theorem 3**

If $A$ is an $n \times n$ matrix, either $A$ can be reduced to $I$ by elementary row operations or it cannot. In the first case, the algorithm produces $A^{-1}$; in the second case, $A^{-1}$ does not exist. $\blacksquare$

**Example 6** Use the inversion algorithm to find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 7 & 1 \\ 1 & 4 & -1 \\ 1 & 3 & 0 \end{bmatrix}$$

**Solution** Apply elementary row operations to the double matrix

$$\begin{bmatrix} A & I \end{bmatrix} = \left[\begin{array}{ccr|ccc} 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 4 & -1 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

so as to carry $A$ to $I$. First interchange rows 1 and 2.

$$\left[\begin{array}{ccr|ccc} 1 & 4 & -1 & 0 & 1 & 0 \\ 2 & 7 & 1 & 1 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

Next subtract 2 times row 1 from row 2, and subtract row 1 from row 3.

$$\left[\begin{array}{rrr|rrr} 1 & 4 & -1 & 0 & 1 & 0 \\ 0 & -1 & 3 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 & -1 & 1 \end{array}\right]$$

Continue to reduced row-echelon form.

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 11 & 4 & -7 & 0 \\ 0 & 1 & -3 & -1 & 2 & 0 \\ 0 & 0 & -2 & -1 & 1 & 1 \end{array}\right]$$

$$\left[\begin{array}{rrr|rrr} 1 & 0 & 0 & -\frac{3}{2} & -\frac{3}{2} & \frac{11}{2} \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{array}\right]$$

**Theorem 4**
All the following matrices are square matrices of the same size.

1. $I$ is invertible and $I^{-1} = I$.

2. If $A$ is invertible, so is $A^{-1}$, and $(A^{-1})^{-1} = A$.

3. If $A$ and $B$ are invertible, so is $AB$, and $(AB)^{-1} = B^{-1}A^{-1}$.

4. If $A_1$, $A_2$, ..., $A_k$ are all invertible, so is their product $A_1A_2...A_k$, and $(A_1A_2...A_k)^{-1} = A_k^{-1}...A_2^{-1}A_1^{-1}$.

5. If $A$ is invertible, so is $A^k$ for $k \geq 1$, and $(A^k)^{-1} = (A^{-1})^k$.

6. If $A$ is invertible and $a \neq 0$ is a number, then $aA$ is invertible and $(aA)^{-1} = \frac{1}{a}A^{-1}$.

7. If $A$ is invertible, so is its transpose $A^T$, and $(A^T)^{-1} = (A^{-1})^T$.

■

**Theorem 5**
The following conditions are equivalent for an $n \times n$ matrix $A$:

1. $A$ is invertible.
2. The homogeneous system $AX = 0$ has only the trivial solution $X = 0$.
3. $A$ can be carried to the identity matrix $I_n$ by elementary row operations.
4. The system $AX = B$ has at least one solution $X$ for every choice of column $B$.
5. There exists an $n \times n$ matrix $C$ such that $AC = I_n$.

■

**Corollary**
If $A$ and $C$ are square matrices such that $AC = I$, then also $CA = I$. In particular, both $A$ and $C$ are invertible, $C = A^{-1}$, and $A = C^{-1}$. ■

## Elementary Matrices

It is now clear that elementary row operations are important in linear algebra.

It turns out that they can be performed by left multiplying by certain invertible matrices.

These matrices are the subject of this section.

An $n \times n$ matrix $E$ is called an **elementary matrix** if it can be obtained from the identity matrix $I_n$ by a single elementary row operation.

We say that $E$ is of type I, II, III if the operation is of that type.

**Lemma 1**
If an elementary row operation is performed on an $m \times n$ matrix $A$, the result is $EA$ where $E$ is the elementary matrix obtained by performing the operation on the $m \times m$ identity matrix. ■

The effect of an elementary operation can be reversed by another such operation which is also elementary of the same type.

**Lemma 2**
Every elementary matrix $E$ is invertible, and $E^{-1}$ is also a elementary matrix (of the same type). Moreover, $E^{-1}$ corresponds to the inverse of the row operation that produces $E$. ■

The following table gives the inverse of each type of elementary operations.

| **Type** | **Operation** | **Inverse Operation** |
|---|---|---|
| I | Interchange rows $p$ and $q$ | Interchange rows $q$ and $p$ |
| II | Multiply row $p$ by $k \neq 0$ | Multiply row $p$ by $1/k$ |
| III | Add $k$ times row $p$ to row $q \neq p$ | Subtract $k$ times row $p$ from row $q$ |

## Inverses and Rank

**Theorem 1**
Suppose $A$ is $m \times n$ and $A \to B$ by elementary row operations.

1. $B = UA$ where $U$ is an $m \times m$ invertible matrix.
2. $U$ can be computed by $\left[\begin{array}{cc} A & I_m \end{array}\right] \to \left[\begin{array}{cc} B & U \end{array}\right]$ using the operations carrying $A \to B$.
3. $U = E_k E_{k-1} ... E_2 E_1$ where $E_1, E_2, ..., E_k$ are the elementary matrices corresponding (in order) to the elementary row operations carrying $A$ to $B$. ■

**Theorem 2**
A square matrix is invertible if and only if it is a product of elementary matrices.■

## Uniqueness of the Reduced Row-echelon form

**Theorem 4**
If a matrix $A$ is carried to reduced row-echelon matrices $R$ and $S$ by row operations, then $R = S$. ■