Theory of Sets

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Definition

- A set is an unordered collection of objects.
- This description of a set as a collection of objects, based on the intuitive notion of an object, was first stated by the German mathematician Georg Cantor in 1895.
- This theory is also known as naive set theory
- Examples:
  - The cities in India
  - Even natural numbers less than 15

- The objects in a set are called the elements, or members, of the set.
- A set is said to contain its elements.
- If a is an element of the set A, we say a belongs to A and is denoted by $a \in A$.
- If a is not an element of the set A, it is denoted by $a \notin A$.

Representation of a Set

- Roster or Tabular Form

  In roster form, all the elements of a set are listed and are separated by commas and enclosed within curly brackets, $\{\ \}$.

  Examples:

  - Set of odd integers less than 10.

    $$\mathbf{D} = \{1, 3, 5, 7, 9\} \tag{1}$$

  - Set of last 5 alphabets in English.

    $$\mathbf{E} = \{v, w, z, y, z\} \tag{2}$$

Remarks:

- The order in which elements are listed is immaterial.

  Example:

$$\mathbf{E} = \{v, w, z, y, z\} \tag{3}$$

$$\mathbf{E'} = \{z, v, y, w, z\} \tag{4}$$

- Elements are not repeated, i.e., all elements that are distinct are listed.
  Let P be the set of all alphabets of the word programme.

$$\mathbf{P} = \{p, r, o, g, a, m, e\} \tag{5}$$

- Set-Builder Form:

  In this form, all the elements of a set possess a single common property which is not possessed by an element outside of the set.

  Example:

$$\mathbf{E} = \left\{ x : \text{x belongs to last five alphabets in English} \right\} \tag{6}$$

$$\mathbf{D} = \left\{ x : \text{x is an odd positive number and } 0 < x < 10 \right\} \tag{7}$$

Number System

Set of Natural Numbers, $\mathbb{N}$:

- Also known as counting numbers

$$\mathbb{N} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, ...\} \tag{8}$$

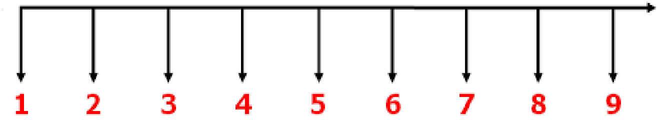


Figure: 1 Representation of Natural numbers
Courtesy: http://mathsfans.blogspot.com

- Set of Whole Numbers, $\mathbb{W}$:

$$\mathbb{W} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, ...\} \tag{9}$$

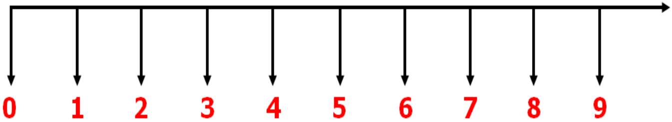


Figure: 2: Representation of Whole numbers
Courtesy: http://mathsfans.blogspot.com

- Set of Integers, $\mathbb{Z}$:

$$\mathbb{Z} = \{..., -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, ...\} \quad (10)$$

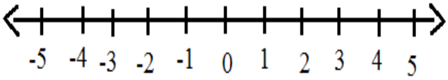
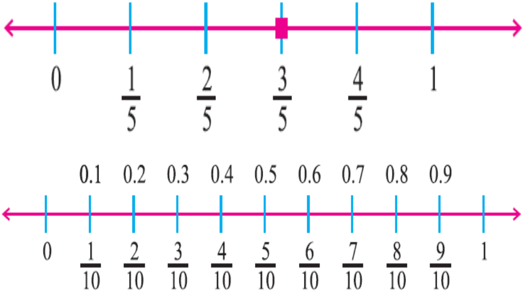


Figure: 3: Representation of IntegersCourtesy: https://www.study.com

- Set of Rational Numbers, $\mathbb{Q}$:
  A number r is called a rational number if r can be represented as $r = \dfrac{p}{q}$ where p and $q \in \mathbb{Z}$, and $q \neq 0$.
  i.e.,
  $\mathbb{Q} = \{ \dfrac{p}{q} : p, q \in Z, q \neq 0 \}$.
  Examples:



Is that all?

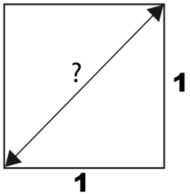


Figure: Courtesy: http://www.pythagoras.nu/

By Pythagoras' Theorem, we know the length of the diagonal is given by

$$\sqrt{1^2 + 1^2} = \sqrt{2}. \tag{11}$$

- $\sqrt{2} \notin \mathbb{Q}$ as it cannot be written in the form of $\dfrac{p}{q}$ where p and $q \in \mathbb{Z}$, and $q \neq 0$.
  Then it cannot belong to $\mathbb{N}$, $\mathbb{W}$ and $\mathbb{Z}$.

- Then where is $\sqrt{2}$ in the number line?

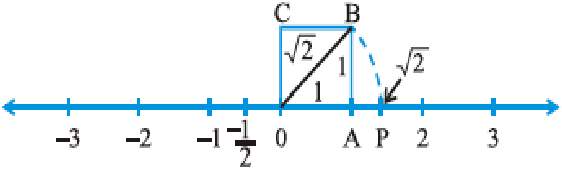


Figure: Courtesy: NCERT

- $\sqrt{2}$ and such numbers are called irrational numbers, denoted by $\mathbb{Q}'$.

- Set of Real Numbers, $\mathbb{R}$:

  Real numbers are defined as the collection of all rational numbers and irrational numbers, denoted by $\mathbb{R}$.

Equality of Sets

- Definition: Two sets are equal if and only if they have the same elements.
  i.e. If A and B are sets, then A and B are equal iff
  $\forall x(x \in A \longleftrightarrow x \in B)$.

- Notation: $\mathbf{A}=\mathbf{B}$.

- Otherwise, the sets **A** and **B** are said to be not equal, denoted by $\mathbf{A} \neq \mathbf{B}$.

- Examples

Universal Set and Empty Set

- Definition: The **universal set**, U, contains all the objects under consideration.
- Let V be the set of vowels in English alphabets. Then what can vbe an universal set of V.
- A set which does not contain any element is called **an empty set or null set or void set**, denoted by Φ and $\{\ \}$.
- Example:

$$\mathbf{B} = \{x : 2 < x < 3, \text{x is a natural number}\}$$

Singleton Set

- A set with one element is called a singleton set.
- Example: $X = \{0\}$, $B = \{a\}$.
- Remark: $\{0\}$ is different from $\{\}$.

Subset

- Definition: A set **A** is said to be a subset of a set **B** iff every element of **A** is also an element of **B**.
- Notation: **A** $\subseteq$**B**.

Subset

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- Notation: **A** $\subseteq$**B**.
- $A \subseteq B$ if and only if the quantification $\forall x\ (x \in A \rightarrow x \in B)$ is true
- Examples
- **A** not a subset of **B** is denoted by **A** $\not\subseteq$**B**.
- Examples

Proper Subset

- Let A be a subset of B but $A \neq B$, then A is said to be a proper subset of B.
- Notation: $A \subset B$.

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- Notation: $A \subset B$.
- A is a proper subset of B if
  $\forall x\ (x \in A \rightarrow x \in B) \wedge \exists x\ (x \in B \wedge x \notin A)$ is true.

Remarks:

- Every set **A** is a subset of itself.
  i.e. $A \subseteq A \ \forall A$.
- $\Phi \subseteq A \ \forall A$.
- $A \subseteq B$ and $B \subseteq A$ if and only if $A = B$.

Cardinality of a Set

- Let S be a set. If there are exactly n distinct elements in S where n is a non-negative integer, then we say that S is a finite set and that n is the cardinality of S.
- The cardinality of S is denoted by the $|S|$.
- Examples:
  - $\mathbf{D} = \{1, 3, 5, 7, 9\}$, $|\mathbf{D}| =$
  - $|\phi| =$
- A set is said to be infinite if it is not finite.
- Example:
  - The cardinality of the set of positive integers.

The Power Set

- Given a set S, the power set of S is the set of all subsets of the set S. The power set of S is denoted by $P(S)$.
- What is the power set of the set $\{0, 1, 2\}$?
- What is the power set of the empty set?
- What is the power set of the set $\{\Phi\}$?
- If a set has n elements, then its power set has 2^n elements.

Ordered n-tuple

- The **ordered n-tuple**, $(a_1, a_2, ..., a_n)$, is the ordered collection that has a_1 as its first element, a_2 as its second element, . . . , and a_n as its nth element.

- $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ if and only if $a_i = b_i$ for $i = 1, 2, ..., n$.

Cartesian Product

- Let A and B be sets. The cartesian product of A and B, denoted by $A \times B$, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$.
- Hence, $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.

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- $(a_1, a_2, ..., a_n) = (b_1, b_2, ..., b_n)$ iff $a_i = b_i \ \forall \ i = 1, 2, ..., n$.
- 2-tuples are called ordered pairs.

Cartesian Product

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- Hence, $A \times B = \{(a, b) | a \in A \wedge b \in B\}$.
- Example: What is the Cartesian product of $A = \{1, 2\}$ and $B = \{a, b, c\}$?
- **Note:** $A \times B \neq B \times A$.

- The Cartesian product of the sets $A_1, A_2, ..., A_n$, denoted by $A_1 \times A_2 \times .... \times A_n$ is the set of ordered n-tuples $(a_1, a_2, ..., a_n)$, where $a_i \in A_i$ for $i = 1, 2, ..., n$.
- $A_1 \times A_2 \times .... \times A_n = \{(a_1, a_2, ..., a_n) | a_i \in A_i \text{ for } i = 1, 2, ..., n\}$.
- What is the Cartesian product $A \times B \times C$, where $A = \{0, 1\}, B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

Set Notation and Truth Sets of Quantifiers

- What do the statements $\forall\ x \in \mathbb{R}(x^2 \geq 0)$ and $\exists\ x \in \mathbb{Z}(x^2 = 1)$ mean?

- What are the truth sets of the predicates $P(x)$, $Q(x)$, and $R(x)$, where the domain is the set of integers and
  1. $P(x) is" |x| = 1,"$
  2. $Q(x) is" x^2 = 2,"$
  3. $R(x) is" |x| = x."$

Set Operations-Union

- Let A and B be sets. The union of the sets A and B, denoted by $A \cup B$, is the set that contains those elements that are either in A or in B, or in both.

- $A \cup B = \{x : x \in A \vee x \in B\}$.

- The union of a collection of sets is the set that contains those elements that are members of at least one set in the collection.

- $A_1 \cup A_2 \cup .... \cup A_n = \bigcup_{i=1}^{n} A_i$.
- $A_1 \cup A_2 \cup .... \cup A_n \cup .... = \bigcup_{i=1}^{\infty} A_i$
- $\bigcup_{i=1}^{\infty} A_i = \{x : \exists i \in \mathbb{Z}^+ (x \in A_i)\}$.

Set Operations-Intersection

- Let A and B be sets. The union of the sets A and B, denoted by $A \cap B$, is the set that contains those elements in both A and B.

- $A \cap B = \{x : x \in A \wedge x \in B\}$.

- The intersection of a collection of sets is the set that contains those elements that are members of all the sets in the collection.

- $A_1 \cap A_2 \cap .... \cap A_n = \bigcup_{i=1}^{n} A_i$

- $A_1 \cap A_2 \cap .... \cap A_n \cap .... = \bigcap_{i=1}^{\infty} A_i$

- $\bigcap_{i=1}^{\infty} A_i = \{x : \forall i \in \mathbb{Z}^+ (x \in A_i)\}$.

Example

- Suppose that $A_i = \{1, 2, 3, ..., i\}$ for $i = 1, 2, 3, ...$ Then find

- $\bigcup_{i=1}^{\infty} A_i$

- $\bigcap_{i=1}^{\infty} A_i$

Disjoint Sets

- Two sets are called disjoint if their intersection is the empty set.
- i.e. $A \cap B = \Phi$.
- $|A \cup B| = |A| + |B| - |A \cap B|$. (Proof by Venn diagram)
- The generalization of this result to unions of an arbitrary number of sets is called the **principle of inclusion-exclusion**.

Difference of Two Sets

- Let A and B be sets. The difference of A and B, denoted by $A - B$, is the set containing those elements that are in A but not in B . The difference of A and B is also called the complement of B with respect to A.

- i.e. $A - B = \{x : x \in A \wedge x \notin B\}$.

Complement of a Set

- Let U be the universal set. The complement of the set A, denoted by A, is the complement of A with respect to U.
- In other words, the complement of the set A is $U - A$.
- i.e. $\bar{A} = \{x : x \notin A\}$.

Relation

- A subset R of the Cartesian product $A \times B$ is called a relation from the set A to the set B.
- The elements of R are ordered pairs, where the first element belongs to A and the second to B.
- Example: $A = \{0, 1\}$, $B = \{1, 2\}$. Then $A \times B = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$.

Set Identities

Identity	*Name*
$A \cup \emptyset = A$ $A \cap U = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws
$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	Distributive laws
$\overline{A \cup B} = \overline{A} \cap \overline{B}$ $\overline{A \cap B} = \overline{A} \cup \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

FUNCTIONS

- Let A and B be nonempty sets. A function f from A to B is an assignment of exactly one element of B to each element of A.
- We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A .
- If f is a function from A to B, we write $f : A \longrightarrow B$.
- Also known as mappings or transformations.

- A function $f : A \longrightarrow B$ can also be defined in terms of a relation from A to B.
- A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defines a function f from A to B.

Definition

- If f is a function from A to B, we say that A is the domain of f and B is the codomain of f.
- If $f(a) = b$, we say that b is the image of a and a is a preimage of b.
- The range of f is the set of all images of elements of A.
- If f is a function from A to B , we say that f maps A to B.
- **Note: When we define a function we specify its domain, its codomain, and the mapping of elements of the domain to elements in the codomain.**

Equality of Functions

- Two functions are equal when they have the same domain, have the same codomain, and map elements of their common domain to the same elements in their common codomain.

- If we change either the domain or the codomain of a function, then we obtain a different function.

- If we change the mapping of elements, then we also obtain a different function.

Examples

- Let $f : \mathbb{Z} \Longrightarrow \mathbb{Z}$ assign the square of an integer to this integer.
- Then, $f(x) = x^2$, where the domain of f is the set of all integers.
- The codomain of f to be the set of all integers.
- The range of f is the set of all integers that are perfect squares, namely, $\{0, 1, 4, 9, ...\}$.

Definition

- Let f_1 and f_2 be functions from A to $\mathbb{R}$. Then $f_1 + f_2$ and $f_1 f_2$ are also functions from A to $\mathbb{R}$ defined by

$$(f_1 + f_2)(x) = f_1(x) + f_2(x)$$
$$(f_1 f_2)(x) = f_1(x) f_2(x)$$

- Example: Let $f_1 : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_1 = x^2$ and $f_2 : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f_2 = x - x^2$.

- Then

$$(f_1 + f_2)(x) = x^2 + x - x^2 = x$$

-

$$(f_1 f_2)(x) = x^2(x - x^2) = x^3 - x^4.$$

- Let f be a function from the set A to the set B and let $S \subseteq A$. The image of S under the function f is the subset of B that consists of the images of the elements of S.

- We denote the image of S by $f(S)$, so $f(S) = \{t : \exists s \in S(t = f(s))\}$.

- Can also be represented as $\{f(s) : s \in S\}$.

- Let $A = \{a, b, c, d, e\}$ and $B = \{l, 2, 3, 4\}$ with $f(a) = 2, f(b) = 1, f(c) = 4, f(d) = 1$, and $f(e) = 1$. The image of the subset $S = \{b, c, d\}$ is the set $f(S) = \{1, 4\}$.

One-to-One Functions

- A function f is said to be one-to-one, or injective, if and only if $f(a) = f(b) \longrightarrow a = b$ for all a and b in the domain of f.
- A function is said to be an injection if it is one-to-one.
- f is one-to-one if and only if $f(a) \neq f(b)$ whenever $a \neq b$.
- Examples:
  - $f(x) = x + 1$ from the set of integers to the set of integers.
  - $f(x) = x^2$ from the set of integers to the set of integers.

Increasing and Decreasing Functions

Let f be a function whose domain and codomain are subsets of the set of real numbers. Then

- f is increasing if $\forall x \forall y (x \leq y \longrightarrow f(x) \leq f(y))$ where the universe of discourse is the domain of f.
- f is strictly increasing if $\forall x \forall y (x < y \longrightarrow f(x) < f(y))$. where the universe of discourse is the domain of f.
- f is called decreasing if $\forall x \forall y (x \leq y \longrightarrow f(x) \geq f(y))$ where the universe of discourse is the domain of f.
-
- f is strictly decreasing if $\forall x \forall y (x < y \longrightarrow f(x) > f(y))$ where the universe of discourse is the domain of f.

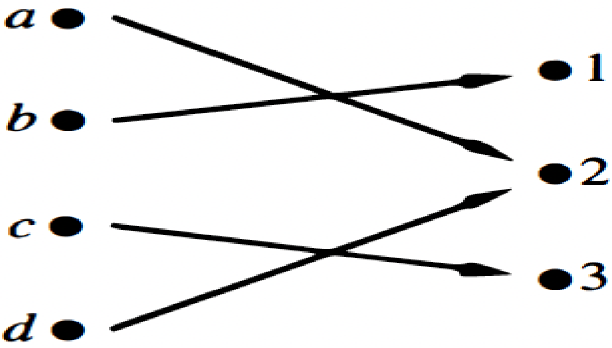
-
- Note:
  1. a function that is either strictly increasing or strictly decreasing must be one-to-one.
  2. A function that is increasing, but not strictly increasing, or decreasing, but not strictly decreasing, is not necessarily one-to-one.

Onto Functions

- A function f is said to be onto, or surjective, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.
- A function f is called a surjection if it is onto.
- f is onto if $\forall y \; \exists x \; (f(x) = y)$ where the domain for x is the domain of the function and the domain for y is the codomain of the function.
- Examples:
  - $f(x) = x + 1$ from the set of integers to the set of integers.
  - $f(x) = x^2$ from the set of integers to the set of integers.(not onto)
  - $f(x) = x^2$ from the set of integers to the set of non negative integers.(onto)

Bijective Function

- The function f is a one-to-one correspondence, or a bijection, if it is both one-to-one and onto.
- Examples:
  - $f(x) = x + 1$ from the set of integers to the set of integers.
- Let f is a function from a set A to itself. If A is finite, then f is one-to-one if and only if it is onto.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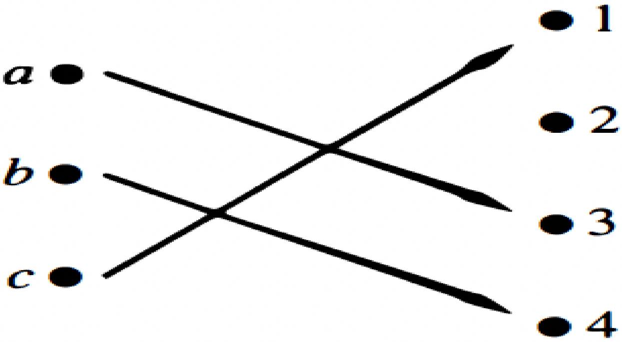

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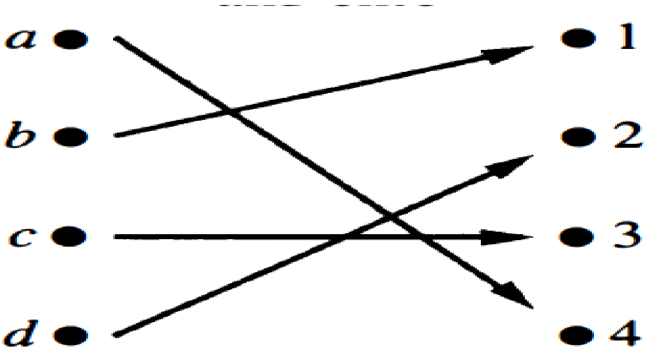
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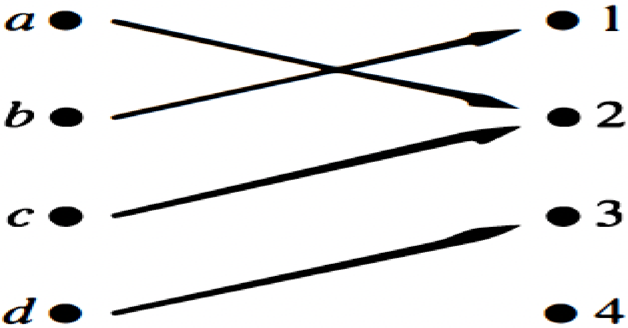

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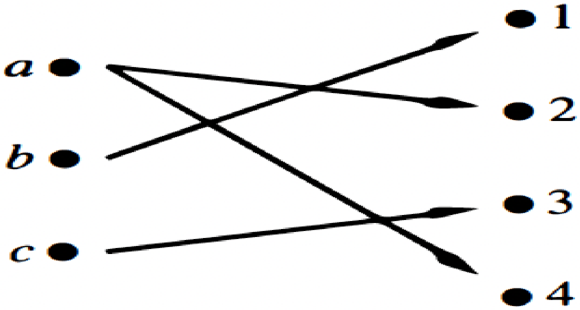
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Inverse Functions

- Let f be a one-to-one correspondence from the set A to the set B.
- The inverse function of f is the function that assigns to an element b belonging to B the unique element a in A such that $f(a) = b$.
- The inverse function of f is denoted by f^{-1}. Hence, $f^{-1}(b) = a$ when $f(a) = b$.
- A one-to-one correspondence is called invertible.

- **Note:** f^{-1} is not the same as $\frac{1}{f}$.
- $\frac{1}{f}$ is the function that assigns to each x in the domain the value $\frac{1}{f(x)}$.
- Example: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x + 1$. Then
  - $f(2) = 2 + 1 = 3$.
  - $\frac{1}{f}(2) = \frac{1}{f(2)} = \frac{1}{3}$.
  - $f^{-1}(2) = 1$ (Since $f(1) = 1 + 1 = 2$, $\therefore$ $f^{-1}(2) = 1$).

- **Note:** If there is no one-to-one correspondence from the set A to the set B, then f^{-1} will not exists.

- **Note:** If there is no one-to-one correspondence from the set A to the set B, then f^{-1} will not exists.
- Example: Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that $f(x) = x^2$. Then
  - $f(2) = 2^2 = 4$.
  - But $f^{-1}(4)$ has two images $x = 2$ and $x = -2$. Therefore it fails to be a function.
- A function is not invertible if it is not a one-to-one correspondence, because the inverse of such a function does not exist

Composition of Functions

- Let g be a function from the set A to the set B and let f be a function from the set B to the set C.
- The composition of the functions f and g, denoted by $f \circ g$, is defined by $(f \circ g)(a) = f(g(a))$.
- Example: Let f and g be the functions from the set of integers to the set of integers defined by $f(x) = 2x + 3$ and $g(x) = 3x + 2$. What is the composition of f and g? What is the composition of g and f?

- What is $f \circ f^{-1}$ and $f^{-1} \circ f$?

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- Suppose that f is a one-to-one correspondence from the set A to the set B . Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A.

- What is $f \circ f^{-1}$ and $f^{-1} \circ f$?

- Suppose that f is a one-to-one correspondence from the set A to the set B . Then the inverse function f^{-1} exists and is a one-to-one correspondence from B to A.

- $f \circ f^{-1} = I_B$ and $f^{-1} \circ f = I_A$ where I_A and I_B are identity functions on sets A and B.

- $(f^{-1})^{-1} = f$.

Partial Functions

- A partial function f from a set A to a set B is an assignment to each element a in a subset of A, called the domain of definition of f, of a unique element b in B.
- The sets A and B are called the domain and codomain of f, respectively.
- We say that f is undefined for elements in A that are not in the domain of definition of f.
- When the domain of definition of f equals A, we say that f is a **total function**.

Sequences

Consider the following

- 1,3,5,7
- 2,4,6,8,10,......

Consider the following

- 1,3,5,7
- 2,4,6,8,10,......
- A sequence is a function from a subset of the set of integers (usually either the set $\{0, 1, 2, ...\}$ or the set $\{1, 2, 3, ...\}$) to a set S.
- We use the notation a_n to denote the image of the integer n.

- a_n is called a *term* of the sequence.
- A sequence is denoted by $\{a_n\}$, $\{b_n\}$ etc.
- The notation $\{a_n\}$ for a sequence conflicts with the notation for a set.
- Sequences are described by listing the terms of the sequence in order of increasing subscripts.

Geometric Progression

- A geometric progression is of the form

  $a, ar, ar^2, \ldots, ar^n, \ldots$

  where the initial term is a and the common ratio r are real numbers.

- Examples:
  - $b_n = (-1)^n$
  - $c_n = 2.5^n$
  - $d_n = 6.\left(\frac{1}{3}\right)^n$

Arithmetic Progression

- A arithmetic progression is of the form

  $$a, a+d, a+2d, ..., a+nd, ...$$

  where the initial term is a and the common difference d are real numbers.

- Examples:
  - $s_n = -1 + 4n$
  - $t_n = 7 - 3n$

Application in Computer Science

- The sequences $a_1, a_2, ..., a_n$ are often used in computer science.
- These finite sequences are also called **strings**.
- This string is also denoted by $a_1 a_2 ... a_n$.
- The length of the string S is the number of terms in this string.
- The empty string, denoted by A, is the string that has no terms. The empty string has length zero.
- Example: BIT strings

Exercise

- Find formulae for the sequences with the following first five terms:
  1. $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$
  2. $1, 3, 5, 7, \ldots$
  3. $1, -1, 1, -1, 1, \ldots$

- How can we produce the terms of a sequence if the first 10 terms are $1, 2, 2, 3, 3, 3, 4, 4, 4, 4$?

- How can we produce the terms of a sequence if the first 10 terms are $5, 11, 17, 23, 29, 35, 41, 47, 53, 59$?

TABLE 1 Some Useful Sequences.	
nth Term	***First 10 Terms***
n^2	1, 4, 9, 16, 25, 36, 49, 64, 81, 100, ...
n^3	1, 8, 27, 64, 125, 216, 343, 512, 729, 1000, ...
n^4	1, 16, 81, 256, 625, 1296, 2401, 4096, 6561, 10000, ...
2^n	2, 4, 8, 16, 32, 64, 128, 256, 512, 1024, ...
3^n	3, 9, 27, 81, 243, 729, 2187, 6561, 19683, 59049, ...
$n!$	1, 2, 6, 24, 120, 720, 5040, 40320, 362880, 3628800, ...

Summation

- The notation for the sum of the terms a_m, $a_{m+1},\ldots,a_n$ is

$$\sum_{j=m}^{n} a_j$$

  or

$$\sum_{m \leq j \leq n} a_j$$

  .

- The variable j is called the index of summation.

- $$\sum_{j=m}^{n} a_j = \sum_{k=m}^{n} a_k = \sum_{i=m}^{n} a_i$$
  .

- $$\sum_{i=1}^{n} (ax_i + by_i) = a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} y_i$$
  .

Exercise

- $$\sum_{j=1}^{4} j^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$$
  .

- $$\sum_{j=4}^{8} (-1)^k = 1 + (-1) + 1 + (-1) + 1 = 1$$
  .

Exercise

- Suppose we have the sum $\sum_{j=1}^{4} j^2$, but want the index of summation to run between 0 and 4 rather than from 1 to 5.

Exercise

- Suppose we have the sum $\sum_{j=1}^{4} j^2$, but want the index of summation to run between 0 and 4 rather than from 1 to 5.

- Solution: Let $k = j - 1$. Then

$$\sum_{j=1}^{4} j^2 = \sum_{k=0}^{3} (k+1)^2 = 1 + 4 + 9 + 16 = 30$$

.

Important Result

- If a and b are real numbers and $r \neq 0$,then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1}, & \text{if } r \neq 1 \\ (n+1)a, & \text{if } r = 1 \end{cases}$$

Double Summations

- $$\sum_{i=1}^{4}\sum_{j=1}^{3} ij$$

- Solution:

$$\sum_{i=1}^{4}\sum_{j=1}^{3} ij = \sum_{i=1}^{4}(i + 2i + 3i)$$

$$= \sum_{i=1}^{4} 6i$$

$$= 6\sum_{i=1}^{4} i$$

$$= 6(1 + 2 + 3 + 4) = 60$$

Summation over sets

- What is
$$\sum_{s \in \{0,1,2\}} s?$$
- Solution:
$$\sum_{s \in \{0,1,2\}} s = 0 + 1 + 2 = 3$$
.

TABLE 2 Some Useful Summation Formulae.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x \rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty}, kx^{k-1}, \lvert x \rvert < 1$	$\frac{1}{(1-x)^2}$

Summation over sets

- Find
$$\sum_{k=51}^{100} k^2?$$

Summation over sets

- Find

$$\sum_{k=51}^{100} k^2?$$

- Solution:

$$\sum_{k=1}^{100} k^2 = \sum_{k=1}^{50} k^2 + \sum_{k=51}^{100} k^2$$

$$\sum_{k=51}^{100} k^2 = \sum_{k=1}^{100} k^2 - \sum_{k=1}^{50} k^2$$

$$= \frac{100 * 101 * 201}{6} - \frac{50 * 51 * 101}{6}$$

Cardinality

- The sets A and B have the same cardinality if and only if there is a one-to-one correspondence from A to B.
- A set that is either finite or has the same cardinality as the set of positive integers is called countable.
  1. When an infinite set S is countable, we denote the cardinality of S by $\aleph_0$.
- A set that is not countable is called uncountable

Examples

- The set of odd positive integers.
- The set of all integers.
- The set of positive rational numbers.
- The set of real numbers.