

# Temas de Computação Visual

## Modelagem Geométrica

Luiz Henrique de Figueiredo

## Formas geométricas em mundos virtuais



Pixar

PIXAR  
PRESENTS

# Geri's game



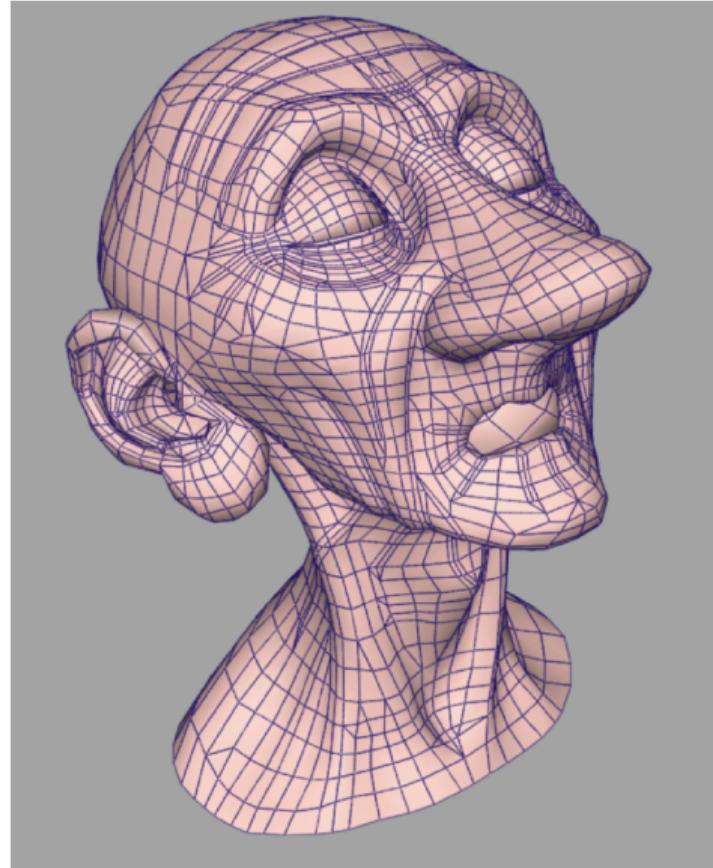
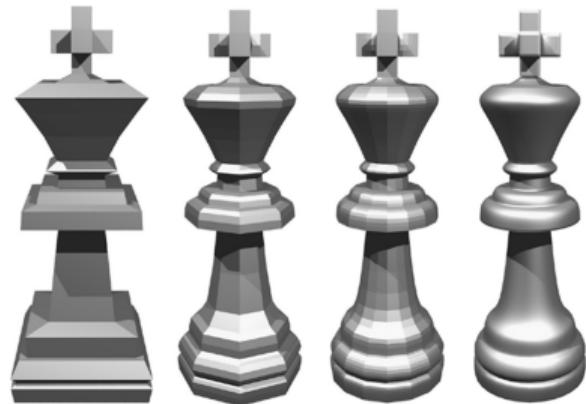
1997

## Formas geométricas em mundos virtuais



Pixar

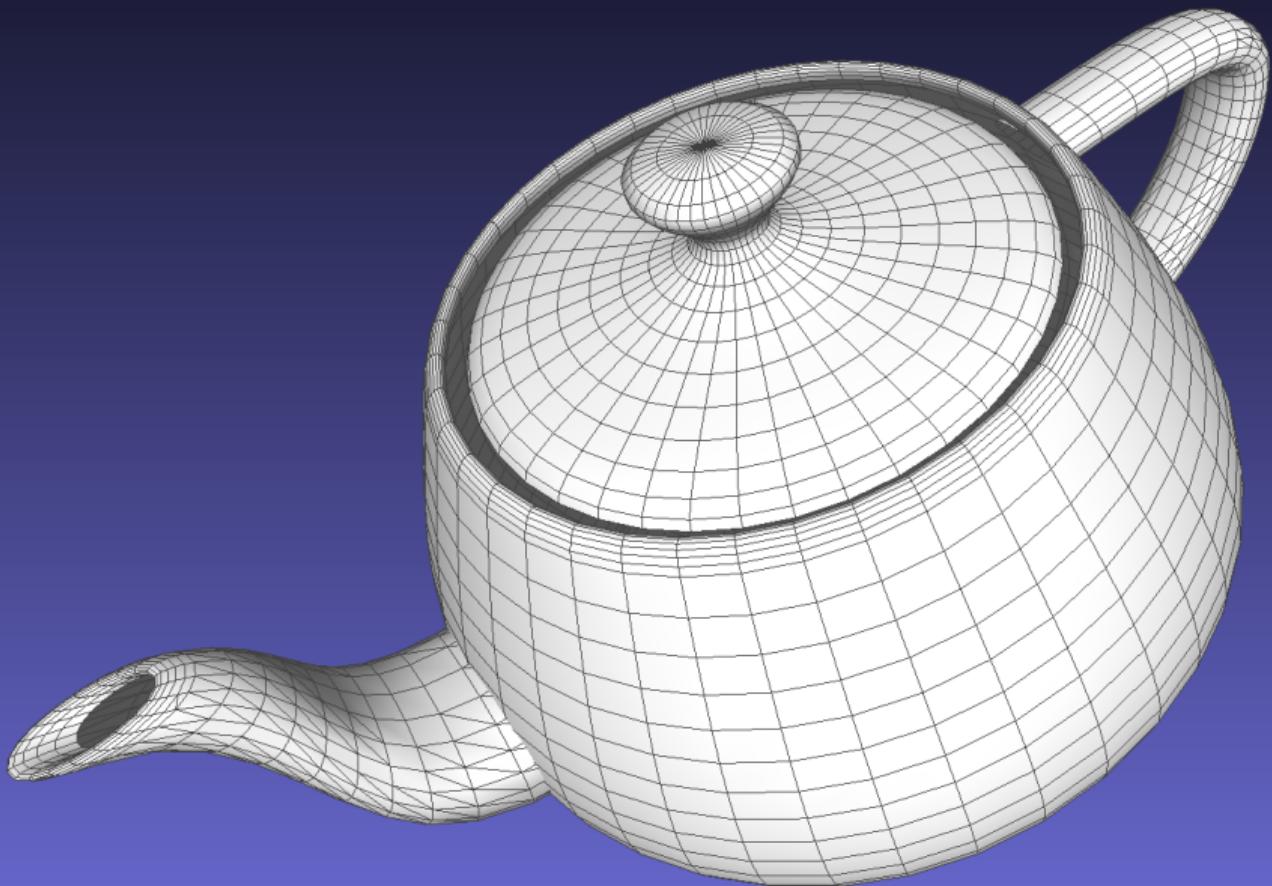
# Anatomia das formas virtuais



Pixar



Newell (1975) / Morrical (2019)



Newell (1975) / Morrical (2019)



Gouraud



Gouraud

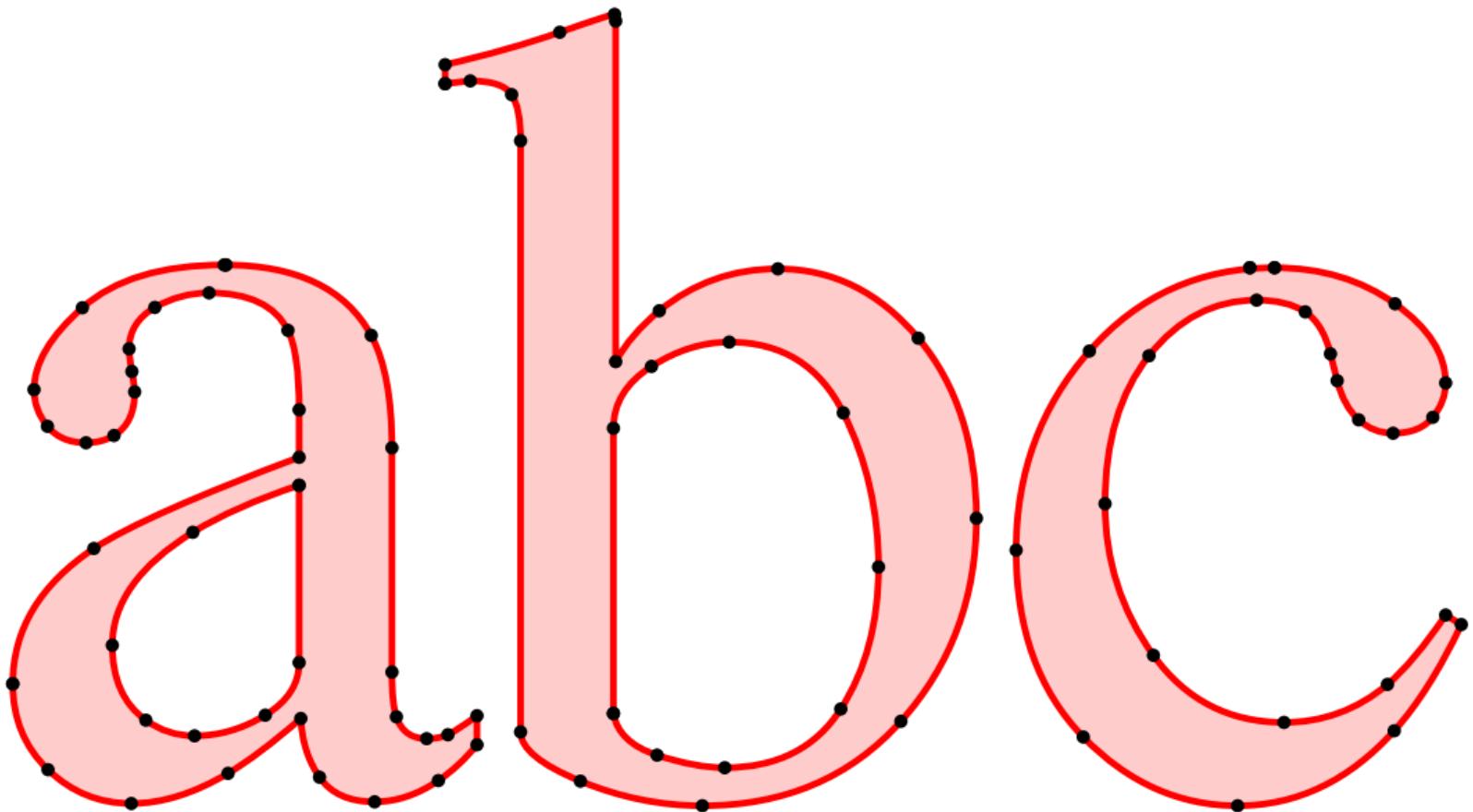
## Anatomia das formas virtuais

a b c 1

## Anatomia das formas virtuais

a b c

## Anatomia das formas virtuais



fundamentos

Jonas Gomes  
Luiz Velho

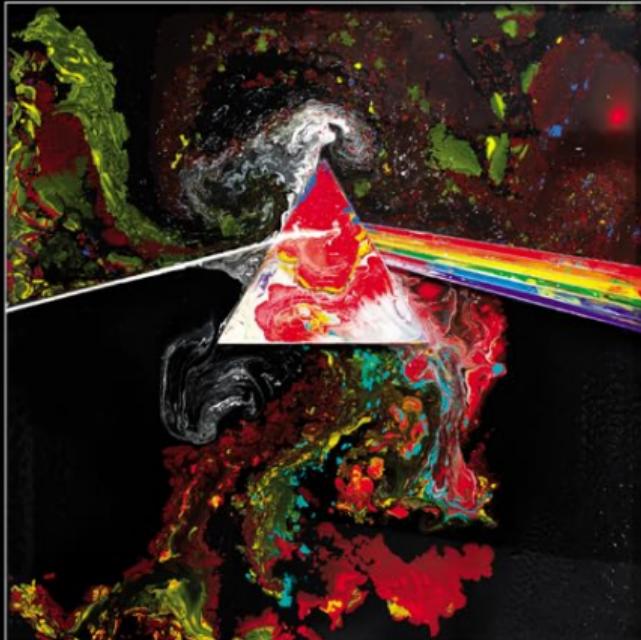
Fundamentos da Computação Gráfica



Instituto Nacional de Matemática Pura e Aplicada - IMPA  
Série de Computação e Matemática

2003

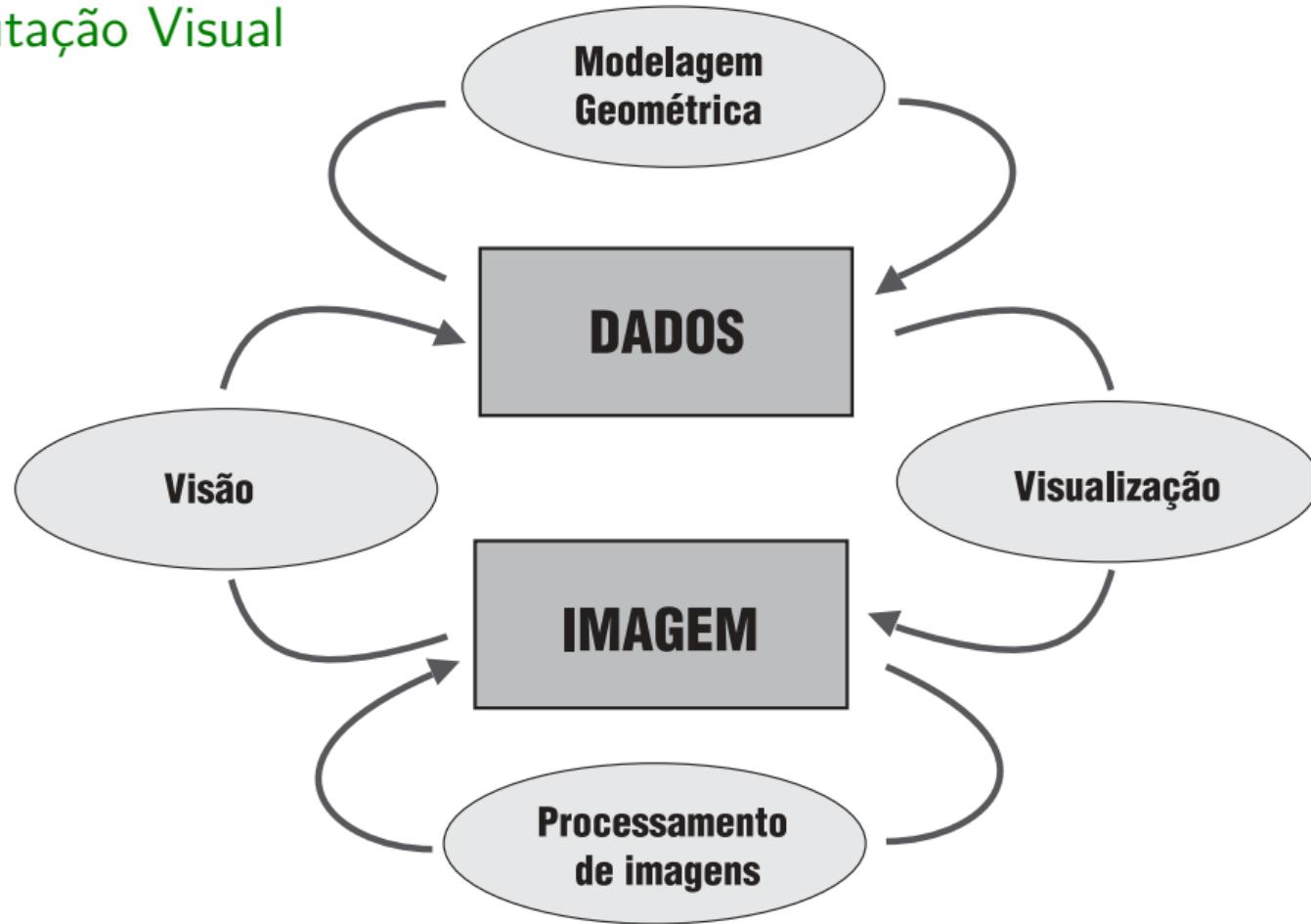
COMPUTER GRAPHICS  
THEORY AND PRACTICE

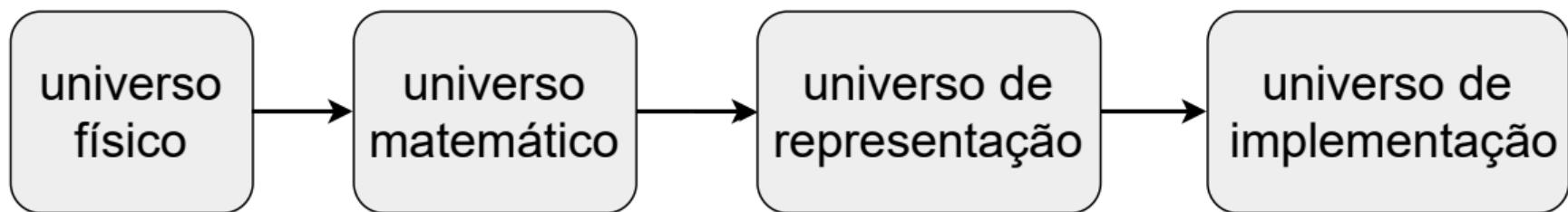


Jonas Gomes • Luiz Velho • Mario Costa Sousa

2012

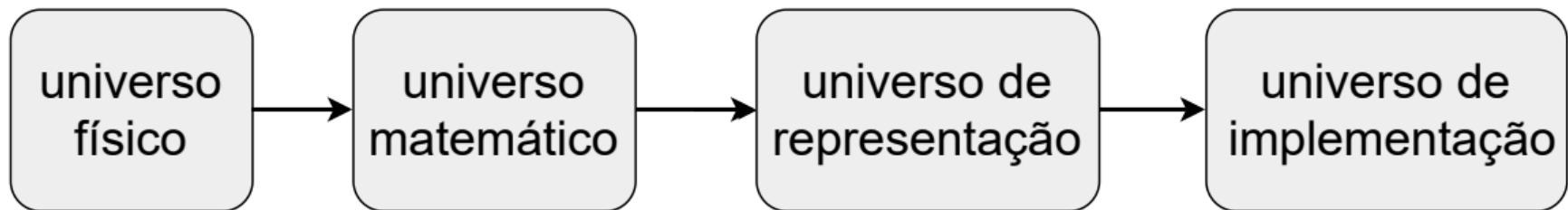
# Computação Visual





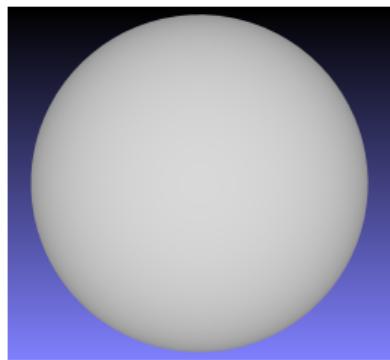
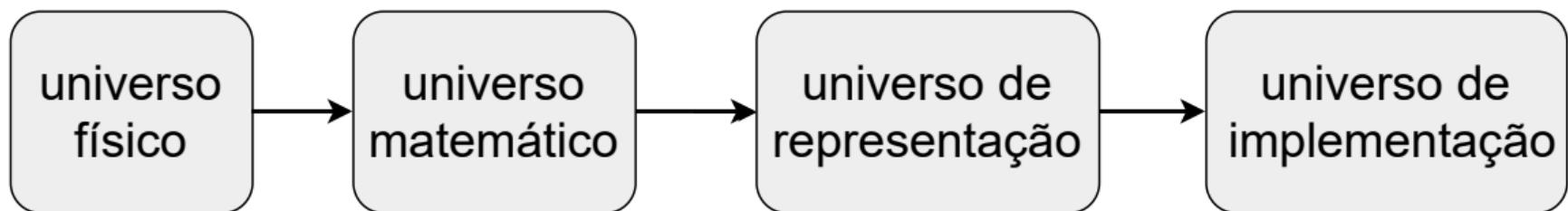
# Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)



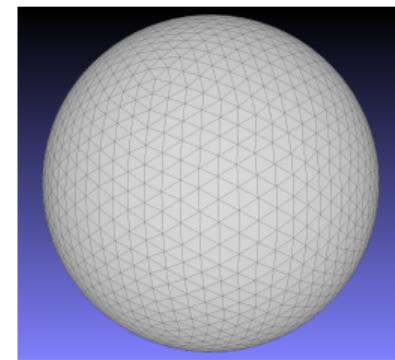
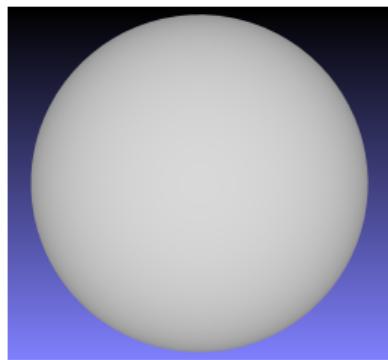
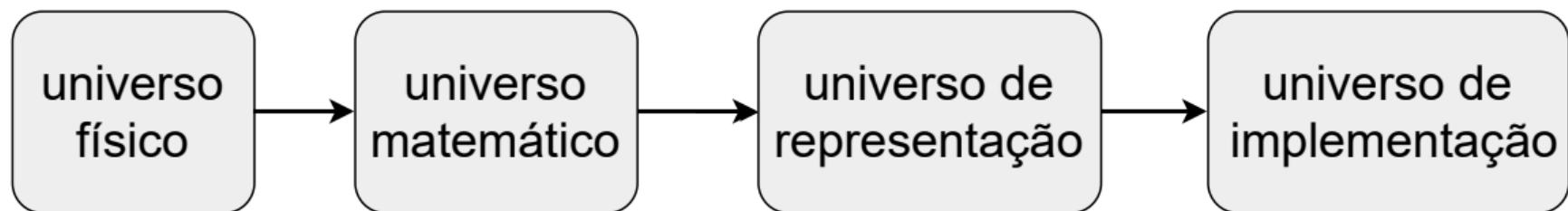
# Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)



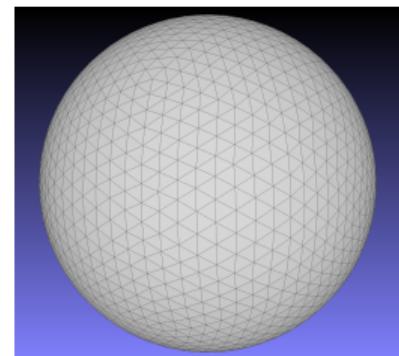
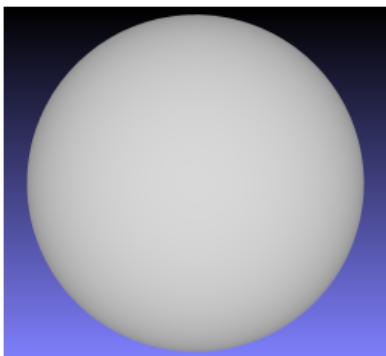
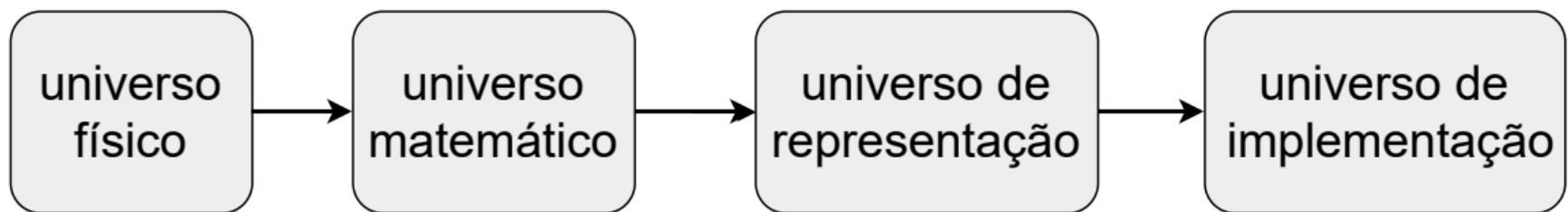
# Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)



# Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)

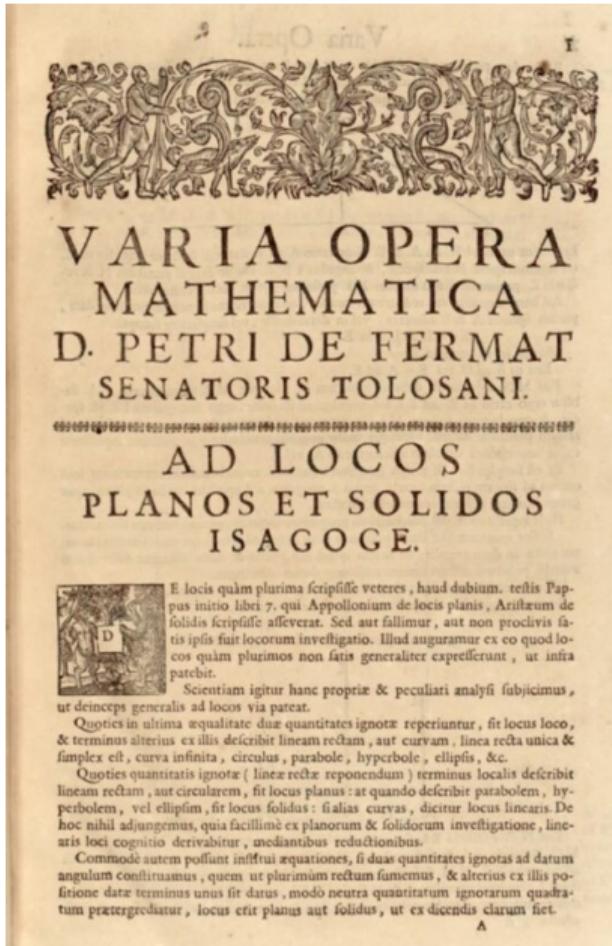


```
v 0.525731 0 -0.850651  
v 0.525731 0 0.850651  
v -0.525731 0 0.850651  
v -0.850651 -0.525731 0  
v -0.850651 0.525731 0  
v 0.850651 0.525731 0  
v 0.850651 -0.525731 0  
v 0 -0.850651 0.525731  
v 0 -0.850651 -0.525731  
v 0 0.850651 -0.525731  
...  
f 23 24 25  
f 25 24 20  
f 25 21 23  
f 22 24 23  
f 37 45 23  
f 17 45 37  
f 23 21 37  
f 23 45 22  
f 31 24 47  
f 28 24 31
```

universo matemático

# Geometria analítica

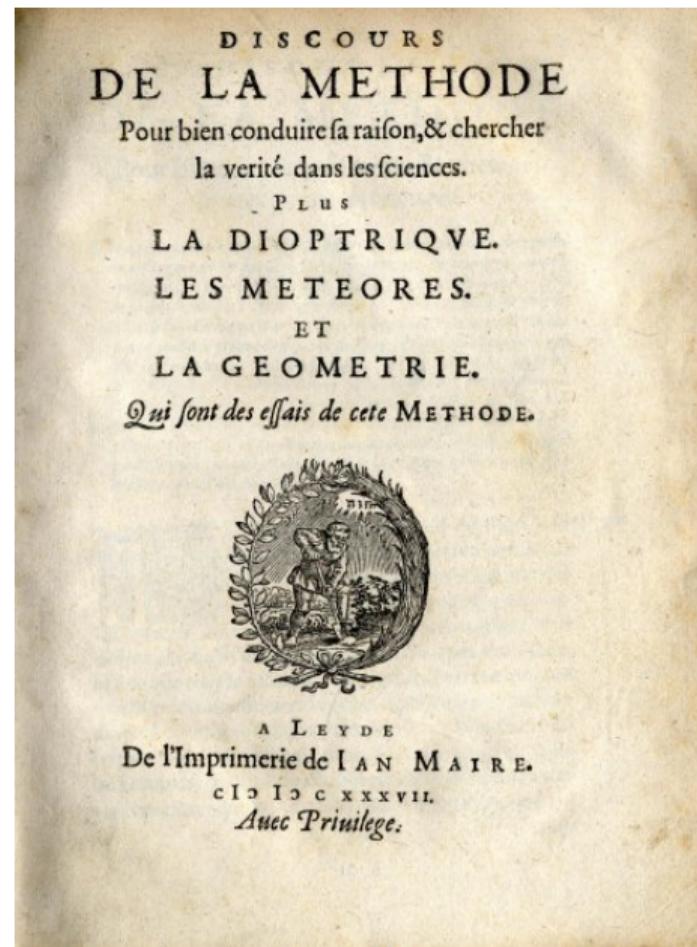
Fermat (1636, 1679)



# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)



# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

# L A G E O M E T R I E. LIVRE PREMIER.

*Des problemes qu'on peut construire sans y employer que des cercles & des lignes droites.*



O u s les Problemes de Geometrie se peuvent facilement reduire a tels termes, qu'il n'est besoin par apres que de connoître la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Division, & l'Extraction des racines, qu'on peut prendre pour vne espece de Division : Ainsi n'at'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnite pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'autre est a l'vnite, ce qui est le mesme que la Multiplication, ou bien en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'vnite

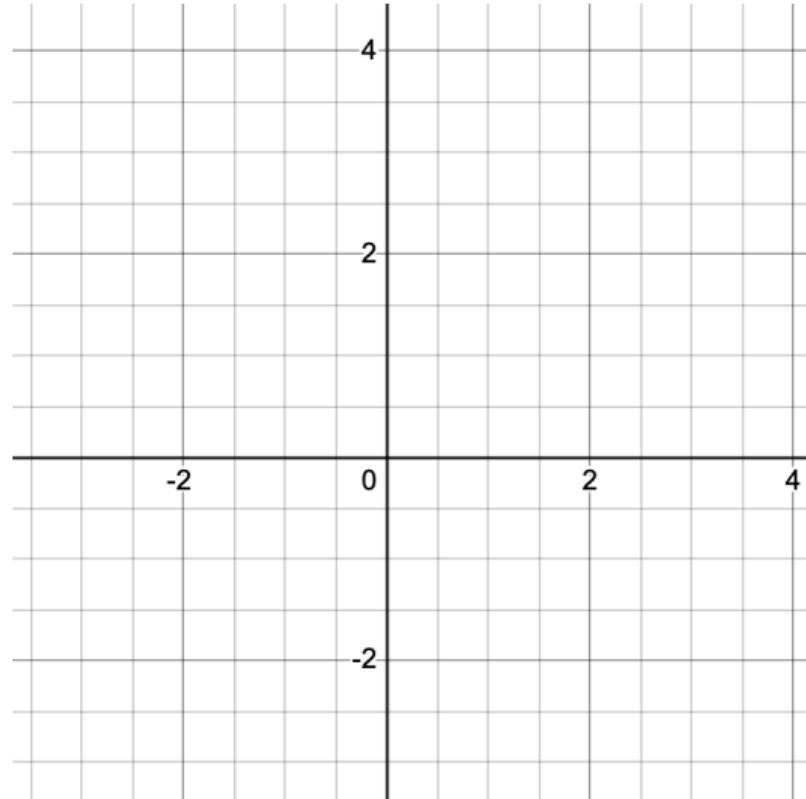
Commencé le calcul d'Addition, la d'At-  
Soustraction, la Multiplication, la Division, & l'Extra-  
ction des racines, que le rapporte aux opérations de  
metrie : Ainsi n'at'on autre chose à faire en Geo-  
metrie touchant les lignes qu'on cherche, pour les pre-  
parer à estre connues, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnite pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'autre est a l'vnite, ce qui est le mesme que la Multiplication, ou bien en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'vnite

# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas      álgebra      +      geometria  
fórmulas       $\longleftrightarrow$       formas  
finito       $\longleftrightarrow$       infinito

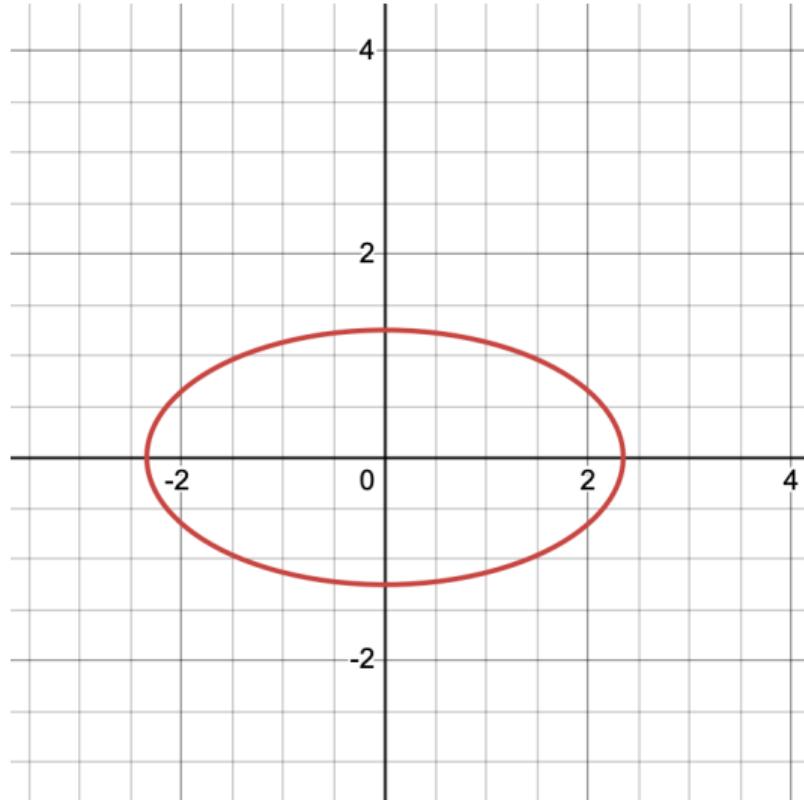


# Geometria analítica

Fermat (1636, 1679)

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coordenadas      álgebra      +      geometria  
fórmulas       $\longleftrightarrow$       formas  
finito       $\longleftrightarrow$       infinito



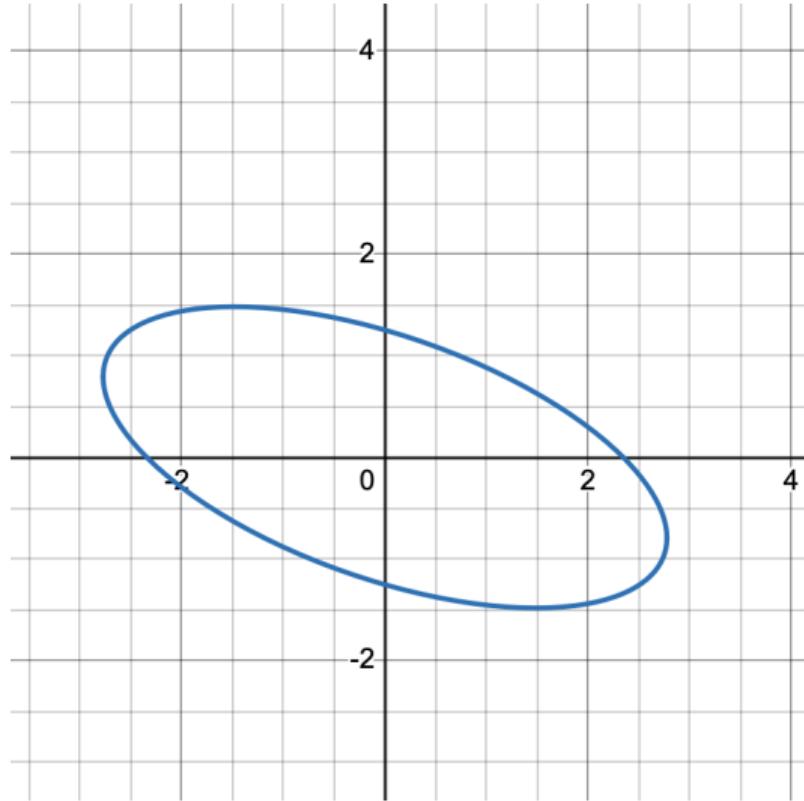
$$2x^2 + 7y^2 = 11$$

# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas      álgebra      +      geometria  
fórmulas       $\longleftrightarrow$       formas  
finito       $\longleftrightarrow$       infinito



$$2x^2 + 4xy + 7y^2 = 11$$

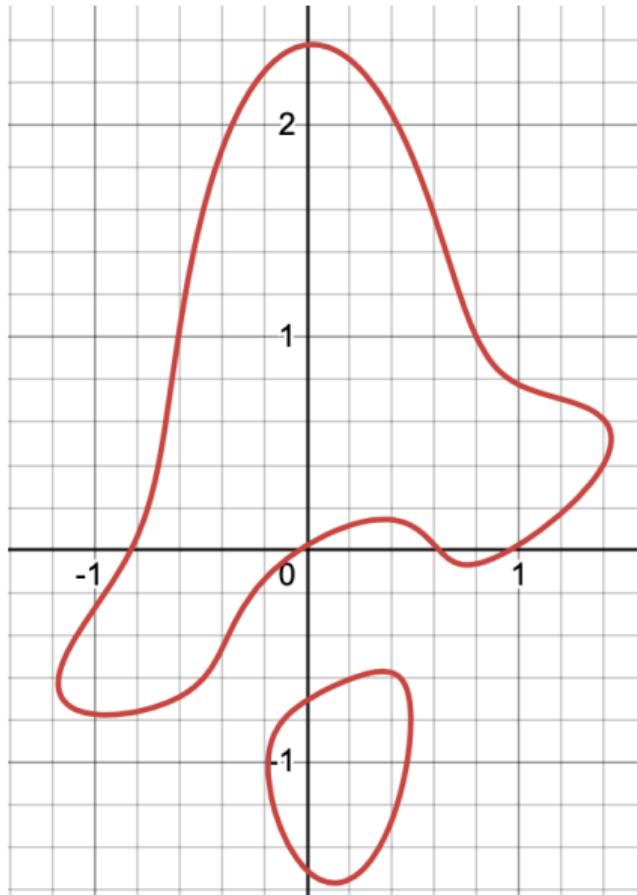
# Geometria analítica

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Descartes (1637)

coordenadas      álgebra      +      geometria  
fórmulas       $\longleftrightarrow$       formas  
finito       $\longleftrightarrow$       infinito

$$\begin{aligned} & 0.004 + 0.110x - 0.177y - 0.174x^2 + 0.224xy - 0.303y^2 \\ & - 0.168x^3 + 0.327x^2y - 0.087xy^2 - 0.013y^3 + 0.235x^4 \\ & - 0.667x^3y + 0.745x^2y^2 - 0.029xy^3 + 0.072y^4 = 0 \end{aligned}$$

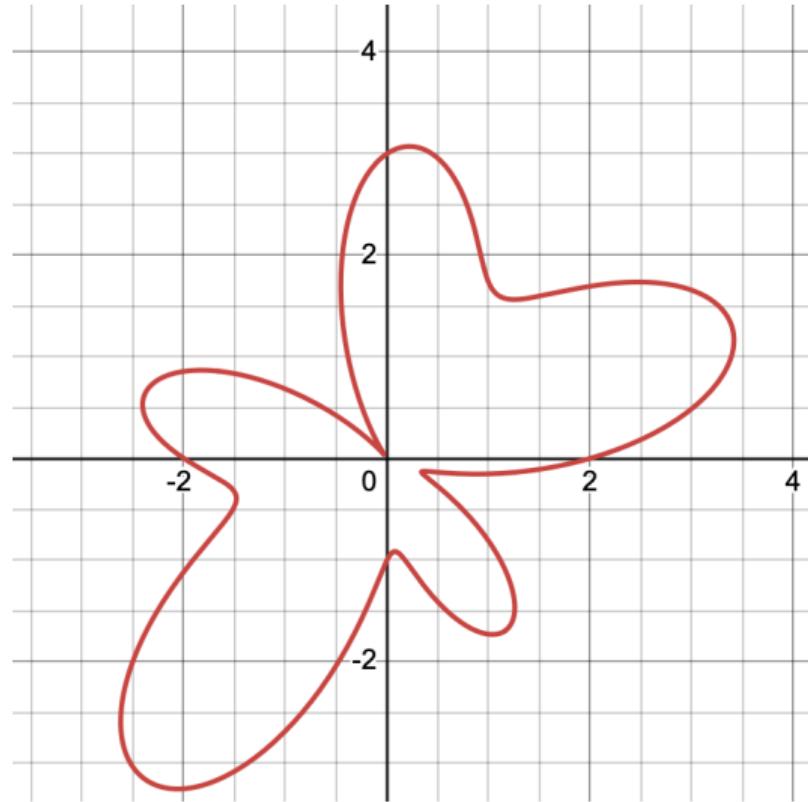


# Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas	álgebra	+	geometria
fórmulas	$\longleftrightarrow$	formas	
finito	$\longleftrightarrow$	infinito	

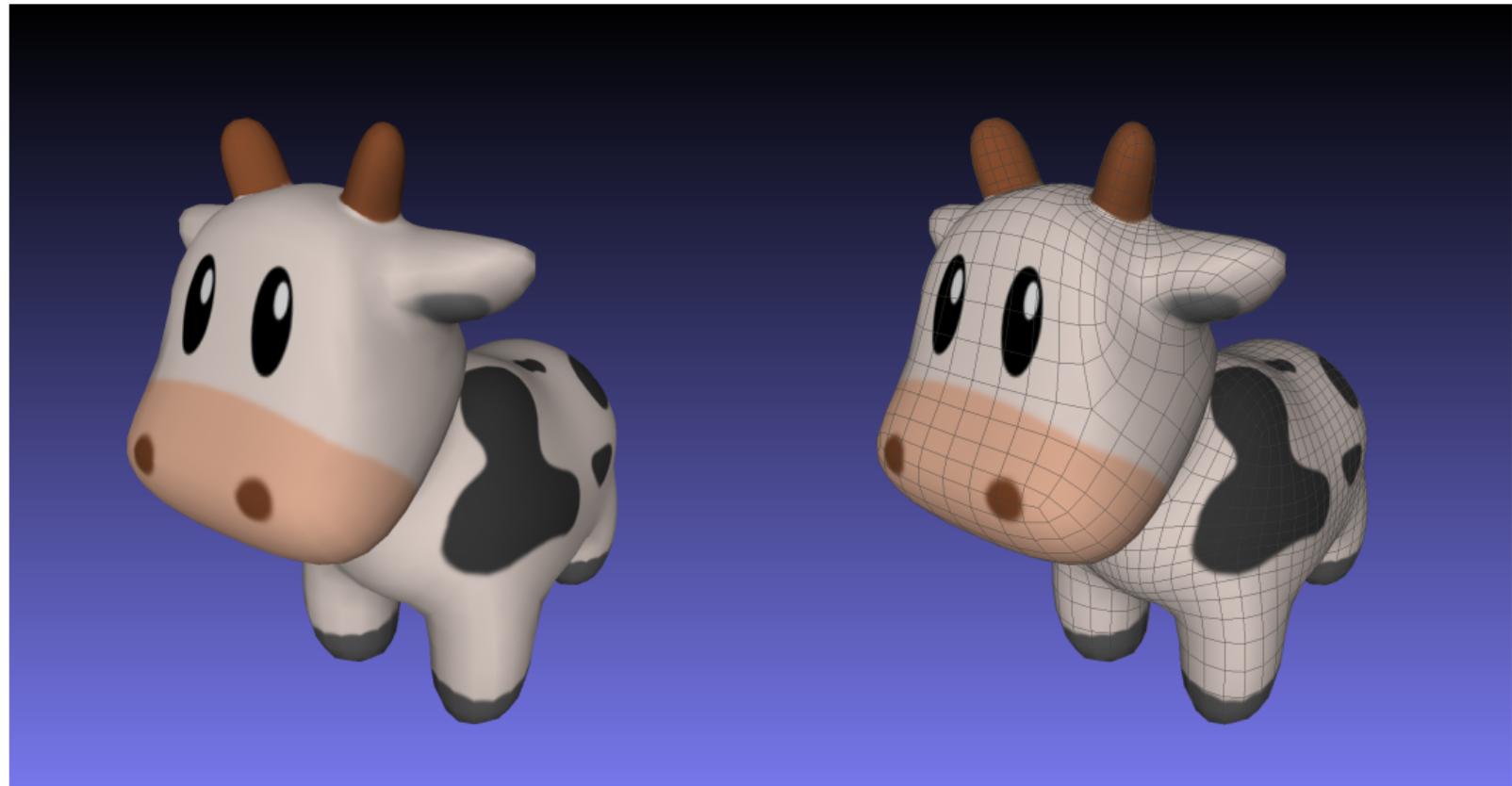


$$r = \sin(2\theta) + \sin(5\theta) + 2 \quad x = r \cos(\theta)$$
$$y = r \sin(\theta)$$

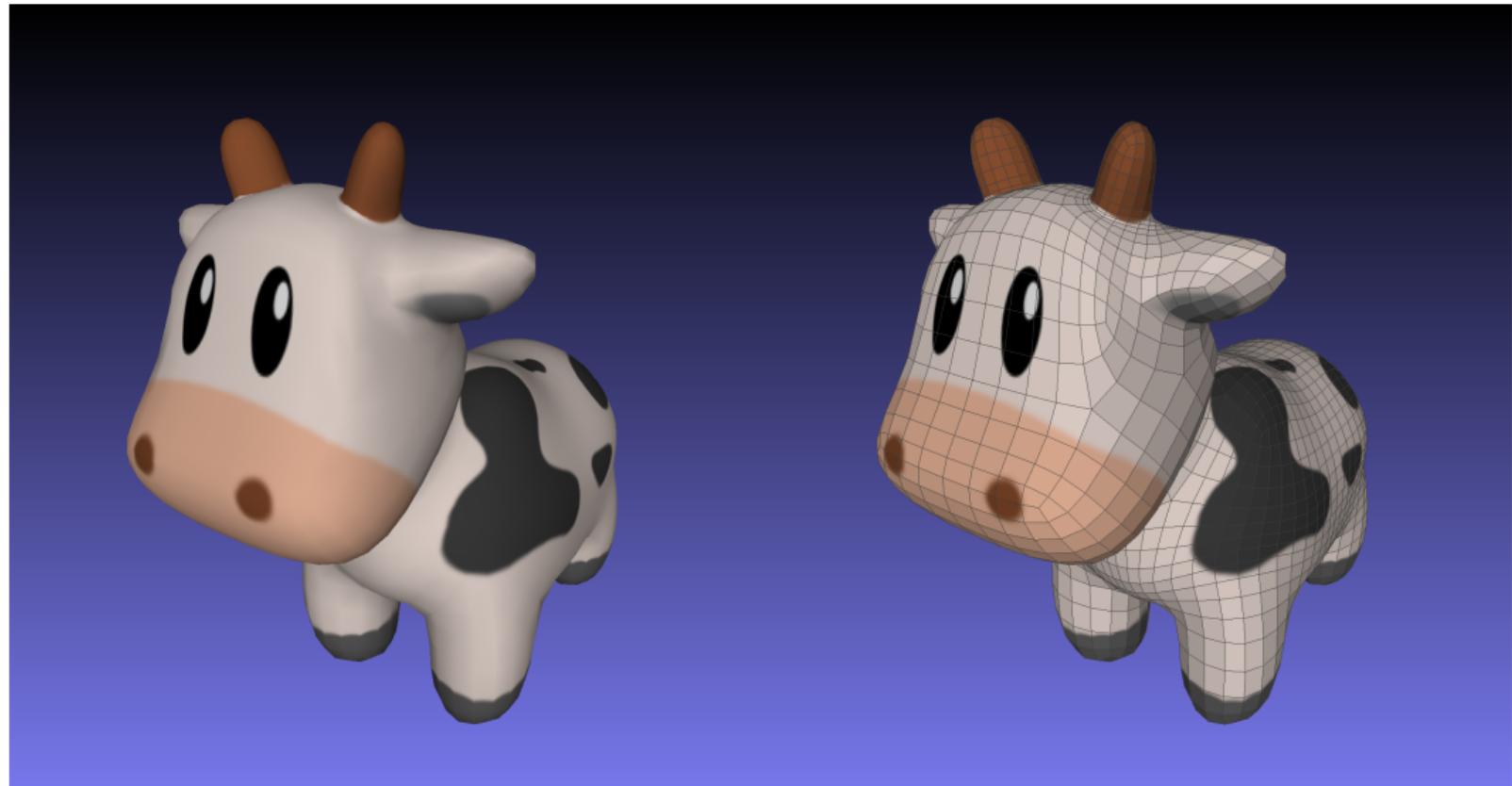
# Geometria analítica



# Geometria analítica



# Geometria analítica



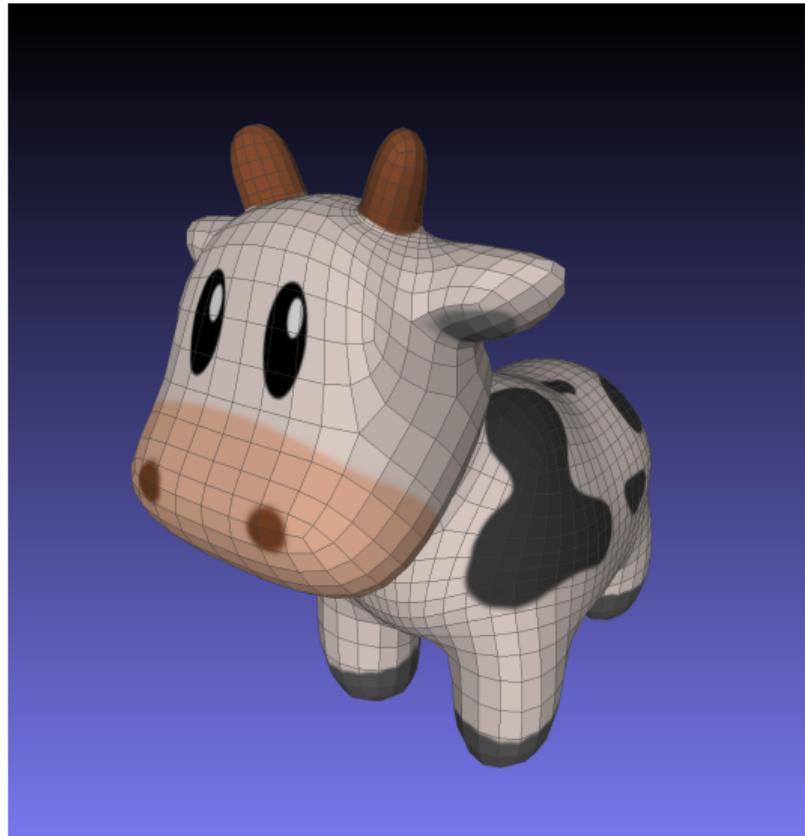
# Geometria analítica

```
v 0.348799 -0.334989 -0.0832331
v 0.313132 -0.399051 0.881192
v 0.266758 0.181628 0.122726
v 0.229555 0.0663178 0.828702
v 0.353623 -0.0486456 0.443969
v 0.335878 -0.384495 0.425693
v 0.305362 0.0307983 -0.00655663
v 0.279611 -0.0552387 0.858547
v 0.266719 0.10578 0.46681
v 0.149341 -0.451522 0.166423
v 0.12606 -0.162036 -0.185668
v 0.114009 -0.358339 -0.0759521
v 0.101546 -0.475382 0.663157
v 0.240757 -0.435635 0.421981
v 0.385825 -0.174501 0.180766
```



# Geometria analítica

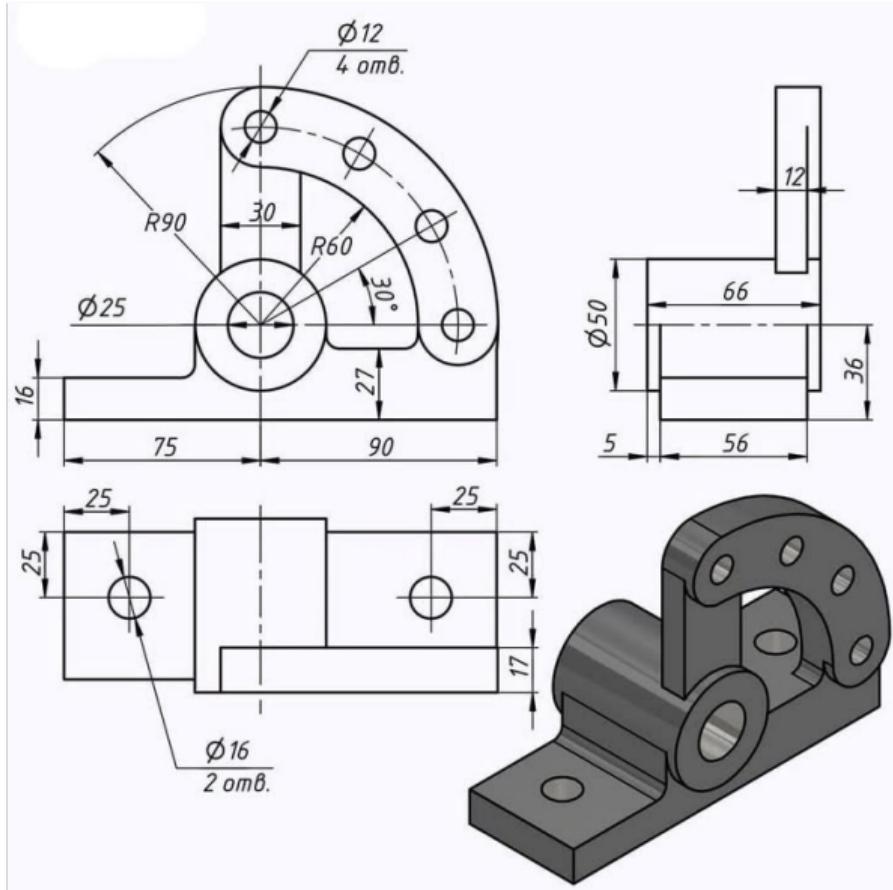
```
f 6 735 739 738
f 189 736 739 735
f 193 737 739 736
f 192 738 739 737
f 14 740 743 742
f 190 741 743 740
f 193 736 743 741
f 189 742 743 736
f 10 744 747 746
f 191 745 747 744
f 193 741 747 745
f 190 746 747 741
f 16 748 750 749
f 192 737 750 748
f 193 745 750 737
```



métodos de modelagem

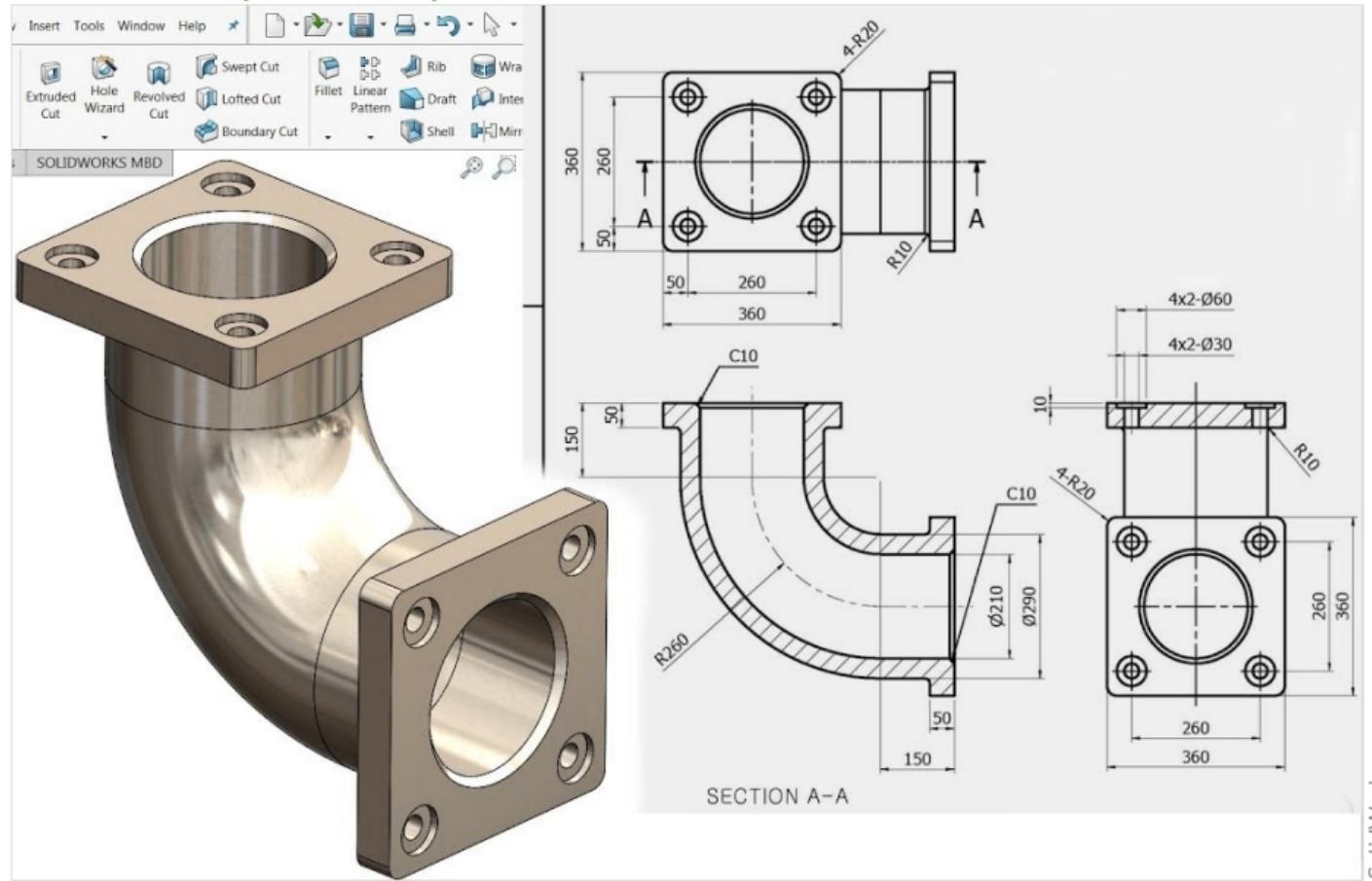
# Projeto assistido por computador

CAD



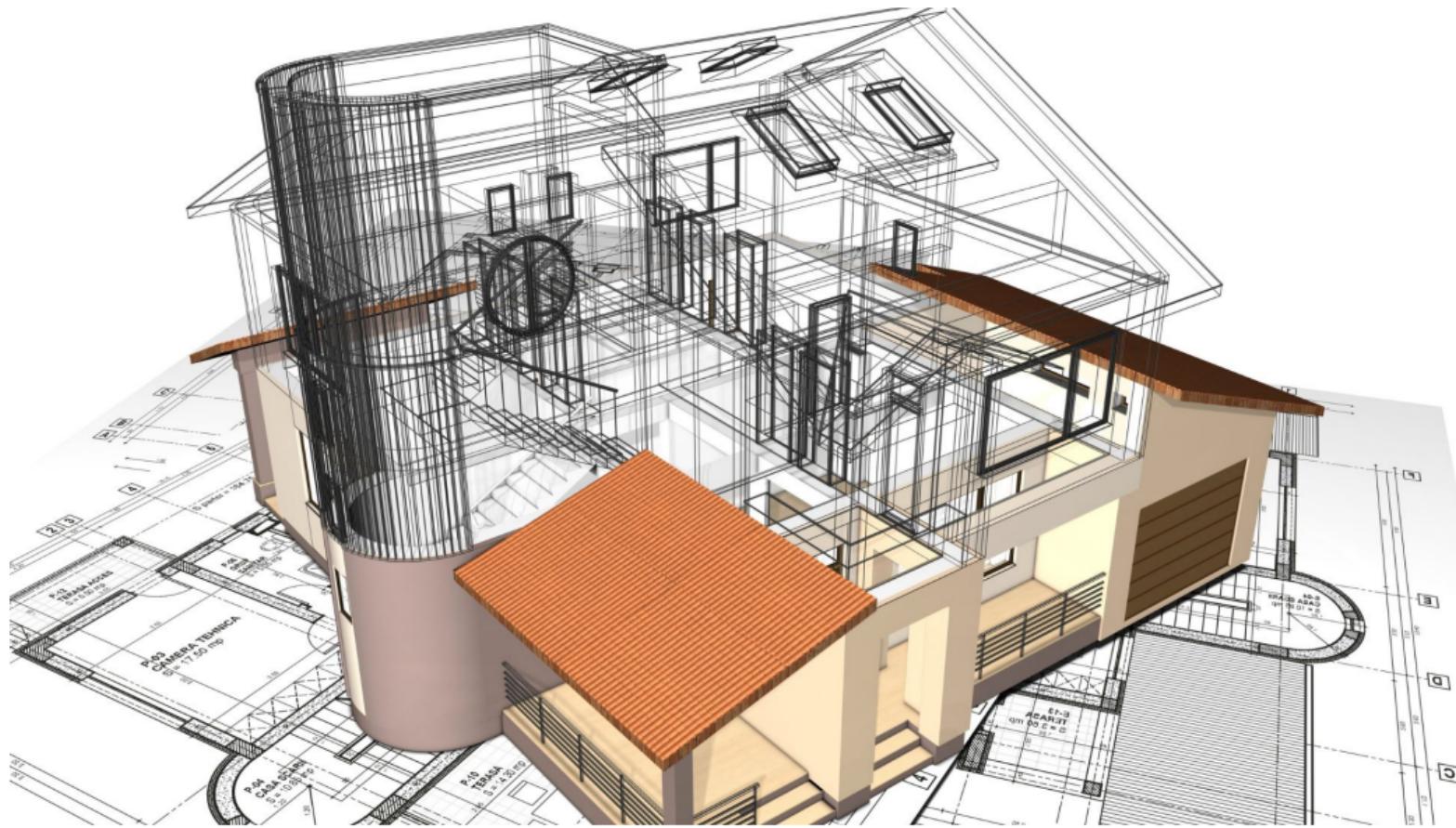
# Projeto assistido por computador

CAD



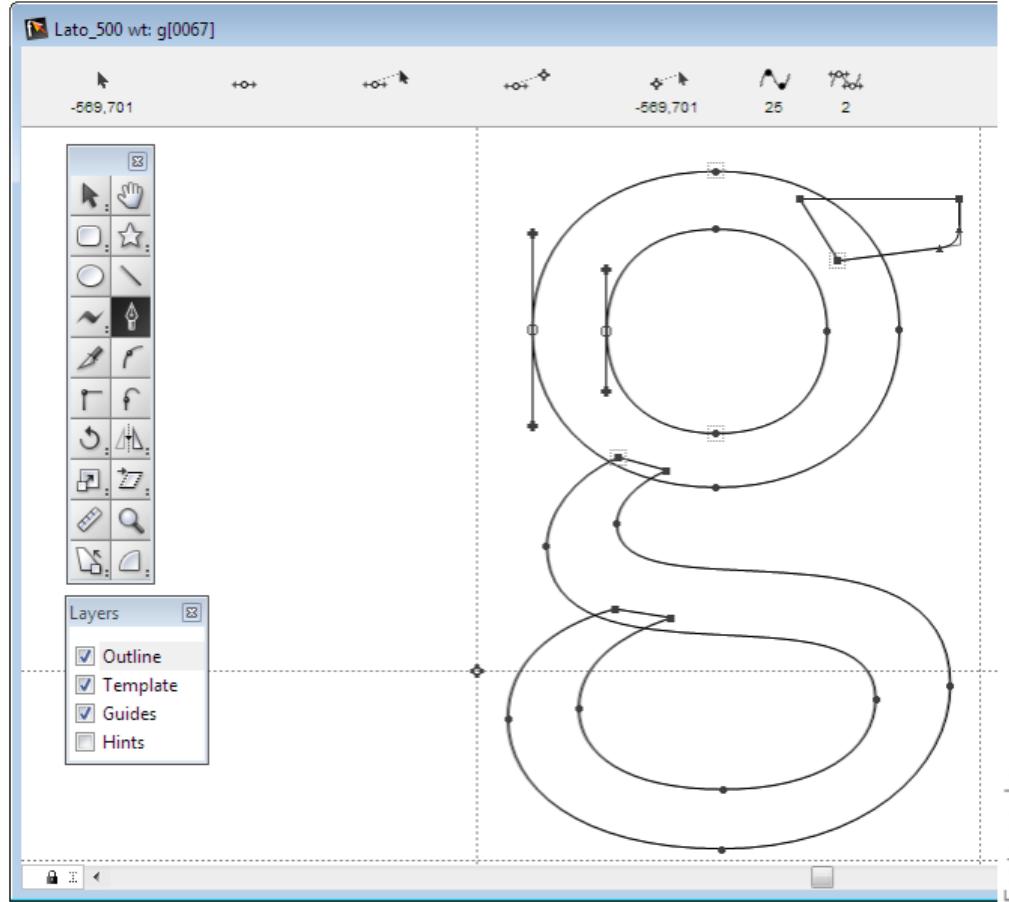
# Projeto assistido por computador

CAD



# Projeto assistido por computador

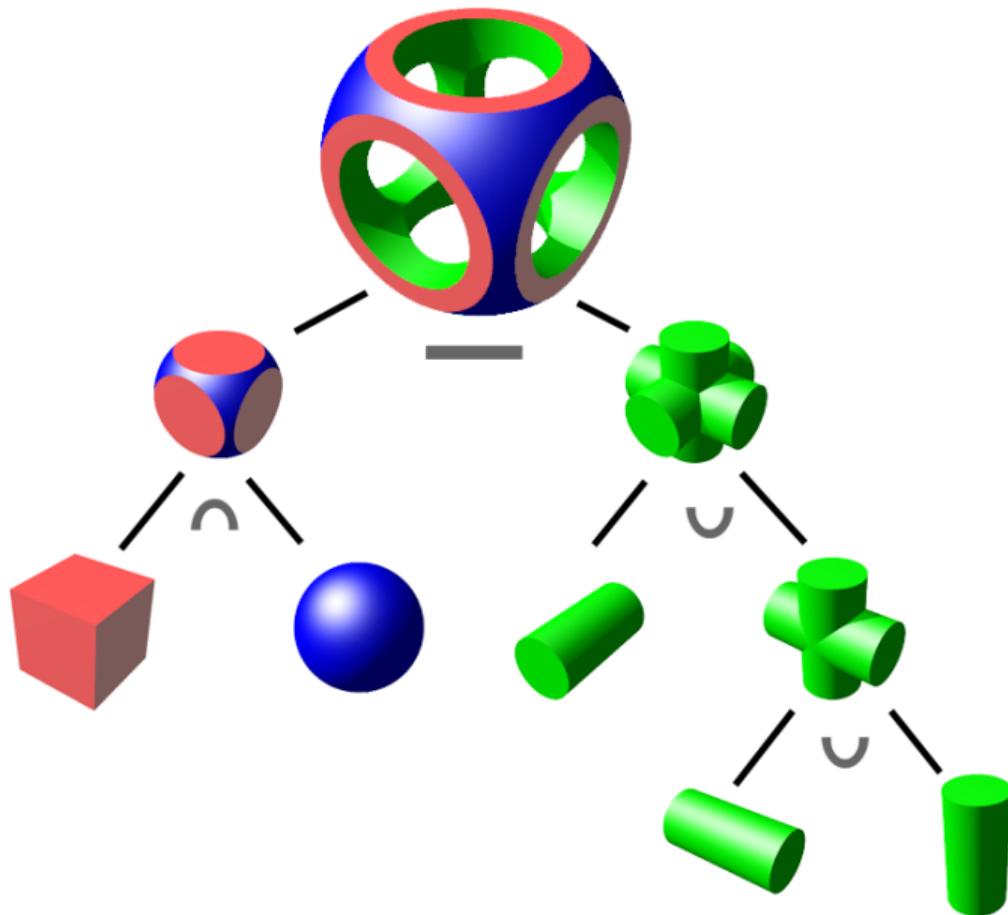
CAD



Fontographer

# Geometria sólida construtiva

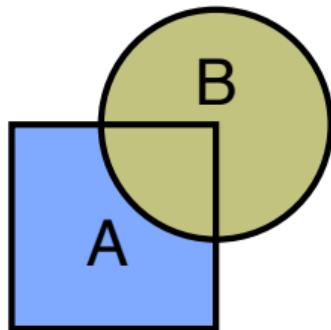
CSG



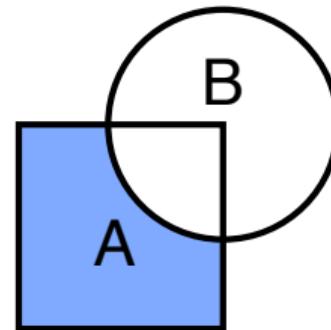
Wikipedia

# Operações booleanas

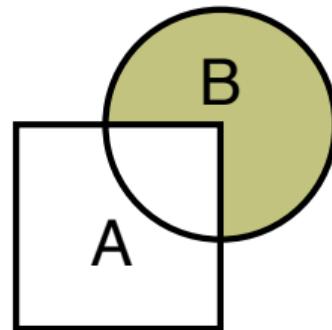
A e B



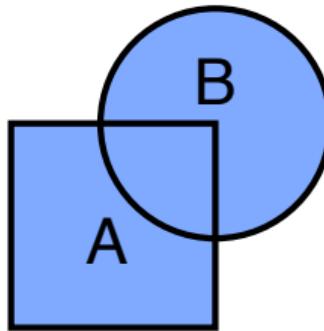
A menos B



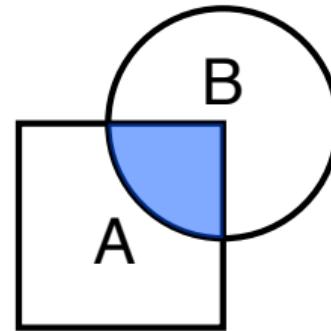
B menos A



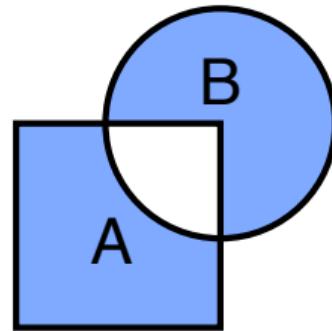
A união B



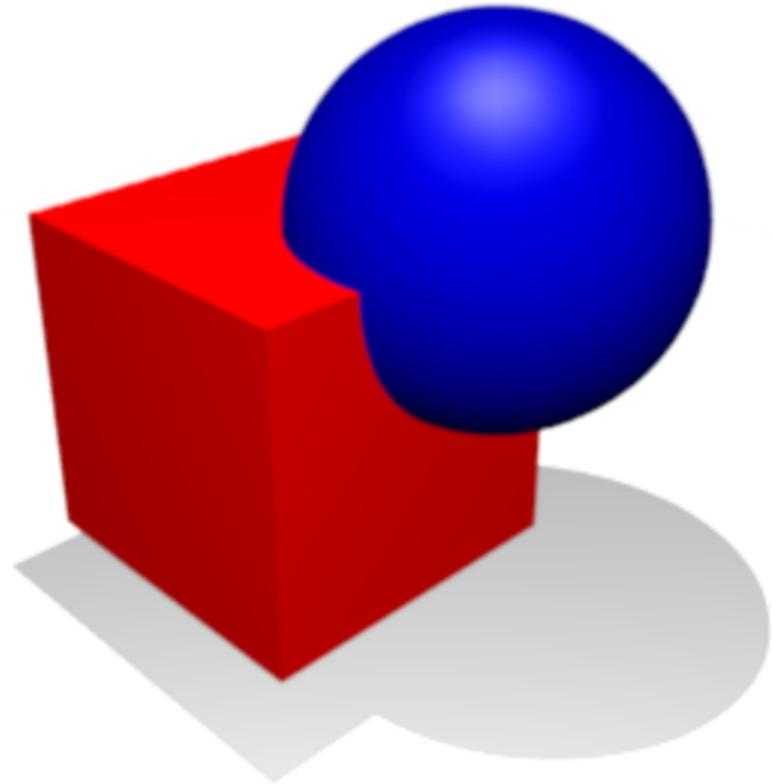
A interseção B



A diff B

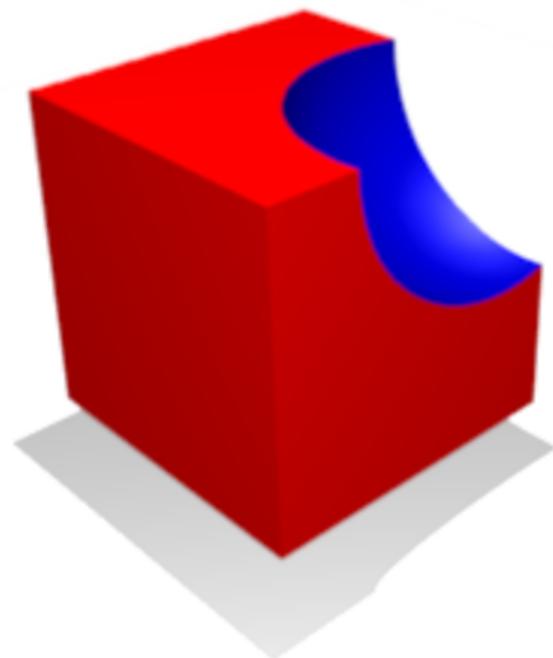


## Operações booleanas



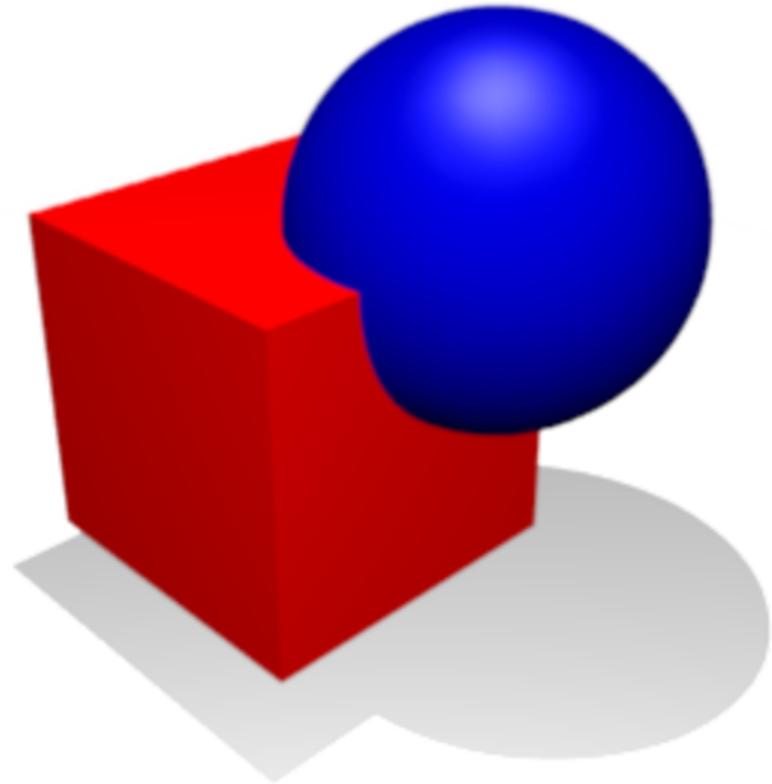
Wikipedia

## Operações booleanas



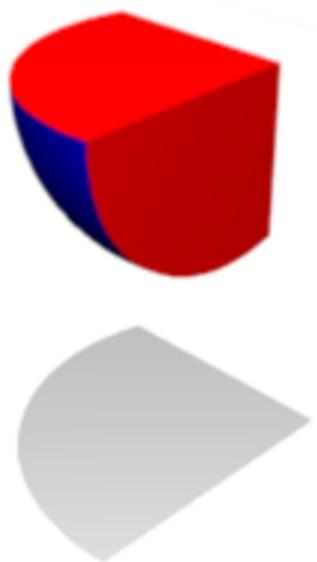
Wikipedia

## Operações booleanas



Wikipedia

## Operações booleanas



Wikipedia

## Operações booleanas

1  
d

## Operações booleanas



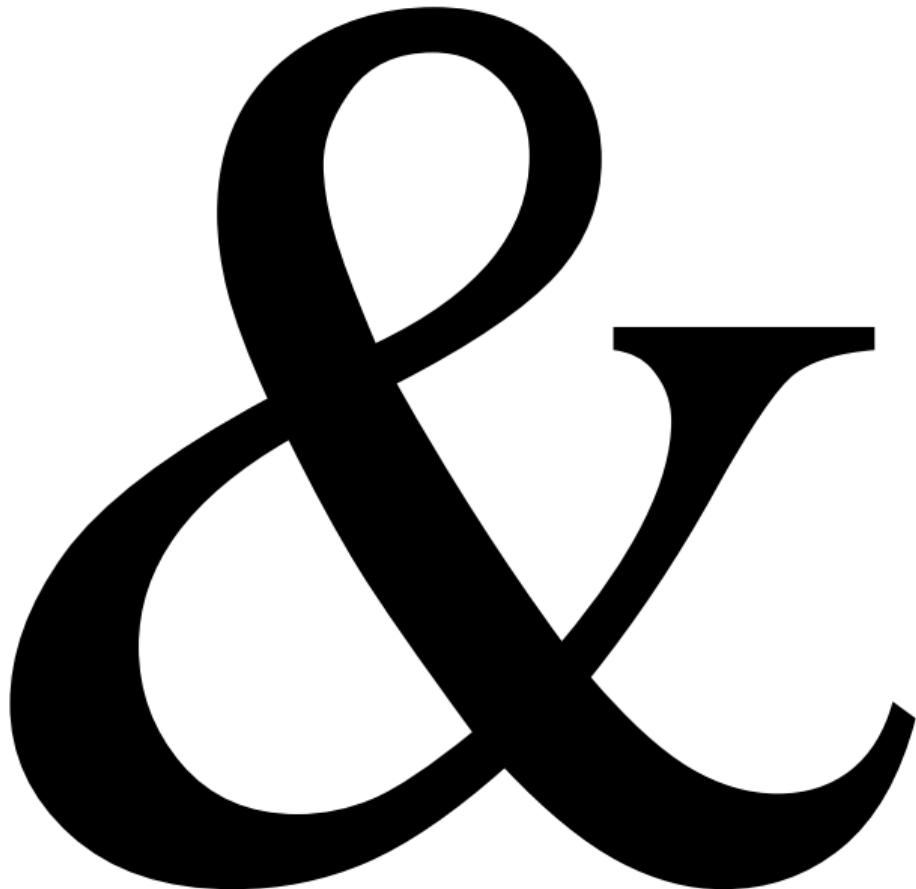
## Operações booleanas

*and*

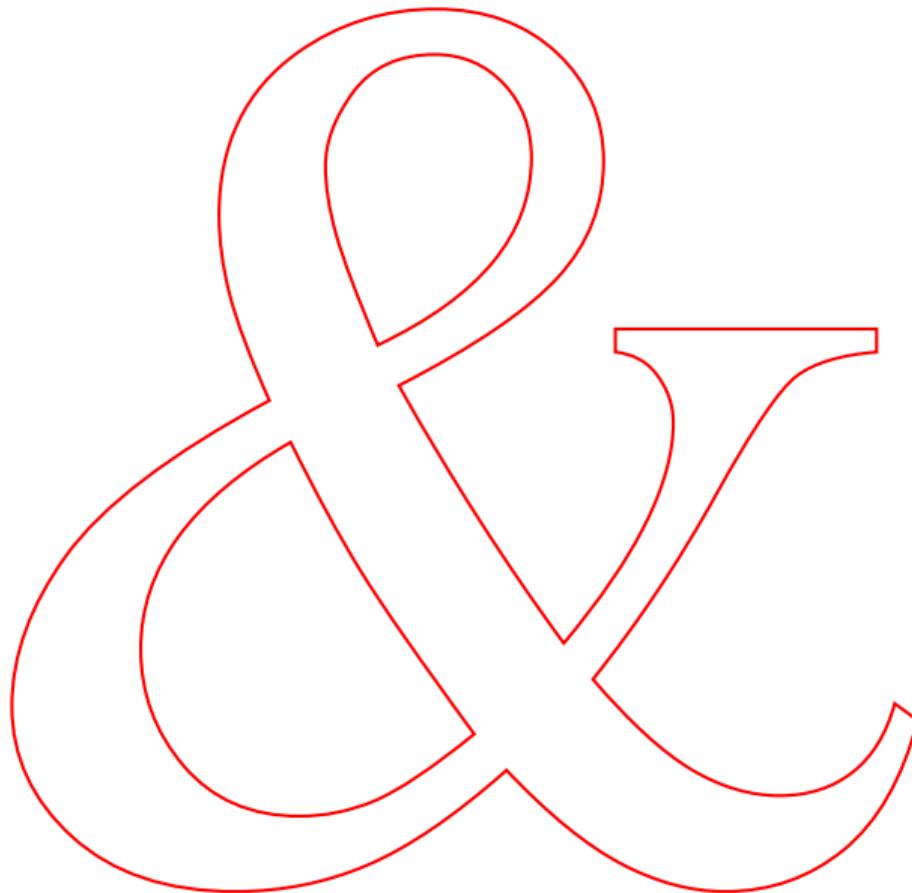
## Operações booleanas



Representação por fronteira

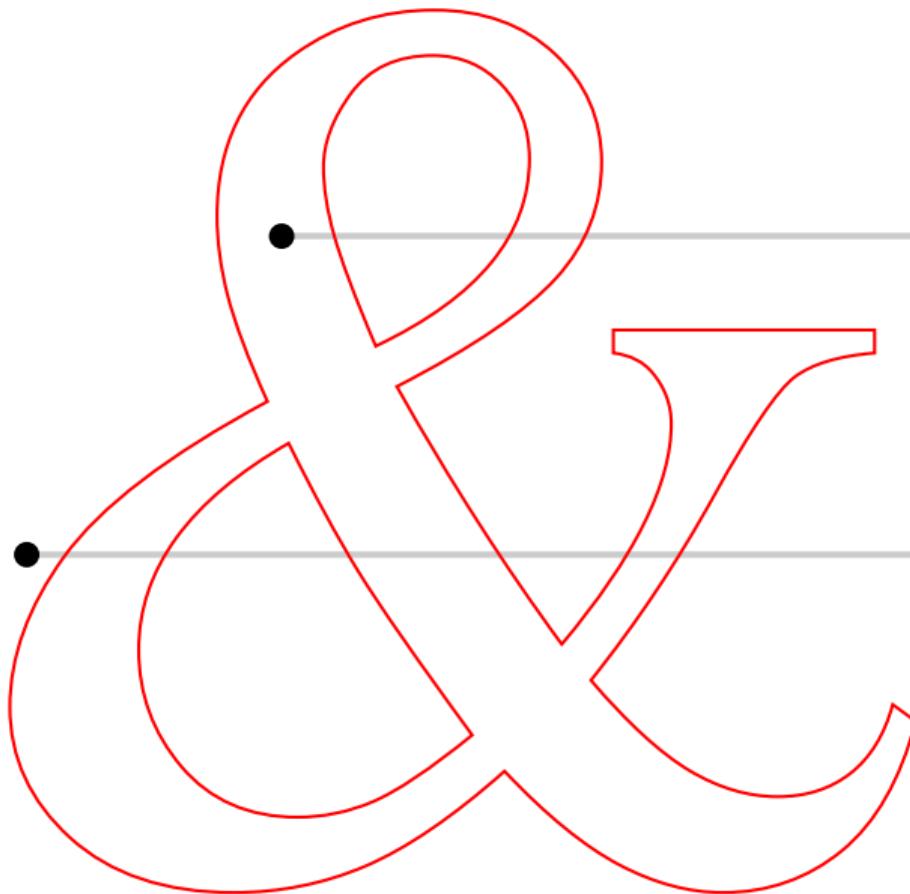


## Representação por fronteira



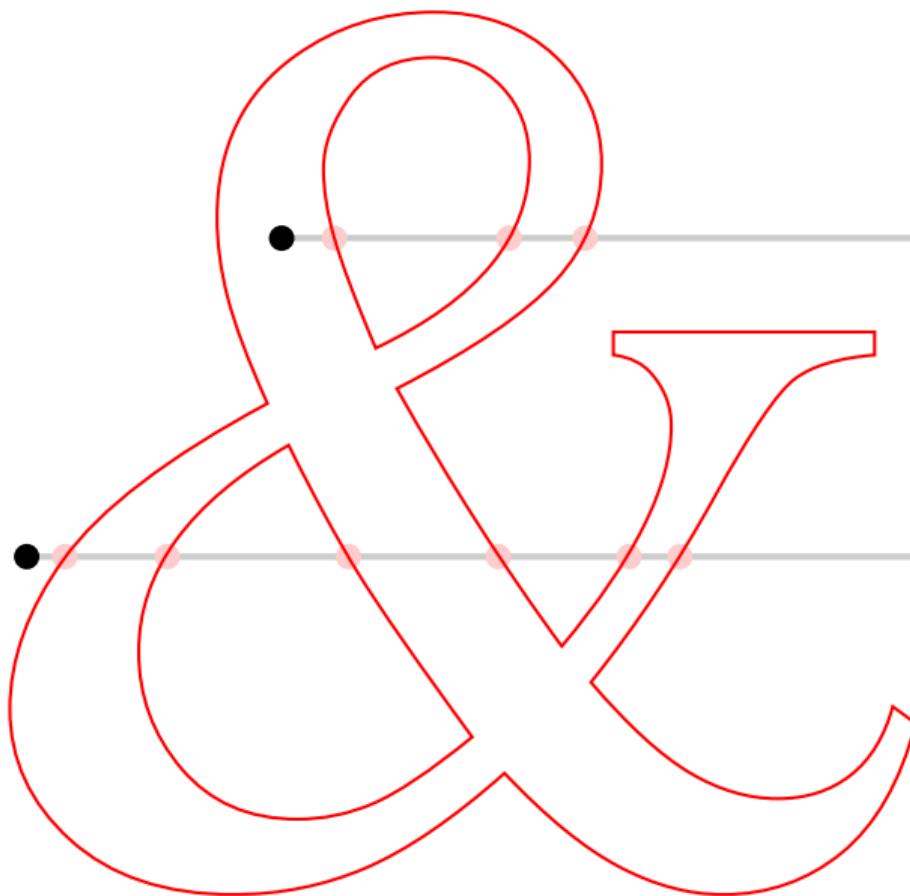
## Representação por fronteira

Jordan (1887)

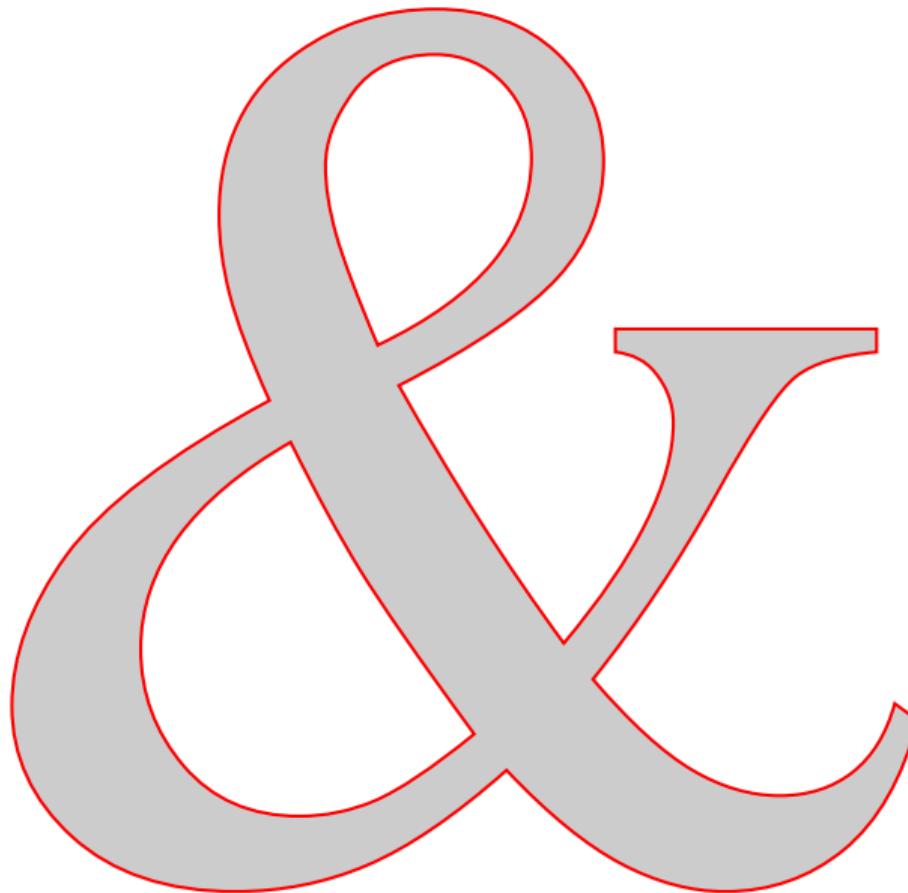


## Representação por fronteira

Jordan (1887)



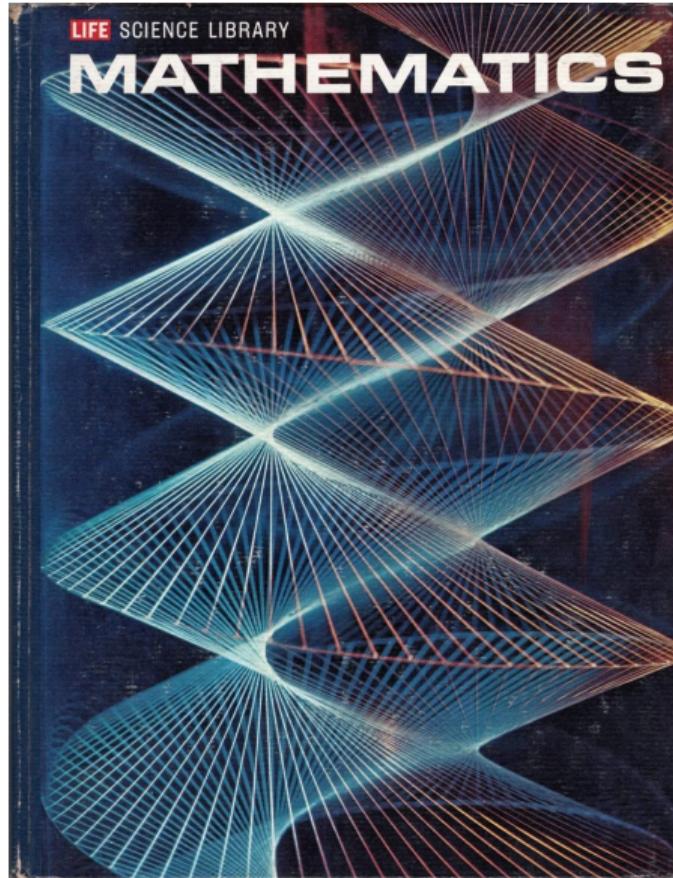
## Representação por fronteira



# Representação por fronteira

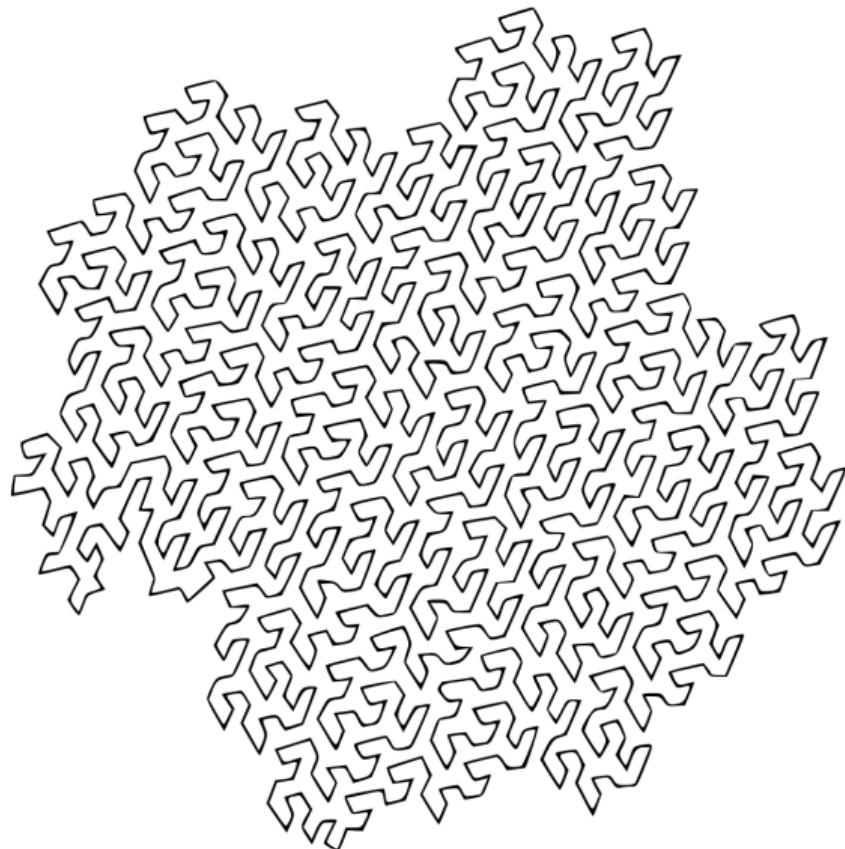


## Representação por fronteira

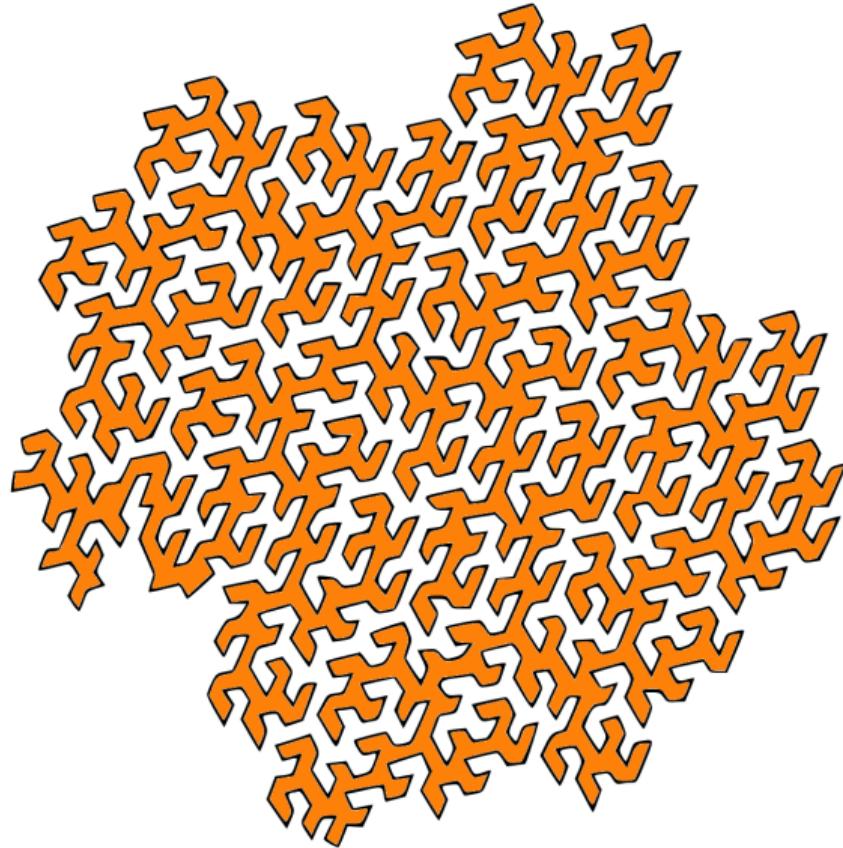


Life Science Library (1963)

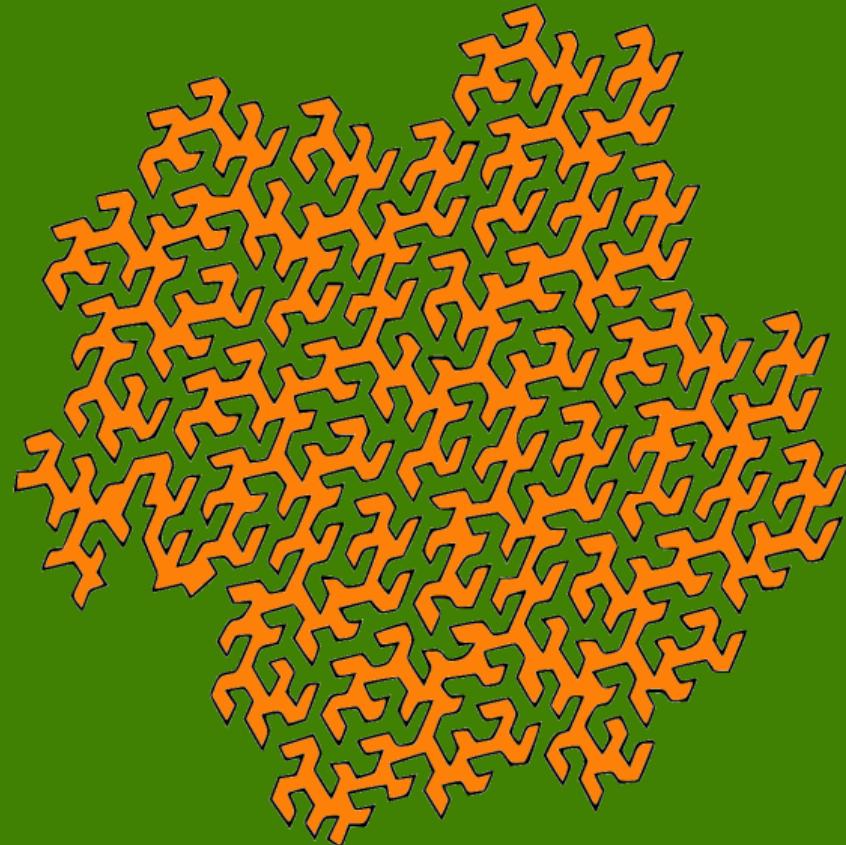
## Representação por fronteira



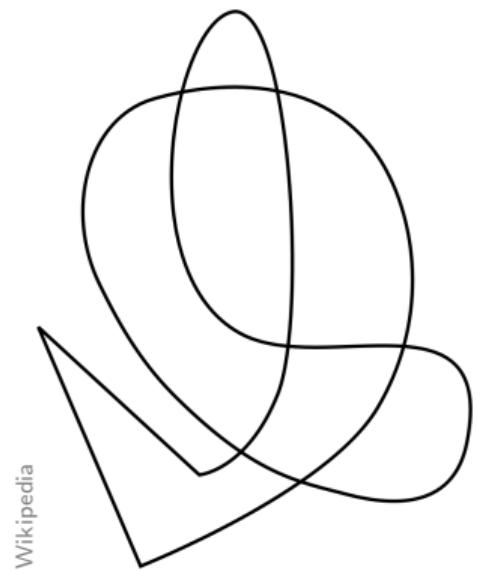
## Representação por fronteira



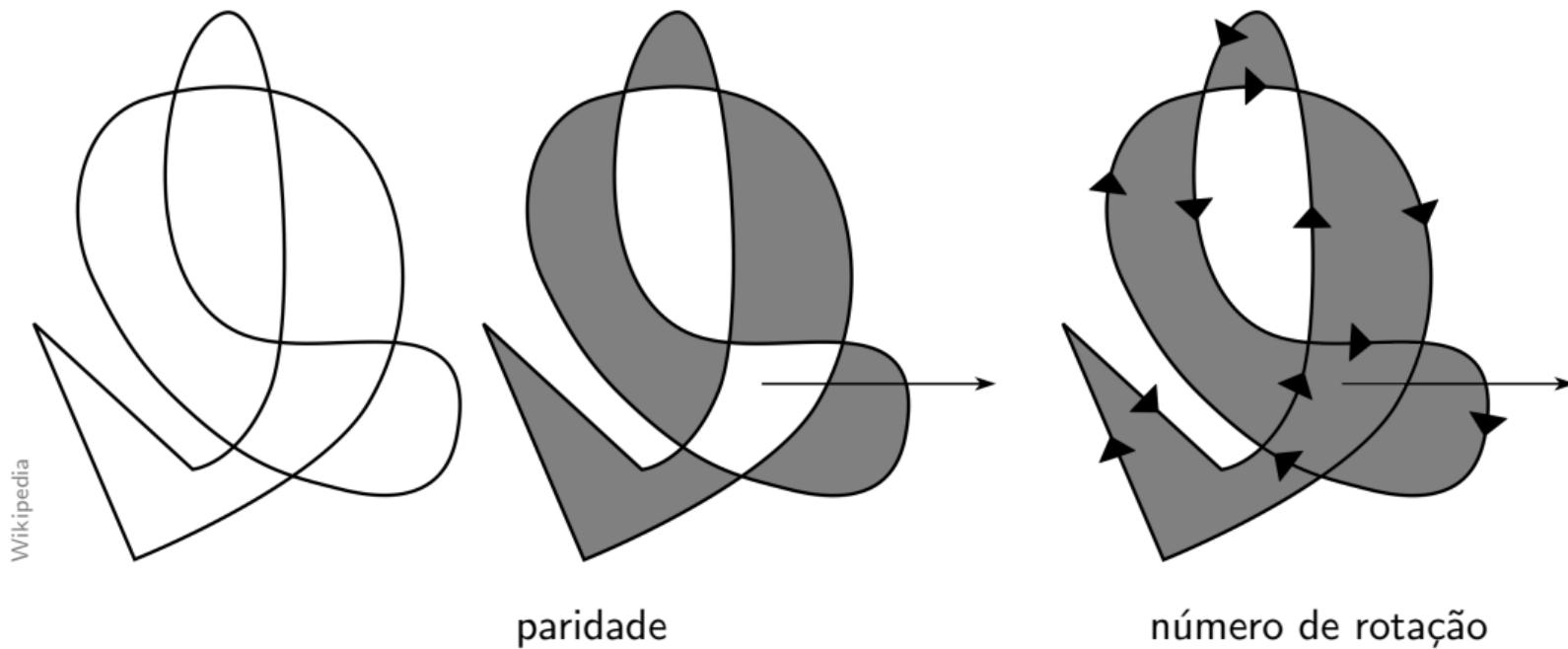
## Representação por fronteira



## Representação por fronteira



## Representação por fronteira



Representação por fronteira

Three stylized lowercase letters 'd' in red, each with a red outline and a light red fill. The first letter has a vertical stroke extending upwards.

# Representação por fronteira

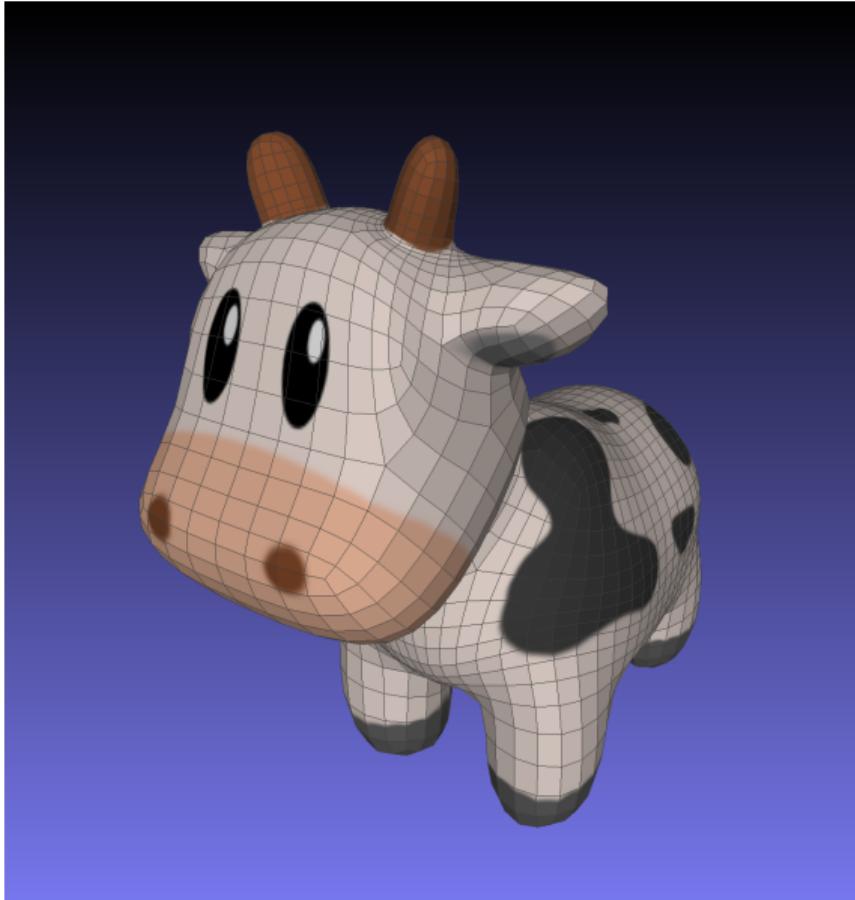
3D



Crane

# Representação por fronteira

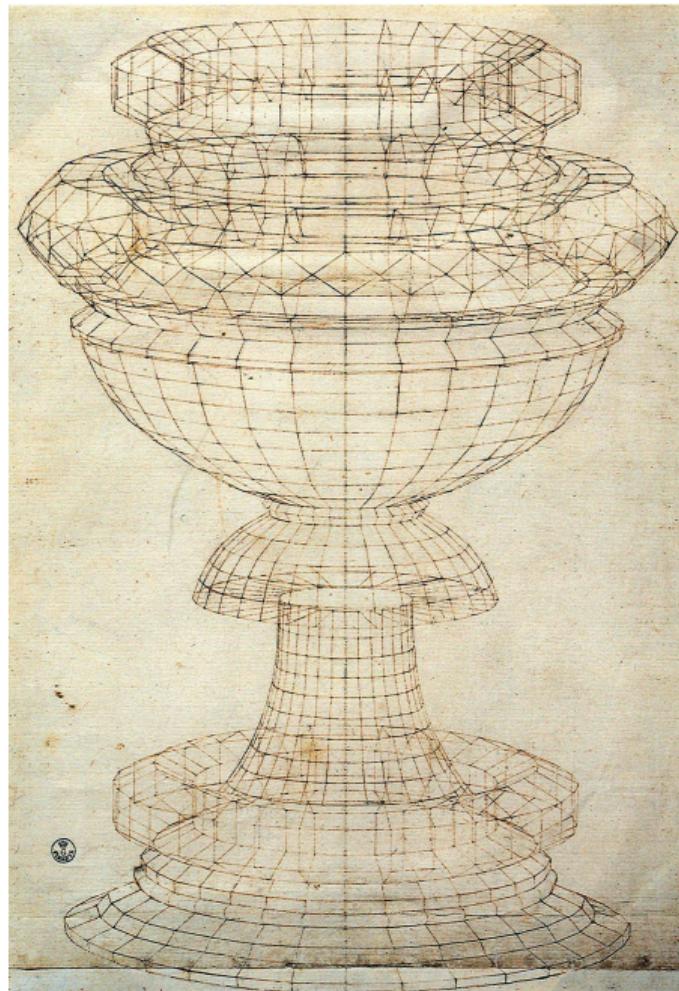
3D



Crane

decomposição

Uccello (1450)



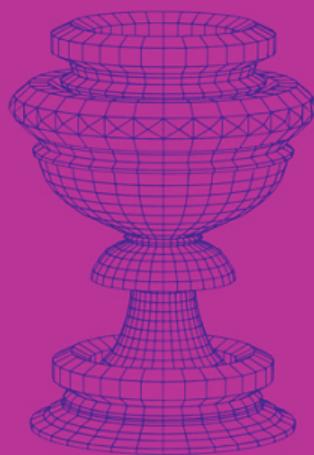


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Uccello (1450)

# COMPUTER AIDED GEOMETRIC DESIGN



## Métodos de decomposição

- decomposição celular
- decomposição intrínseca
- decomposição extrínseca

## Métodos de decomposição

- decomposição celular = discretização  
amostragem do objeto  
estruturação em células
- decomposição intrínseca
- decomposição extrínseca

## Métodos de decomposição

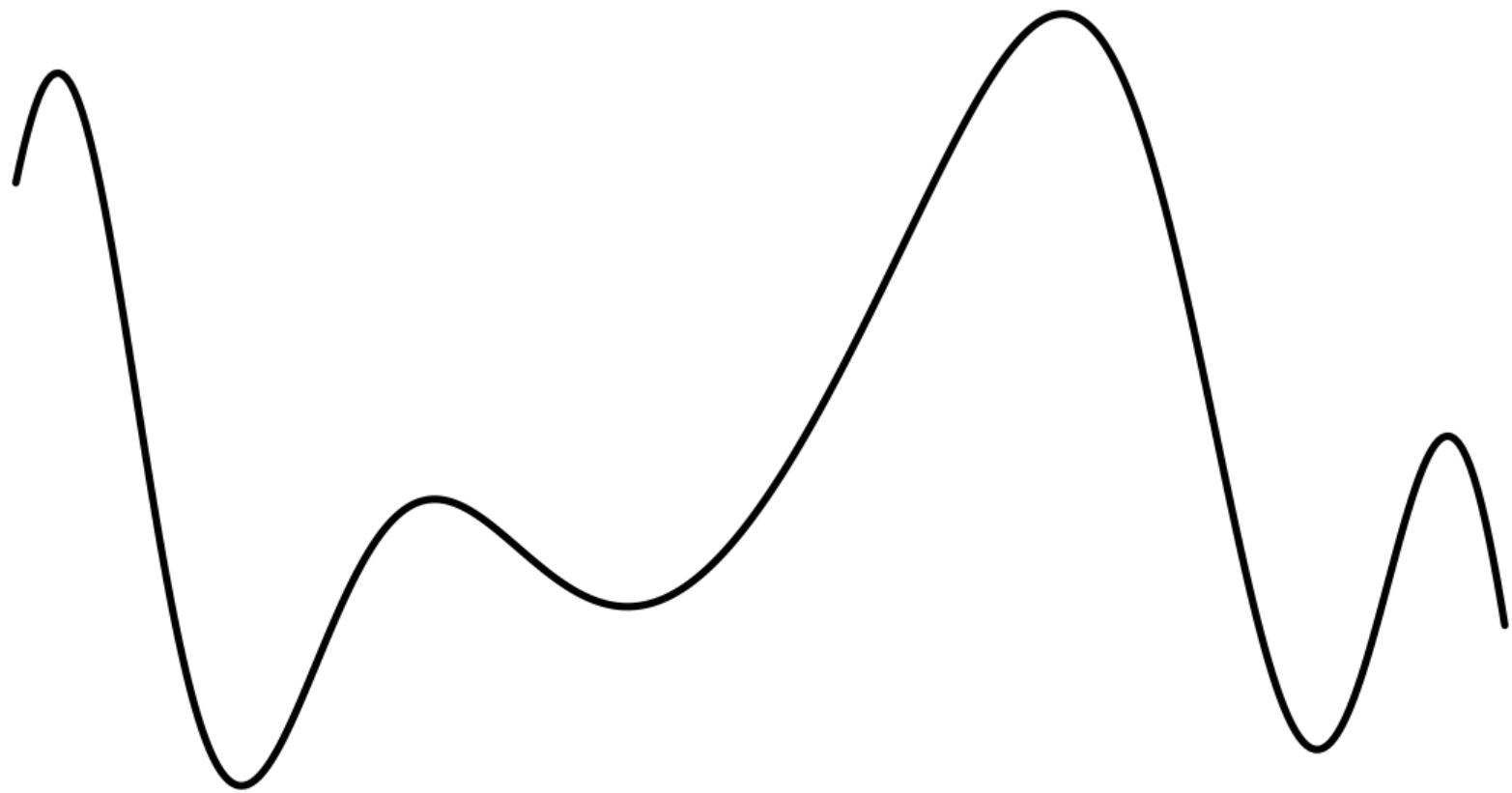
- decomposição celular = discretização  
amostragem do objeto  
estruturação em células
- decomposição intrínseca  
geometria do objeto  
poligonização
- decomposição extrínseca  
localização do objeto  
enumeração espacial

decomposição intrínseca  
poligonização

decomposição intrínseca  
amostragem + estruturação

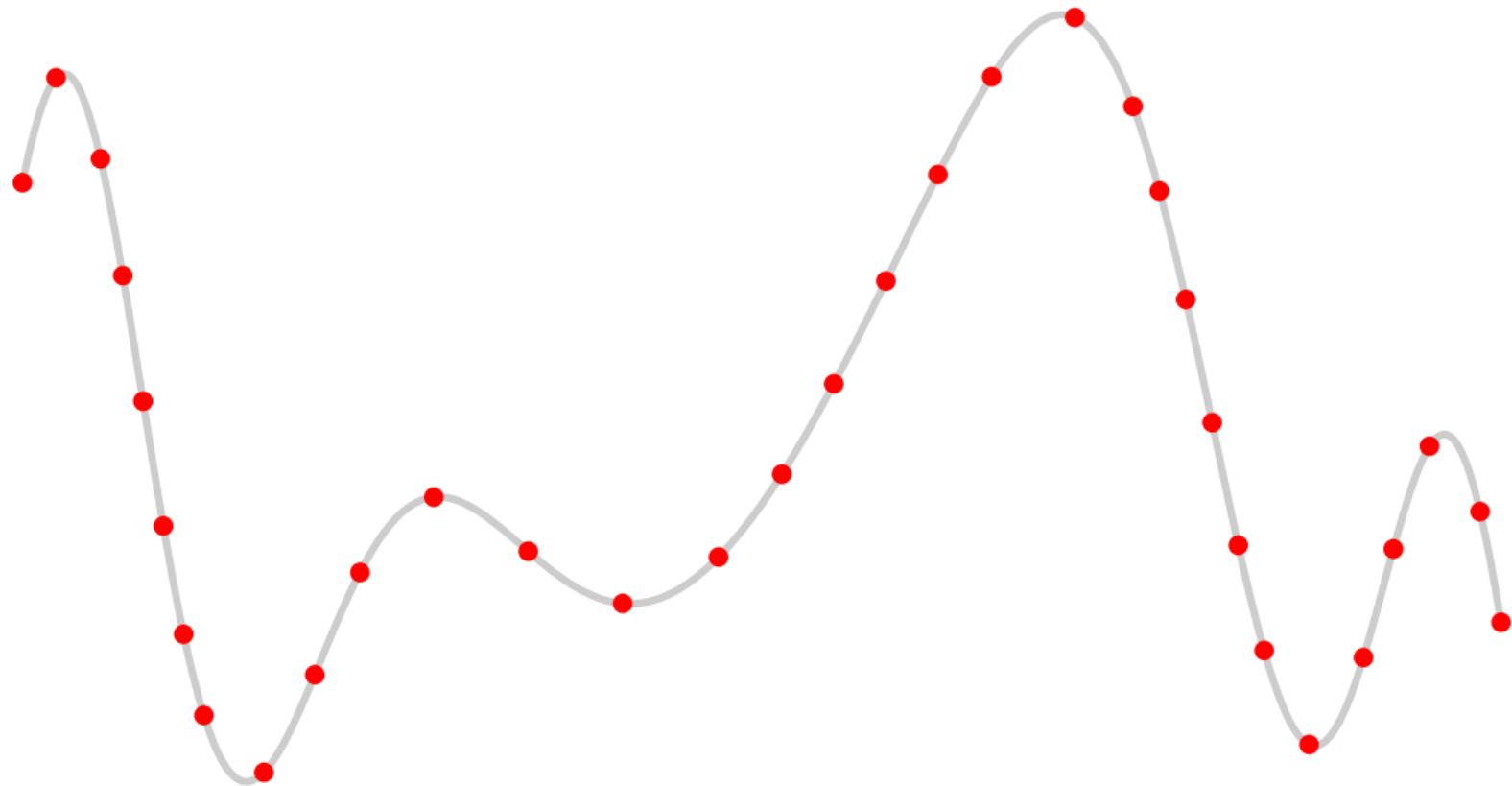
## Decomposição celular intrínseca

1D



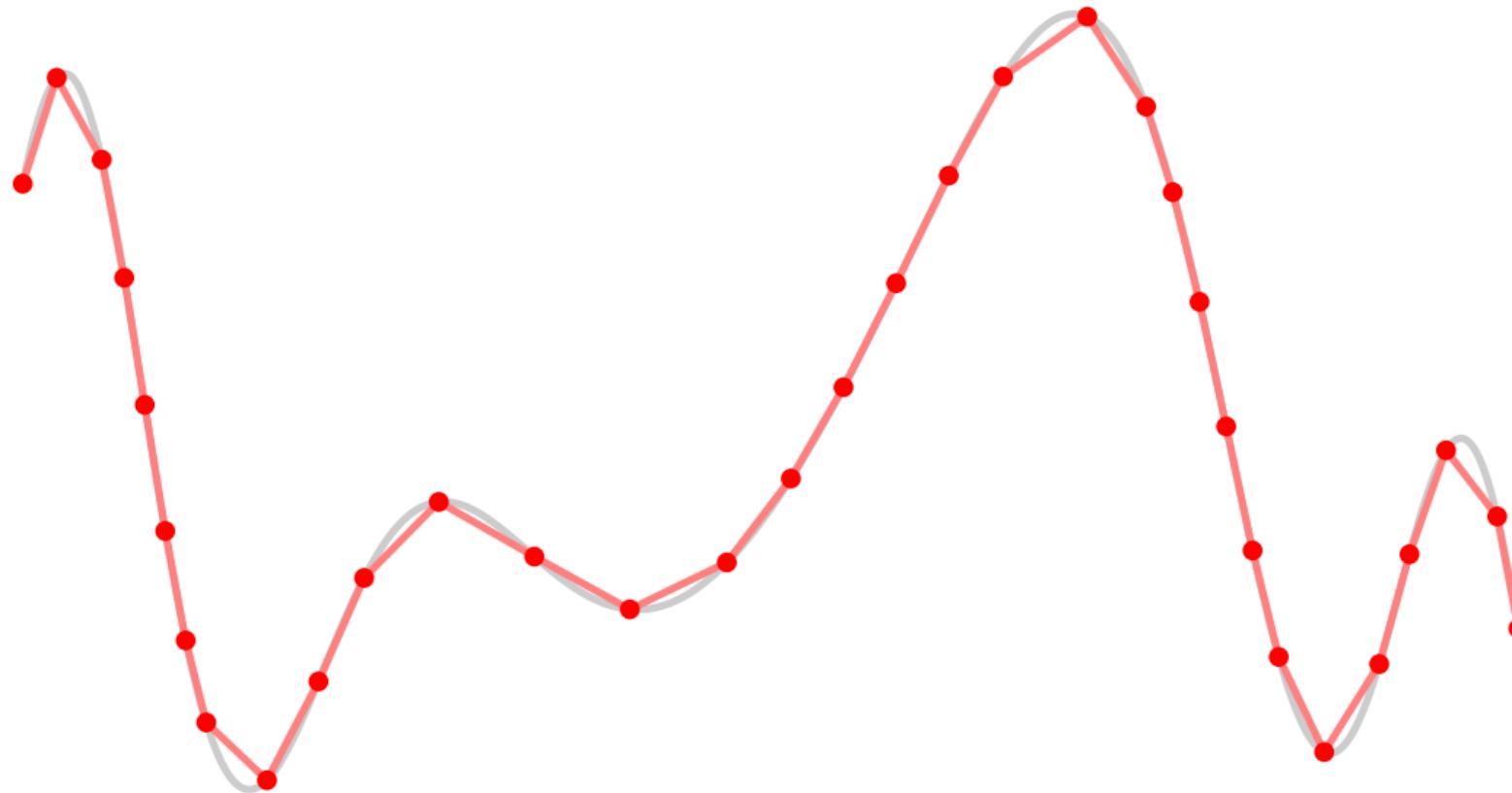
# Decomposição celular intrínseca

1D



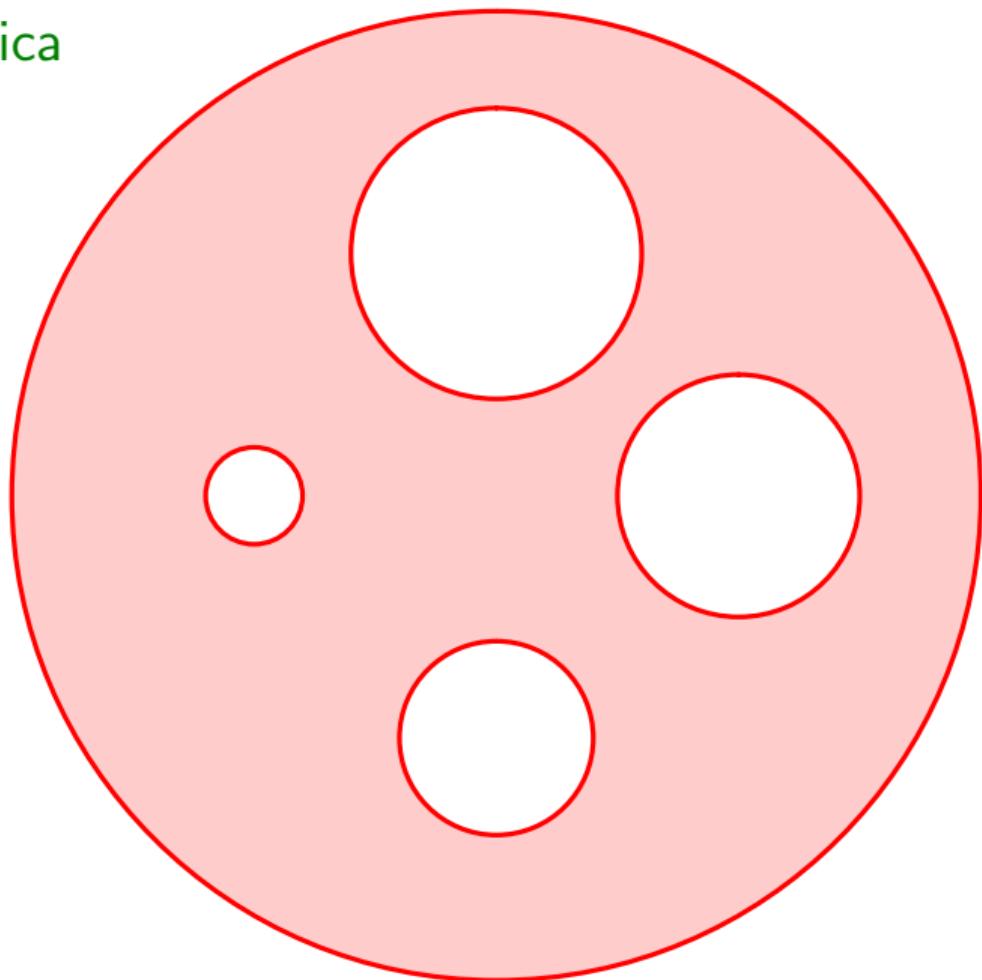
# Decomposição celular intrínseca

1D



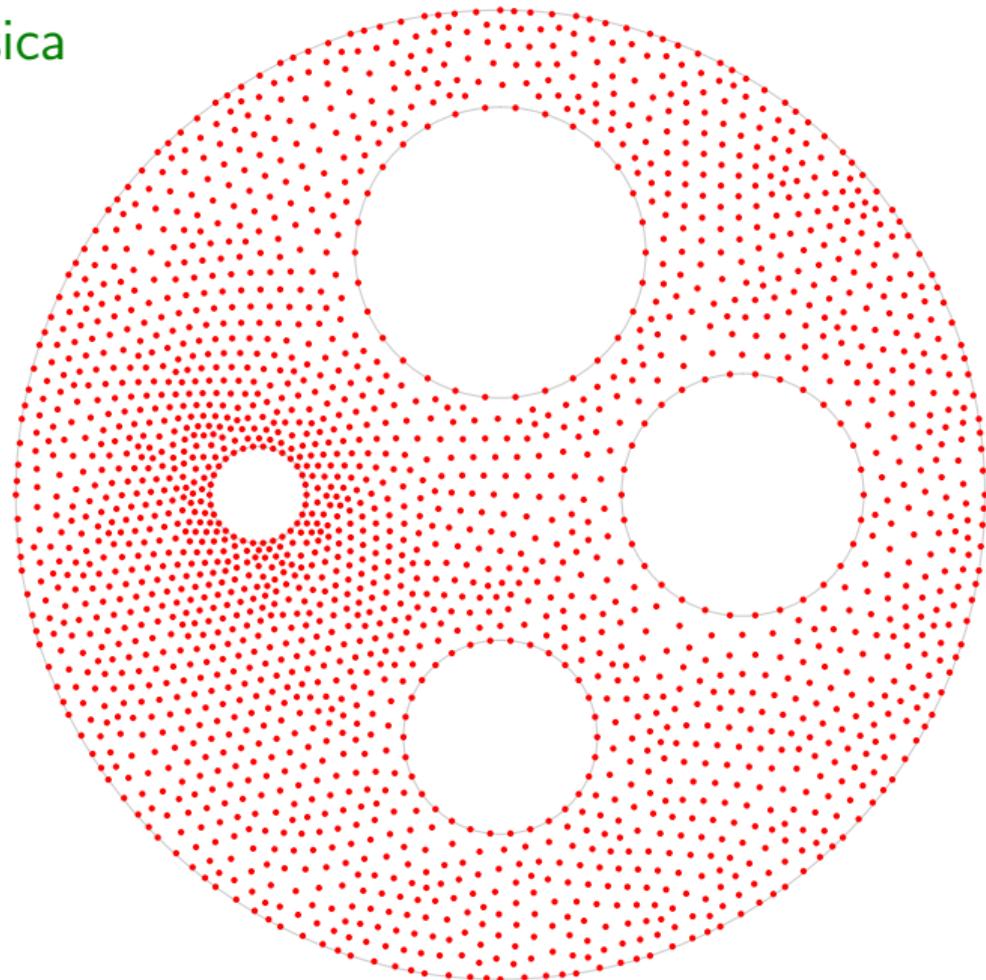
## Decomposição celular intrínseca

2D



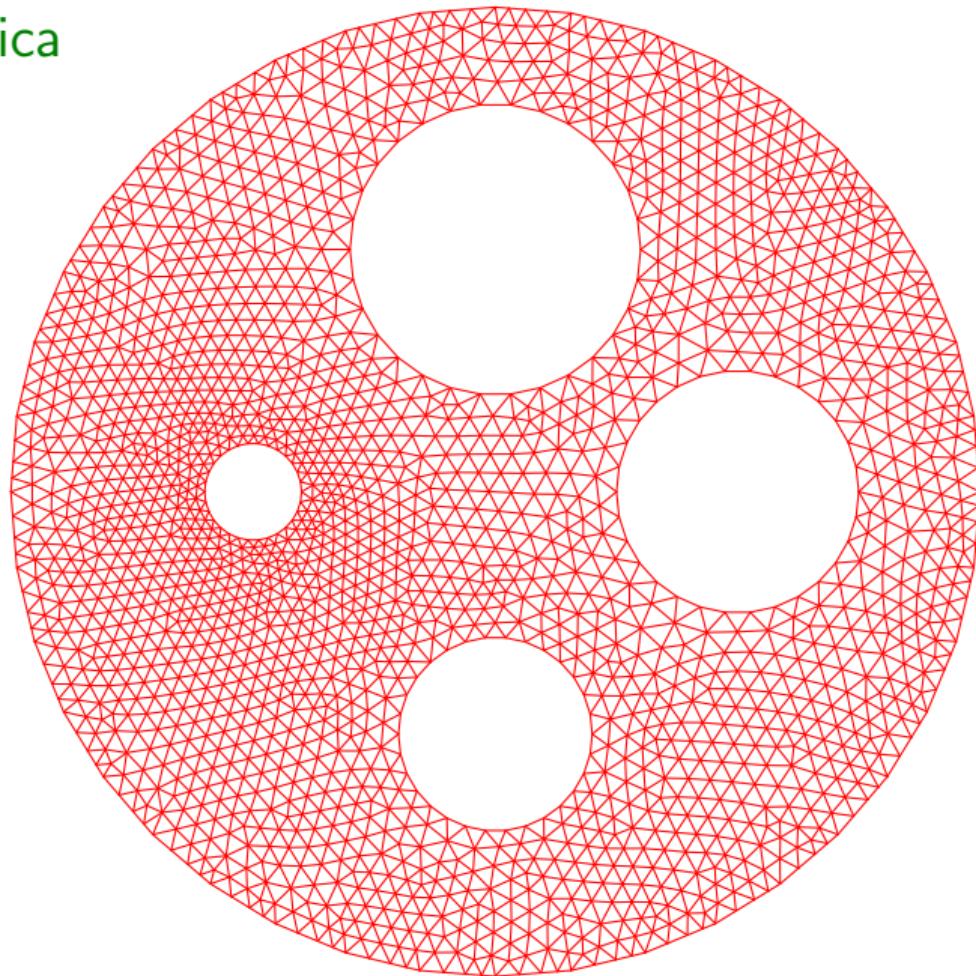
## Decomposição celular intrínseca

2D



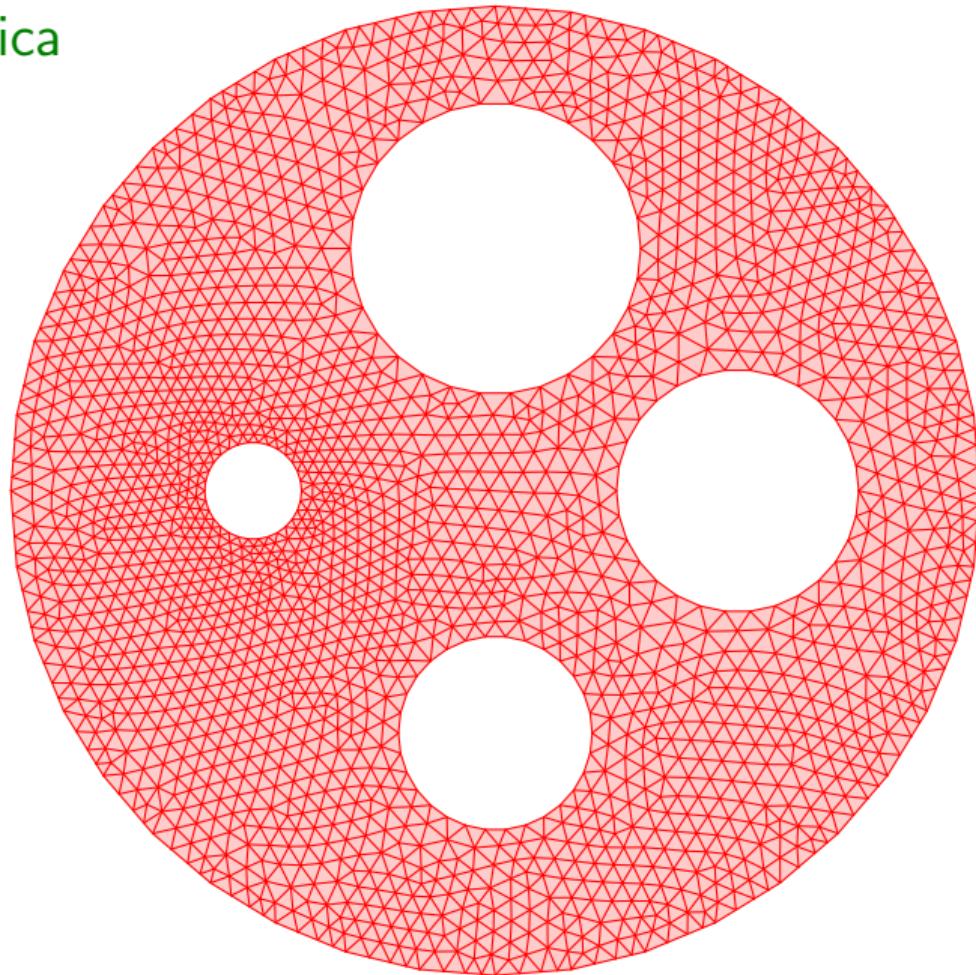
## Decomposição celular intrínseca

2D



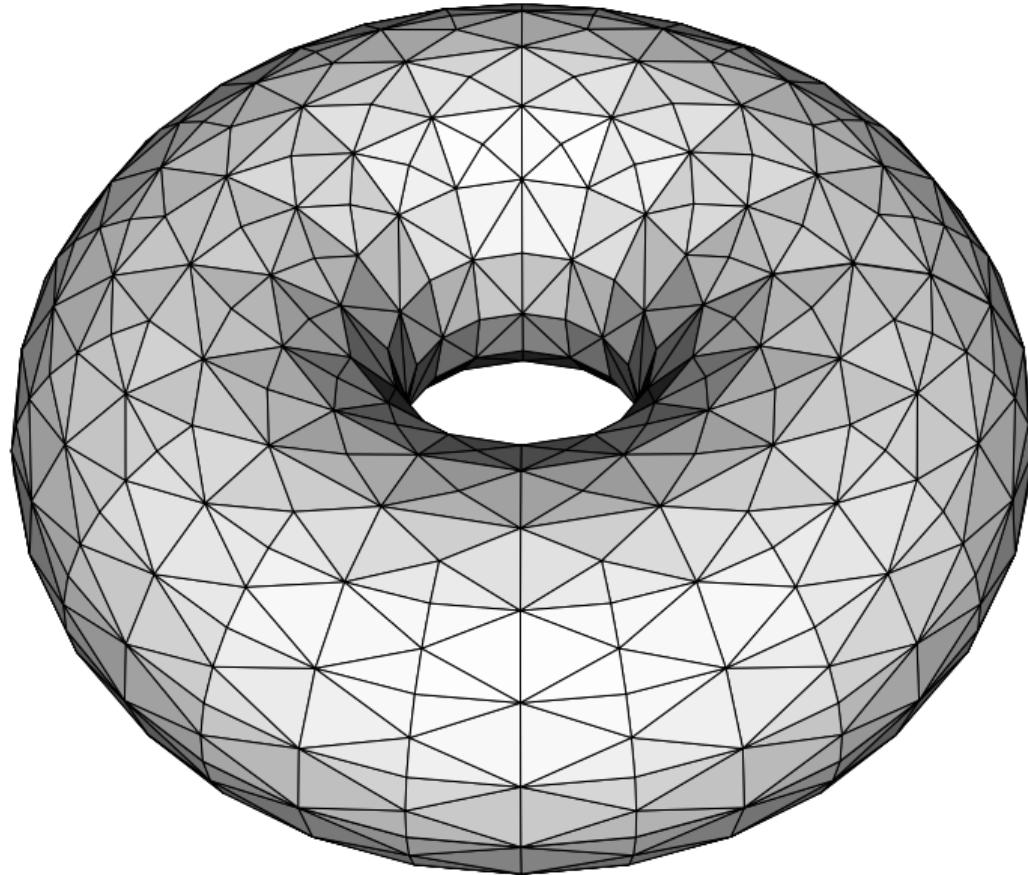
## Decomposição celular intrínseca

2D



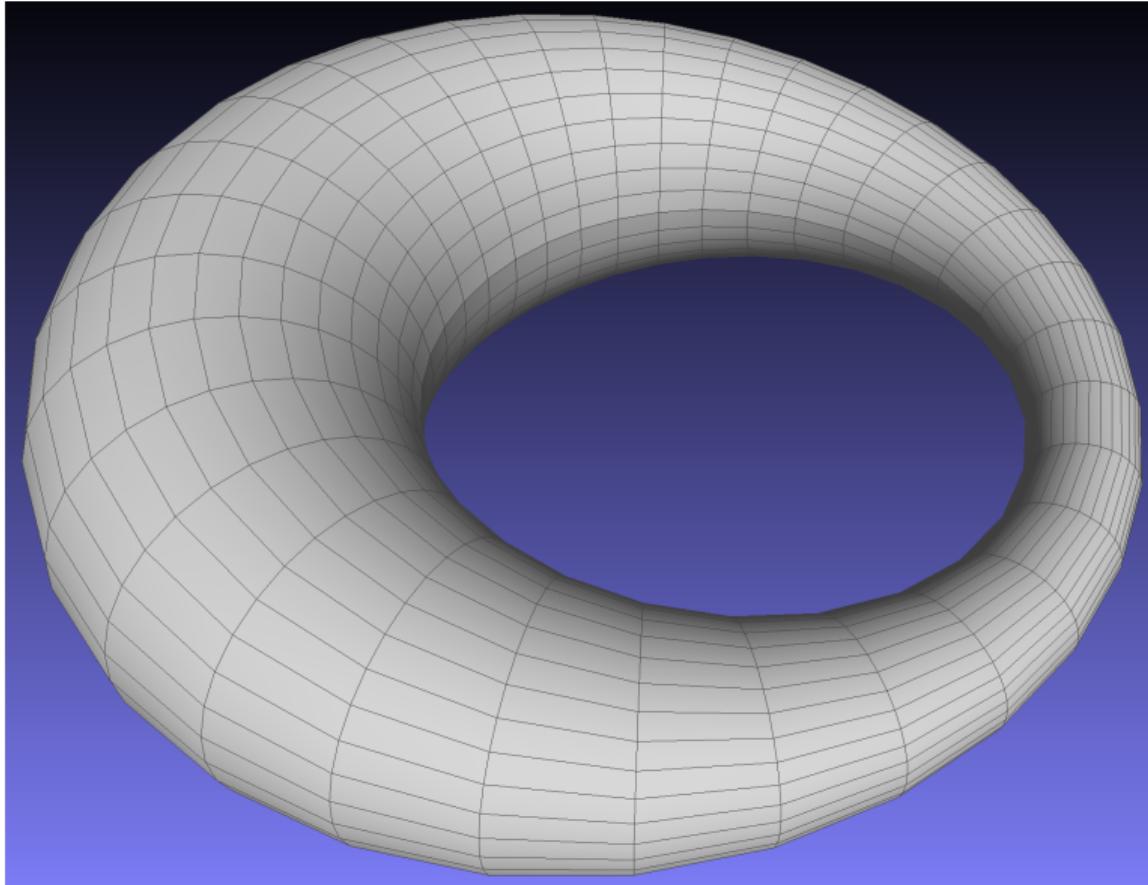
## Decomposição celular intrínseca

2D



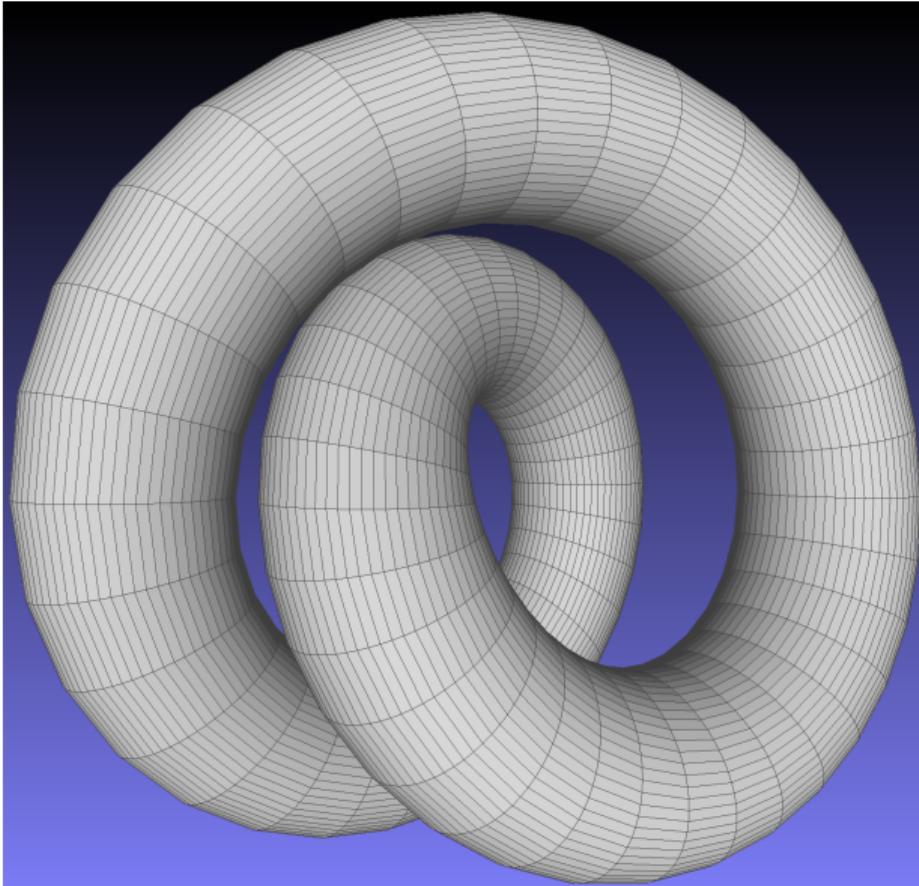
## Decomposição celular intrínseca

2D



## Decomposição celular intrínseca

2D



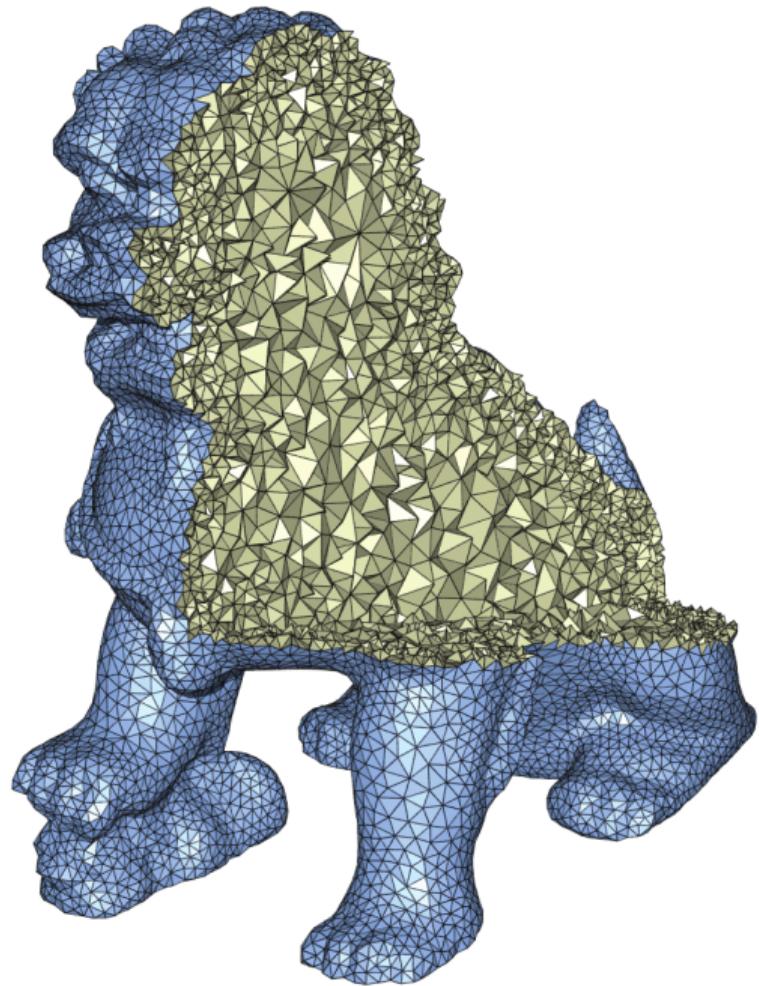
# Decomposição celular intrínseca

3D



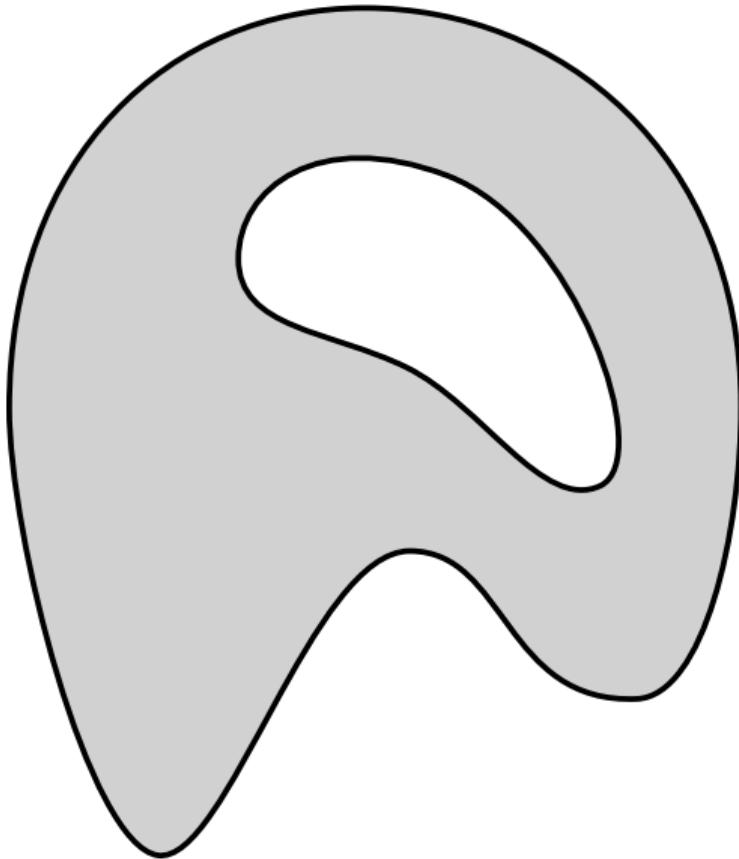
# Decomposição celular intrínseca

3D

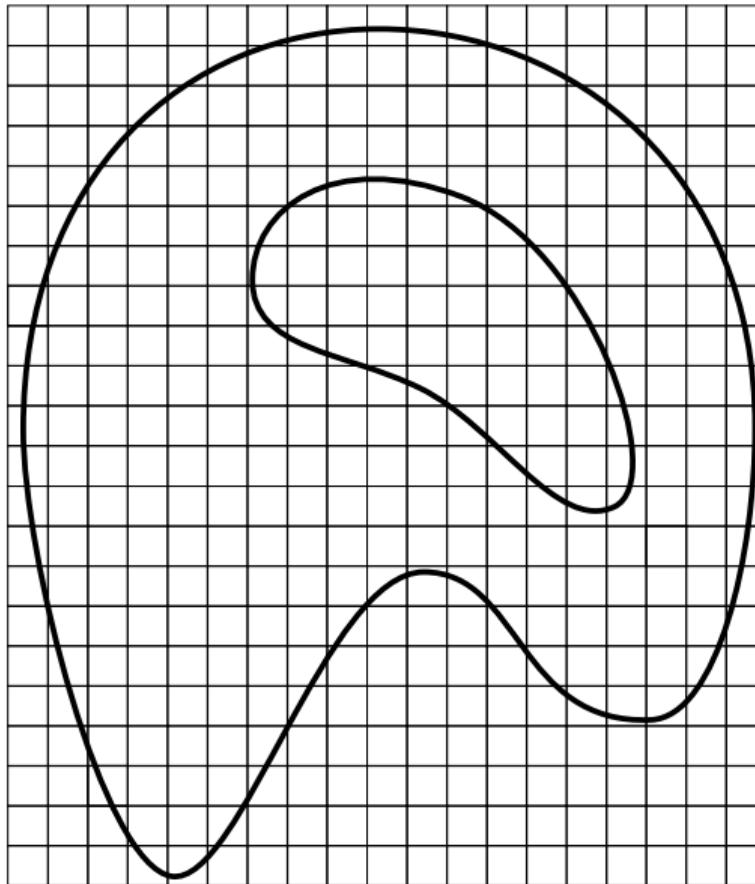


decomposição extrínseca  
enumeração espacial

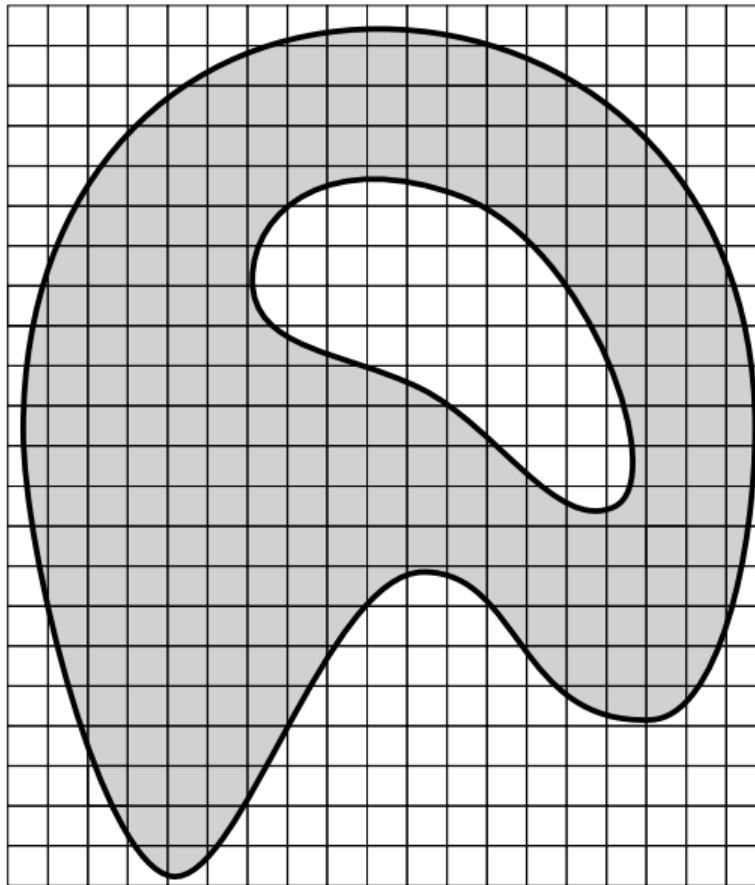
## Decomposição celular extrínseca



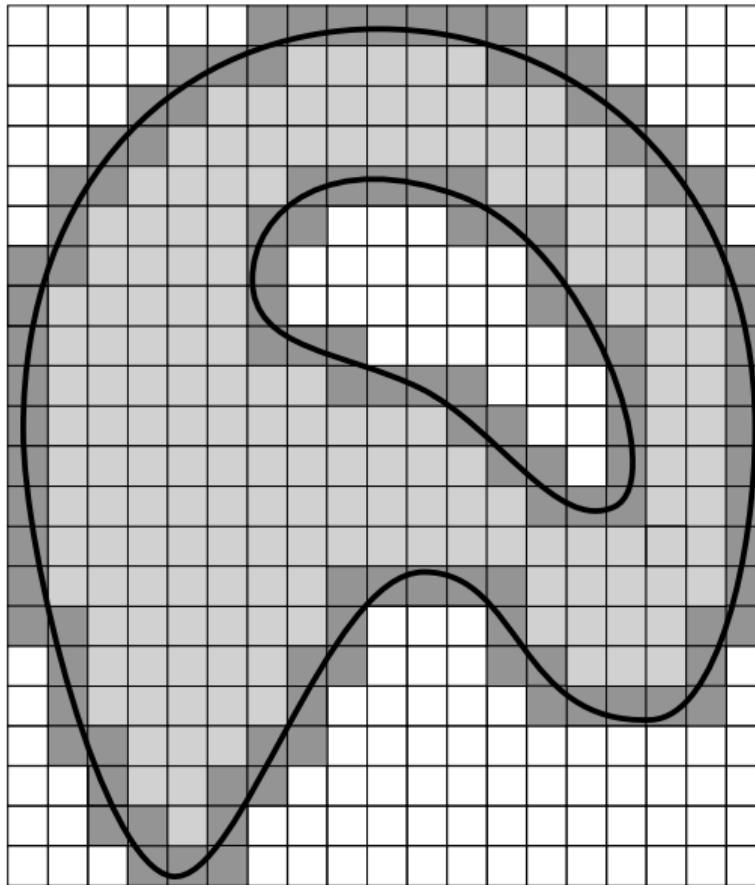
## Decomposição celular extrínseca



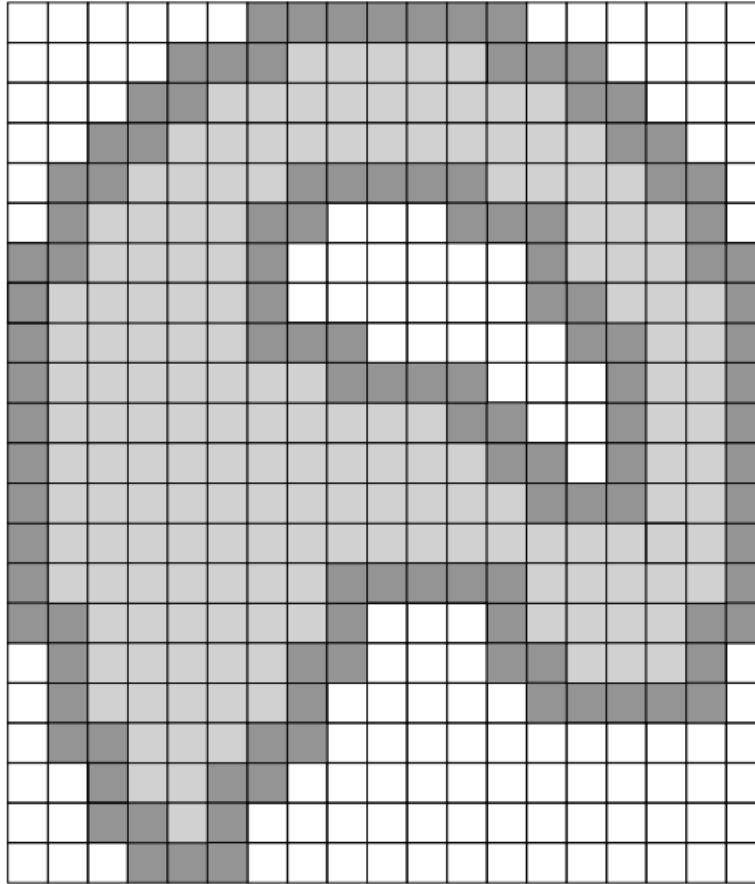
## Decomposição celular extrínseca



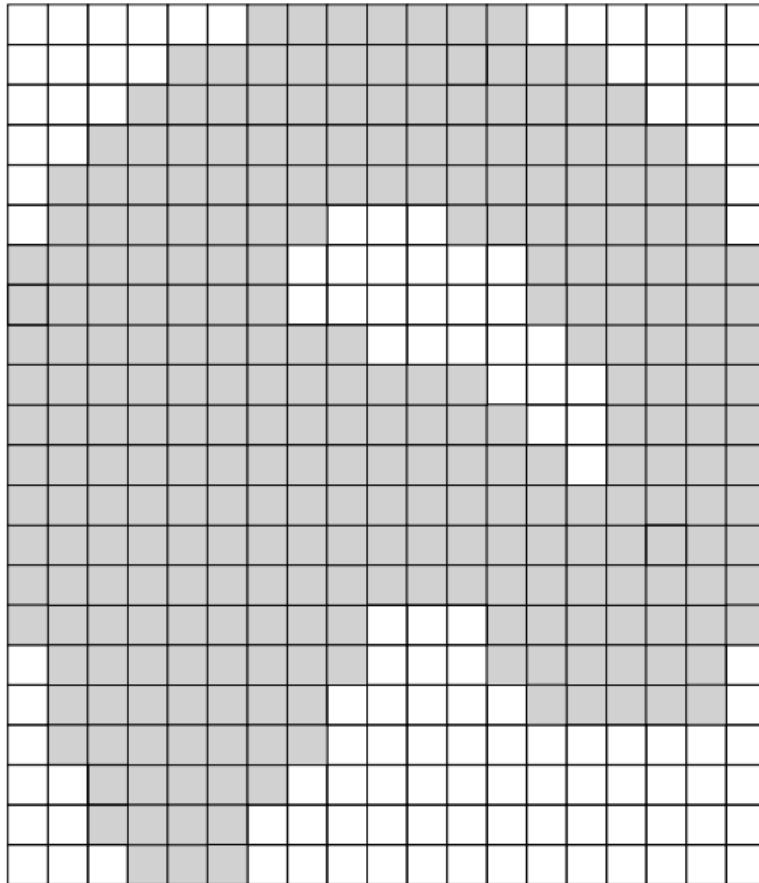
## Decomposição celular extrínseca



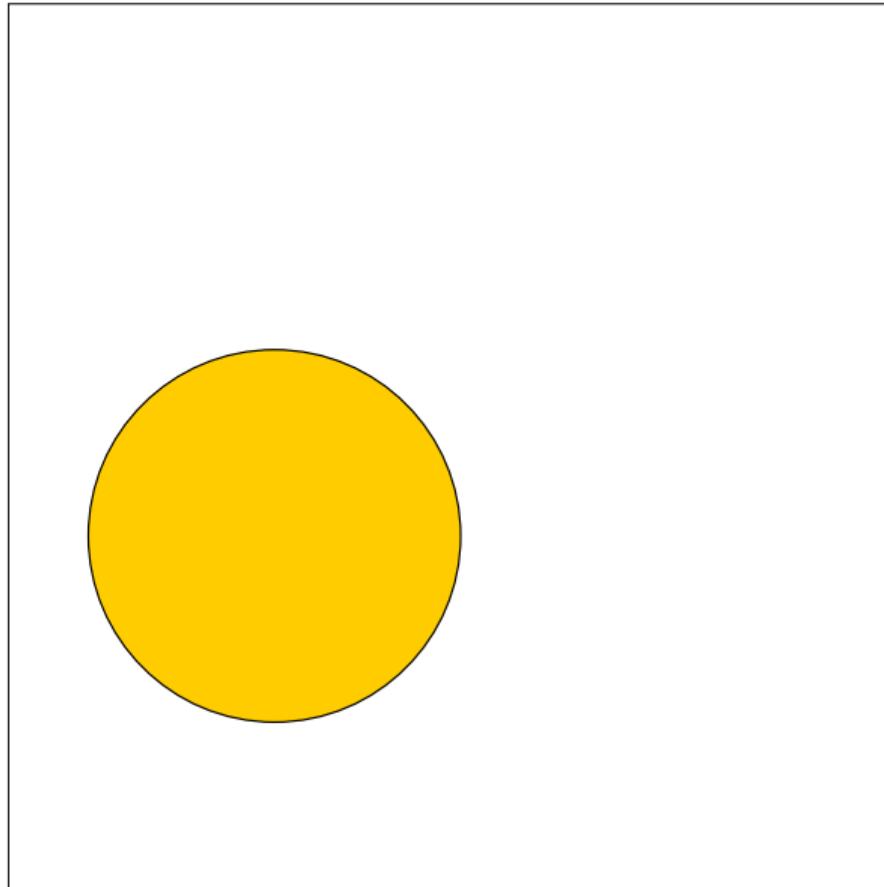
## Decomposição celular extrínsica



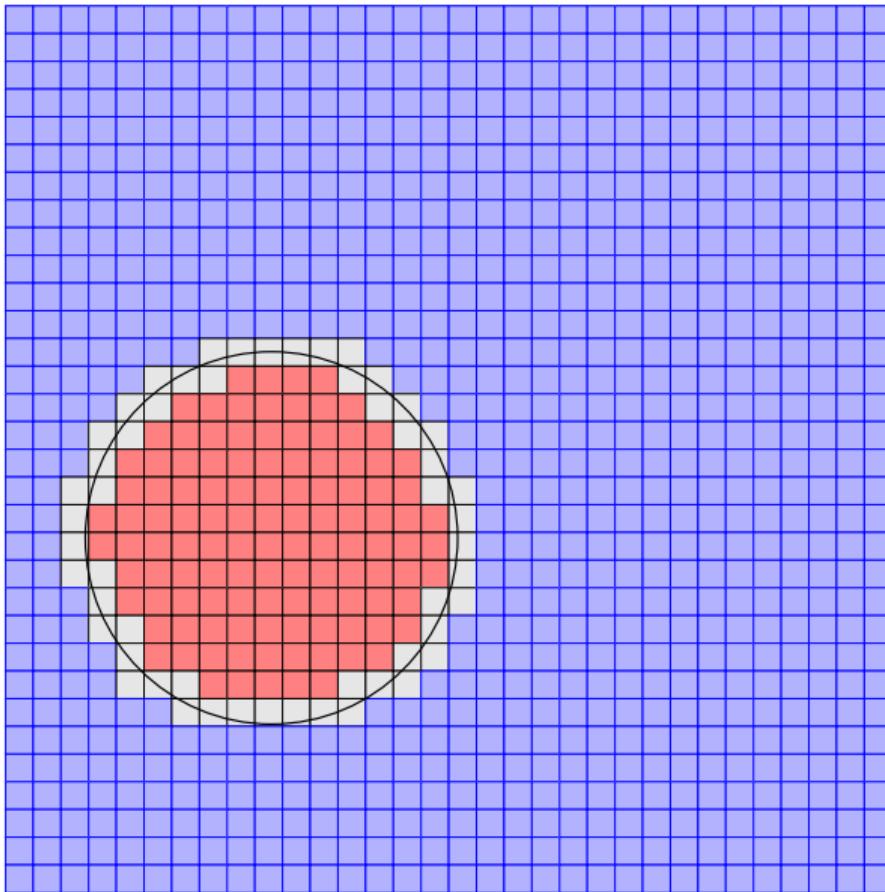
## Decomposição celular extrínsica



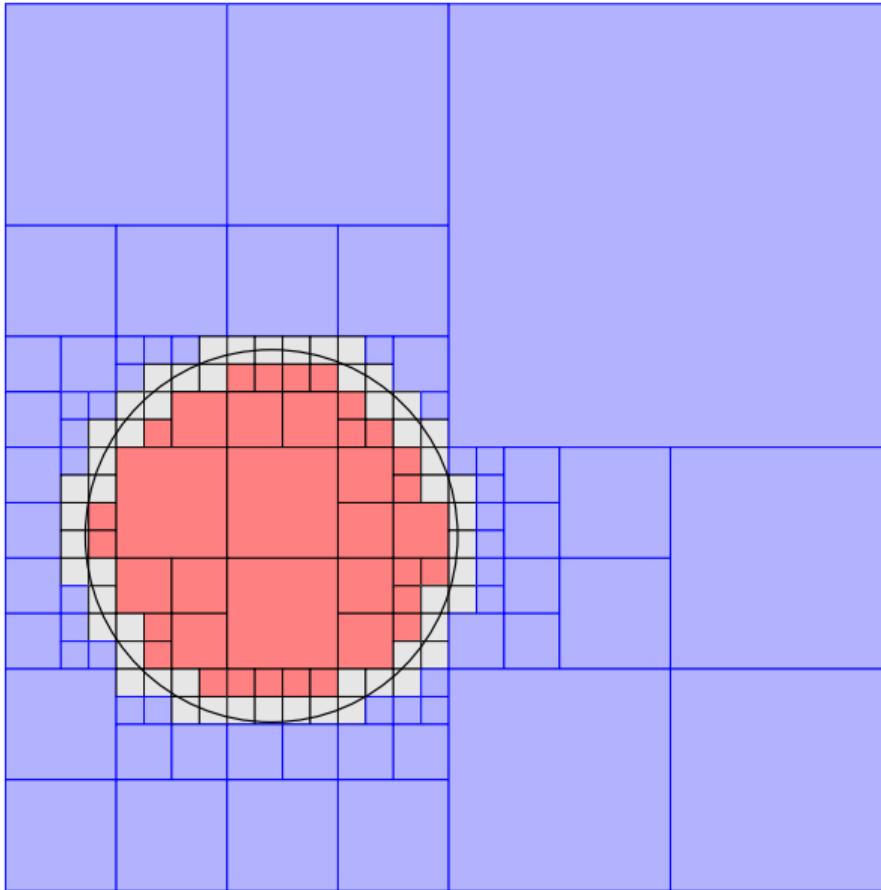
## Enumeração espacial



## Enumeração espacial uniforme



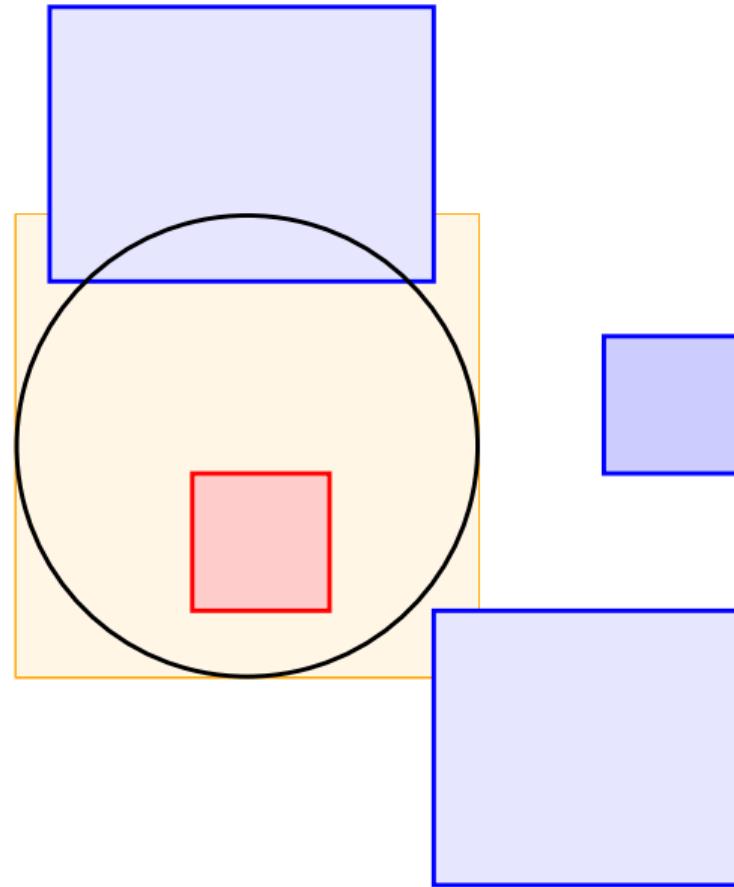
## Enumeração espacial adaptativa



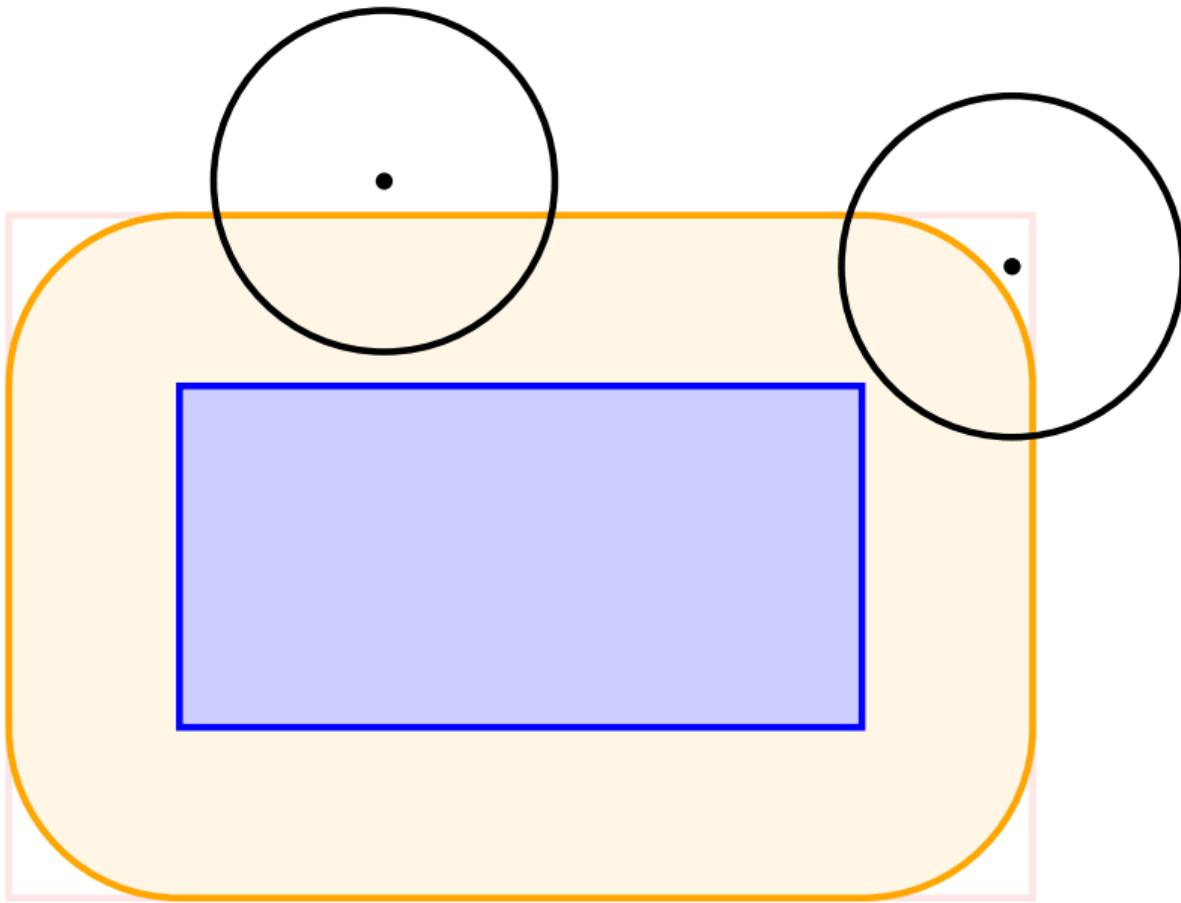
## Enumeração espacial adaptativa

```
function explore(xmin,xmax,ymin,ymax,level)
    local c=classify(xmin,xmax,ymin,ymax)
    if c~="none" or level==0 then
        output(xmin,xmax,ymin,ymax,c)
    else
        local xmid=(xmin+xmax)/2
        local ymid=(ymin+ymax)/2
        explore(xmin,xmid,ymid,ymax,level-1)
        explore(xmid,xmax,ymid,ymax,level-1)
        explore(xmin,xmid,ymin,ymid,level-1)
        explore(xmid,xmax,ymin,ymid,level-1)
    end
end
```

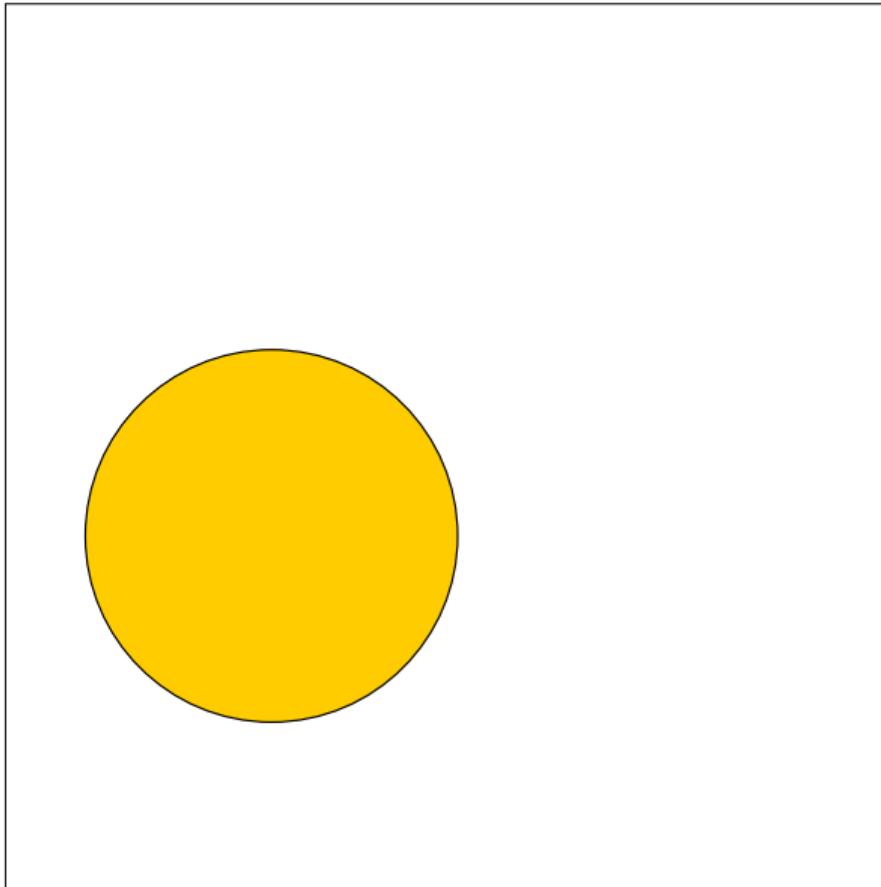
## Enumeração espacial



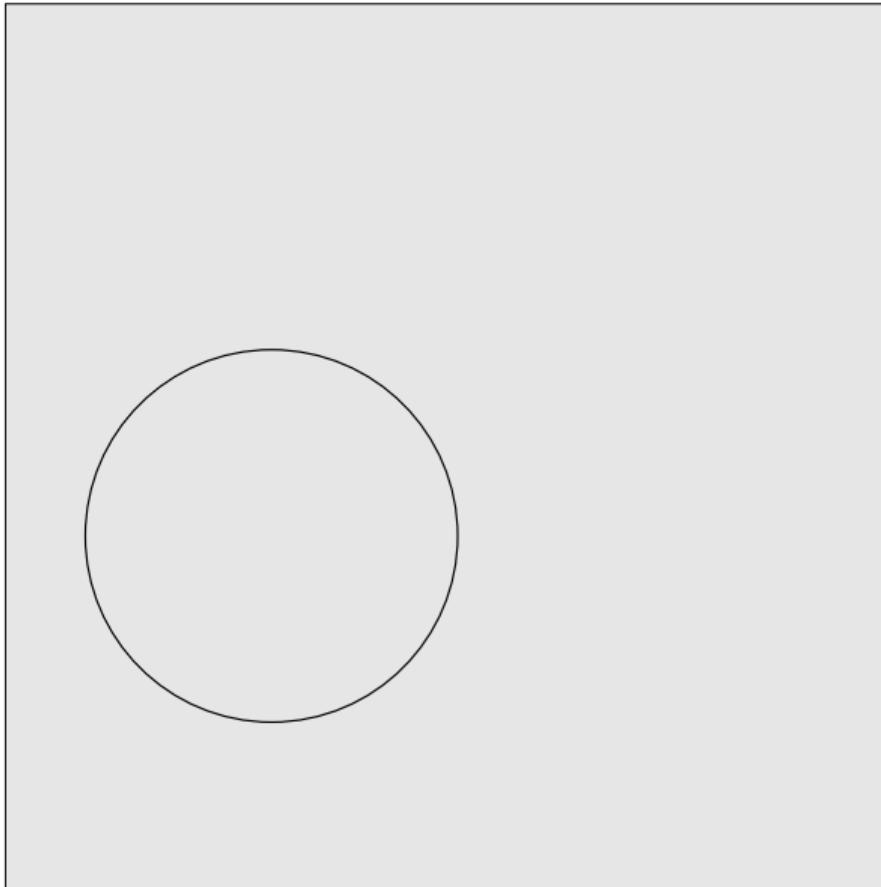
## Enumeração espacial



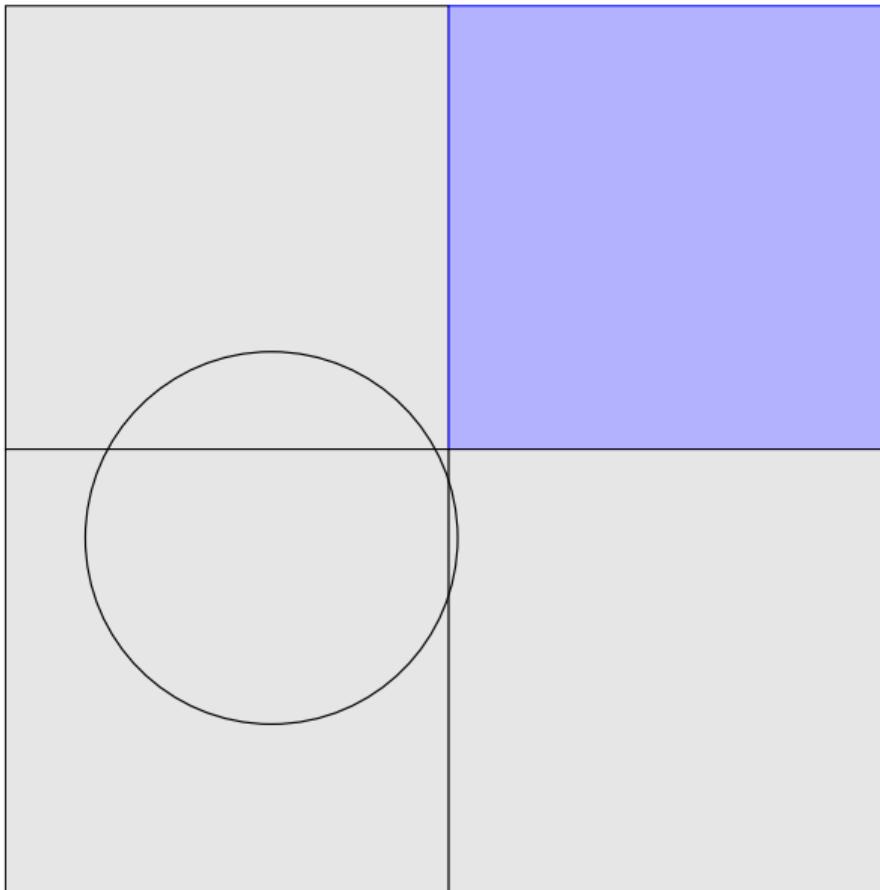
## Enumeração espacial adaptativa



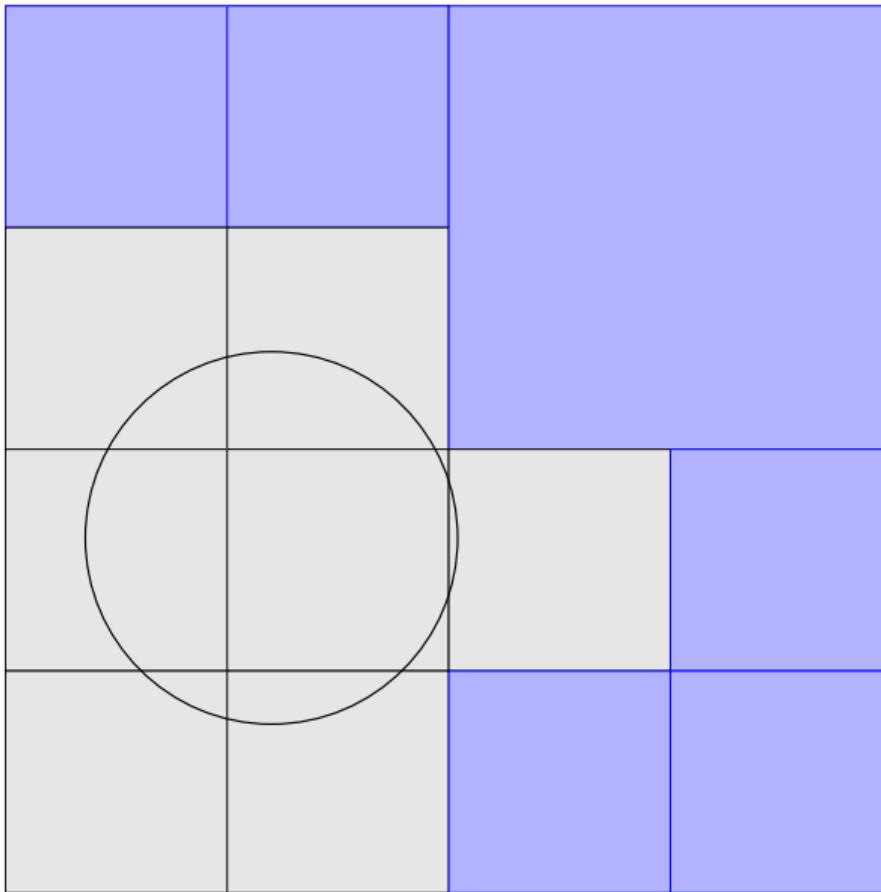
## Enumeração espacial adaptativa



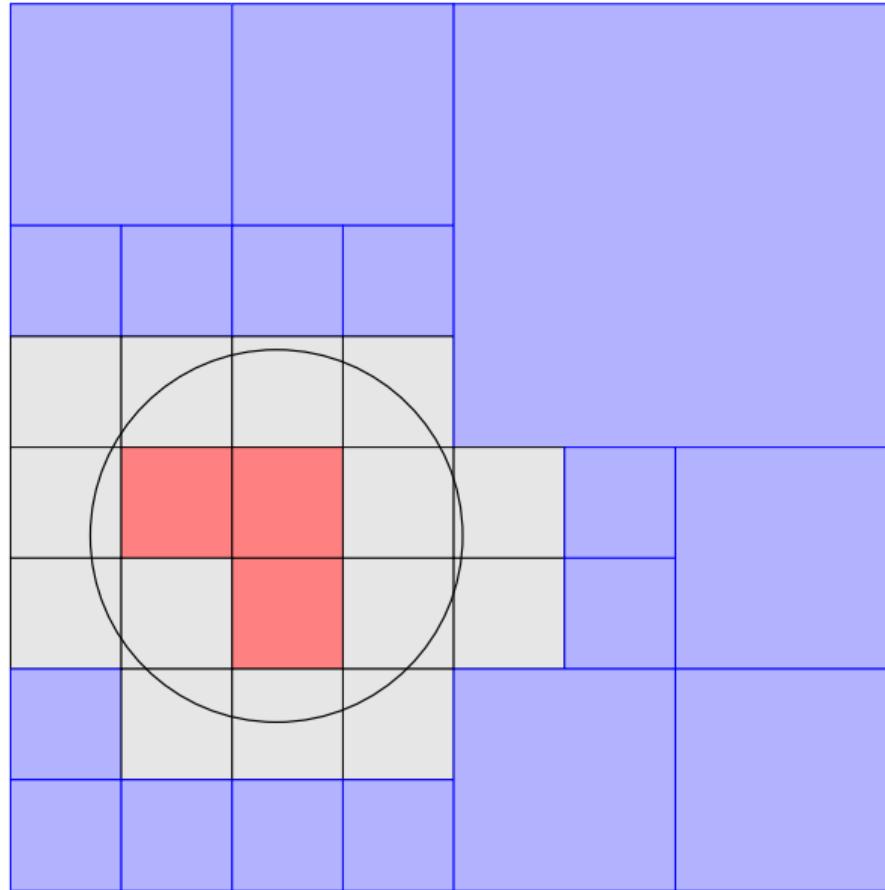
## Enumeração espacial adaptativa



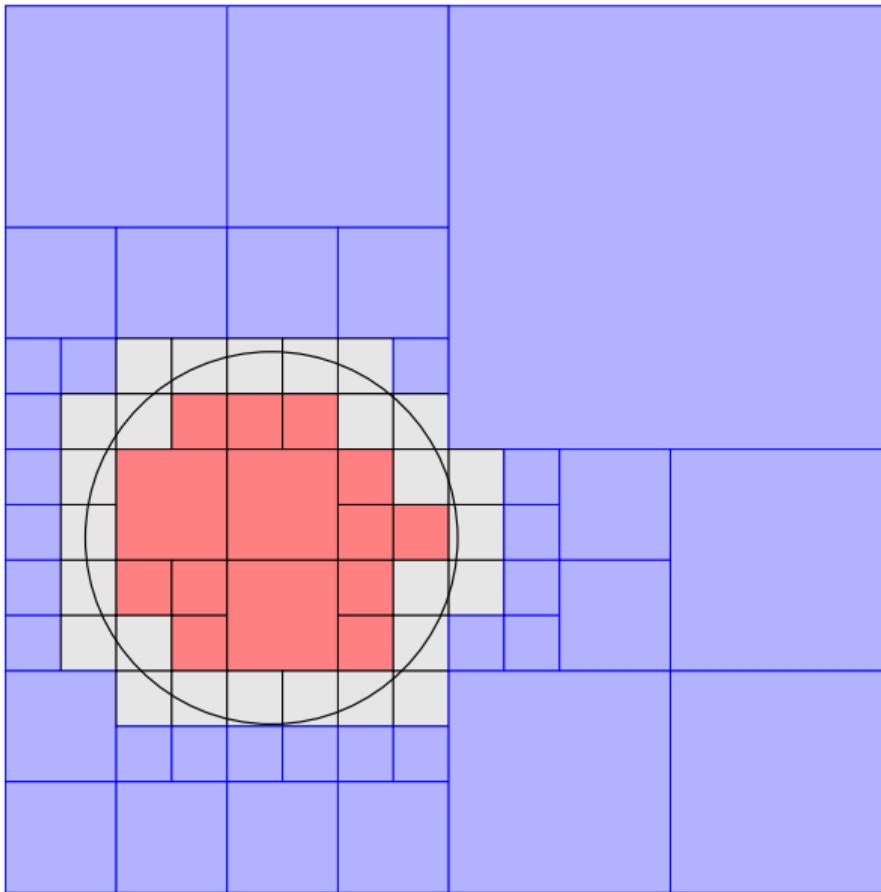
## Enumeração espacial adaptativa



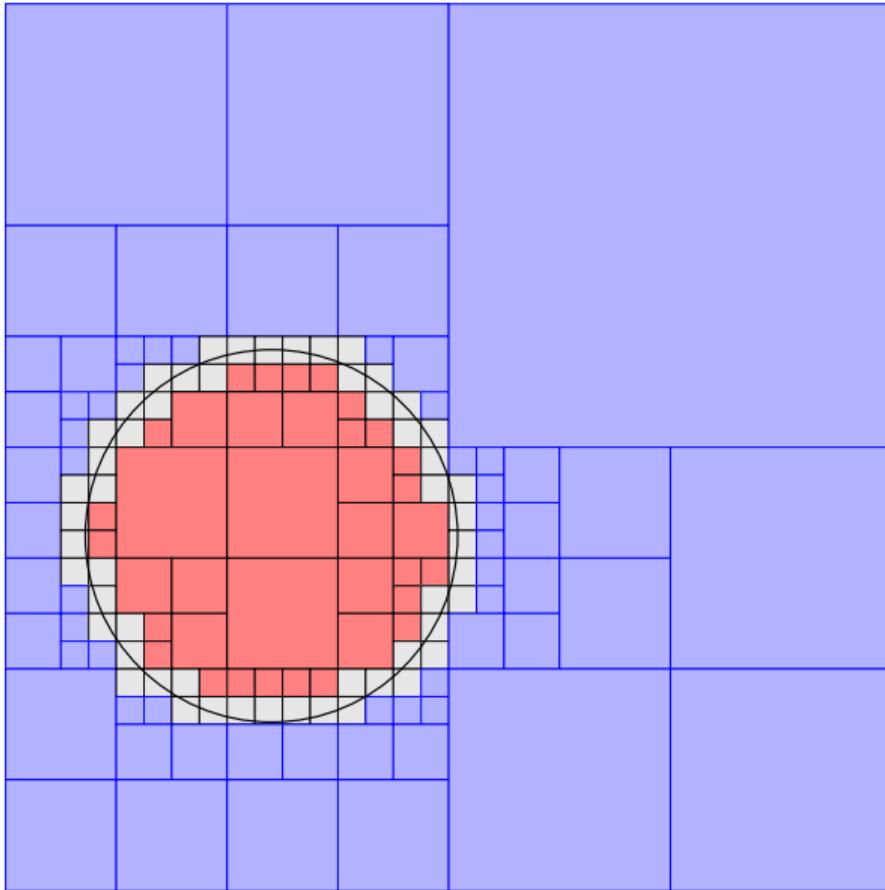
## Enumeração espacial adaptativa



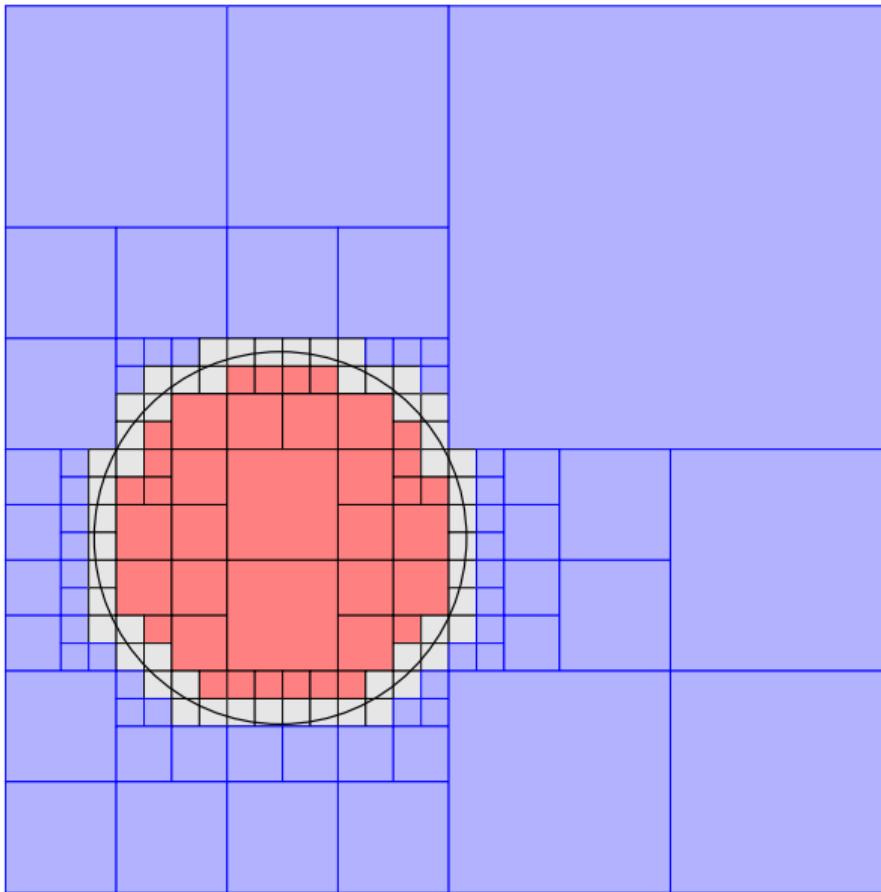
## Enumeração espacial adaptativa



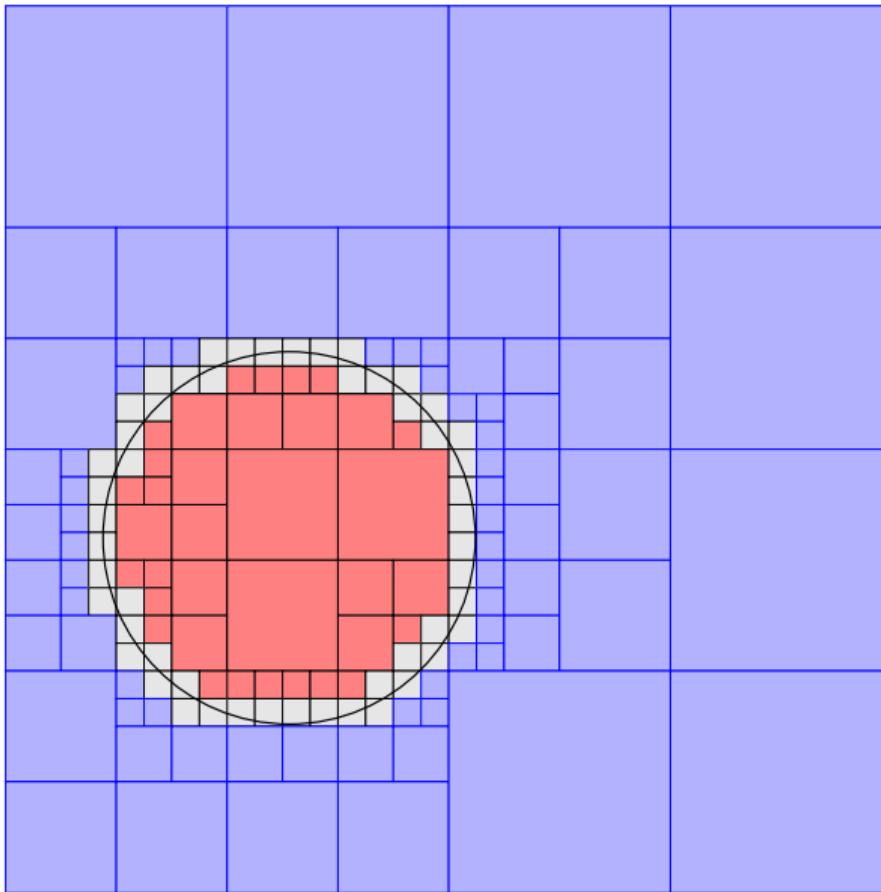
## Enumeração espacial adaptativa



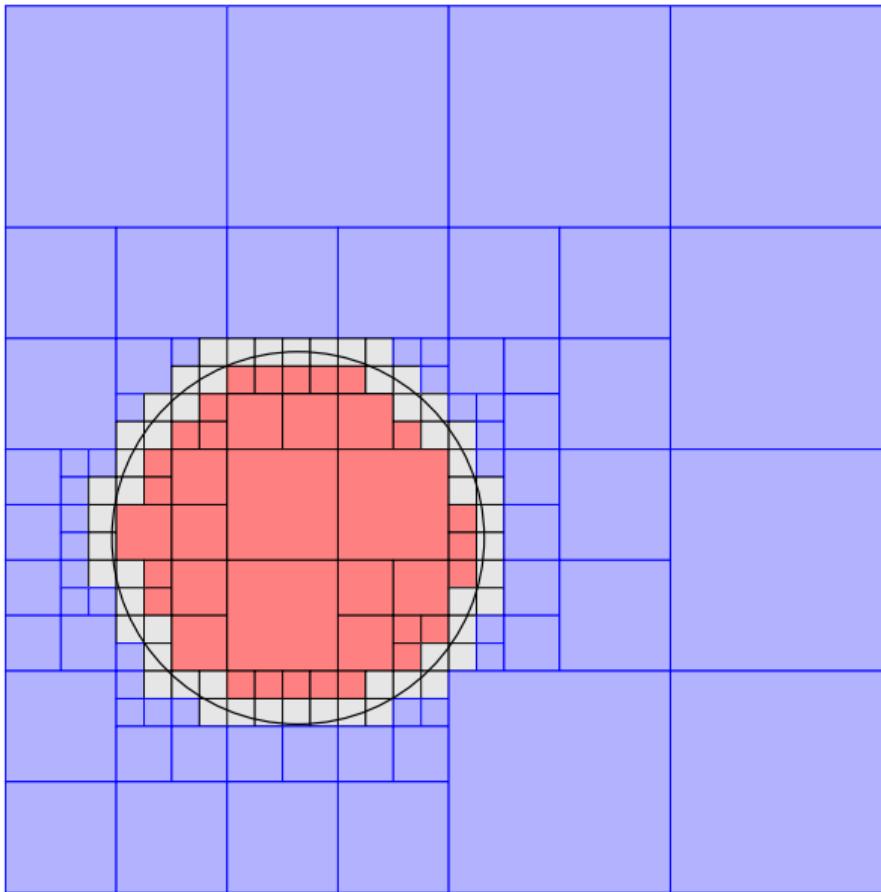
## Enumeração espacial adaptativa



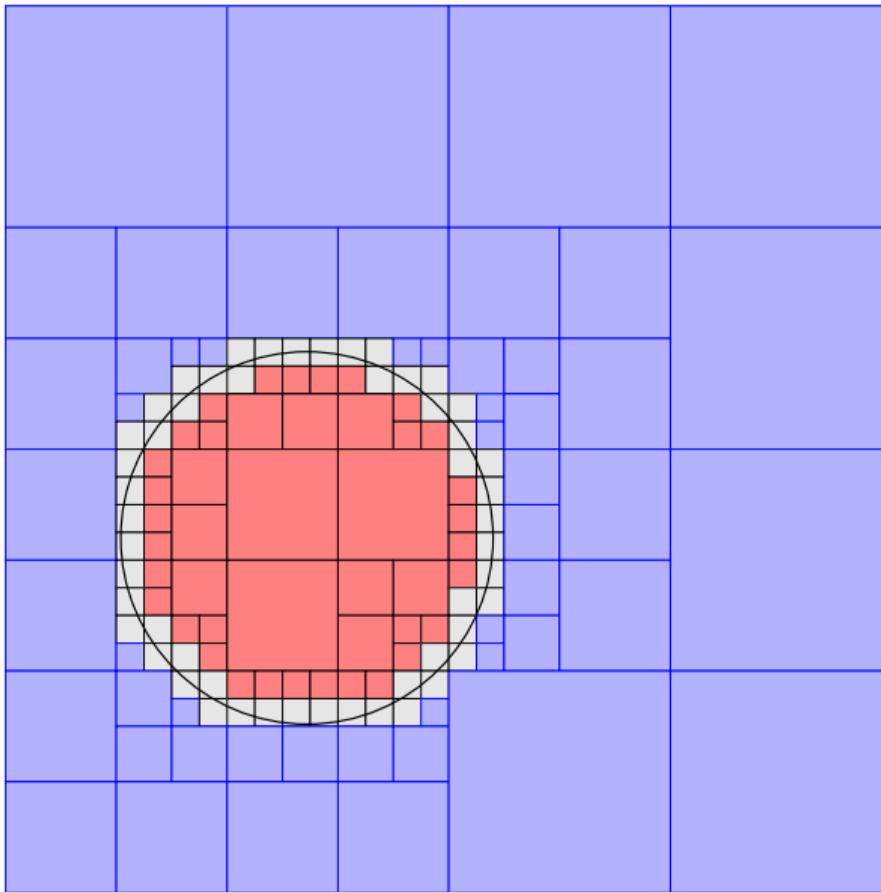
## Enumeração espacial adaptativa



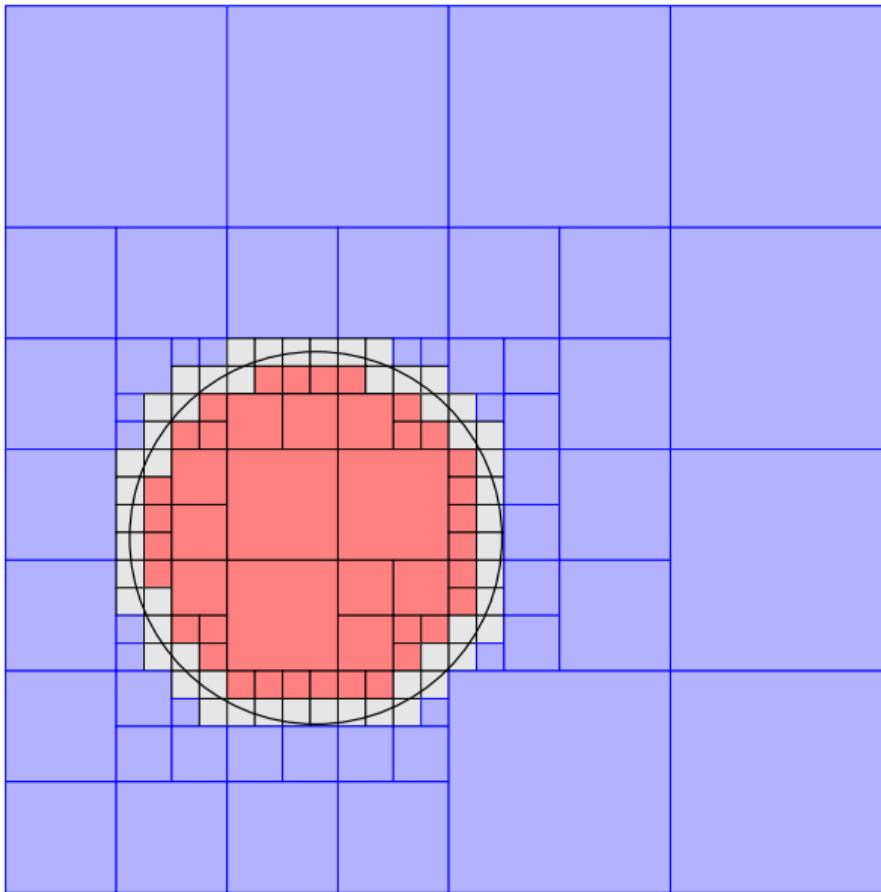
## Enumeração espacial adaptativa



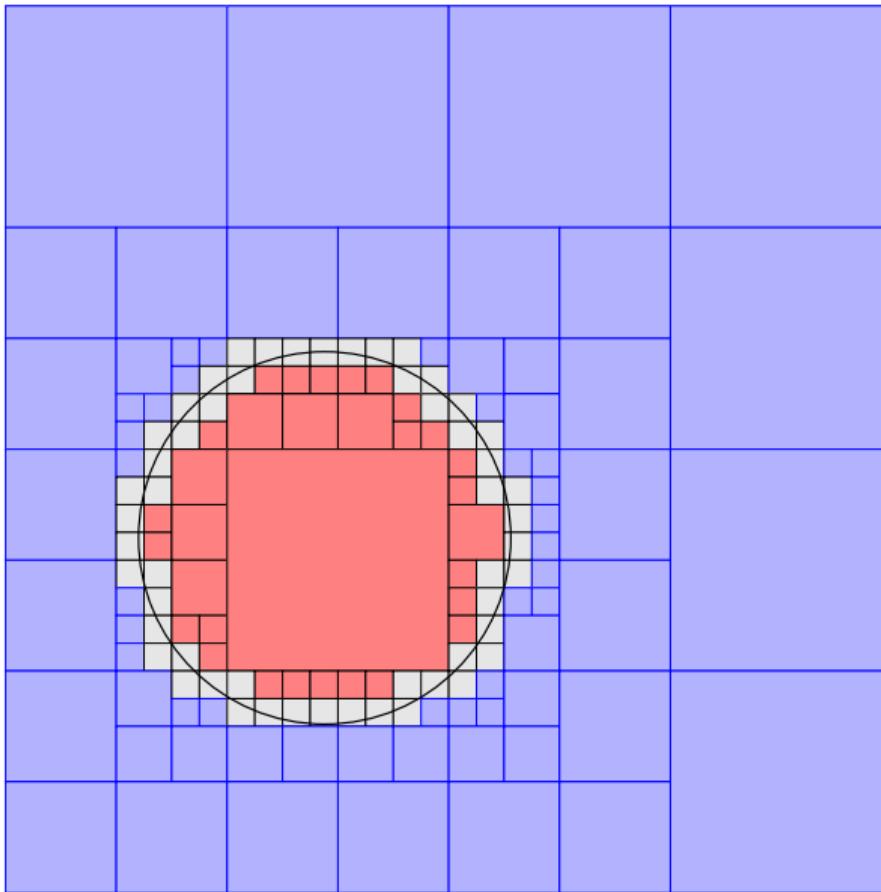
## Enumeração espacial adaptativa



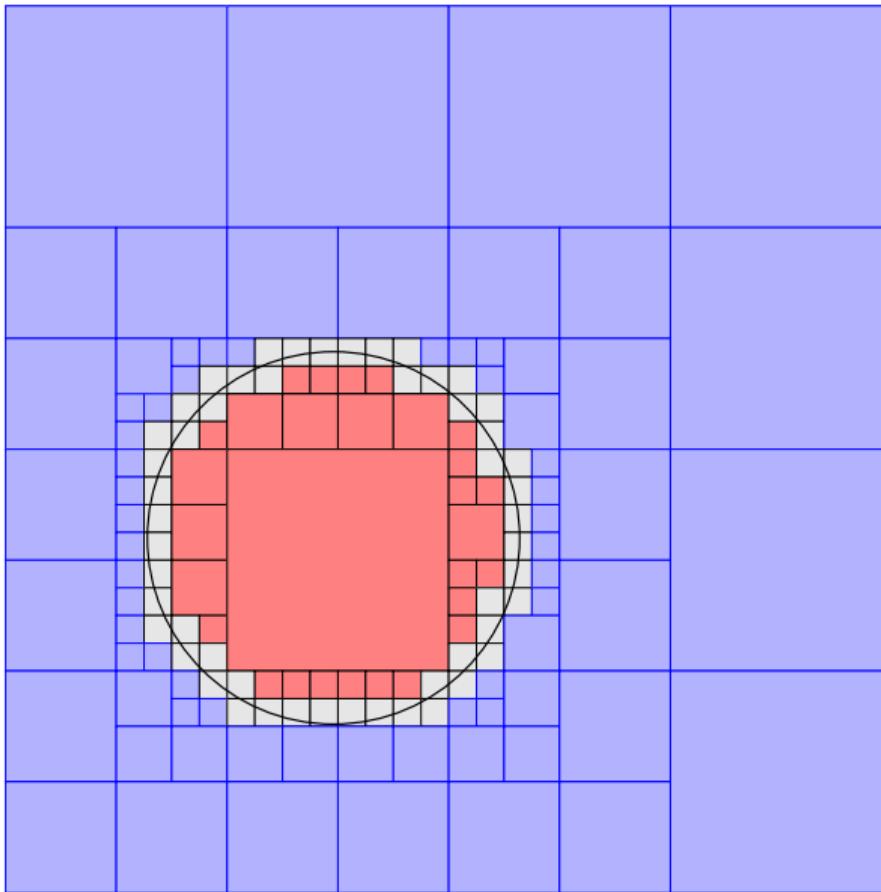
## Enumeração espacial adaptativa



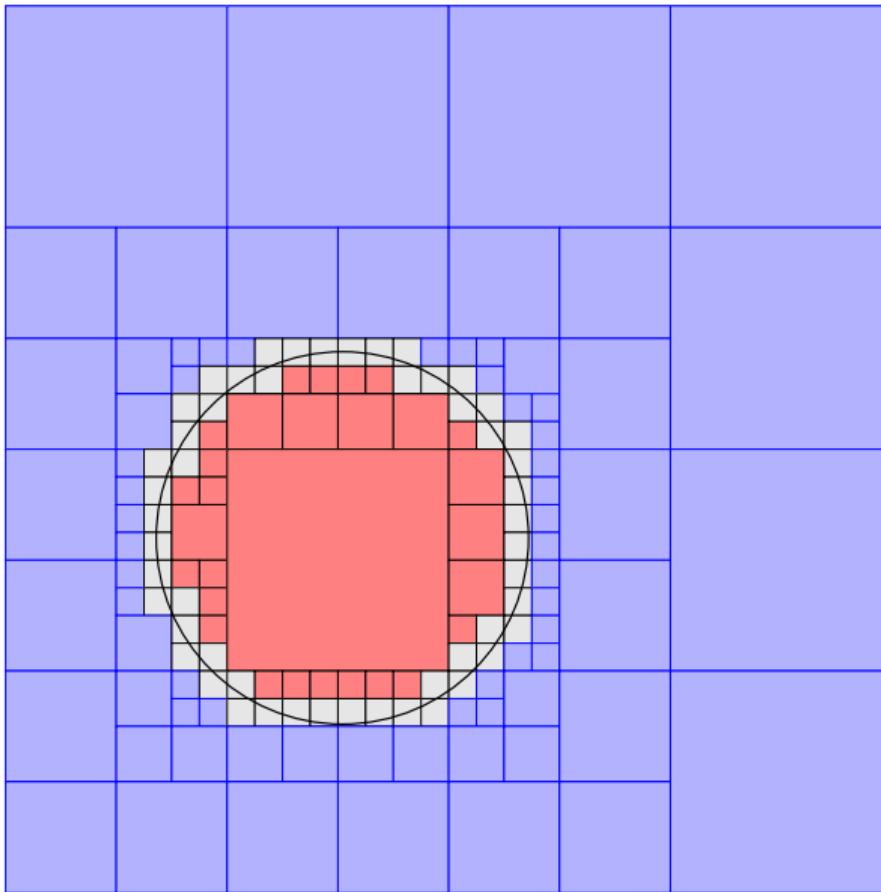
## Enumeração espacial adaptativa



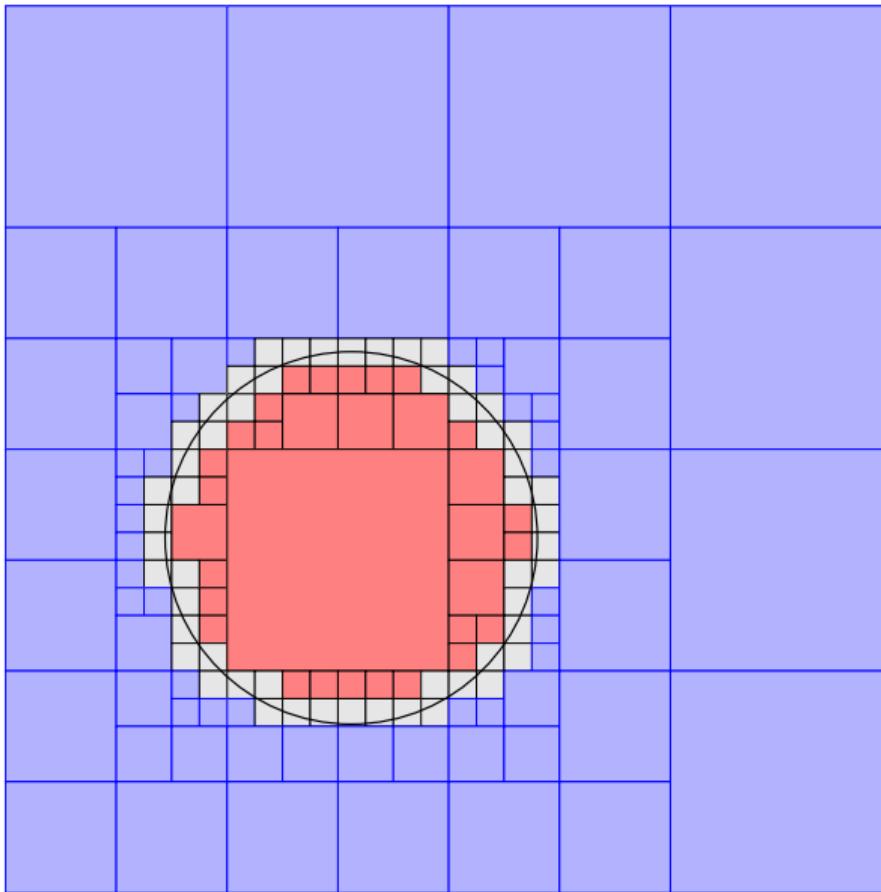
## Enumeração espacial adaptativa



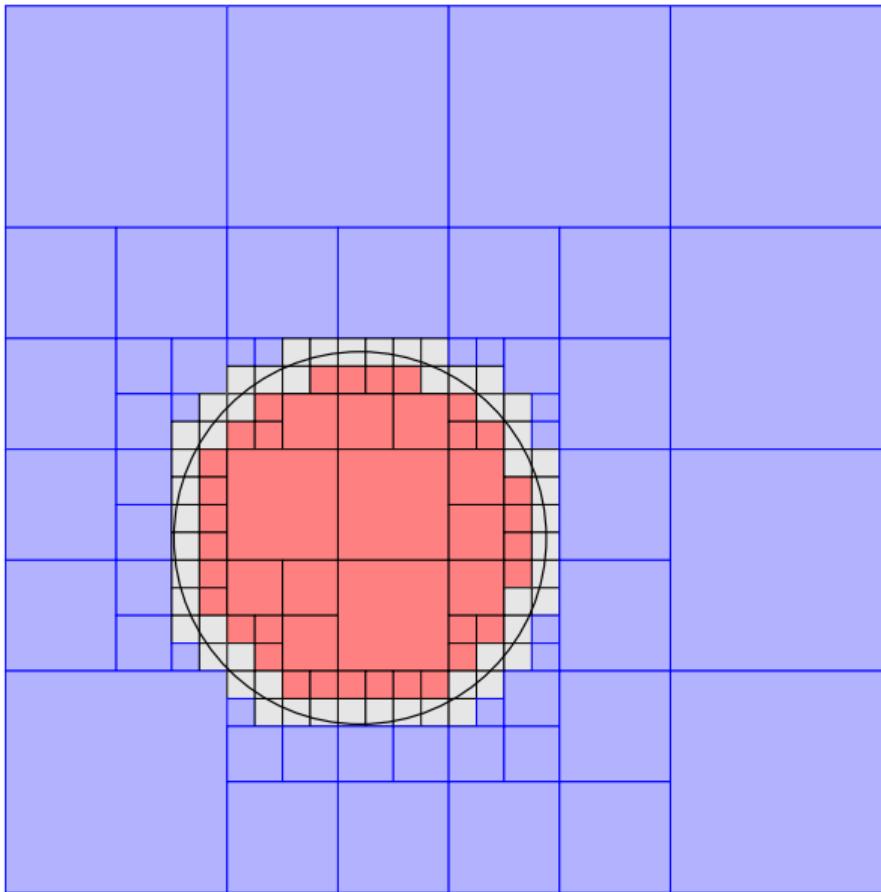
## Enumeração espacial adaptativa



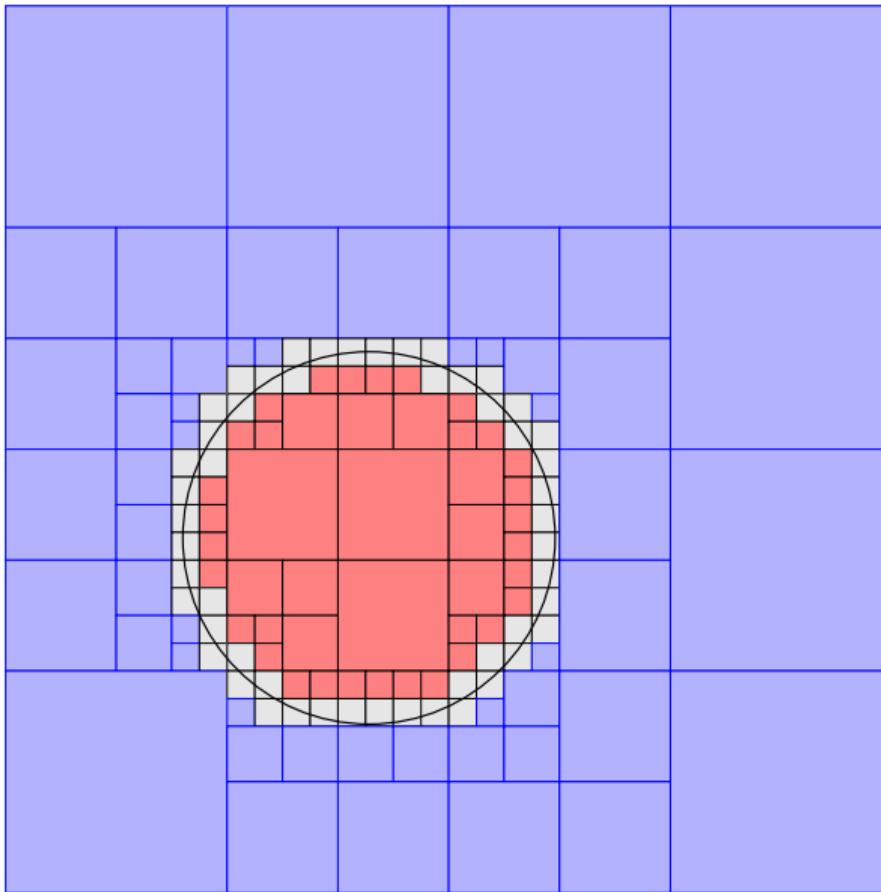
## Enumeração espacial adaptativa



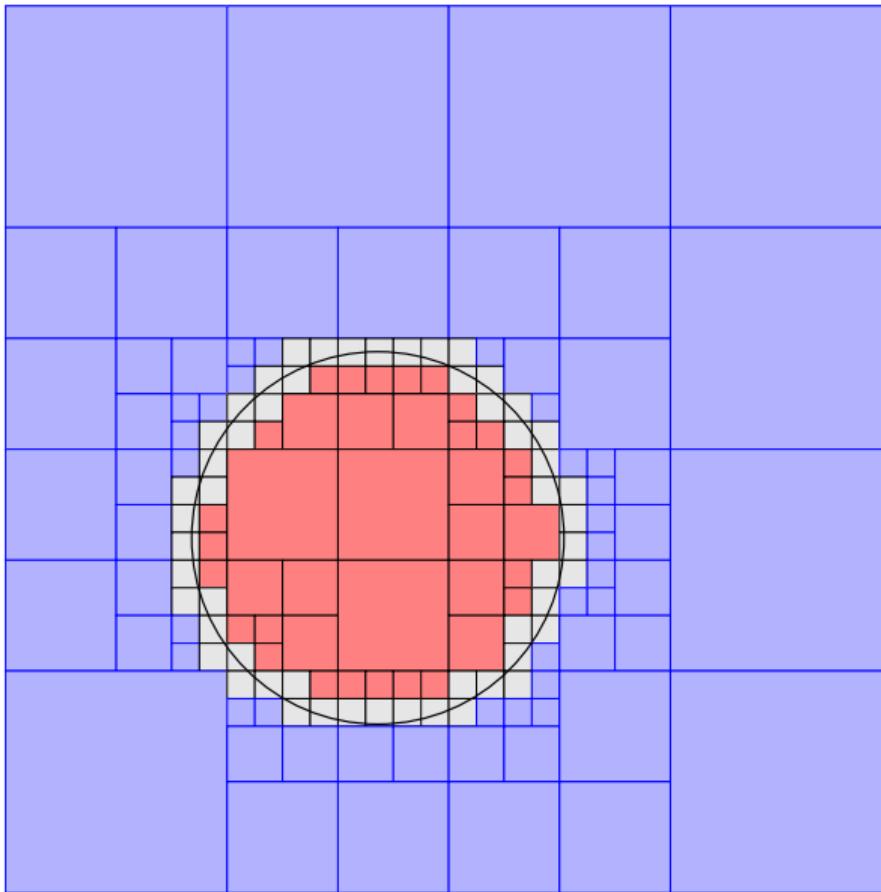
## Enumeração espacial adaptativa



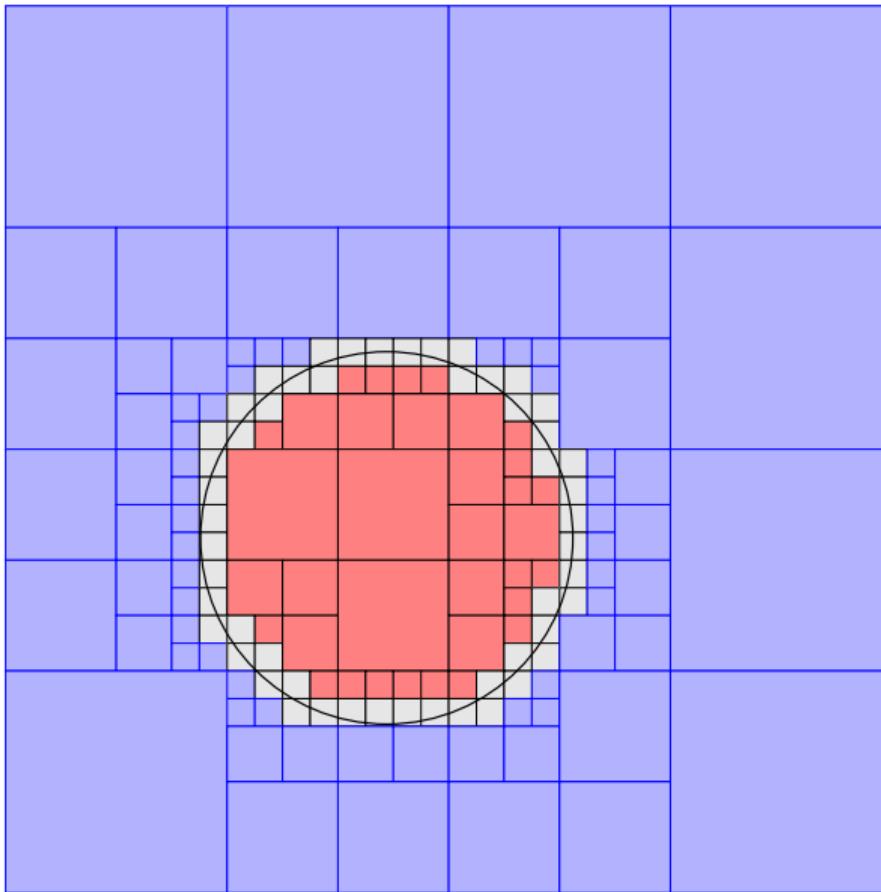
## Enumeração espacial adaptativa



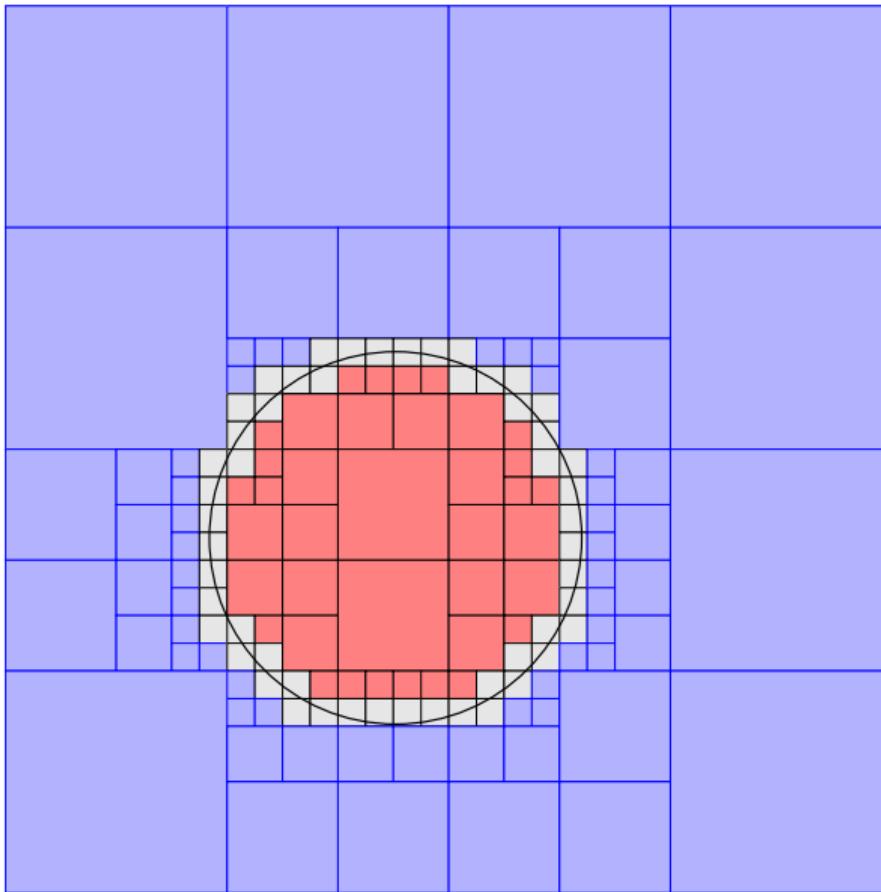
## Enumeração espacial adaptativa



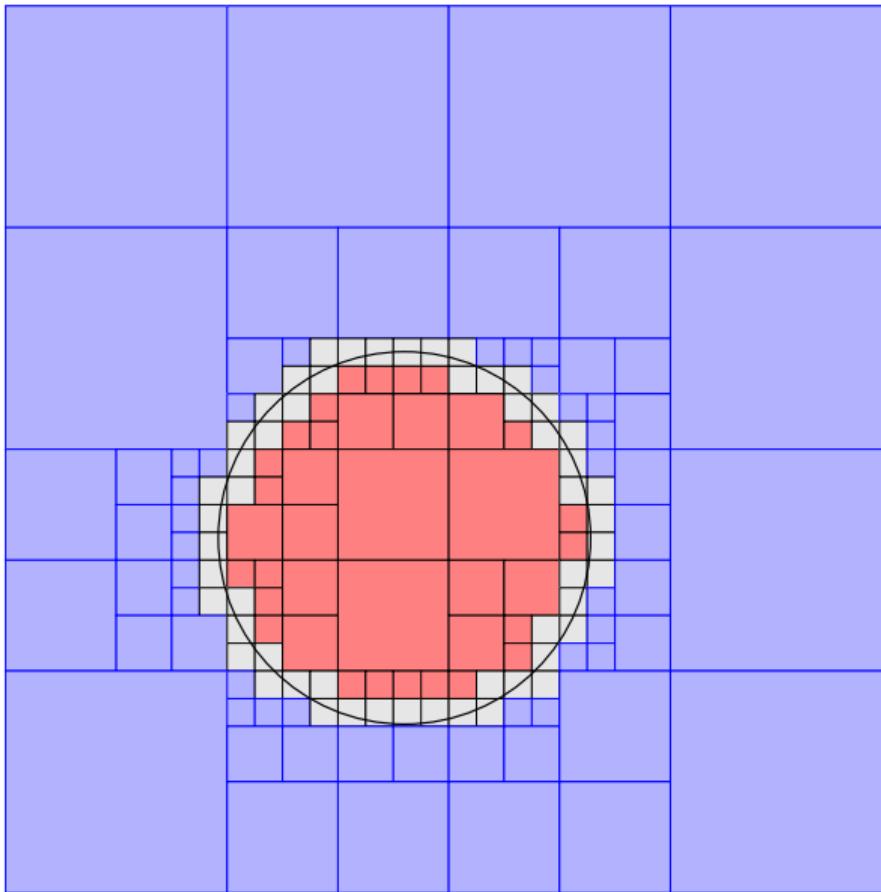
## Enumeração espacial adaptativa



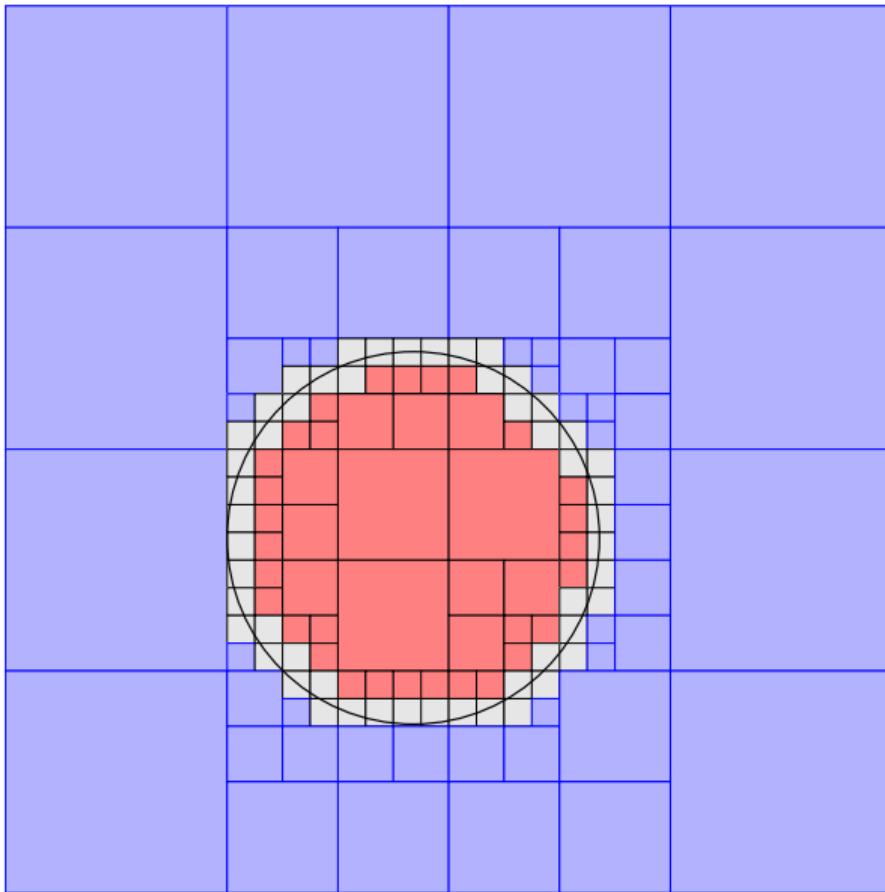
## Enumeração espacial adaptativa



## Enumeração espacial adaptativa

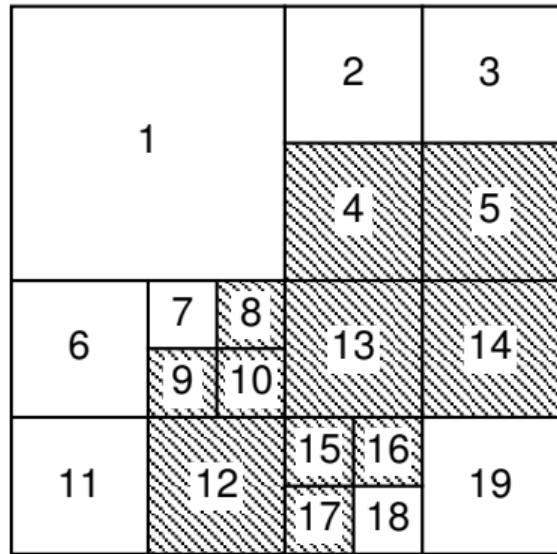


## Enumeração espacial adaptativa

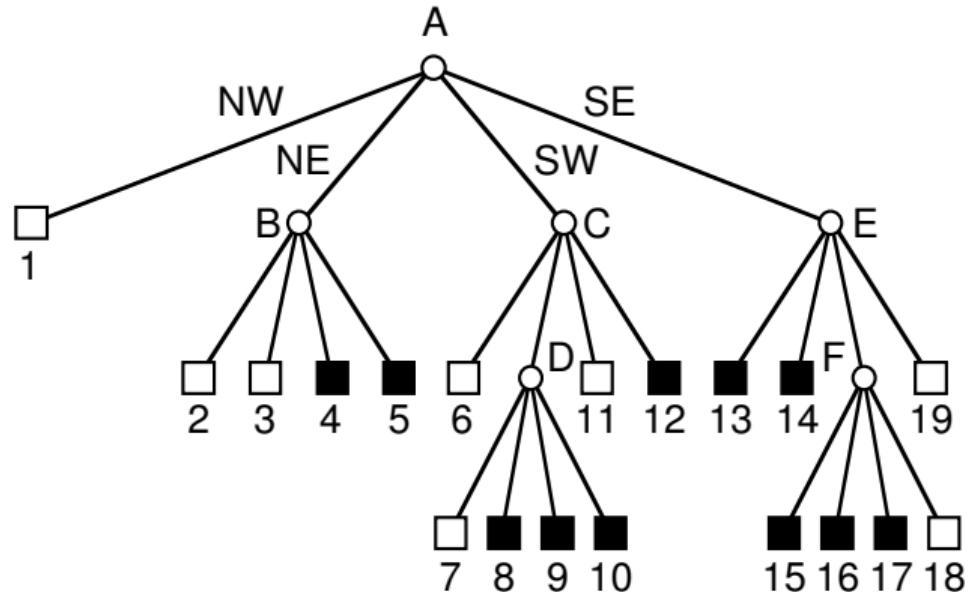


# Quadtree

Finkel–Bentley (1974)

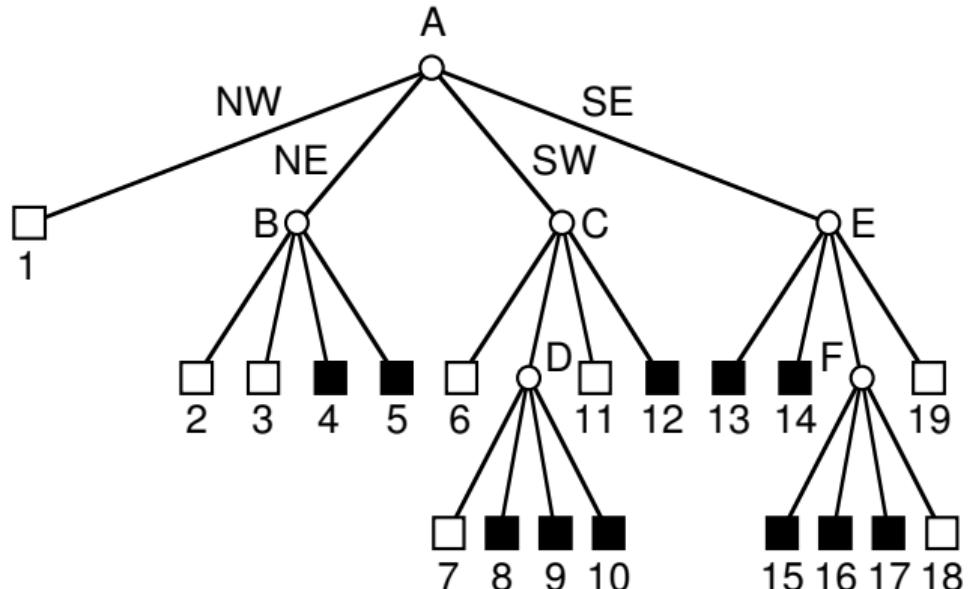
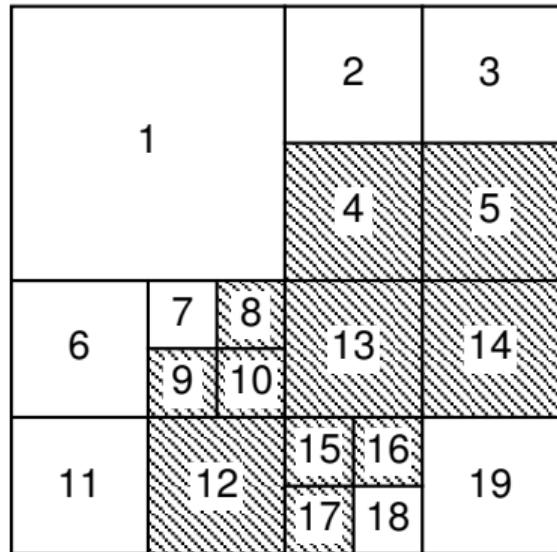


Samet



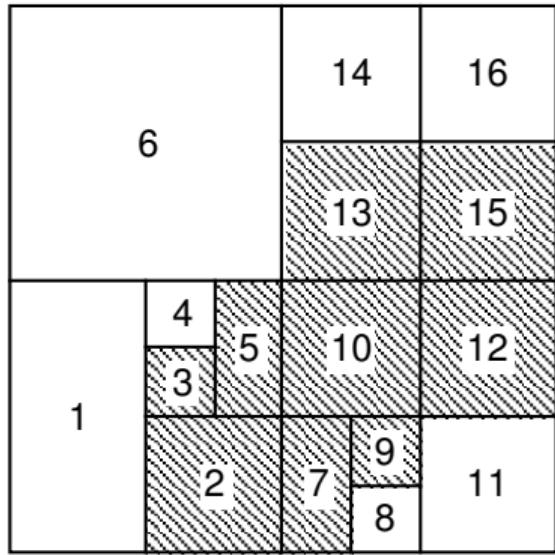
# Quadtree

Finkel–Bentley (1974)

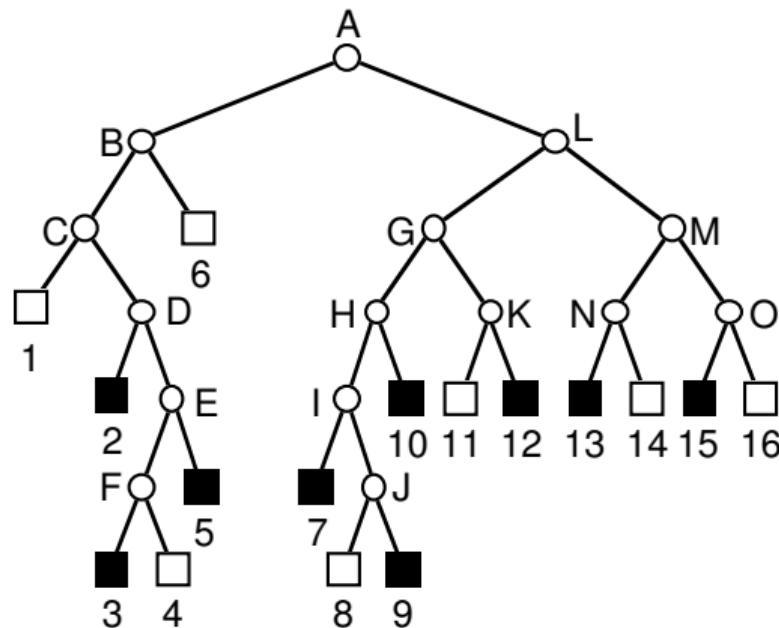


(B(BBPP)(B(BPPP)BP)(PP(PPPBP)B))

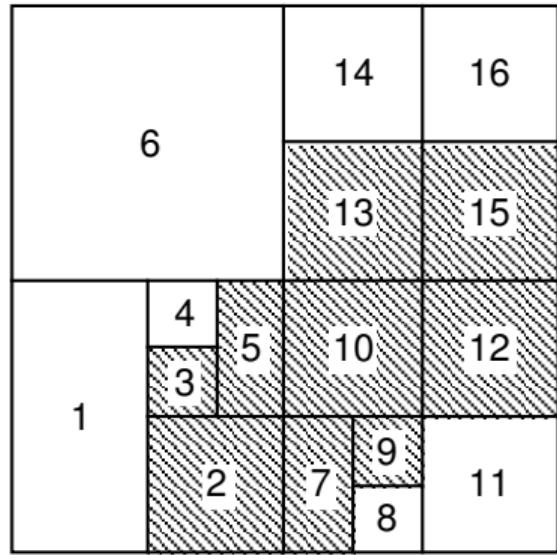
# Bintree



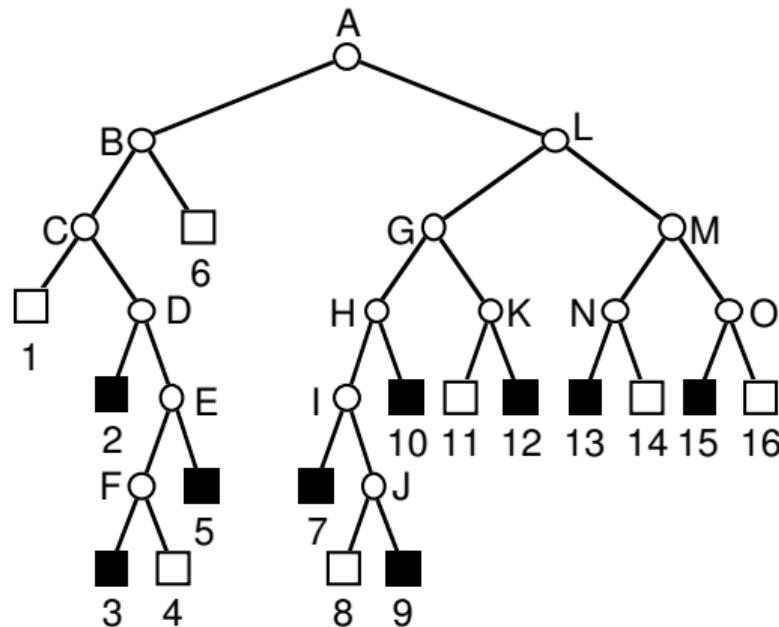
Tamminen et al. (1984)



# Bintree

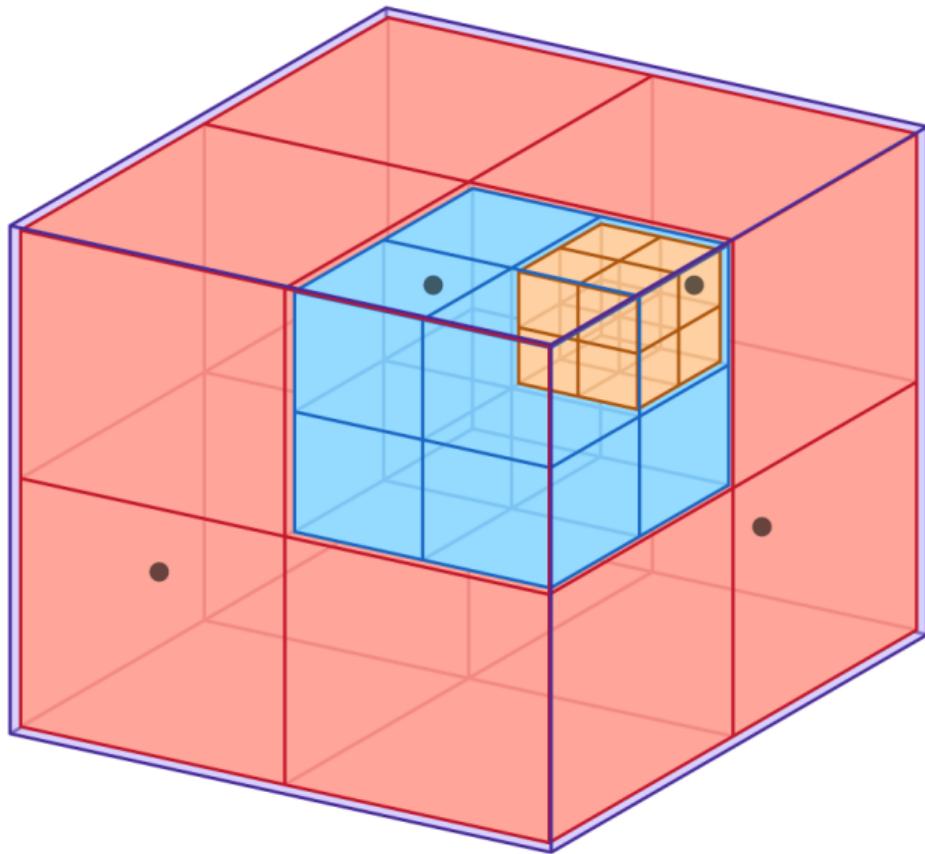


Tamminen et al. (1984)

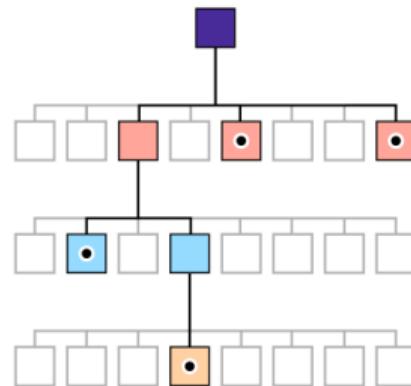


$((((B(P((PB)P)))B))(((P(BP))P)(BP))((PB)(PB))))$

# Octree



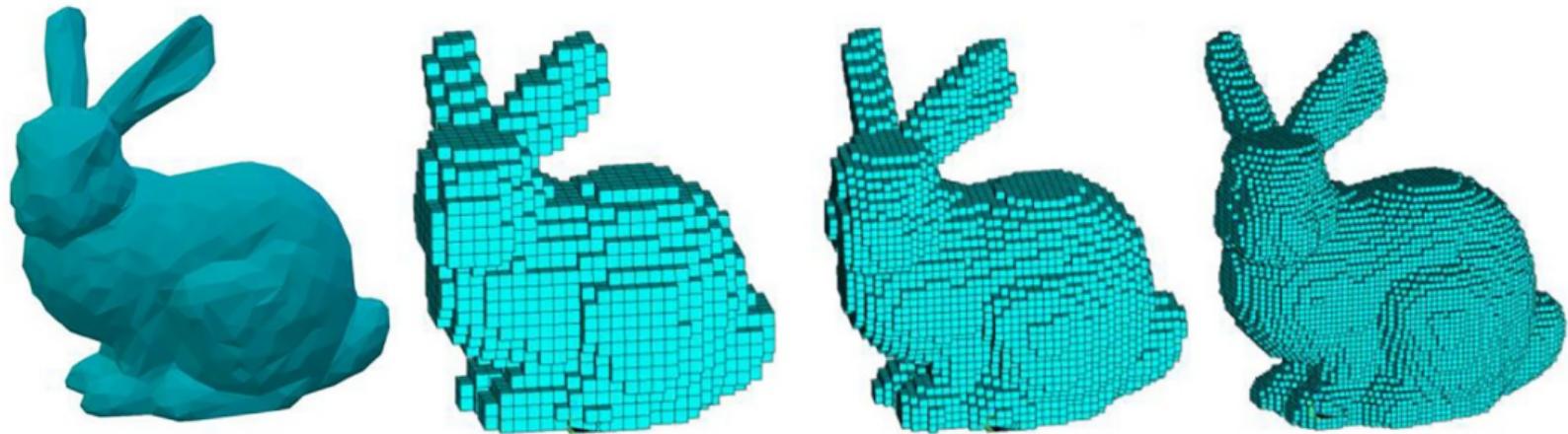
Meagher (1980)



Plasma\_Node

# Enumeração espacial uniforme

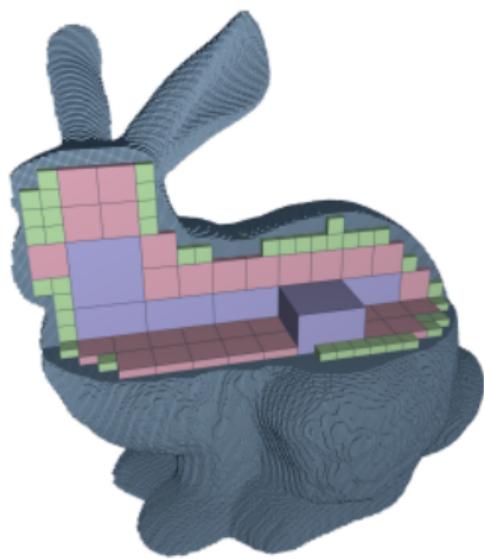
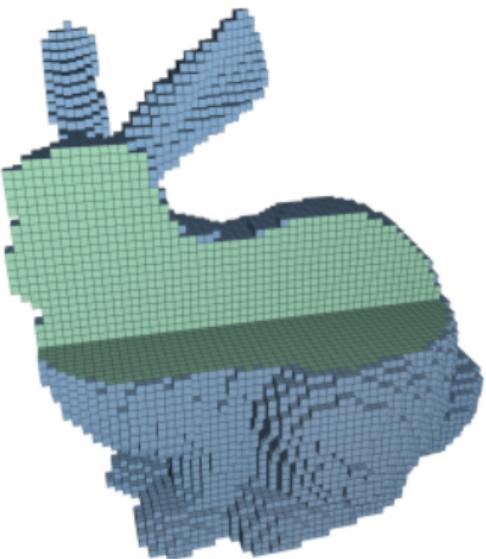
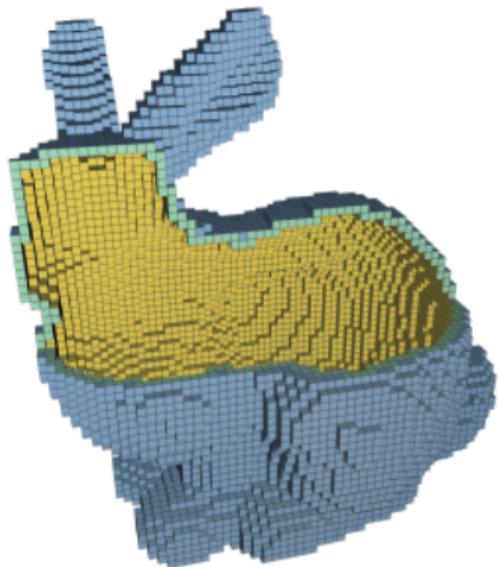
3D



Zhou et al.

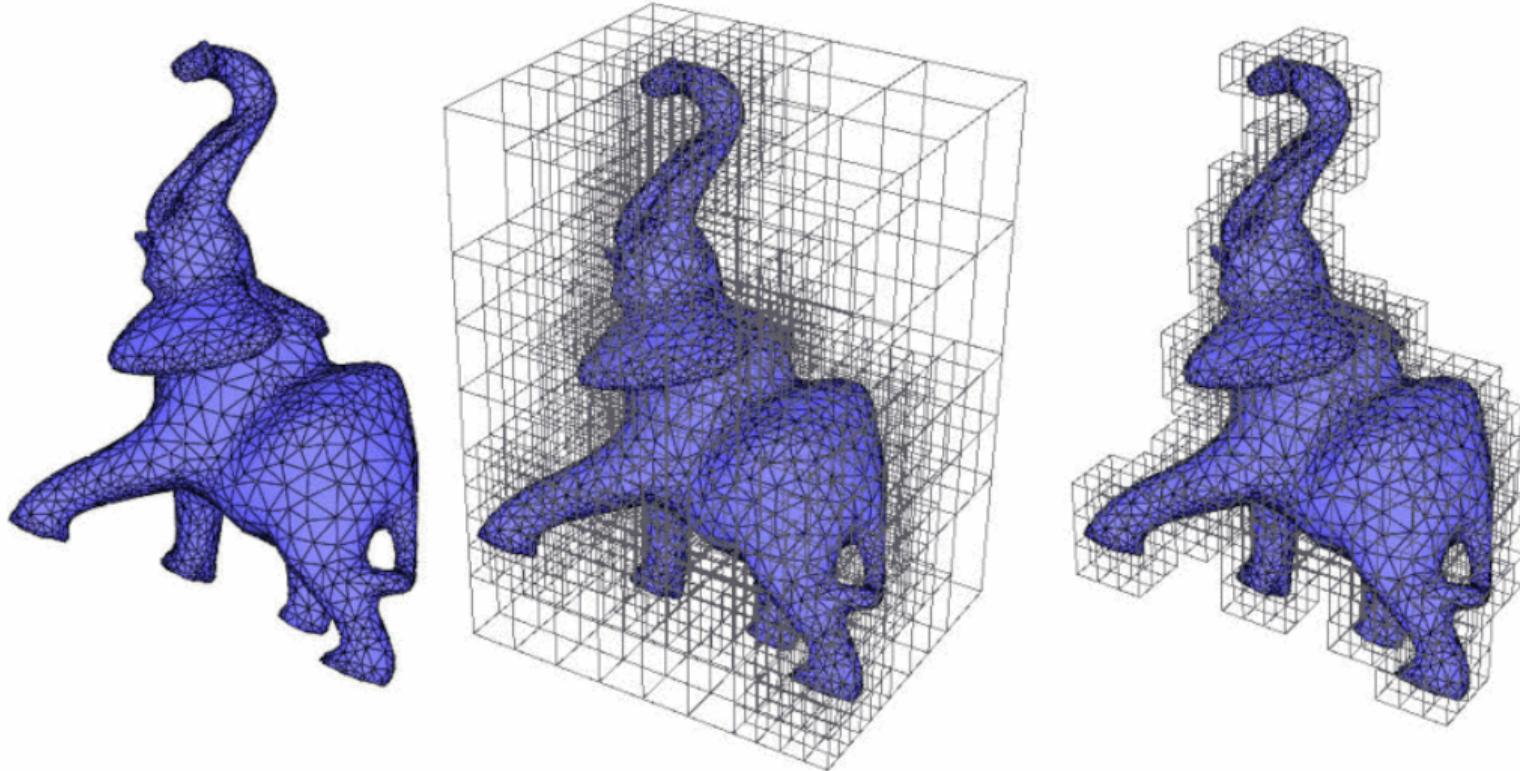
# Enumeração espacial adaptativa

3D



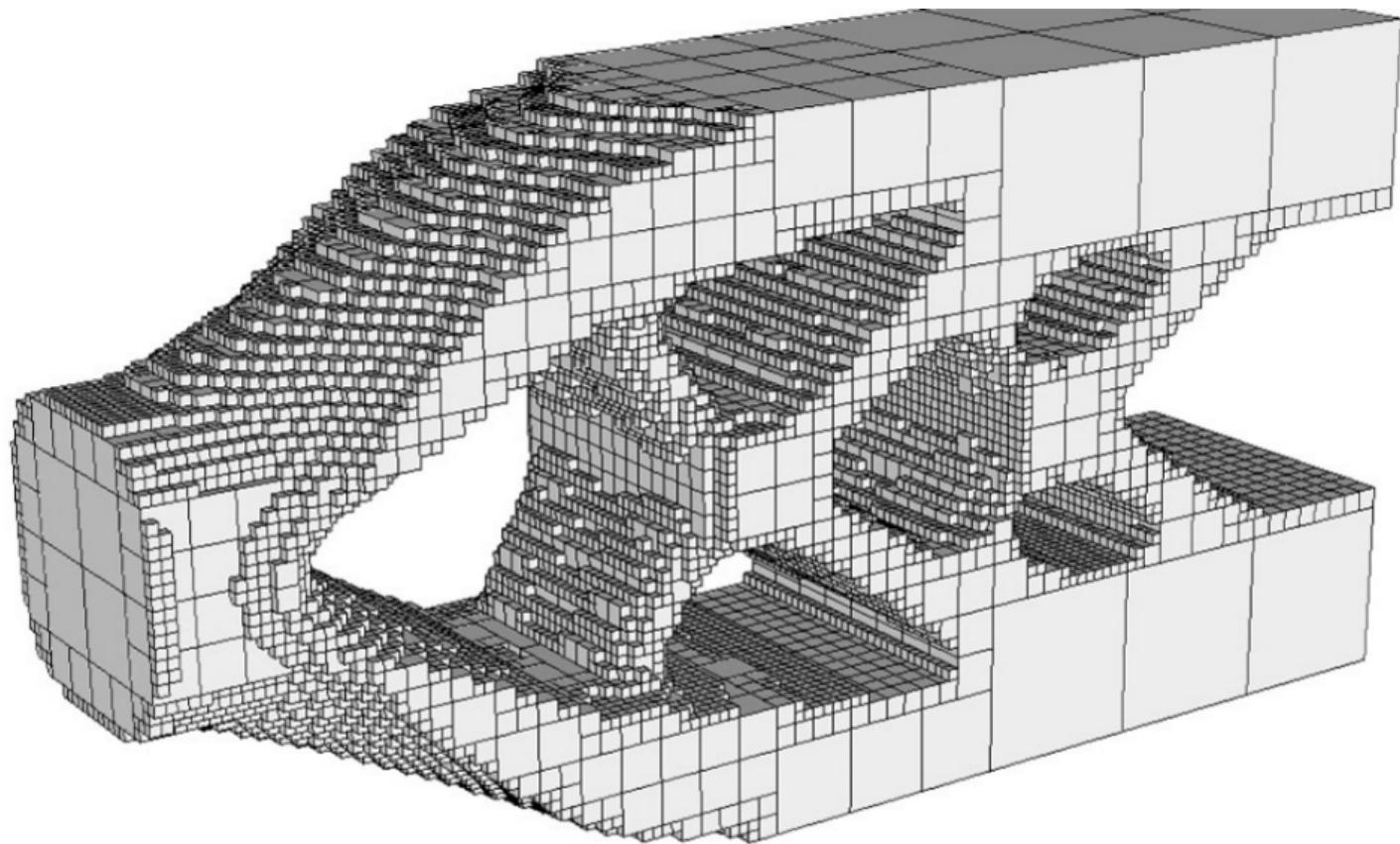
# Enumeração espacial adaptativa

3D



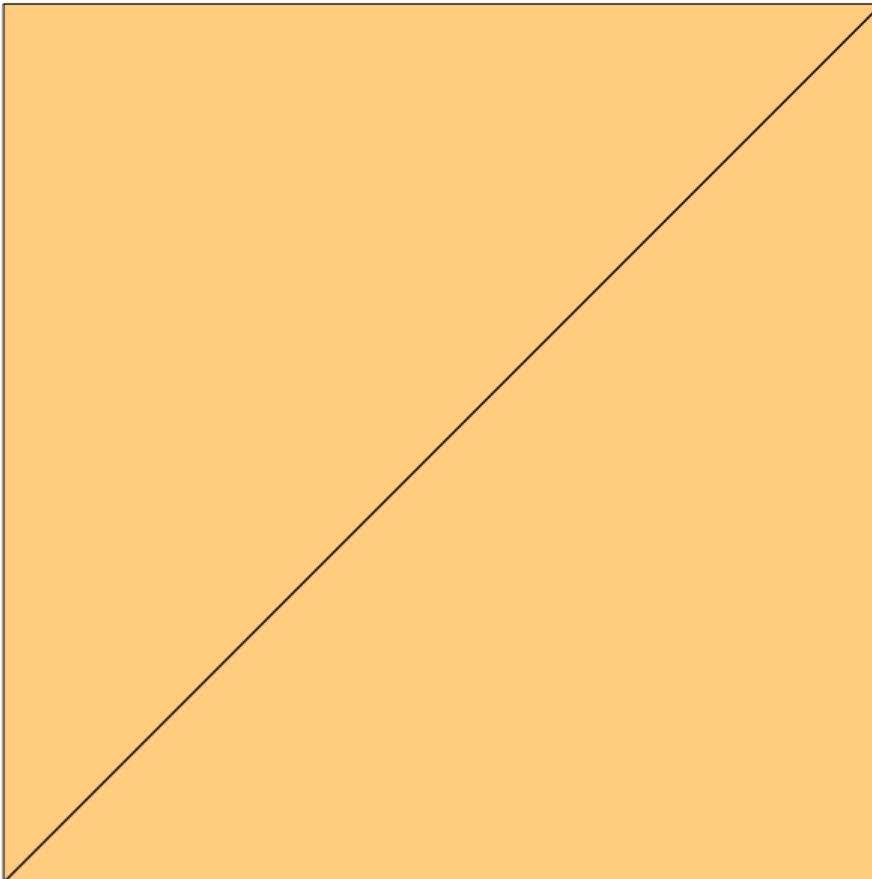
# Enumeração espacial adaptativa

3D



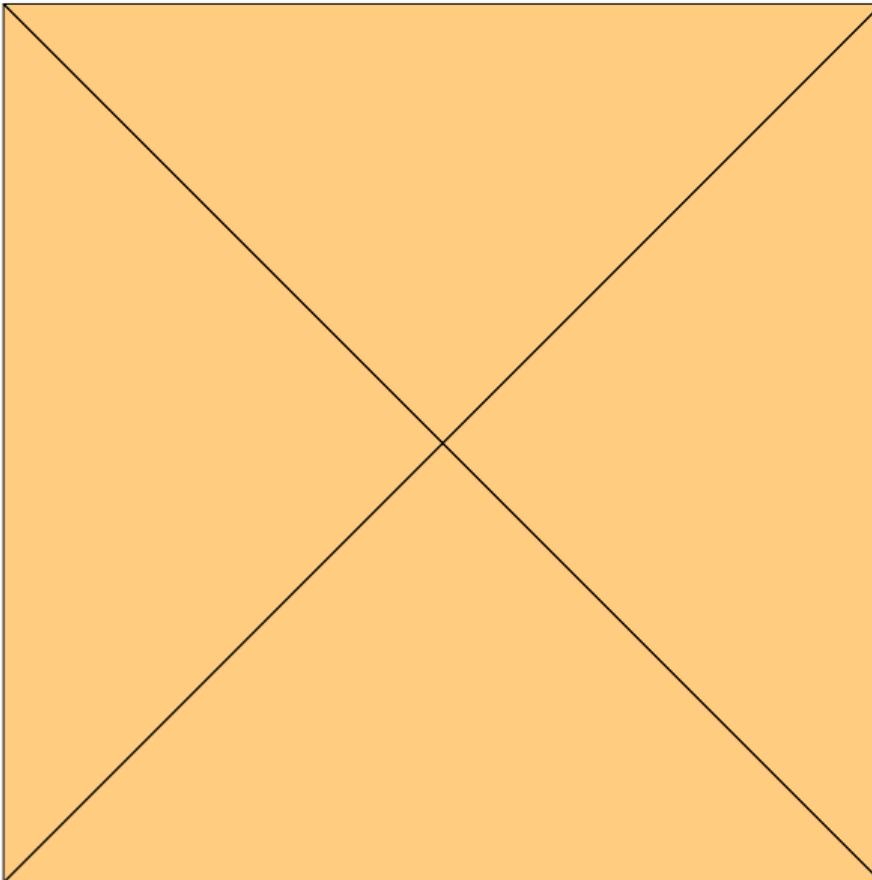
## Enumeração espacial adaptativa

4-8



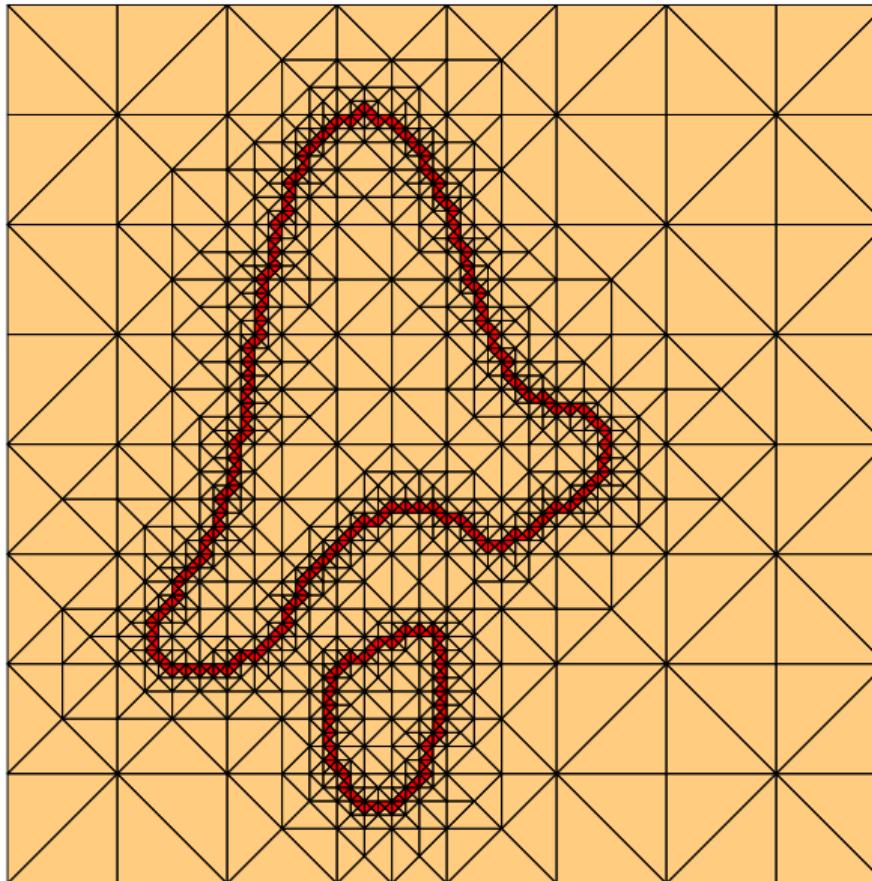
## Enumeração espacial adaptativa

4-8



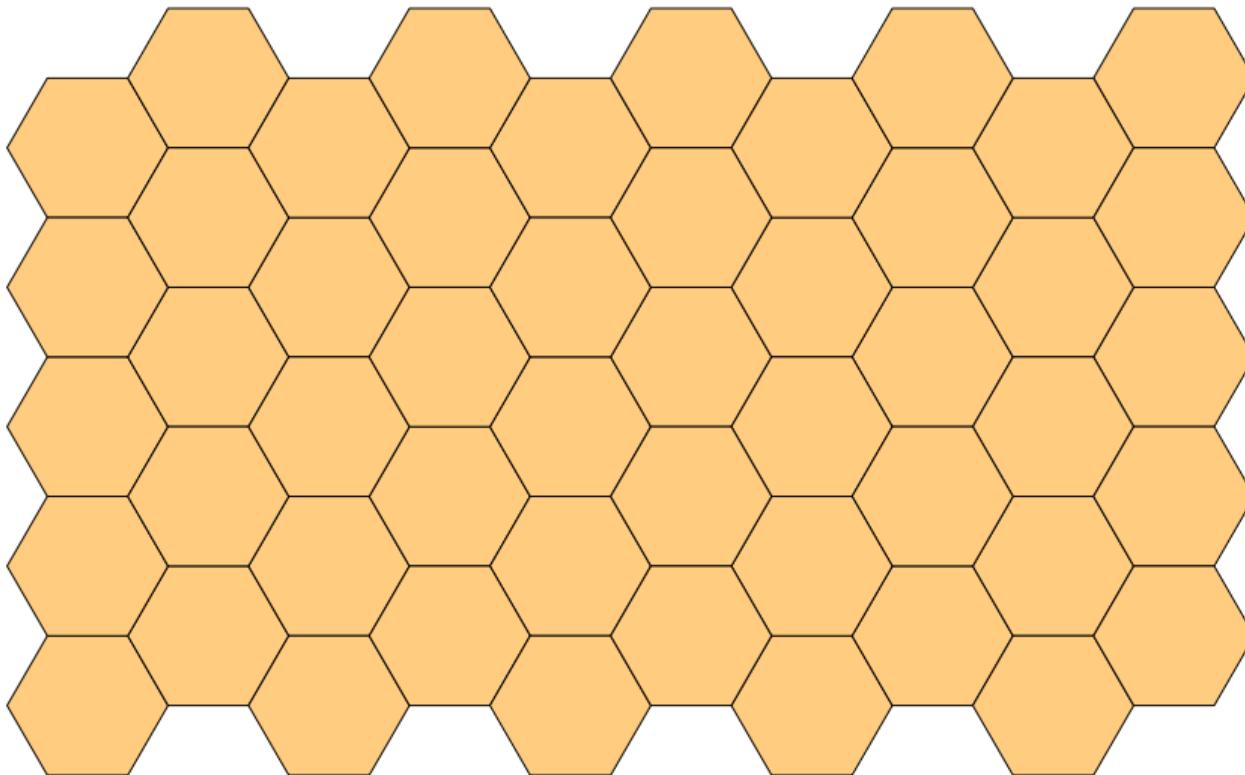
## Enumeração espacial adaptativa

4-8



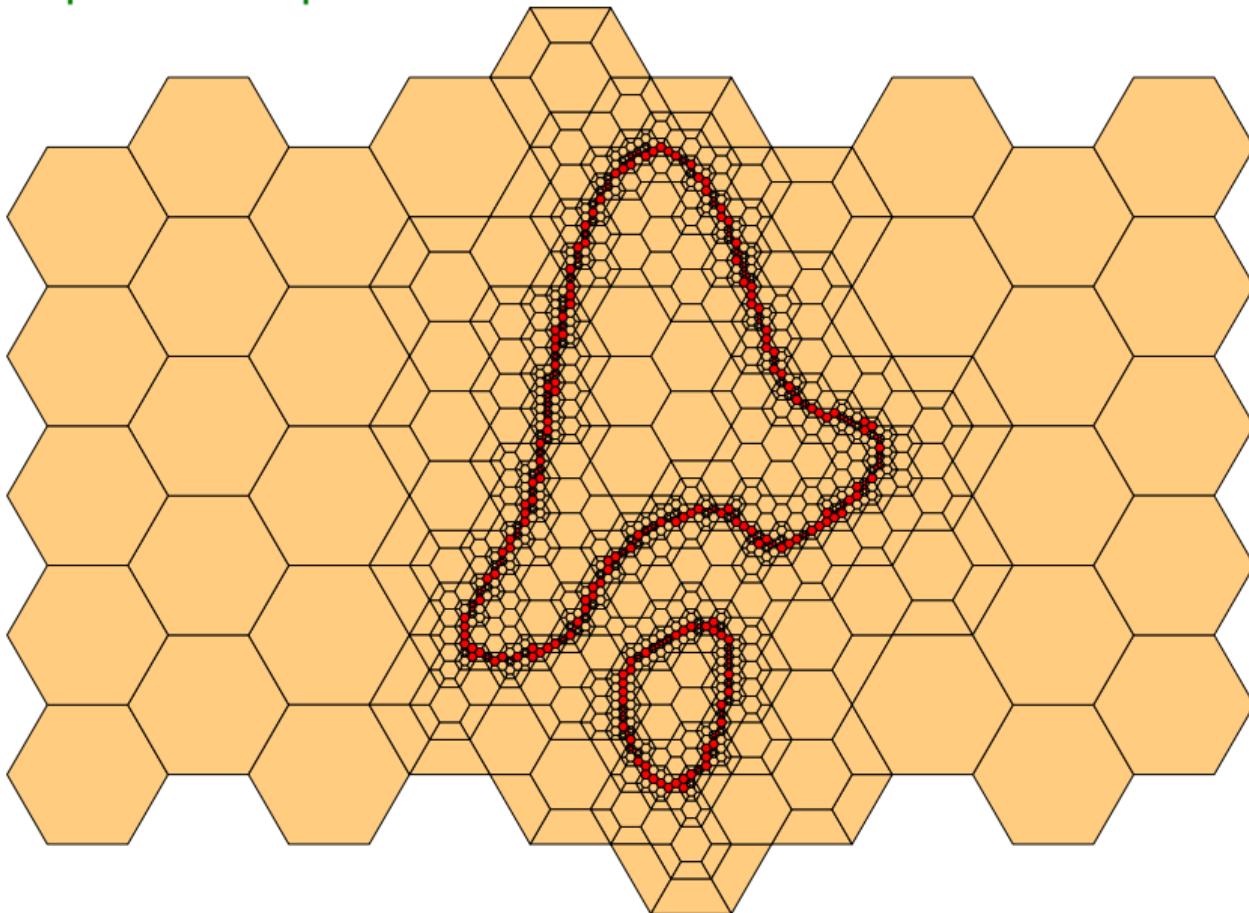
## Enumeração espacial adaptativa

hexagonal



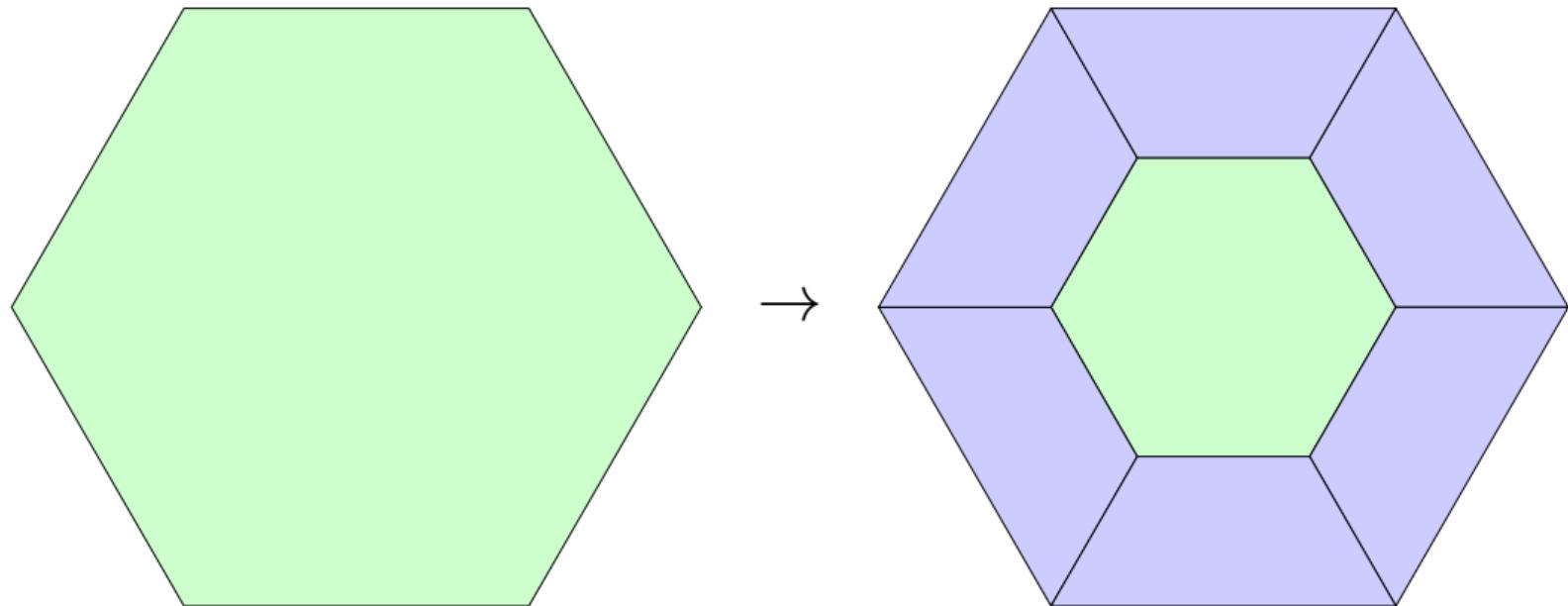
# Enumeração espacial adaptativa

hexagonal



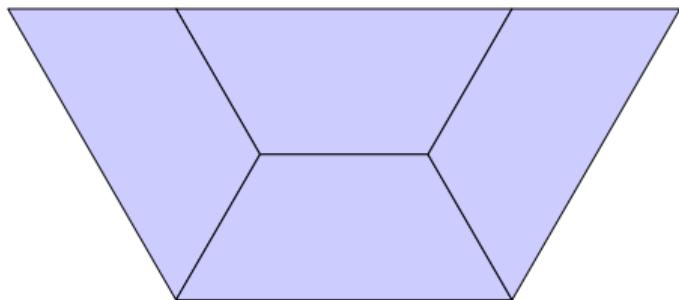
## Enumeração espacial adaptativa

hexagonal



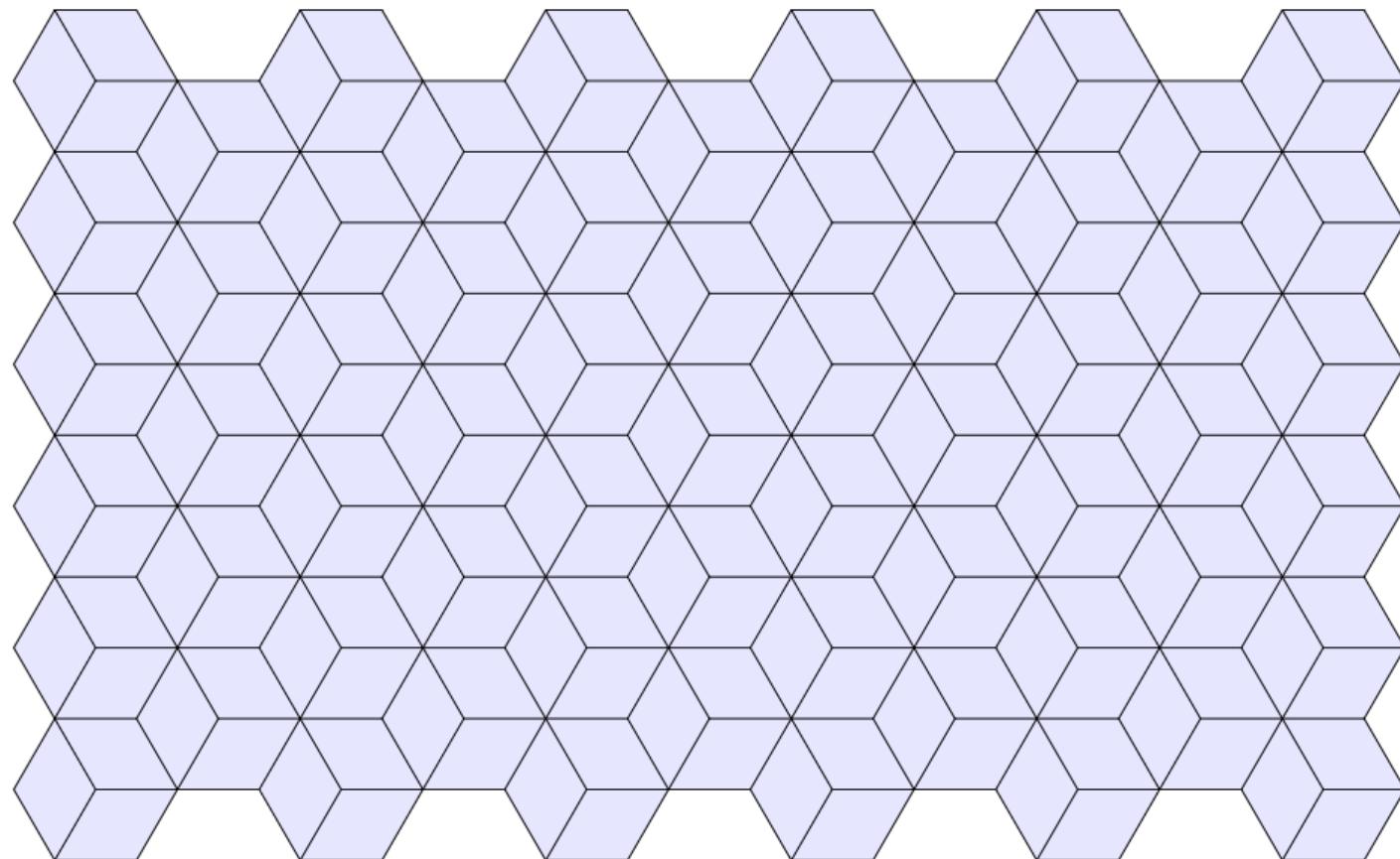
## Enumeração espacial adaptativa

hexagonal



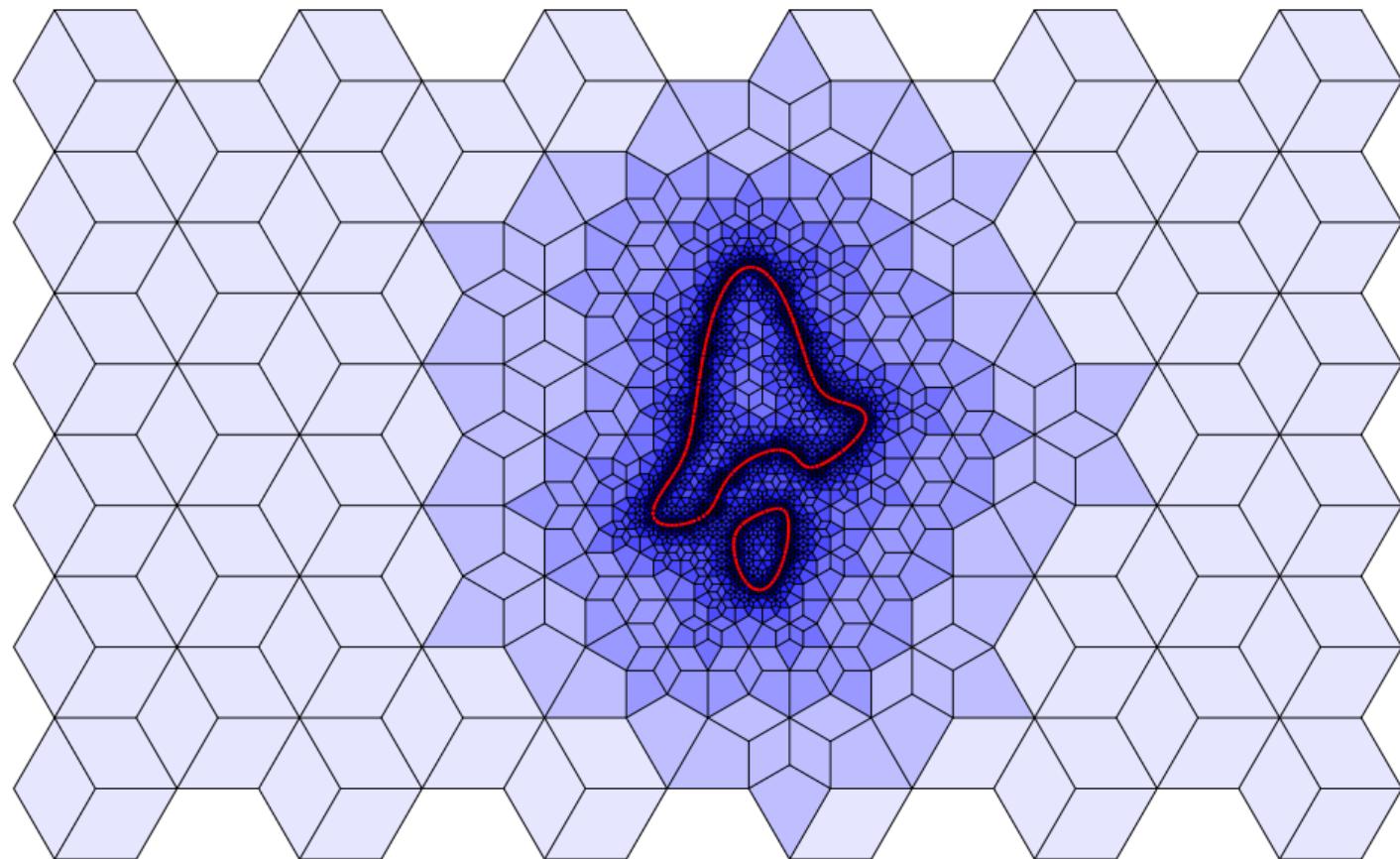
## Enumeração espacial adaptativa

dk



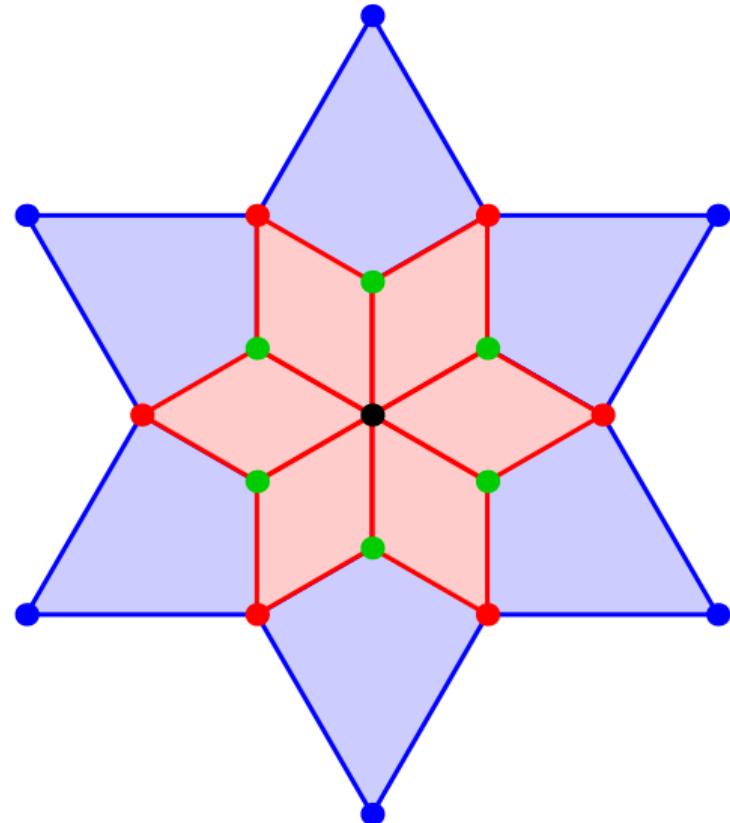
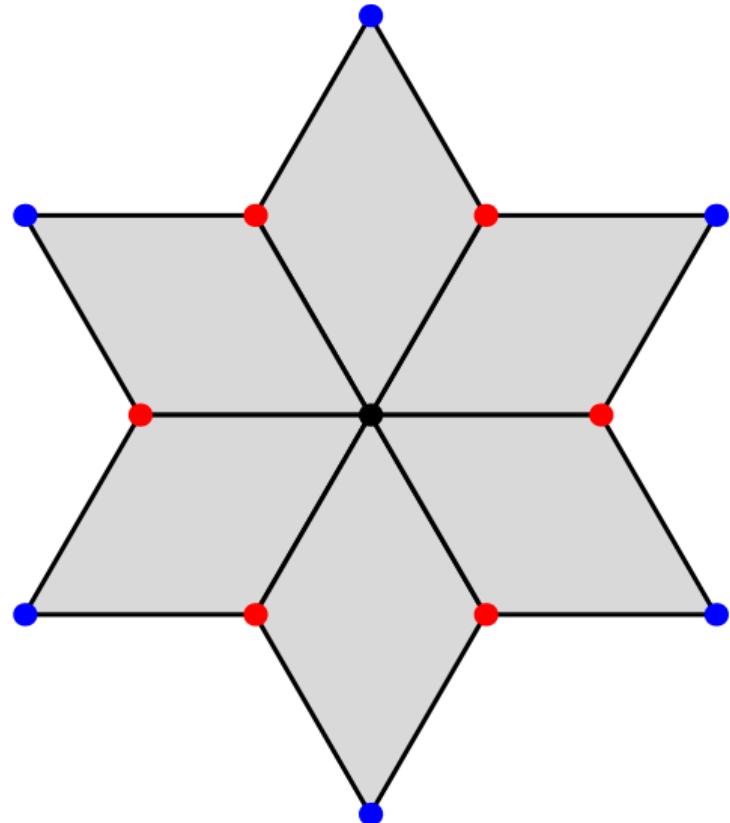
## Enumeração espacial adaptativa

dk



## Enumeração espacial adaptativa

dk



objetos implícitos

## Região implícita

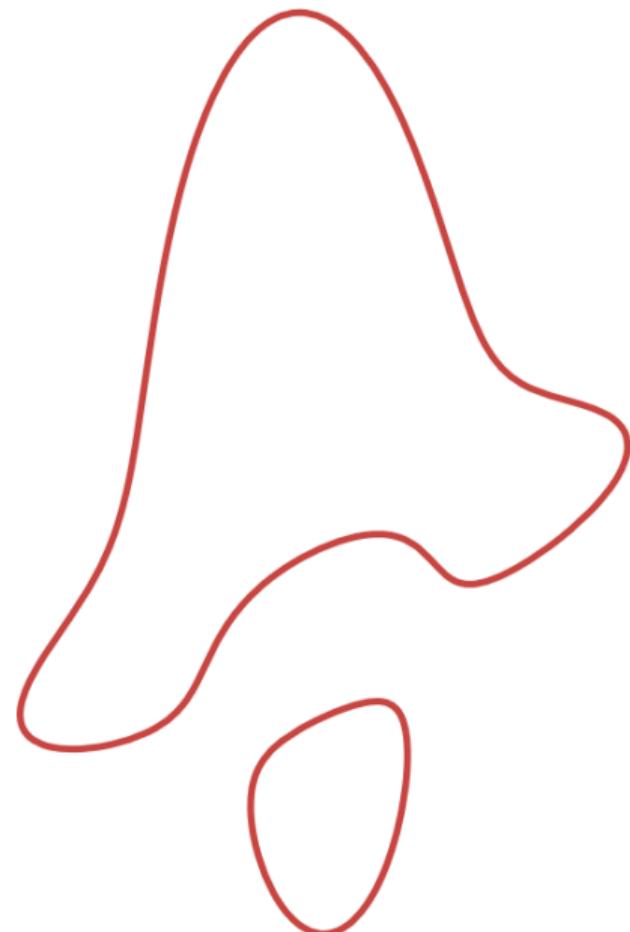
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



## Região implícita

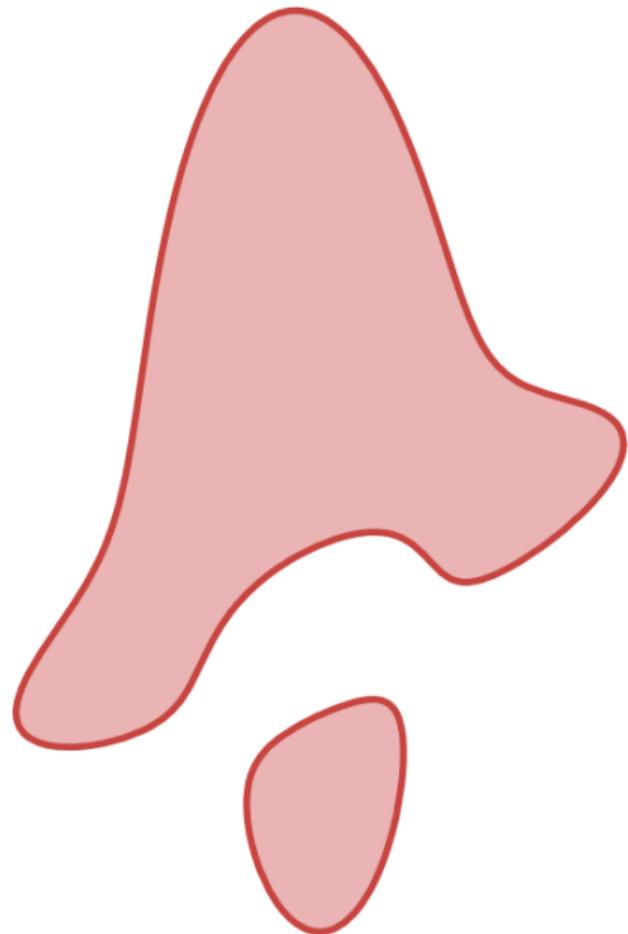
curva = fronteira da região

$$f(x, y) = 0$$

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- amostrar fronteira
- amostrar interior
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# Região implícita

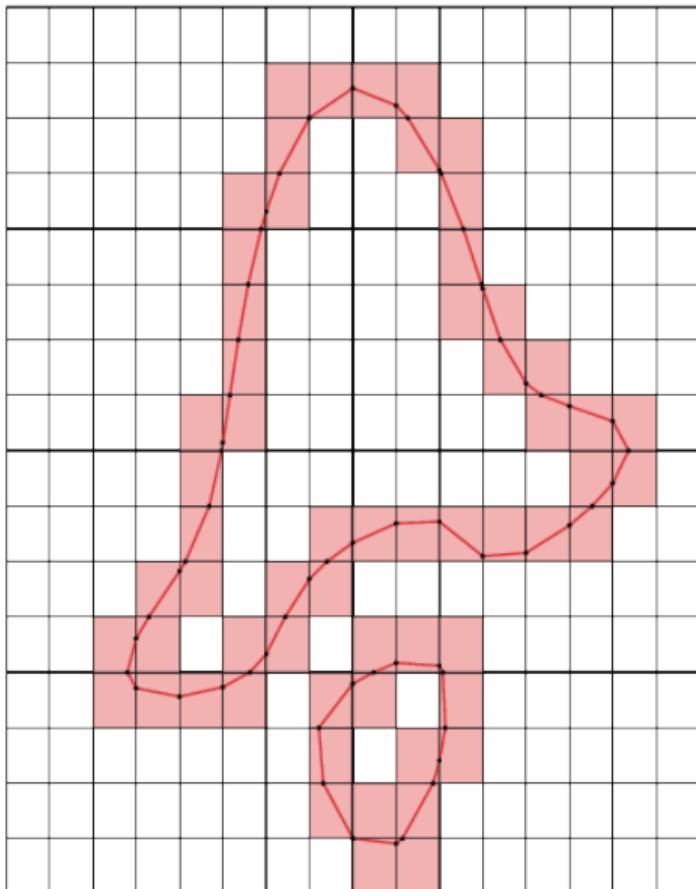
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



# Região implícita

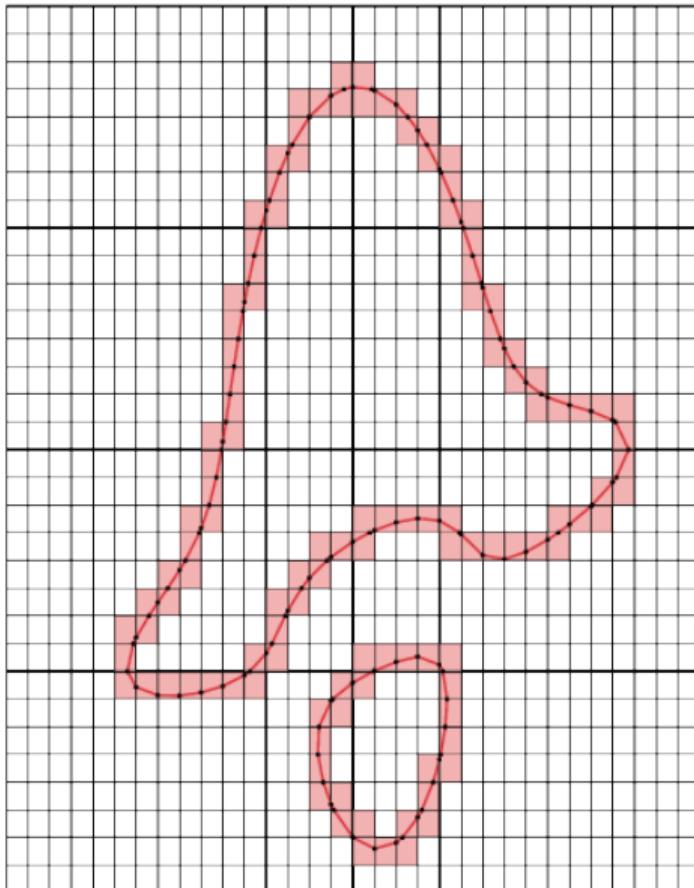
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



# Região implícita

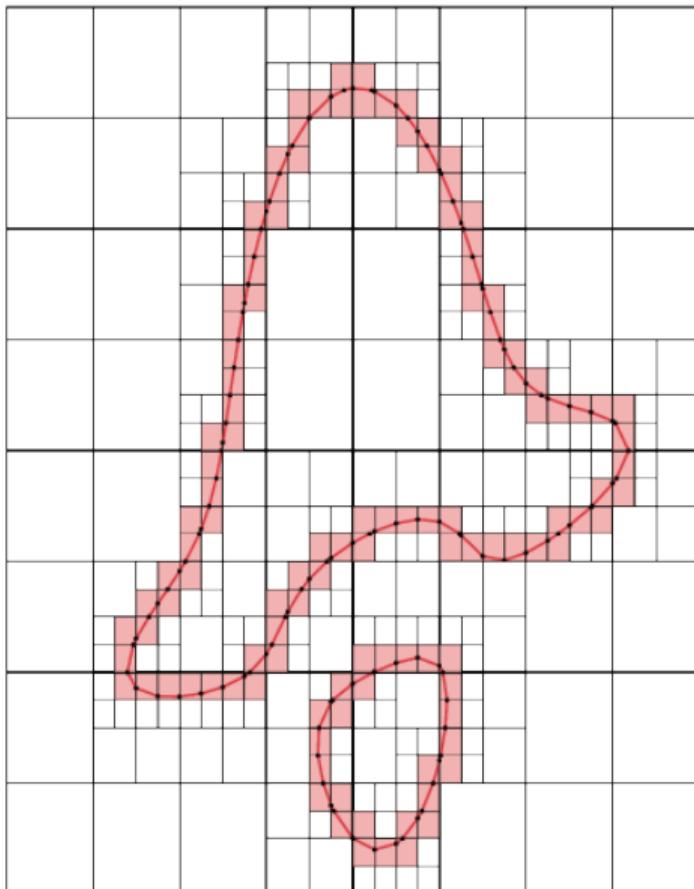
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

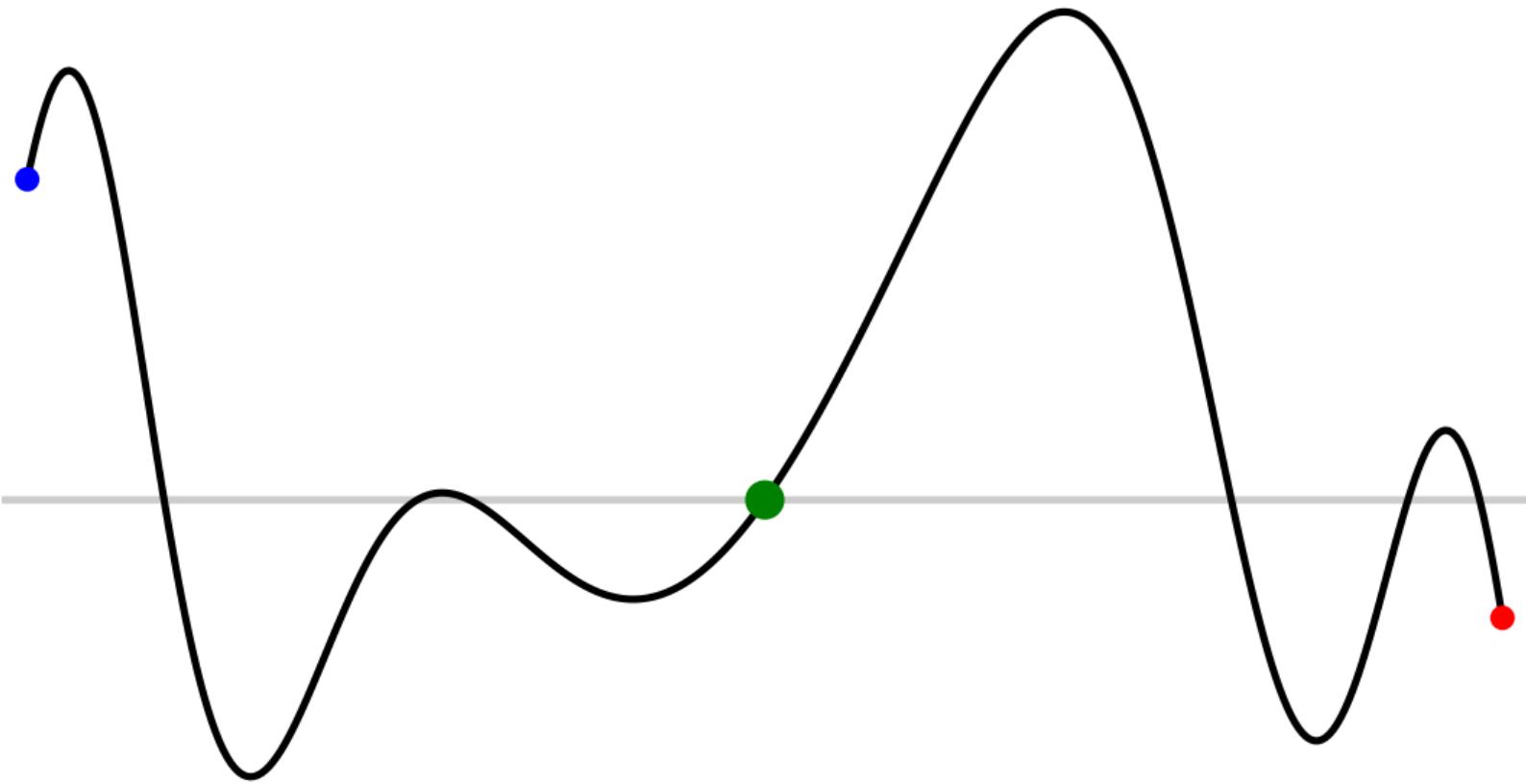
- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua
  - $[a, b] \subseteq \Omega$
  - $f(a) > 0$
  - $f(b) < 0$
- ⇒ existe  $x^* \in (a, b)$  tal que  $f(x^*) = 0$

## Teorema do valor intermediário

Bolzano (1817)



- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua
  - $[a, b] \subseteq \Omega$
  - $f(a) > 0$
  - $f(b) < 0$
- ⇒ existe  $x^* \in (a, b)$  tal que  $f(x^*) = 0$

- $f: \Omega = \mathbf{R} \rightarrow \mathbf{R}$  dada por  $f(x) = x^2 - 2$  é contínua
- $[0, 2] \subseteq \Omega$
- $f(0) < 0$
- $f(2) > 0$

$\implies$  existe  $r \in (0, 2)$  tal que  $f(r) = 0$ , isto é,  $r^2 = 2$

$$r = \sqrt{2}$$

- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$  contínua
  - $[a, b] \subseteq \Omega$
  - $f(a) > 0$
  - $f(b) < 0$
- ⇒ existe  $x^* \in (a, b)$  tal que  $f(x^*) = 0$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b$   $f(a) > 0, \quad f(b) < 0$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se  $f(m) = 0: \quad x^* = m \quad \text{fim}$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se  $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se  $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se  $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se  $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se  $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se  $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se  $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se  $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
- $f(a_n) > 0, \quad f(b_n) < 0 \quad \text{invariante}$

## Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se  $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se  $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se  $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
- $f(a_n) > 0, \quad f(b_n) < 0 \quad \text{invariante}$
- $a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1 \leq b_0 = b$

## Teorema do valor intermediário – prova construtiva: método da bisseção

■  $a_0 = a, \quad b_0 = b$   $f(a) > 0, \quad f(b) < 0$

■  $m = \frac{a_n + b_n}{2}$

■ se  $f(m) = 0: \quad x^* = m$  fim

■ se  $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n$   $b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

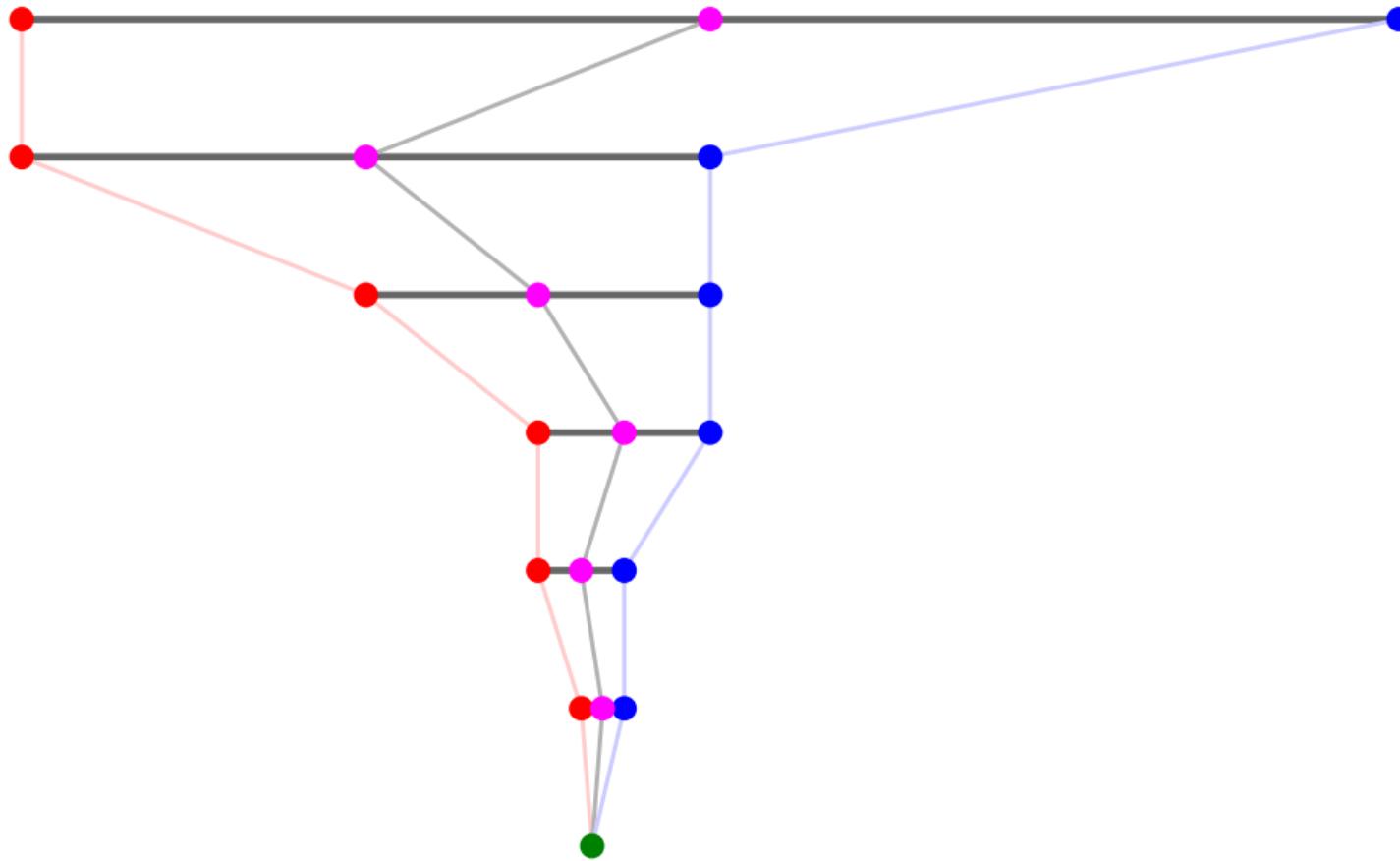
■ se  $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m$   $[a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

■  $f(a_n) > 0, \quad f(b_n) < 0$  invariante

■  $a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1 \leq b_0 = b$

$\Rightarrow x^* = \lim a_n = \lim b_n, \quad f(x^*) = 0$   $b_n - a_n = \frac{b - a}{2^n} \rightarrow 0$

## Teorema do valor intermediário – método da bisseção para $\sqrt{2}$



## Método da bisseção

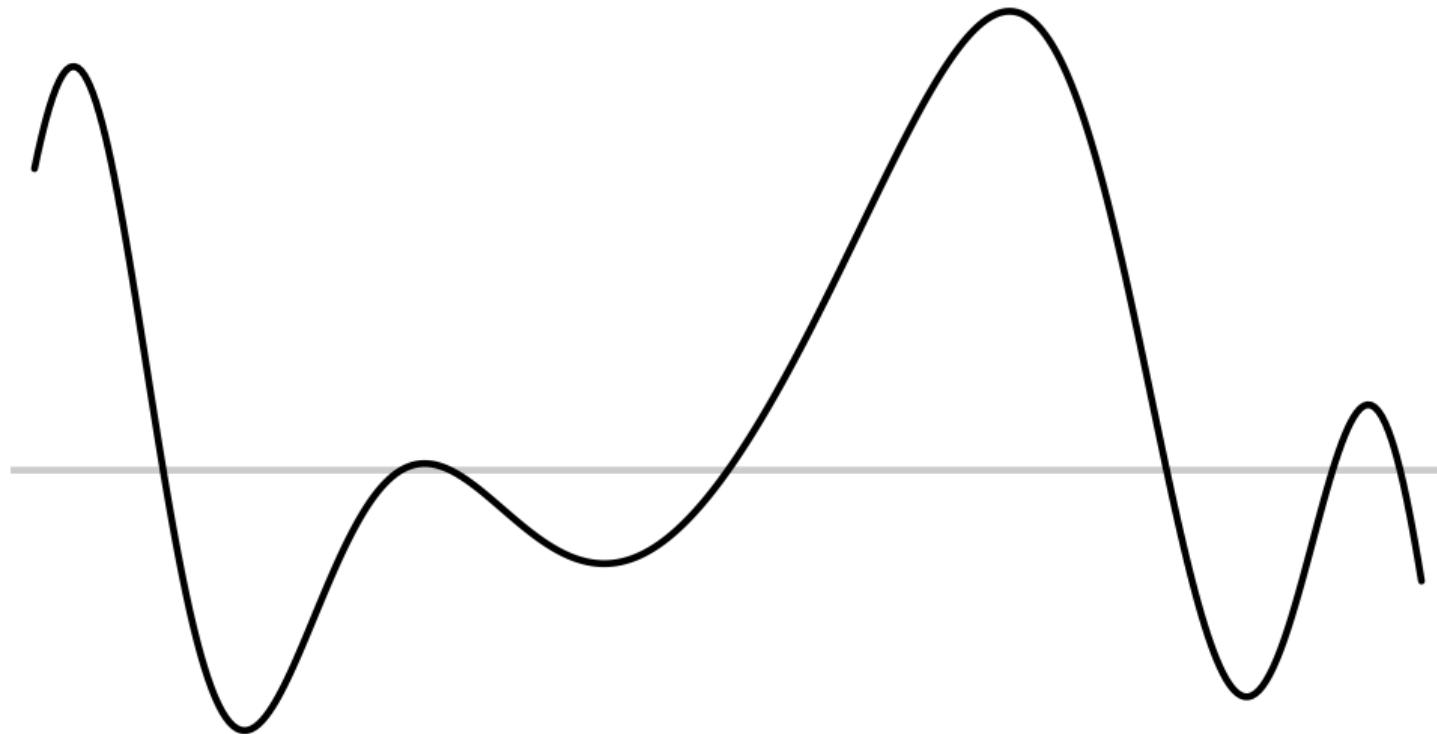
```
function bisect(f,a,b)
    local c=(a+b)/2
    if abs(a-b)<delta or c==a or c==b then
        return c
    end
    if f(c)>0 then
        return bisect(f,a,c)
    else
        return bisect(f,c,b)
    end
end
```

## Método da bisseção

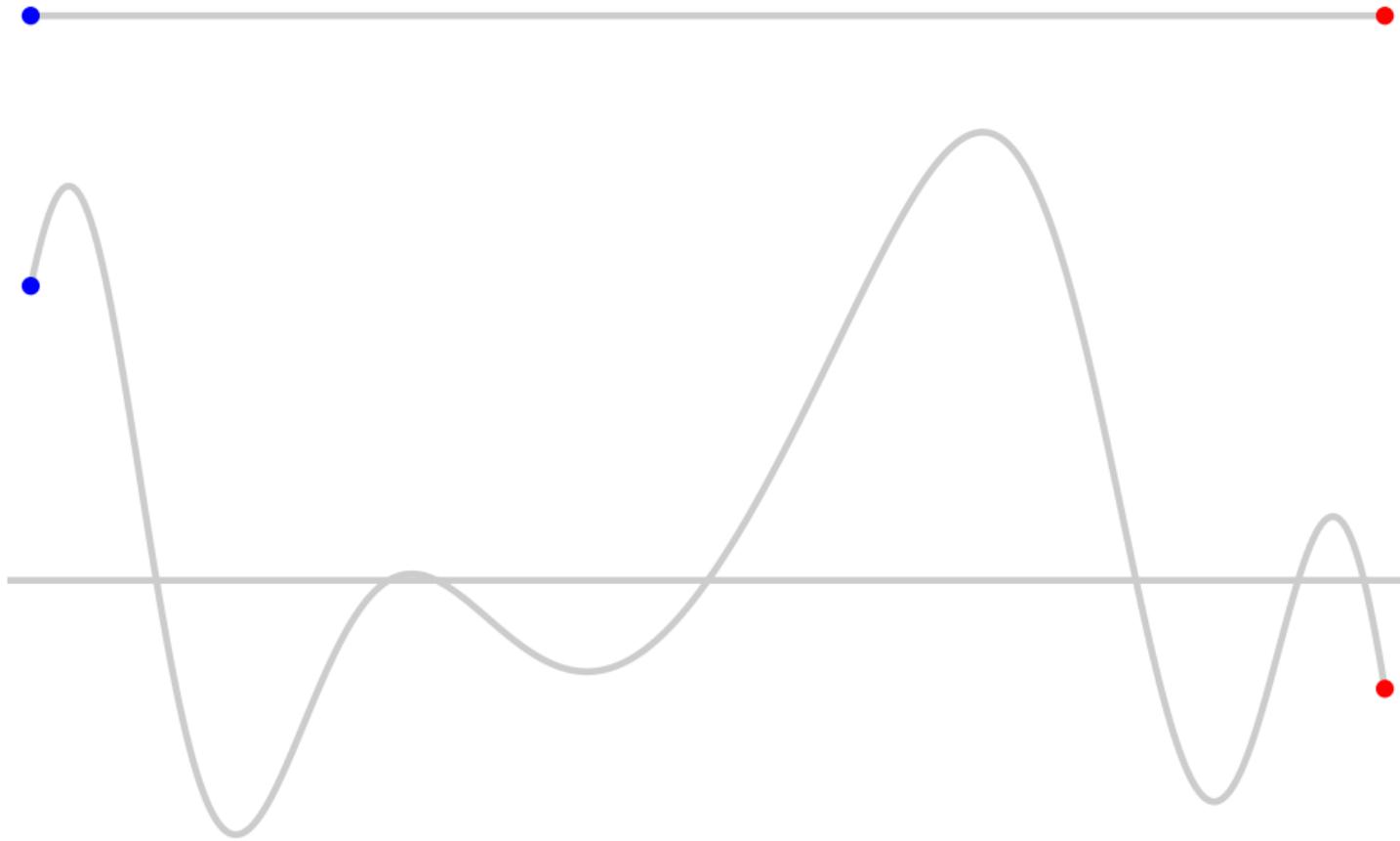
```
function bisect(f,a,b)
    while true do
        local c=(a+b)/2
        if abs(a-b)<delta or c==a or c==b then
            return c
        end
        if f(c)>0 then
            b=c
        else
            a=c
        end
    end
end
```

## Amostragem em grade

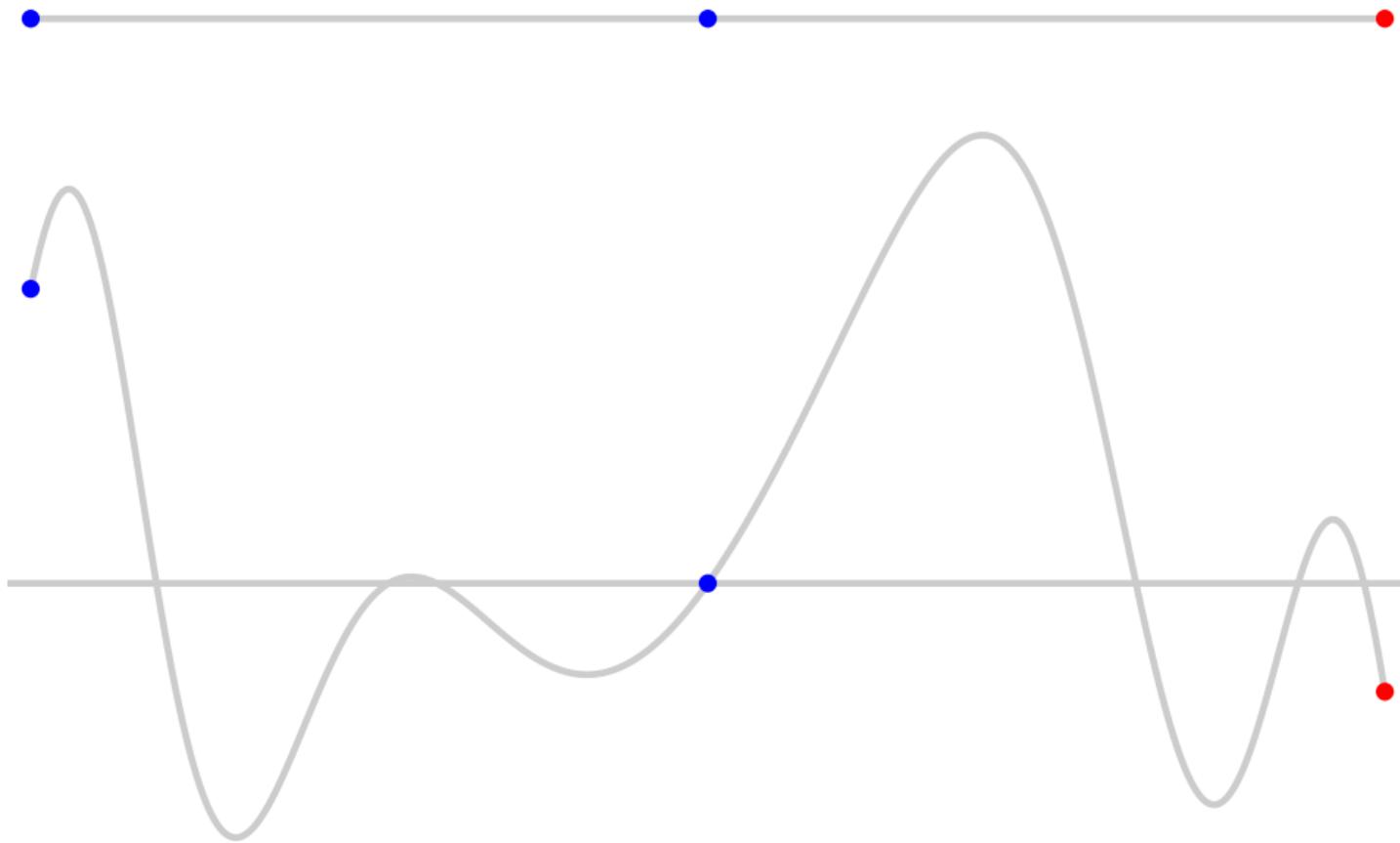
---



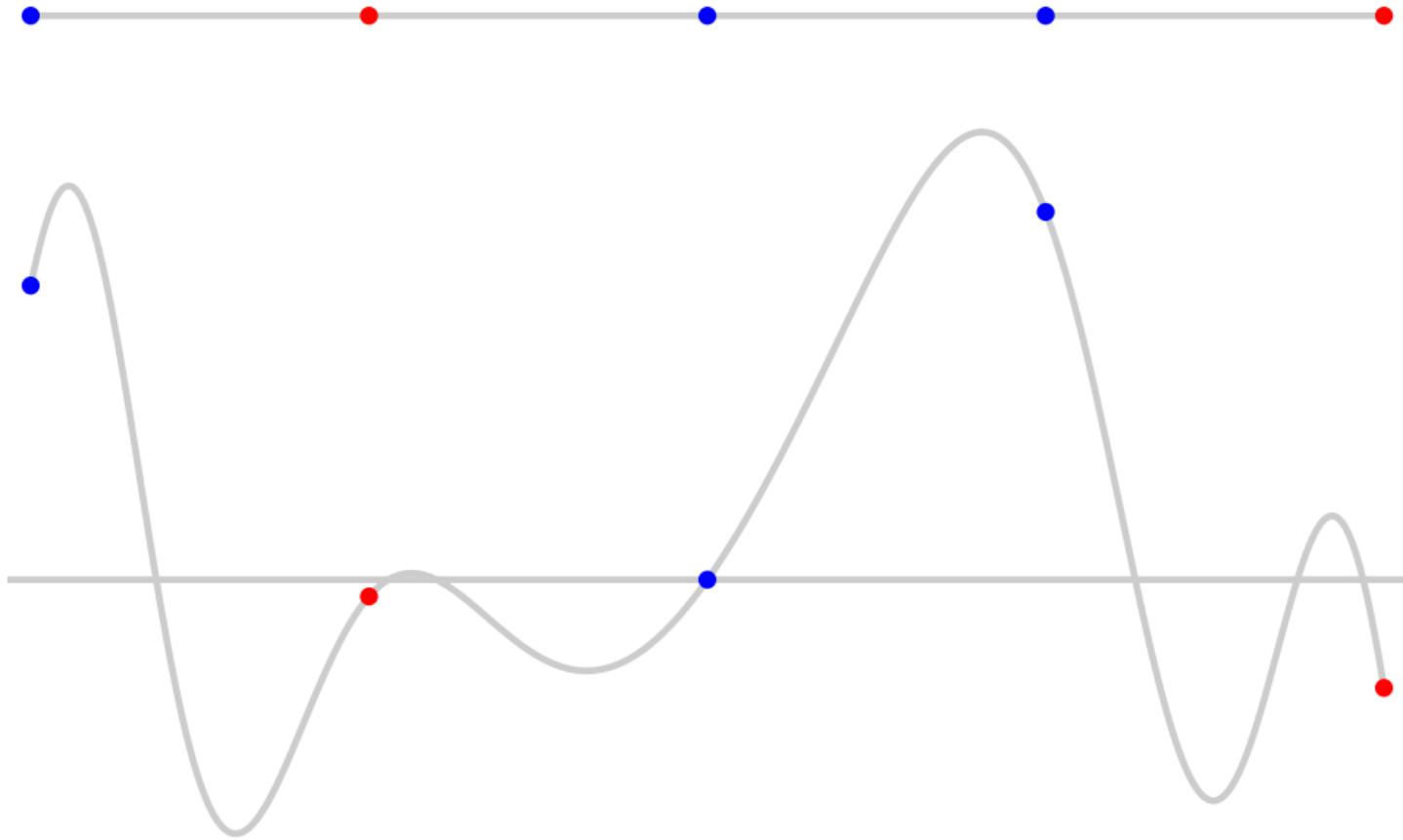
## Amostragem em grade



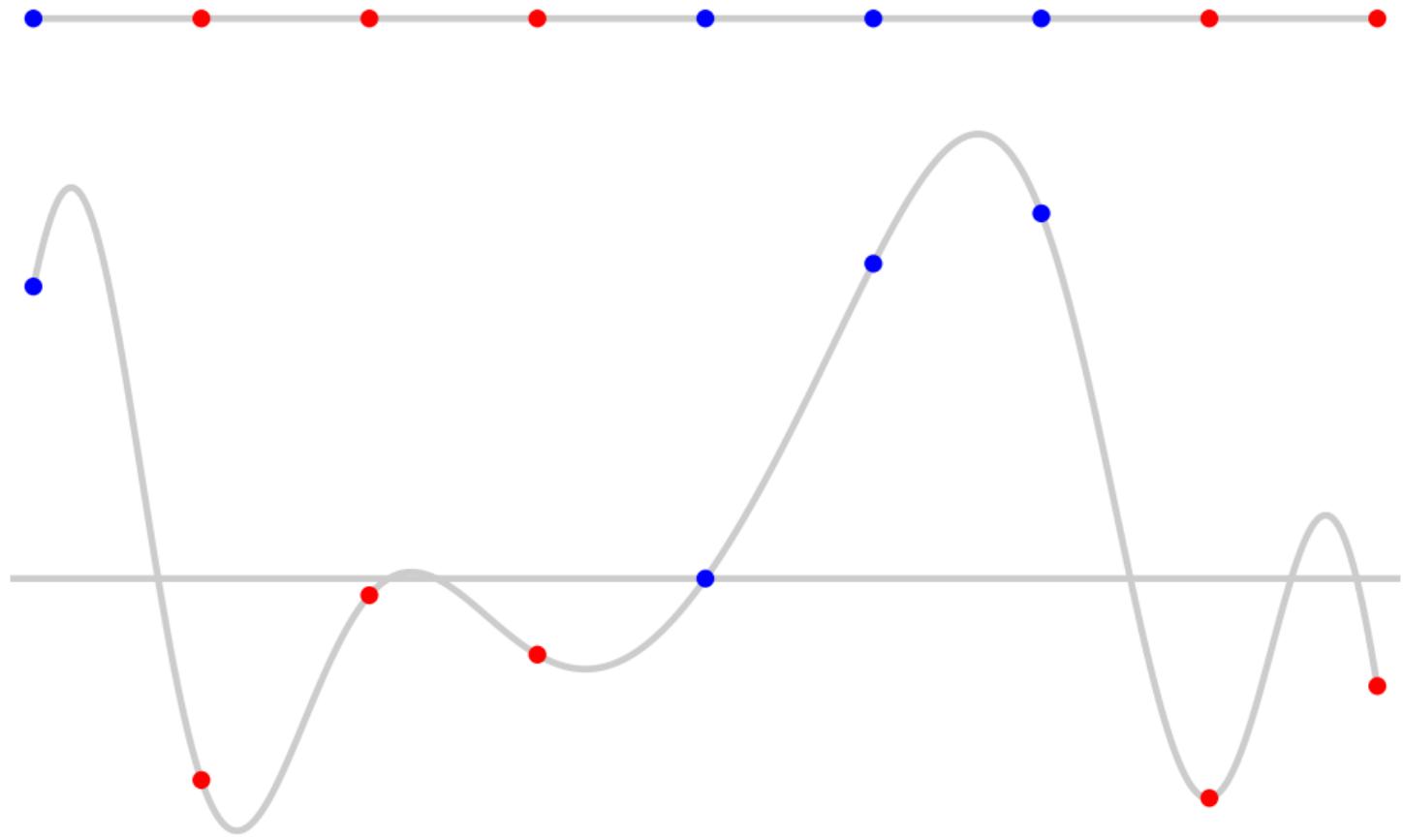
## Amostragem em grade



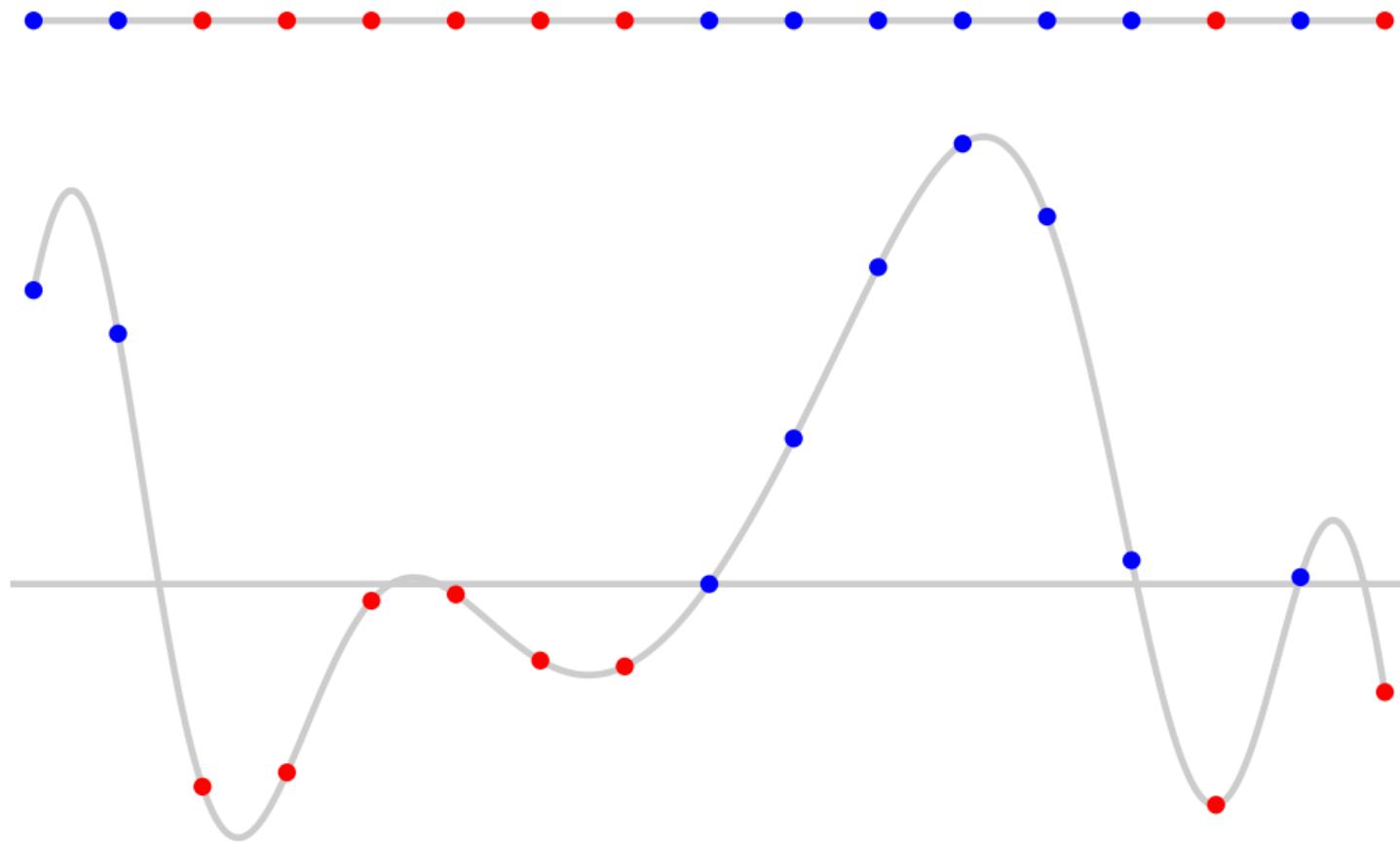
## Amostragem em grade



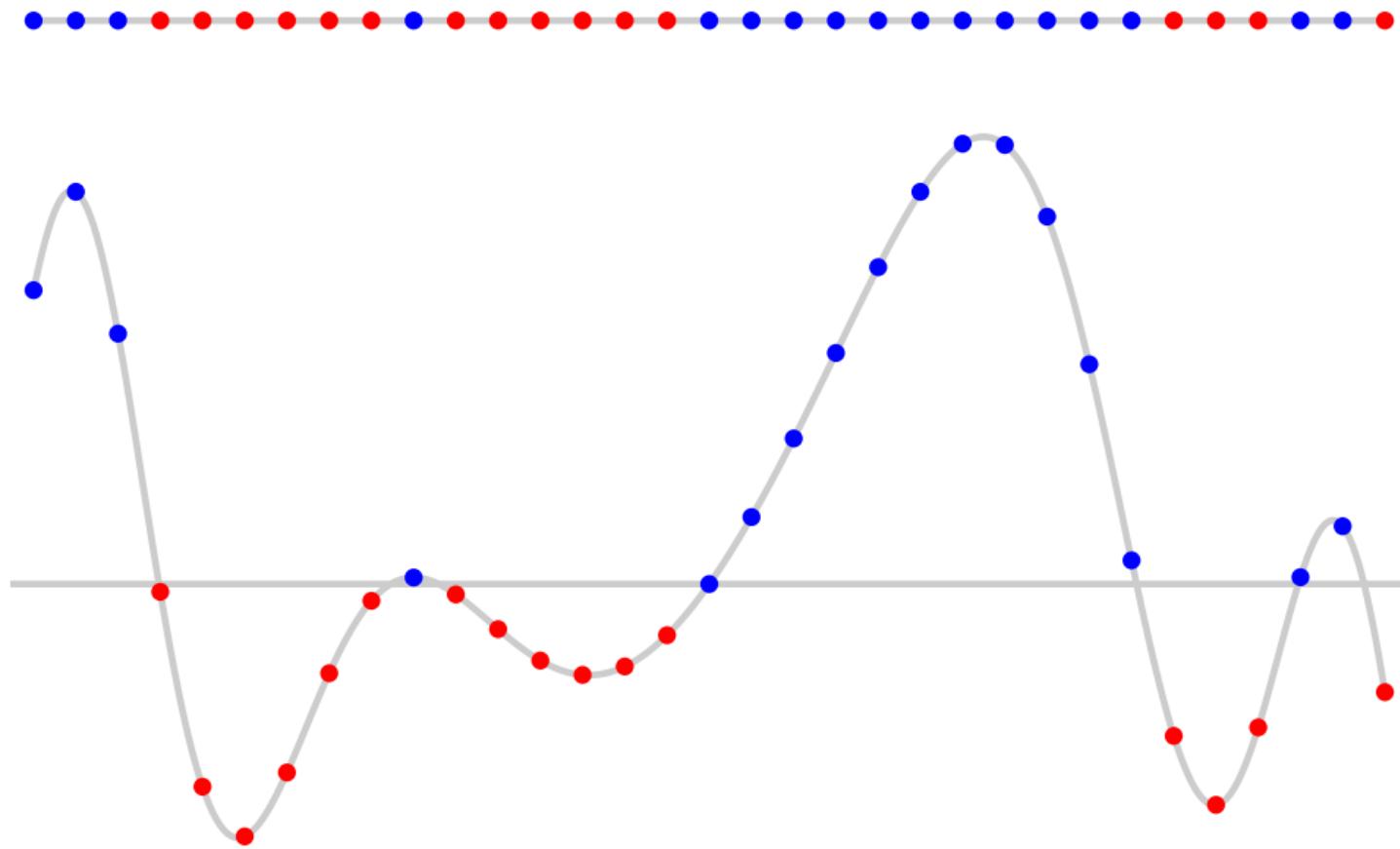
## Amostragem em grade



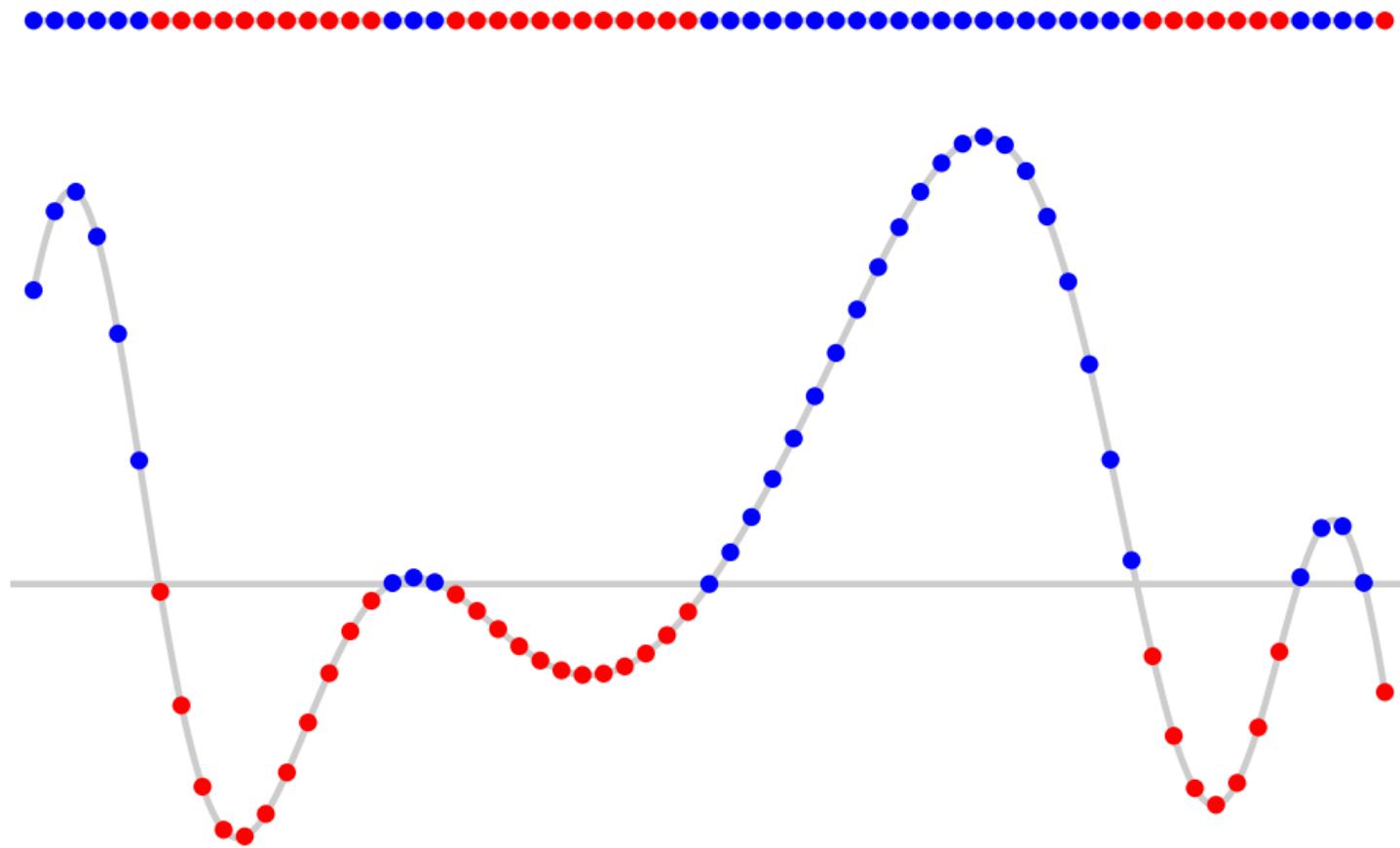
## Amostragem em grade



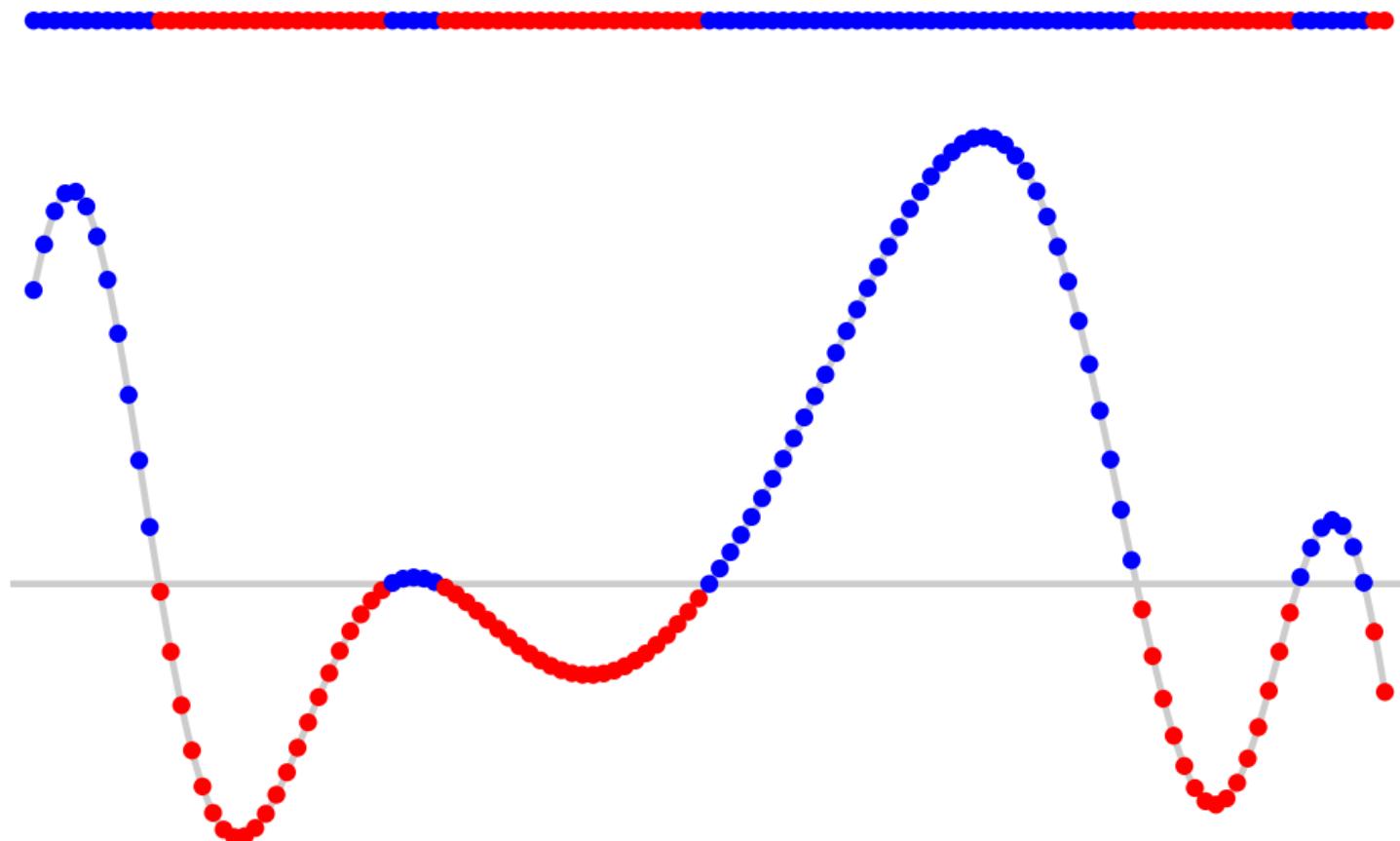
## Amostragem em grade



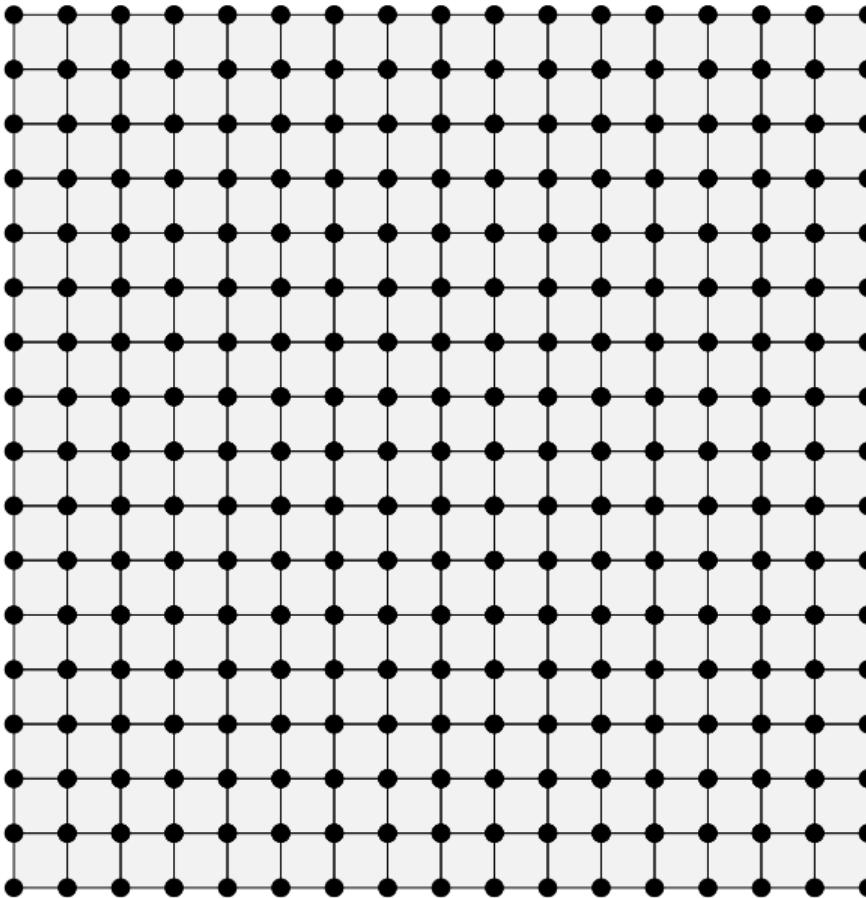
## Amostragem em grade



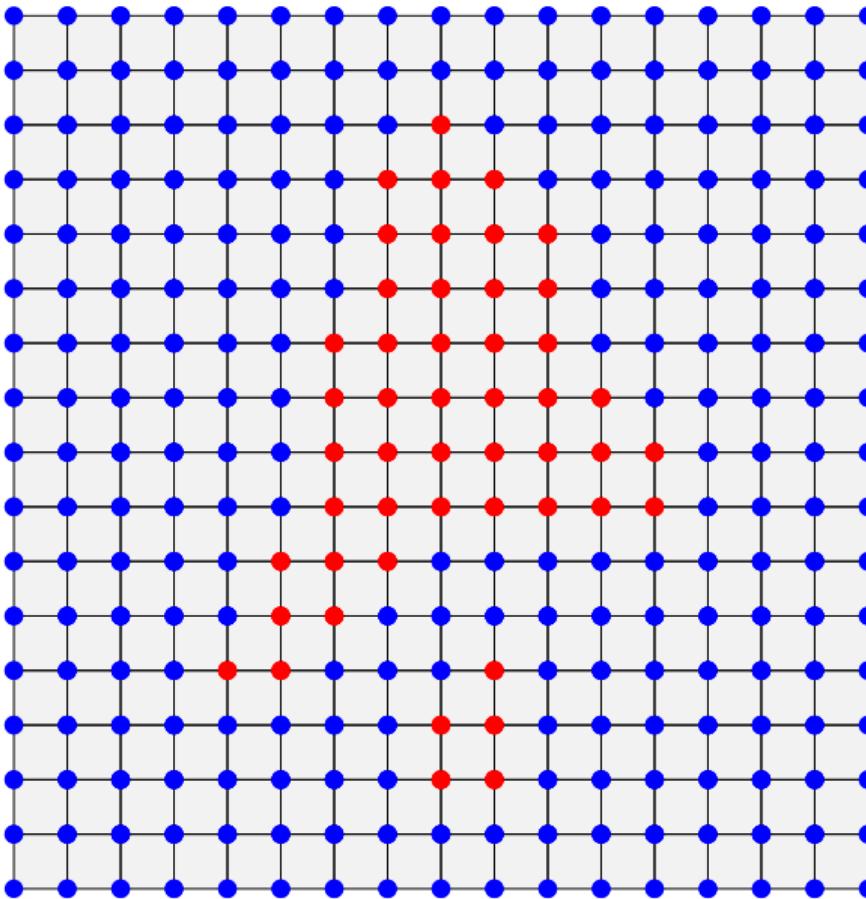
## Amostragem em grade



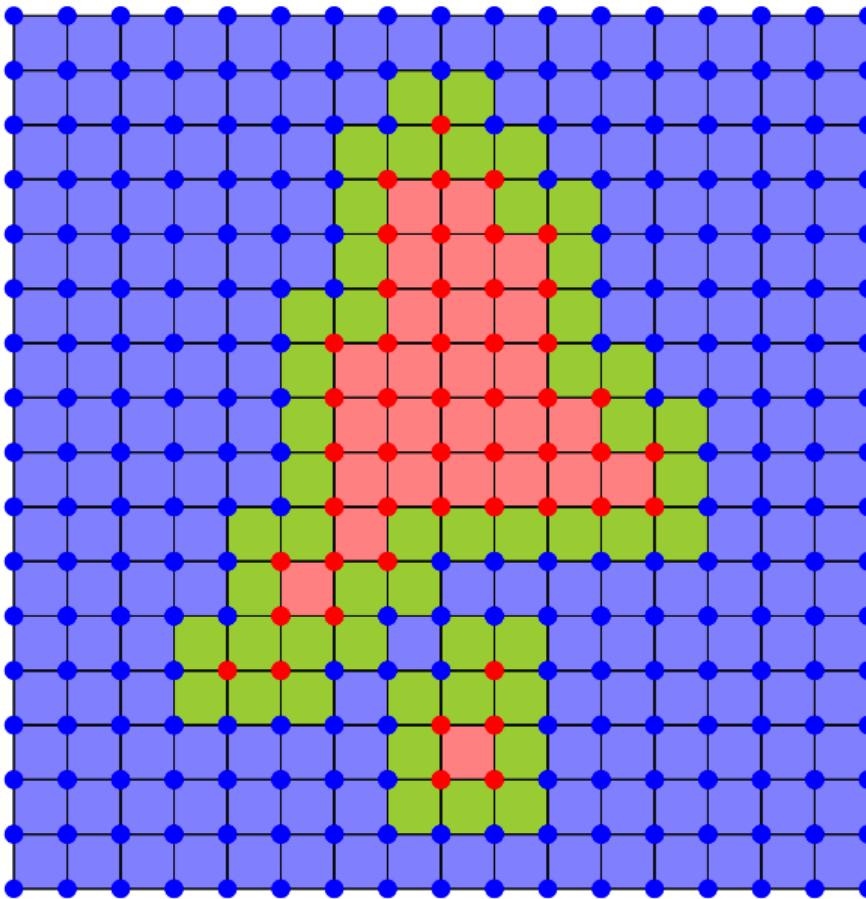
## Região implícita – classificação por amostragem



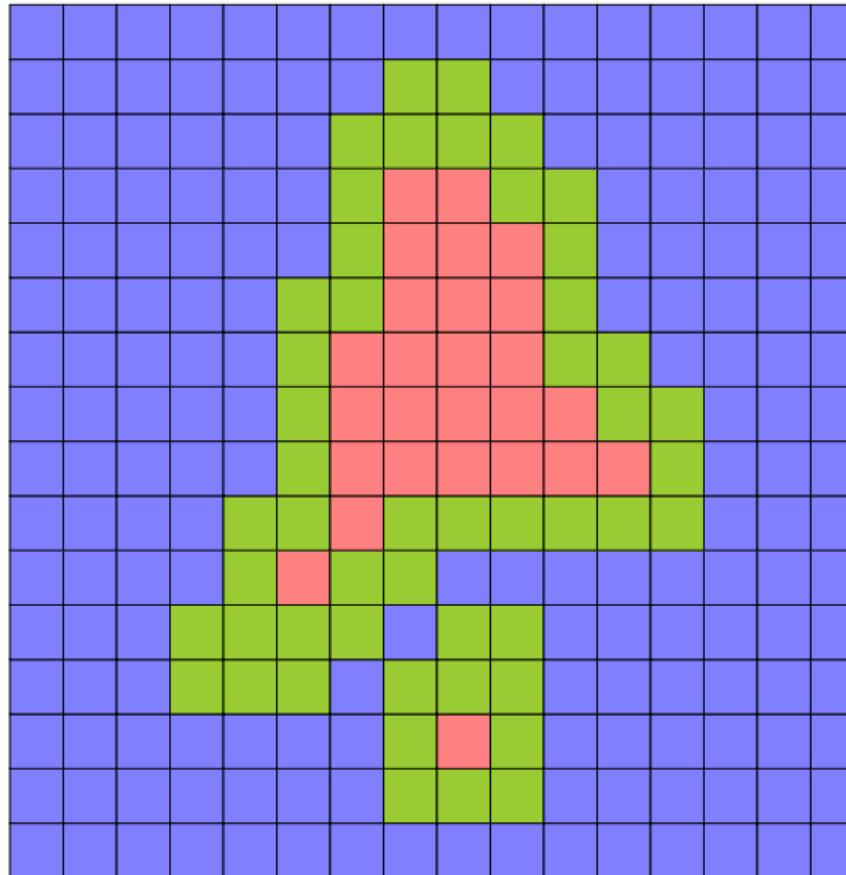
## Região implícita – classificação por amostragem



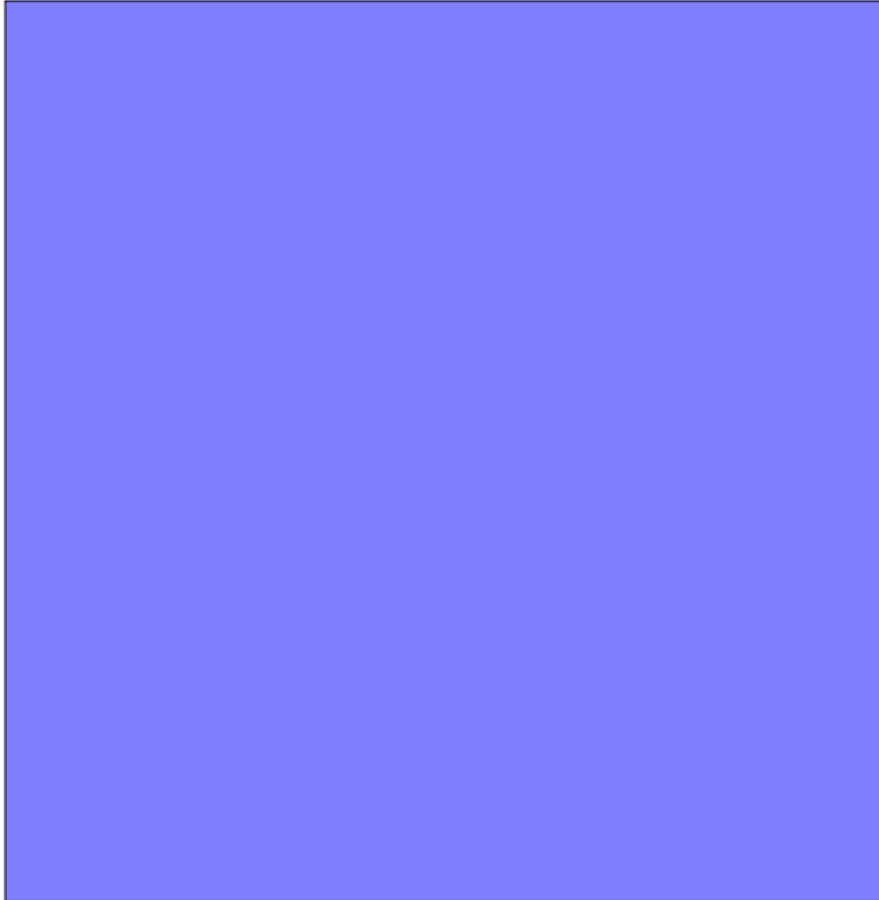
## Região implícita – classificação por amostragem



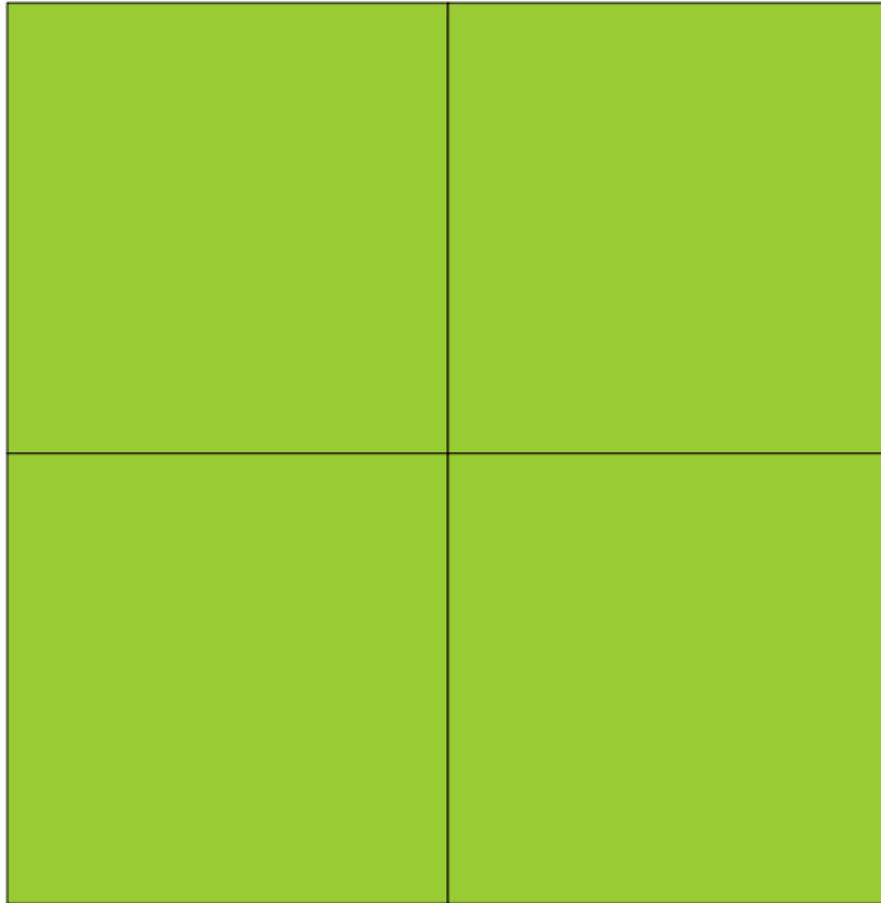
## Região implícita – classificação por amostragem



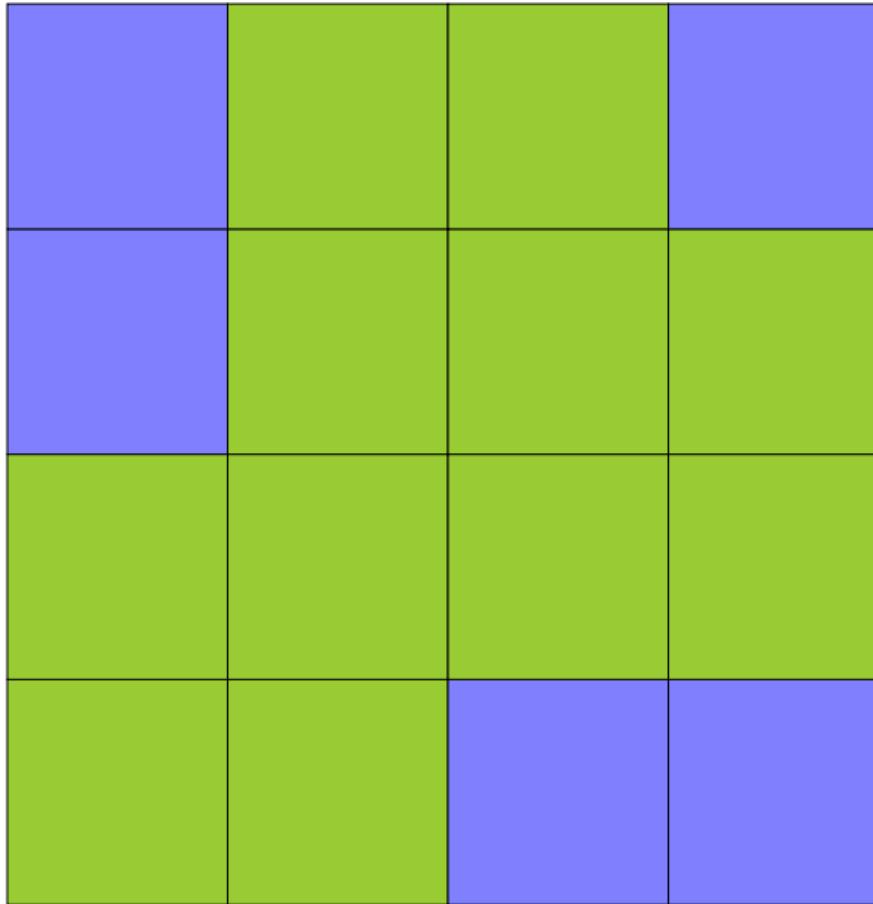
## Região implícita – classificação por amostragem



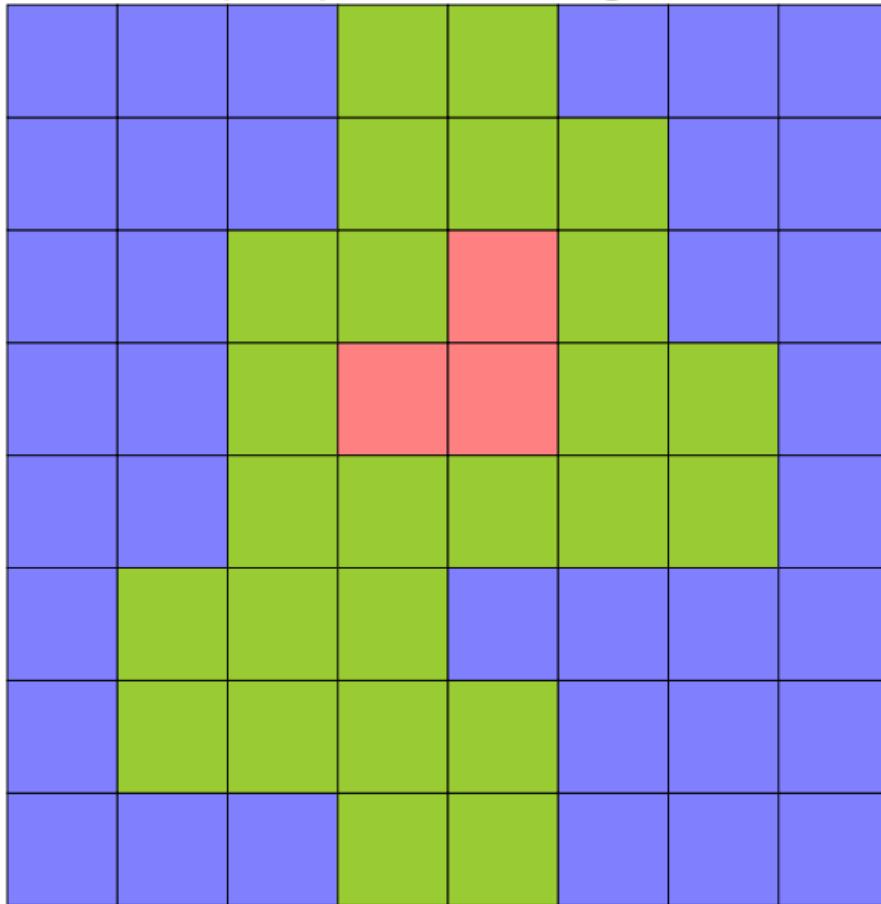
## Região implícita – classificação por amostragem



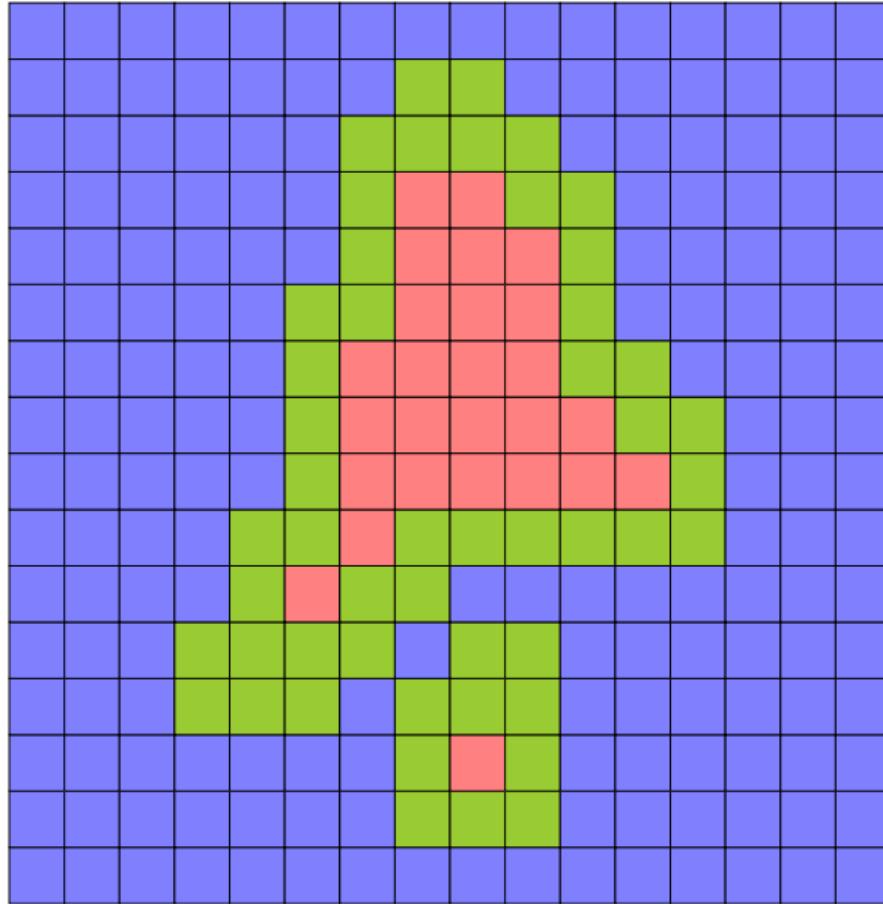
## Região implícita – classificação por amostragem



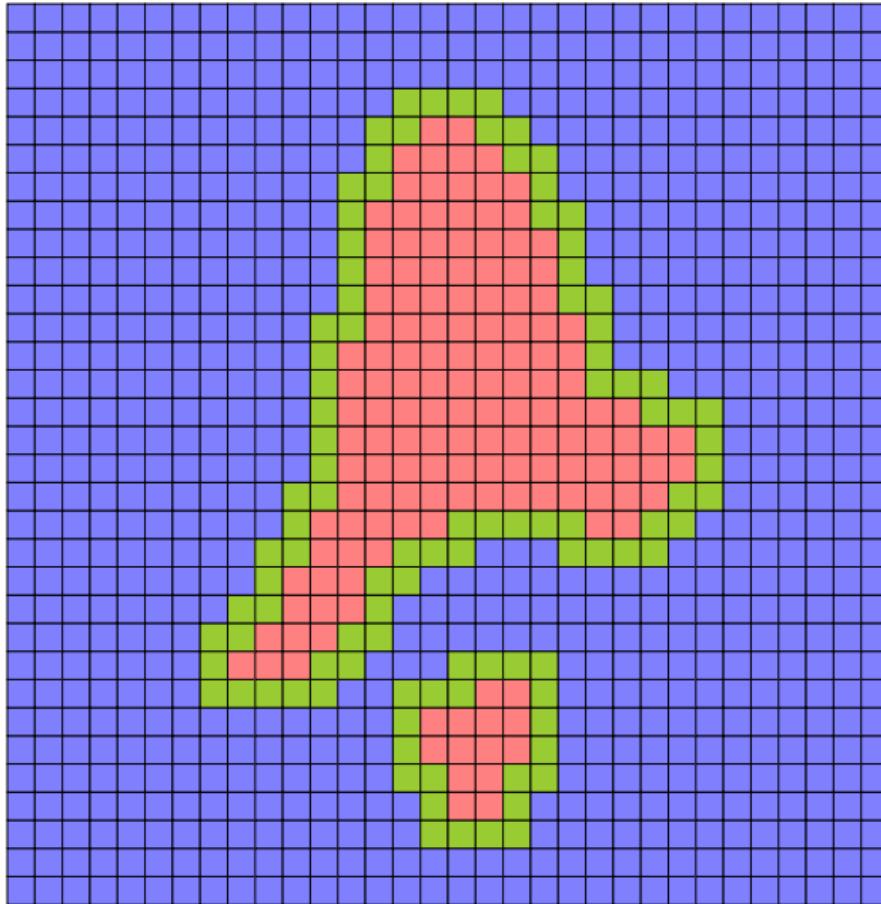
## Região implícita – classificação por amostragem



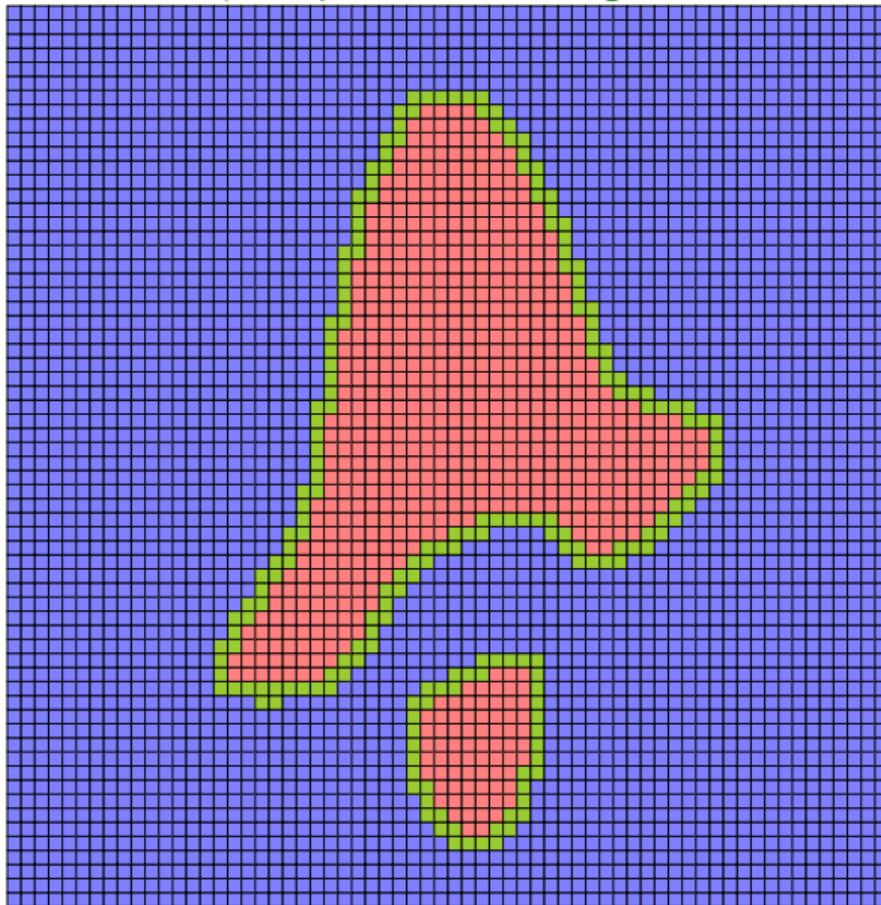
## Região implícita – classificação por amostragem



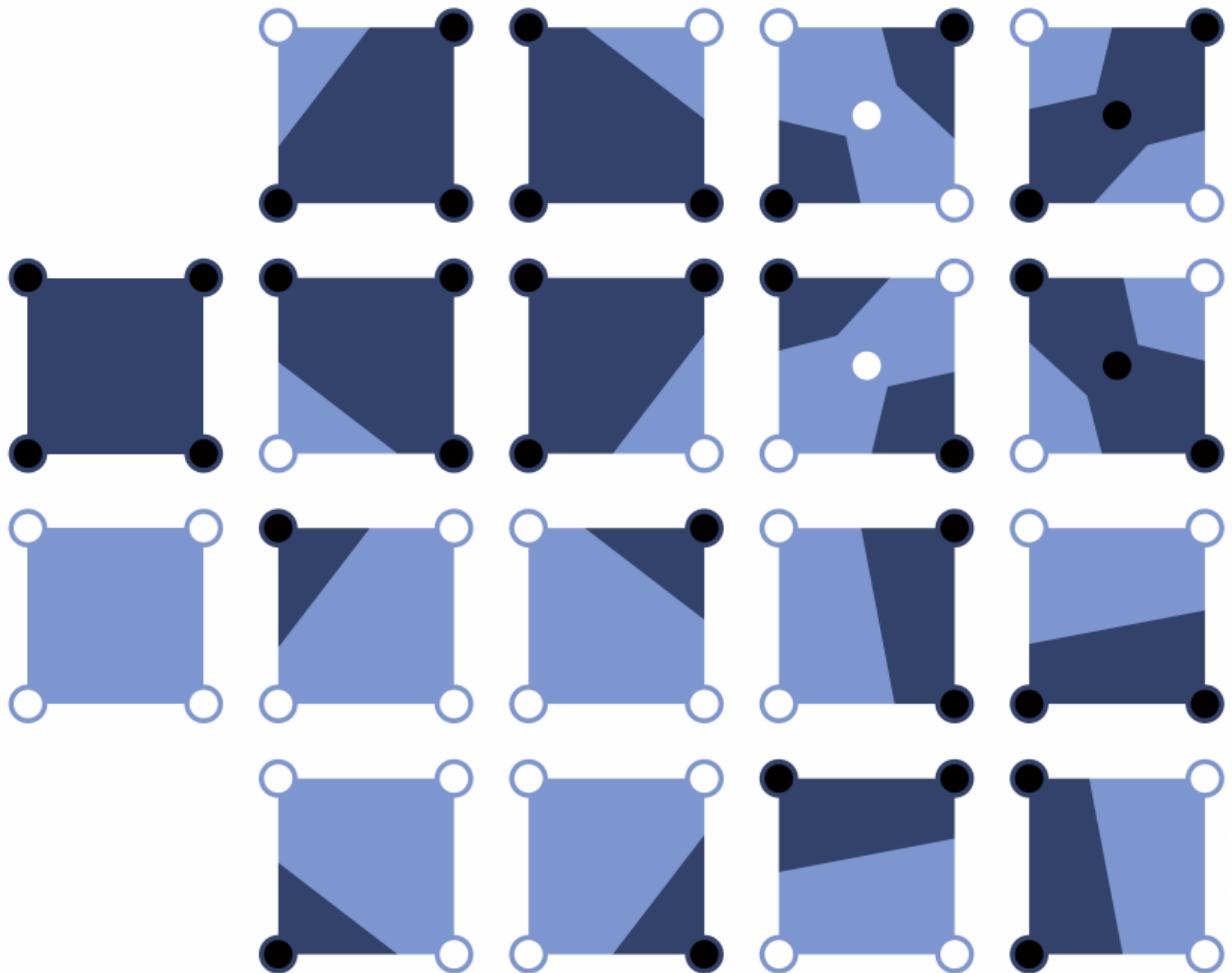
## Região implícita – classificação por amostragem



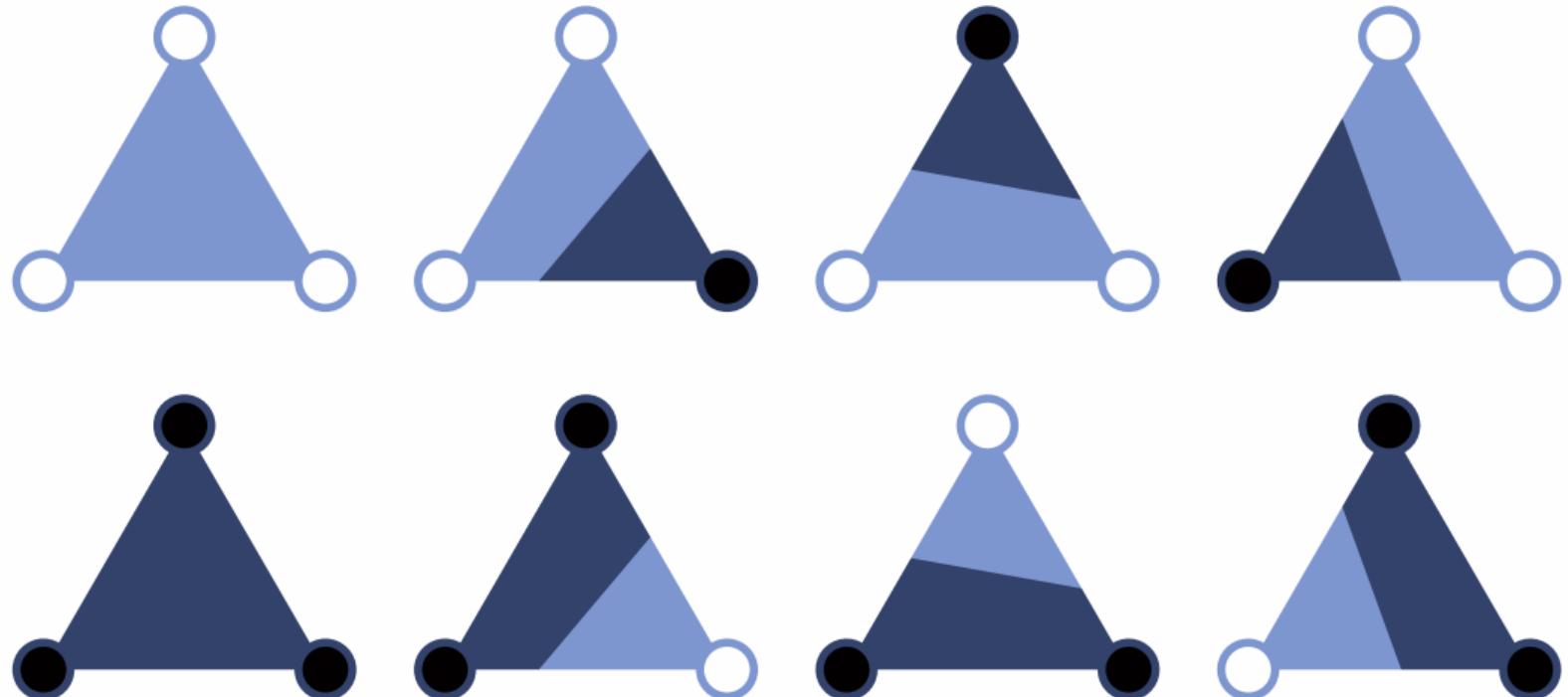
## Região implícita – classificação por amostragem



## Marching squares

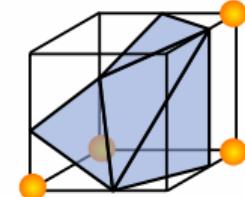
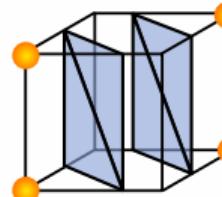
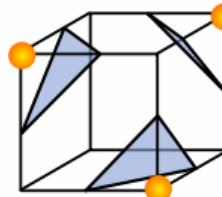
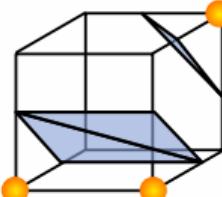
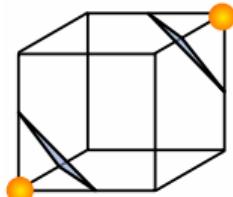
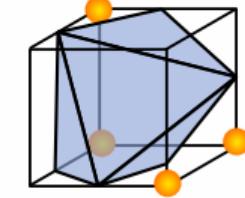
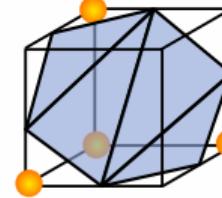
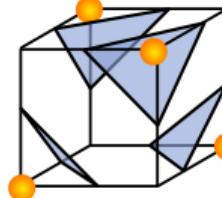
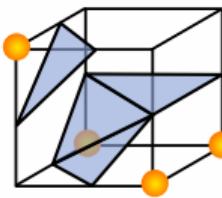
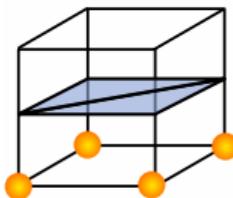
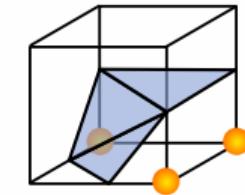
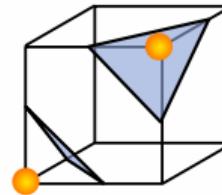
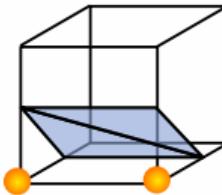
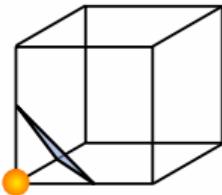
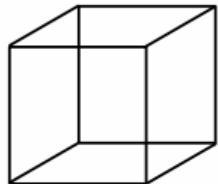


## Marching triangles



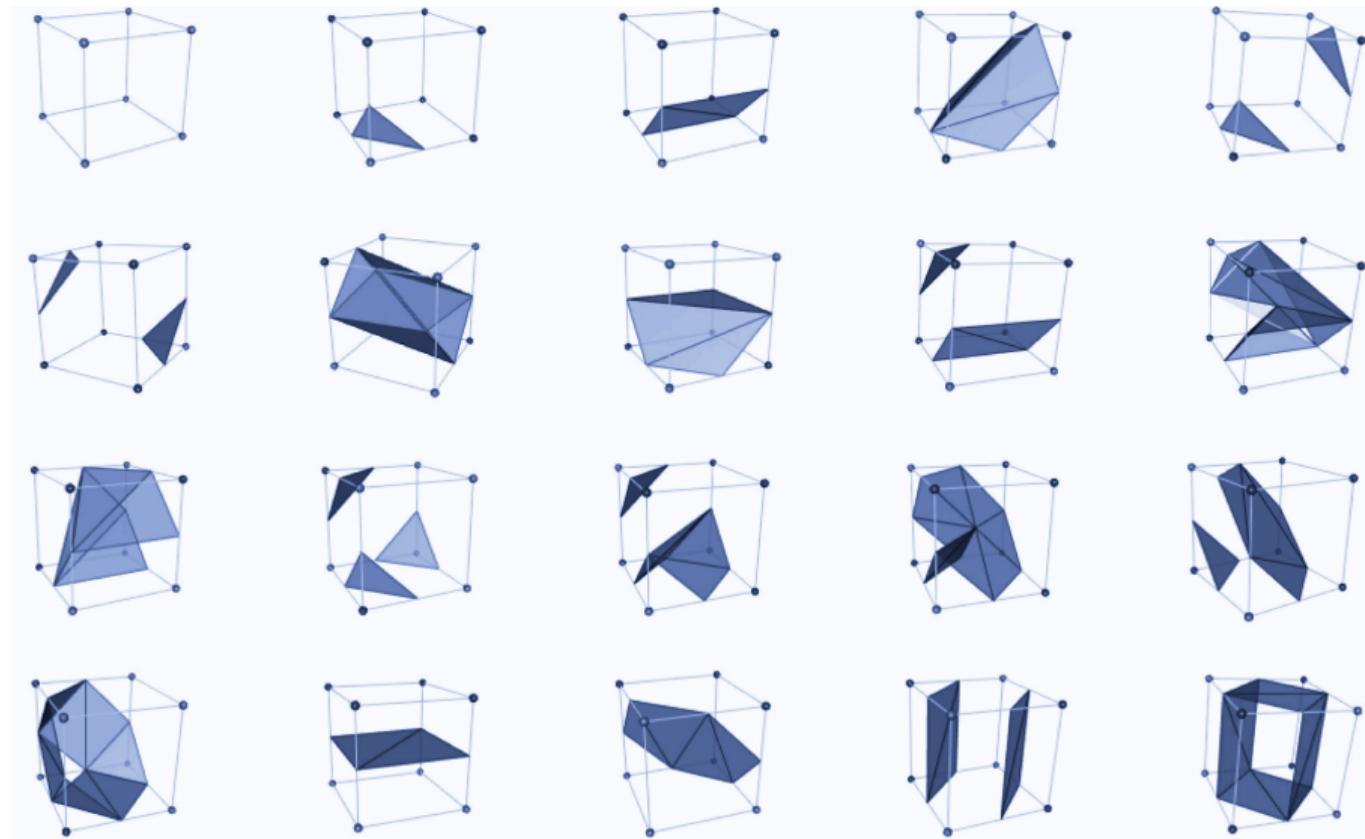
# Marching cubes

Lorensen–Cline (1987)



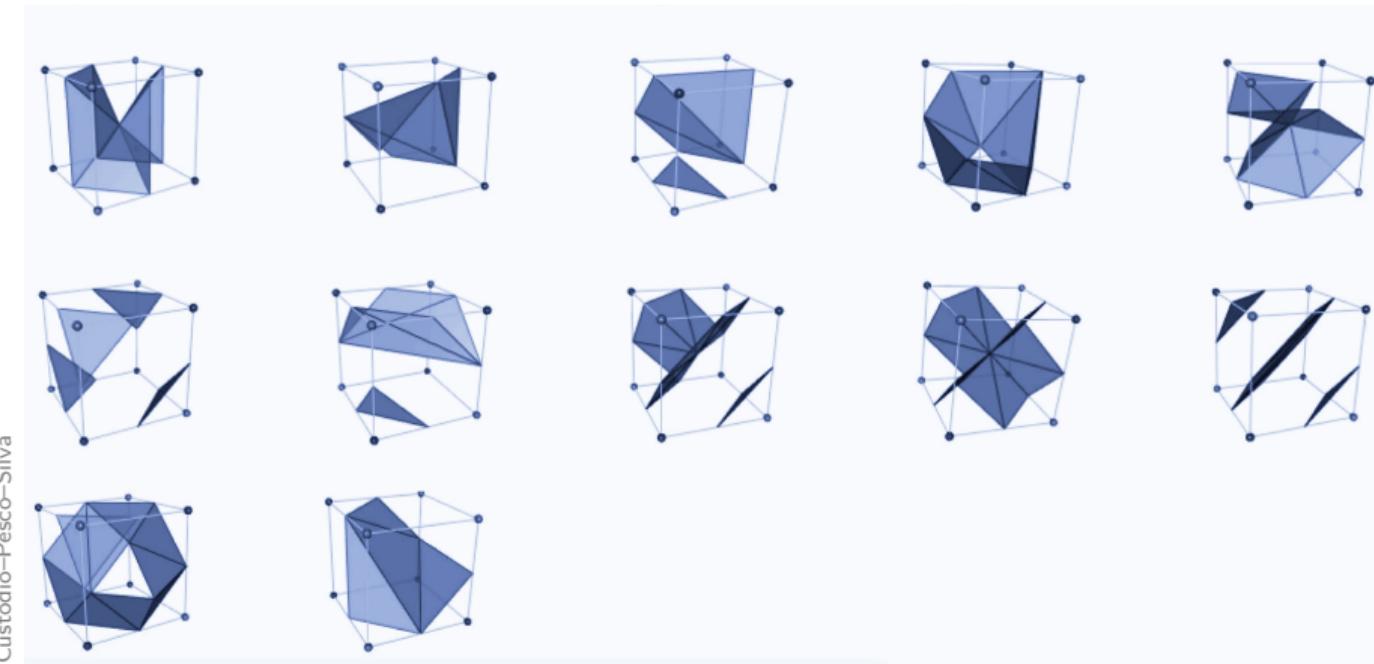
# Marching cubes

Chernyaev (1995)

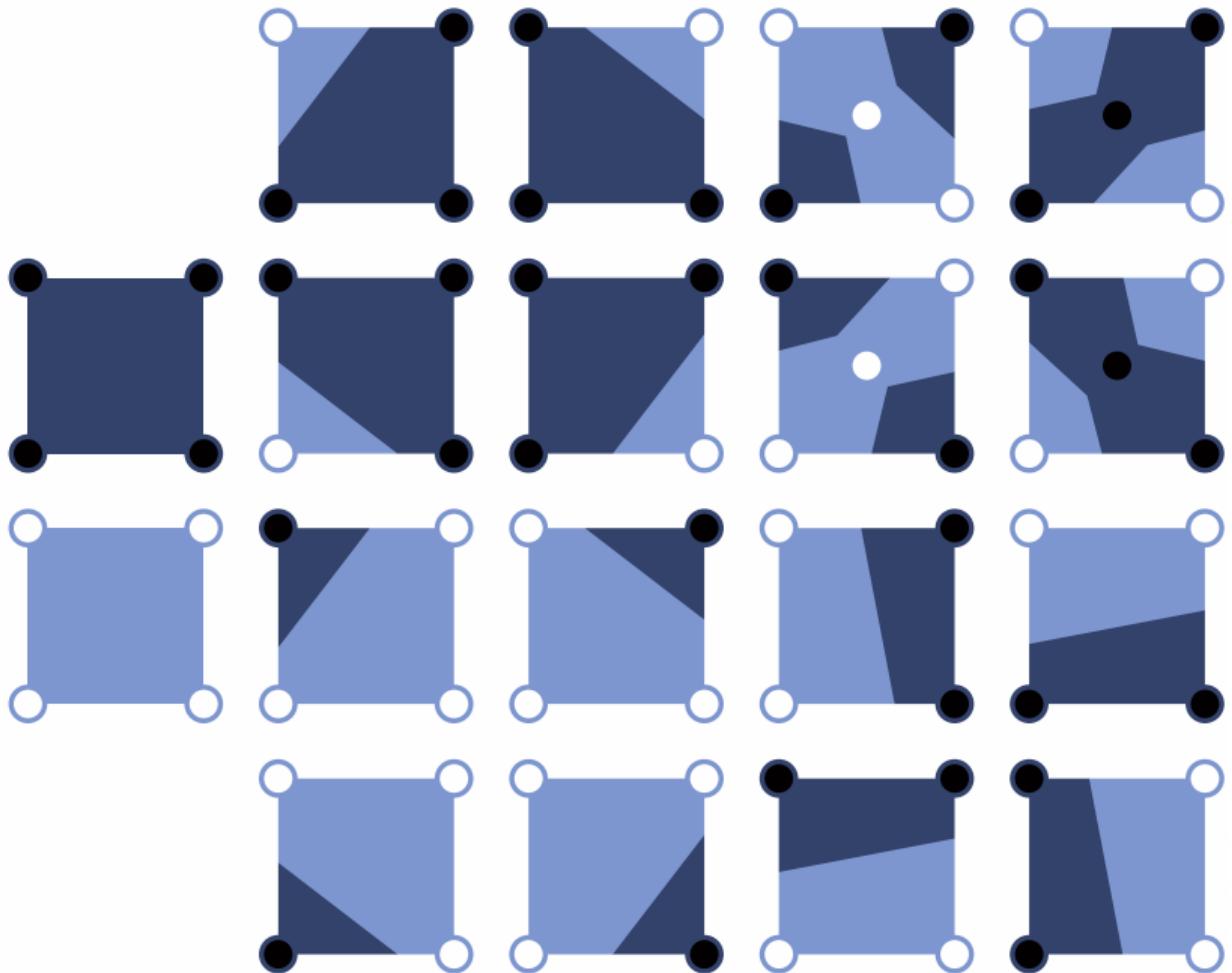


# Marching cubes

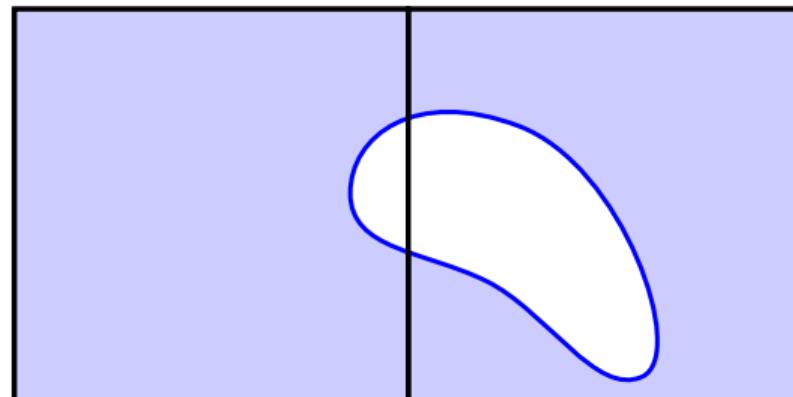
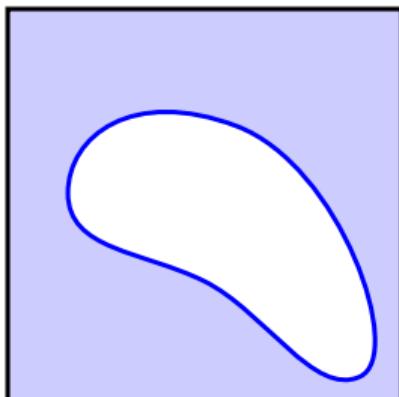
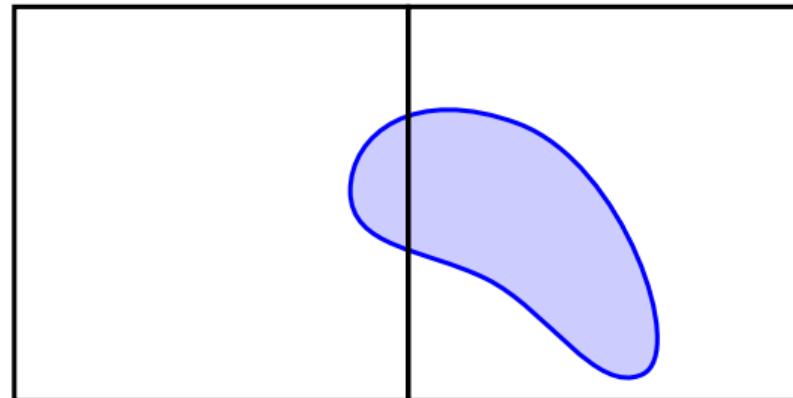
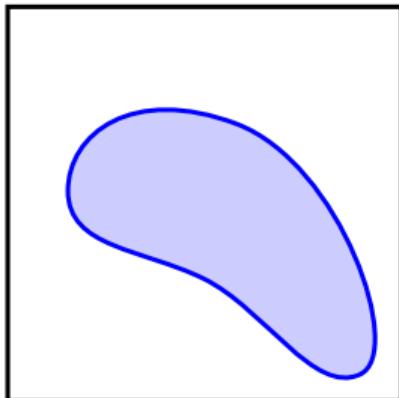
Chernyaev (1995)



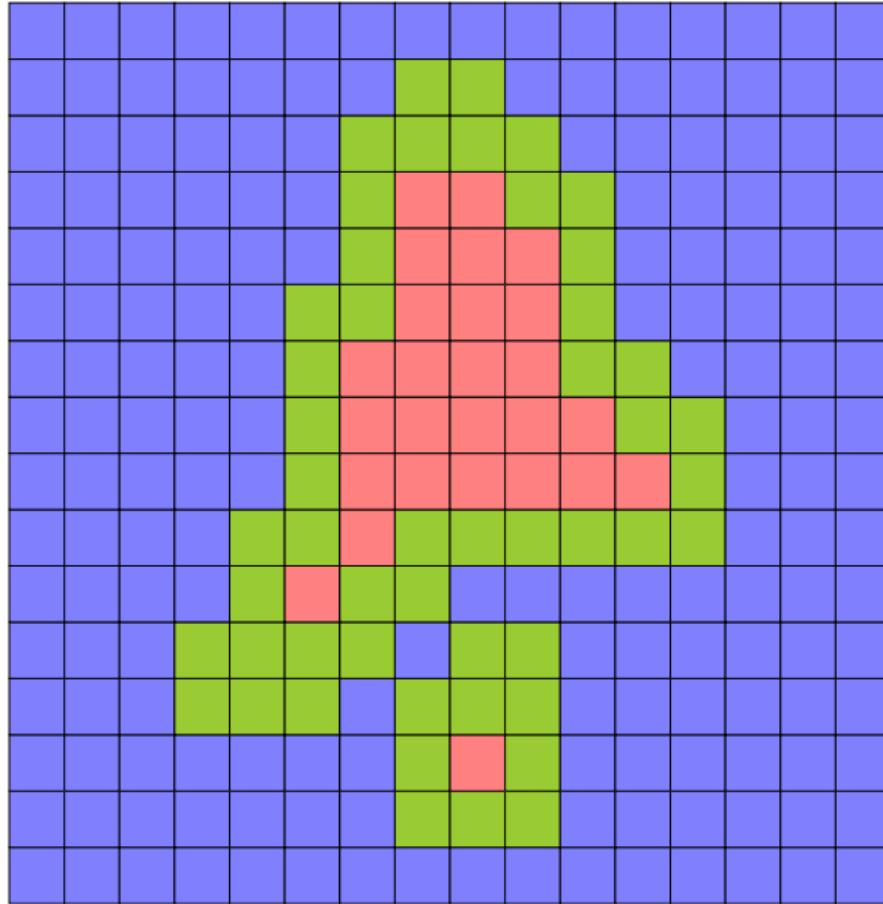
## Marching squares



## Marching squares – casos omissos

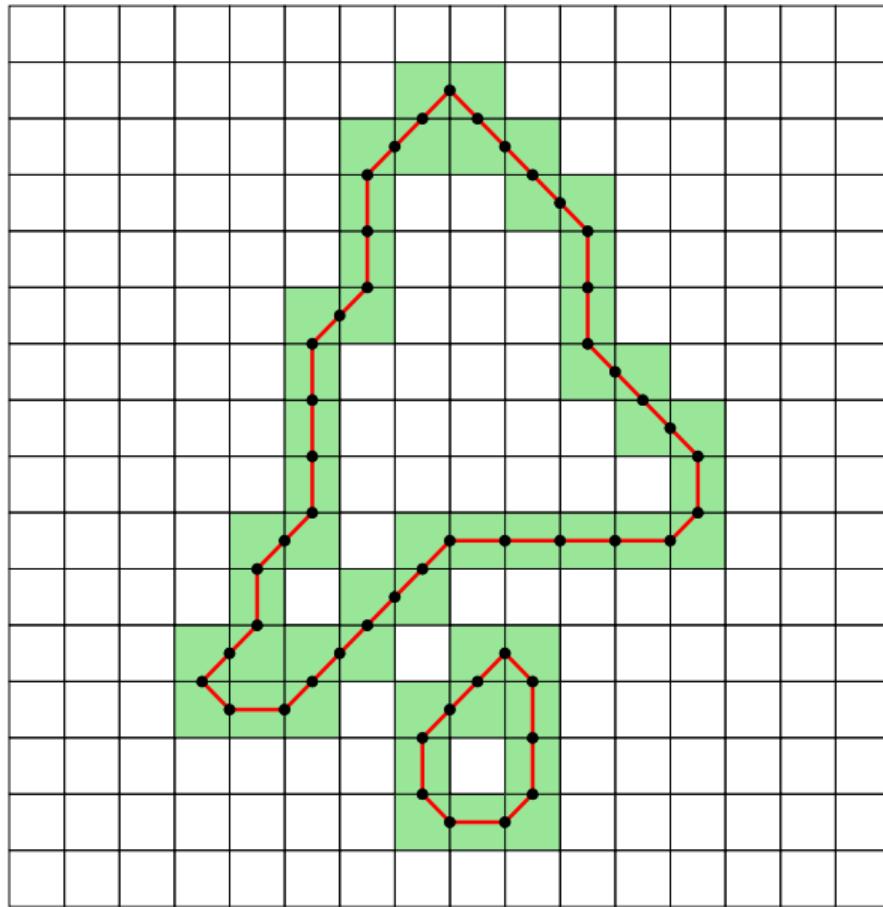


## Região implícita – fronteira



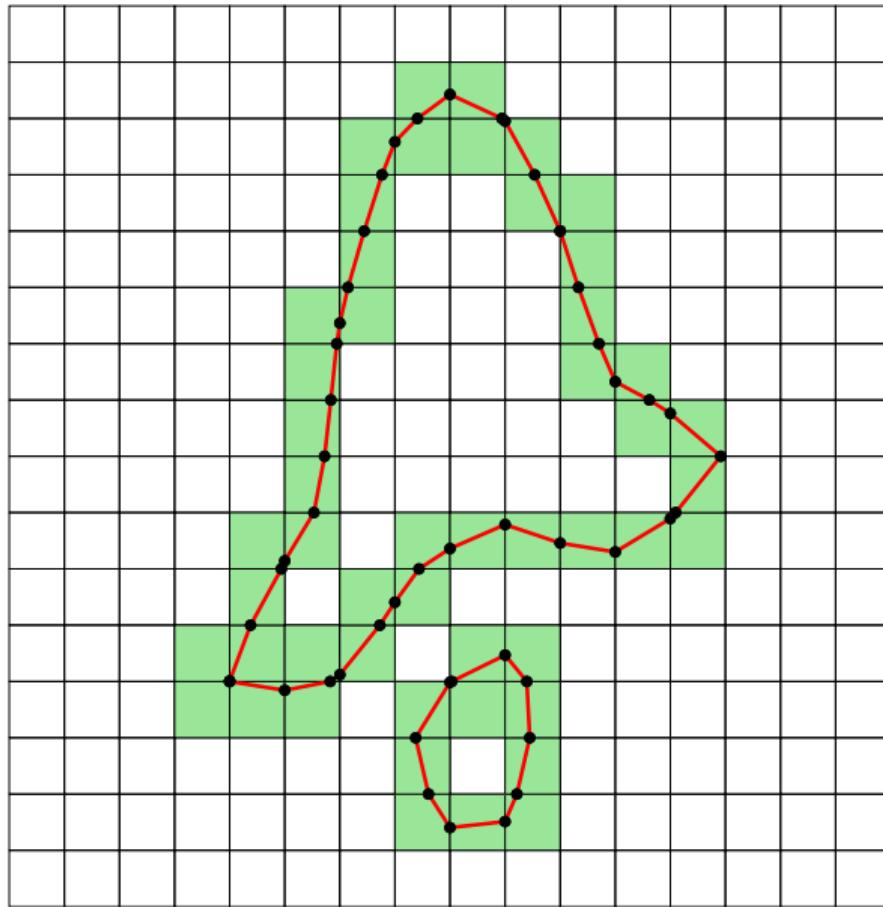
## Região implícita – fronteira

ponto médio



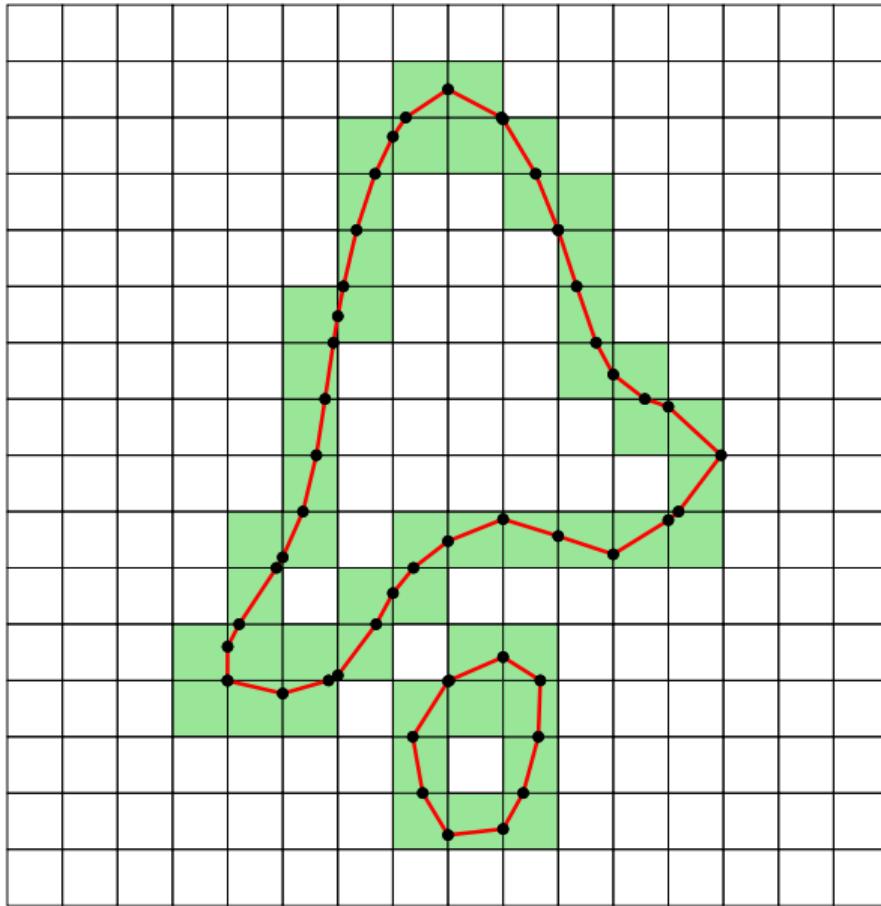
## Região implícita – fronteira

interpolação linear

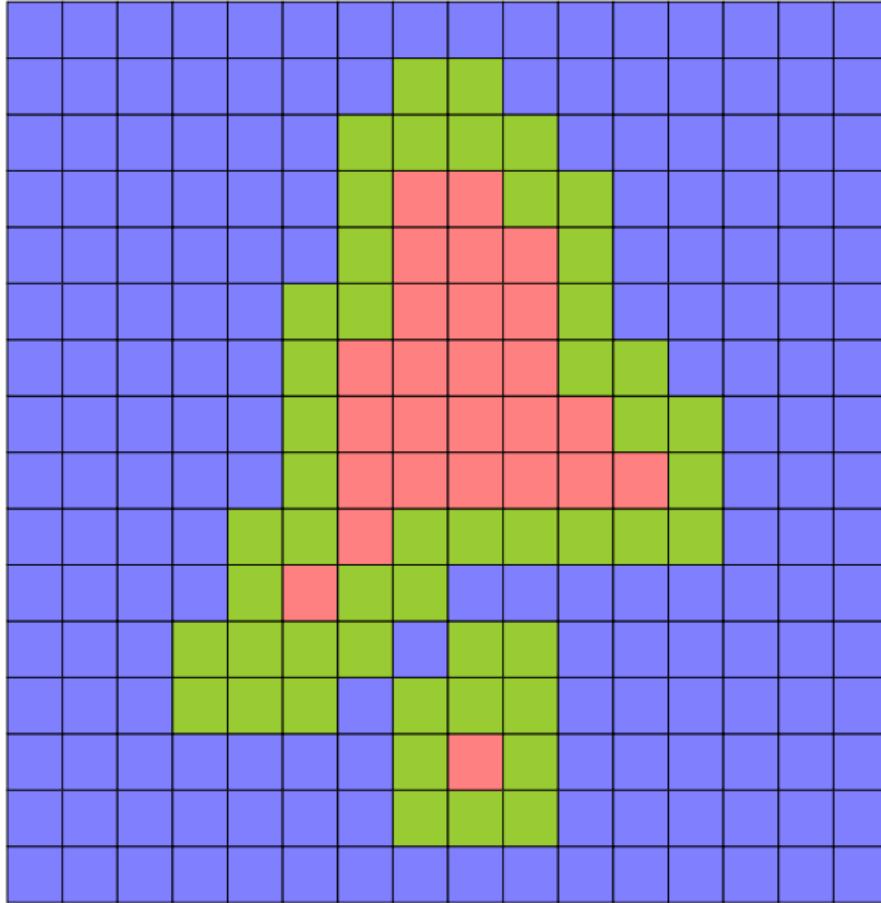


## Região implícita – fronteira

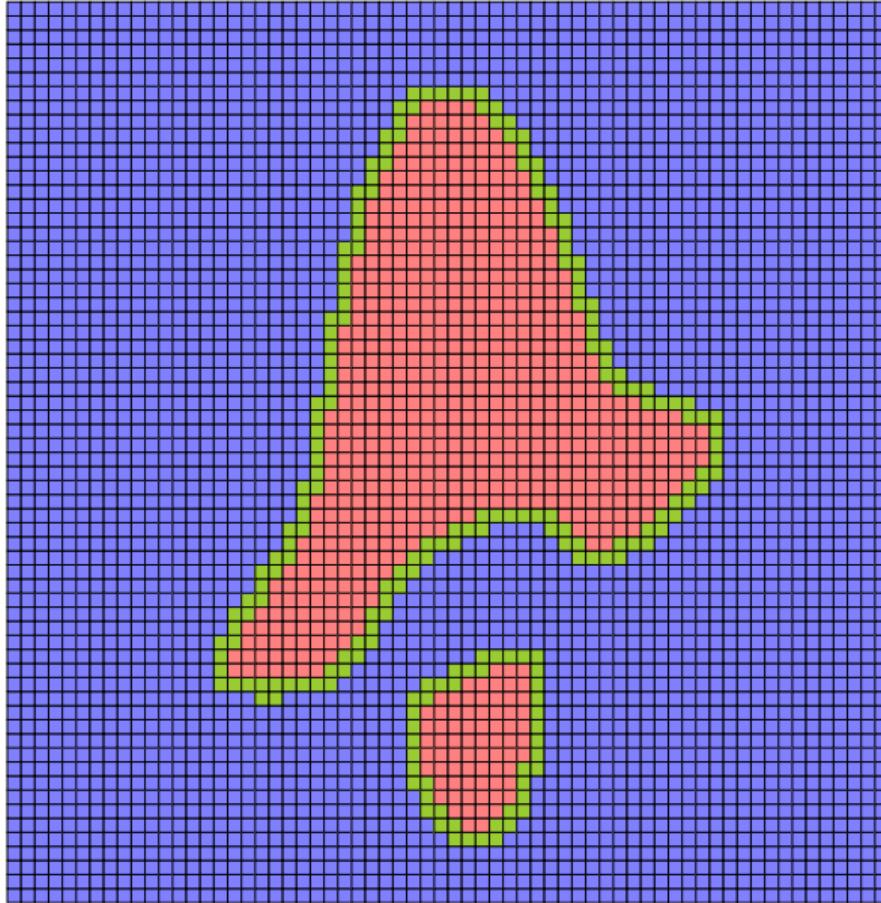
bisseção



## Enumeração espacial uniforme

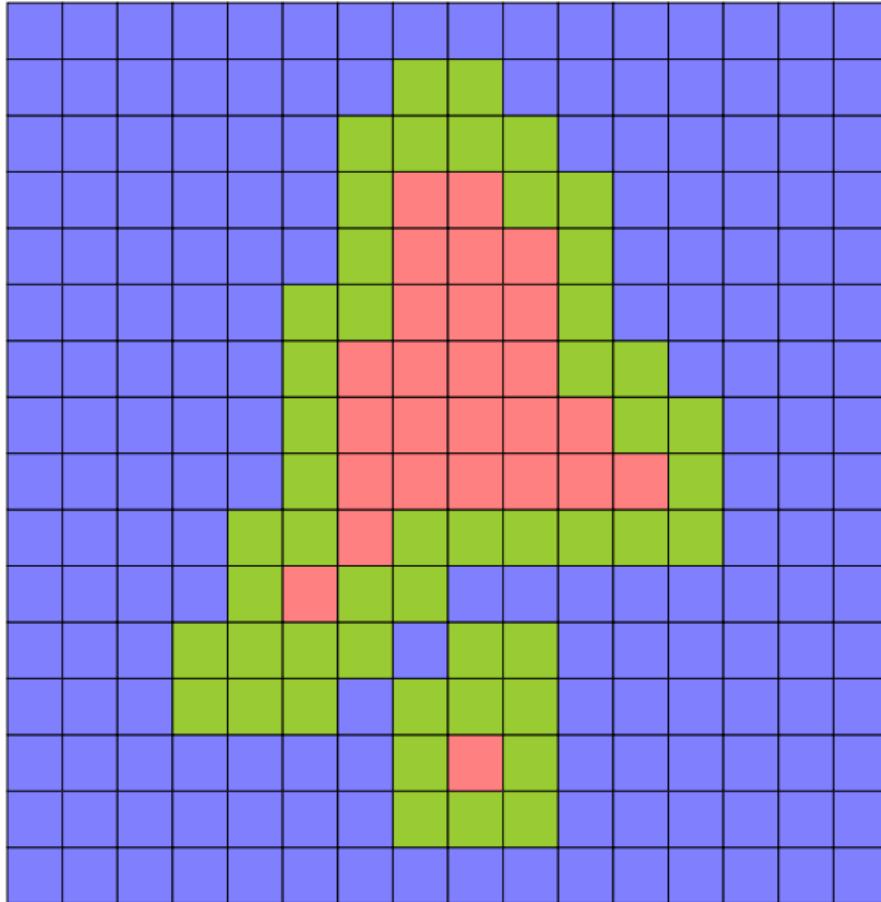


## Enumeração espacial uniforme



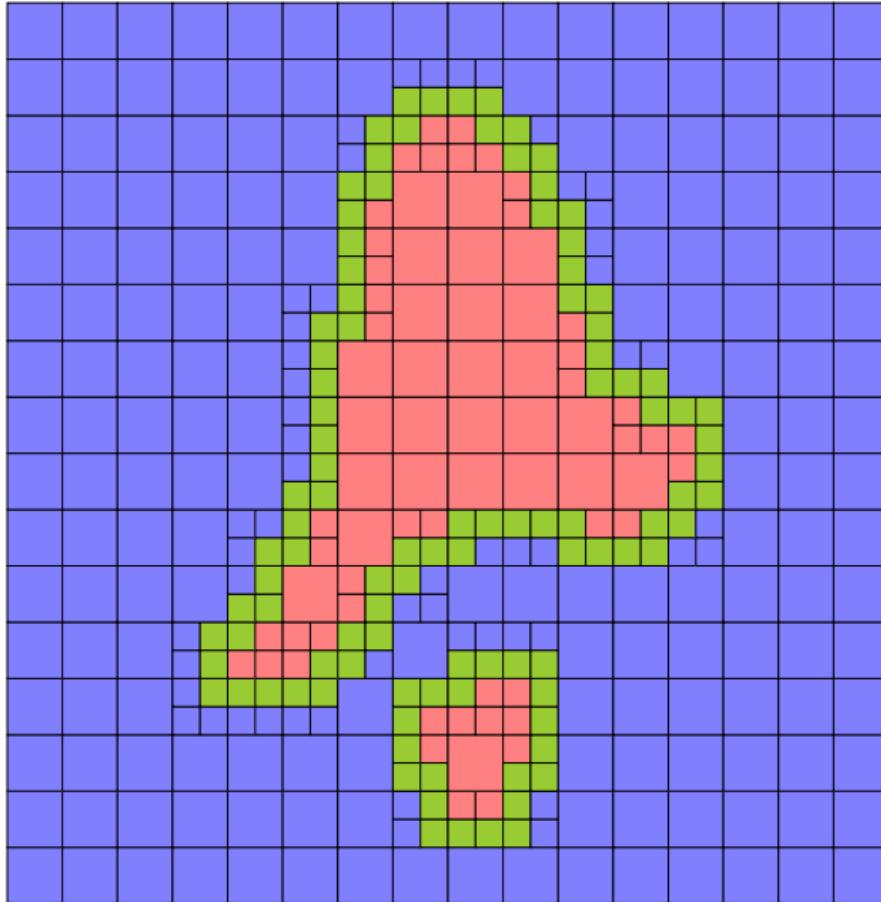
# Enumeração espacial semi-adaptativa

Suffern (1990)



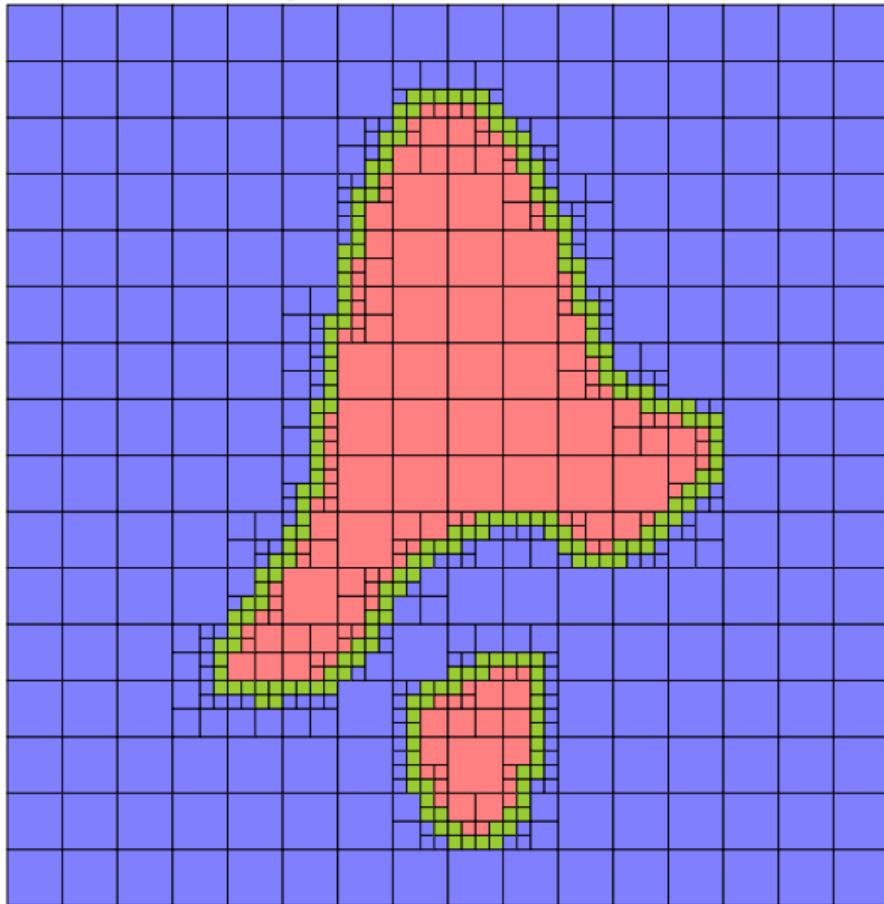
# Enumeração espacial semi-adaptativa

Suffern (1990)



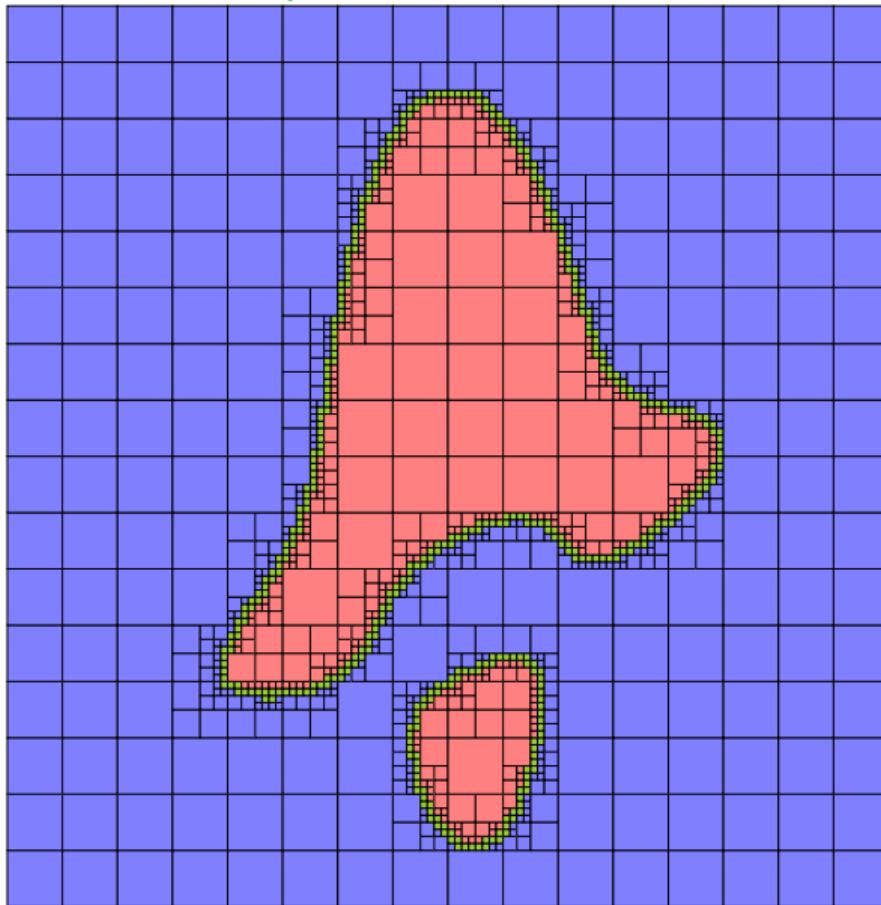
# Enumeração espacial semi-adaptativa

Suffern (1990)



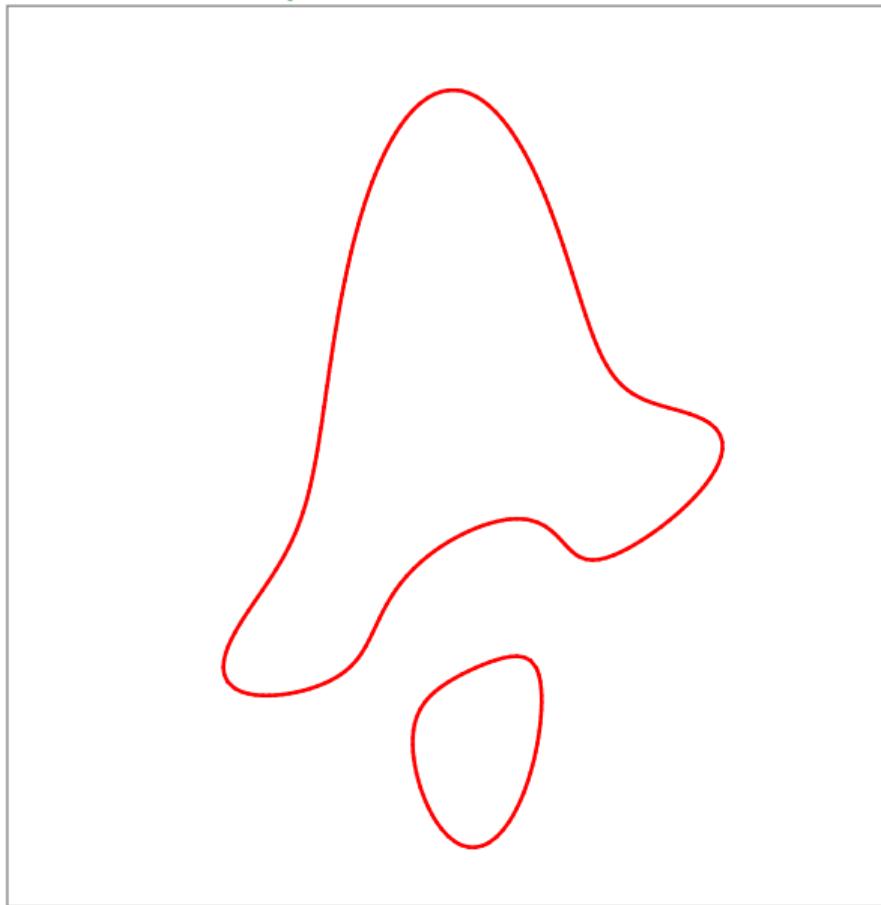
# Enumeração espacial semi-adaptativa

Suffern (1990)



## Enumeração espacial semi-adaptativa

Suffern (1990)

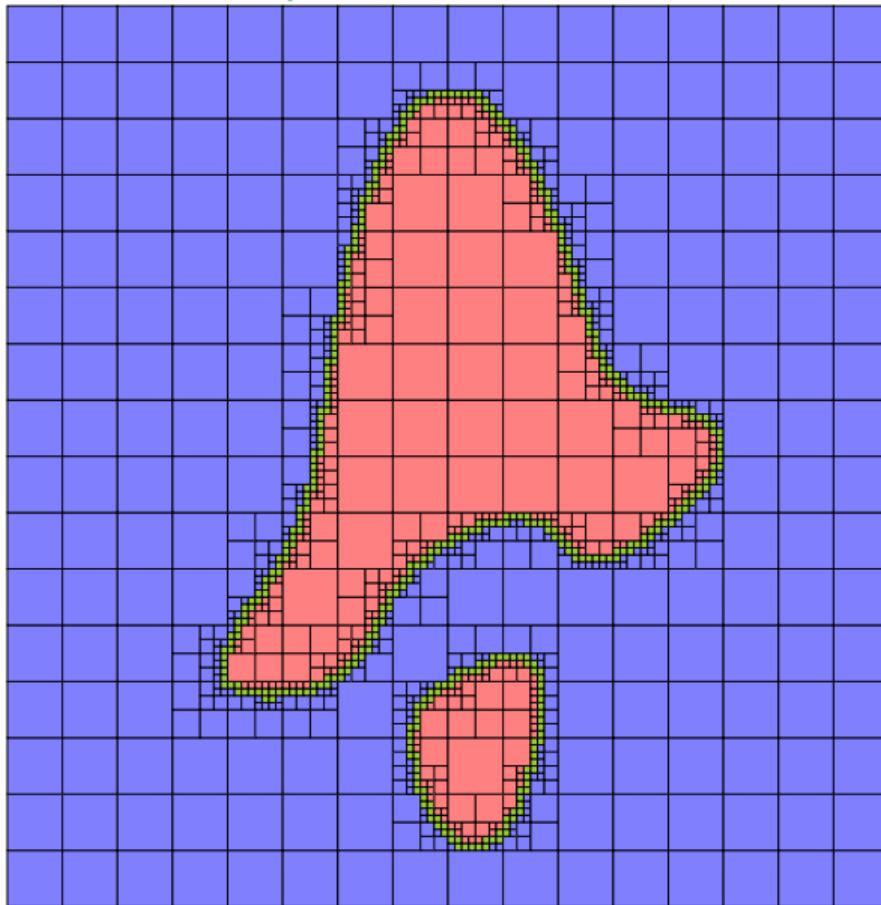


```
function explore(xmin,xmax,ymin,ymax,level)
    if level < searchlevel then
        subdivide(xmin,xmax,ymin,ymax,level)
    elseif contour_present(xmin,xmax,ymin,ymax) then
        if level < plotlevel then
            subdivide(xmin,xmax,ymin,ymax,level)
        else
            draw(xmin,xmax,ymin,ymax)
        end
    else
        draw(xmin,xmax,ymin,ymax)
    end
end
```

```
function subdivide(xmin,xmax,ymin,ymax,level)
    local xmid=(xmin+xmax)/2
    local ymid=(ymin+ymax)/2
    explore(xmin,xmid,ymin,ymid,level+1)
    explore(xmid,xmax,ymin,ymid,level+1)
    explore(xmin,xmid,ymid,ymax,level+1)
    explore(xmid,xmax,ymid,ymax,level+1)
end
```

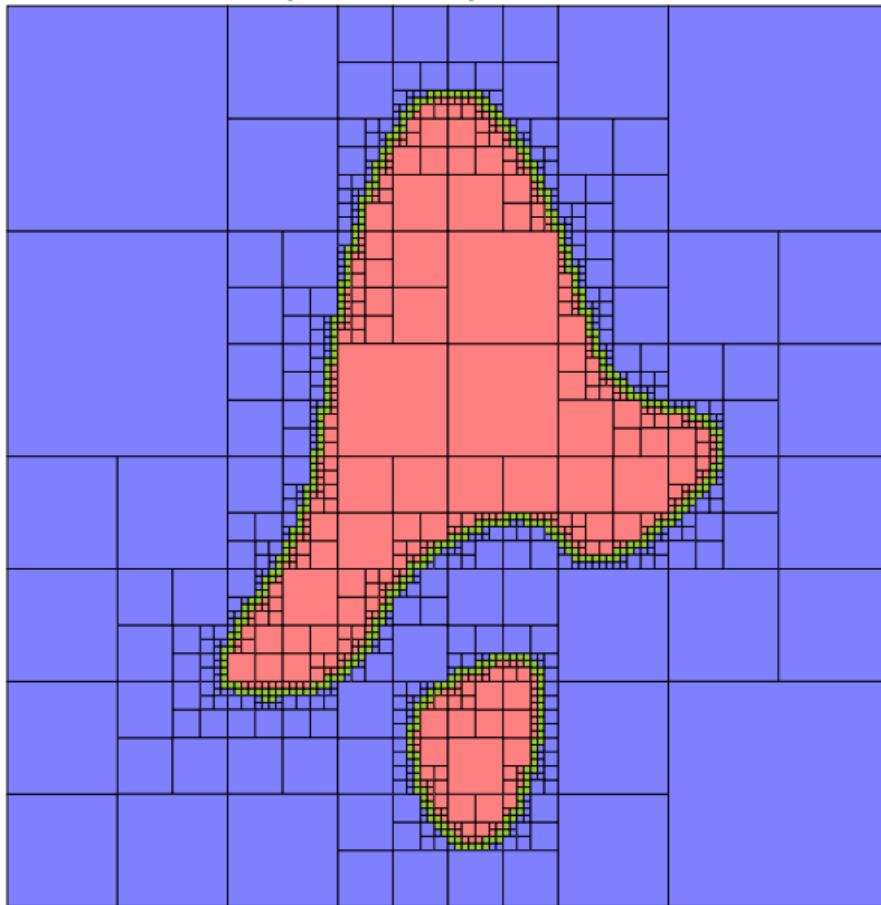
# Enumeração espacial semi-adaptativa

Suffern (1990)



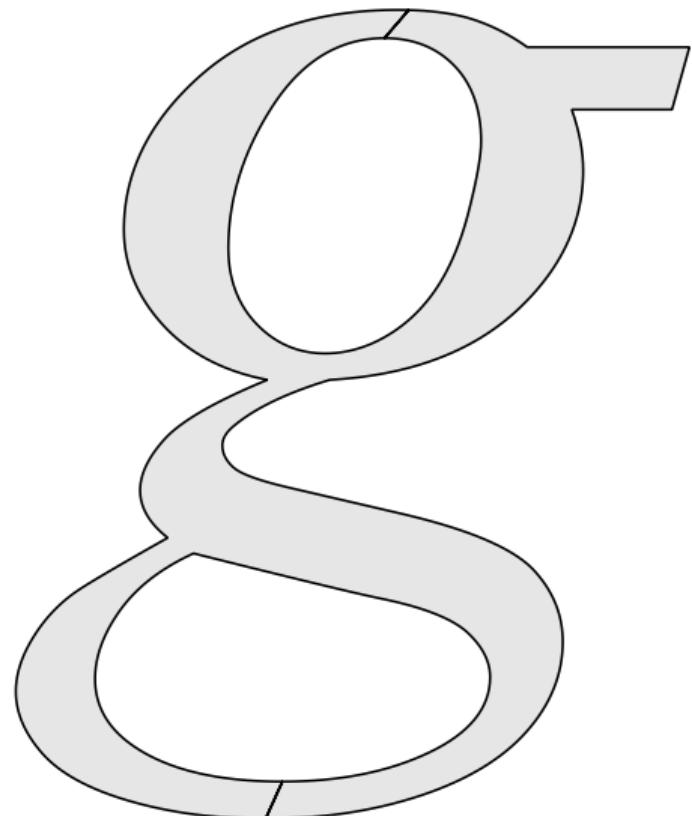
# Enumeração espacial semi-adaptativa podada

Suffern (1990)



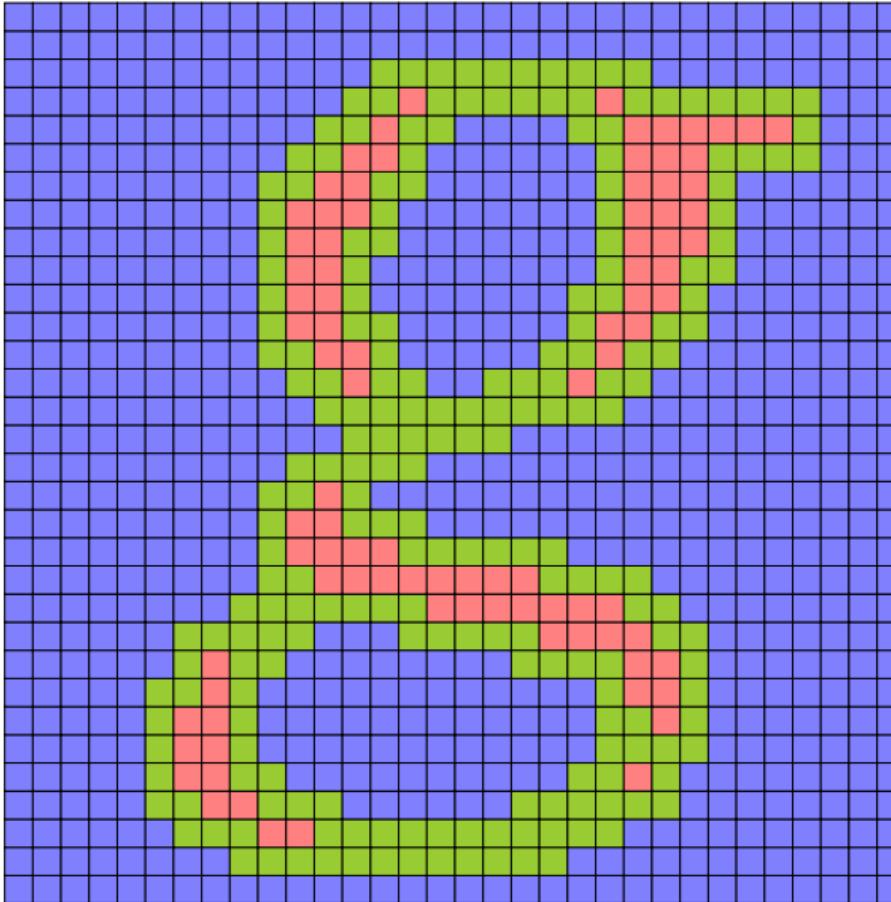
# Enumeração espacial semi-adaptativa

Suffern (1990)



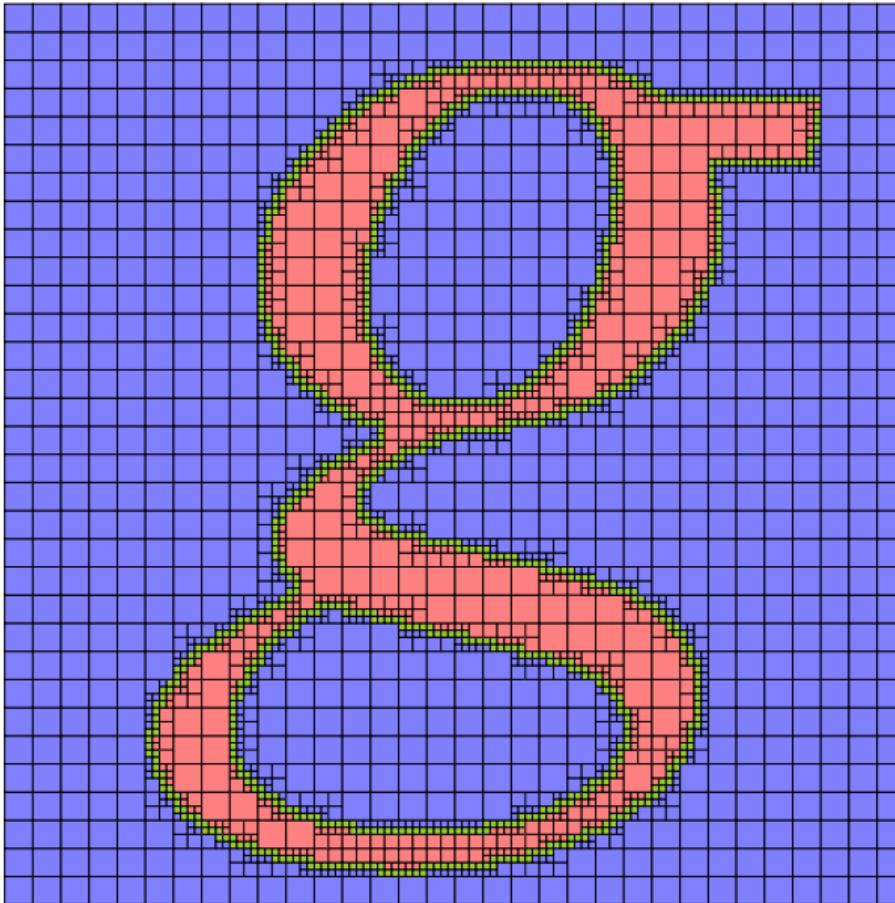
# Enumeração espacial semi-adaptativa

Suffern (1990)



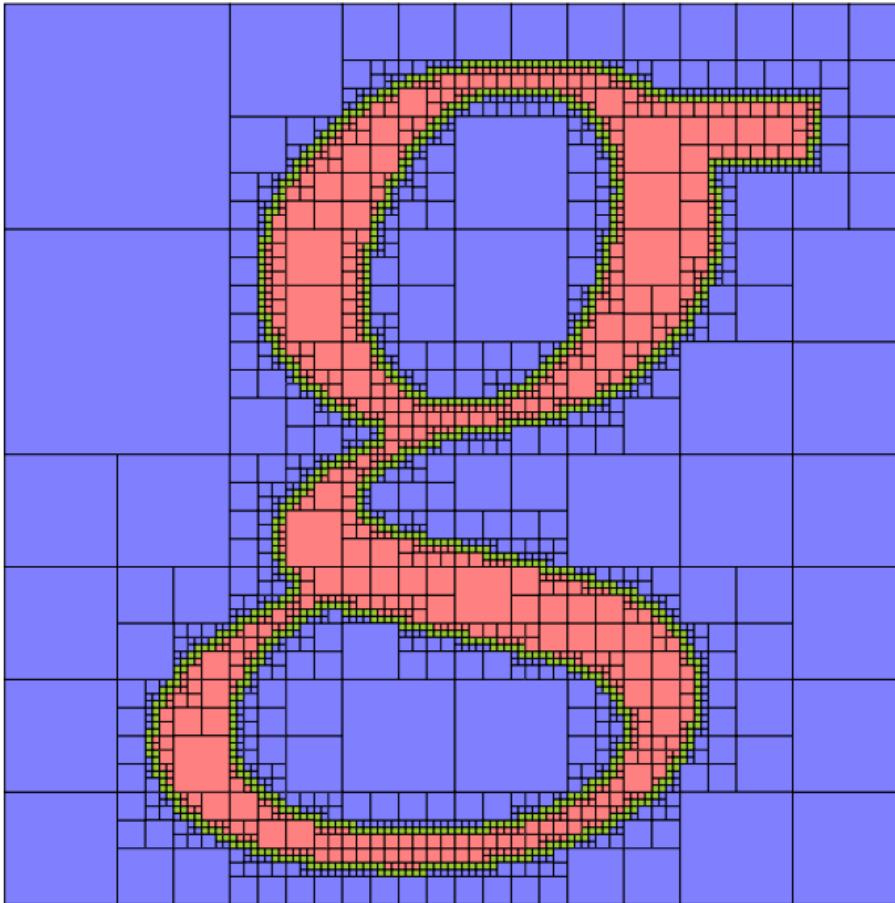
# Enumeração espacial semi-adaptativa

Suffern (1990)



# Enumeração espacial semi-adaptativa

Suffern (1990)



## Enumeração espacial semi-adaptativa

Suffern (1990)



## Localizando ponto em relação a polígono

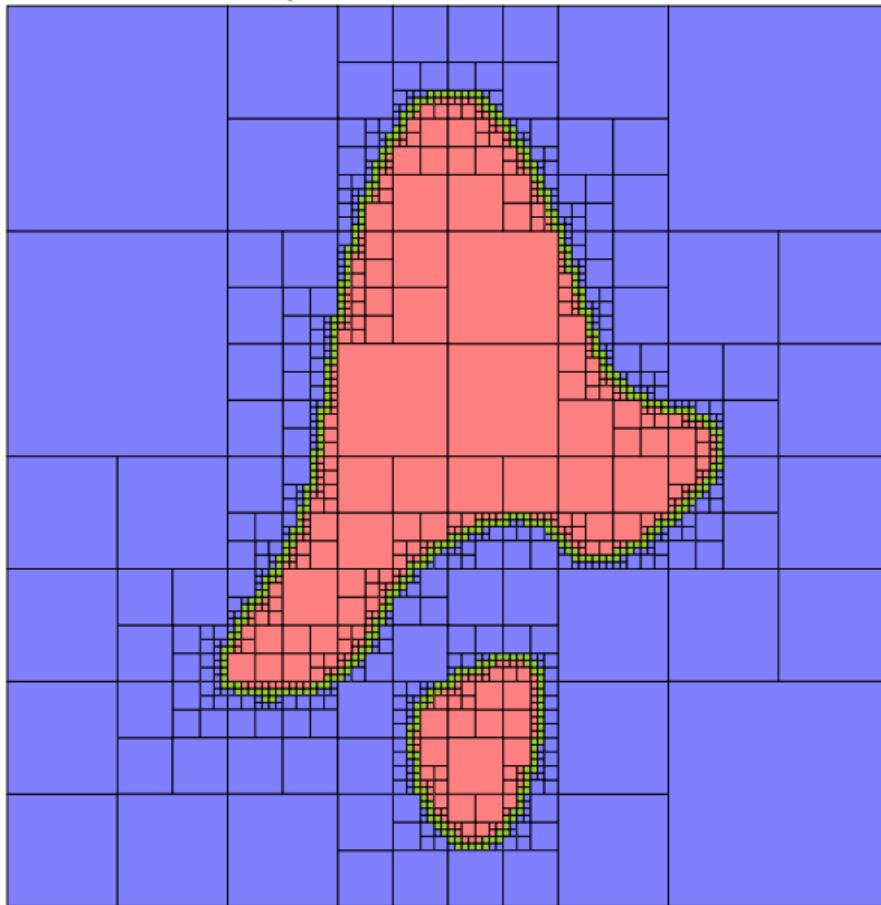
W. Randolph Franklin

```
function f(x,y)
    return inside(x,y) and -1 or 1
end

function inside(x,y)
    local inside=false
    for i=1,N do
        local j=i+1
        if ((Y[i]>y)~=(Y[j]>y)) and
            (x<((X[j]-X[i])*(y-Y[i]))/(Y[j]-Y[i])+X[i])) then
            inside = not inside
        end
    end
    return inside
end
```

# Enumeração espacial semi-adaptativa

Suffern (1990)



## Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
- $X = X_1 \times \cdots \times X_n \subseteq \Omega$
- oráculo: intervalo  $F(X) \subseteq \mathbf{R}$

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

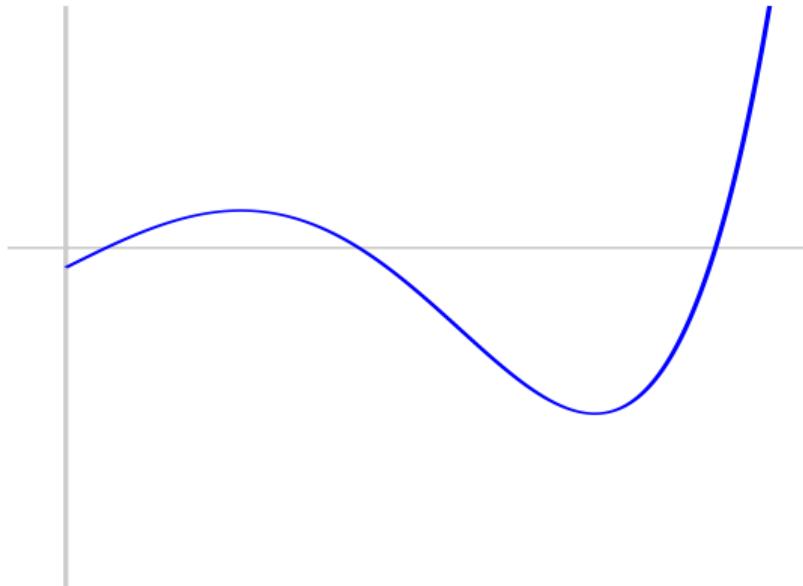
$$x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$$

## Oráculos intervalares

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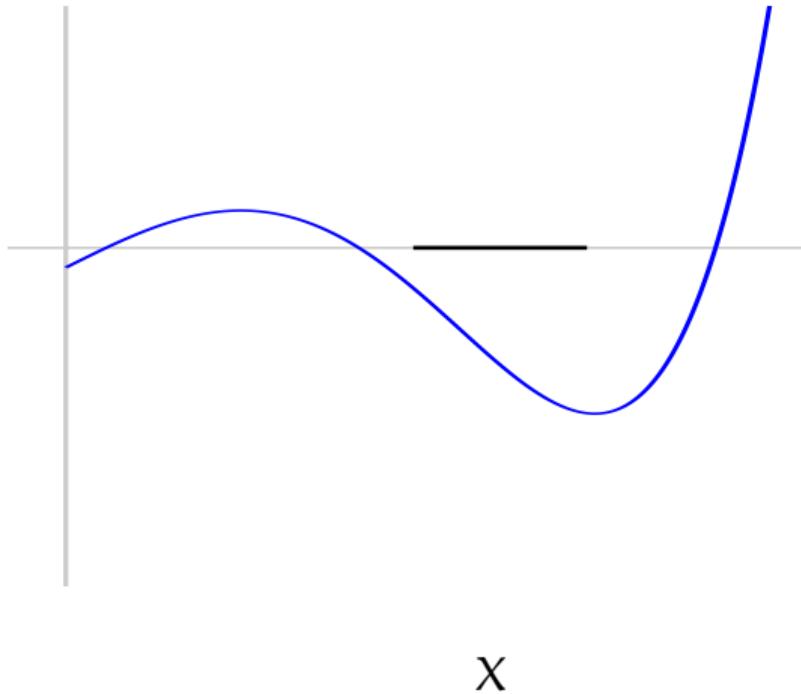
$f$

## Oráculos intervalares

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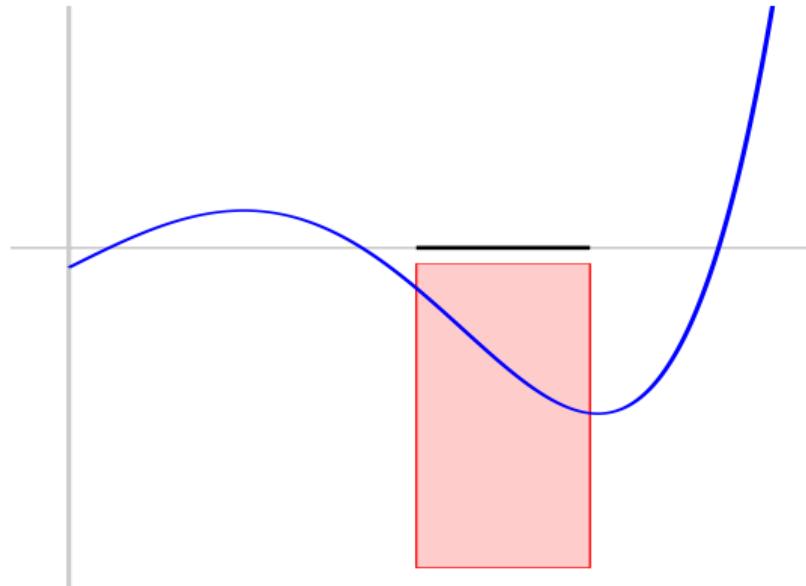


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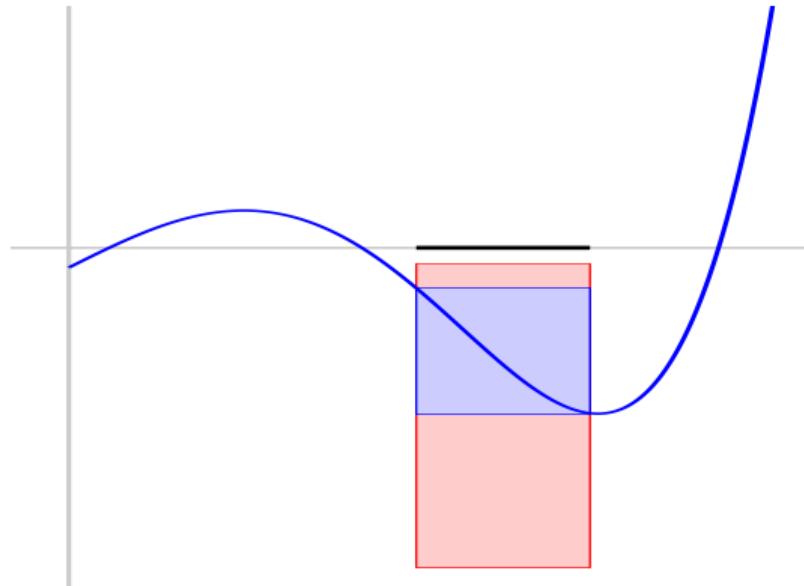
$F(X)$

## Oráculos intervalares

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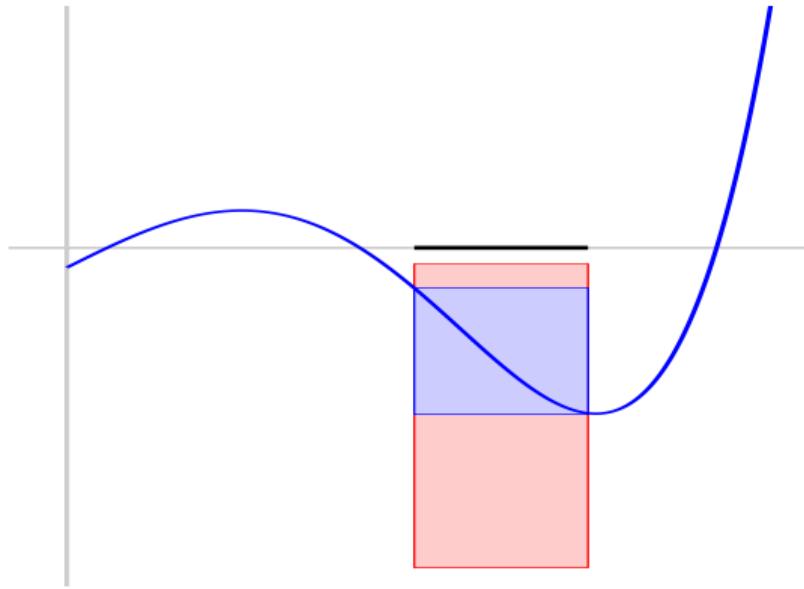


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## Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
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  - oráculo: intervalo  $F(X) \subseteq \mathbf{R}$
- $F(X) \supseteq f(X) = \{f(x) : x \in X\}$
- $x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$
- $0 \notin F(X) \implies 0 \notin f(X)$
  - prova computational

$f(x) = 0$  não tem solução em  $X$

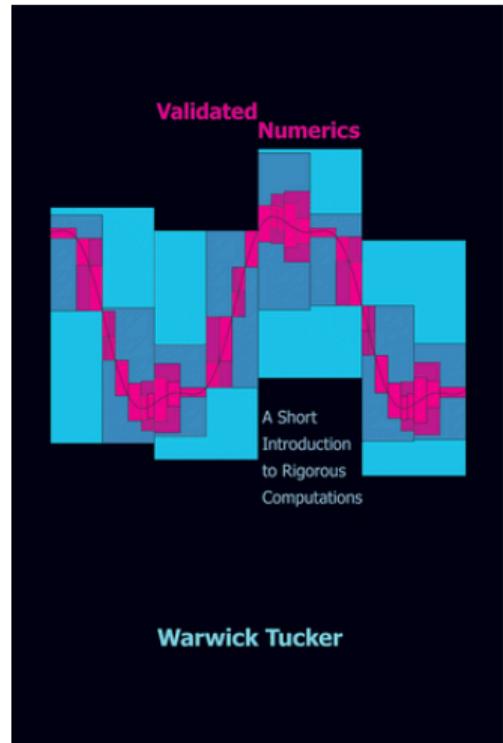
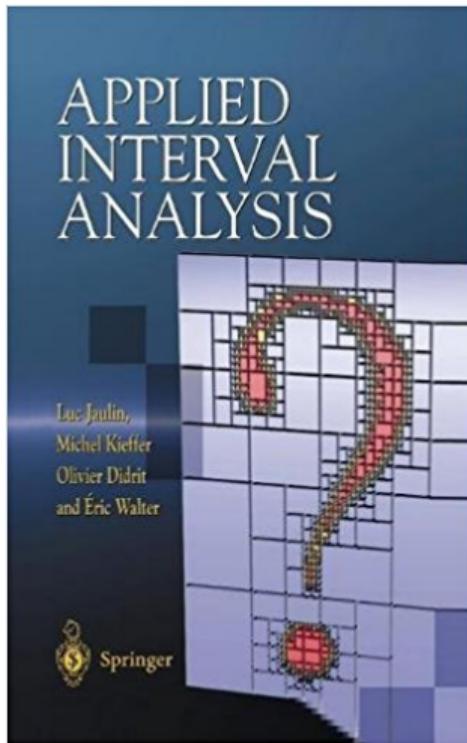
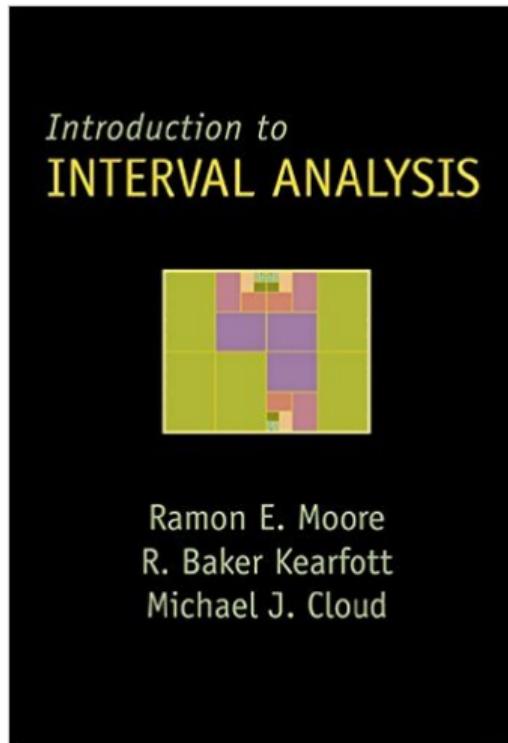


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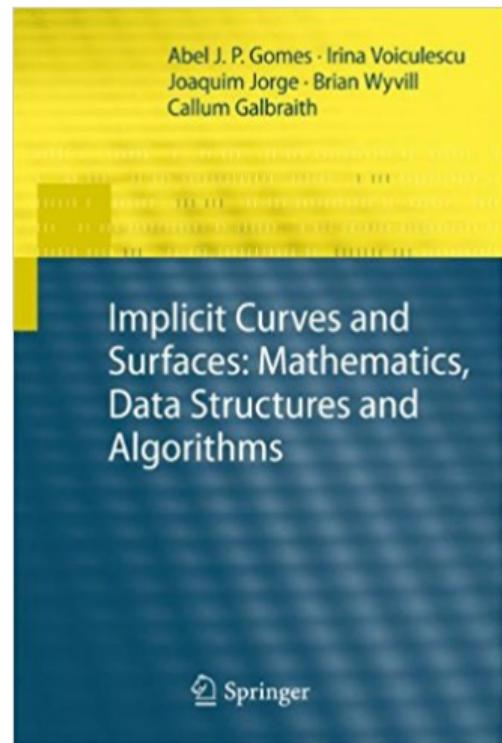
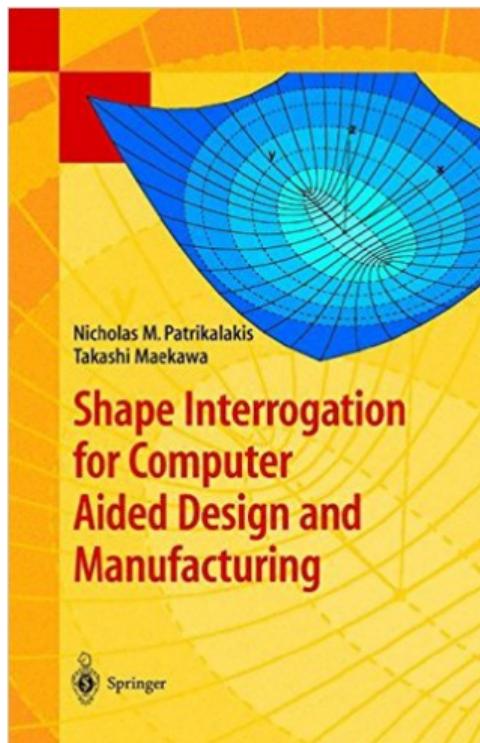
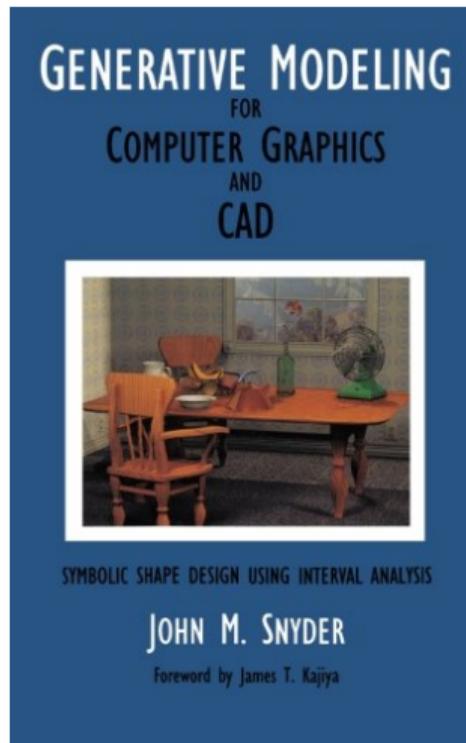
# Oráculos intervalares

- aritmética intervalar Moore (1966)
- formas de valor médio Caprani–Madsen (1980)
- aritmética afim Comba–Stolfi (1993)
- formas de Taylor Berz et al. (1996)

# Métodos intervalares



# Métodos intervalares



## Aritmética intervalar

$$\begin{aligned}[a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)]\end{aligned}$$

## Aritmética intervalar

$$\begin{aligned}[a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)]\end{aligned}$$

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] / [c, d] = [a, b] \times [1/d, 1/c]$$

$$[a, b]^2 = [\min(a^2, b^2), \max(a^2, b^2)] \text{ se } 0 \notin [a, b]$$

$$[a, b]^2 = [0, \max(a^2, b^2)] \text{ se } 0 \in [a, b]$$

$$\exp[a, b] = [\exp(a), \exp(b)]$$

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$$\exp[a, b] = [\exp(a), \exp(b)]$$

uso conveniente via sobrecarga de operadores

## Região implícita – método intervalar

Suffern–Fackerell (1991), Snyder (1992)

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return show(x,y,"0") end
    if z<0 then return show(x,y,"I") end
    if level==0 then return show(x,y,"U") end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    explore(xlo,ylo,level-1)
    explore(xhi,ylo,level-1)
    explore(xlo,yhi,level-1)
    explore(xhi,yhi,level-1)
    return nil
end
```

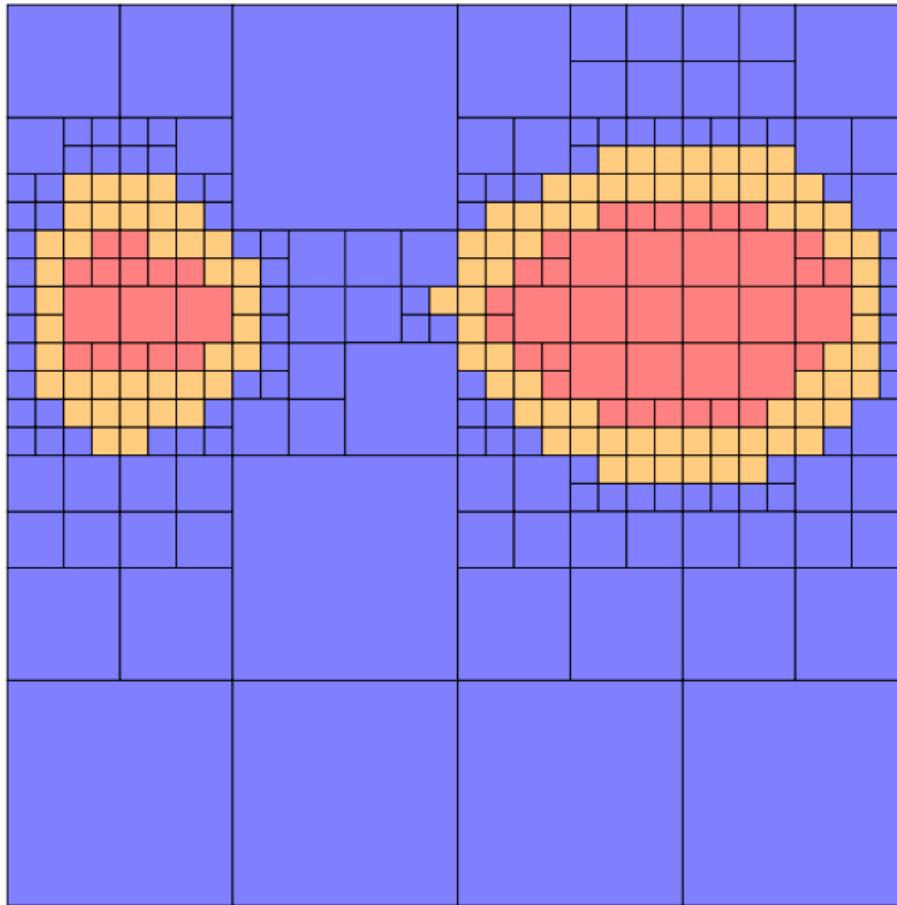
## Região implícita – método intervalar



“When you have eliminated the impossible,  
whatever remains, however improbable,  
must be the truth.”

—Sherlock Holmes in *The Sign of Four*

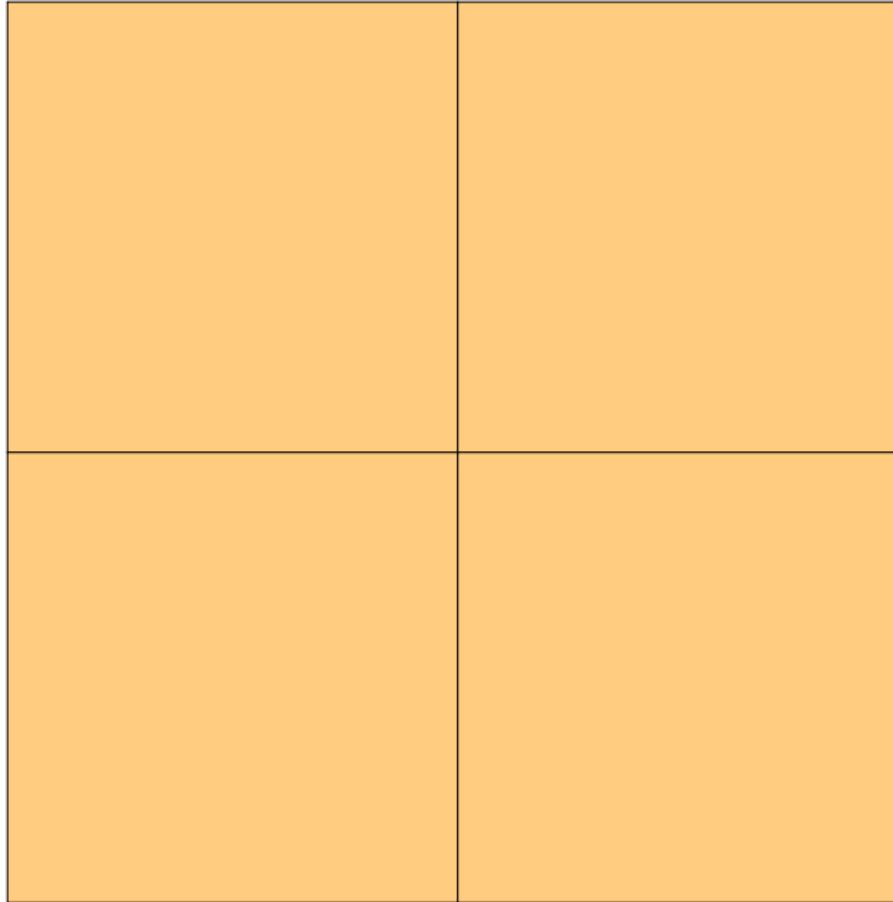
## Região implícita – método intervalar



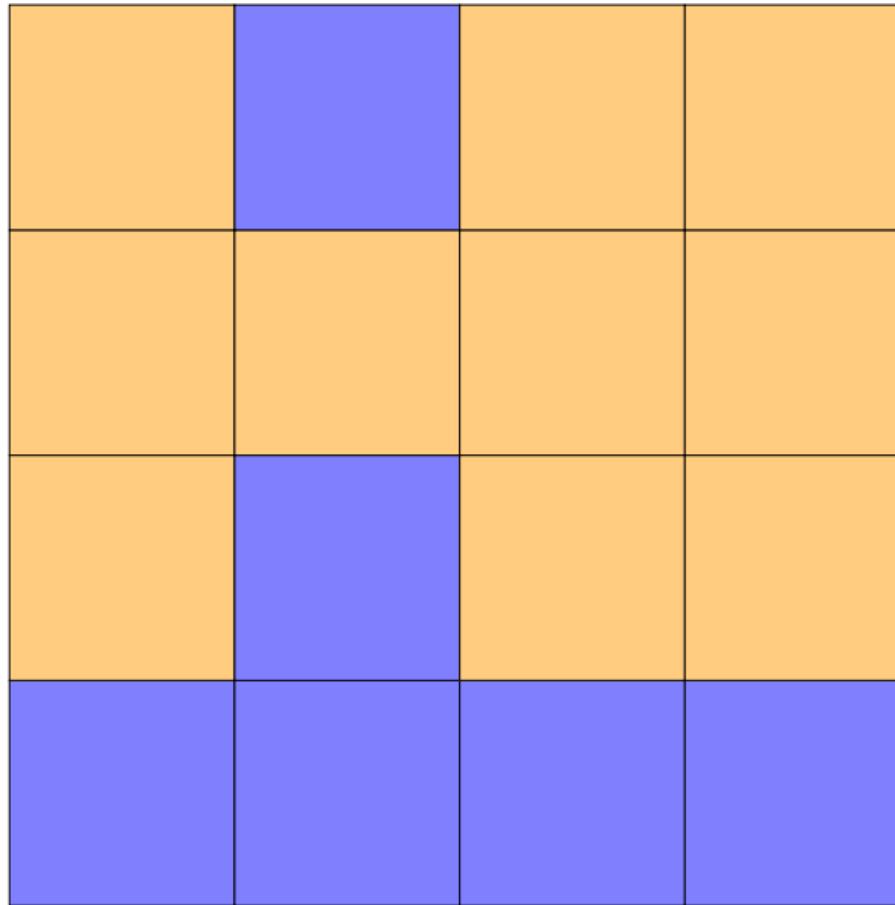
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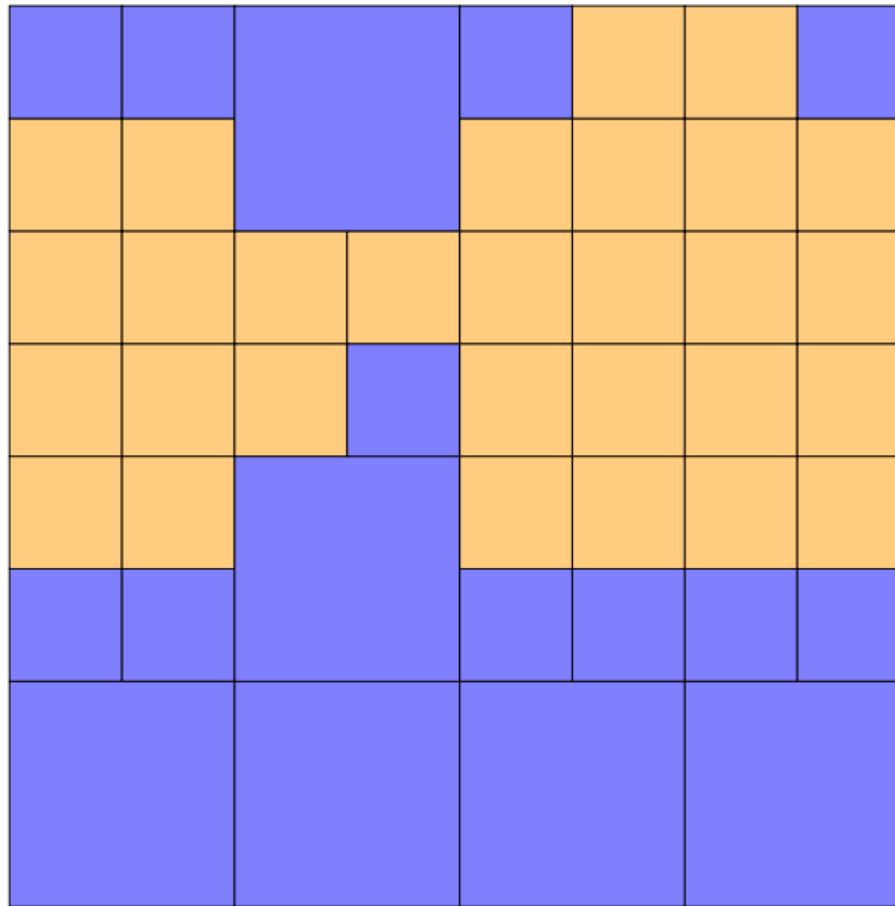
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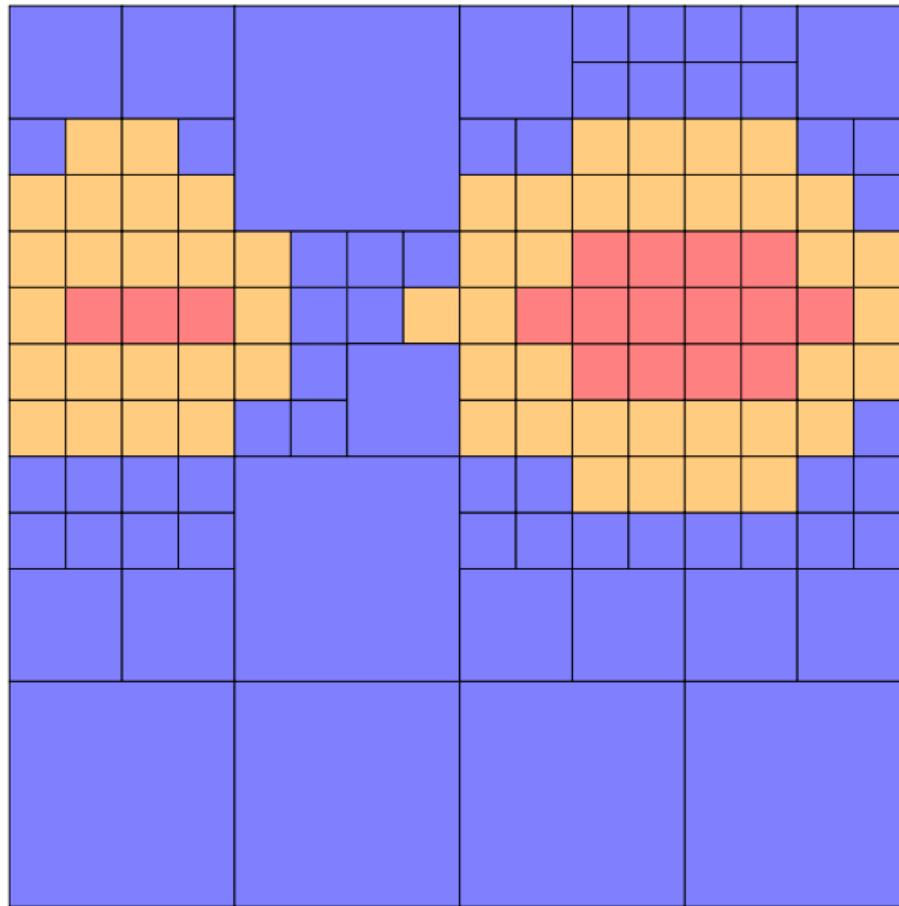
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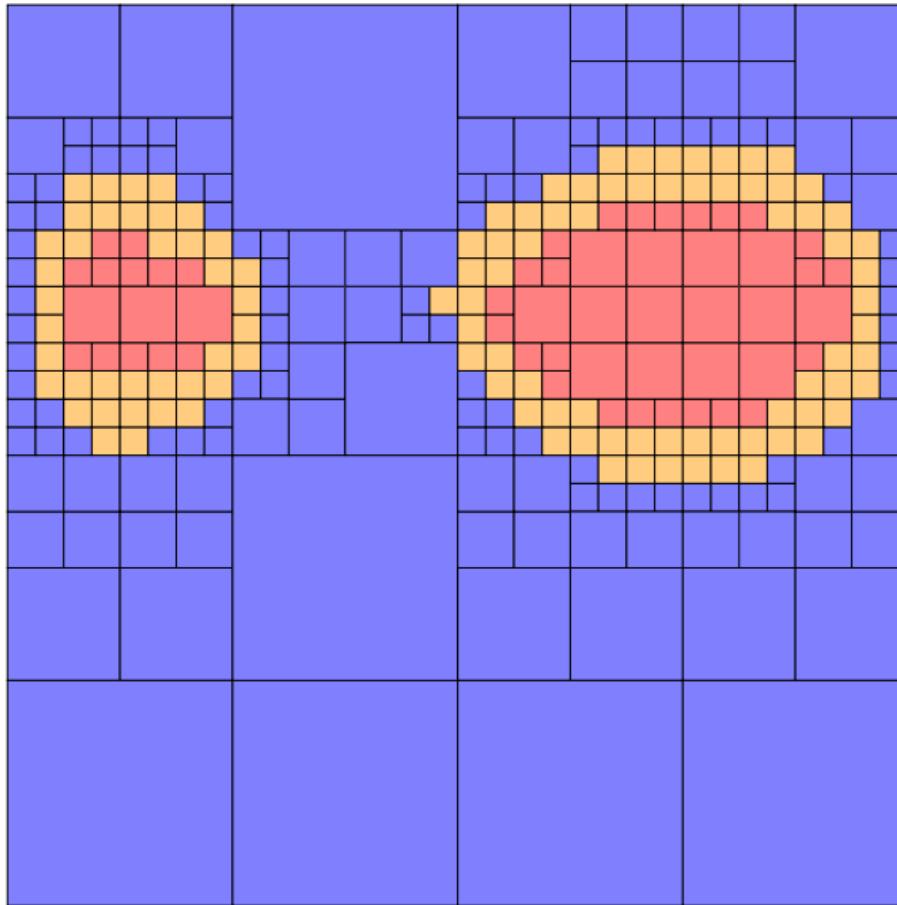
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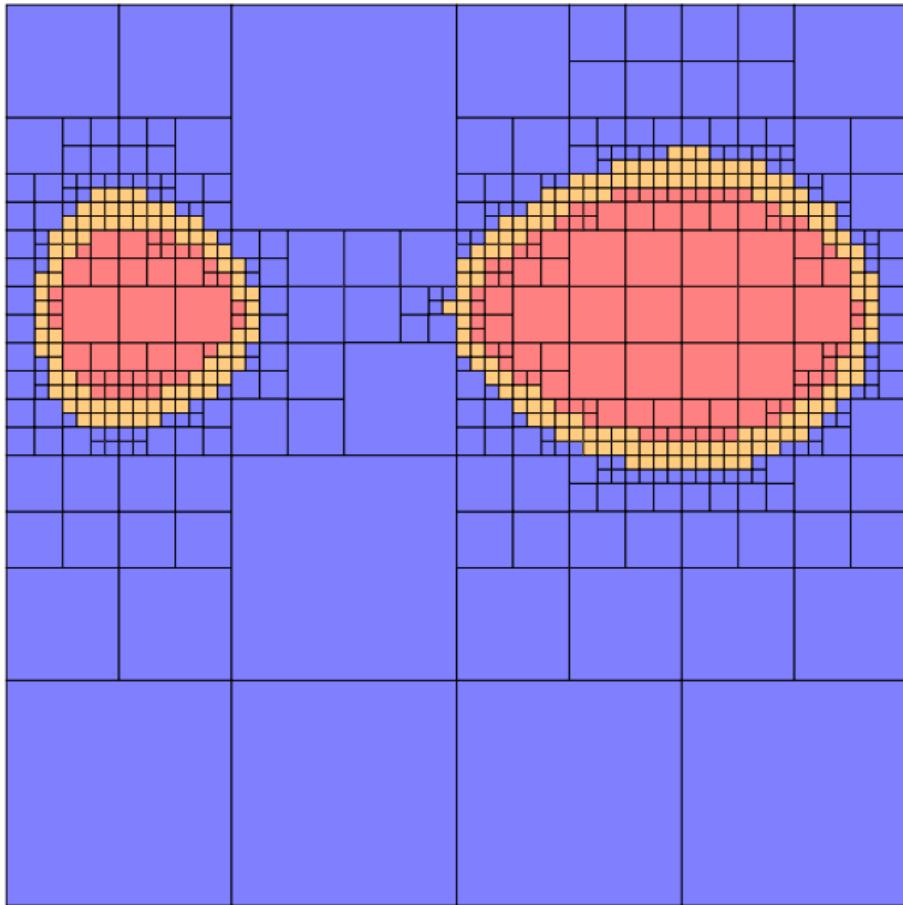
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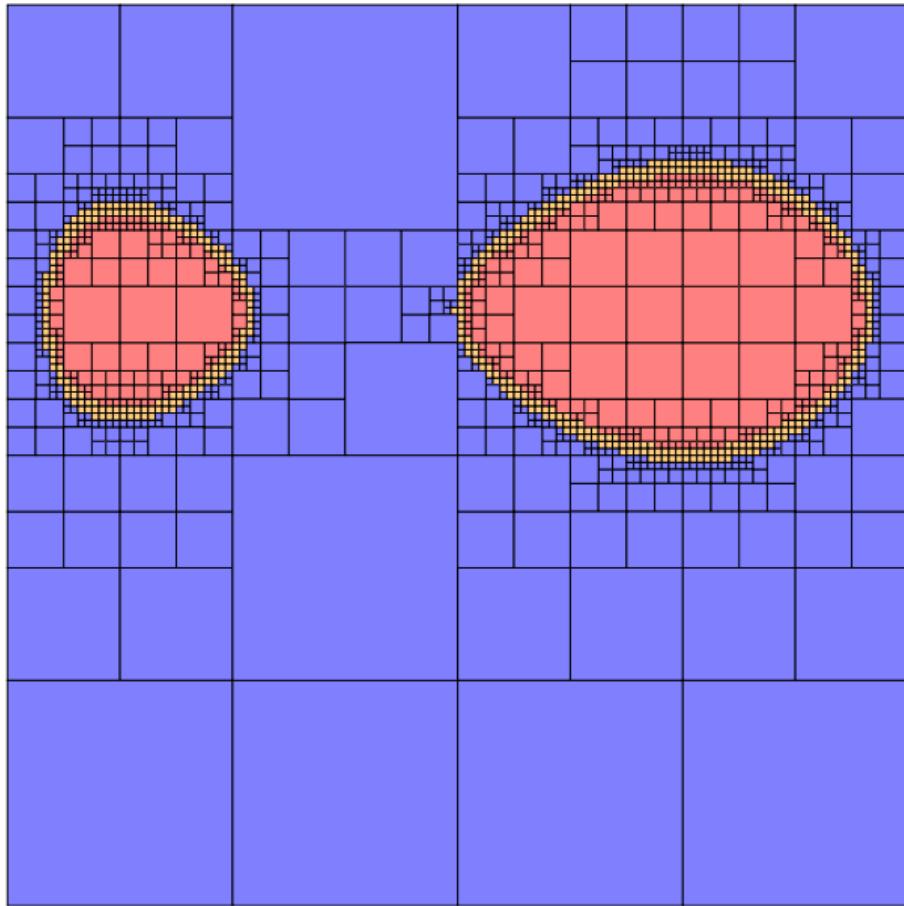
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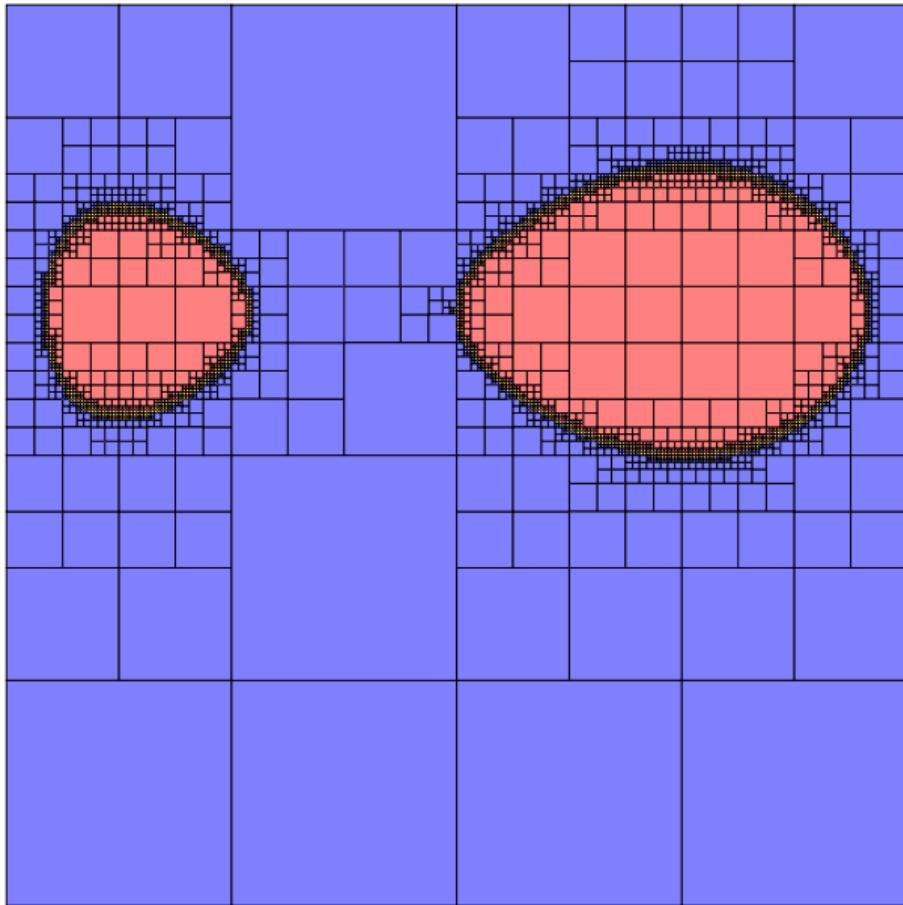
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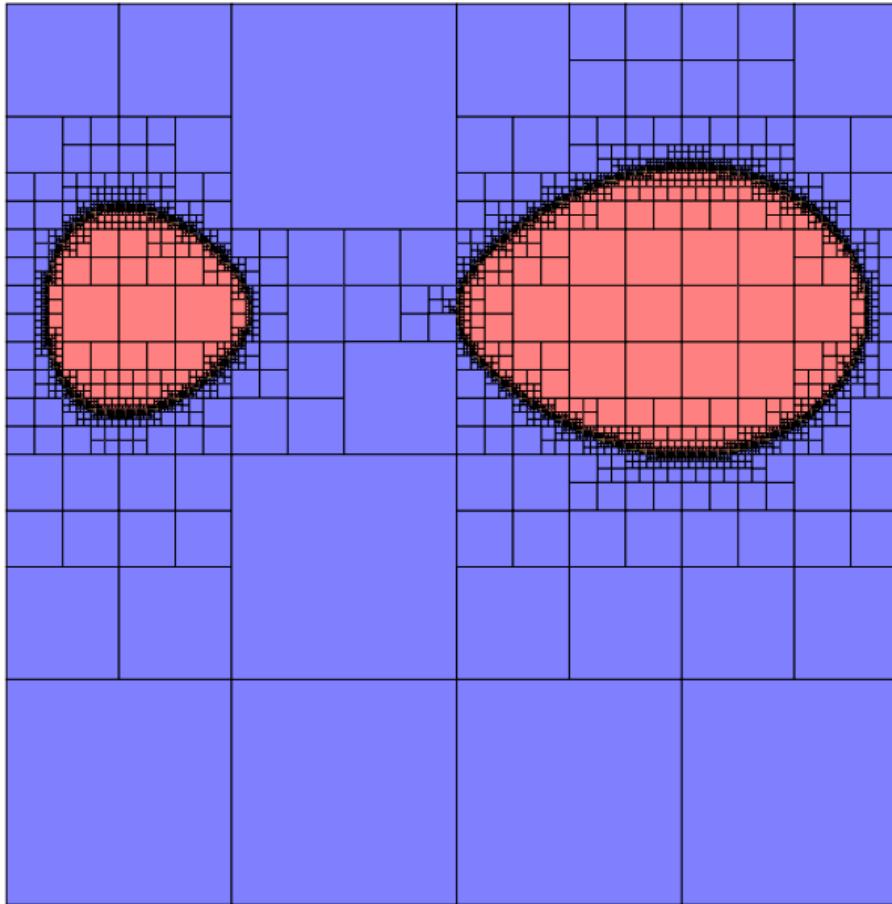
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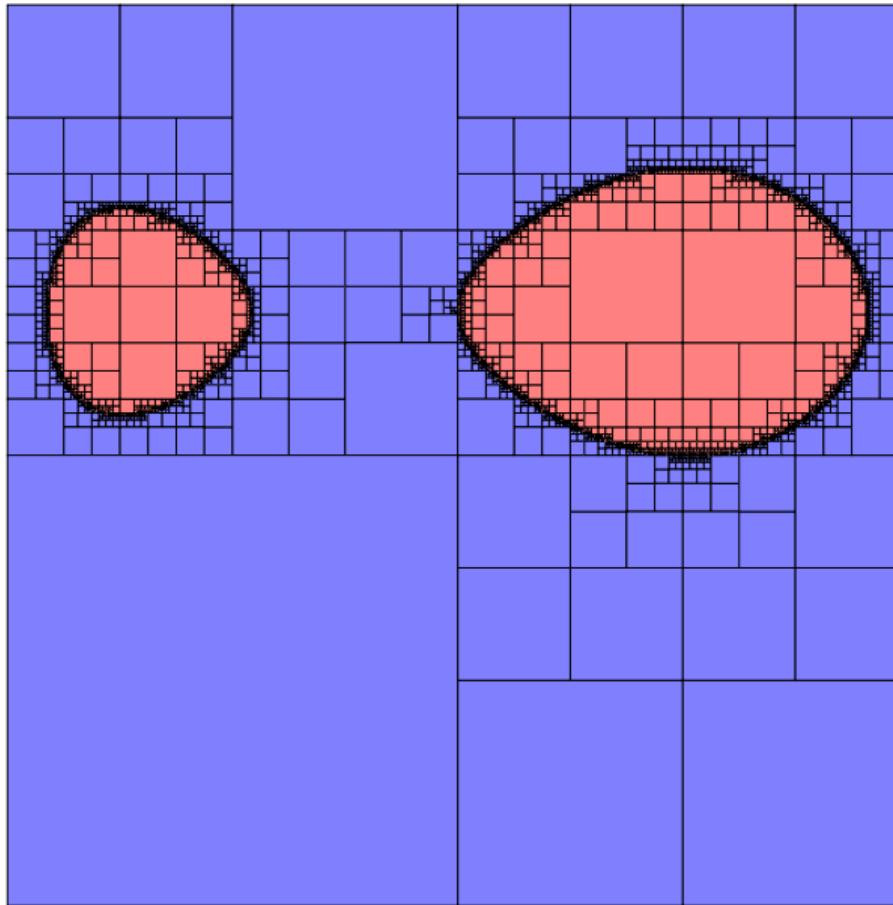
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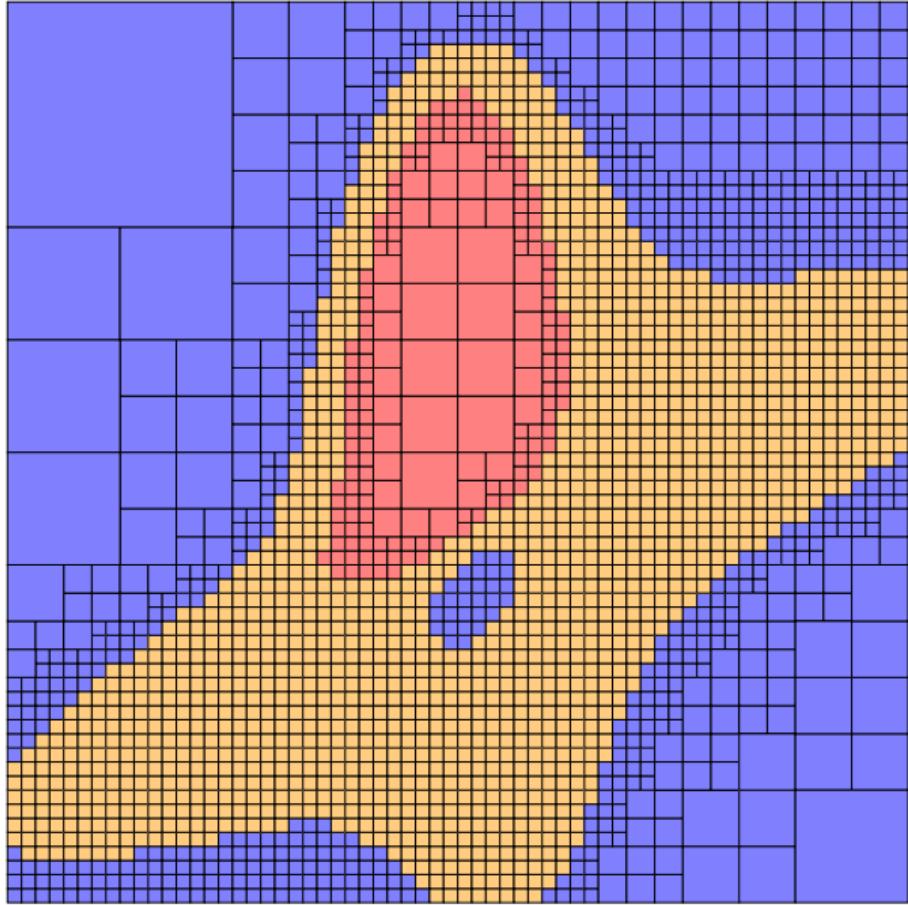
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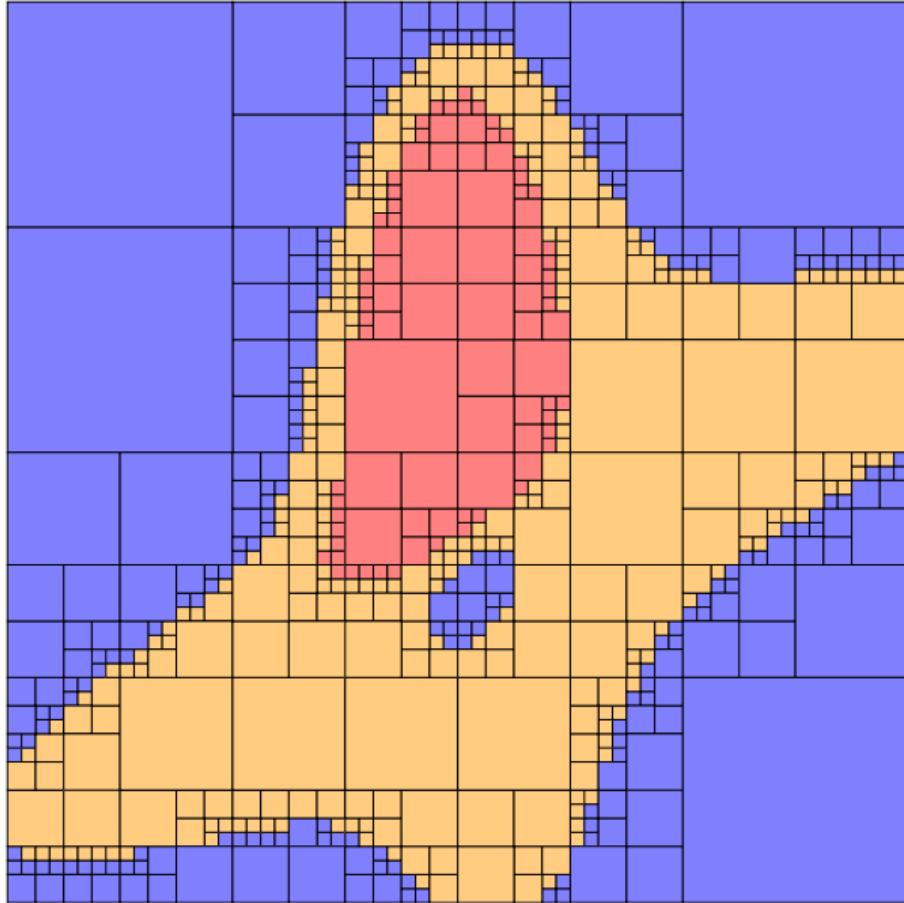
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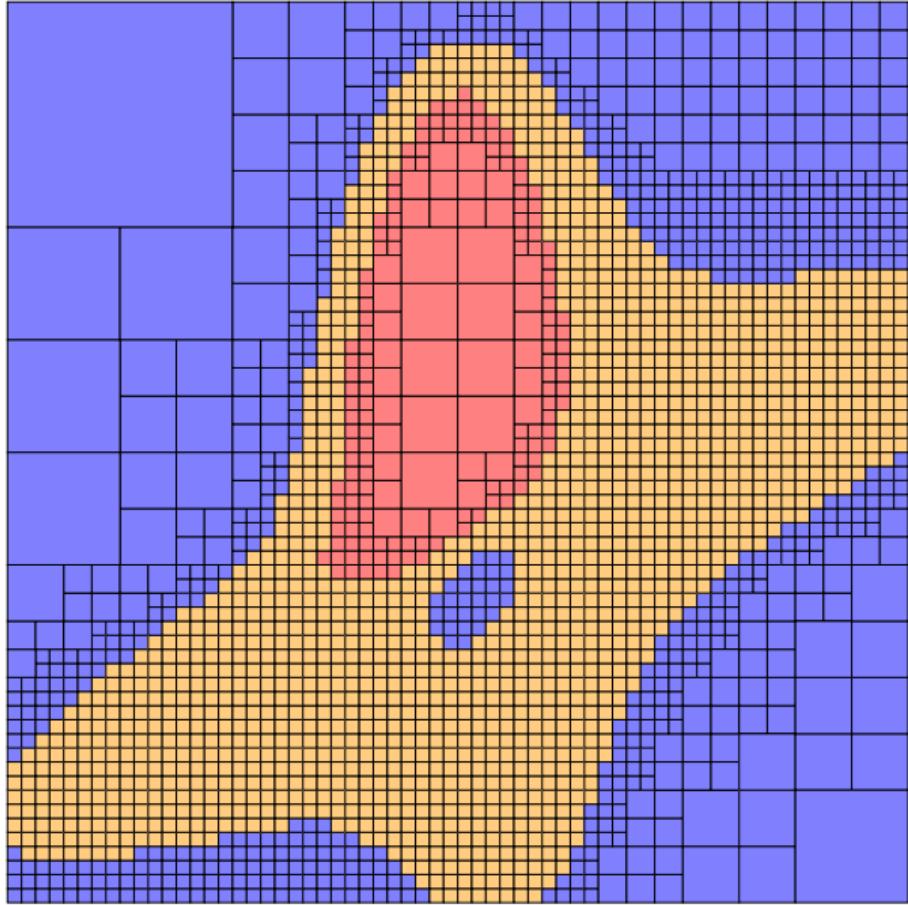
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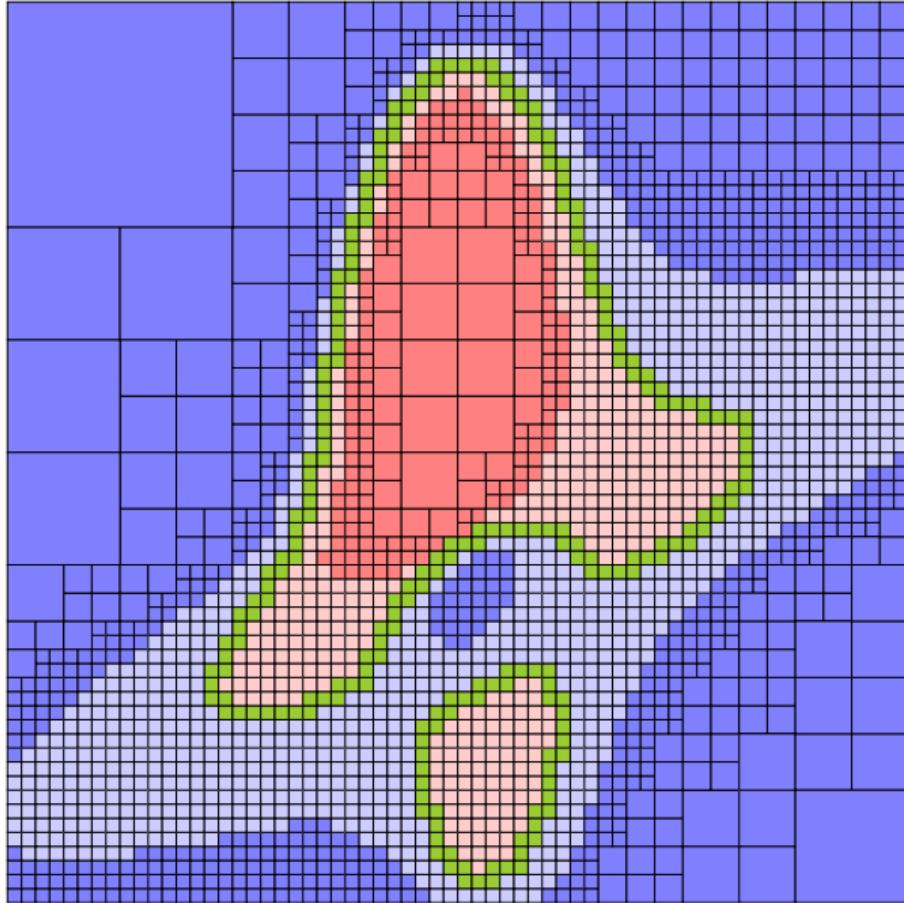


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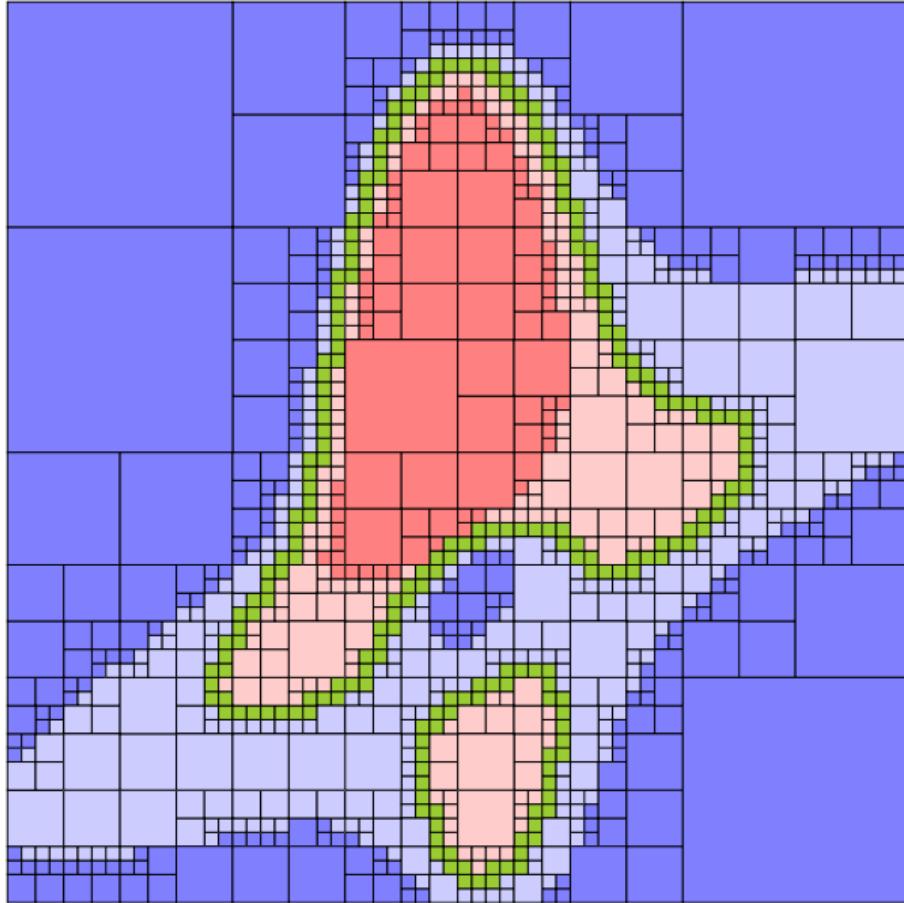


2423 leaves

## Região implícita – método intervalar híbrido

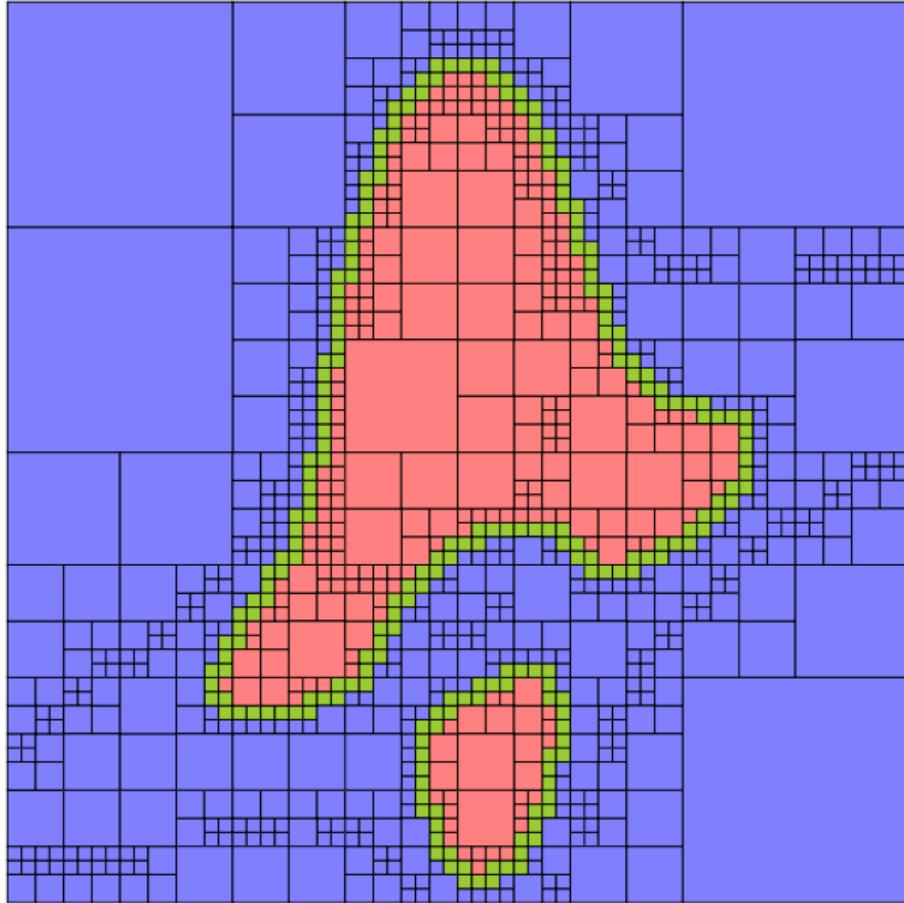


## Região implícita – método intervalar híbrido

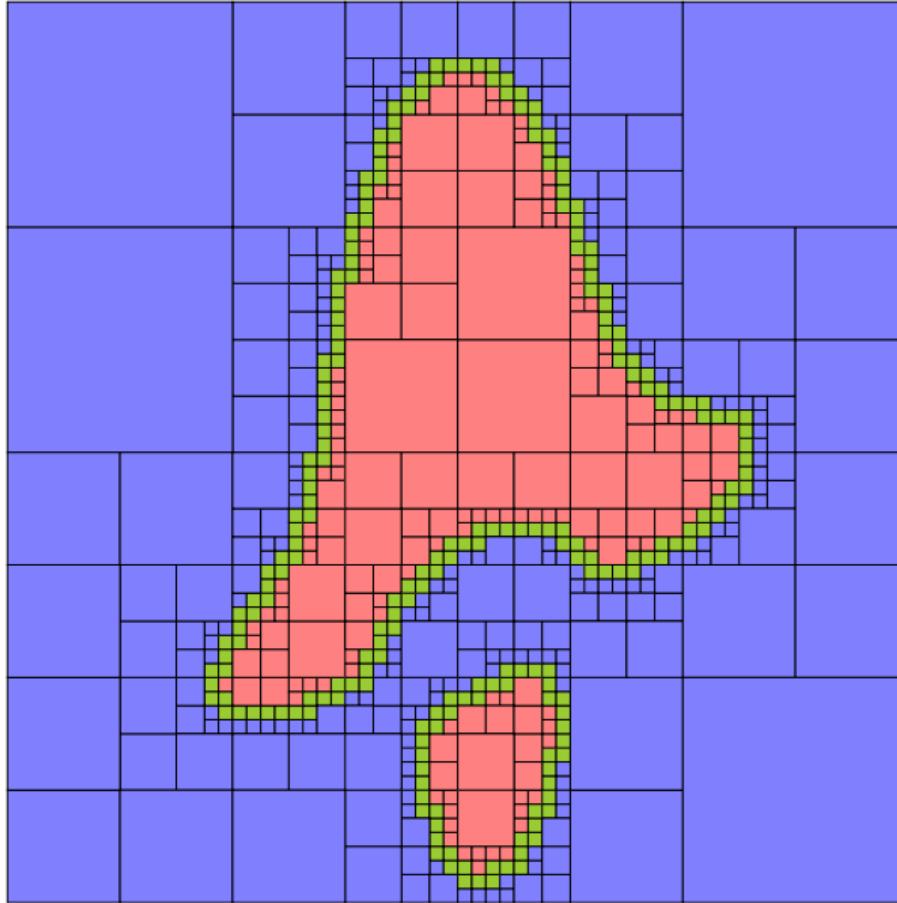


1061 leaves

## Região implícita – método intervalar híbrido otimista



## Região implícita – método intervalar híbrido otimista



644 leaves

## Região implícita – método intervalar

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return show(x,y,"O") end
    if z<0 then return show(x,y,"I") end
    if level==0 then return show(x,y,"U") end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    explore(xlo,ylo,level-1)
    explore(xhi,ylo,level-1)
    explore(xlo,yhi,level-1)
    explore(xhi,yhi,level-1)
    return nil
end
```

## Região implícita – método intervalar híbrido otimista

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explode(xlo,ylo,level-1)
    local c2=explode(xhi,ylo,level-1)
    local c3=explode(xlo,yhi,level-1)
    local c4=explode(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
```

## Região implícita – método intervalar híbrido otimista

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explode(xlo,ylo,level-1)
    local c2=explode(xhi,ylo,level-1)
    local c3=explode(xlo,yhi,level-1)
    local c4=explode(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
```

```
function classify(x,y)
    local xmin,xmax=x:extremes()
    local ymin,ymax=y:extremes()
    local f1=f(xmin,ymin)
    local f2=f(xmax,ymin)
    local f3=f(xmax,ymax)
    local f4=f(xmin,ymax)
    if (f1>0 and f2>0 and f3>0 and f4>0) then
        return optimistic and "O" or "PO"
    end
    if (f1<0 and f2<0 and f3<0 and f4<0) then
        return optimistic and "I" or "PI"
    end
    return "V"
end
```

objetos paramétricos

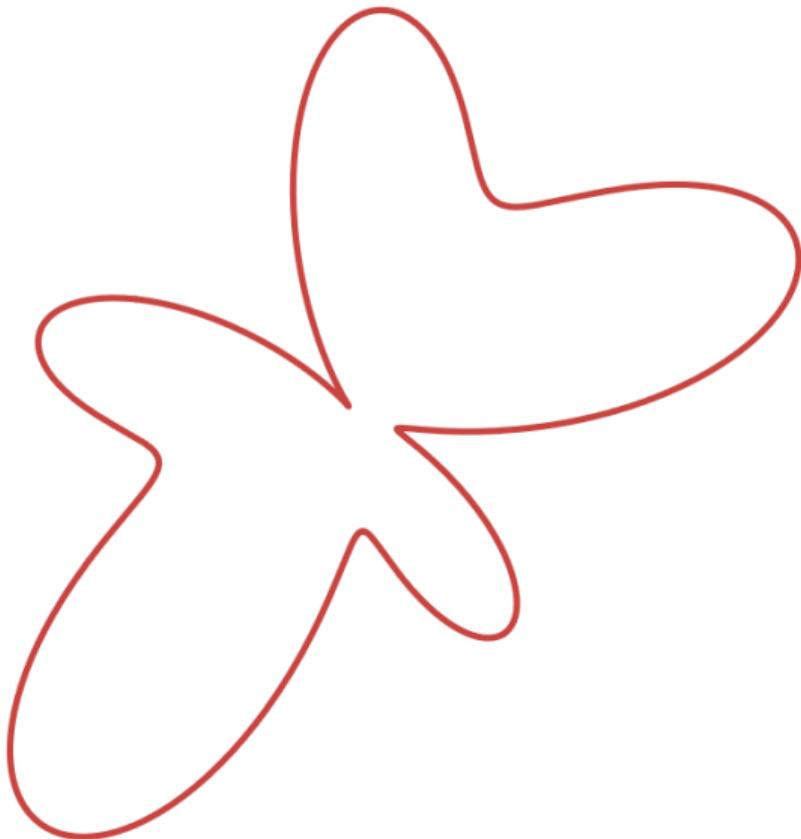
## Curva paramétrica

curva = fronteira da região

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



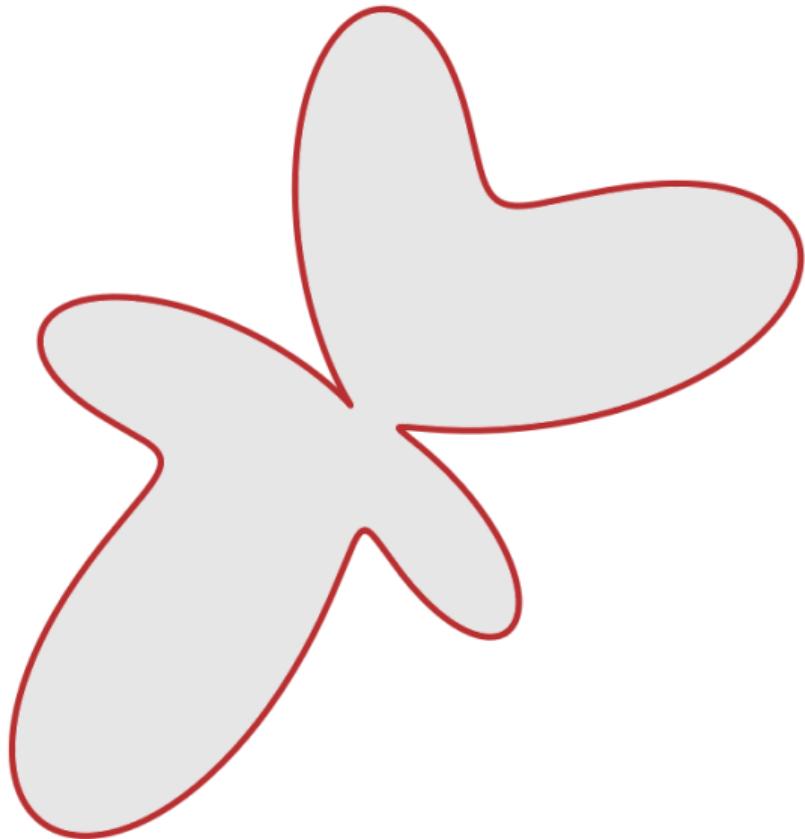
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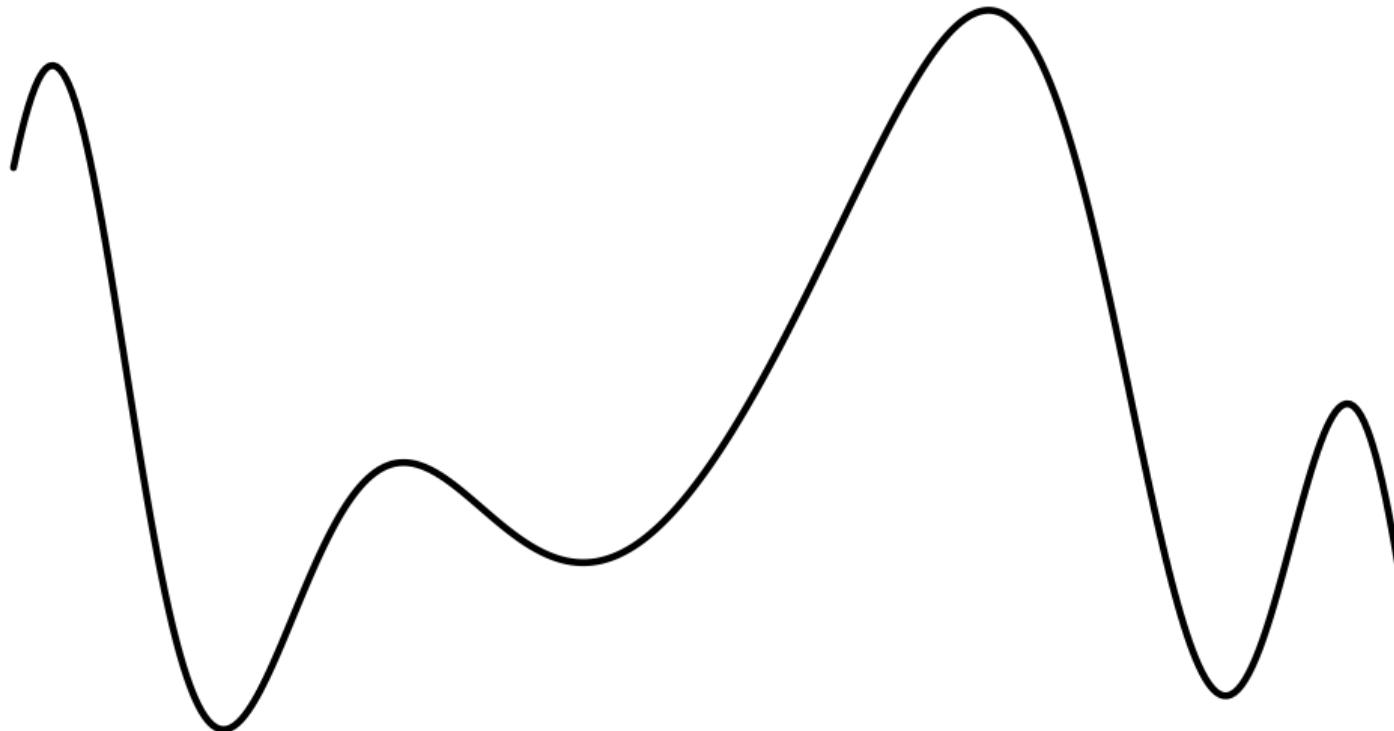
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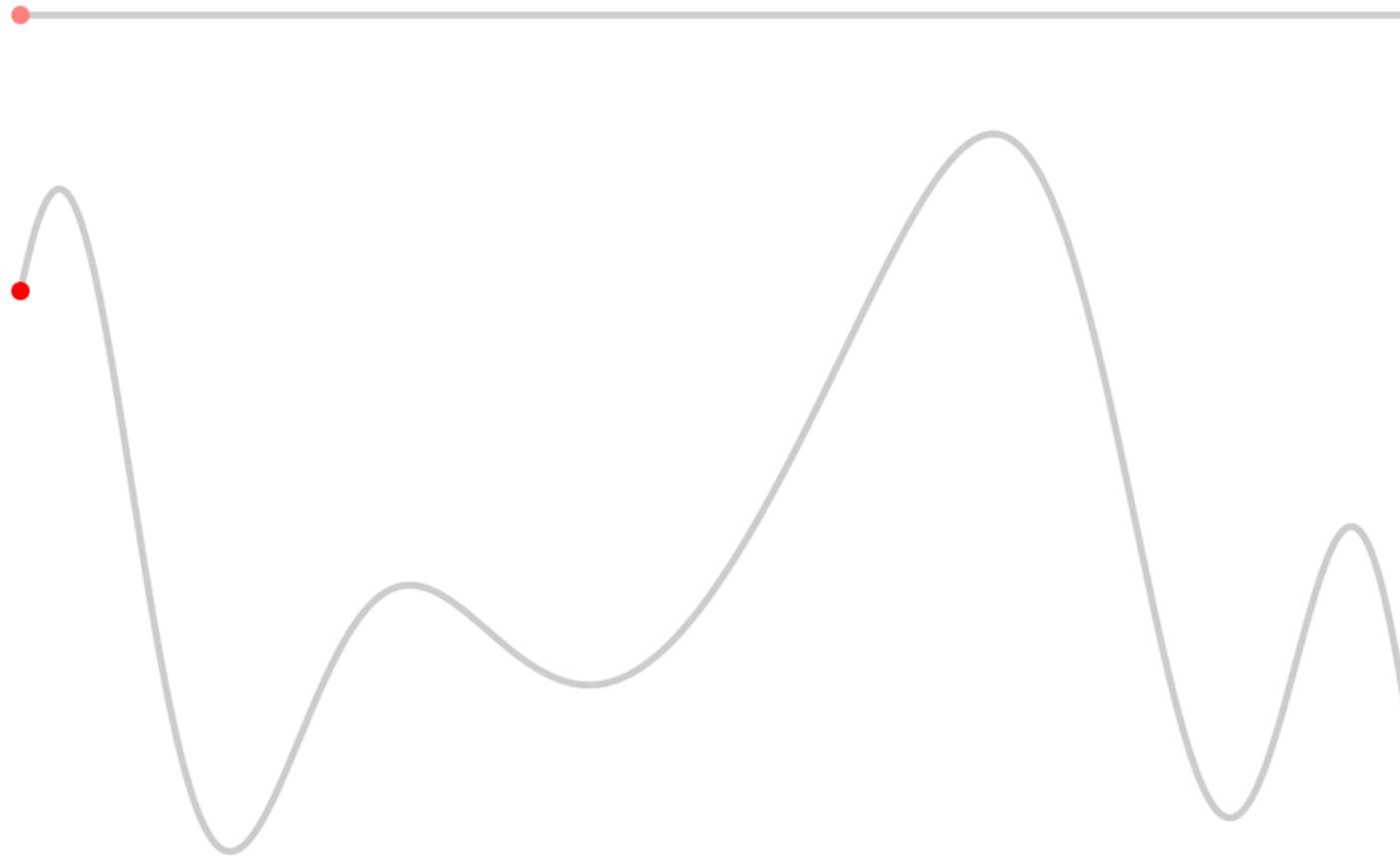


## Curva paramétrica – amostragem uniforme no domínio

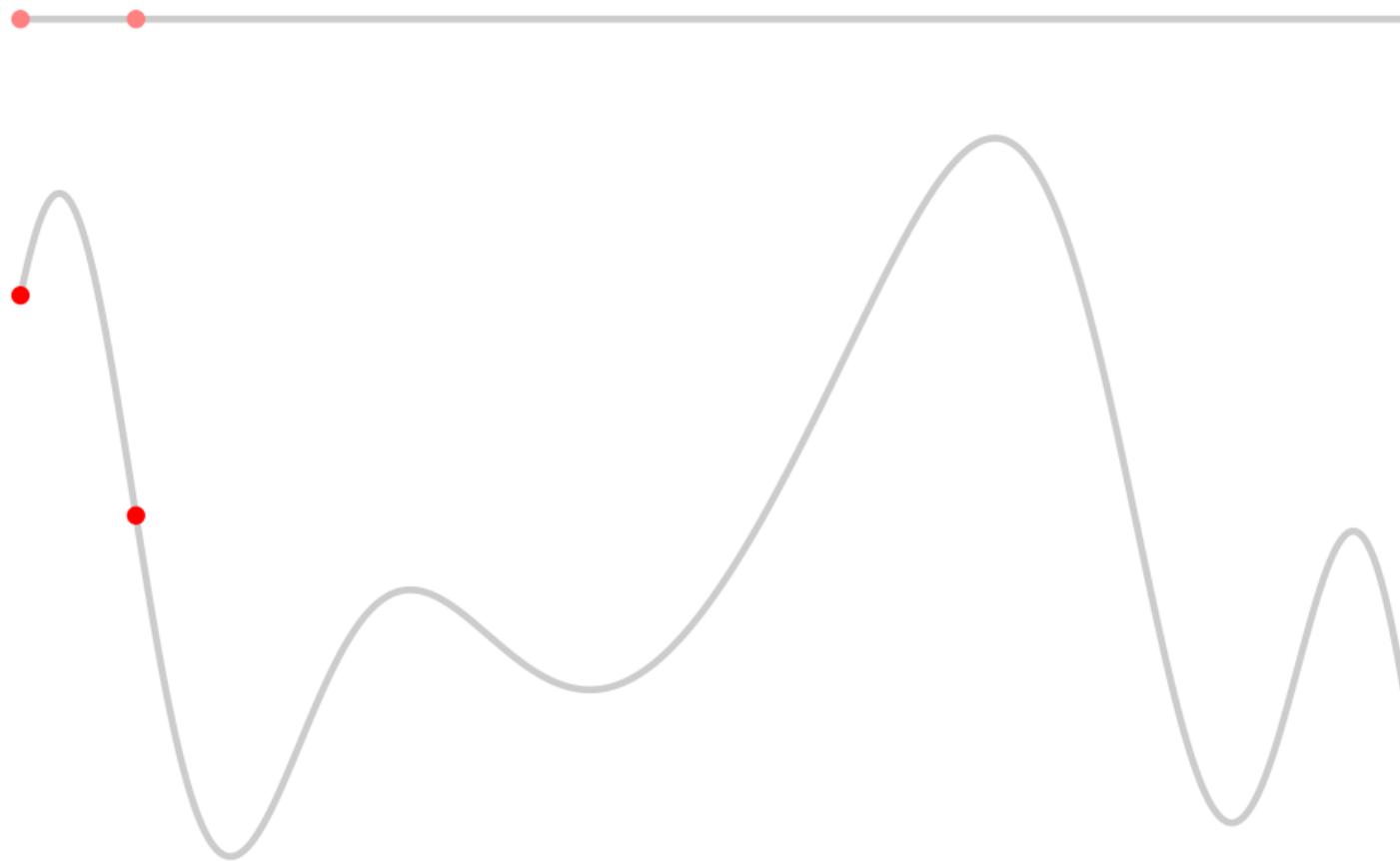
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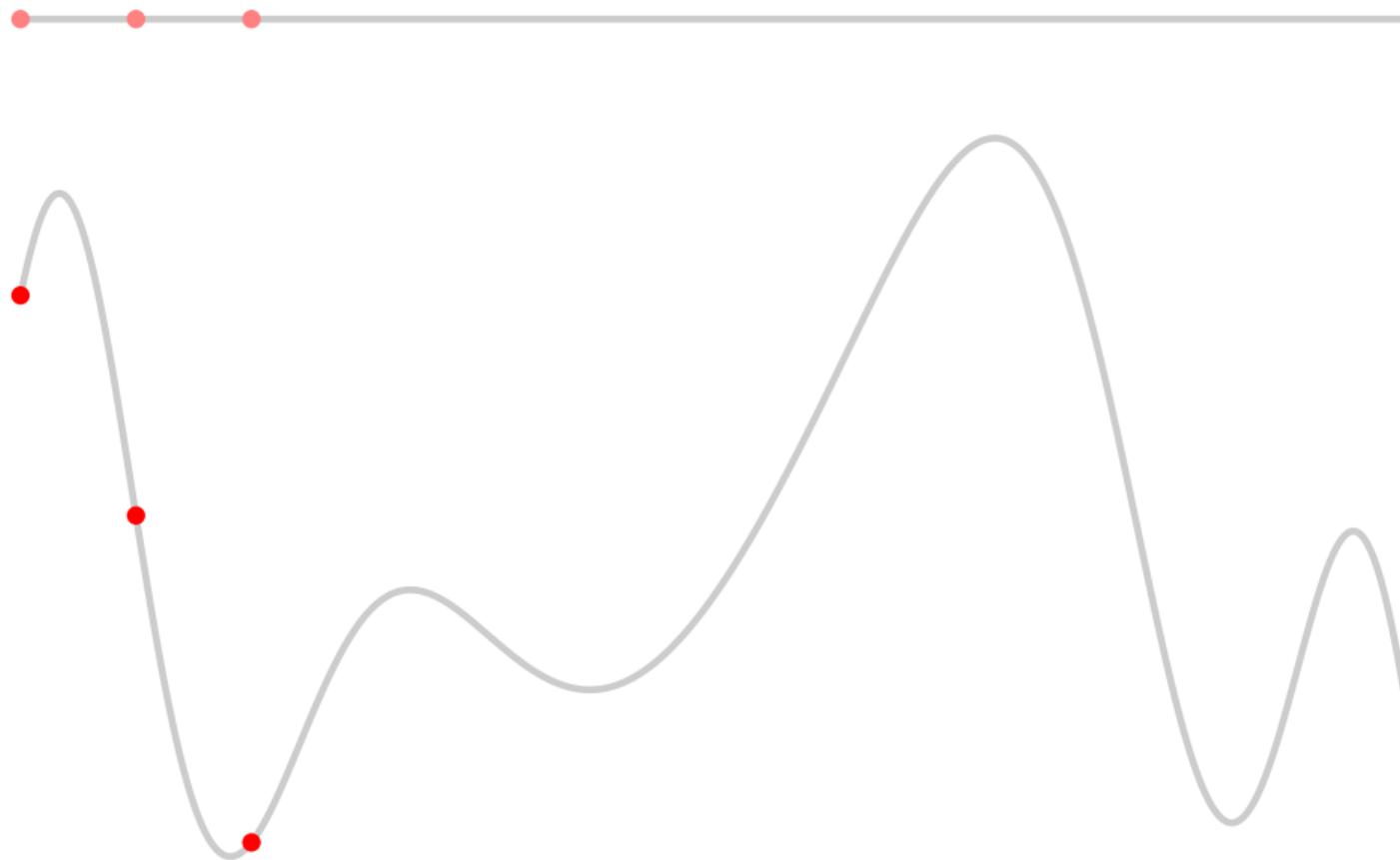
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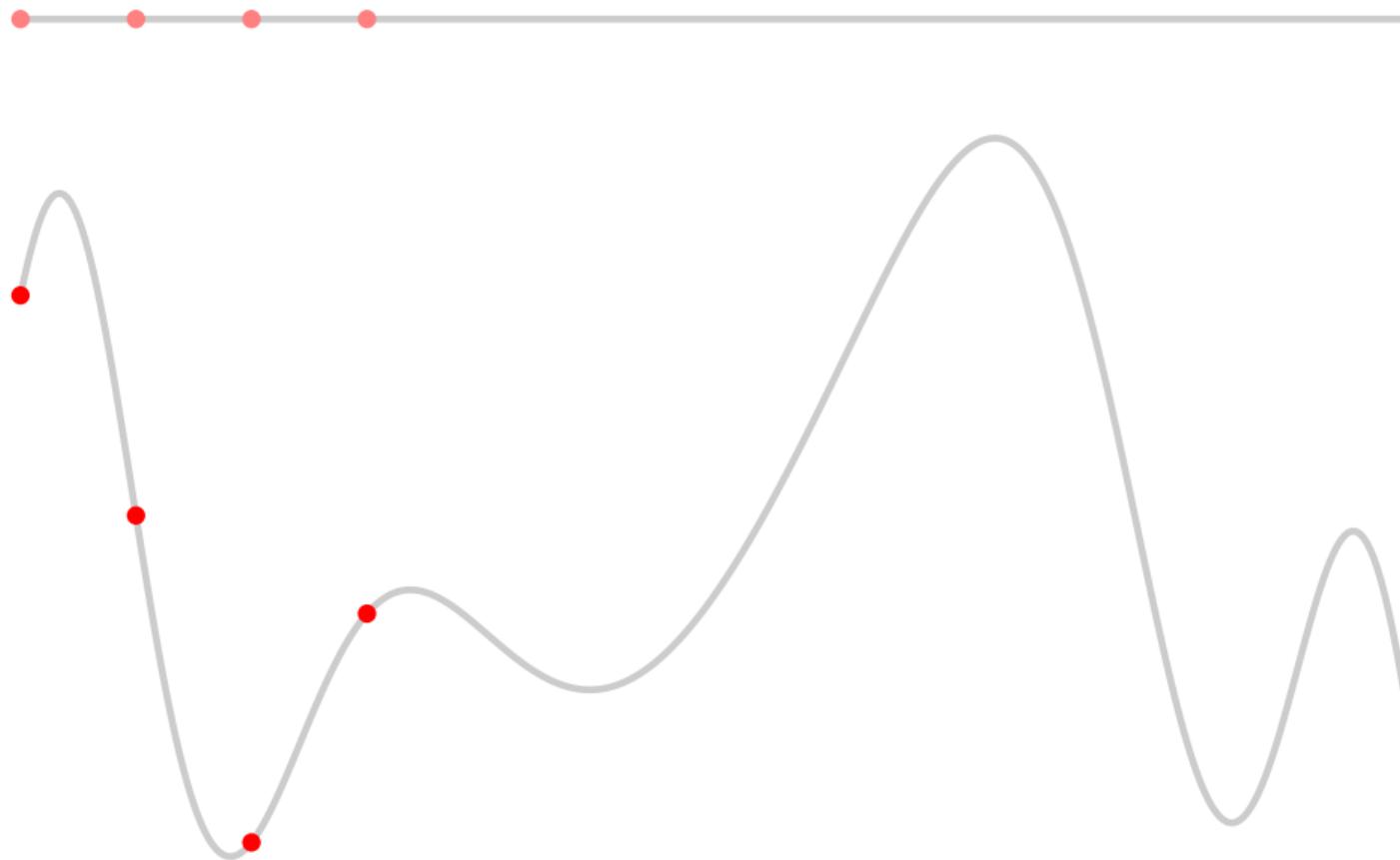
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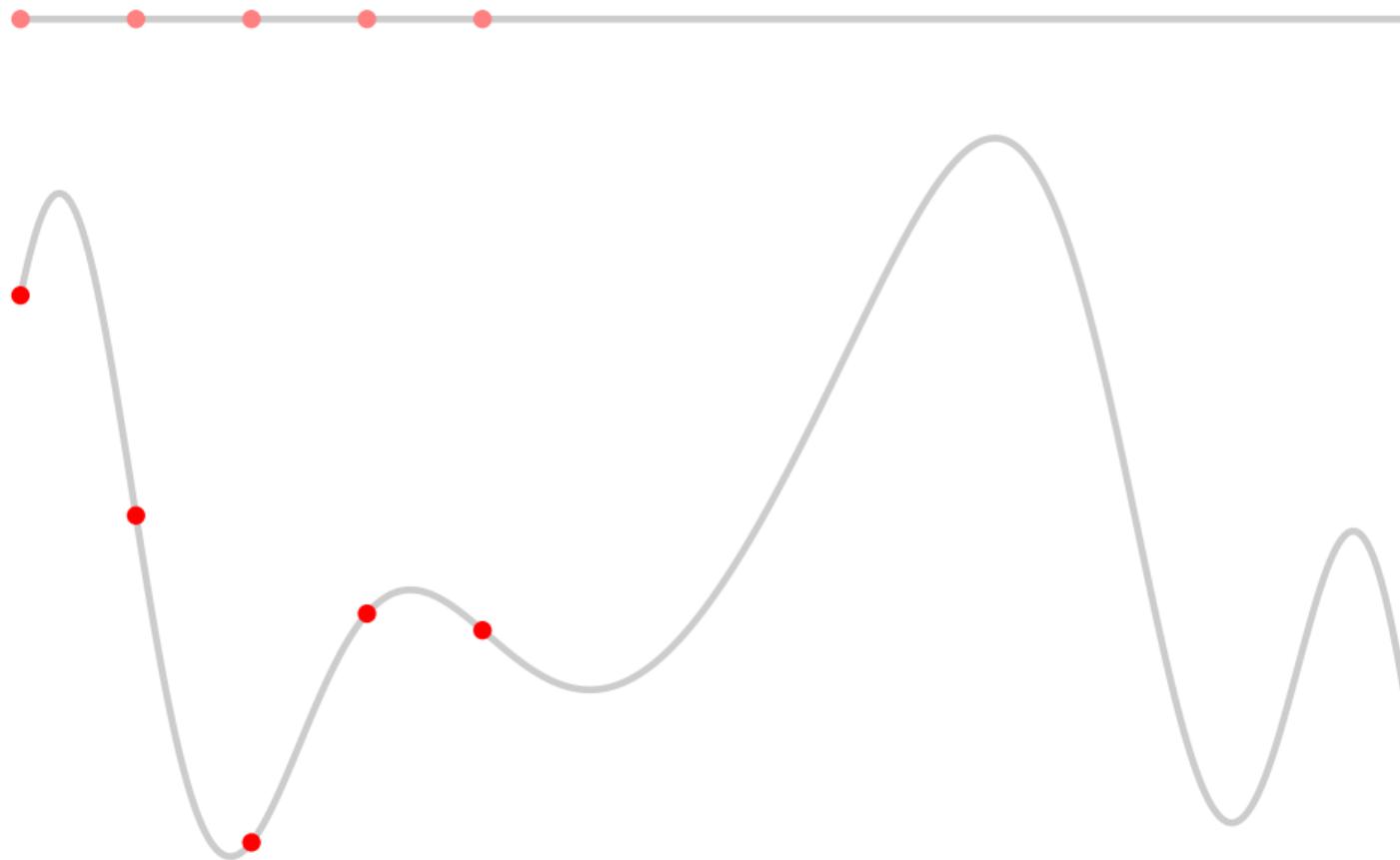
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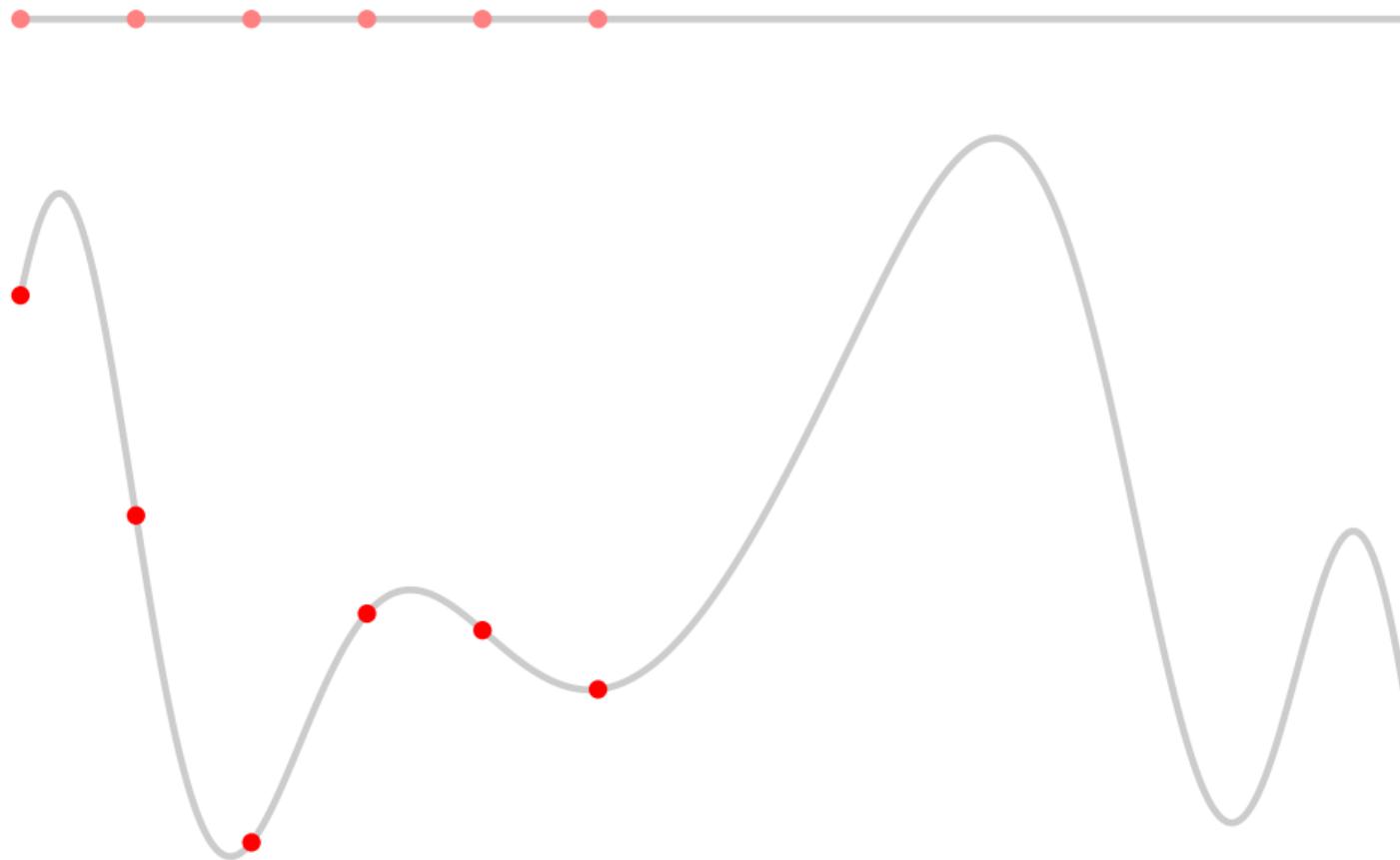
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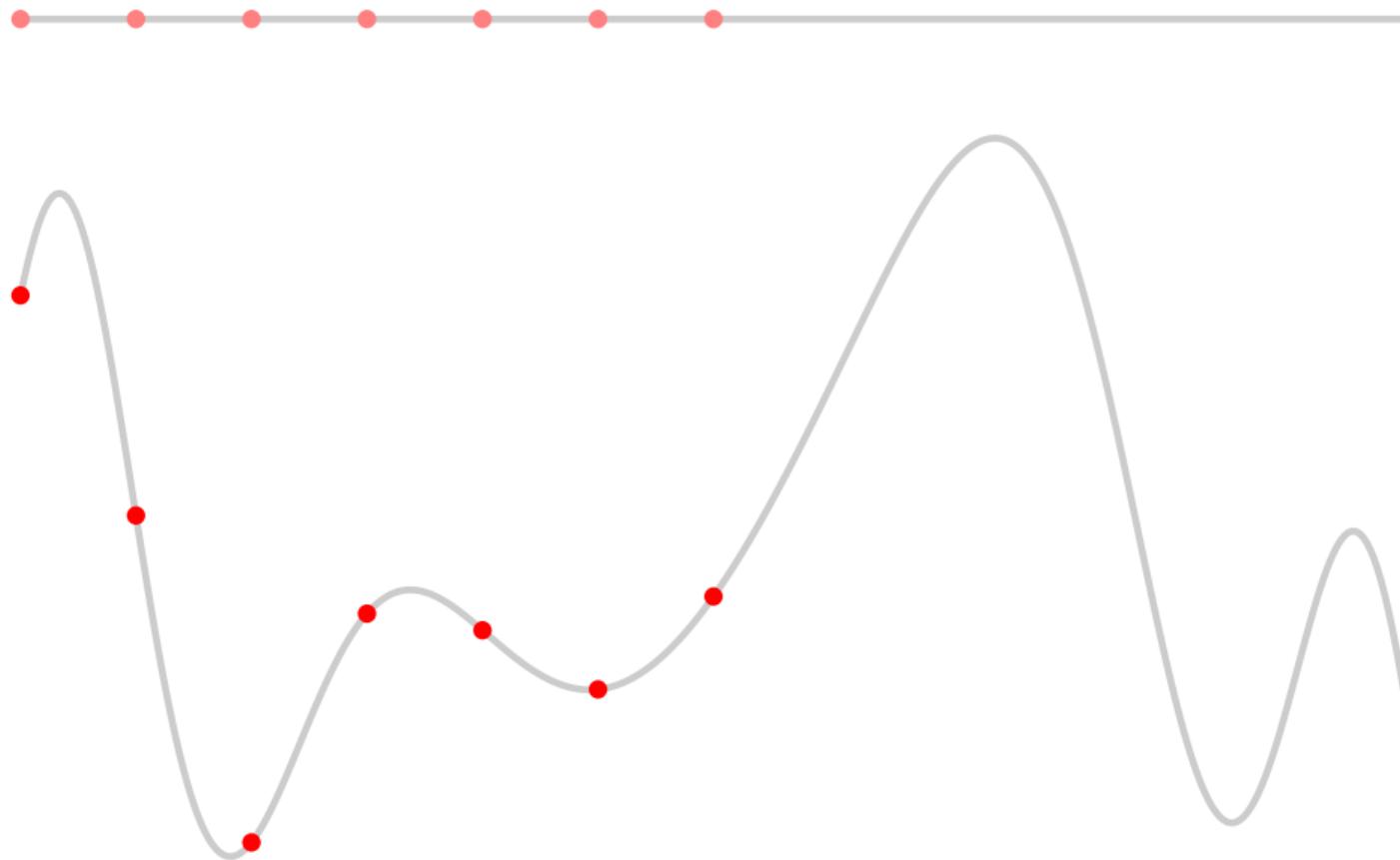
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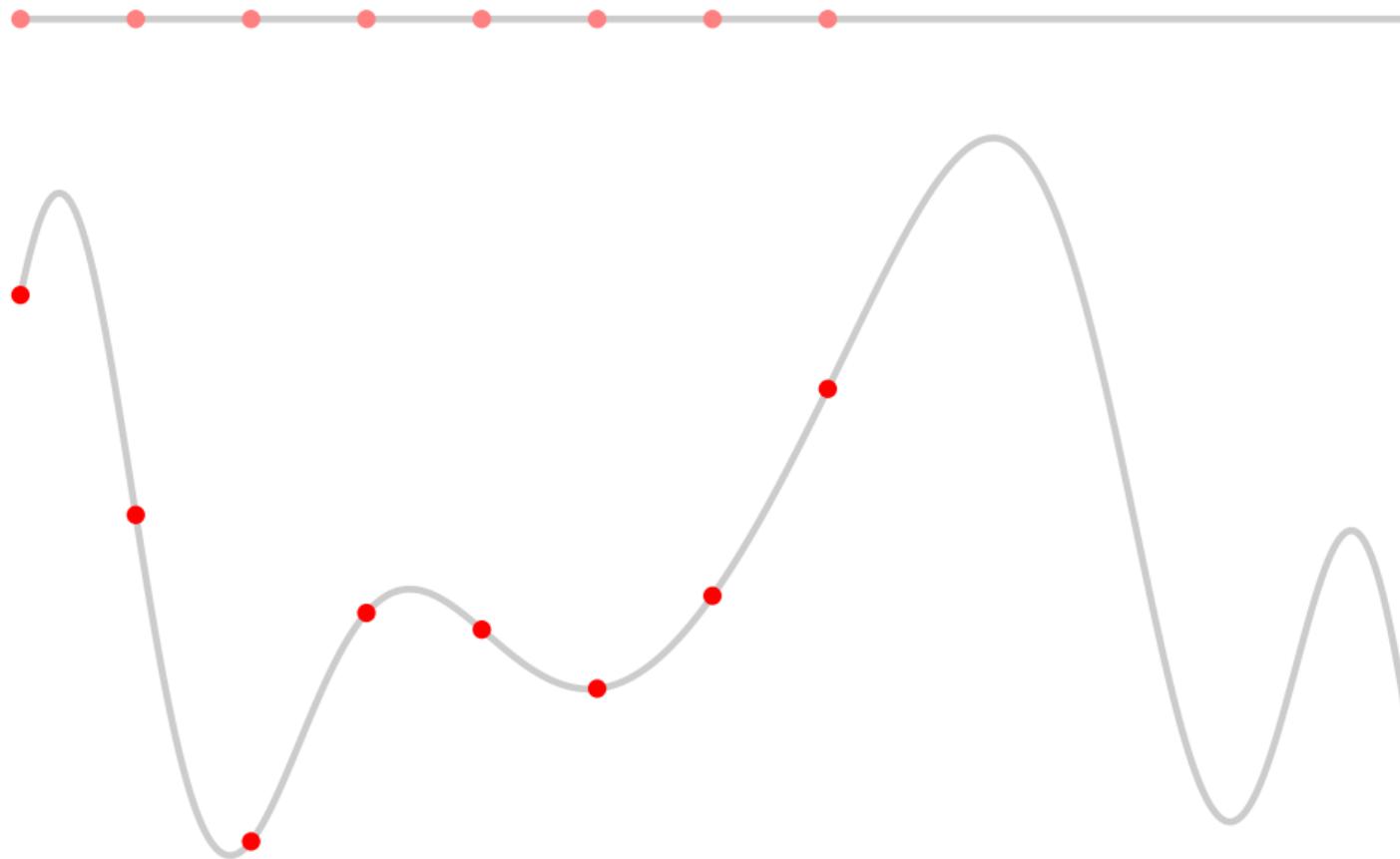
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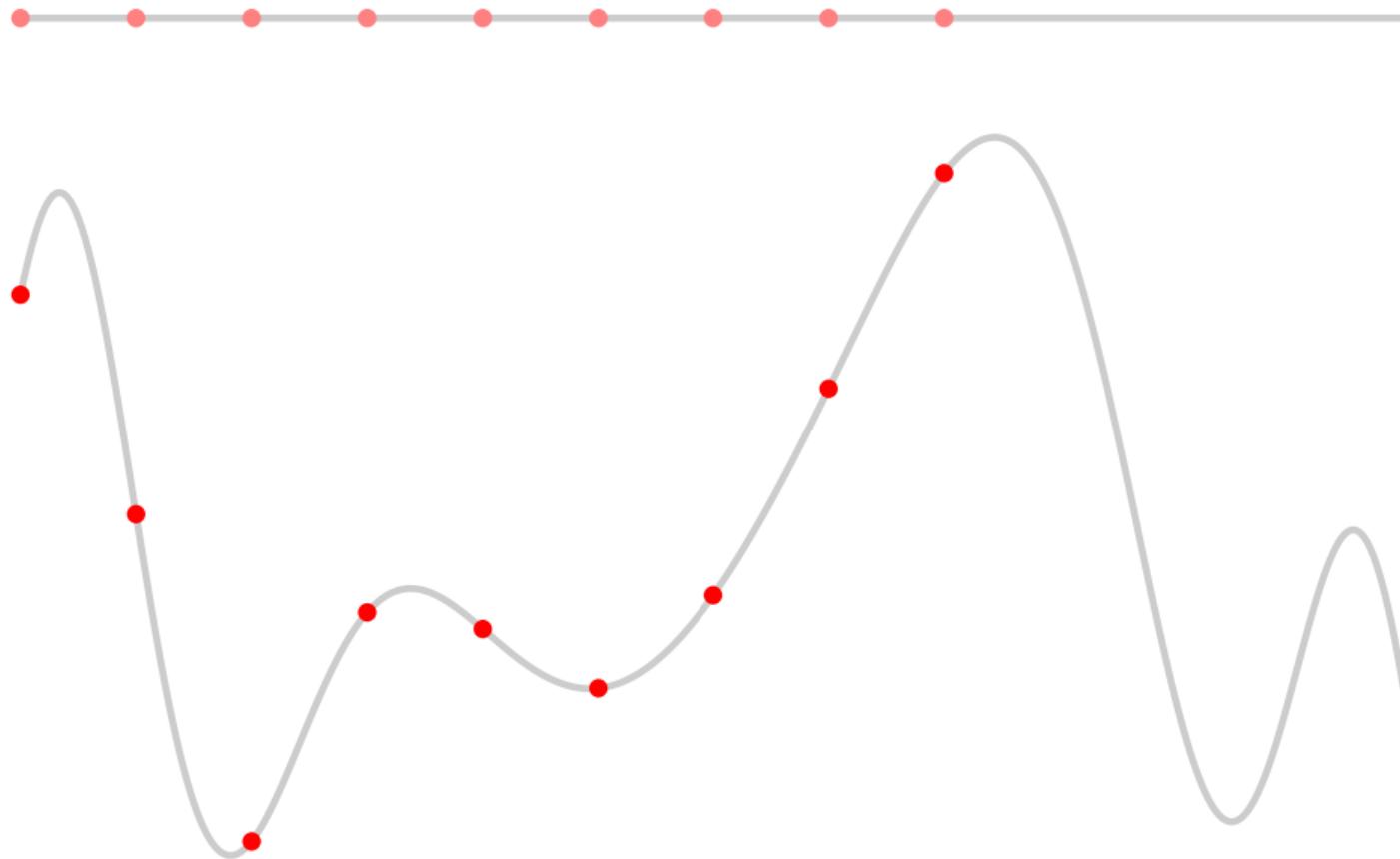
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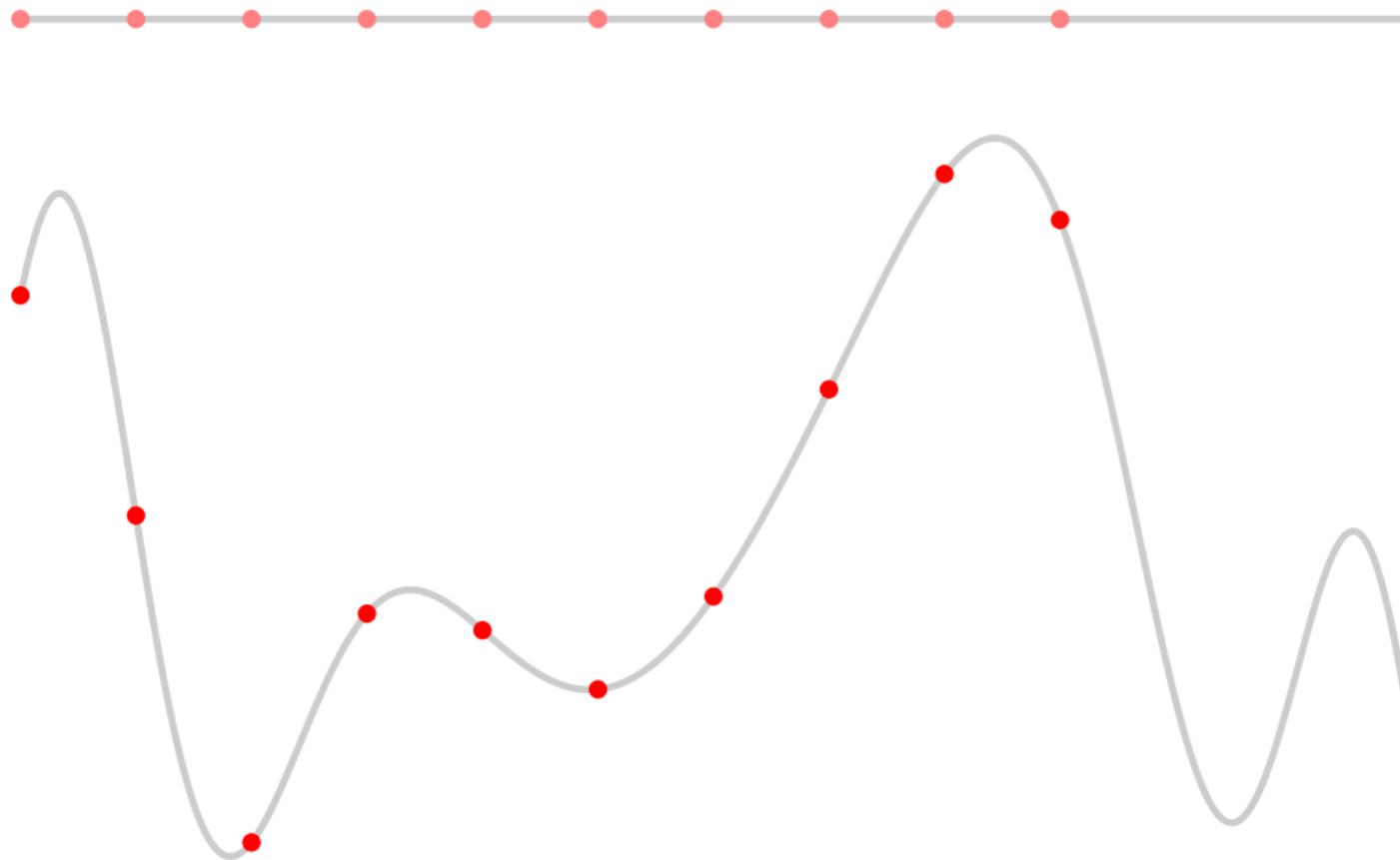
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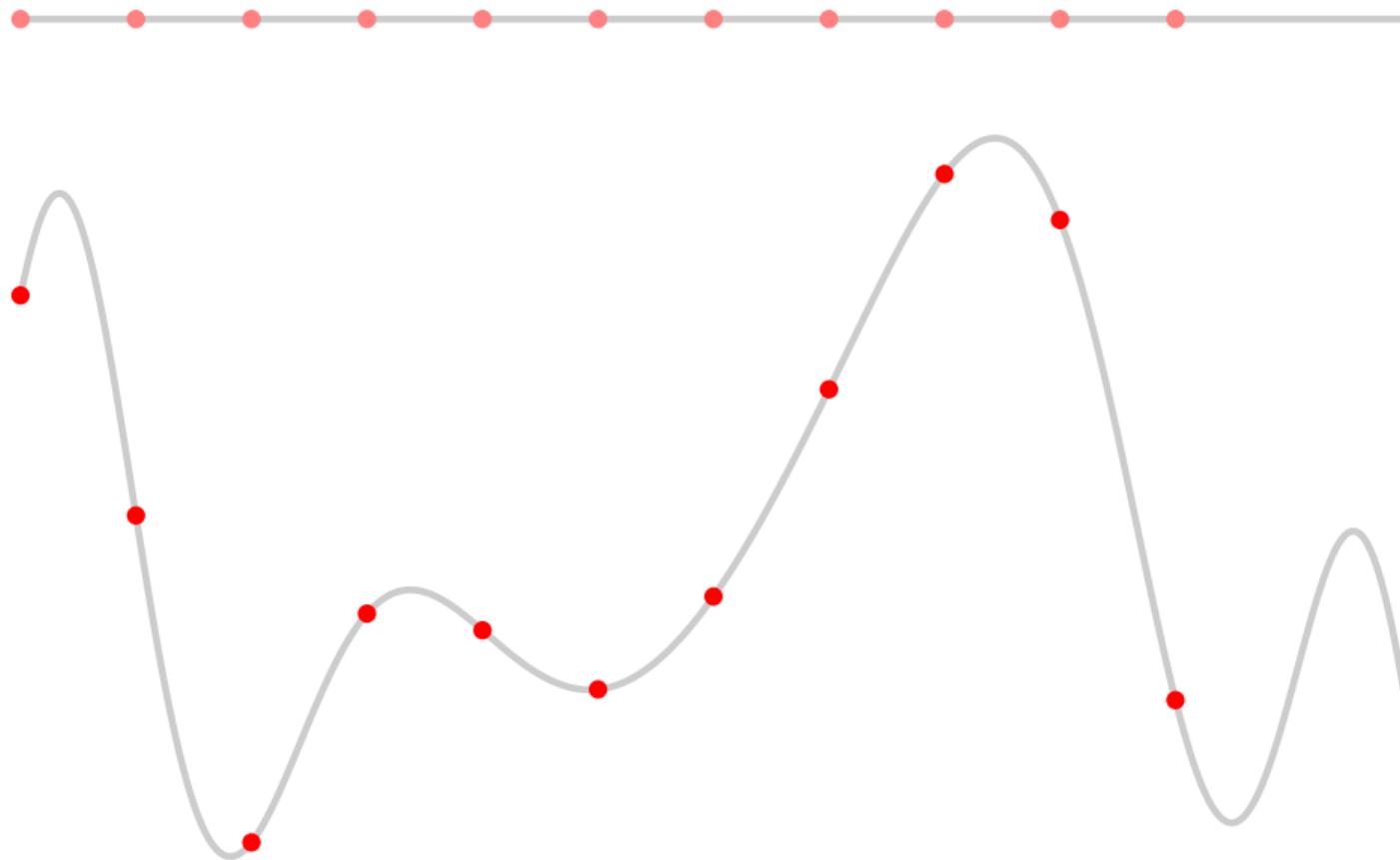
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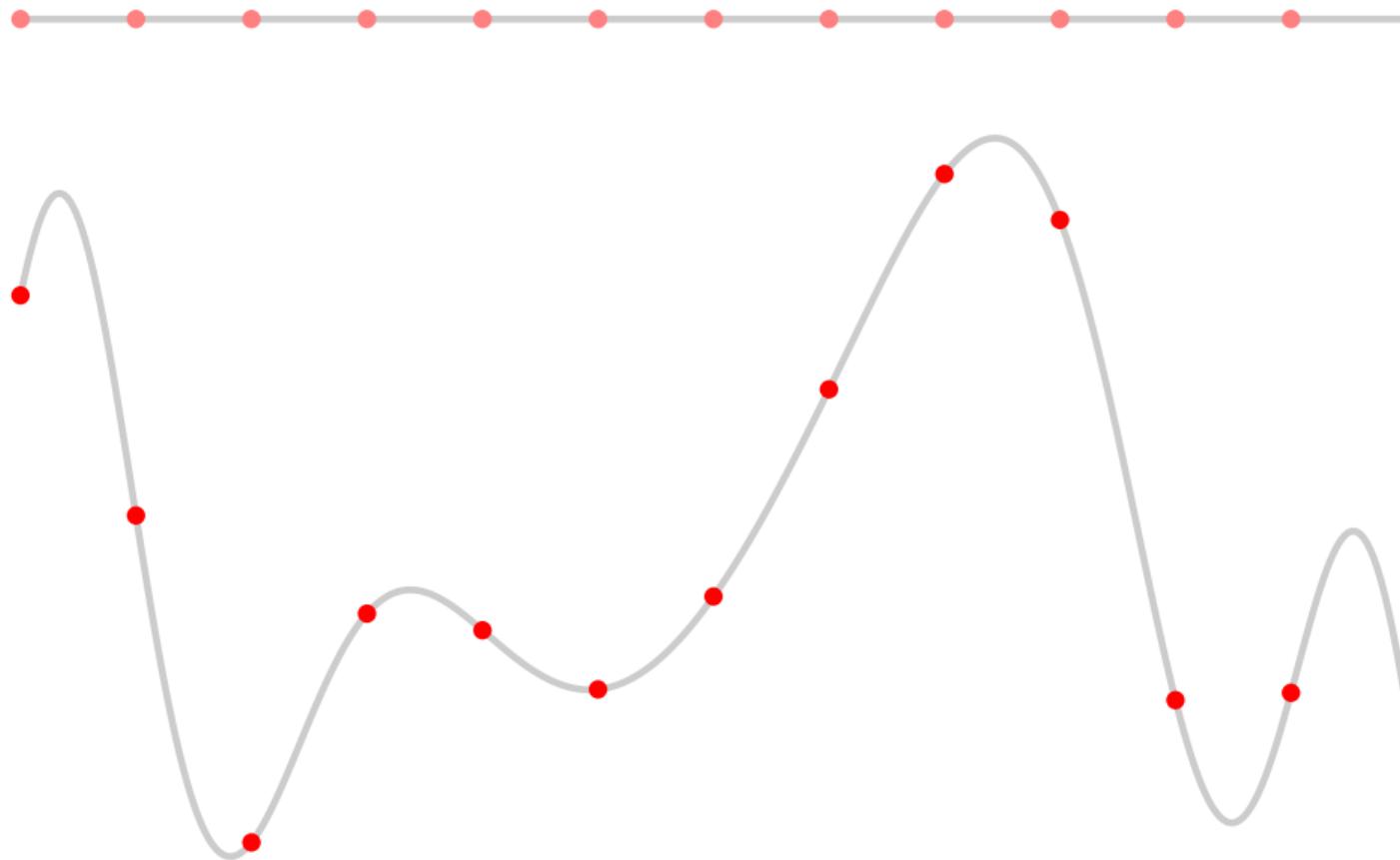
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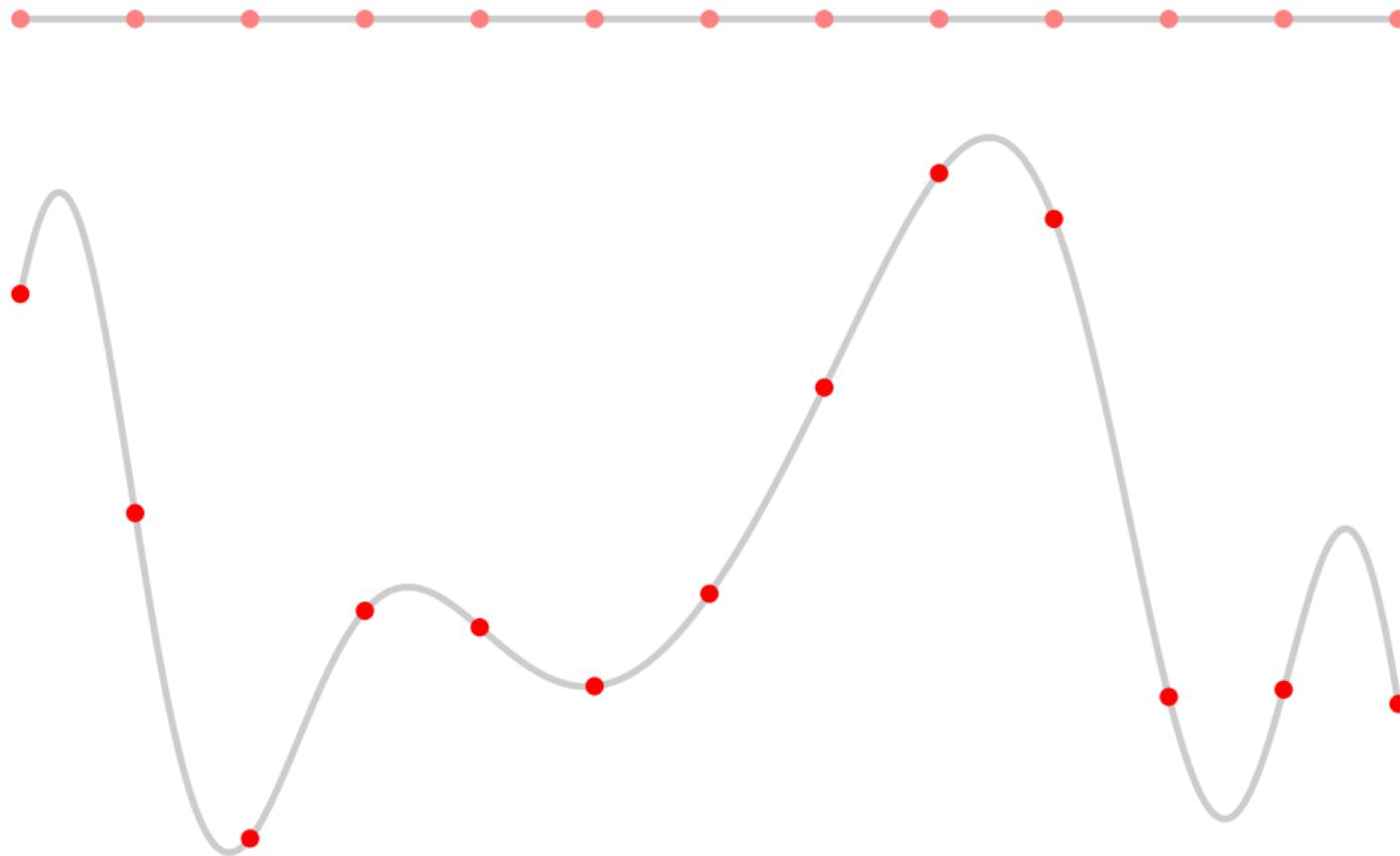
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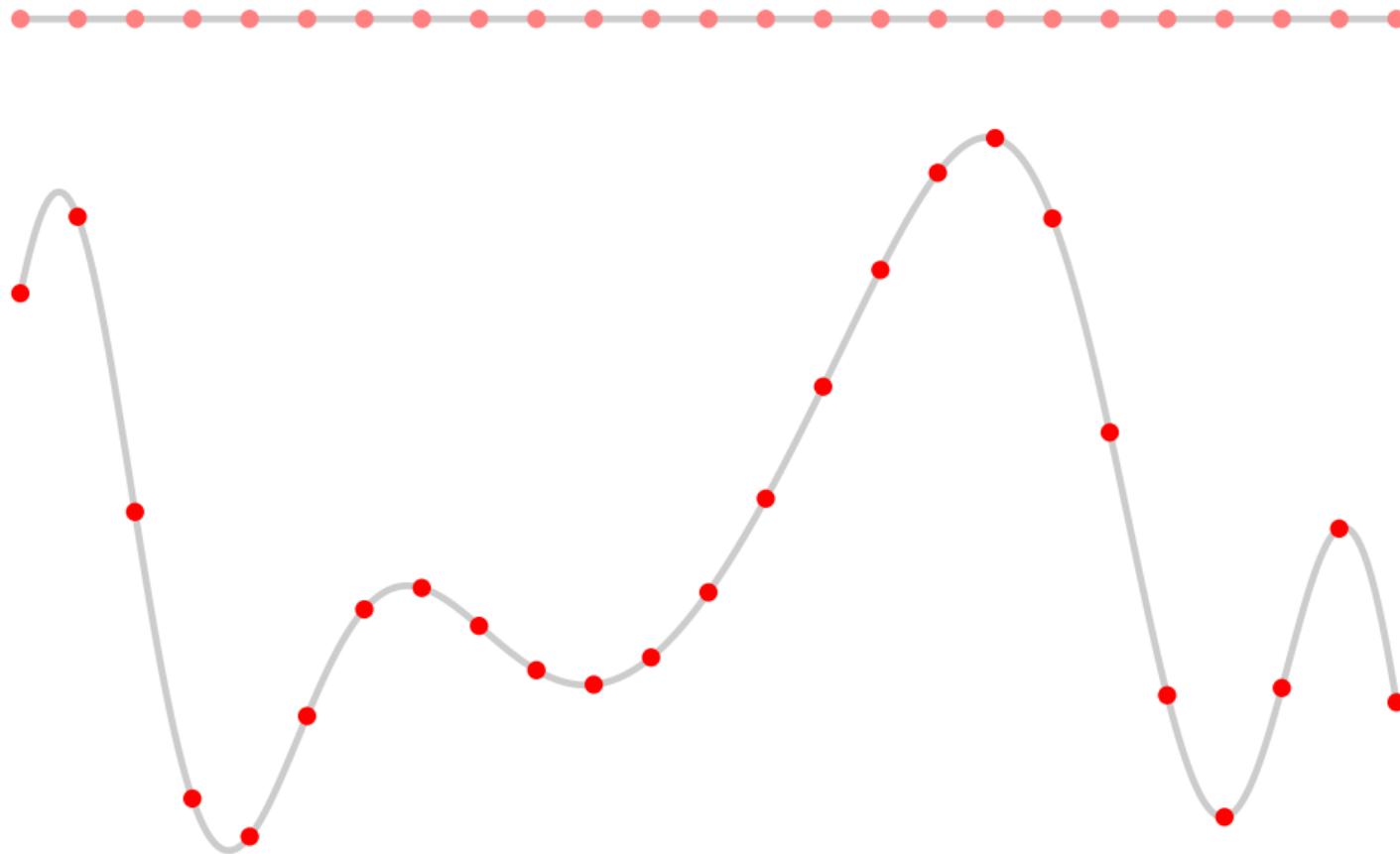
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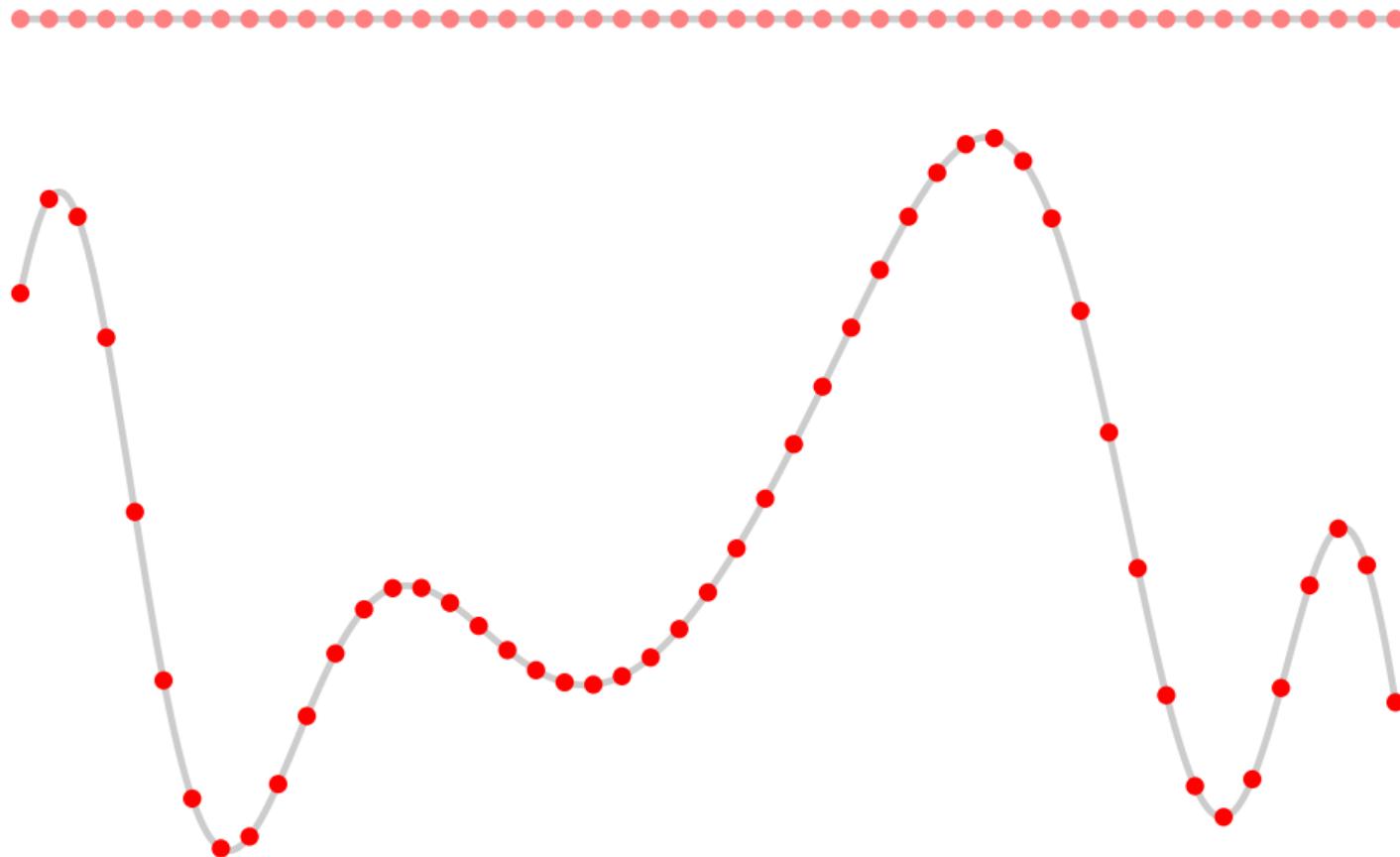
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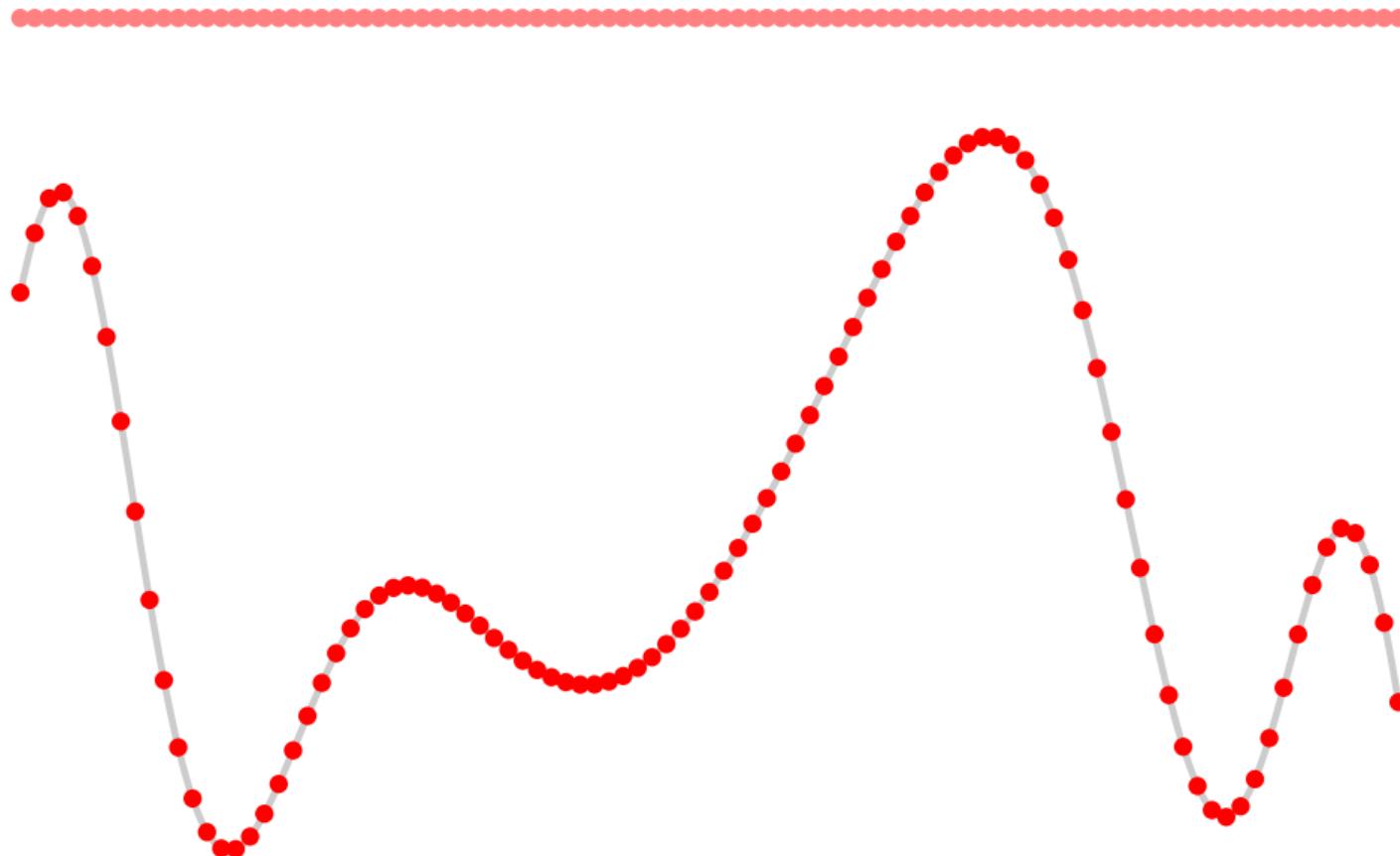
## Curva paramétrica – amostragem uniforme no domínio



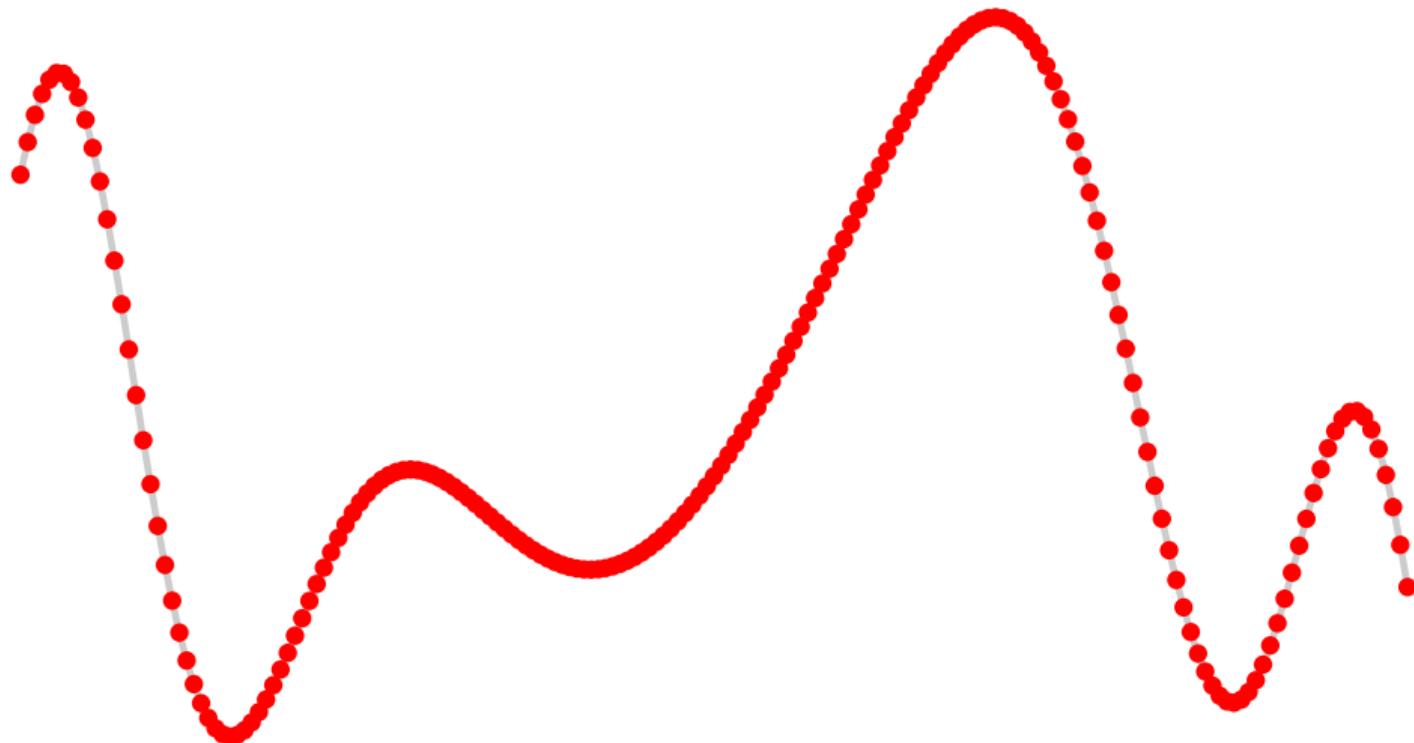
## Curva paramétrica – amostragem uniforme no domínio



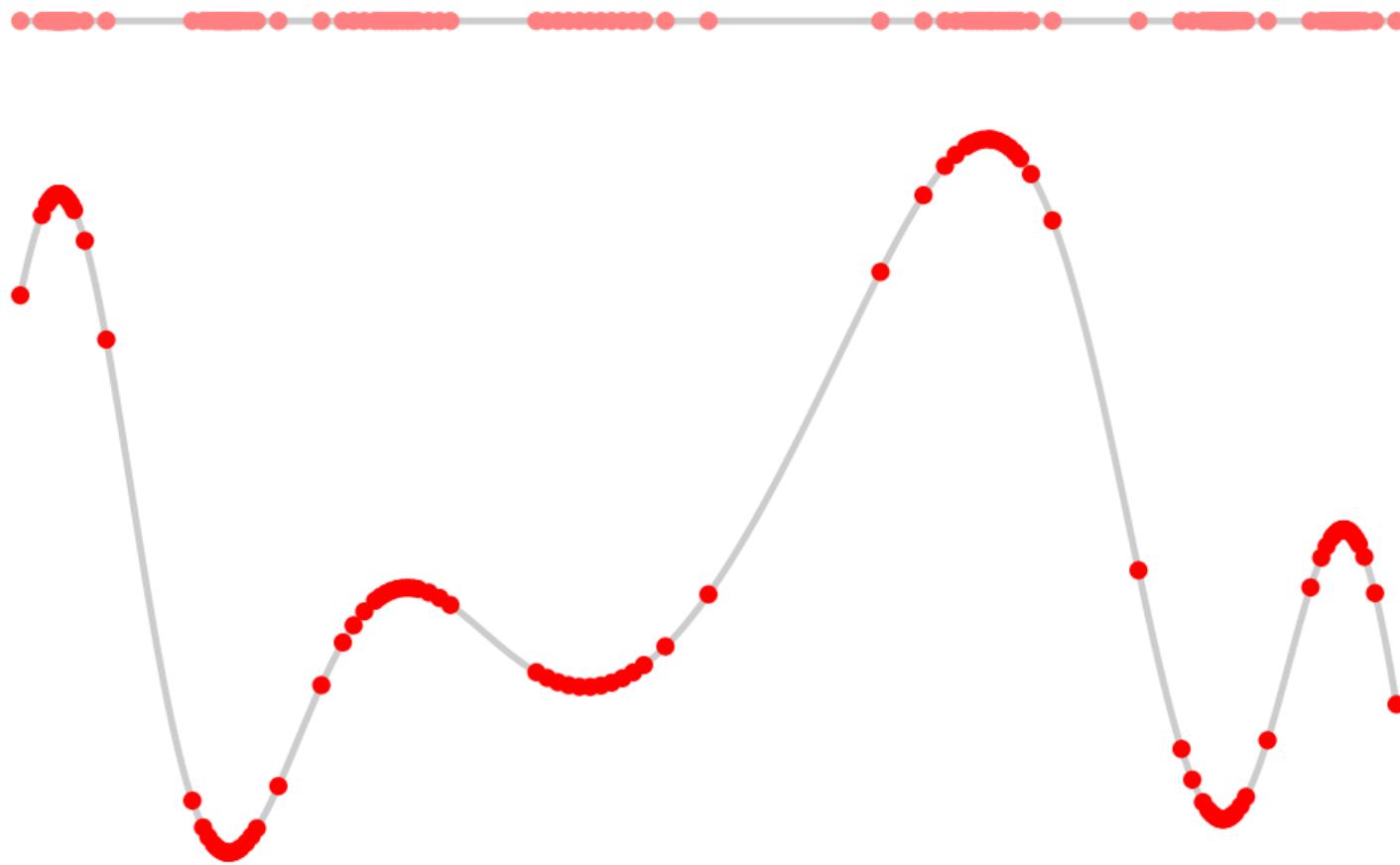
## Curva paramétrica – amostragem uniforme no domínio



## Curva paramétrica – amostragem uniforme no domínio

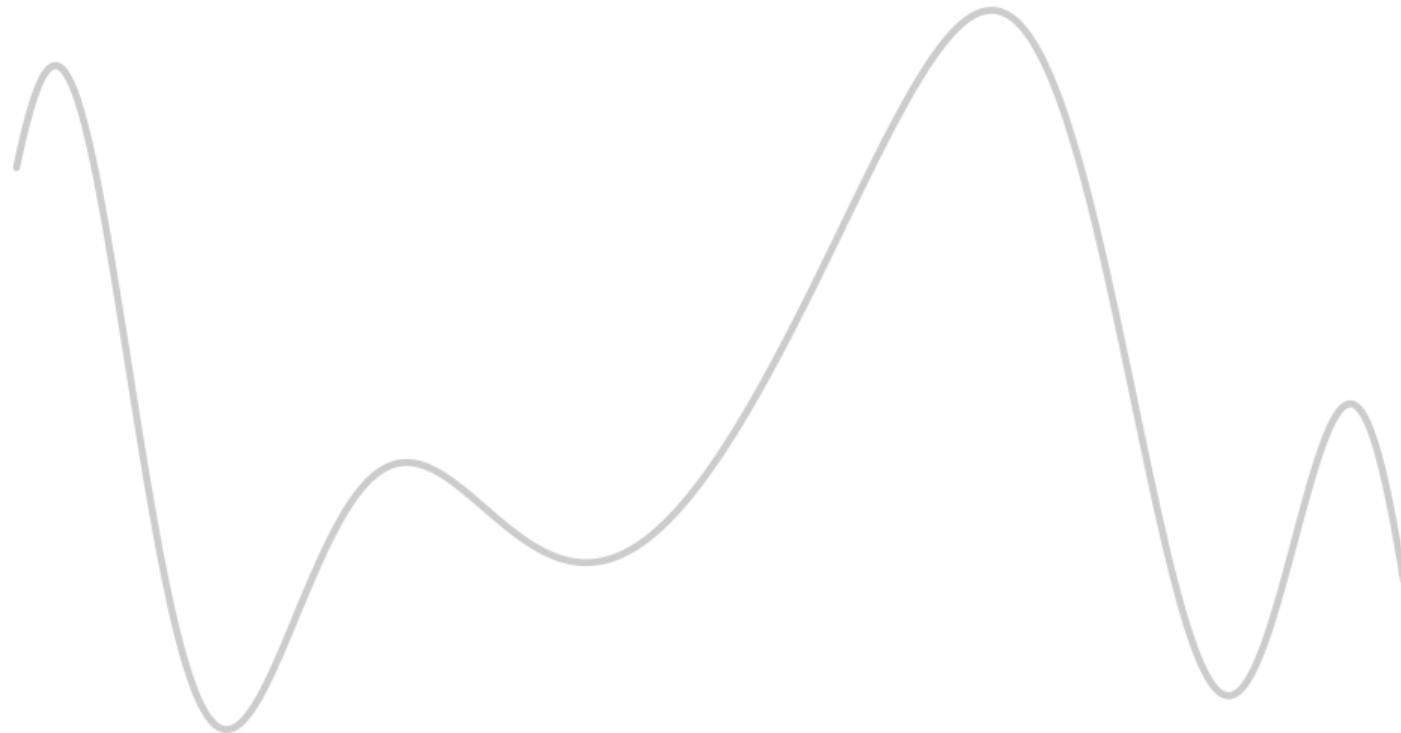


## Curva paramétrica – amostragem adaptativa

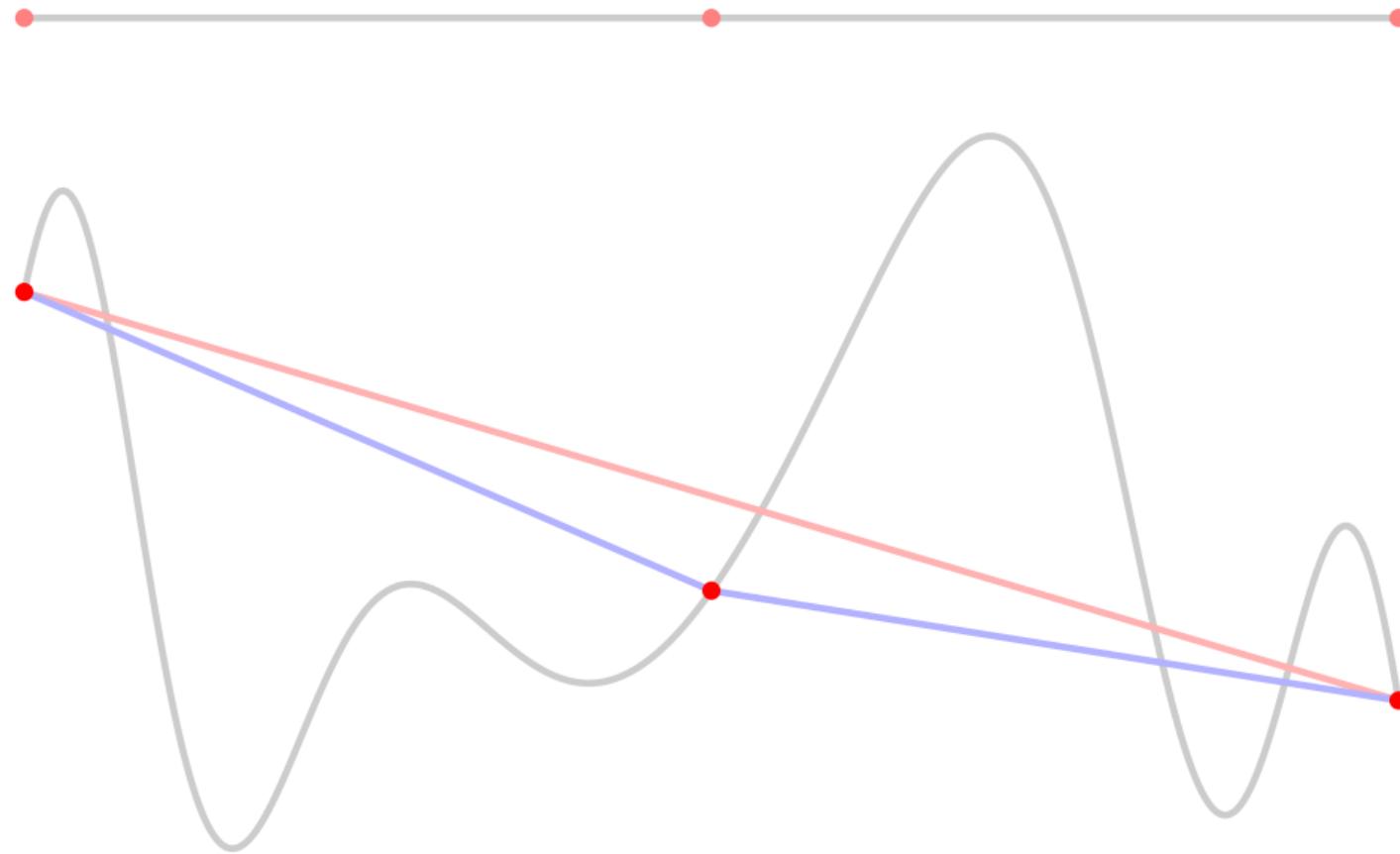


## Curva paramétrica – amostragem adaptativa

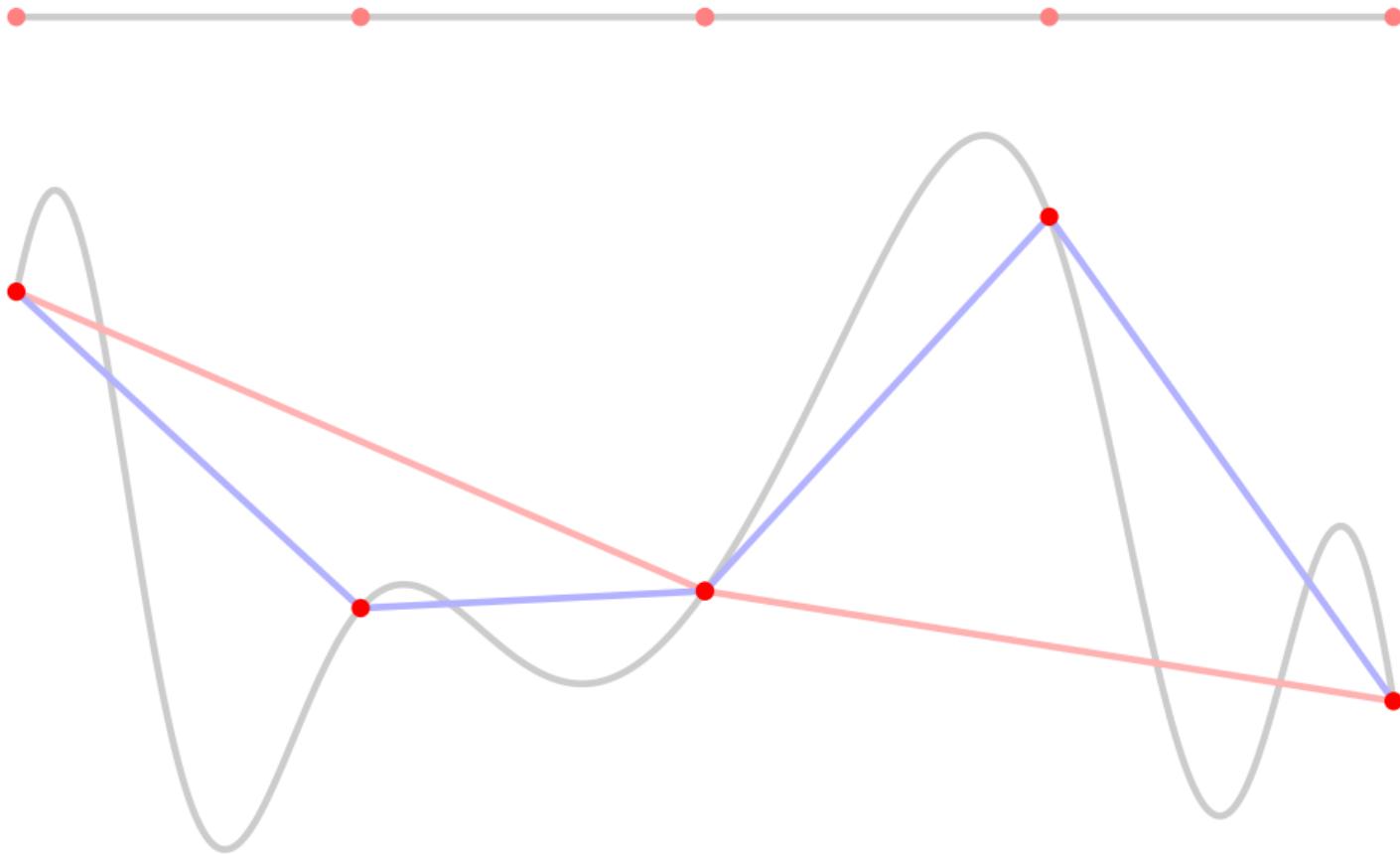
---



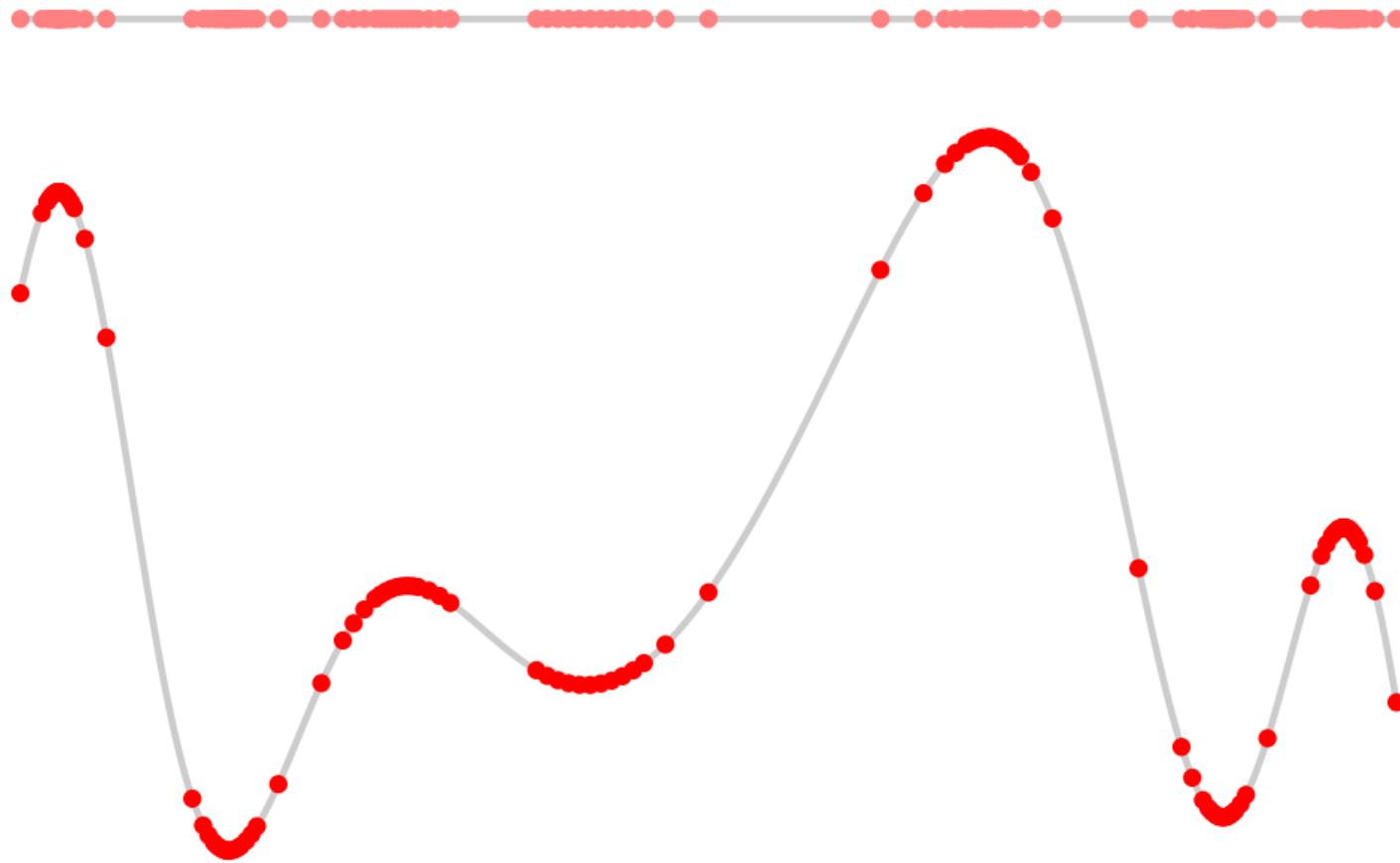
## Curva paramétrica – amostragem adaptativa



## Curva paramétrica – amostragem adaptativa



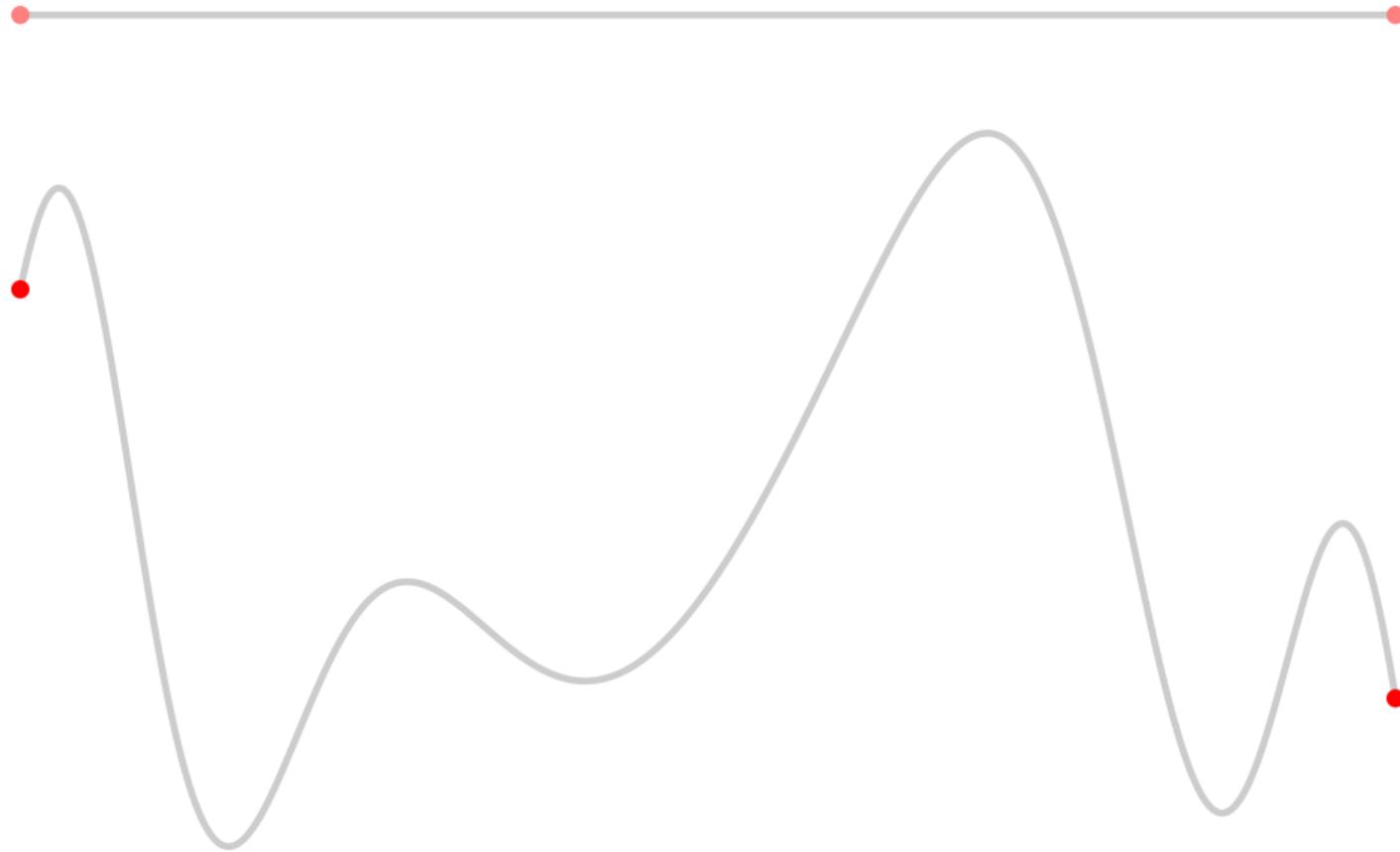
## Curva paramétrica – amostragem adaptativa



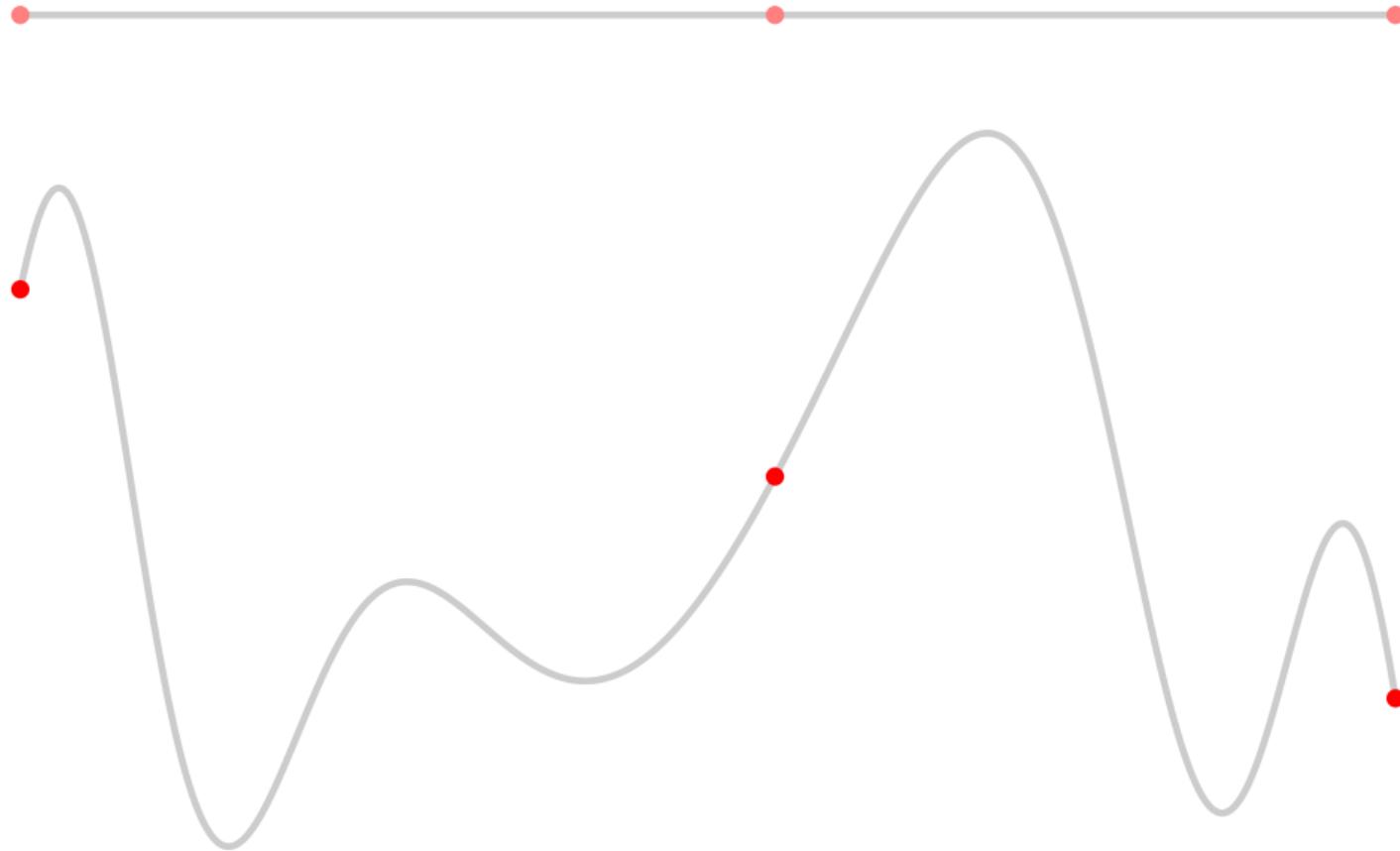
## Curva paramétrica – amostragem adaptativa

```
function sample(t0,t1)
    local tm=(t0+t1)/2
    local x0,y0=gamma(t0)
    local x1,y1=gamma(t1)
    local xm,ym=gamma(tm)
    if flat(x0,y0,x1,y1,xm,ym) then
        save(t0,x0,y0)
    else
        sample(t0,tm)
        sample(tm,t1)
    end
end
```

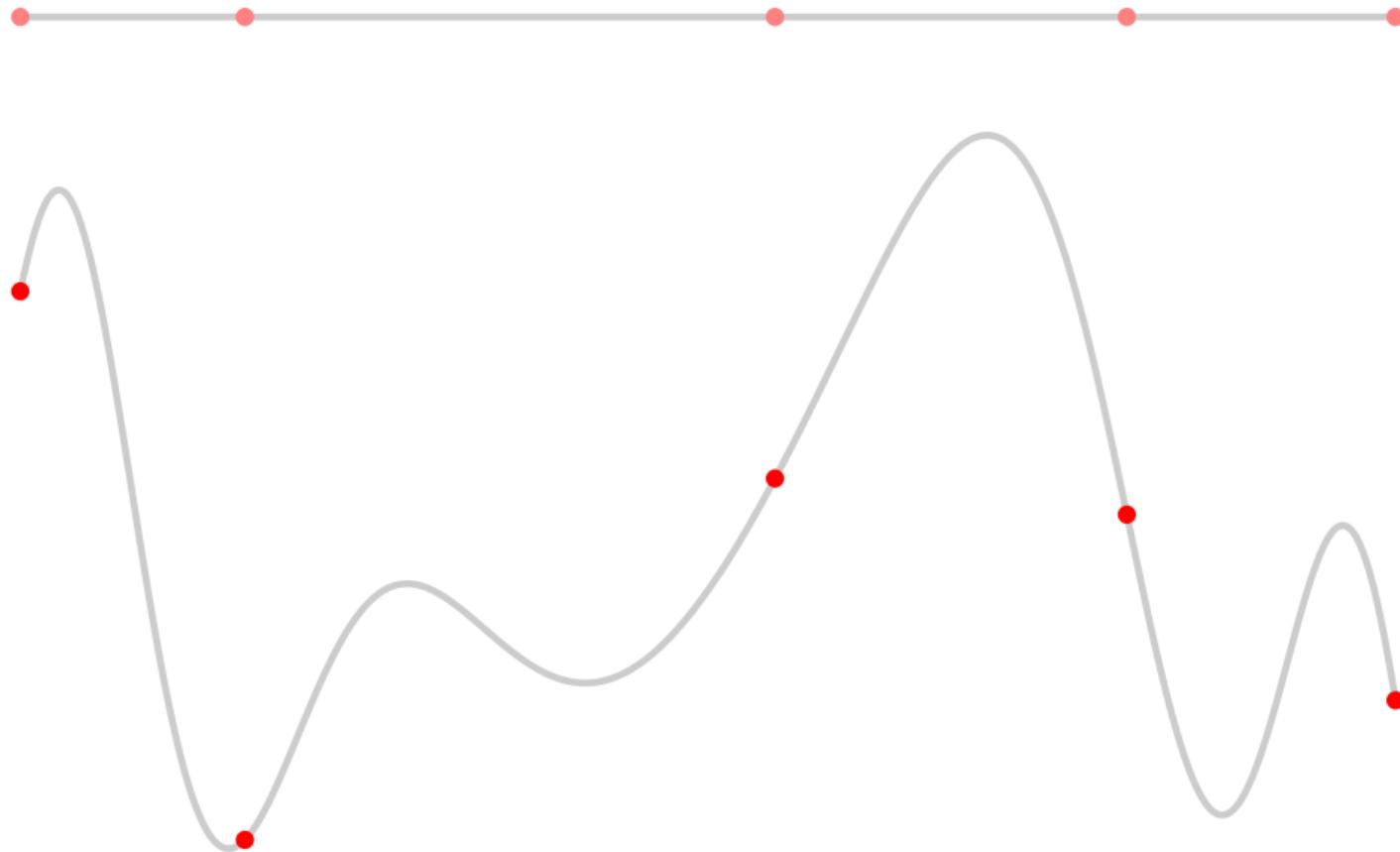
## Curva paramétrica – amostragem uniforme na curva



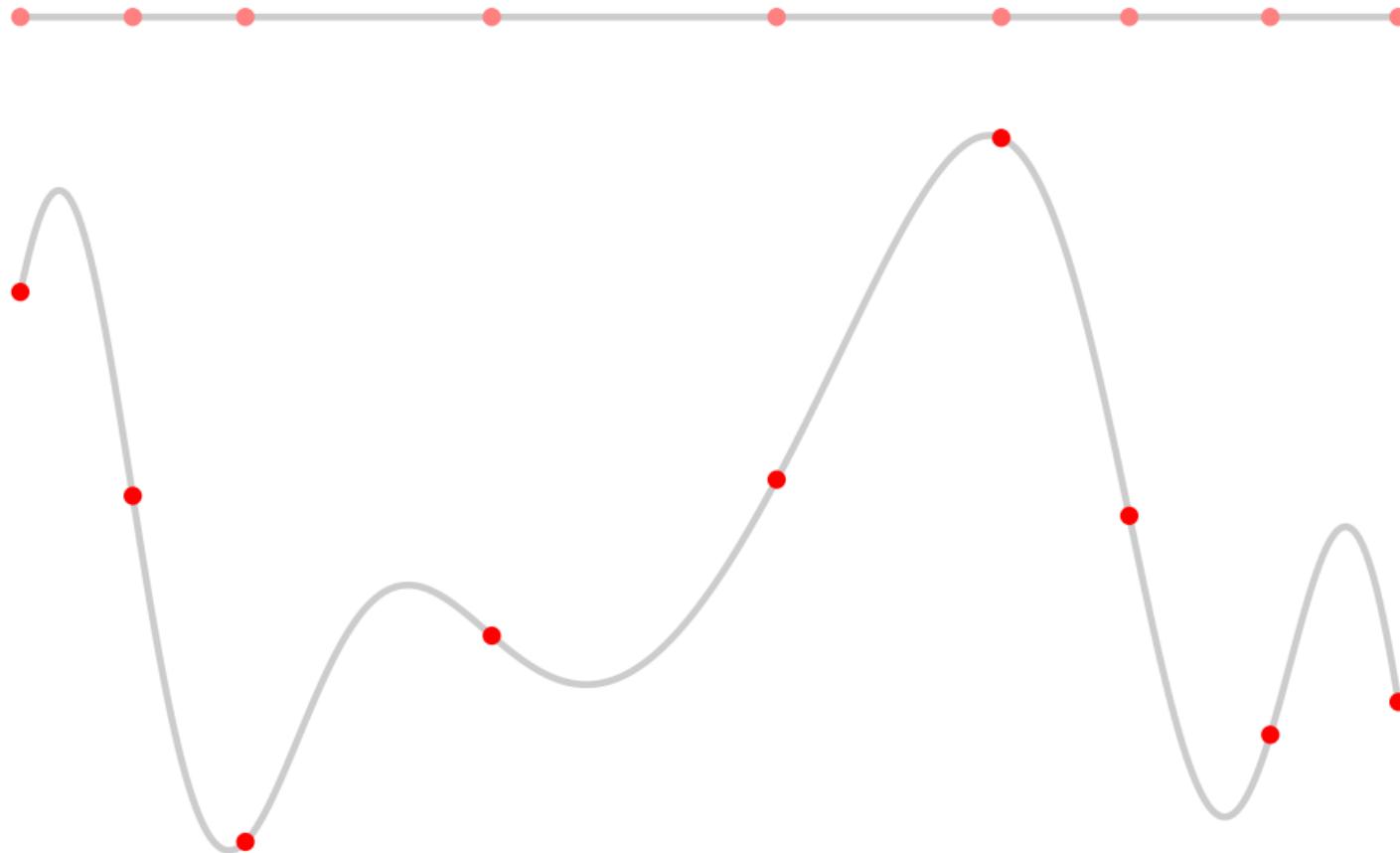
## Curva paramétrica – amostragem uniforme na curva



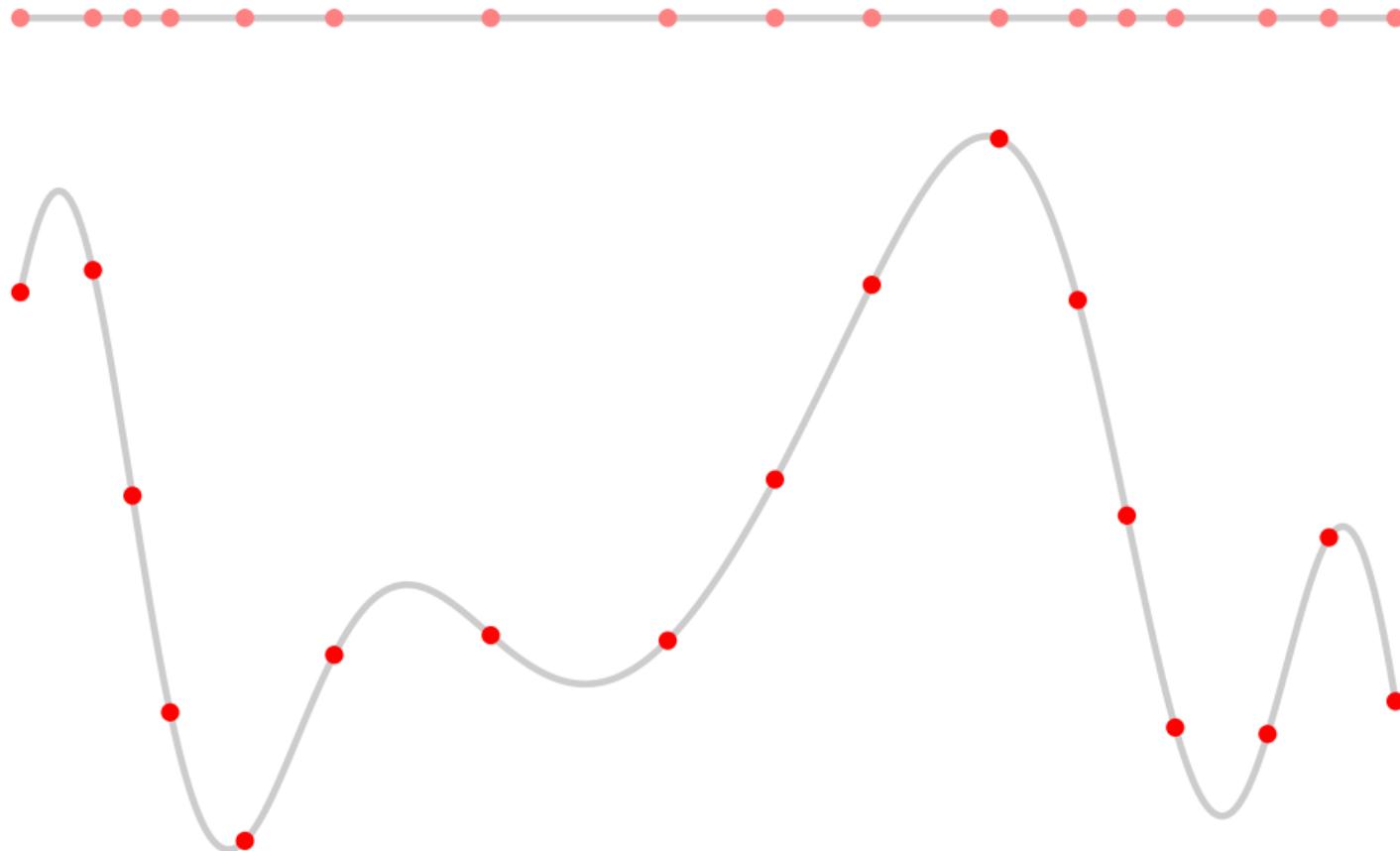
## Curva paramétrica – amostragem uniforme na curva



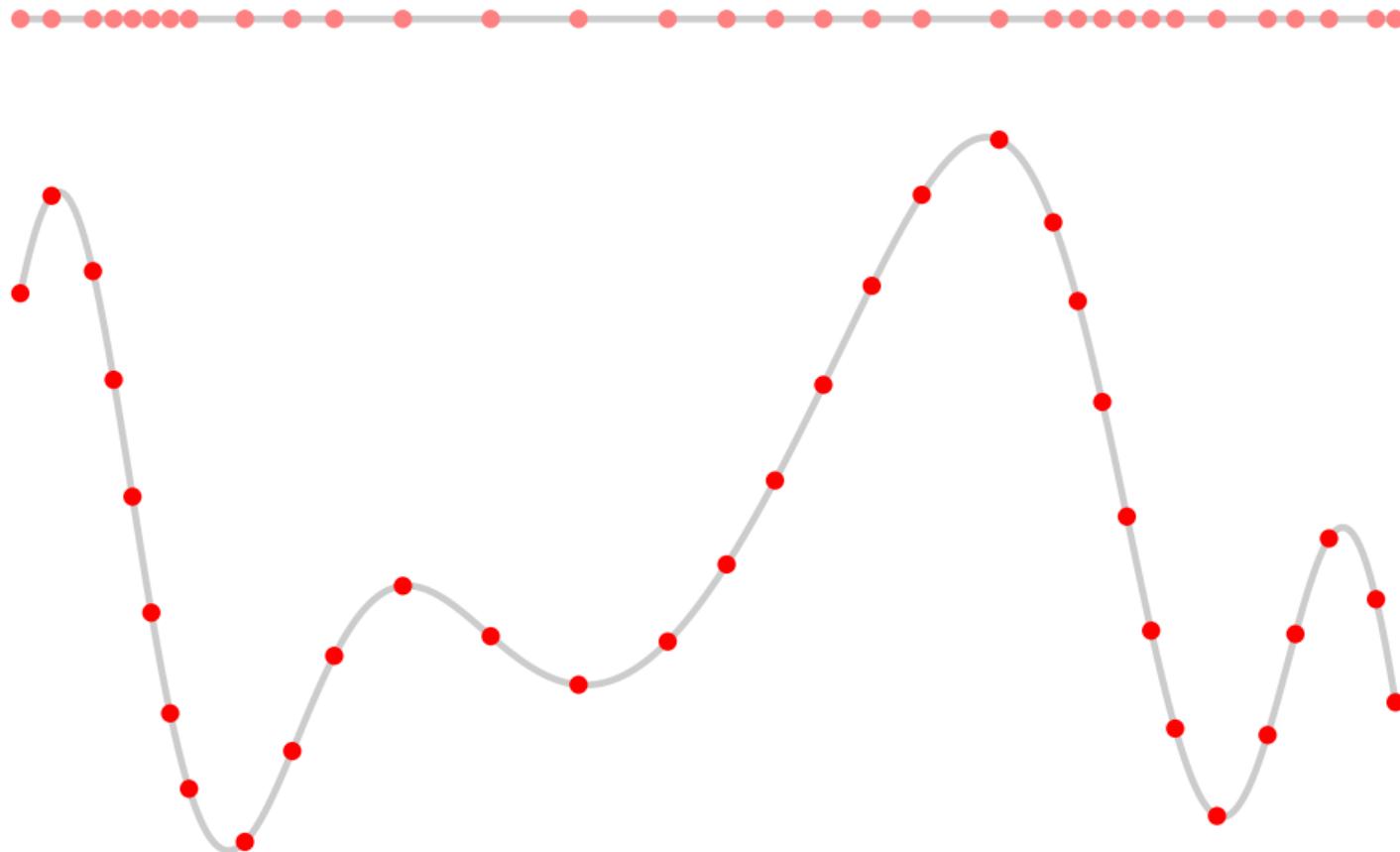
## Curva paramétrica – amostragem uniforme na curva



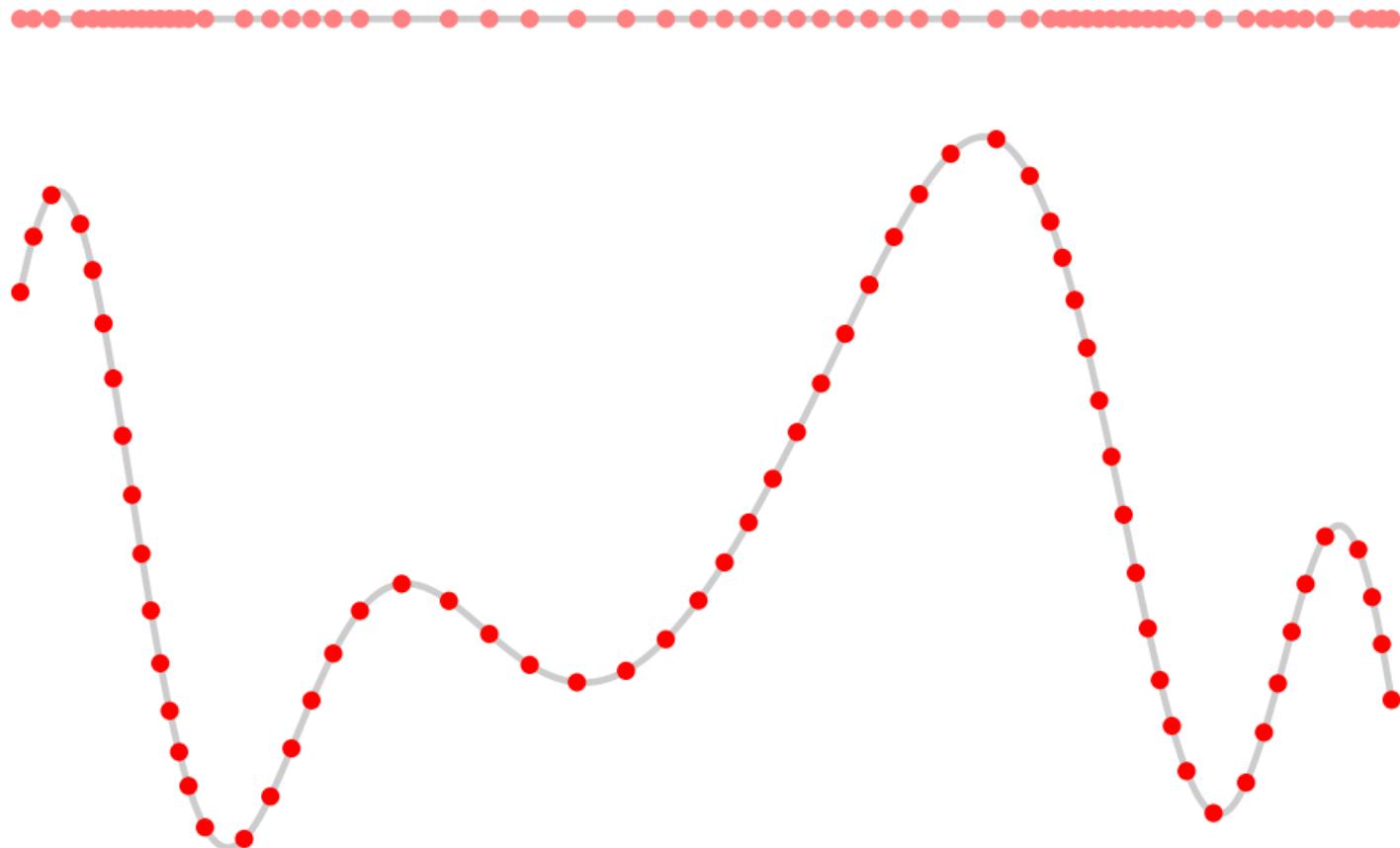
## Curva paramétrica – amostragem uniforme na curva



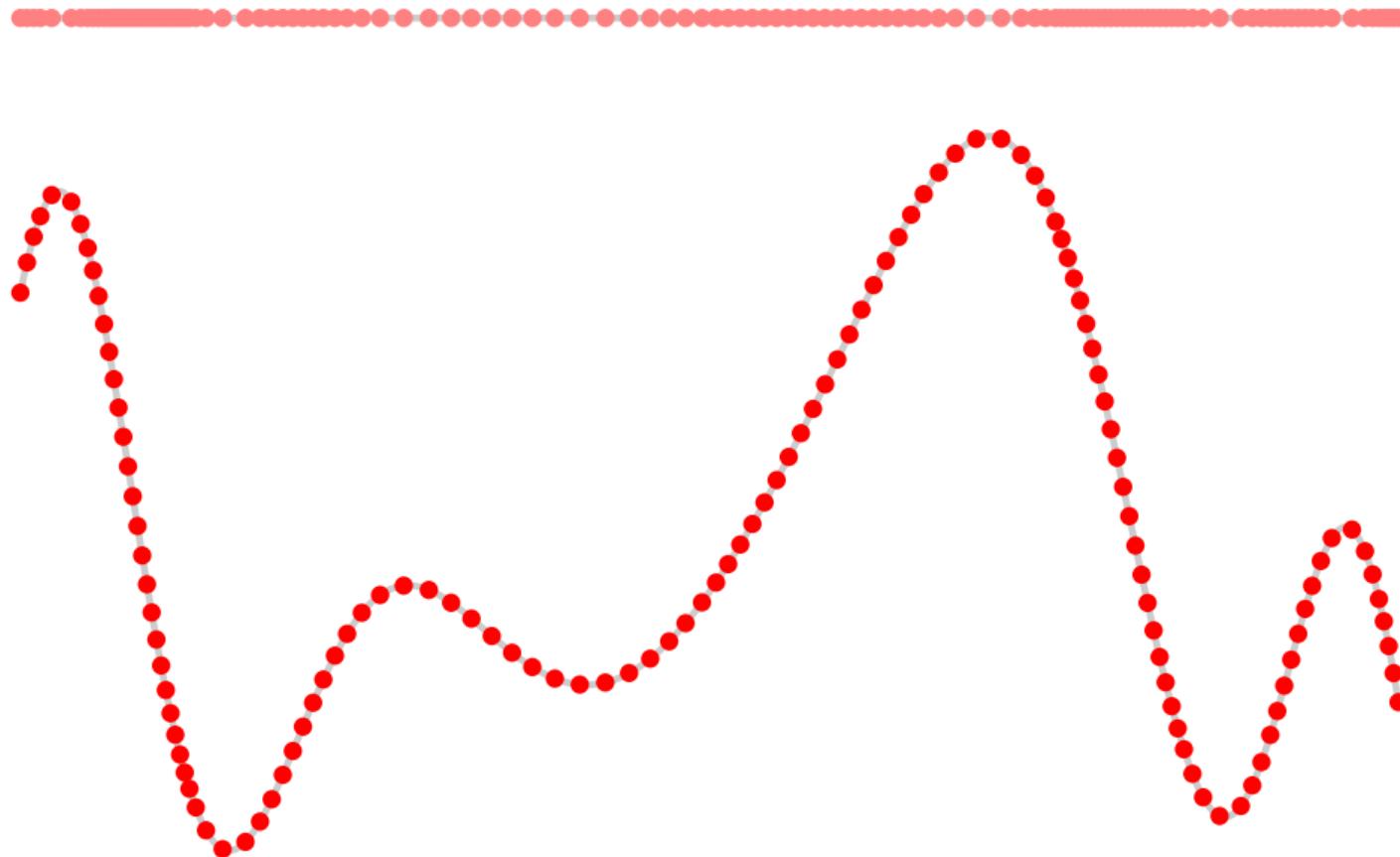
## Curva paramétrica – amostragem uniforme na curva



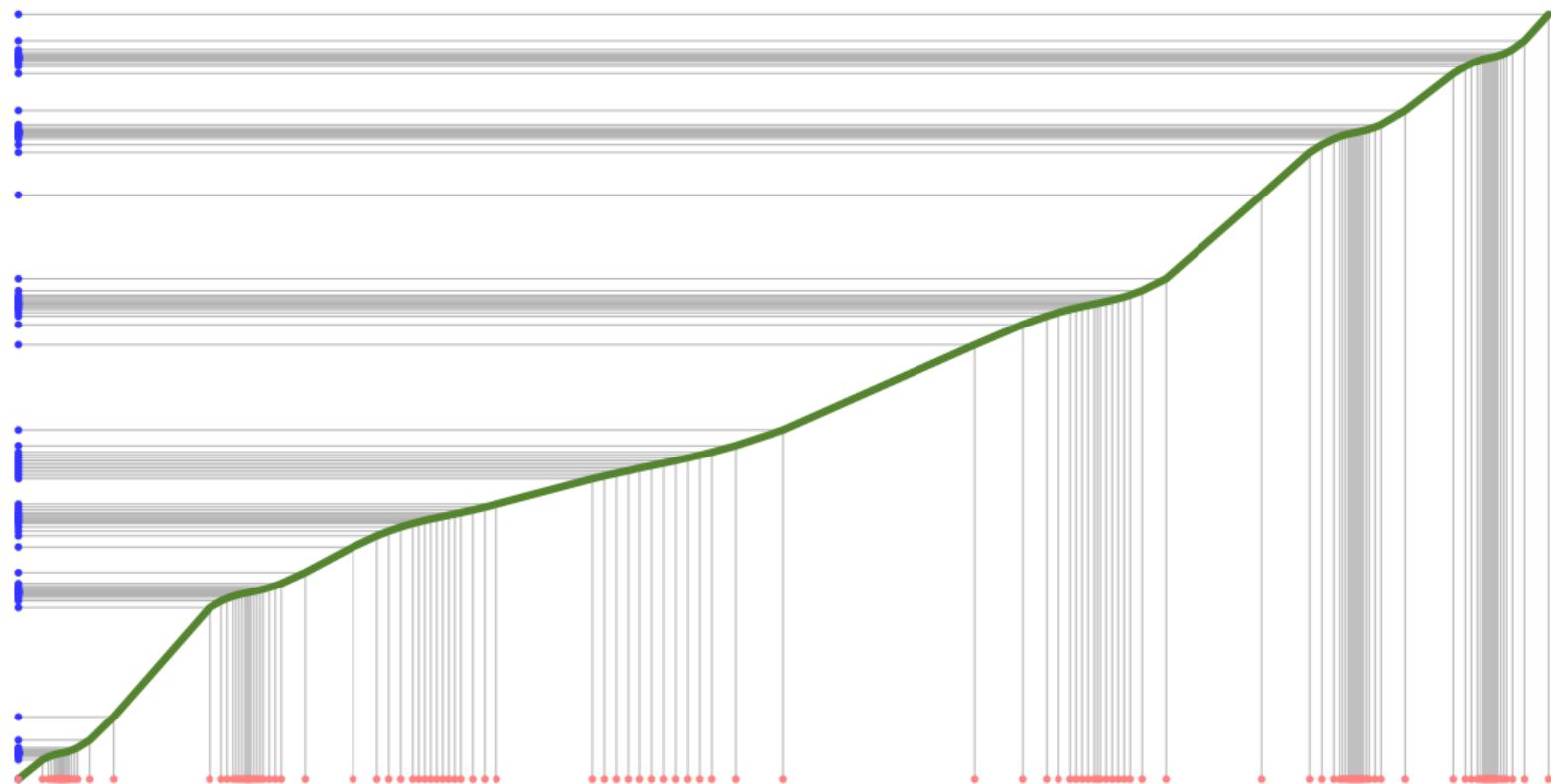
## Curva paramétrica – amostragem uniforme na curva



## Curva paramétrica – amostragem uniforme na curva



## Curva paramétrica – amostragem uniforme na curva

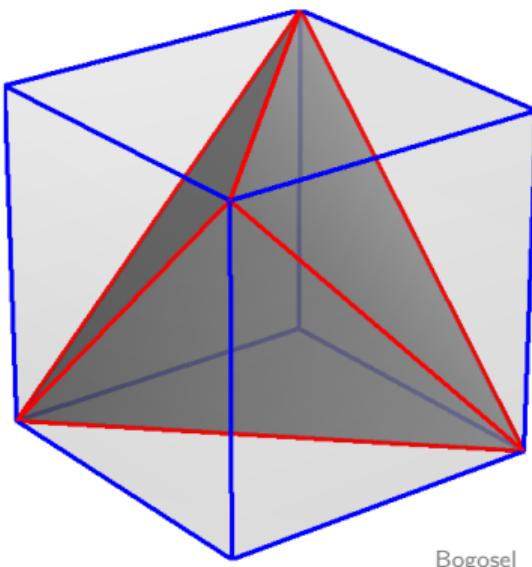


representação de malhas

# Representação de malhas: sopa de polígonos

STL

- conjunto de faces sem organização
- vértices não compartilhados
- difícil de editar
- geometria das faces

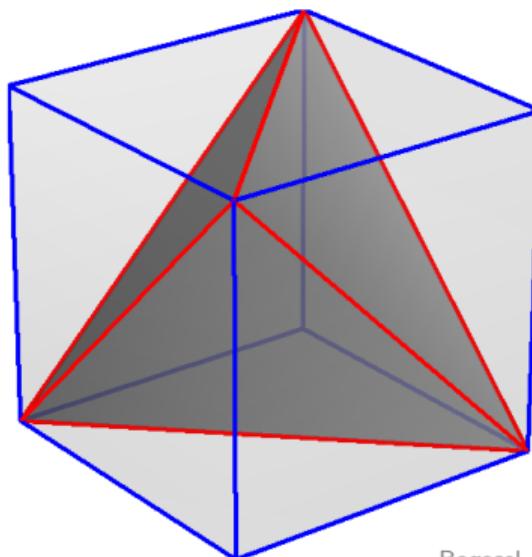


f			
-1	1	-1	
1	1	1	
1	-1	-1	
f			
-1	-1	1	
-1	1	-1	
1	-1	-1	
f			
1	-1	-1	
1	1	1	
-1	-1	1	
f			
-1	-1	1	
1	1	1	
-1	1	-1	

# Representação de malhas: geometria + topologia

OBJ

- conjunto de faces sem organização
- vértices compartilhados
- mais fácil de editar
- geometria dos vértices
- topologia das faces



Bogošel

geometria

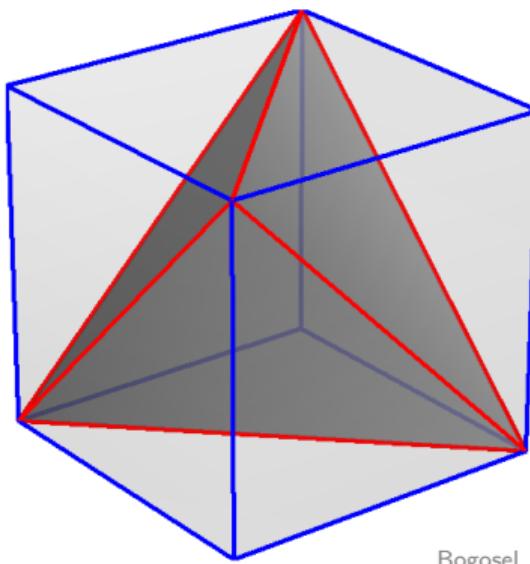
v	1	1	1
v	-1	1	-1
v	1	-1	-1
v	-1	-1	1

topologia

f	1	2	3
f	2	4	3
f	1	3	4
f	1	4	2

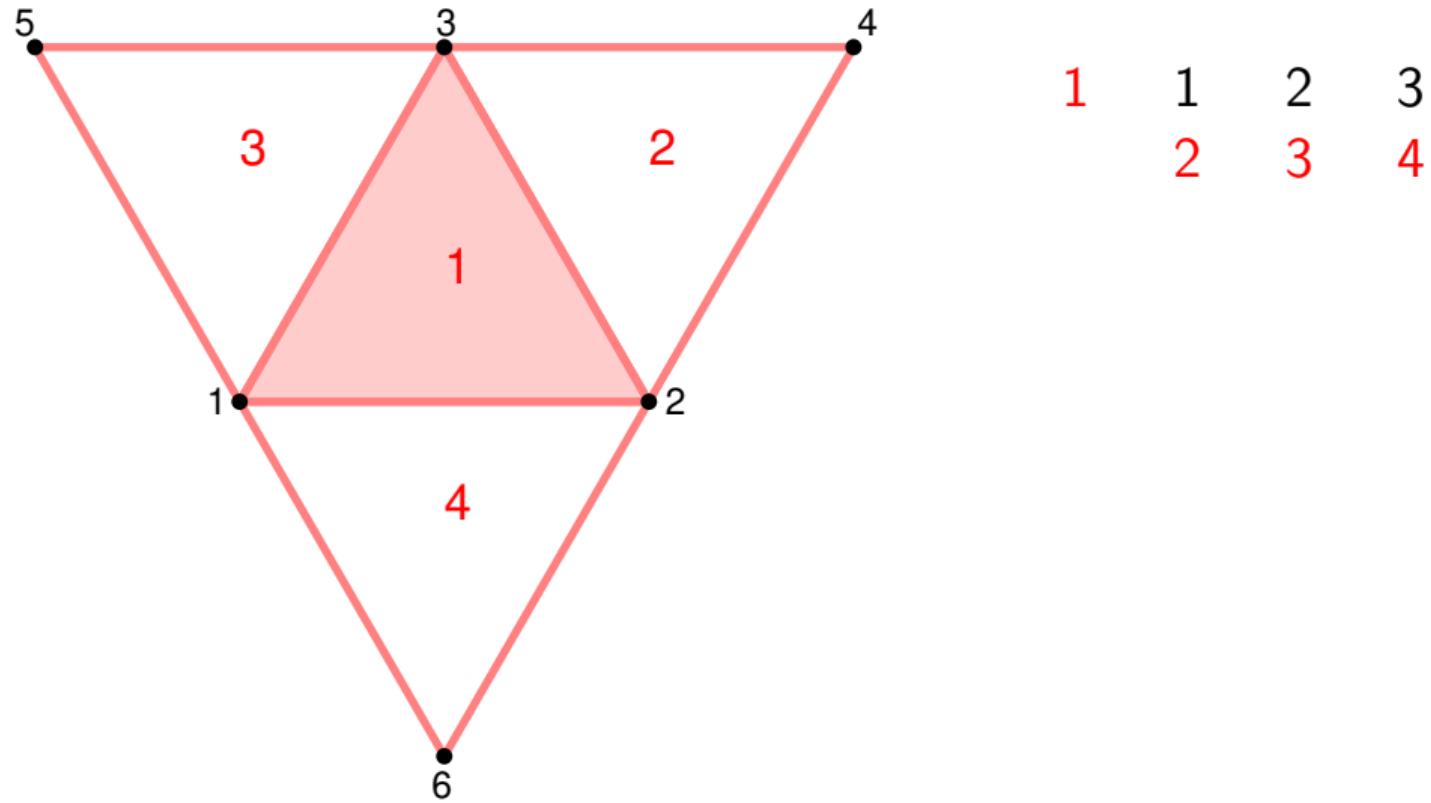
# Representação de malhas: geometria + topologia + adjacências

- conjunto de faces com organização
- vértices compartilhados
- menos fácil de editar
- geometria dos vértices
- topologia das faces

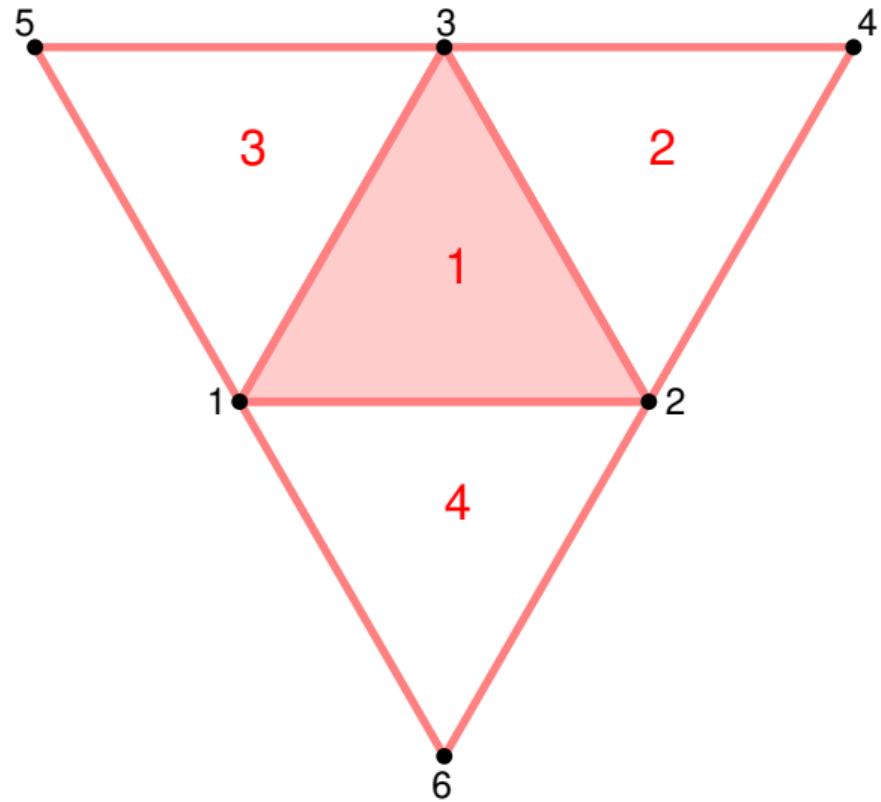


geometria				
v	1	1	1	1
v	-1	1	-1	
v	1	-1	-1	
v	-1	-1	1	
topologia				
f	1	2	3	
	2	3	4	
f	2	4	3	
	3	1	4	
f	1	3	4	
	2	4	1	
f	1	4	2	
	2	1	3	

## Representação de malhas de triângulos



## Representação de malhas de triângulos



1	1	2	3
2	3	4	
2	2	4	3
	1		
3	1	3	5
	1		1
4	1	6	2
	1		

superfícies paramétricas

## Superfície paramétrica

superfície = fronteira da região

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

## Superfície paramétrica

superfície = fronteira da região (retalho)

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

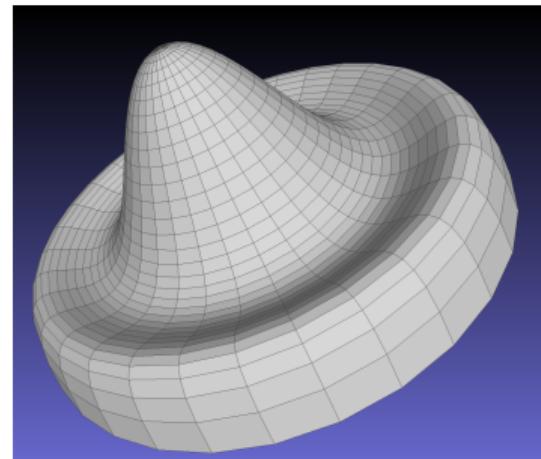
## Superfície paramétrica

superfície = fronteira da região (retalho)

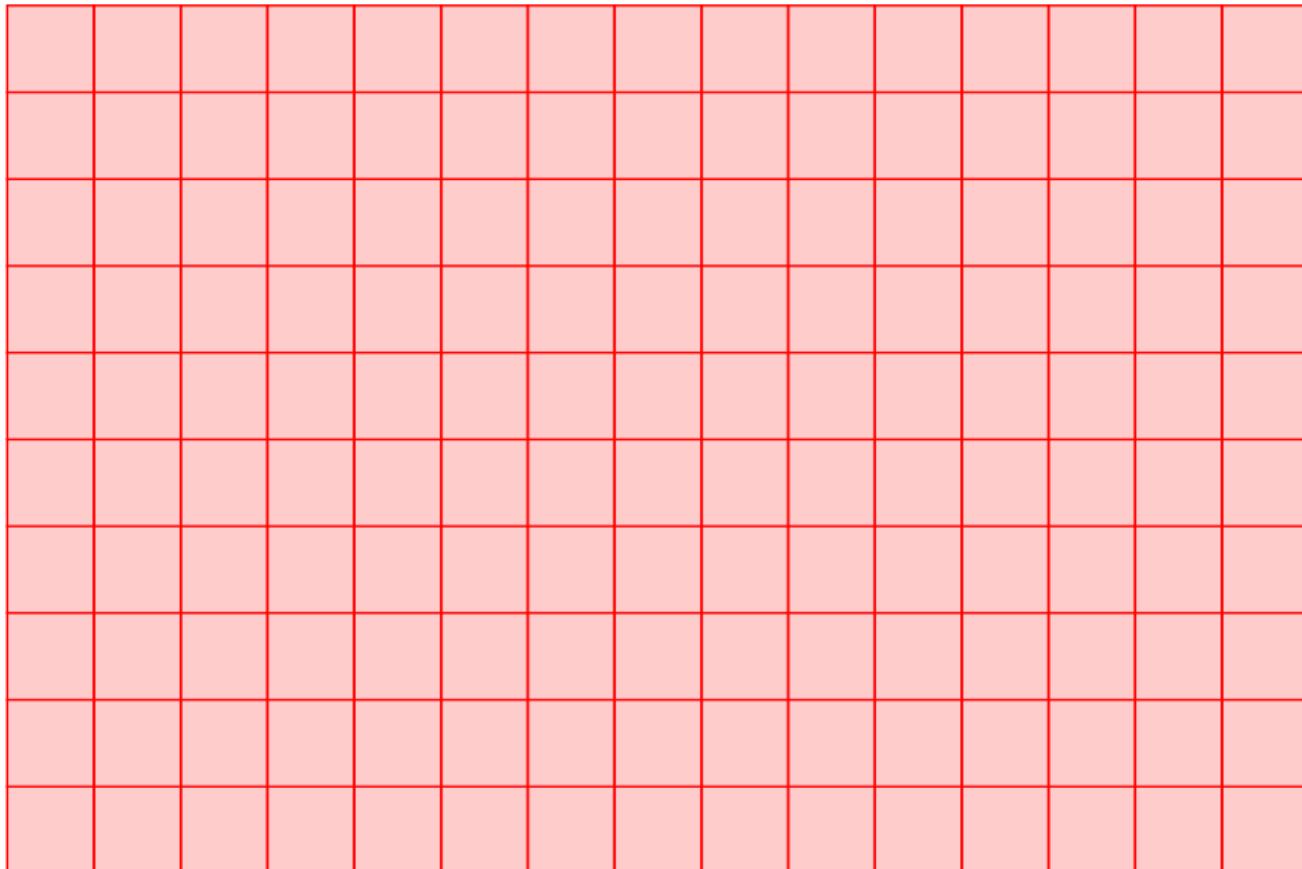
$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

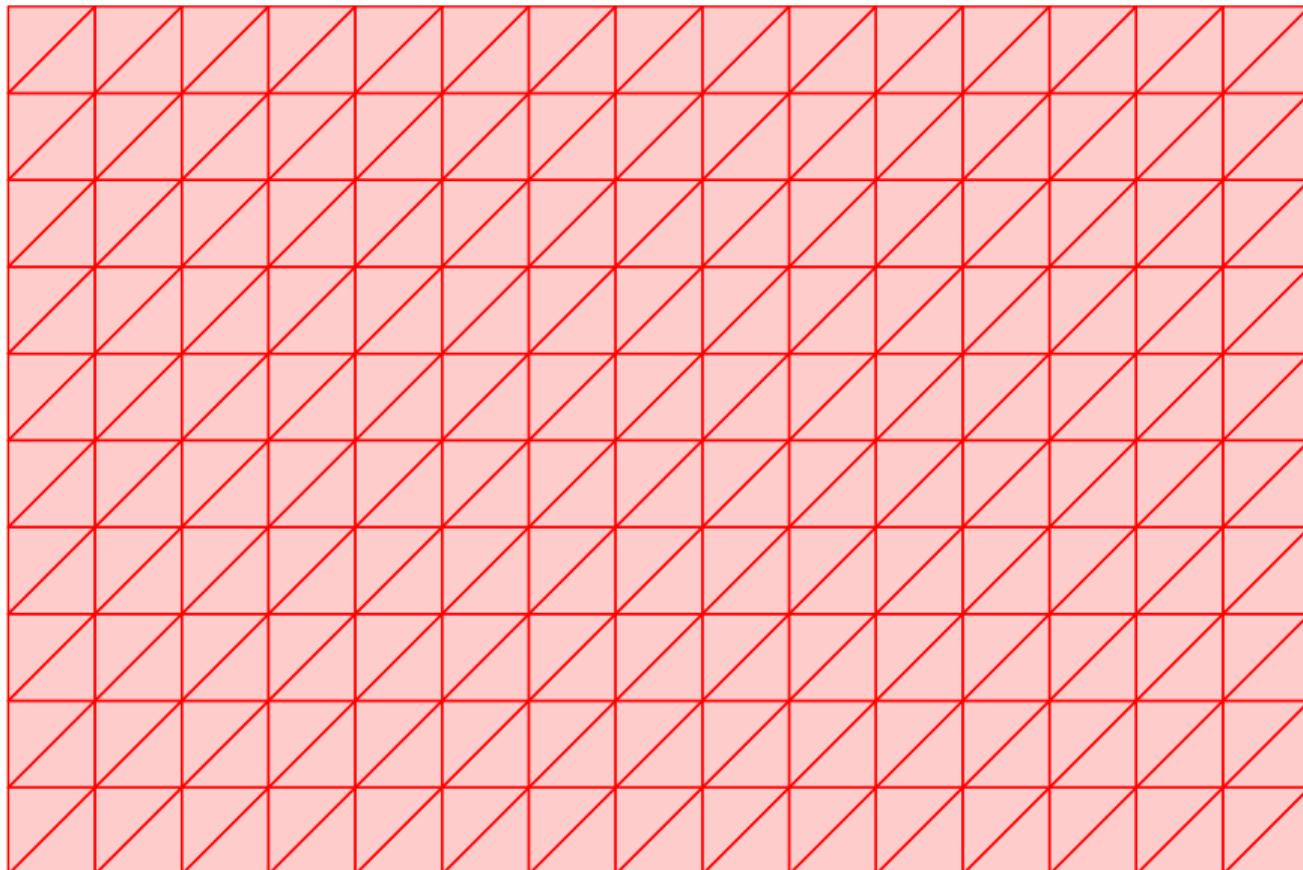
discretização da superfície = discretização do domínio



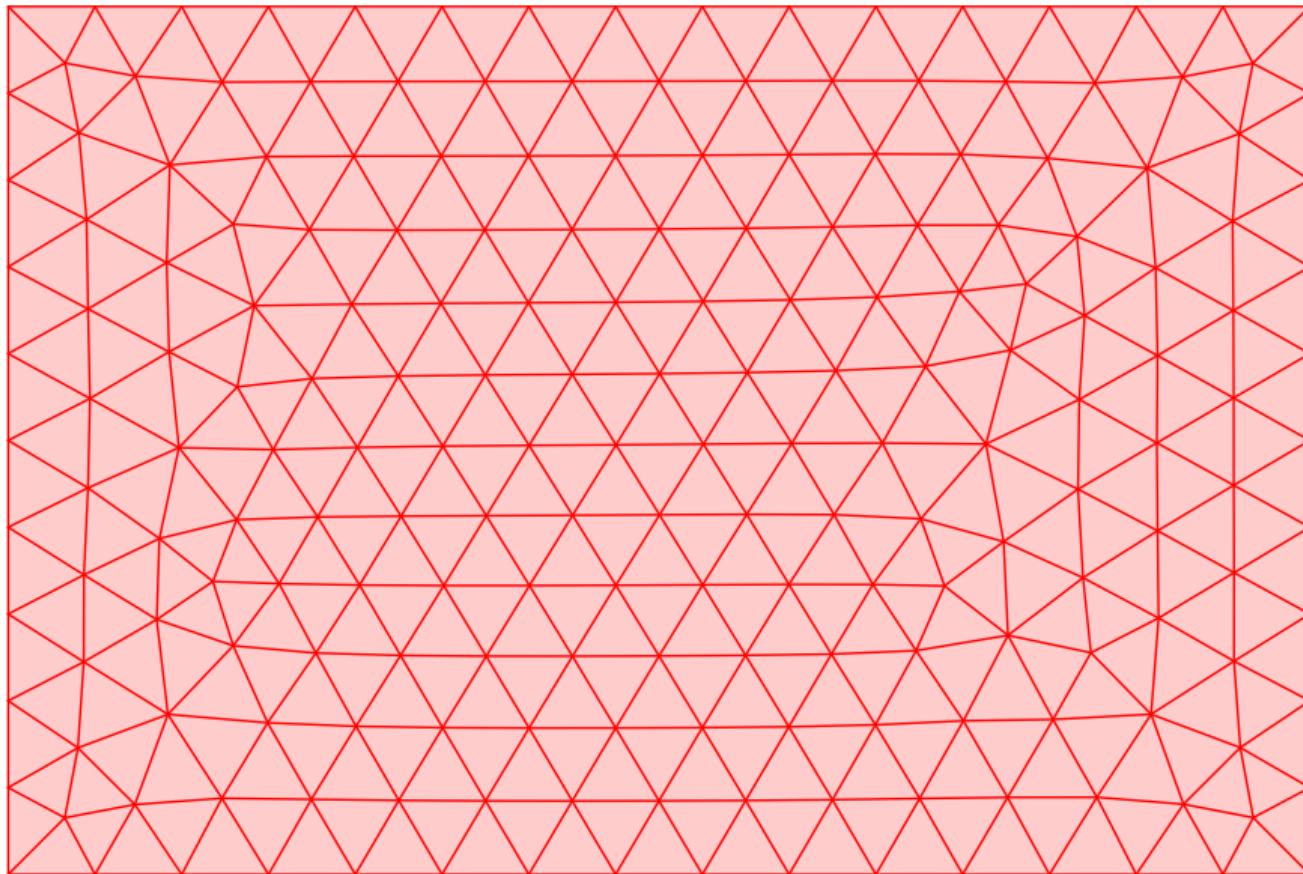
## Superfície paramétrica – discretização do domínio



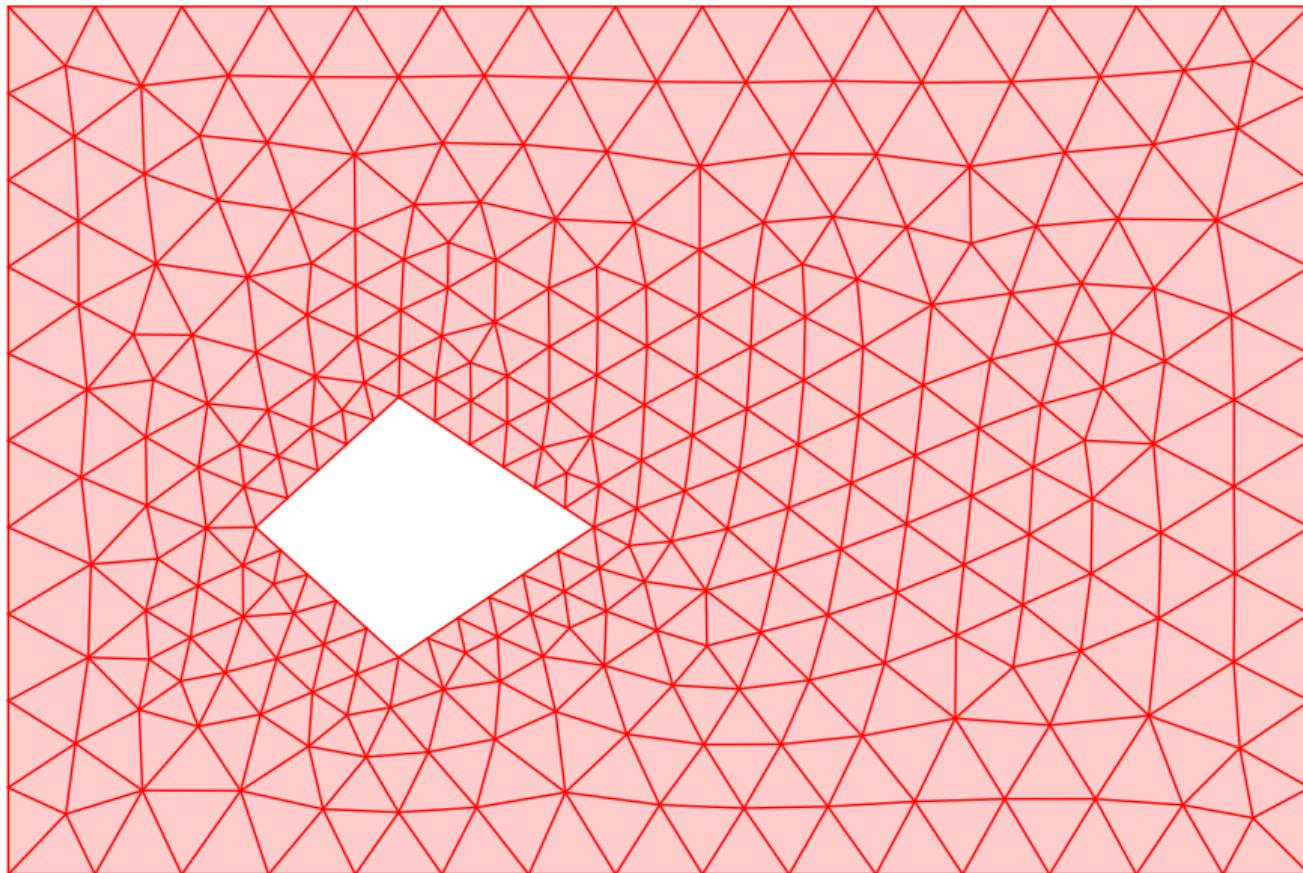
## Superfície paramétrica – discretização do domínio



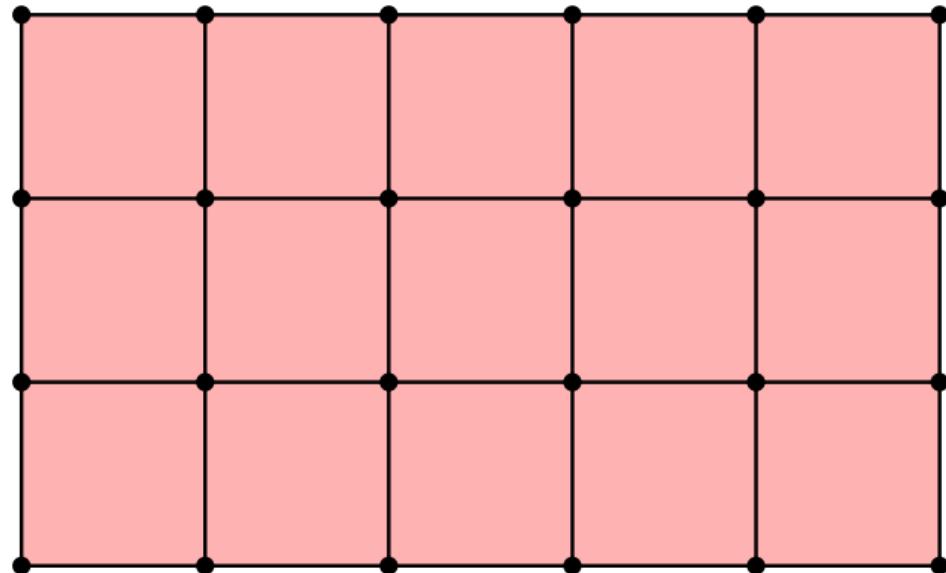
## Superfície paramétrica – discretização do domínio



## Superfície paramétrica – discretização do domínio



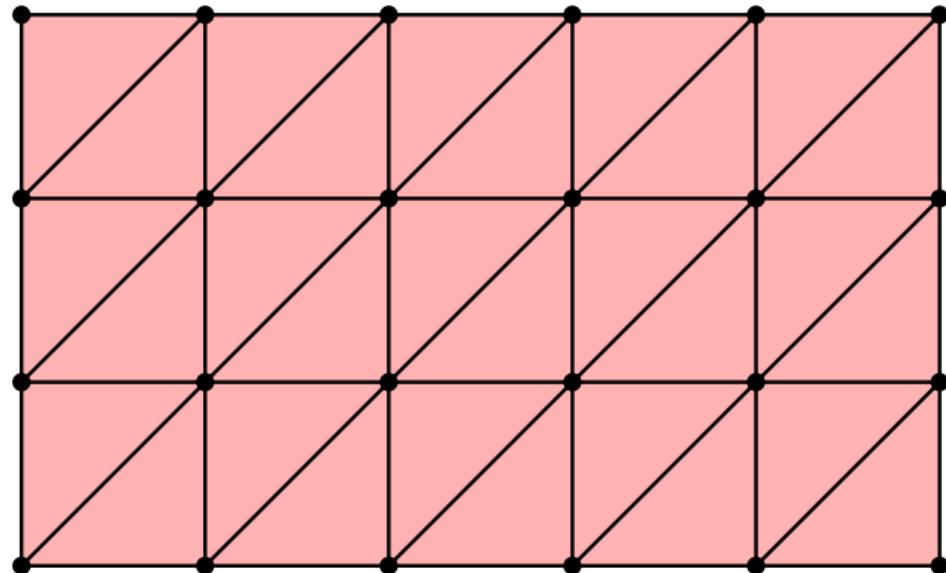
## Superfície paramétrica – representação



19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

f	1	2	8	7
f	2	3	9	8
f	3	4	10	9
f	4	5	11	10
f	5	6	12	11
f	7	8	14	13
f	8	9	15	14
f	9	10	16	15
f	10	11	17	16
f	11	12	18	17
f	13	14	20	19
f	14	15	21	20
f	15	16	22	21
f	16	17	23	22
f	17	18	24	23

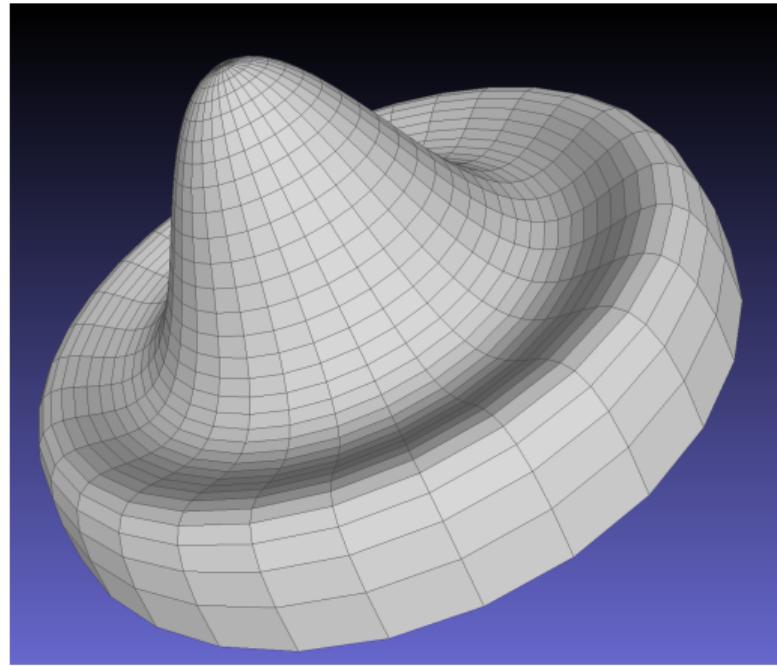
## Superfície paramétrica – representação



19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

f	1	2	8
f	1	8	7
f	2	3	9
f	2	9	8
f	3	4	10
f	3	10	9
f	4	5	11
f	4	11	10
f	5	6	12
f	5	12	11
f	7	8	14
f	7	14	13
f	8	9	15
f	8	15	14
f	...		

## Superfície de revolução

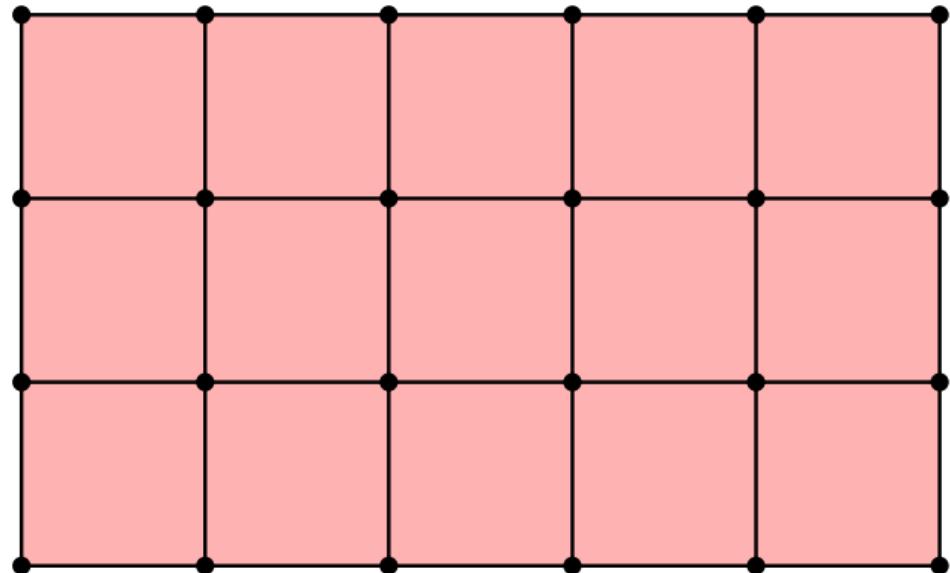


$$f(u, v) = (v \cos(u), v \sin(u), g(v))$$
$$u \in [0, 2\pi], v \in [0, 1.6]$$



$$g(x) = -\frac{4}{9}x^6 + \frac{17}{9}x^4 - \frac{22}{9}x^2 + 1$$

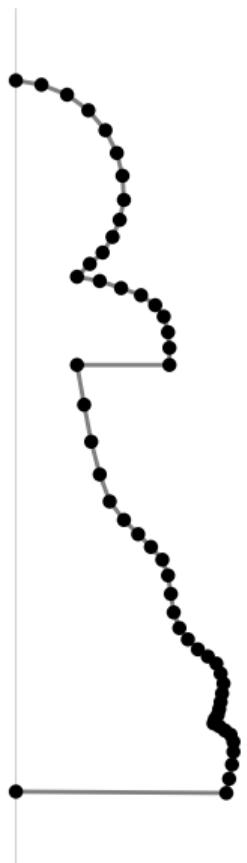
## Superfície de revolução – representação



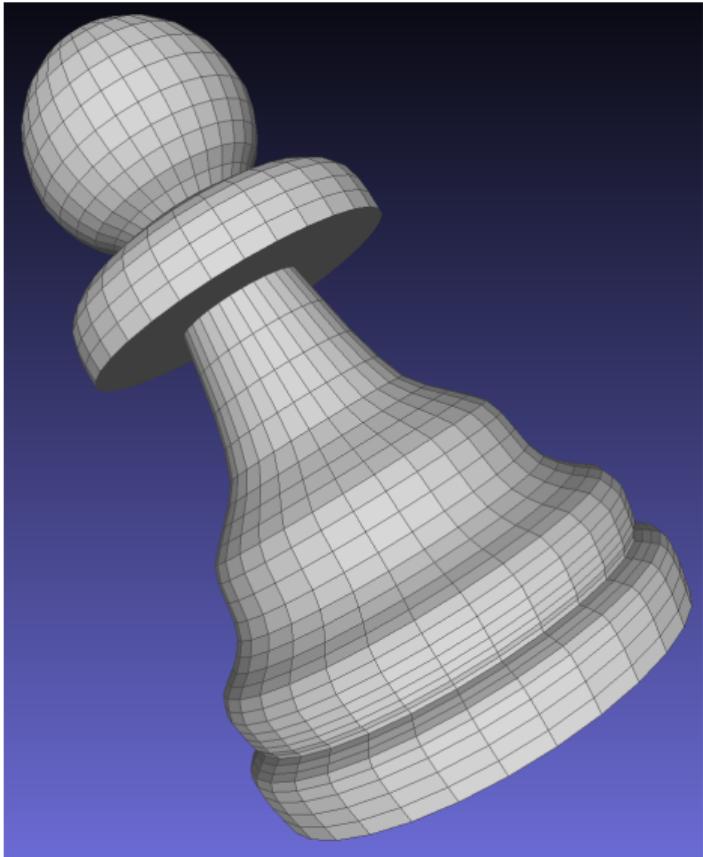
16	17	18	19	20	16
11	12	13	14	15	11
6	7	8	9	10	6
1	2	3	4	5	1

f	1	2	7	6
f	2	3	8	7
f	3	4	9	8
f	4	5	10	9
f	5	1	6	10
f	6	7	12	11
f	7	8	13	12
f	8	9	14	13
f	9	10	15	14
f	10	6	11	15
f	11	12	17	16
f	12	13	18	17
f	13	14	19	18
f	14	15	20	19
f	15	11	16	20

# Superfície de revolução



Warren–Weimer



## Retalho bilinear

- vértices do bordo

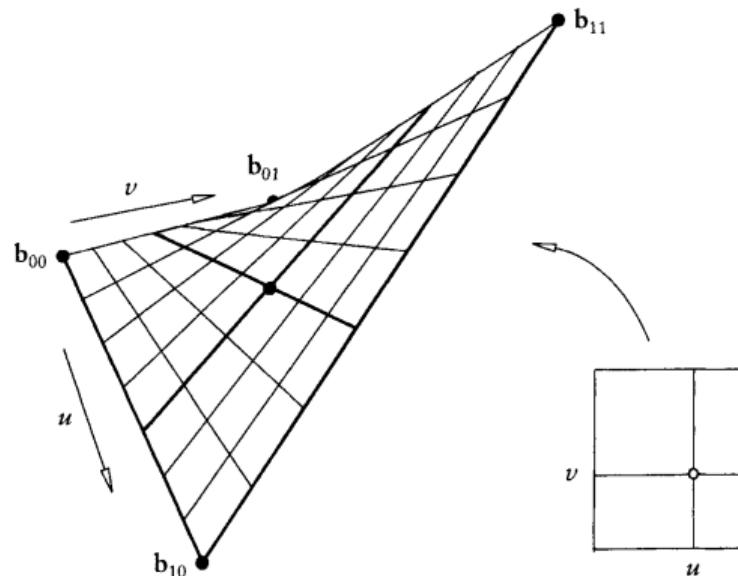
$p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$$u, v \in [0, 1]$$

- paraboloide hiperbólico



# Retalho de Coons

Coons (1967)

- curvas de bordo

$$\alpha_0, \alpha_1, \beta_0, \beta_1: [0, 1] \rightarrow \mathbf{R}^3$$

- vértices do bordo

$$p_{00}, p_{10}, p_{01}, p_{11}$$

- superfícies regradas

$$A(u, v) = (1 - v)\alpha_0(u) + v\alpha_1(u)$$

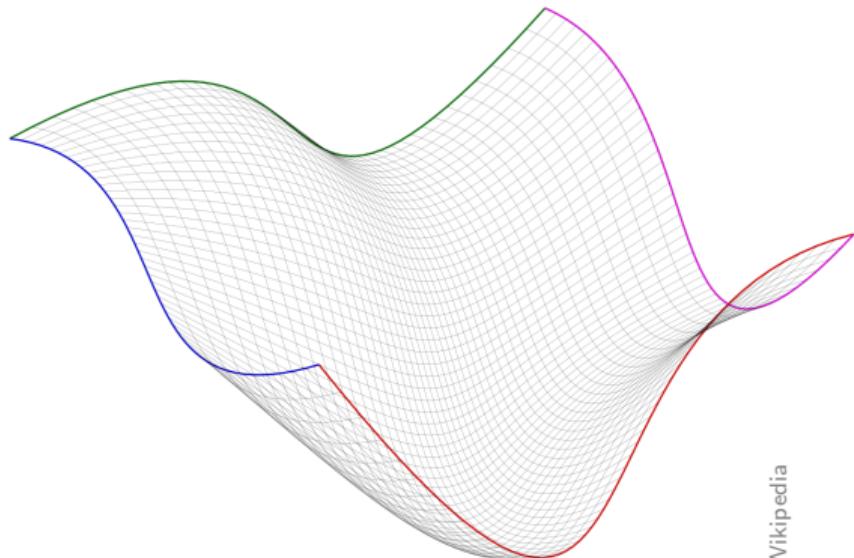
$$B(u, v) = (1 - u)\beta_0(v) + u\beta_1(v)$$

- interpolação bilinear

$$C(u, v) = (1 - u)(1 - v)p_{00} + u(1 - v)p_{10} + (1 - u)v p_{01} + uv p_{11}$$

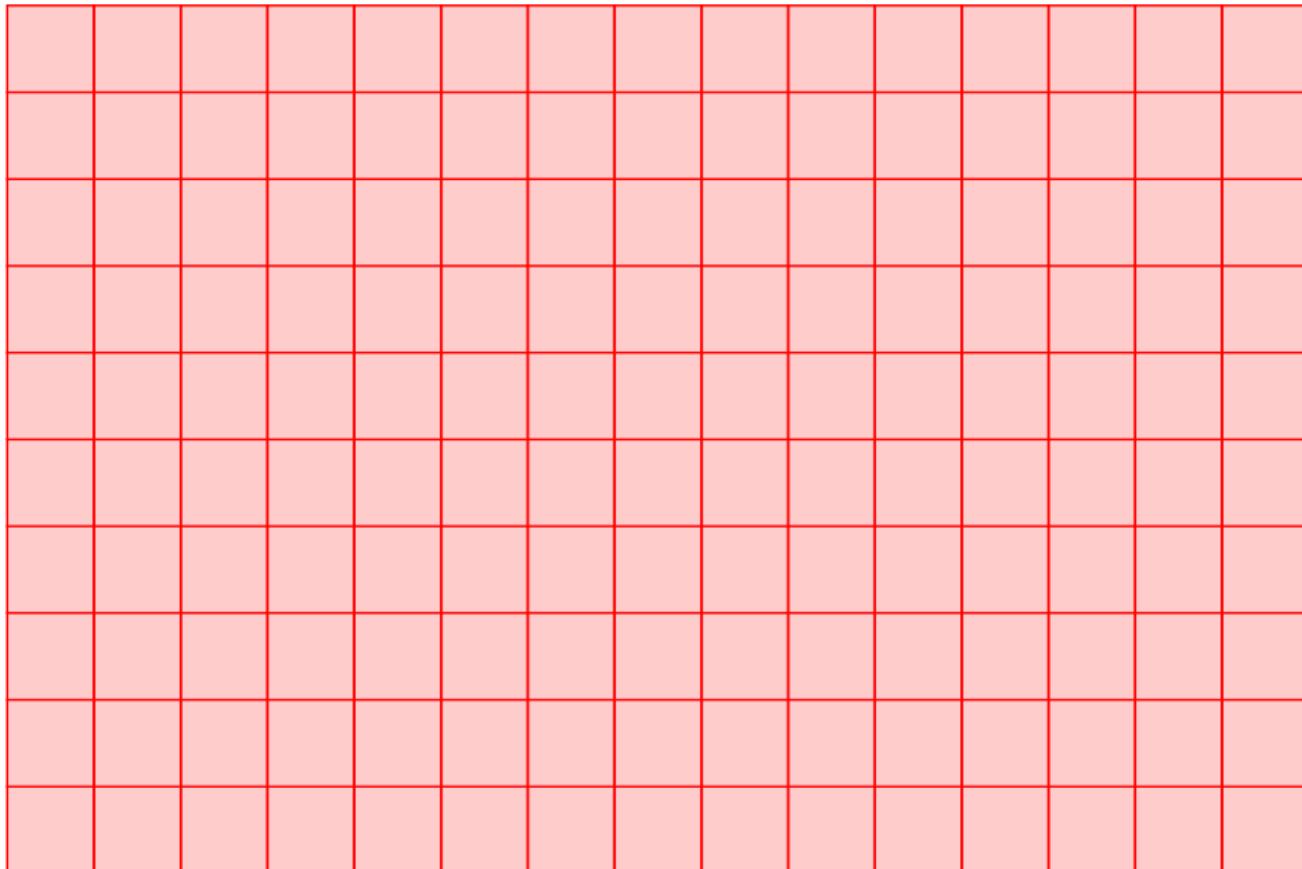
- $f(u, v) = A(u, v) + B(u, v) - C(u, v)$

$$u, v \in [0, 1]$$

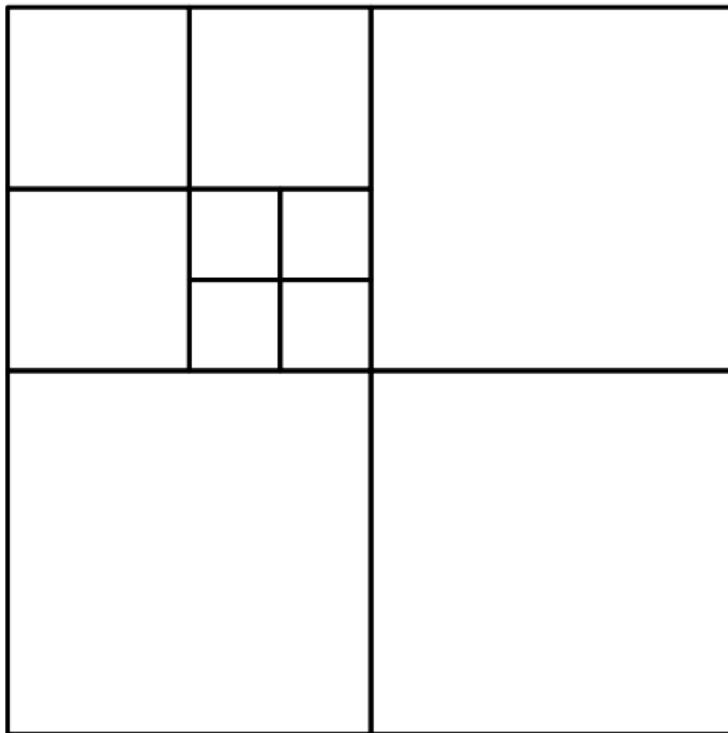


Wikipedia

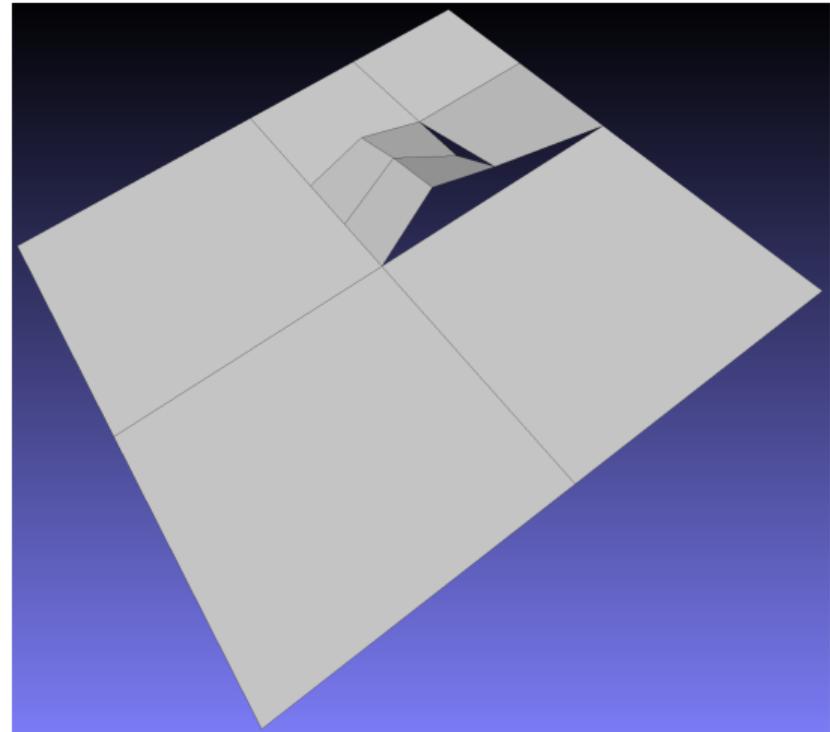
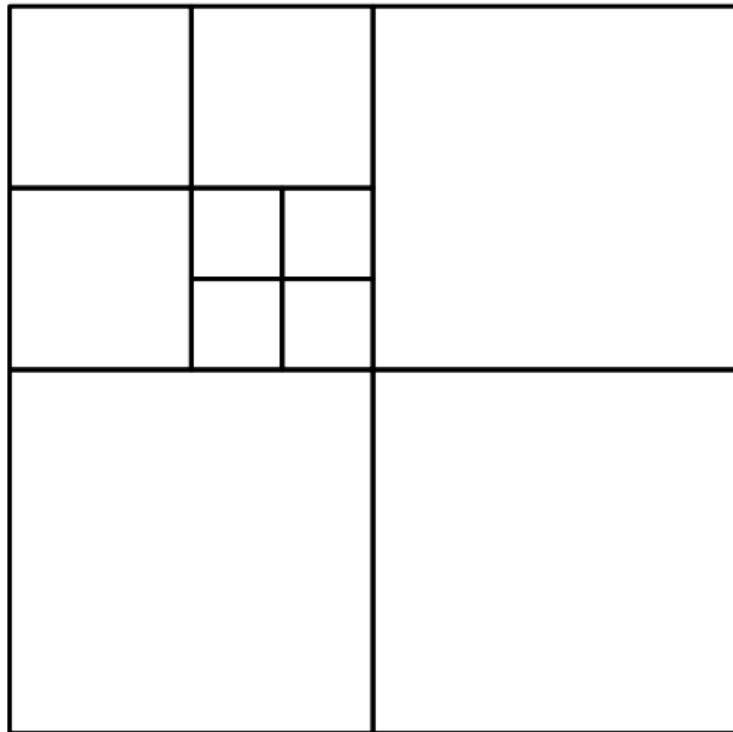
## Superfície paramétrica – discretização do domínio



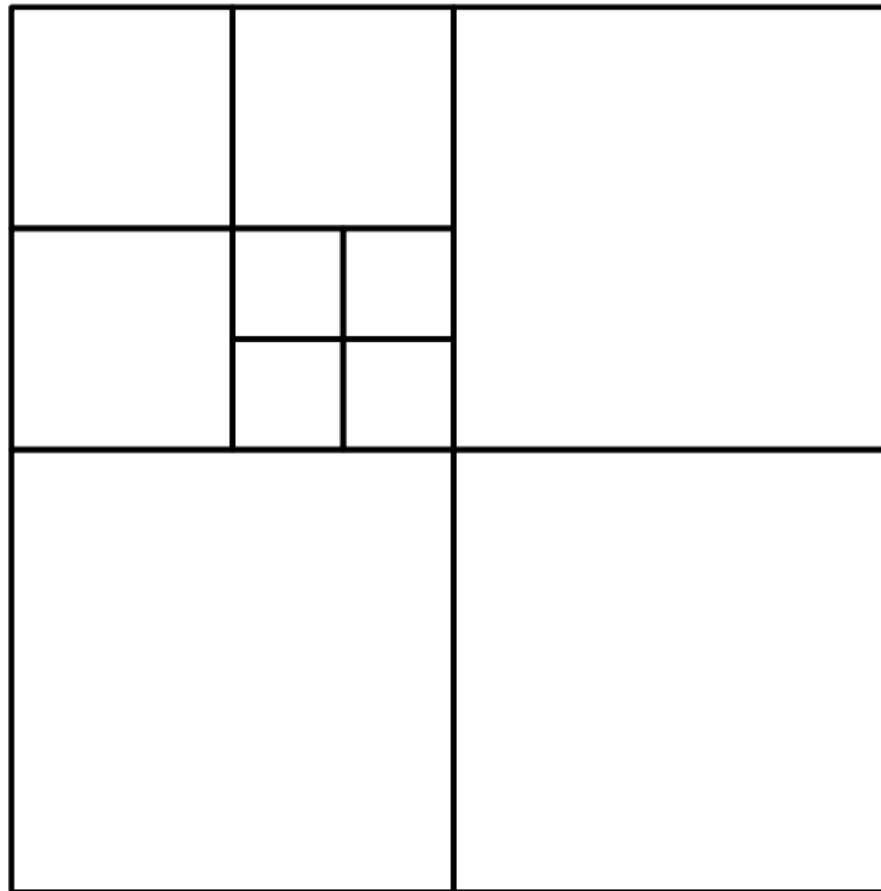
## Superfície paramétrica – discretização do domínio: quadtree



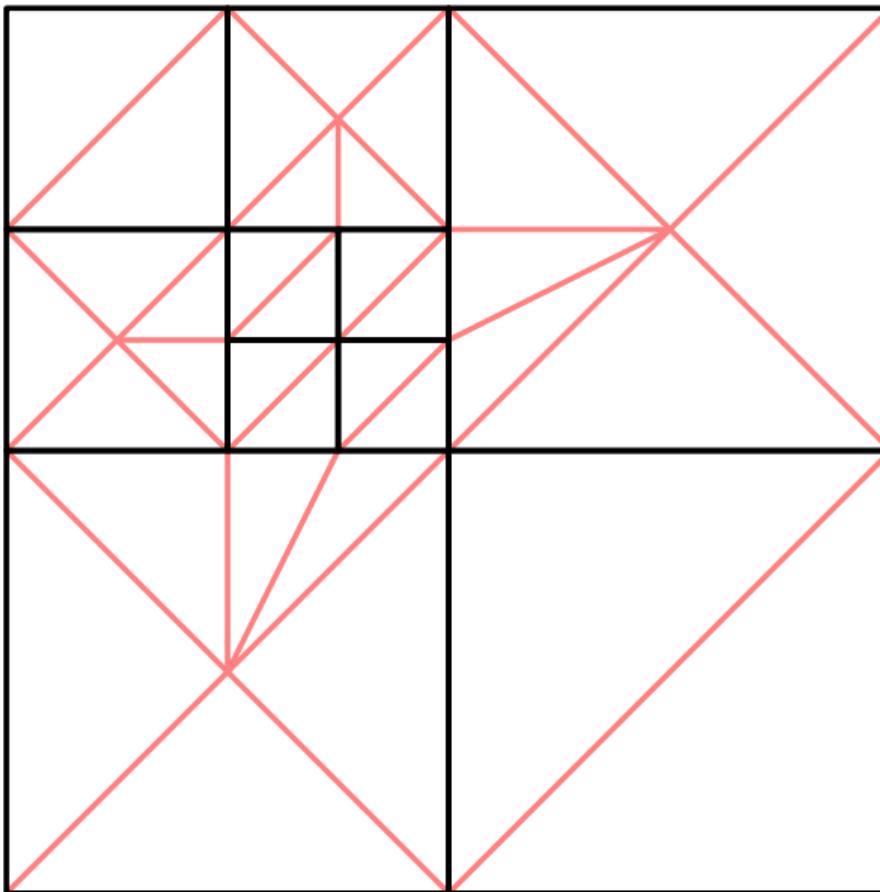
## Superfície paramétrica – discretização do domínio: quadtree



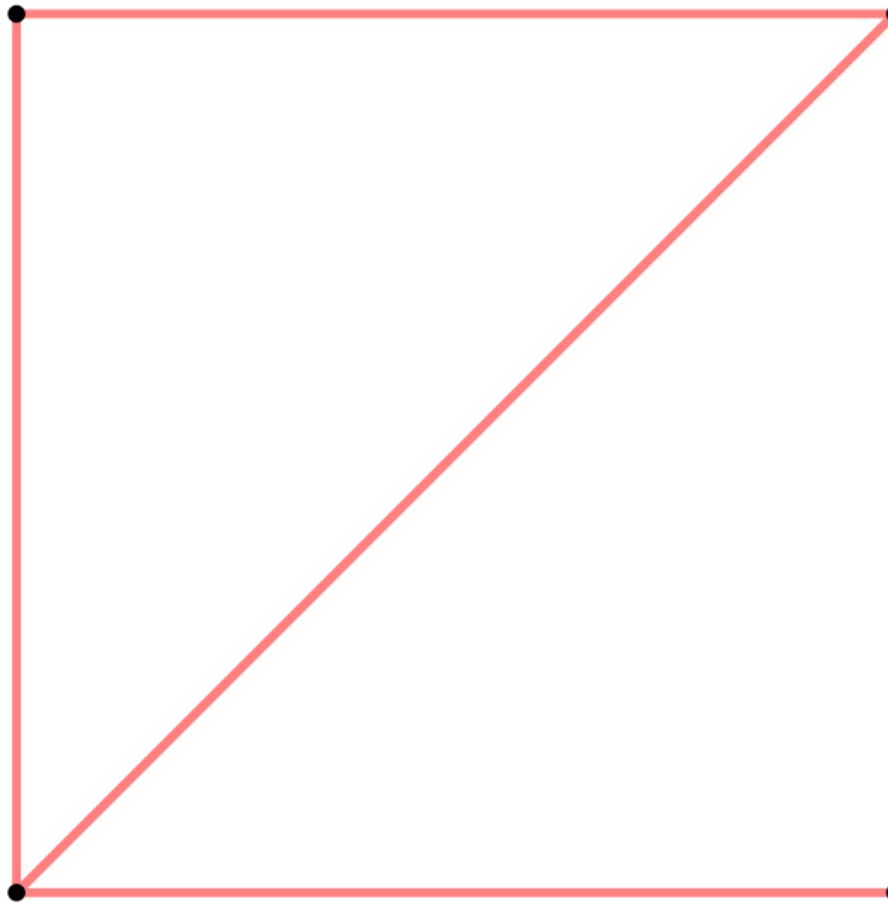
## Superfície paramétrica – discretização do domínio: quadtree



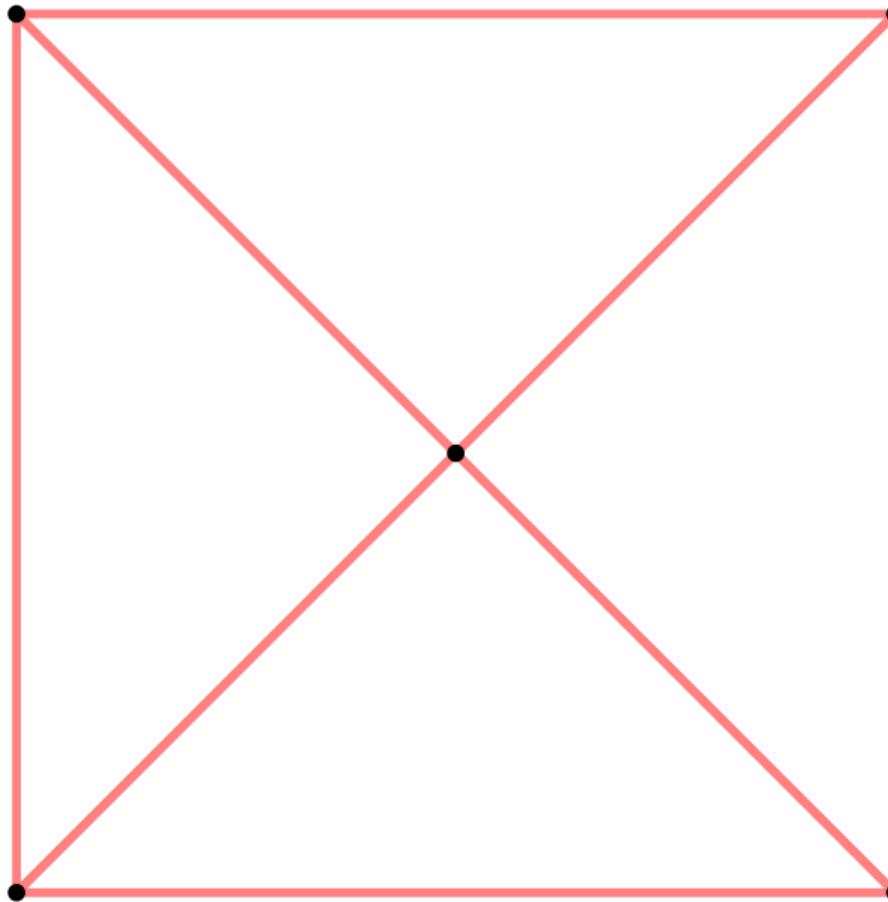
Superfície paramétrica – discretização do domínio: quadtree triangulada



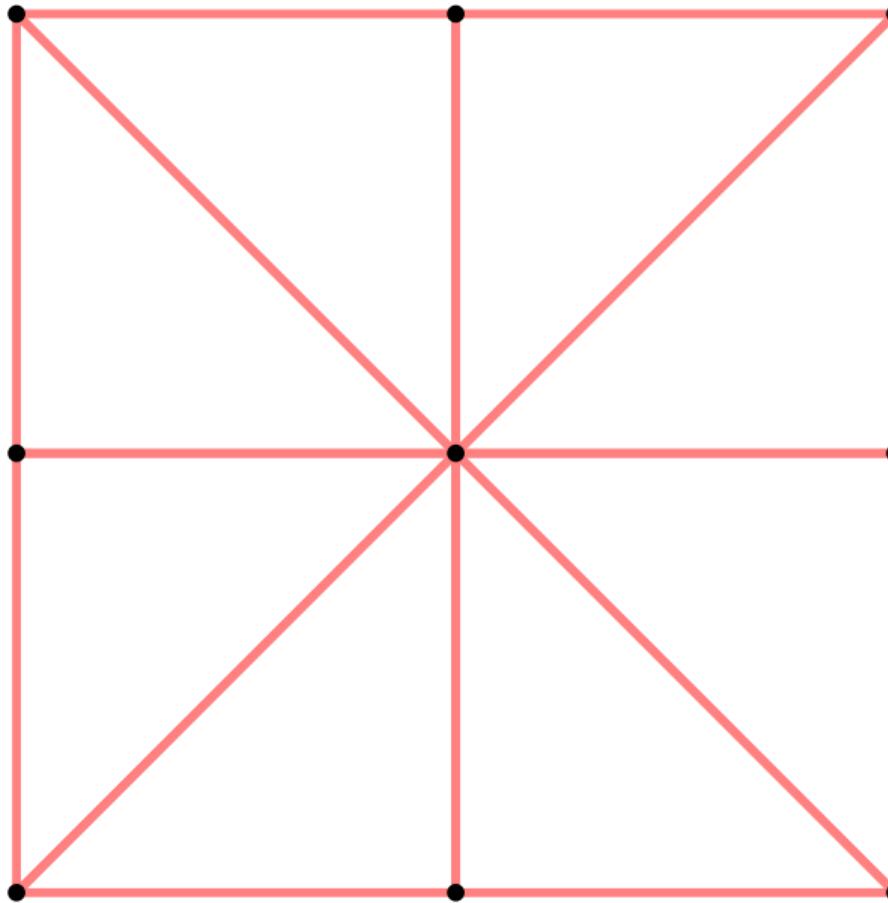
## Malha 4-8 uniforme



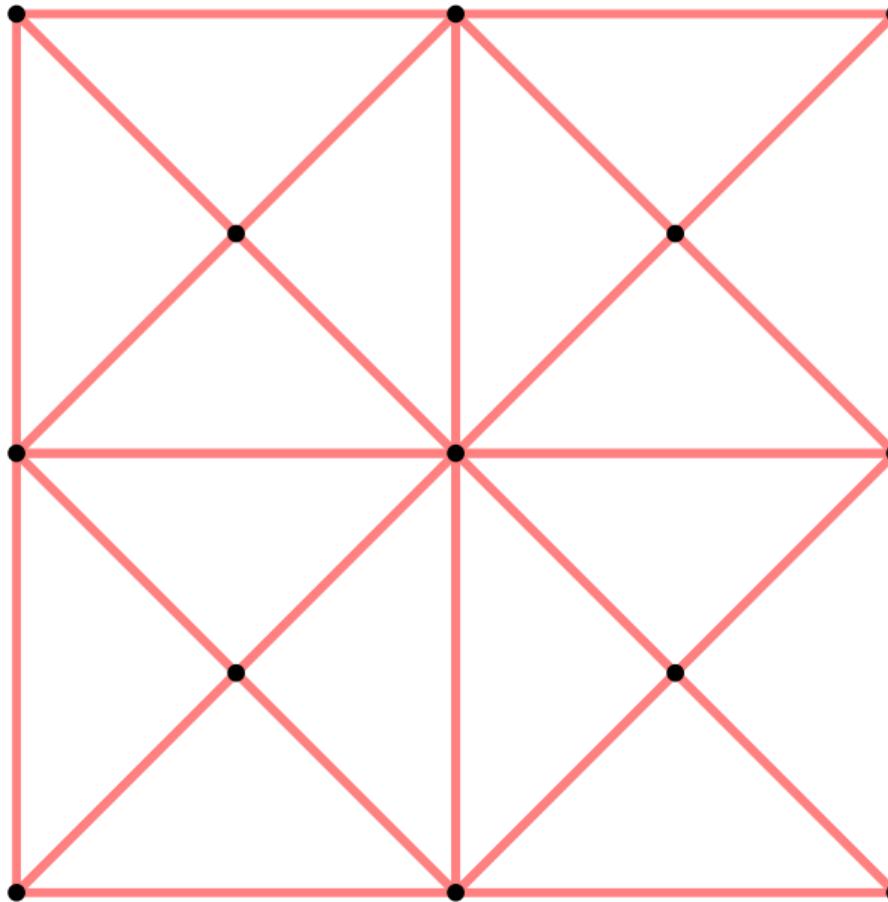
## Malha 4-8 uniforme



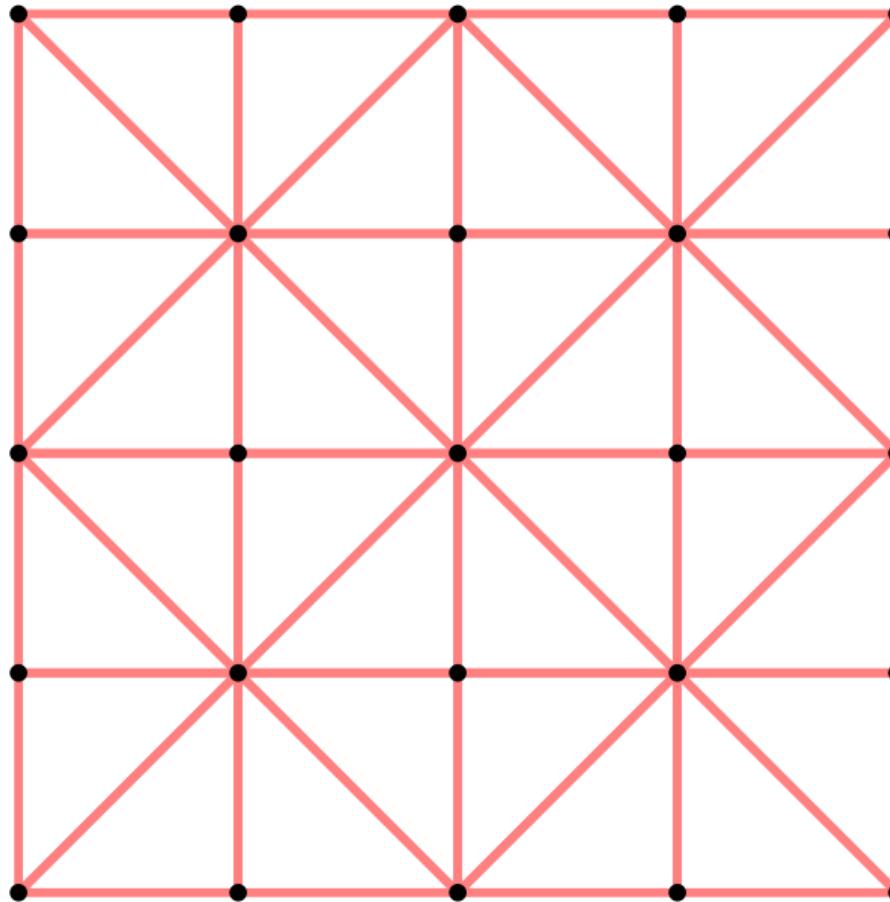
Malha 4-8 uniforme



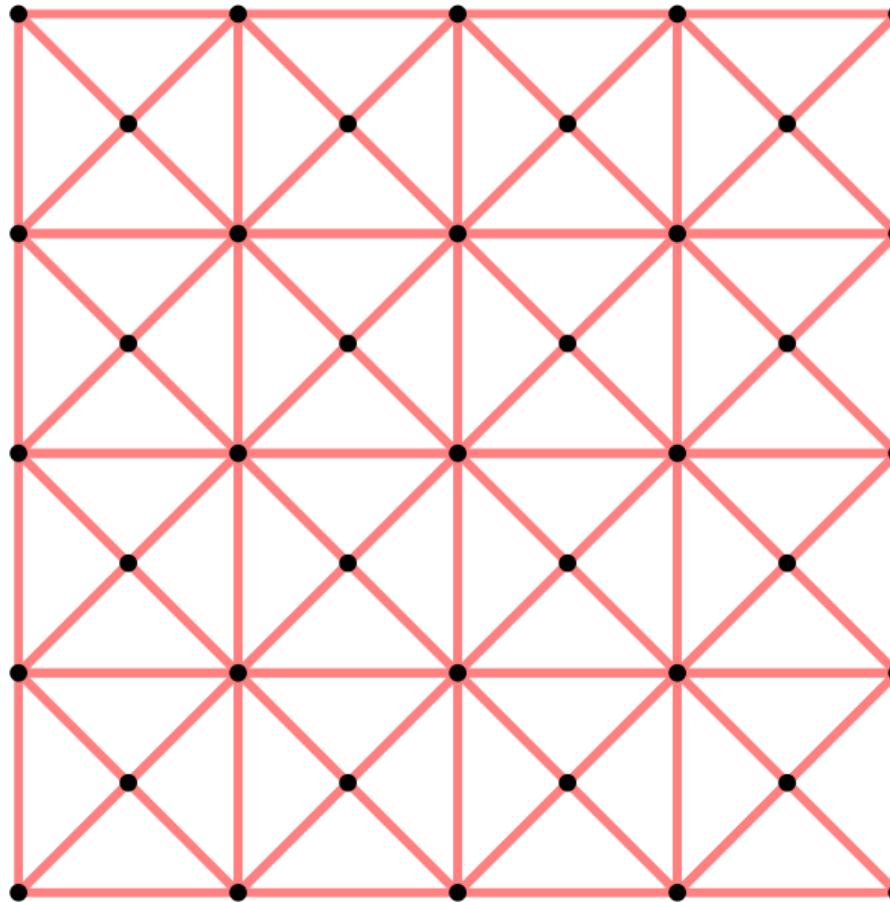
Malha 4-8 uniforme



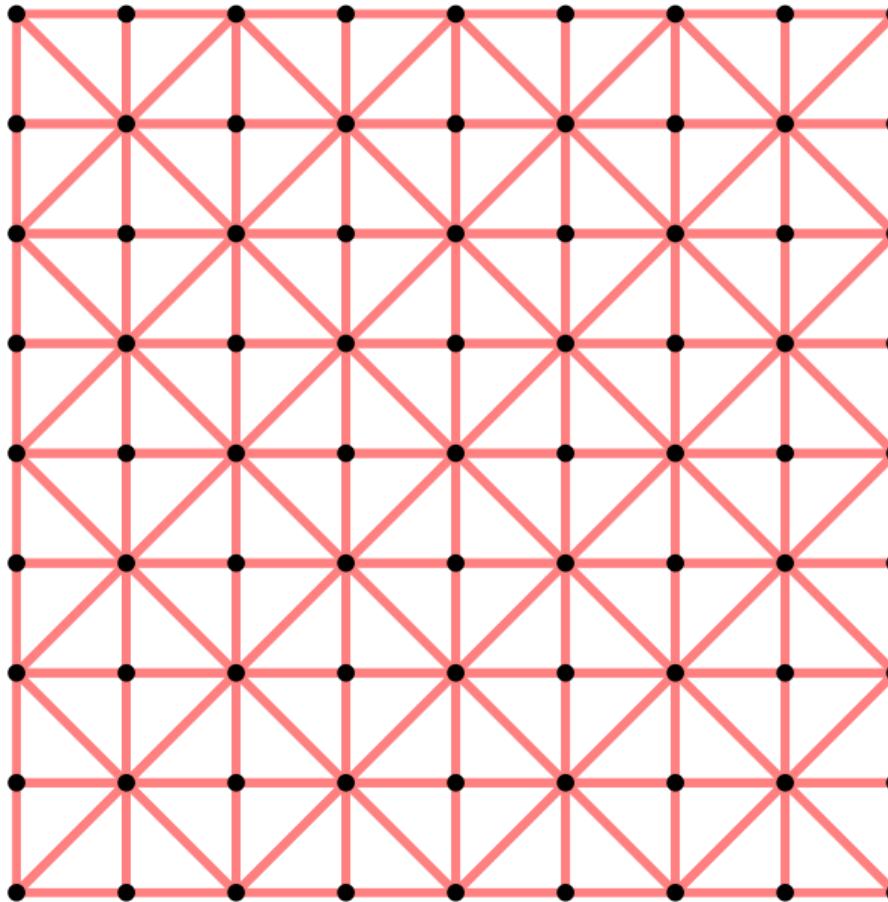
Malha 4-8 uniforme



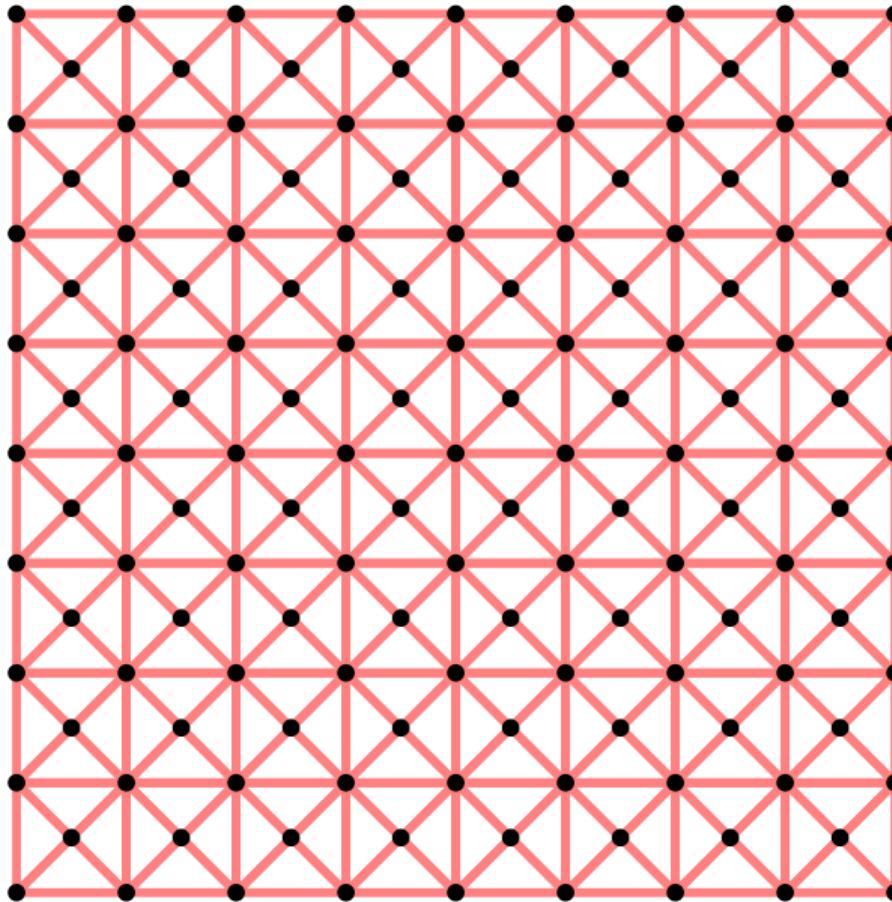
Malha 4-8 uniforme



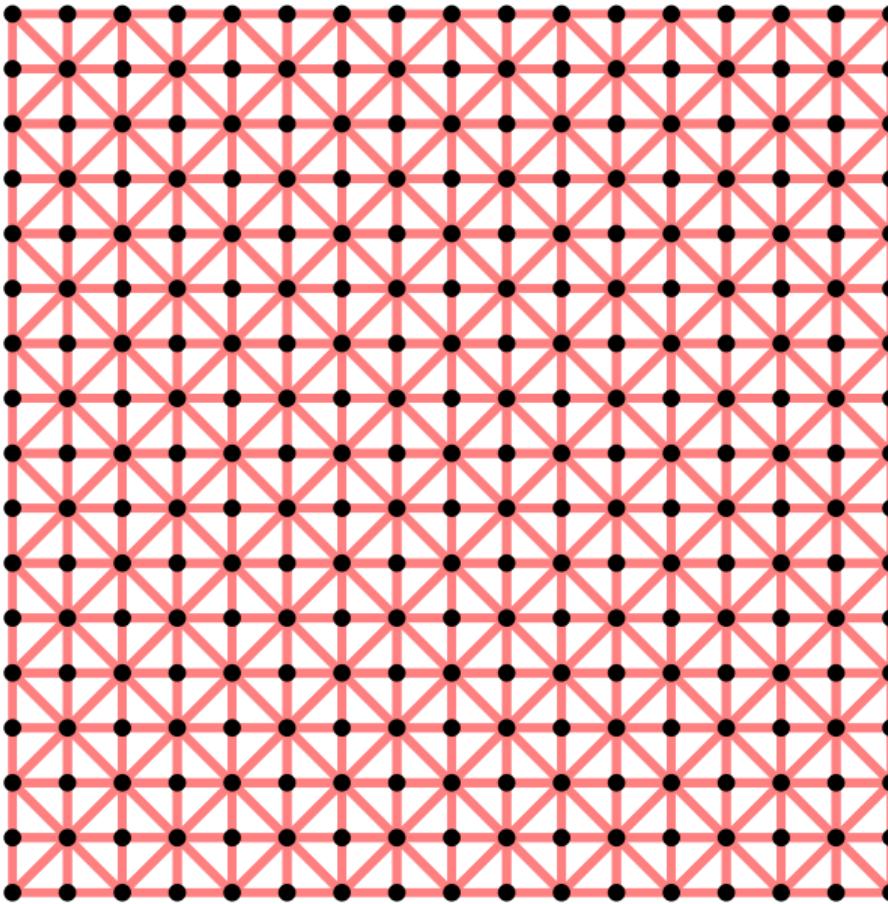
Malha 4-8 uniforme



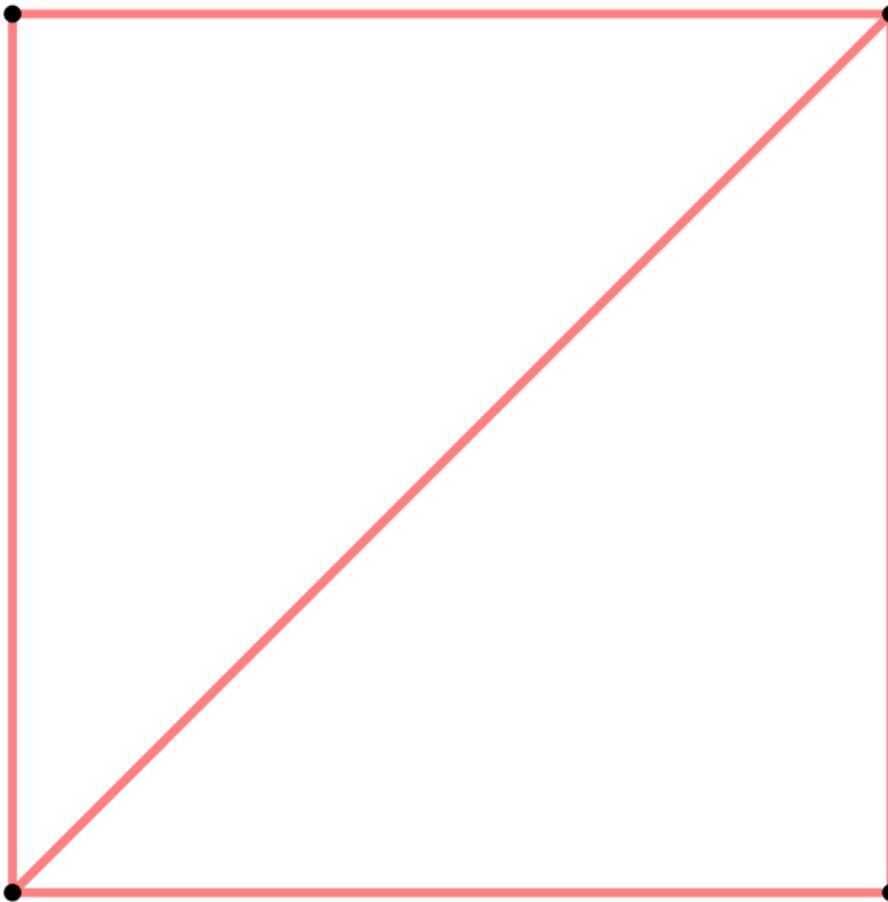
Malha 4-8 uniforme



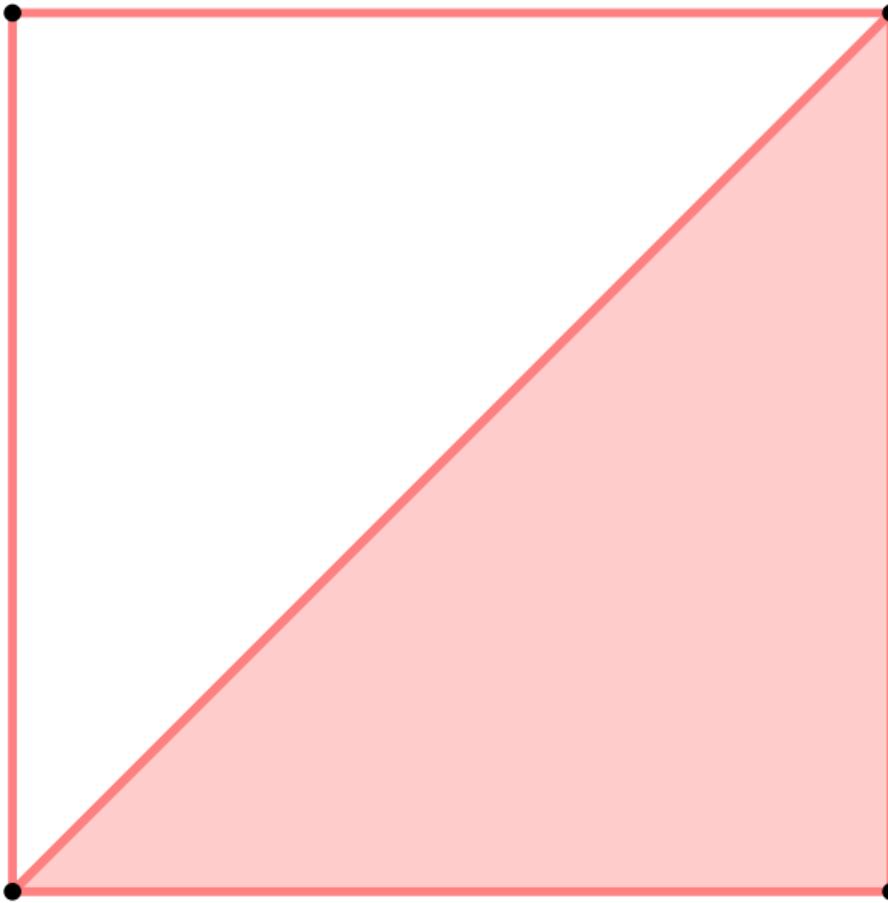
## Malha 4-8 uniforme



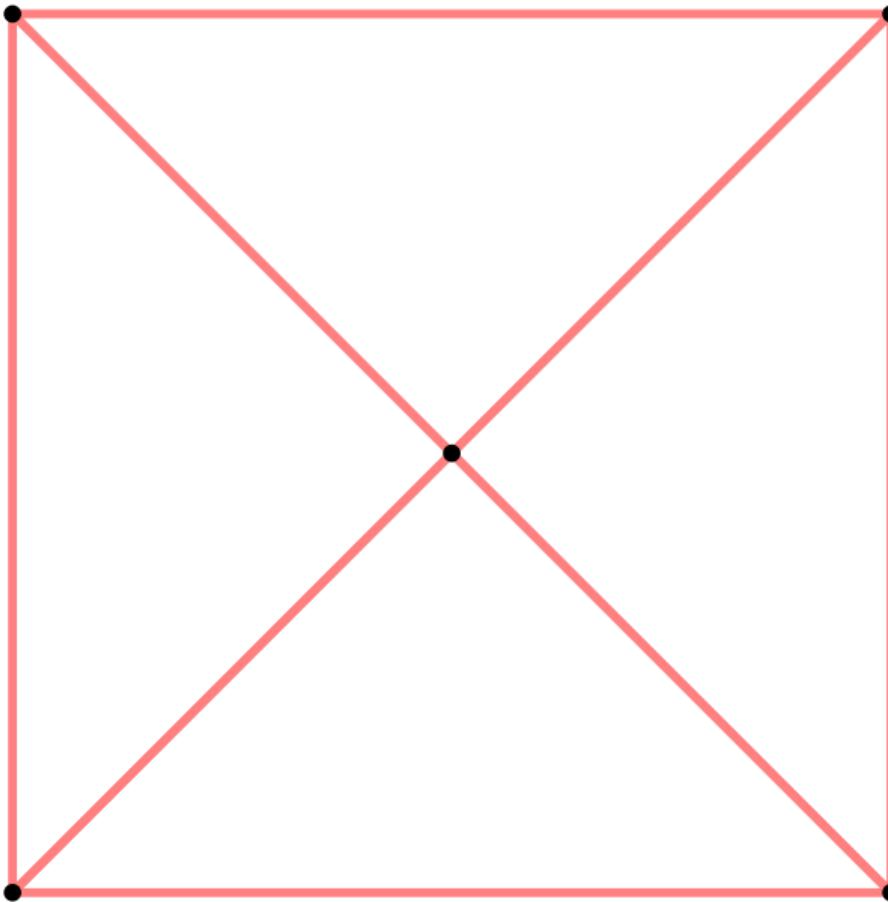
## Malha 4-8 adaptativa



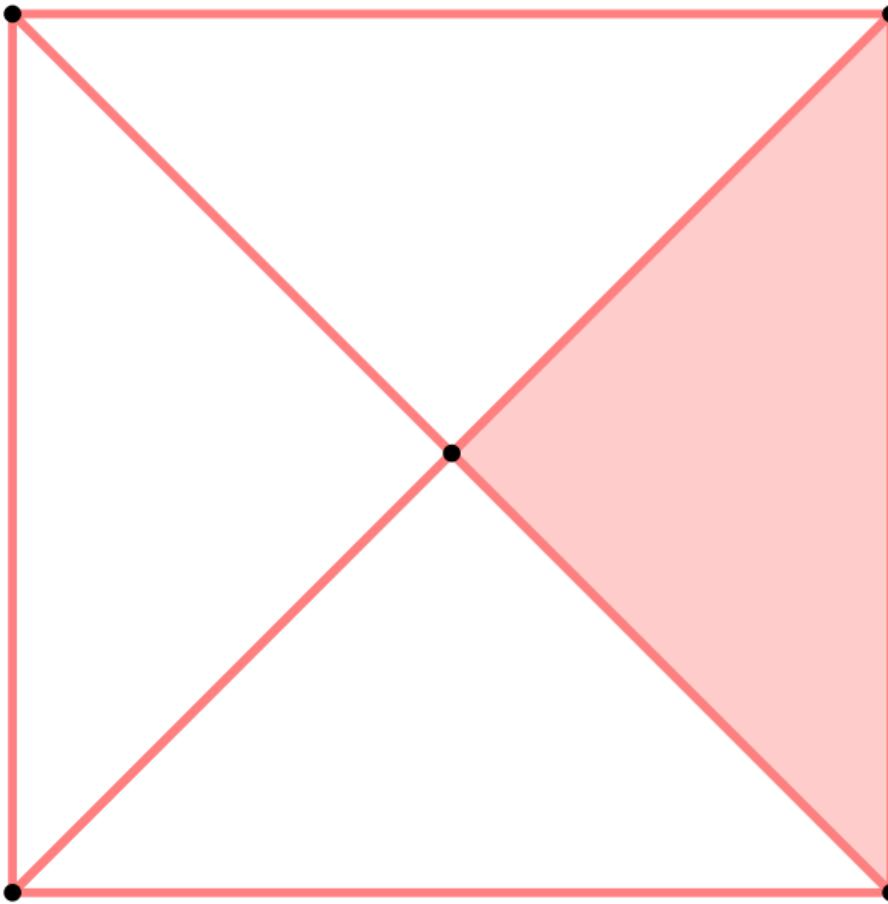
## Malha 4-8 adaptativa



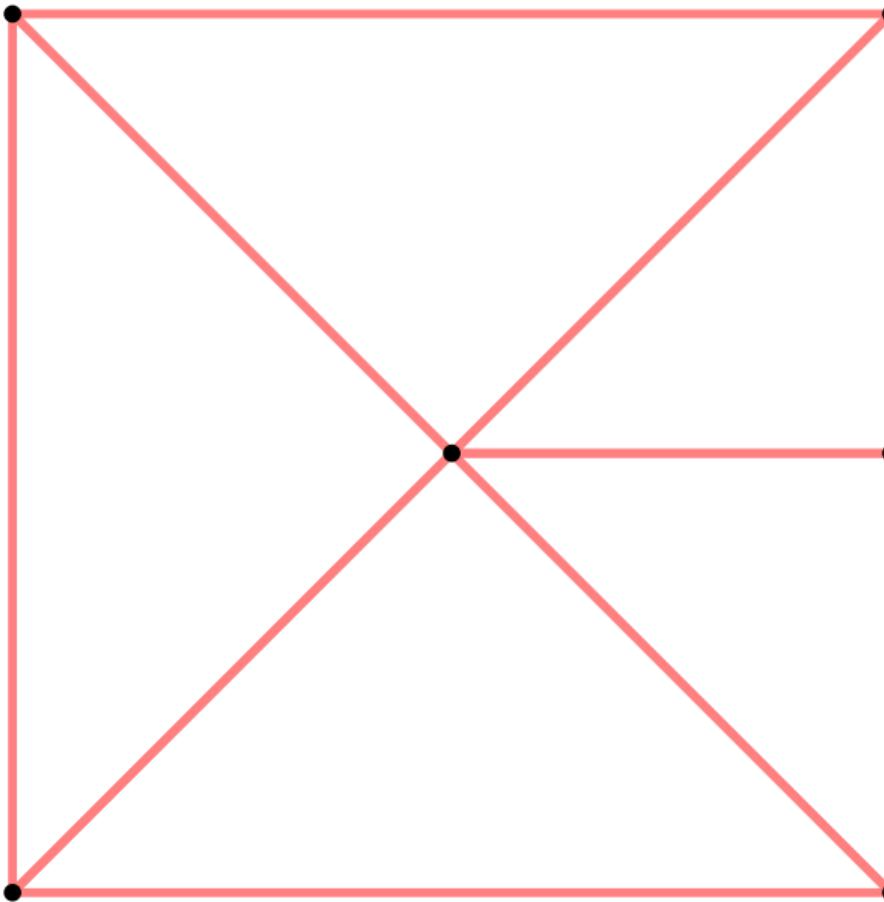
## Malha 4-8 adaptativa



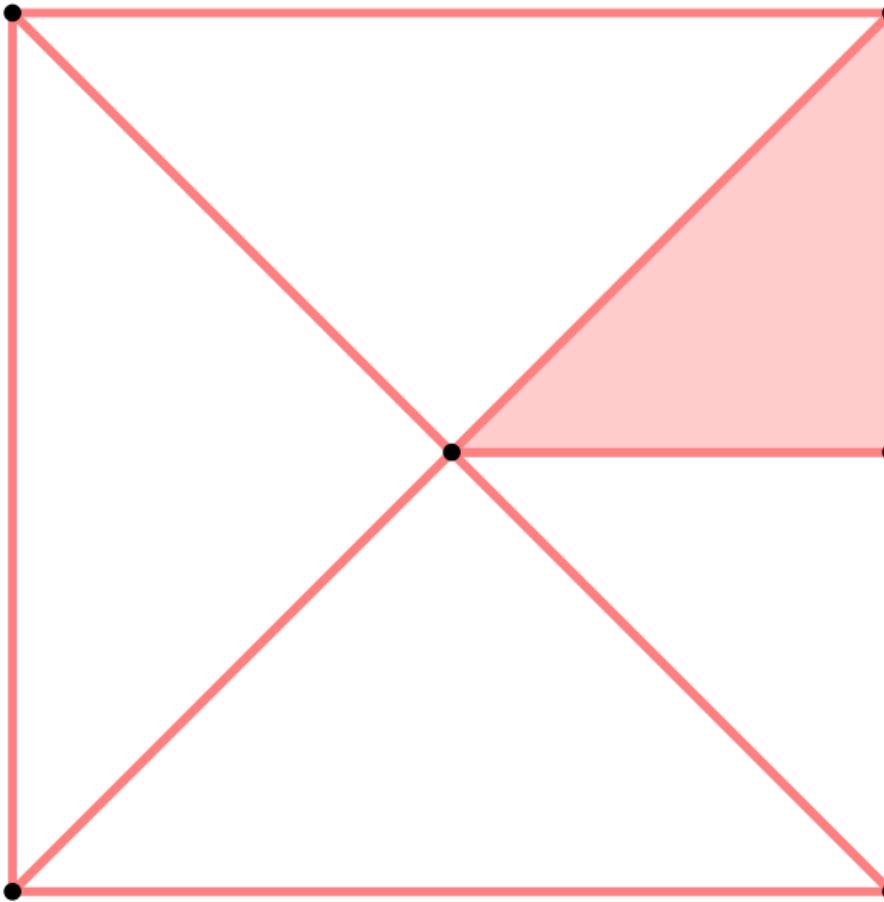
Malha 4-8 adaptativa



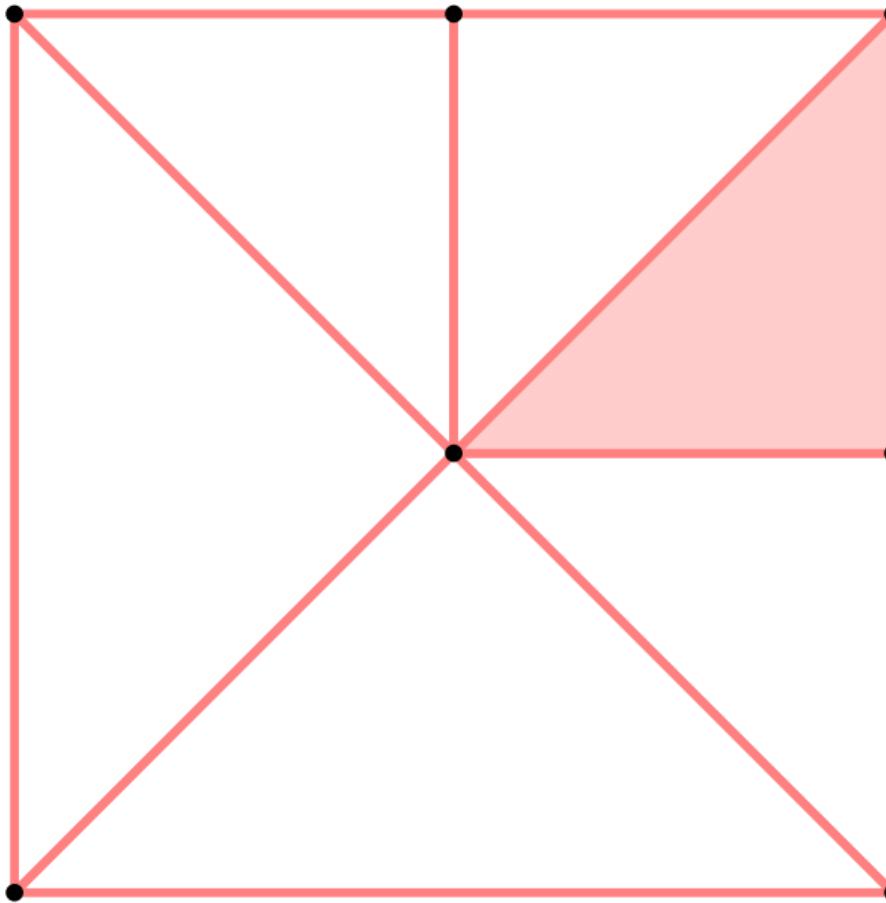
## Malha 4-8 adaptativa



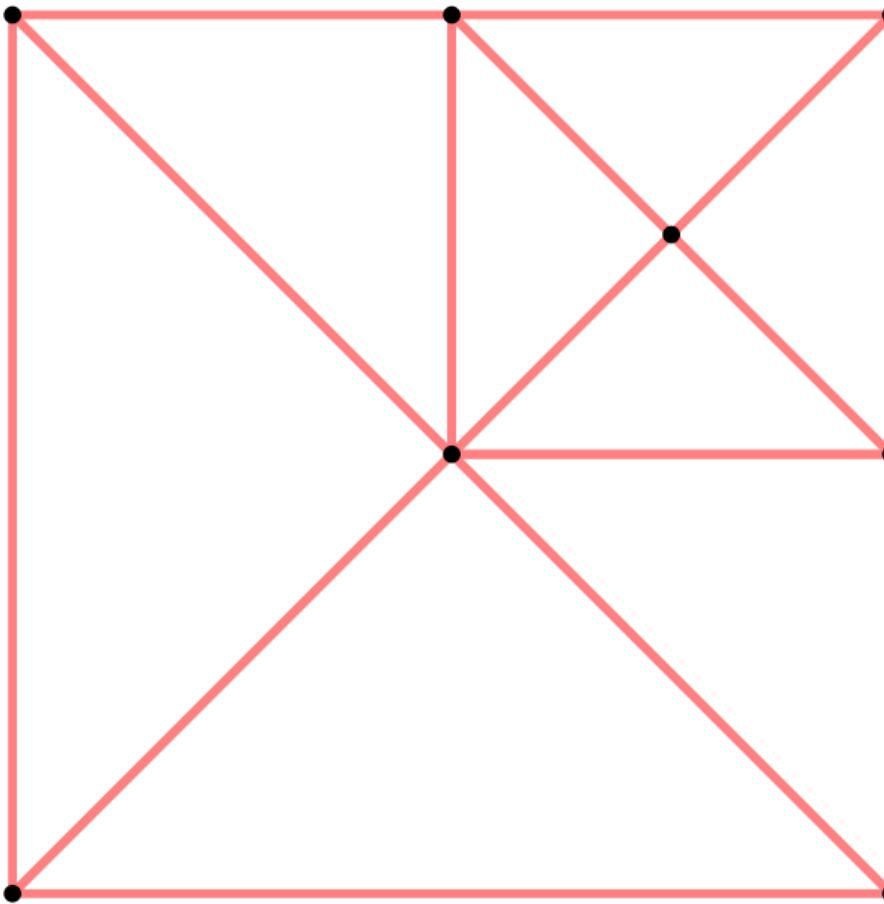
## Malha 4-8 adaptativa



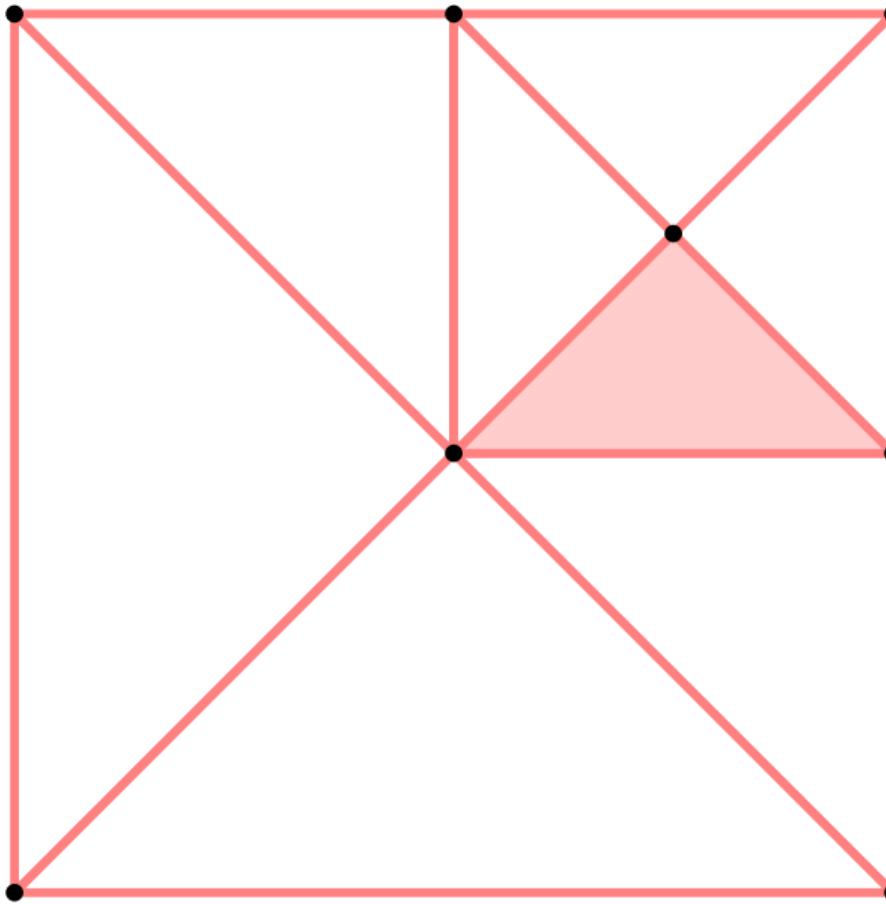
## Malha 4-8 adaptativa



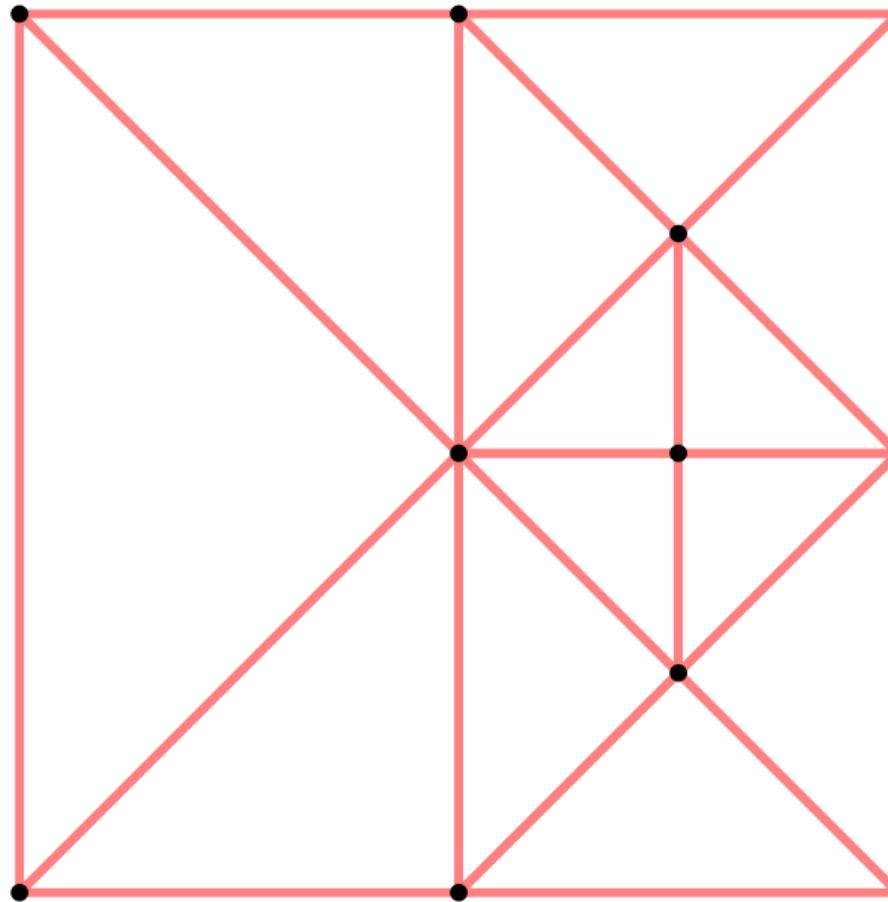
## Malha 4-8 adaptativa



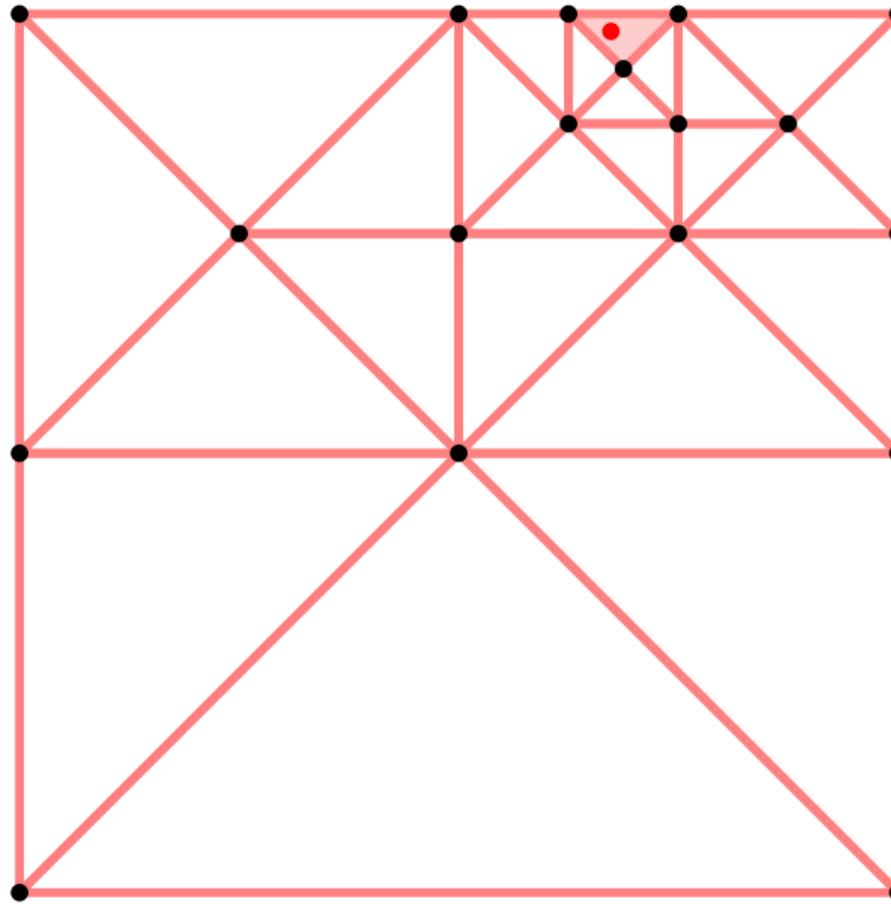
## Malha 4-8 adaptativa



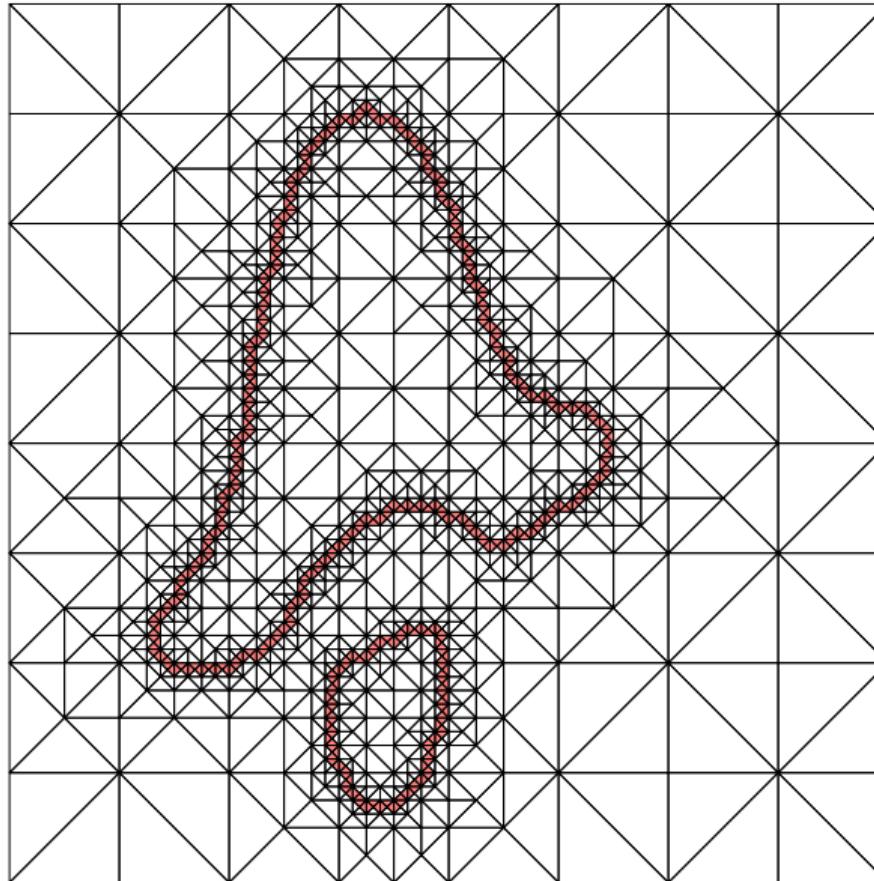
## Malha 4-8 adaptativa



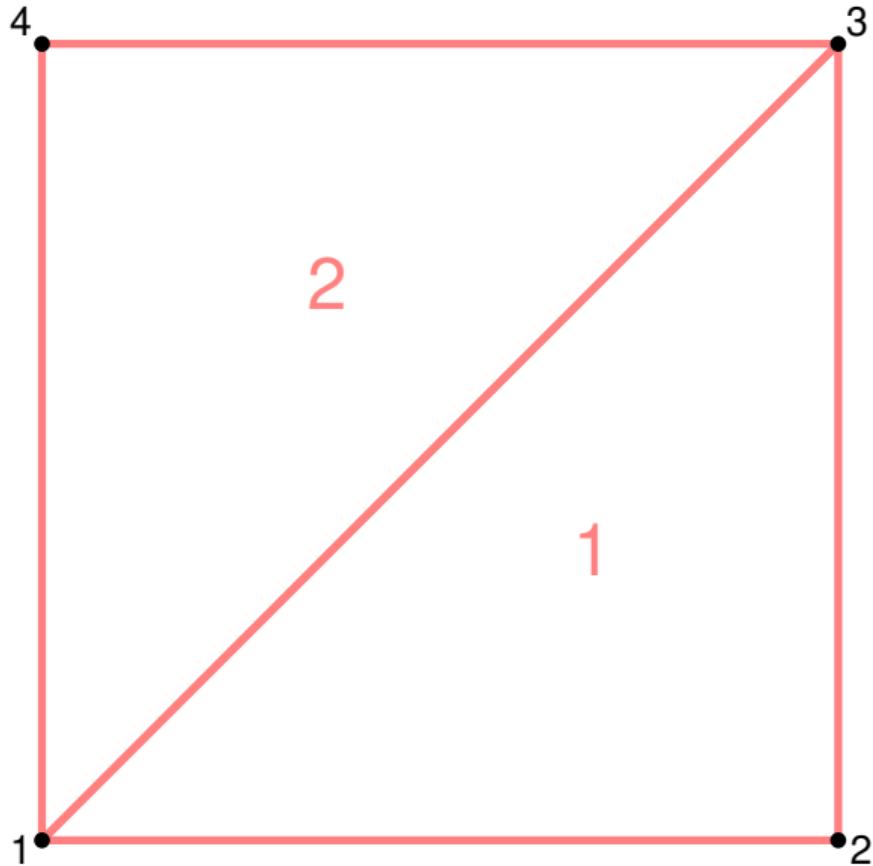
## Malha 4-8 adaptativa



## Malha 4-8 adaptativa – curva implícita

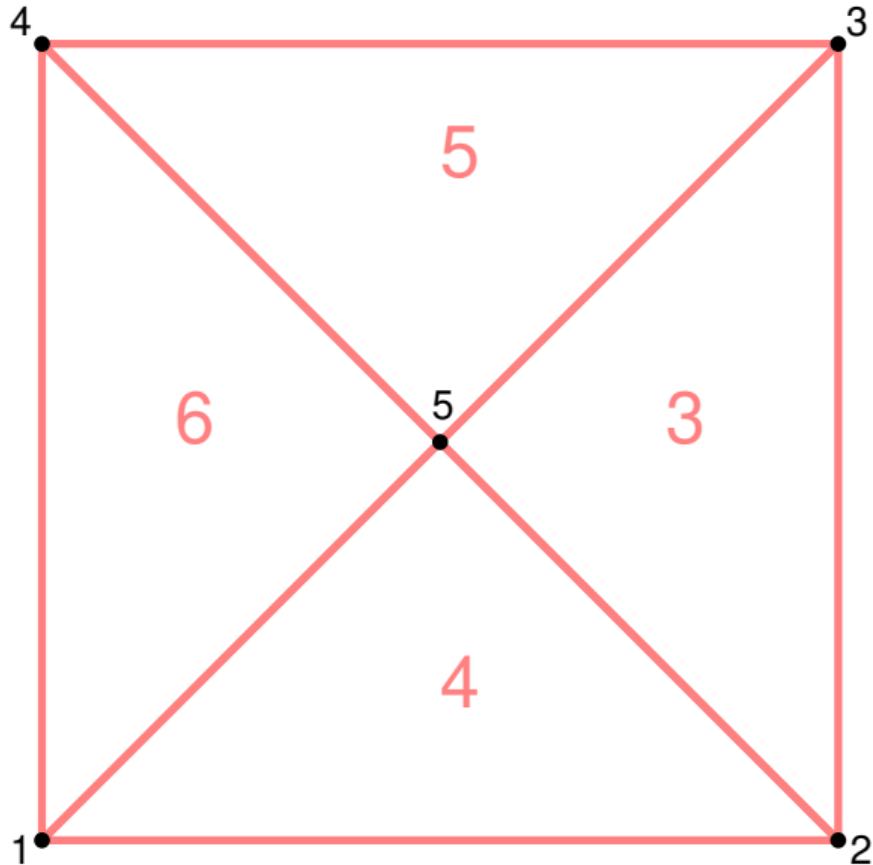


## Malha 4-8 – representação



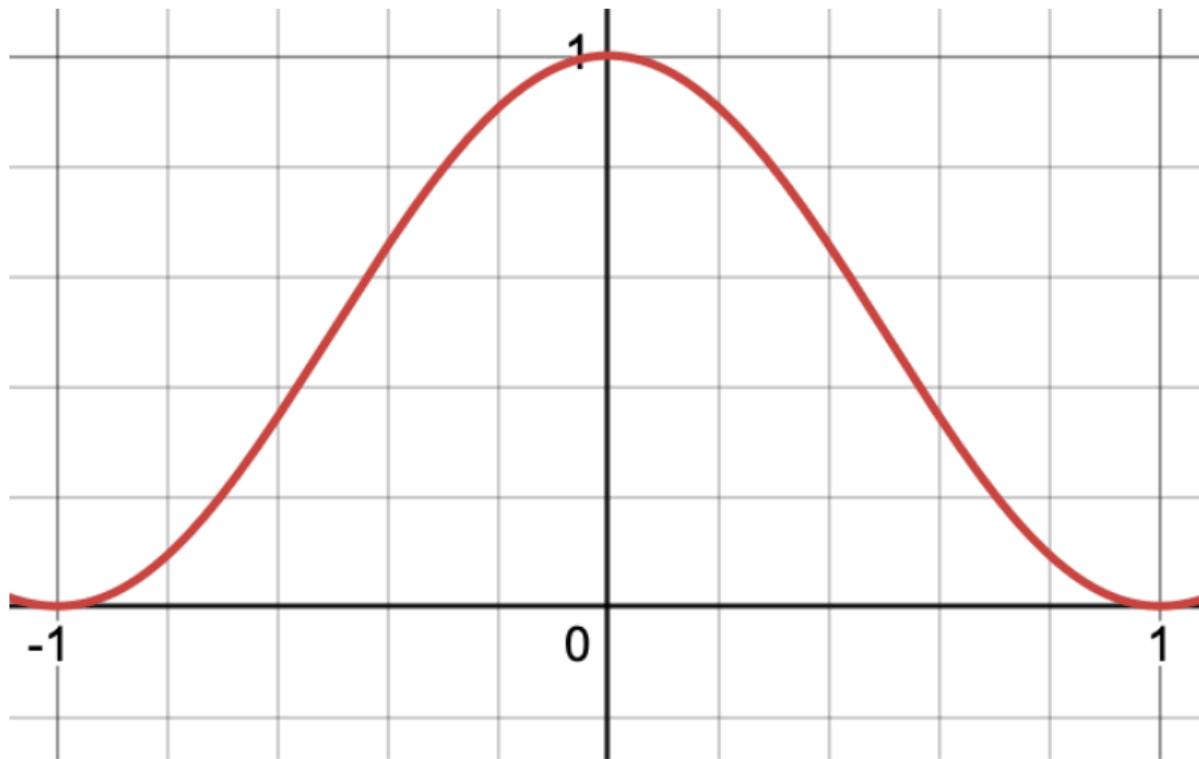
v	1	0	0	0
v	2	1	0	0
v	3	1	1	0
v	4	0	1	0
f	1	2	3	1
		2	—	—
f	2	4	1	3
		1	—	—

## Malha 4-8 – representação



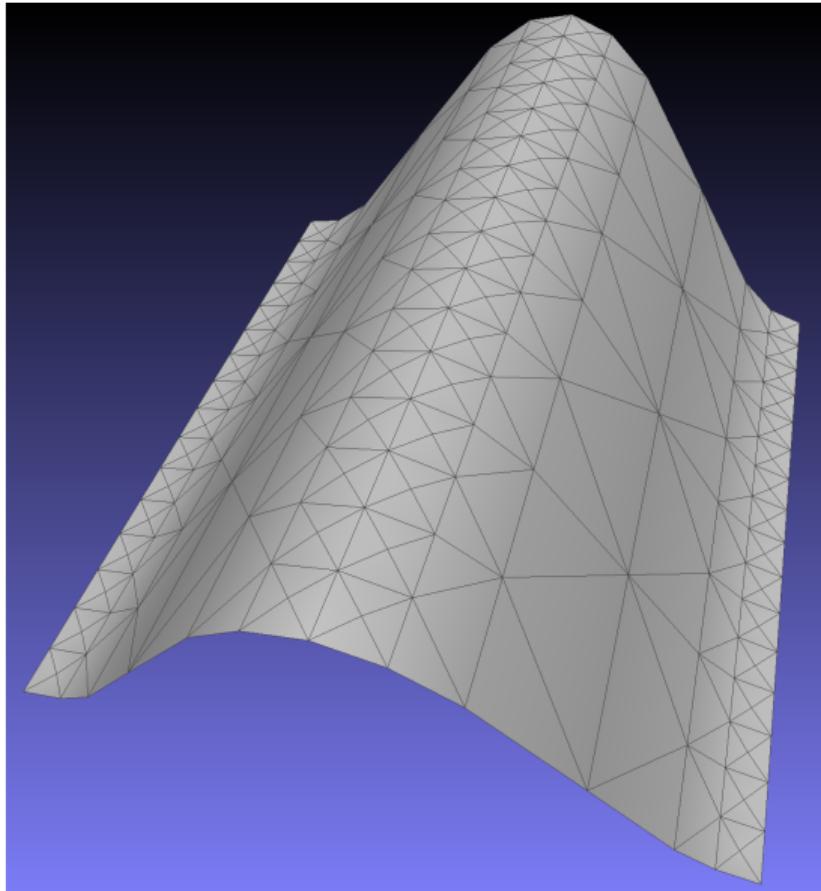
v	1	0	0	0	
v	2	1	0	0	
v	3	1	1	0	
v	4	0	1	0	
v	5	0.5	0.5	0	
f	1	2	3	1	x
f		2	—	—	
f	2	4	1	3	x
f		1	—	—	
f	3	5	2	3	
f		—	5	4	
f	4	5	1	2	
f		—	3	6	
f	5	5	3	4	
f		—	6	3	
f	6	5	4	1	
f		—	4	5	

## Malha 4-8 adaptativa – superfície paramétrica

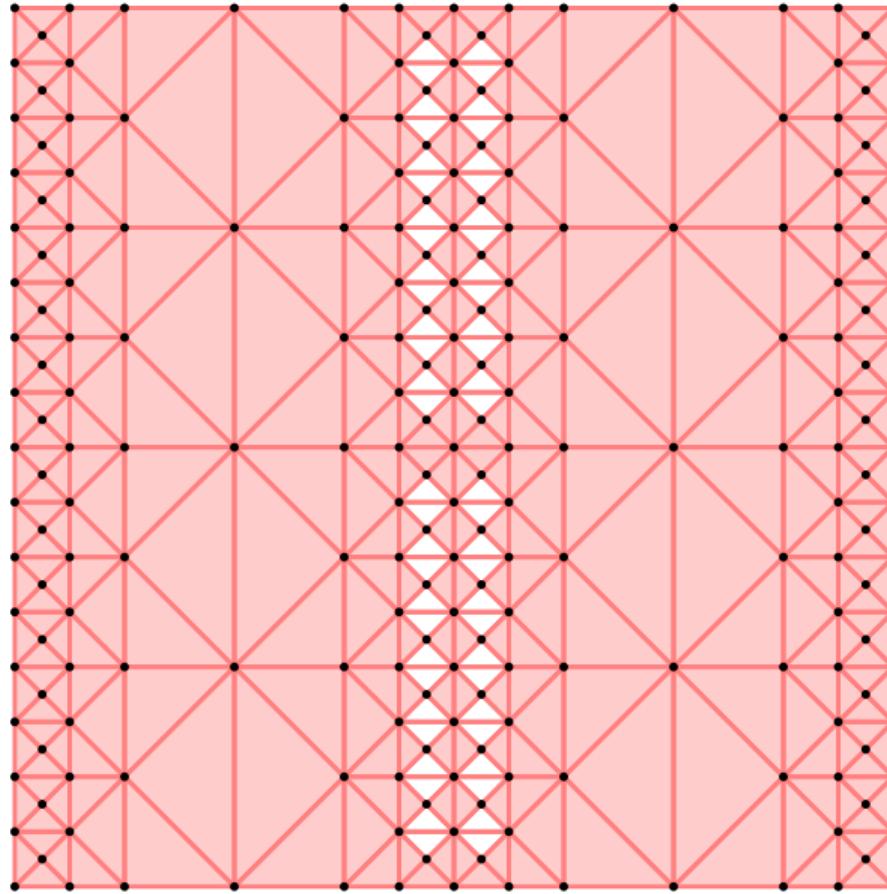


$$f(u, v) = (u, v, g(u)) \quad g(u) = -\frac{4}{9}u^6 + \frac{17}{9}u^4 - \frac{22}{9}u^2 + 1$$

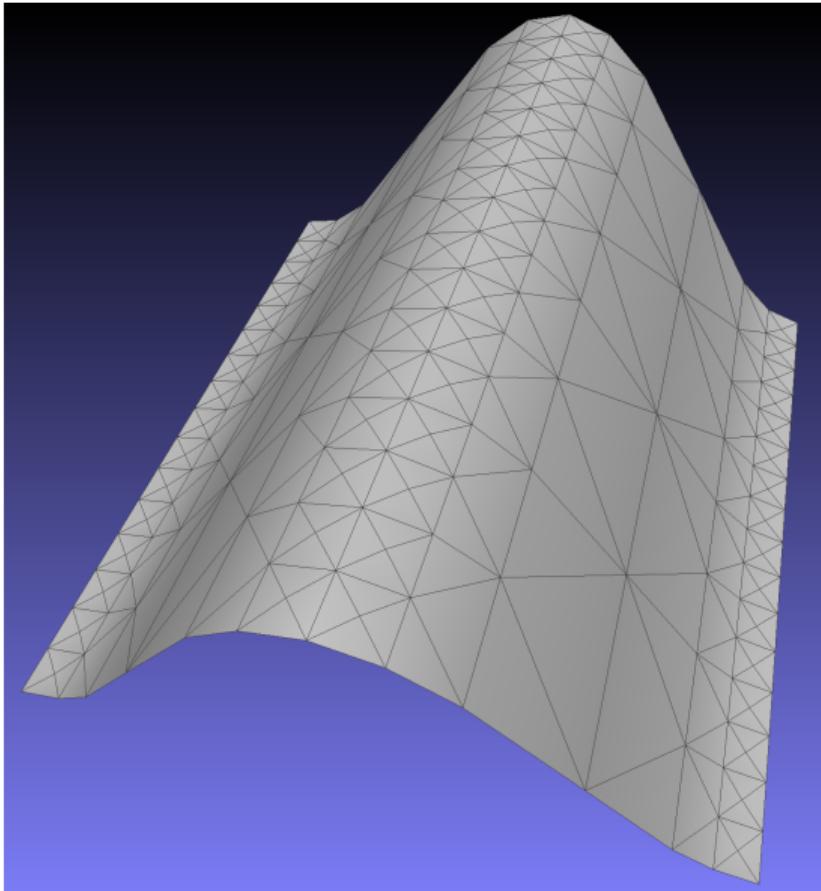
## Malha 4-8 adaptativa – superfície paramétrica



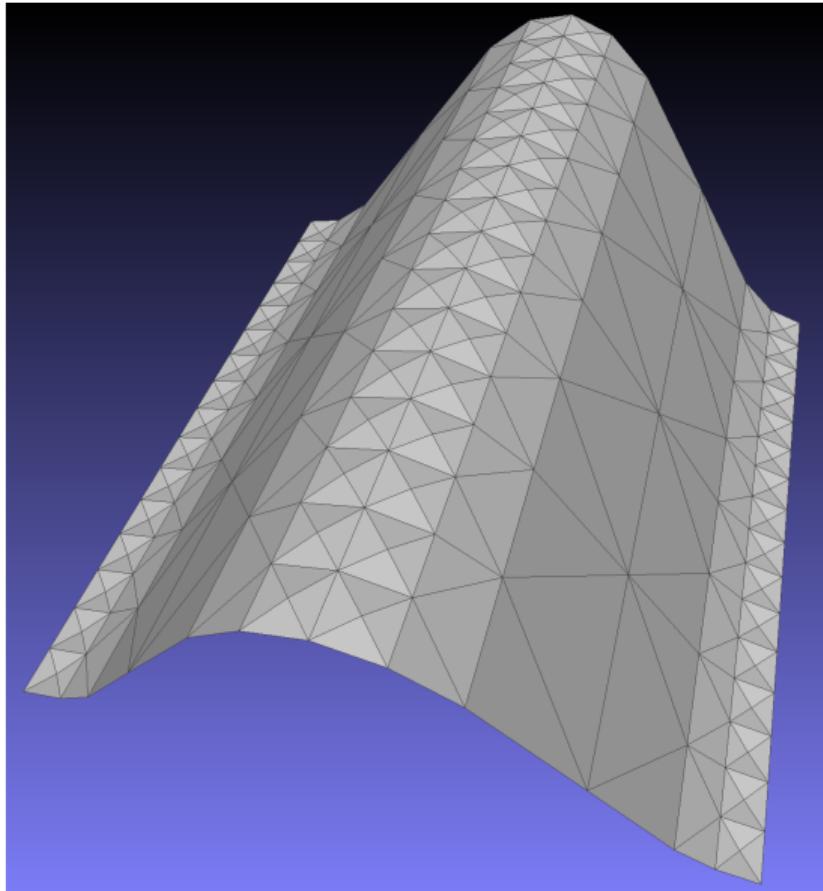
## Malha 4-8 adaptativa – superfície paramétrica



## Malha 4-8 adaptativa – superfície paramétrica

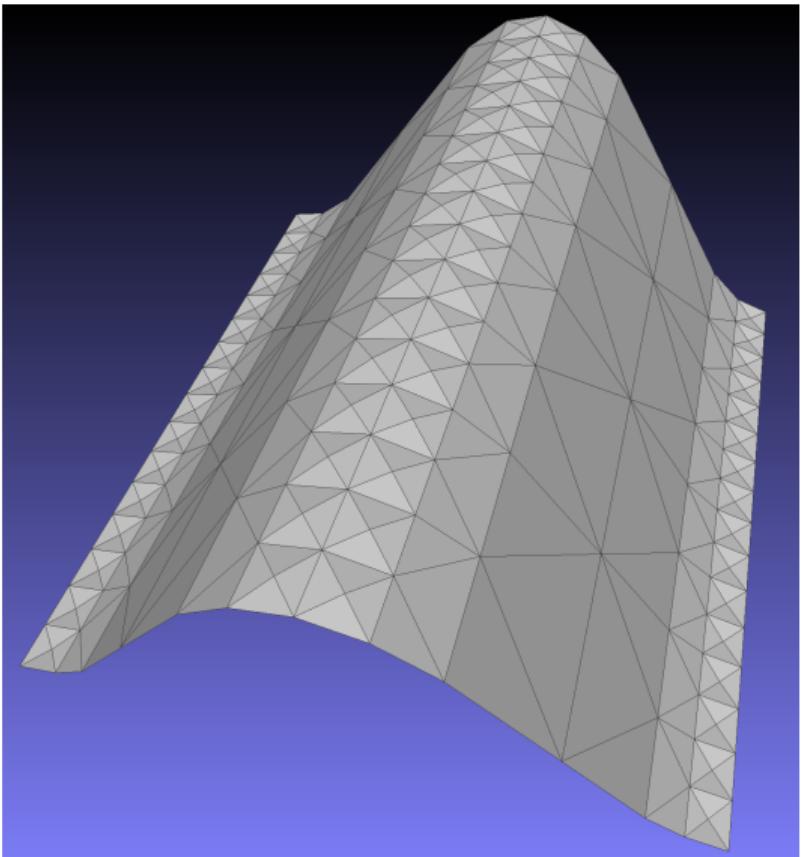


## Malha 4-8 adaptativa – superfície paramétrica



# Superfície paramétrica – representação

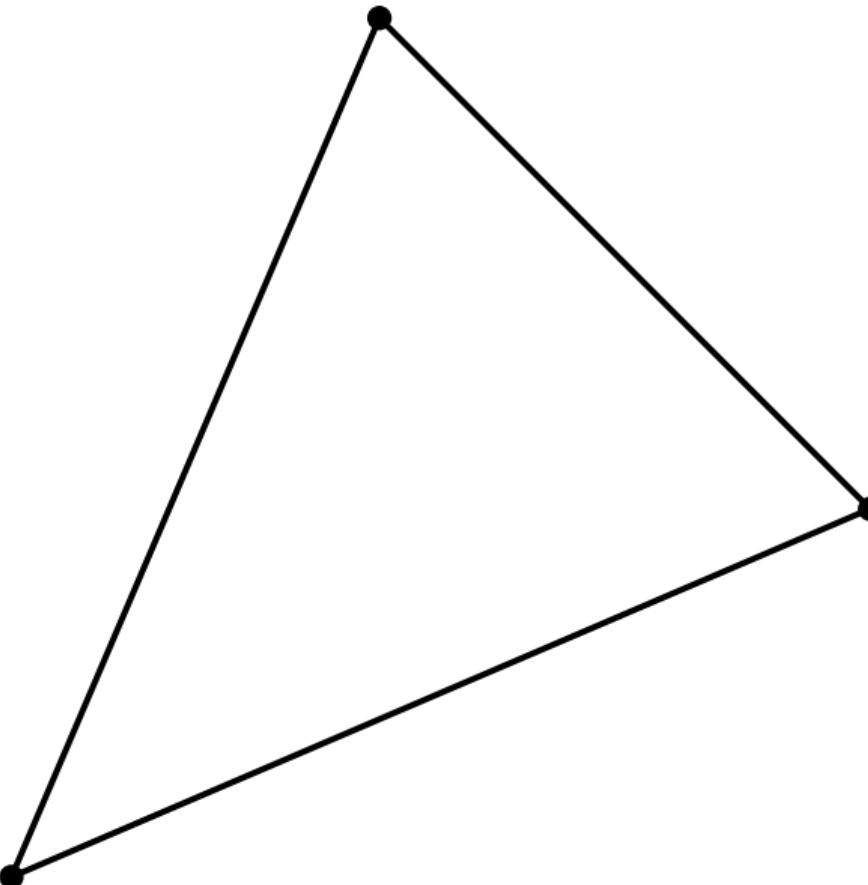
OBJ



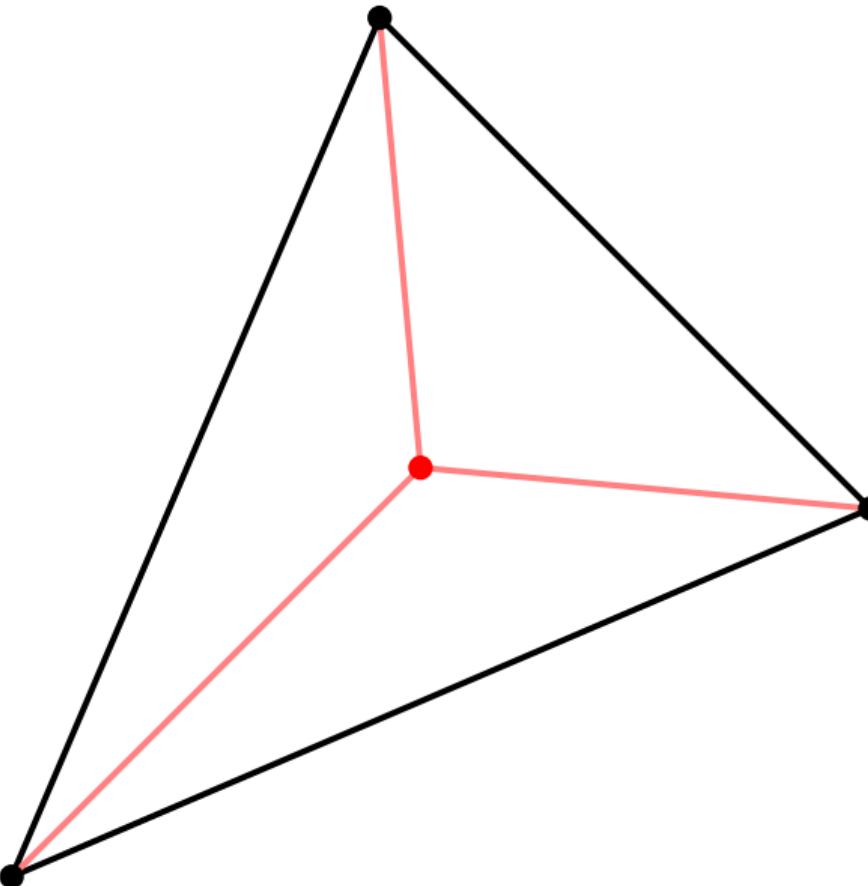
geometria

```
v -0.25 -0.75 0.8544921875  
v -0.5 -0.5 0.5  
v -0.5 -1.0 0.5  
v -0.25 -0.25 0.8544921875  
v -0.5 0.0 0.5  
v 0.25 -0.75 0.8544921875  
v 0.5 -1.0 0.5  
v 0.5 -0.5 0.5  
v 0.25 -0.25 0.8544921875  
v 0.5 0.0 0.5  
v -0.75 -0.25 0.1435546875  
v -0.75 -0.75 0.1435546875  
v 0.75 -0.75 0.1435546875  
v 0.75 -0.25 0.1435546875  
v 0.25 0.75 0.8544921875  
...  
f 1 2 3  
f 4 5 2  
f 6 7 8  
f 9 8 10  
f 11 2 5  
f 12 3 2  
f 13 8 7  
f 14 10 8  
f 15 16 17  
f 18 10 16  
f 19 20 21  
f 22 21 5  
f 23 16 10  
f 24 17 16  
f 25 21 20  
...
```

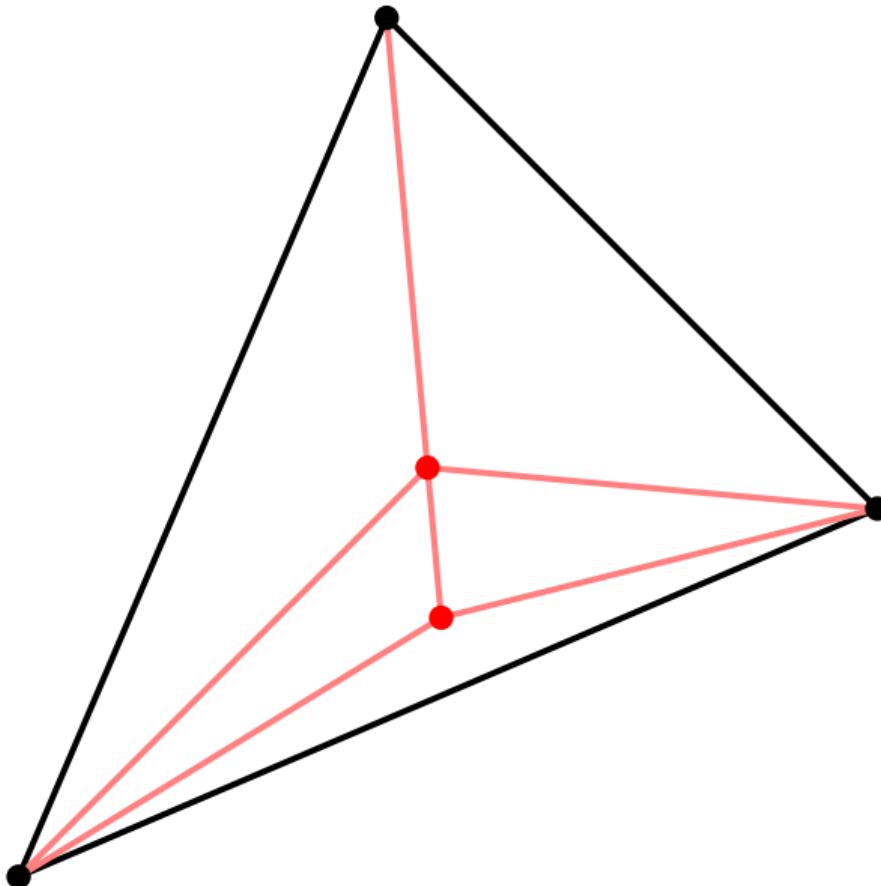
## Subdivisão de triângulos



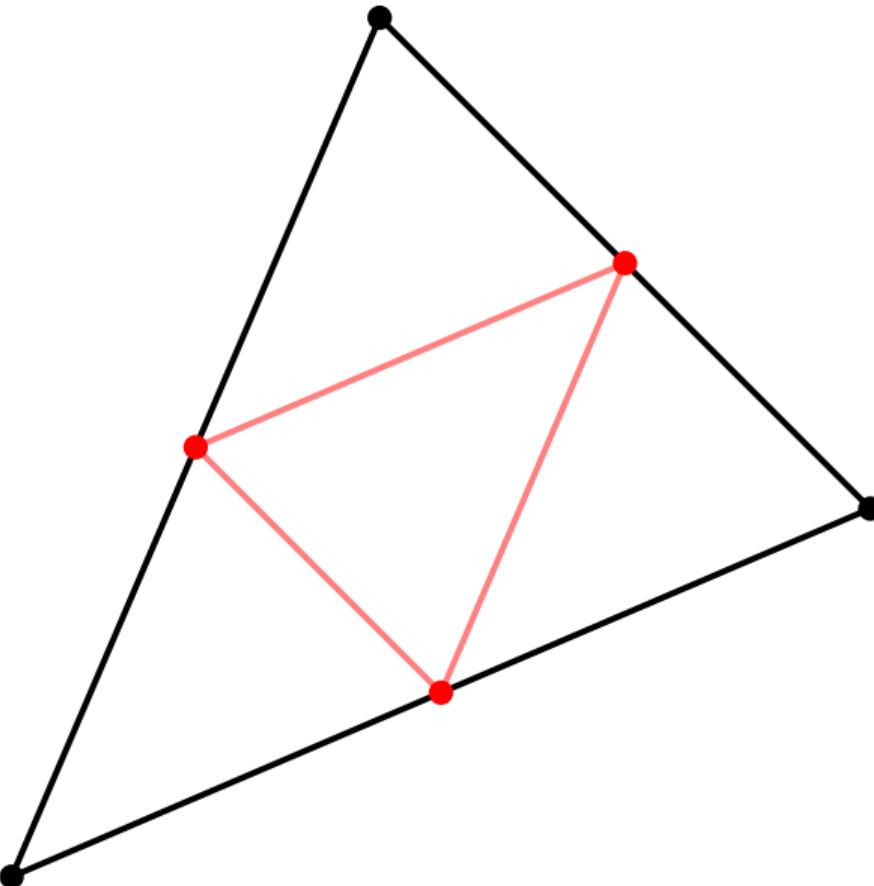
## Subdivisão de triângulos



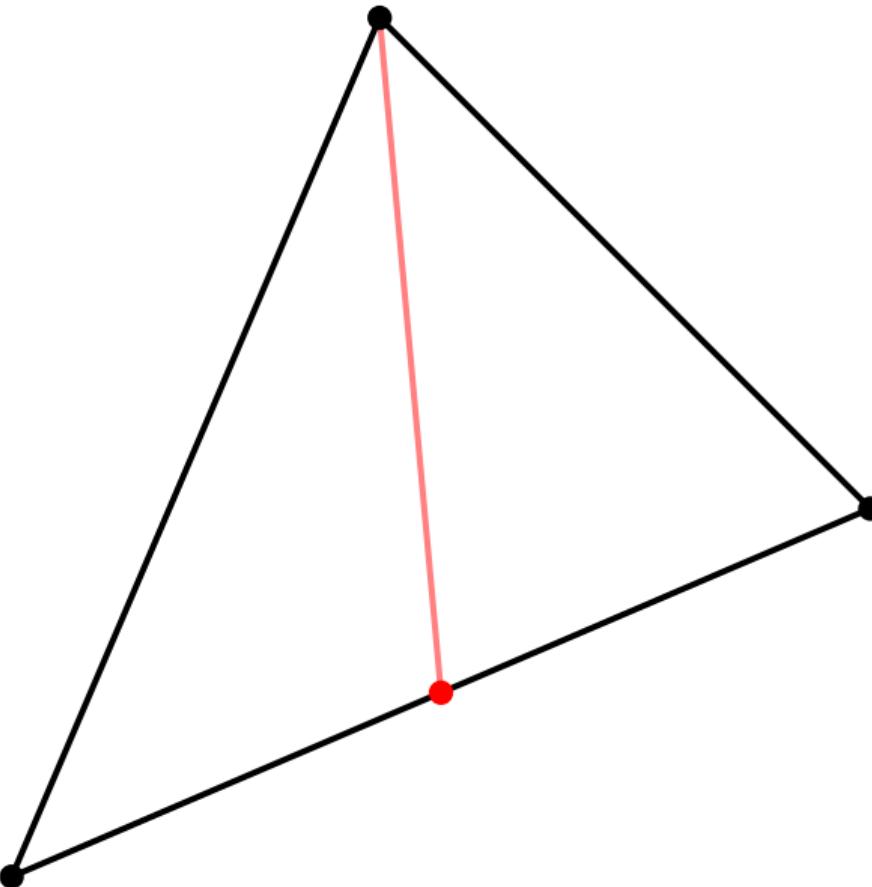
## Subdivisão de triângulos



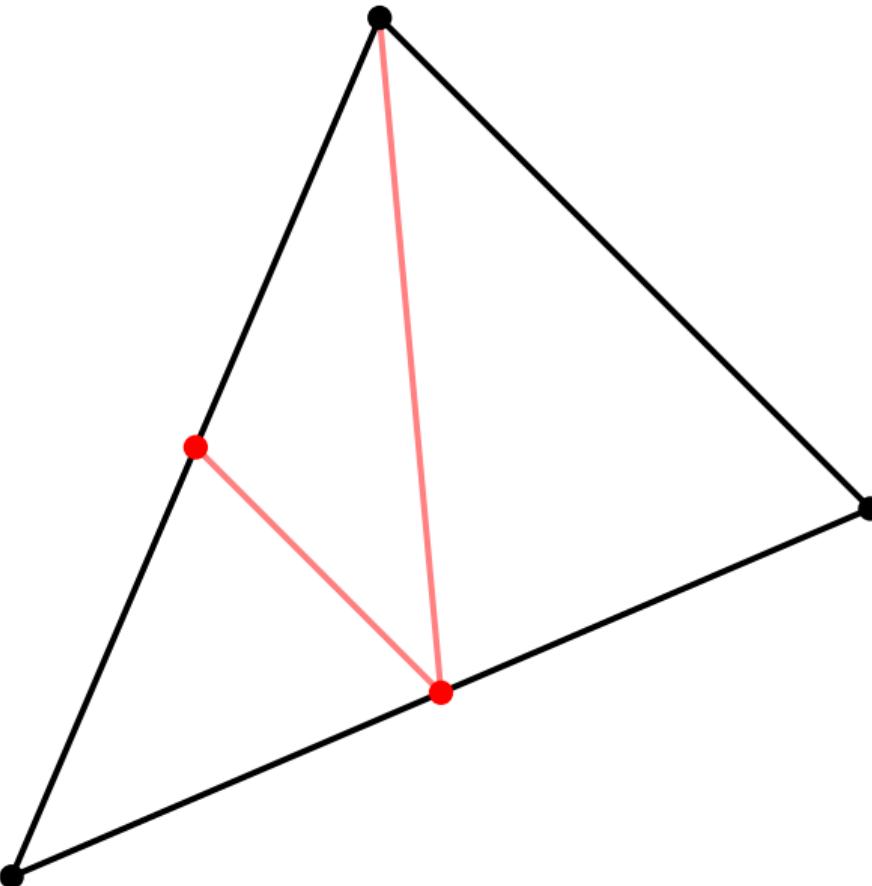
## Subdivisão de triângulos



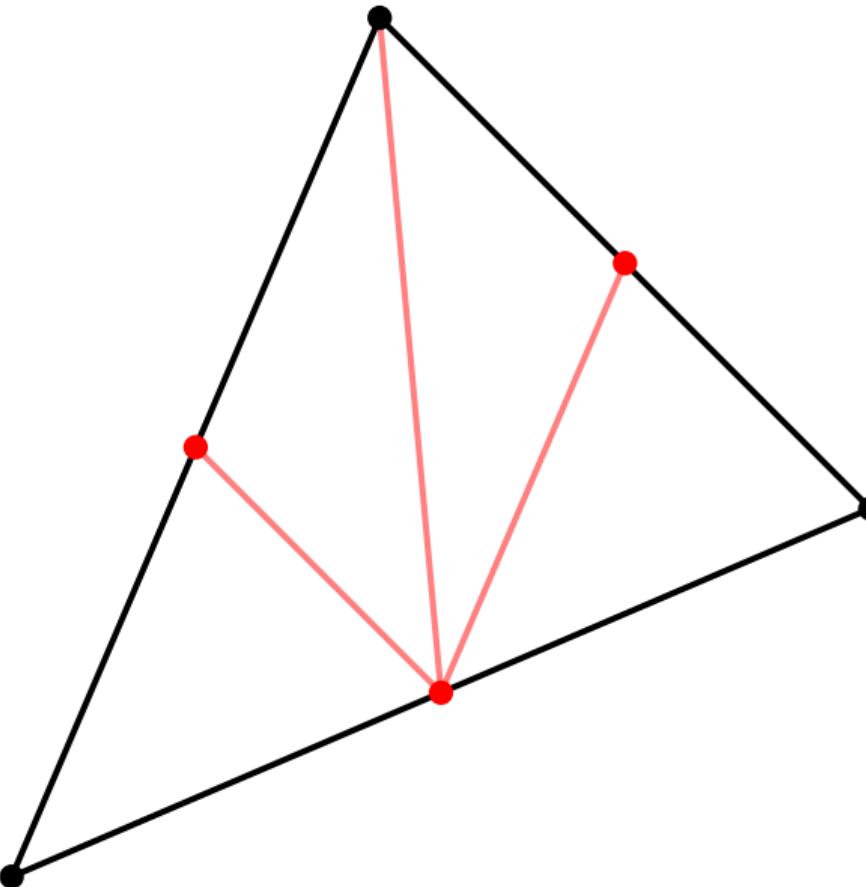
## Subdivisão de triângulos



## Subdivisão de triângulos



## Subdivisão de triângulos



curvas e superfícies de forma livre

## Requisitos para representação de formas

- pontos de controle
- intrínseca
- versatilidade
- controle de continuidade
- controle local
- redução de variação



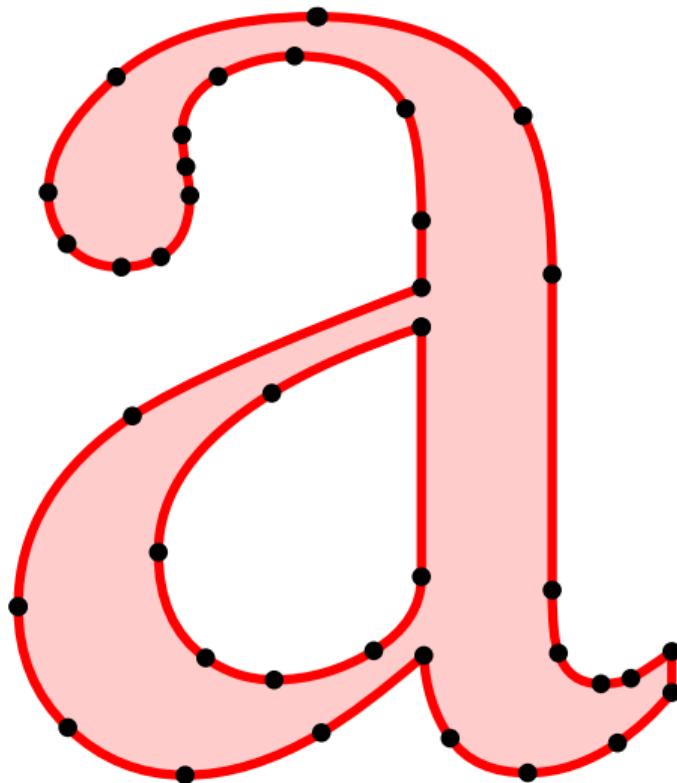
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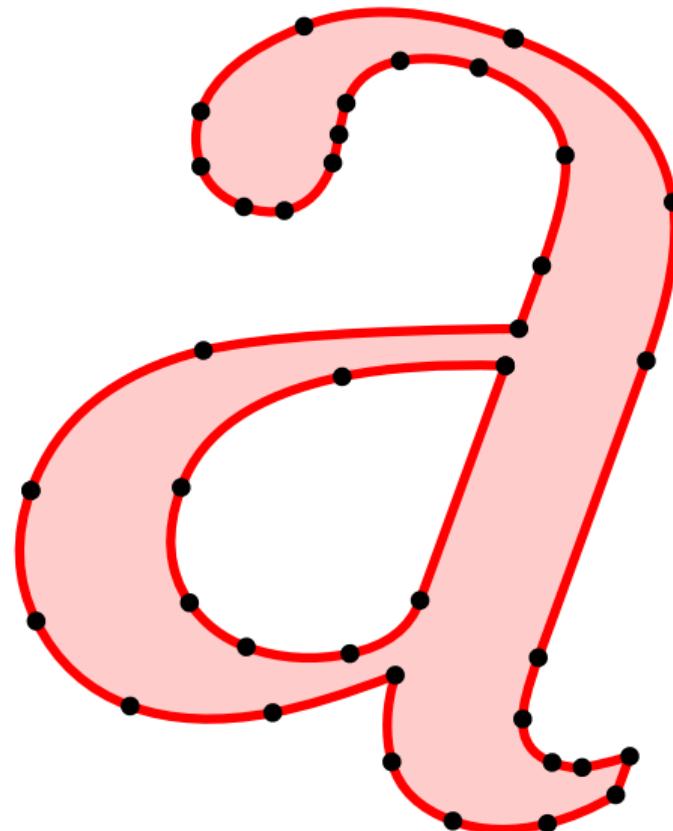
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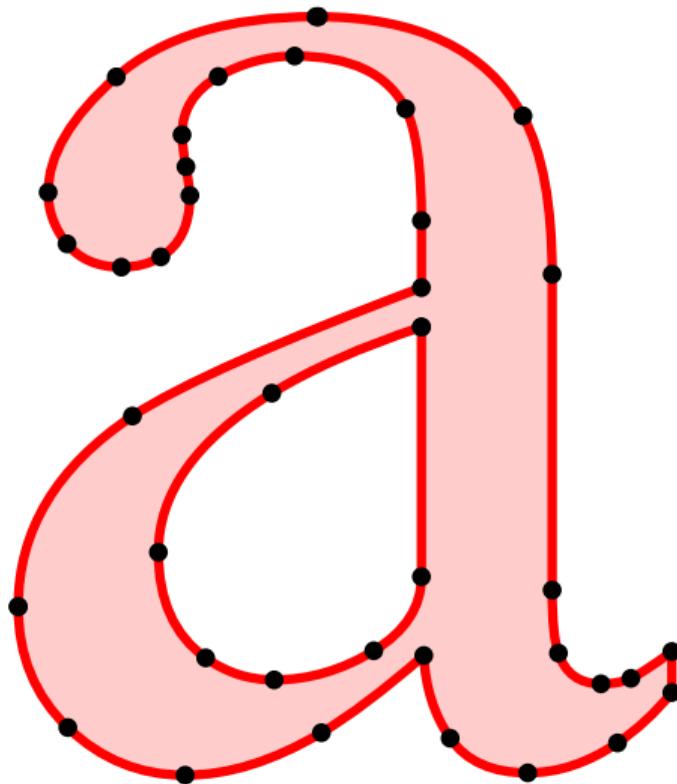
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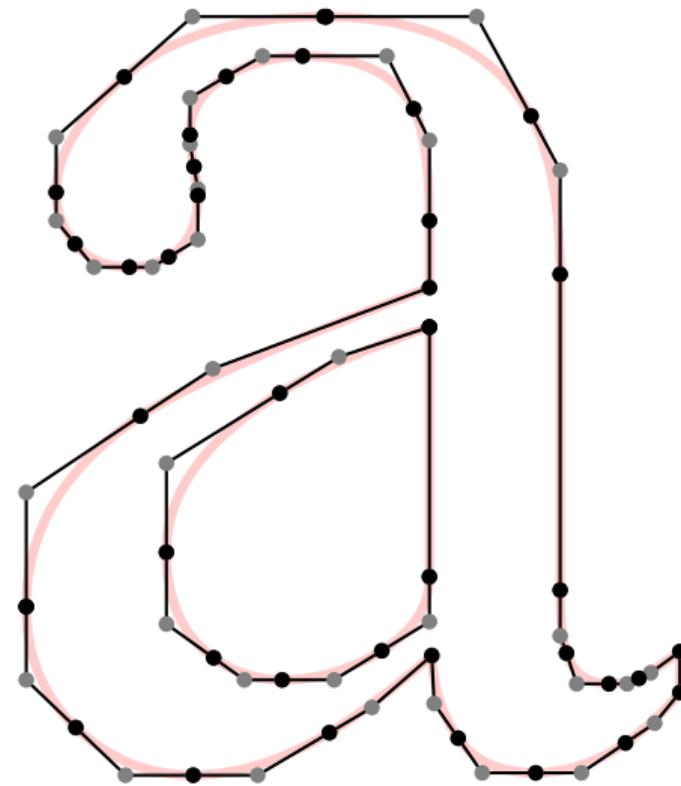
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# Requisitos para representação de formas

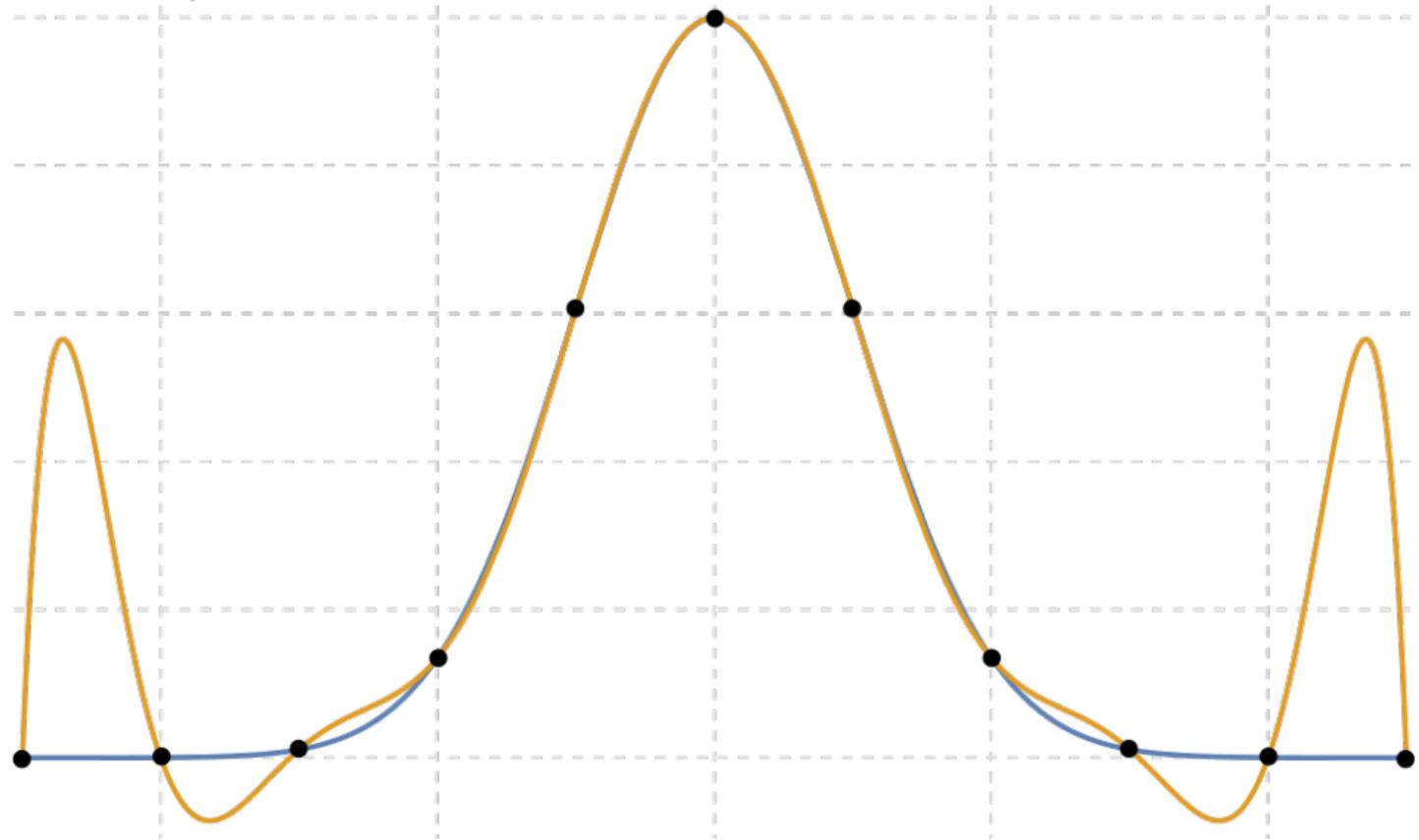
- pontos de controle
- intrínseca
- versatilidade
- controle de continuidade
- controle local
- redução de variação



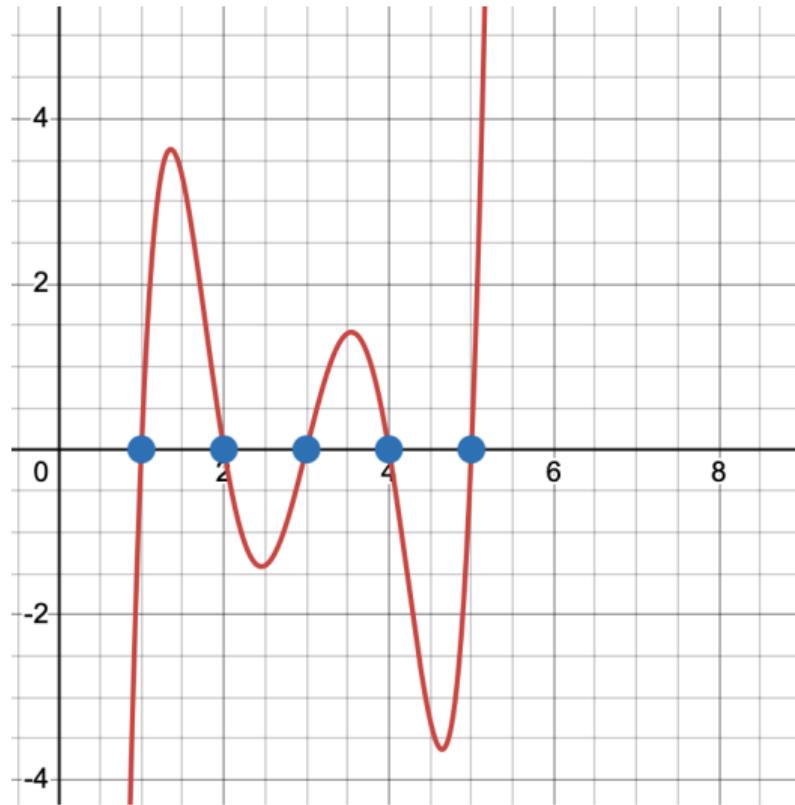
# Requisitos para representação de formas

■ pontos de controle	M 146.75 138.5
	Q 118.0 129.0 99.25 117.5
	Q 63.25 95.25 63.25 67.0
	Q 63.25 44.25 78.25 33.5
	Q 88.0 26.5 100.0 26.5
	Q 116.5 26.5 131.625 35.75
	Q 146.75 45.0 146.75 59.25
■ intrínseca	L 146.75 138.5
■ versatilidade	M 18.75 49.75
■ controle de continuidade	Q 18.75 86.0 55.0 110.25
■ controle local	Q 78.0 125.25 146.75 151.0
■ redução de variação	L 146.75 172.25
	Q 146.75 197.75 141.75 207.75
	Q 133.25 224.5 106.5 224.5
	Q 93.75 224.5 82.25 218.0
	...

## Aumento de variação



## Aumento de variação



# Objetos paramétricos

curvas paramétricas

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\gamma: [a, b] \rightarrow \mathbf{R}^3$$

$$\gamma(t) = (x(t), y(t), z(t))$$

superfícies paramétricas

$$\sigma: [a, b] \times [c, d] \rightarrow \mathbf{R}^3$$

$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

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$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

funções polinomiais e racionais

avaliação computacional exata e eficiente

## Funções polinomiais

avaliação ingênua

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$v \leftarrow 0$$

$$v \leftarrow v + a_0$$

$$v \leftarrow v + a_1 t$$

$$v \leftarrow v + a_2 t^2$$

$$v \leftarrow v + a_3 t^3$$

...

$$v \leftarrow v + a_n t^n$$

$$v = f(t)$$

$$t^k = \text{pow}(t, k)$$

## Funções polinomiais

avaliação menos ingênuas

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$v \leftarrow 0 \qquad s \leftarrow 1$$

$$v \leftarrow v + a_0 s \quad s \leftarrow s t$$

$$v \leftarrow v + a_1 s \quad s \leftarrow s t$$

$$v \leftarrow v + a_2 s \quad s \leftarrow s t$$

$$v \leftarrow v + a_3 s \quad s \leftarrow s t$$

...

$$v \leftarrow v + a_n s \quad s \leftarrow s t$$

$$v = f(t)$$

$$t^{k+1} = t^k \cdot t, \quad t^0 = 1$$

# Funções polinomiais

avaliação eficiente

Horner

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$= a_0 + t(a_1 + t(a_2 + t(a_3 + \cdots + t(a_{n-1} + t a_n) \cdots)))$$

$$v \leftarrow a_n$$

$$v \leftarrow a_{n-1} + t v$$

...

$$v \leftarrow a_3 + t v$$

$$v \leftarrow a_2 + t v$$

$$v \leftarrow a_1 + t v$$

$$v \leftarrow a_0 + t v$$

$$v = f(t)$$

fused multiply-add (FMA)

## Funções polinomiais

interpretação dos coeficientes

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

expansão de Taylor em torno de  $t = 0$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

informação pontual profunda determina forma global

não intuitivo

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots + b_n t^n$$

$$\gamma(t) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots + P_n t^n \quad P_k = (a_k, b_k)$$

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

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$$\begin{aligned}\gamma(t) &= P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots + P_n t^n & P_k &= (a_k, b_k) \\ &= P_0 + tP_1 + t^2P_2 + t^3P_3 + \cdots + t^n P_n\end{aligned}$$

combinação linear de **pontos de controle** com pesos  $1, t, t^2, t^3, \dots, t^n$

não intuitivo

## Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

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combinação linear de **pontos de controle** com pesos  $1, t, t^2, t^3, \dots, t^n$

não intuitivo

como escolher **pontos de controle intuitivos?**

coeficientes geométricos

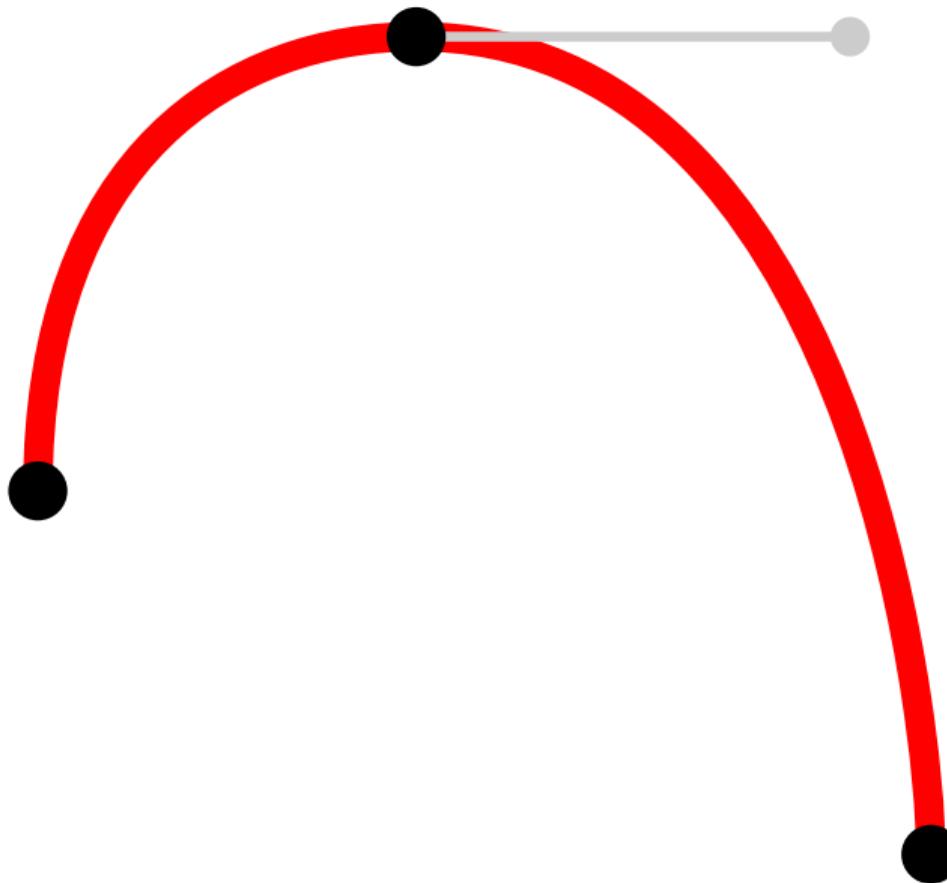
curvas de Hermite

## Continuidade geométrica

- $\alpha, \beta: [0, 1] \rightarrow \mathbf{R}^d$
- continuidade  $C^0$   
 $\alpha(1) = \beta(0)$
- continuidade  $C^1$   
 $\alpha'(1) = \beta'(0)$  paramétrica
- continuidade  $G^1$   
 $\alpha'(1) \parallel \beta'(0)$  geométrica

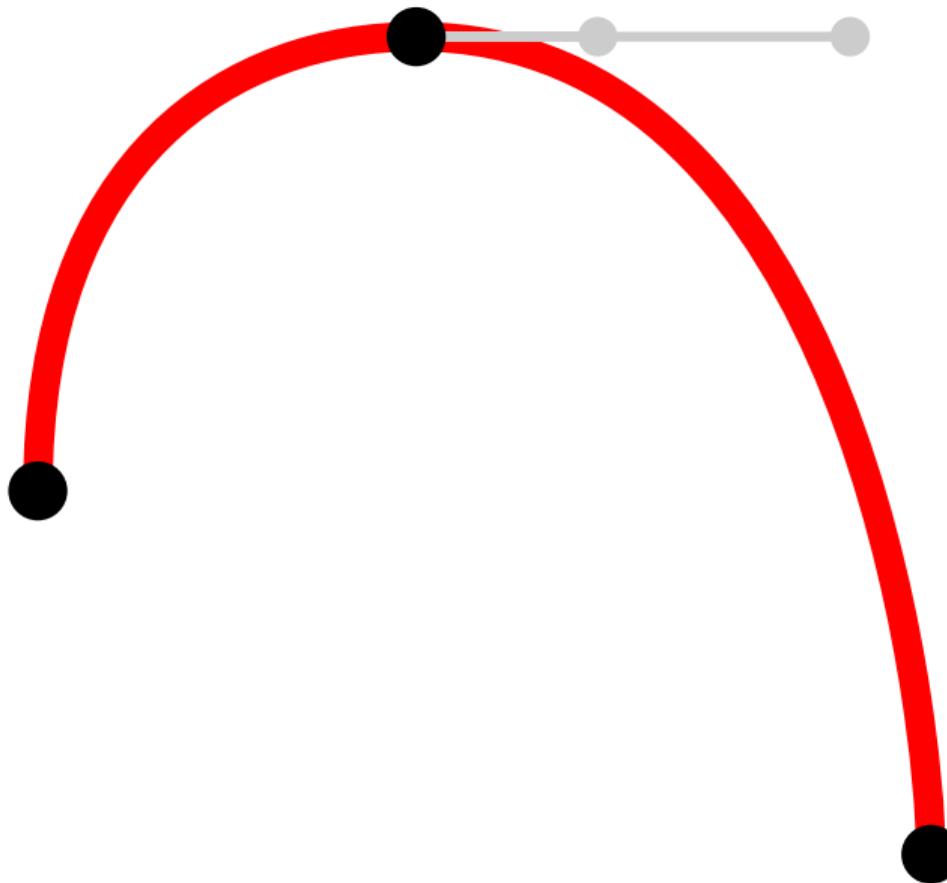
Continuidade paramétrica

$$\alpha'(1) = \beta'(0)$$



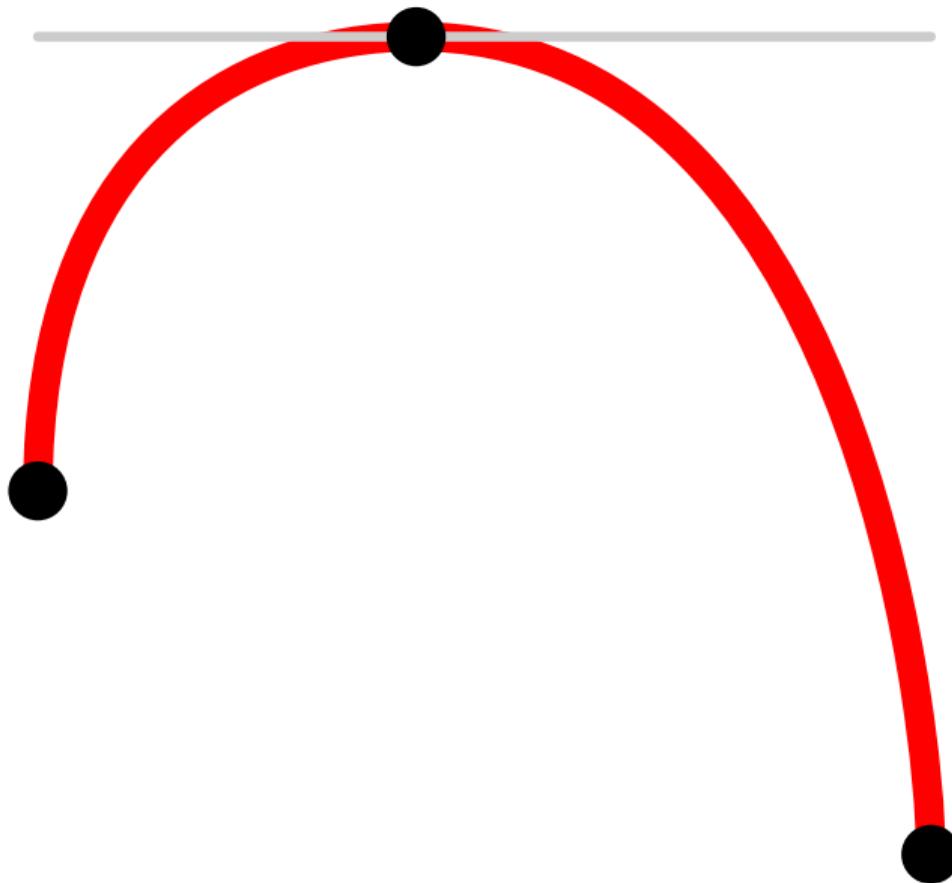
## Continuidade geométrica

$$\alpha'(1) \parallel \beta'(0)$$

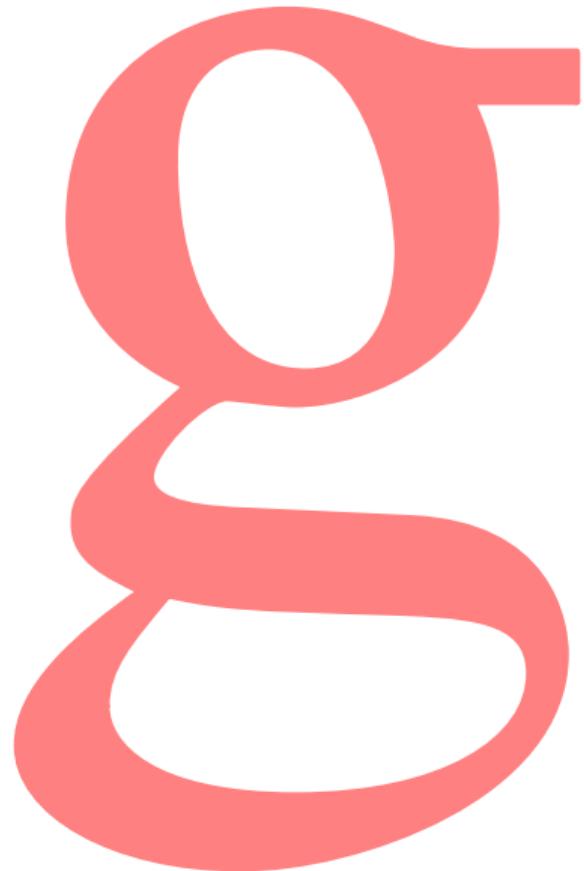


## Continuidade geométrica

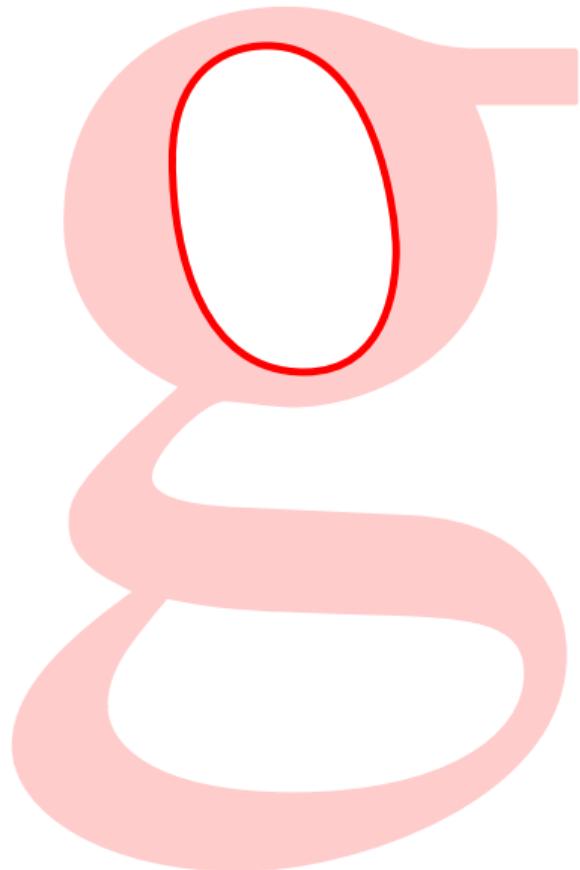
$$\alpha'(1) \parallel \beta'(0)$$



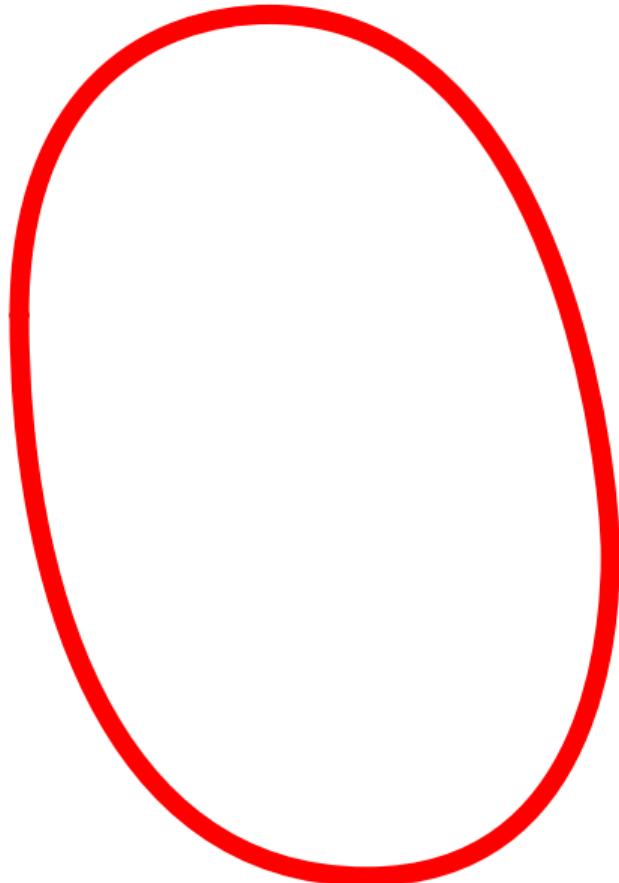
## Continuidade geométrica



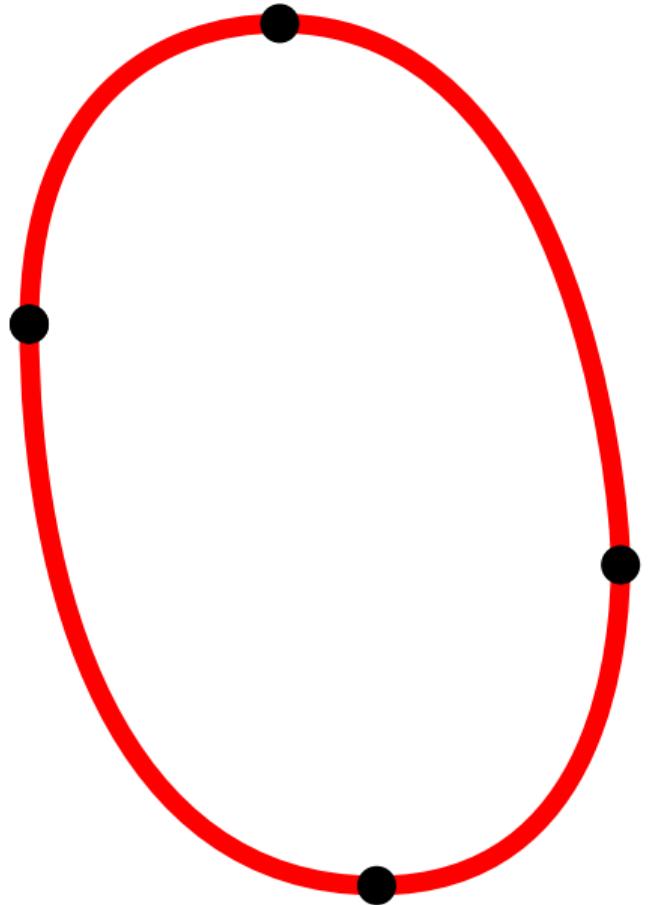
## Continuidade geométrica



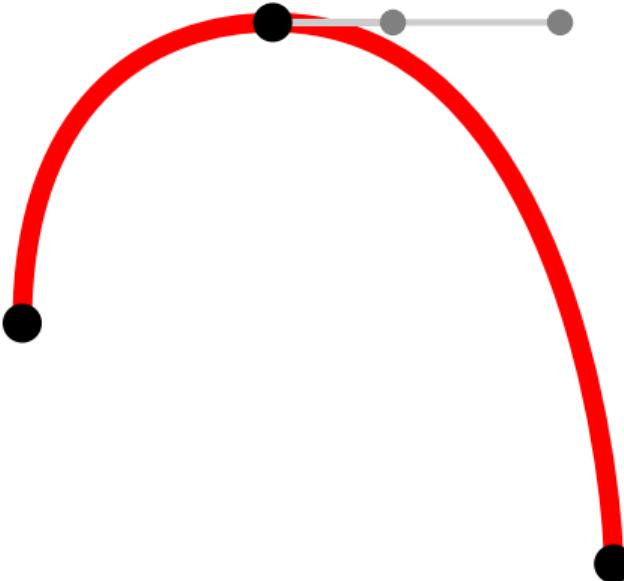
## Continuidade geométrica



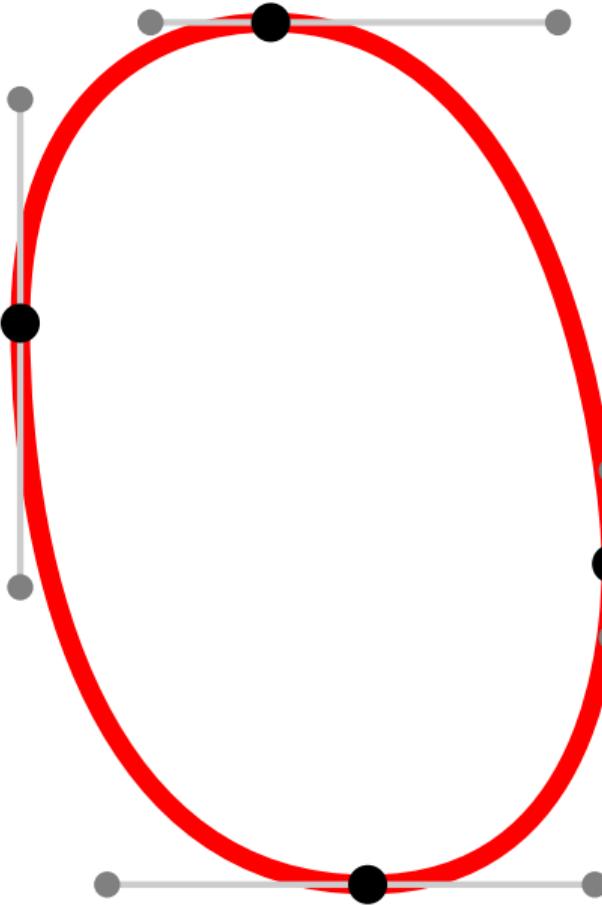
## Continuidade geométrica



## Continuidade geométrica

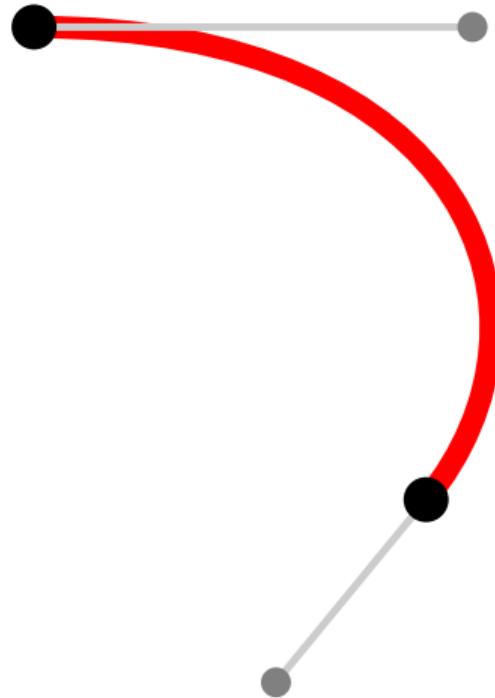


## Continuidade geométrica



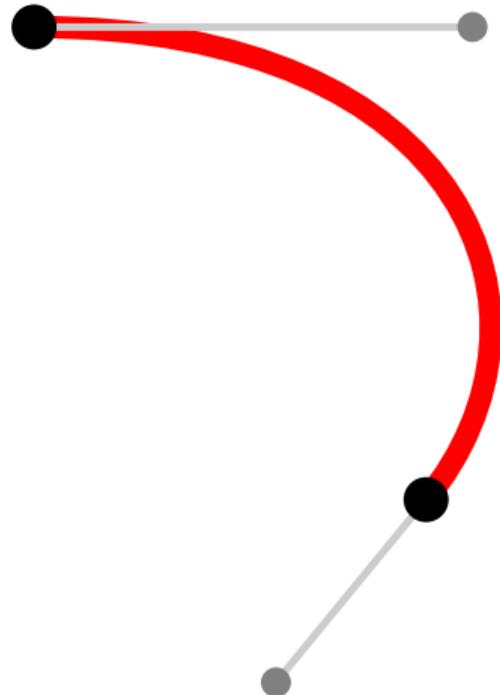
## Cúbica de Hermite

- pontos extremos  $p_0, p_1$
- vetores tangentes  $v_0, v_1$



## Cúbica de Hermite

- pontos extremos  $p_0, p_1$
- vetores tangentes  $v_0, v_1$
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
- $\gamma(1) = p_1$
- $\gamma'(0) = v_0$
- $\gamma'(1) = v_1$



## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

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$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad a_k \in \mathbf{R}^d$$

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$$\gamma \mapsto (\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)) \quad \text{transformação linear}$$

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix}$$

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$$\gamma(t) = p_0 + v_0 t + (-3p_0 + 3p_1 - 2v_0 - v_1)t^2 + (2p_0 - 2p_1 + v_0 + v_1)t^3$$

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\gamma(t) = p_0 + v_0 t + (-3p_0 + 3p_1 - 2v_0 - v_1)t^2 + (2p_0 - 2p_1 + v_0 + v_1)t^3$$

monomial

## Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\gamma(t) = p_0 + v_0 t + (-3p_0 + 3p_1 - 2v_0 - v_1)t^2 + (2p_0 - 2p_1 + v_0 + v_1)t^3$$

monomial

$$= (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

Hermite

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monomial

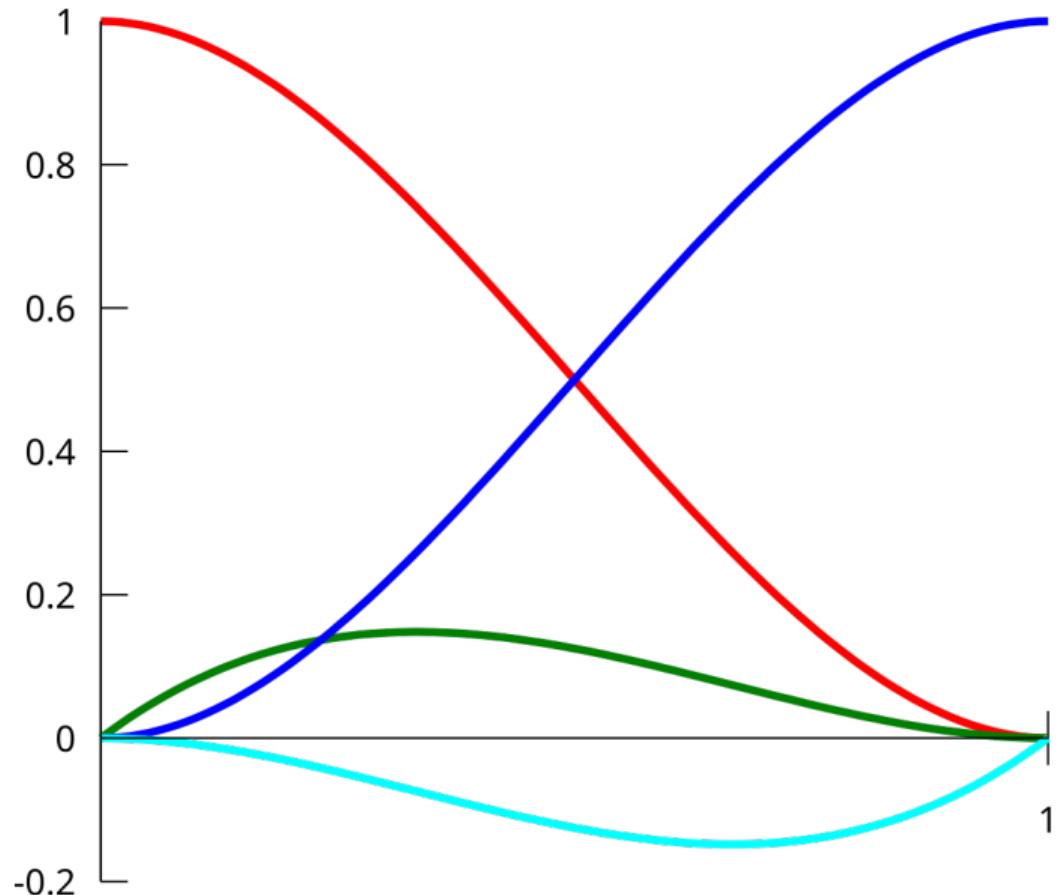
$$= (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

Hermite

$$= B^3[p_0, p_0 + \frac{1}{3}v_0, p_1 - \frac{1}{3}v_1, p_1]$$

Bézier

## Base de Hermite



## Base de Hermite

$$\gamma(t) = (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

$$= H_1(t) p_0 + H_2(t) p_1 + H_3(t) v_0 + H_4(t) v_1$$

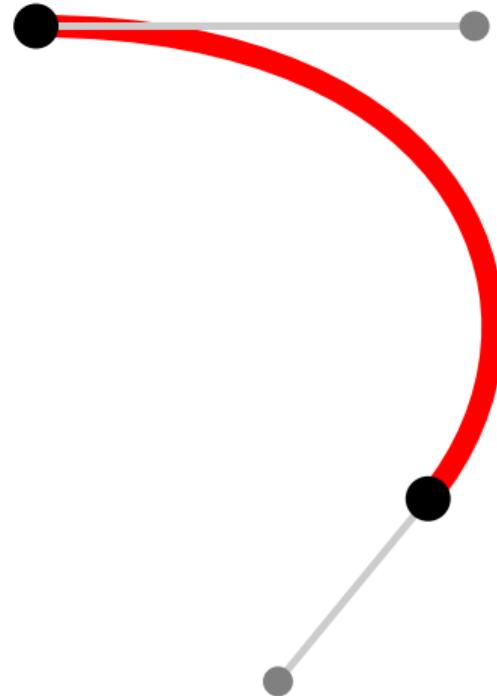
pontos de controle mistos

$f$	$H_1$	$H_2$	$H_3$	$H_4$
$f(0)$	1	0	0	0
$f(1)$	0	1	0	0
$f'(0)$	0	0	1	0
$f'(1)$	0	0	0	1

$$H_1(t) = H_2(1-t), \quad H_3(t) = -H_4(1-t)$$

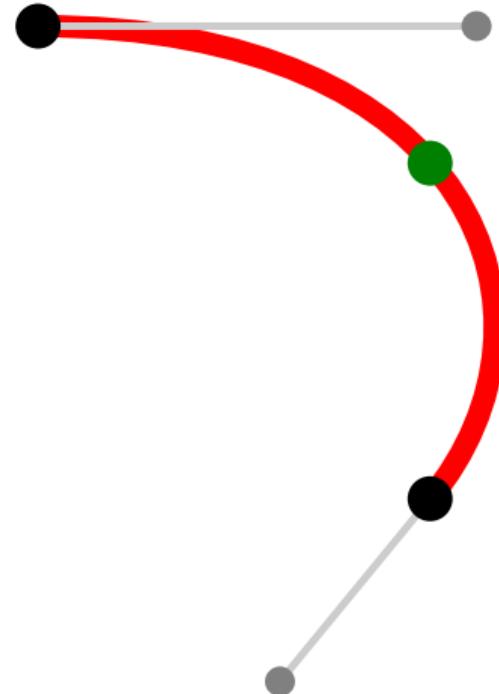
## Cúbica de Hermite geométrica

- pontos extremos  $p_0, p_1$
- direções tangentes  $v_0, v_1$
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
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- $\gamma'(1) = \lambda_1 v_1$



## Cúbica de Hermite geométrica

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- $\gamma(t^*) = p^*$        $d = 2$ :  $t^*$  dado /  $d = 3$ : equação cúbica em  $t^*$



curvas de Bézier

## Formulação geométrica

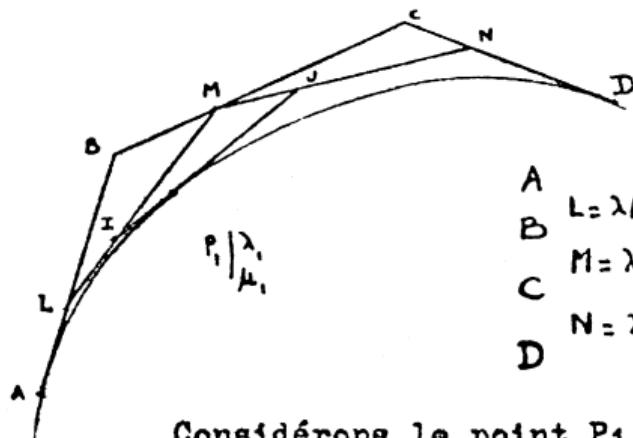
- de Casteljau (Citroën, 1959)      interpolação linear repetida
- Bézier (Renault, 1967)      ponto + deslocamentos
- Forrest (1972)      polinômios de Bernstein

# Formulação geométrica

de Casteljau (Citroën, 1963)

## 1.5.- Sous-Pôles d'une courbe

### 1.5.1.- Définition des sous-pôles.



Considérons une cubique des pôles A, B, C, D. Nous avons vu précédemment que la construction par la méthode des barycentres donnent les différents points L, M, N, I, J, P.

$$A \quad L = \lambda A + \mu B$$

$$B \quad M = \lambda B + \mu C$$

$$C \quad N = \lambda C + \mu D$$

$$D$$

$$I = \lambda^2 A + 2\lambda\mu B + \mu^2 C$$

$$J = \lambda^2 B + 2\lambda\mu C + \mu^2 D$$

$$P_1 = \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D$$

Considérons le point  $P_1$  de paramètres  $\lambda$ , et  $\mu$ , (avec  $\lambda_1 + \mu_1 = 1$ ) ( $\lambda_1$  varie de 0 à 1 et  $\mu_1$  de 0 à 1, lorsque P va en D).

Cherchons les pôles de la cubique  $P_1$ , D. Cette courbe dérive de la cubique initiale AD par changement des paramètres.

$$P = \lambda^3 A + 3\lambda^2\mu B + 3\lambda\mu^2 C + \mu^3 D, \quad \lambda + \mu = 1, \quad \lambda = 1 - \mu, \quad \mu \in [0, 1]$$

# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

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segmento se  $t \in [0, 1]$

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de Casteljau (Citroën, 1963)

interpolação linear

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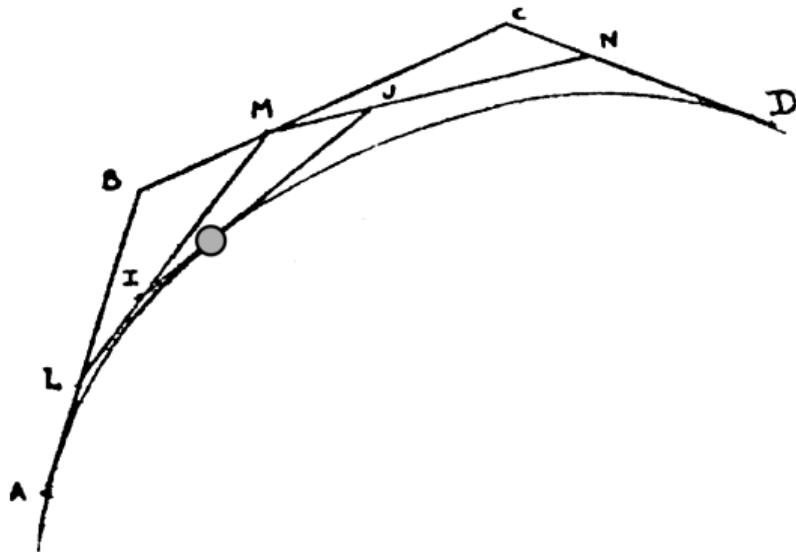
interpolação linear repetida

A

B

C

D



# Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

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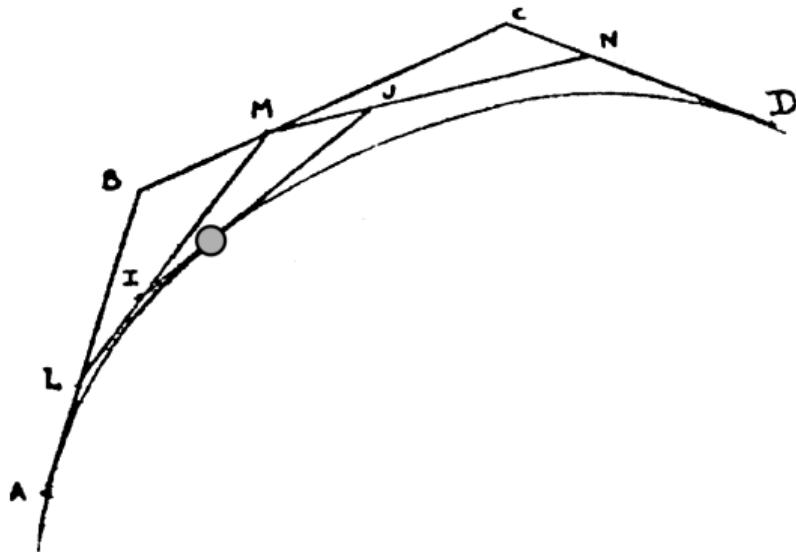
interpolação linear repetida

A       $L = \text{lerp}(t, A, B)$

B       $M = \text{lerp}(t, B, C)$

C       $N = \text{lerp}(t, C, D)$

D



# Formulação geométrica

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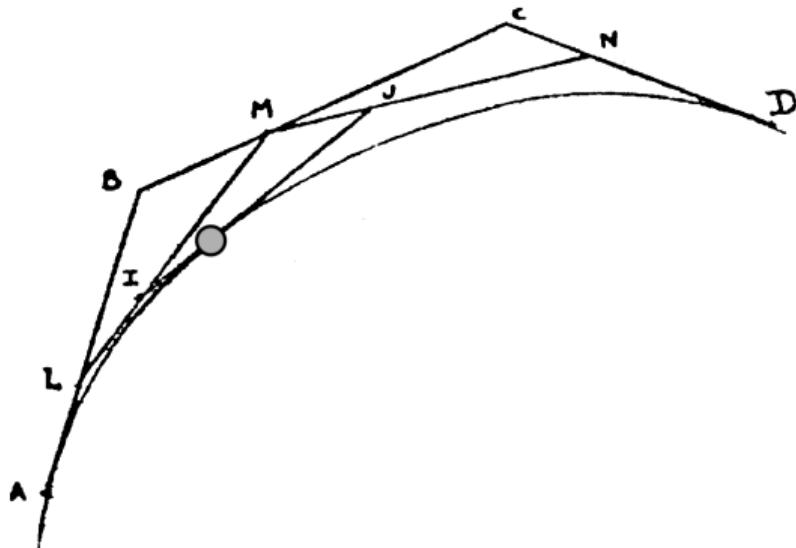
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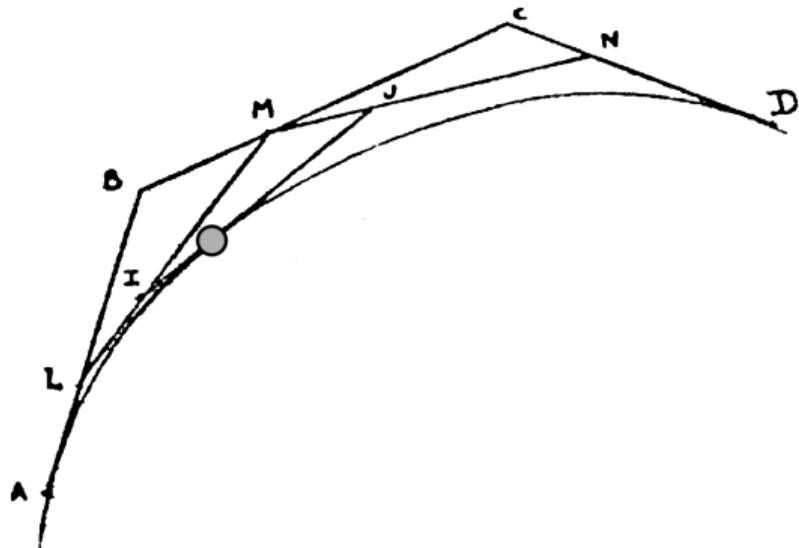
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D



# Formulação geométrica

de Casteljau (Citroën, 1963)

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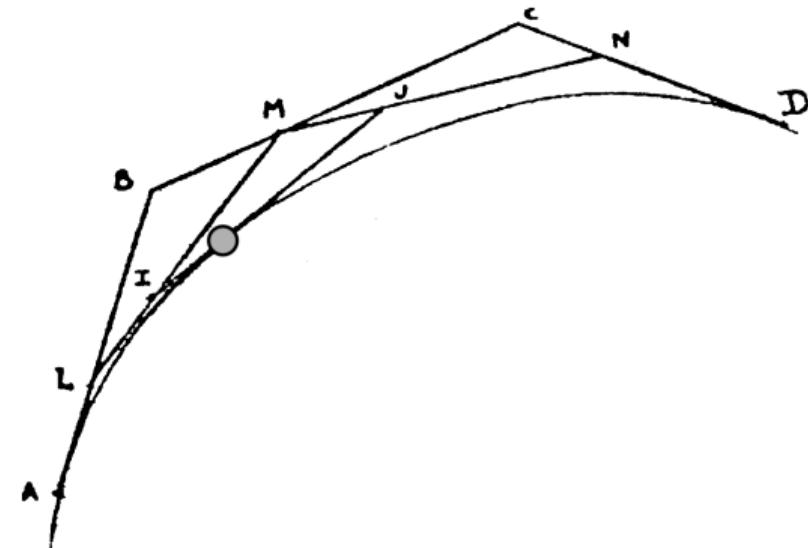
interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

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D



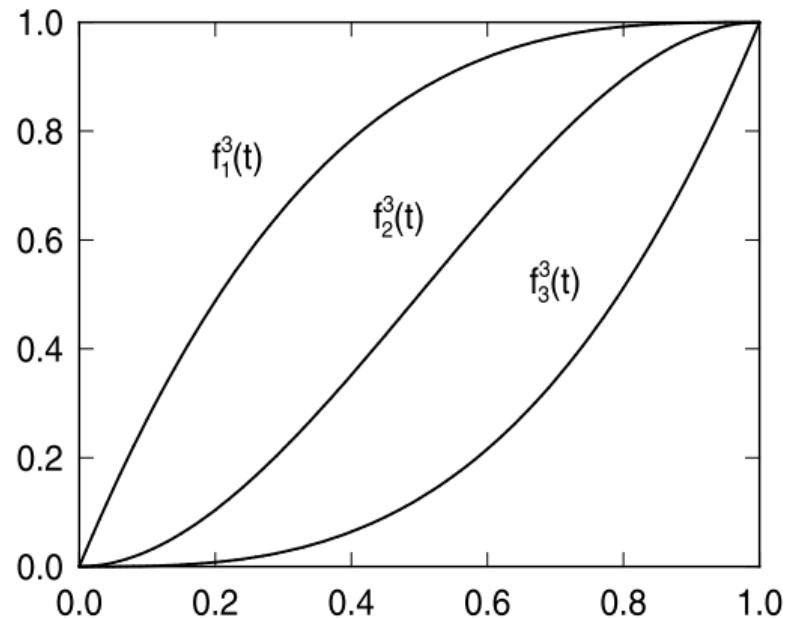
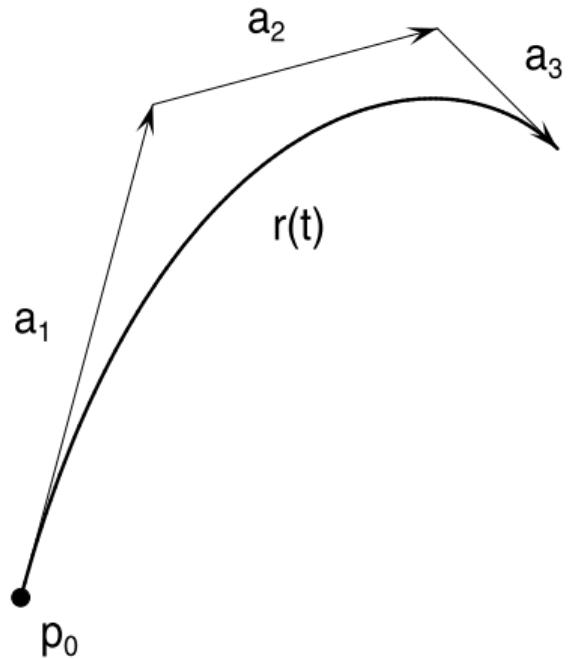
$$P = (1 - t)^3 A + 3(1 - t)^2 t B + 3(1 - t)t^2 C + t^3 D$$

## Formulação geométrica

Bézier (Renault, 1967)

ponto + deslocamentos:

$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$



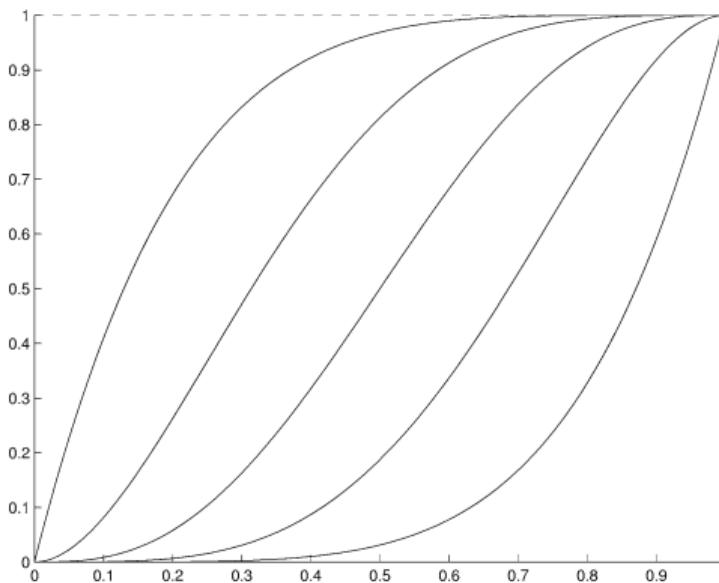
# Formulação geométrica

Bézier (Renault, 1967)

ponto + deslocamentos:  $B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$

- $f_k^n$  polinômio de grau  $n$
- $f_k^n(0) = 0$
- $f_k^n(1) = 1$
- $(f_k^n)^{(r)}(0) = 0, \quad r = 1, \dots, k - 1$
- $(f_k^n)^{(r)}(1) = 0, \quad r = 1, \dots, n - k$

interpolação de Hermite



Laurent-Sablonnière

Onésime Duran

## Formulação geométrica

Forrest (1972)

$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$

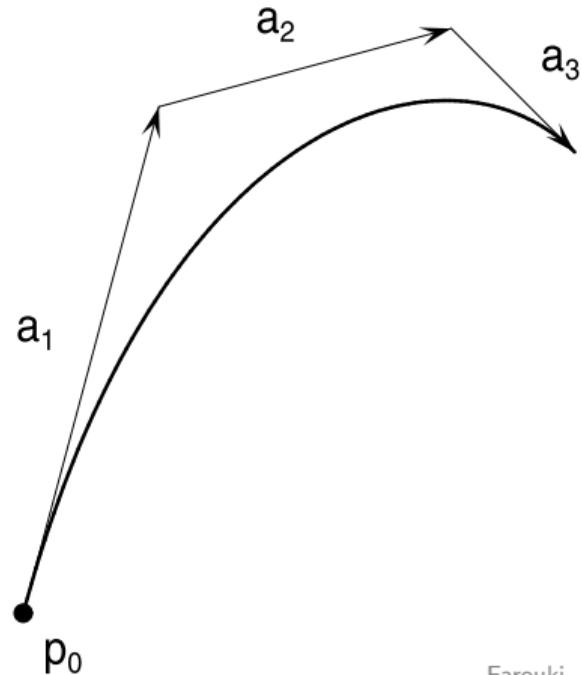
$$p_k = p_0 + \sum_{j=1}^k a_j$$

$$a_k = p_k - p_{k-1}$$

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

curva de Bézier

$$B_k^n(t) = f_k^n(t) - f_{k+1}^n(t)$$



Farouki

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \text{polinômios de Bernstein}$$

## Curvas de Bézier

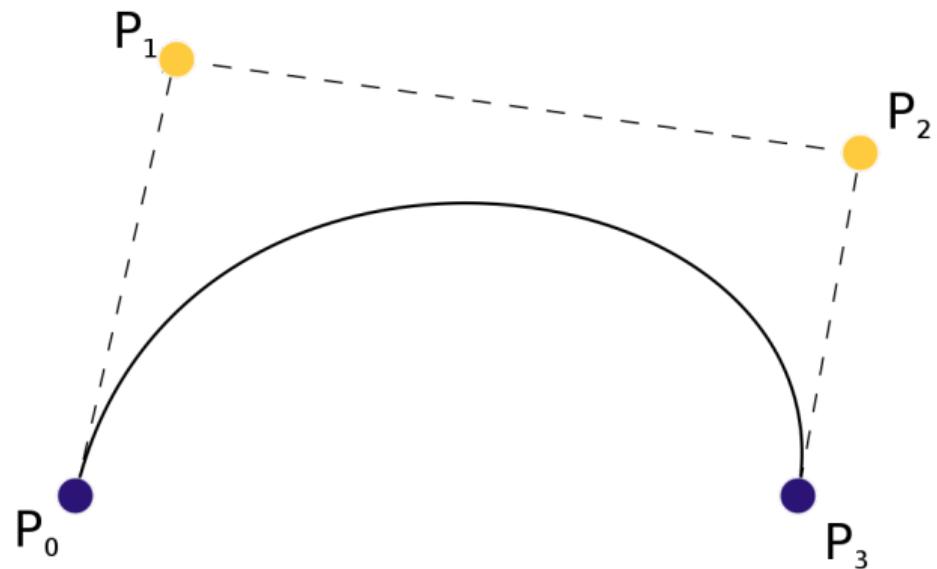
$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

$$B^0(t) = p_0$$

$$B^1(t) = (1-t)p_0 + tp_1$$

$$B^2(t) = (1-t)^2 p_0 + 2(1-t)t p_1 + t^2 p_2$$

$$B^3(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

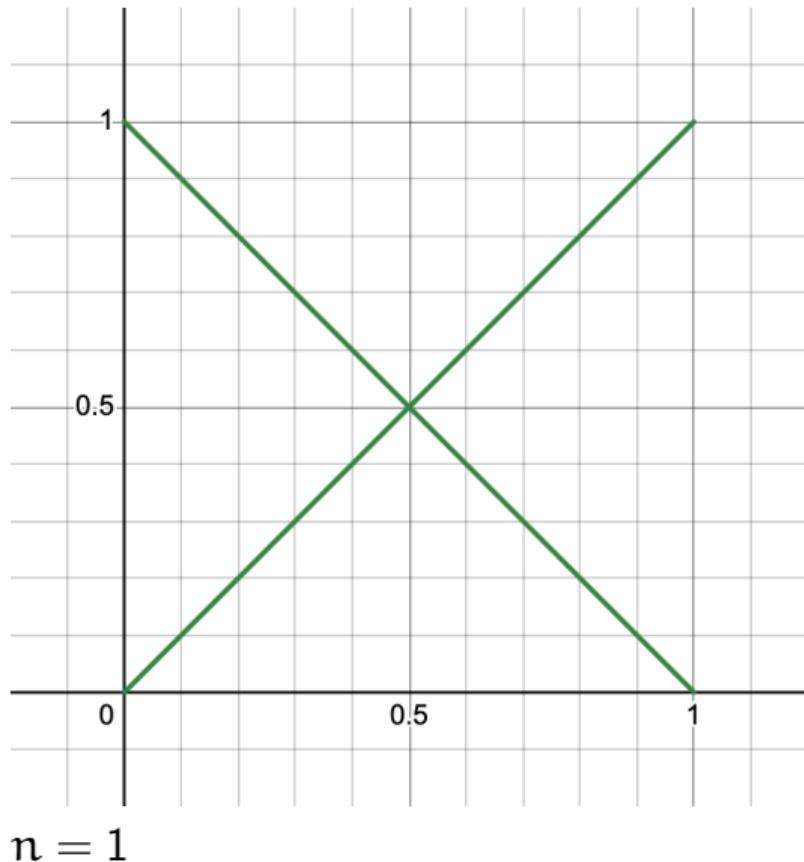


demo

# Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

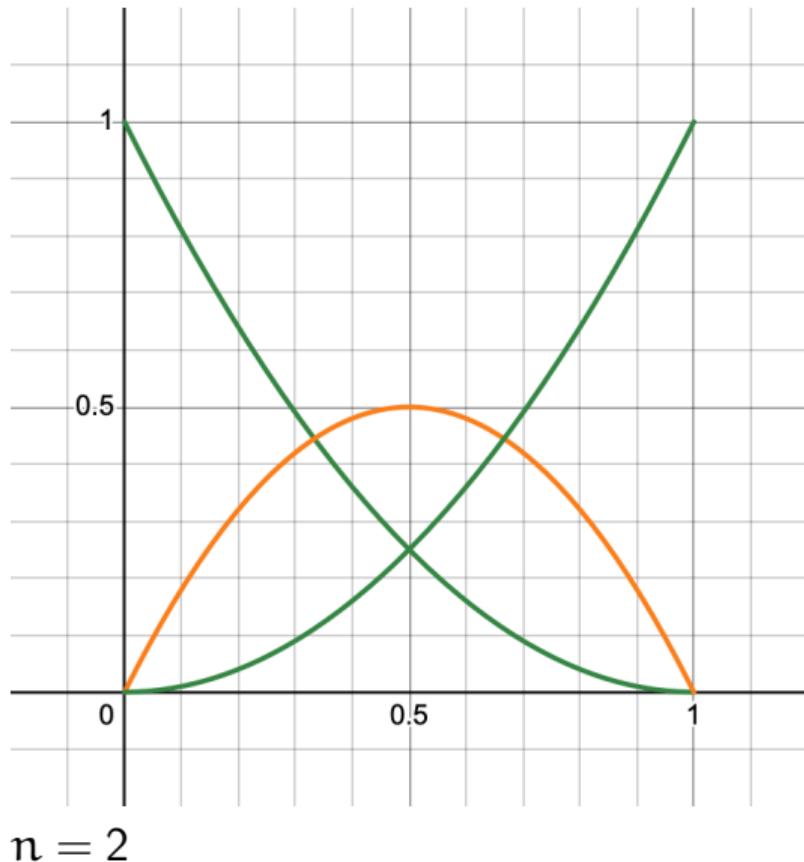
n				
0	1			
1	1 - t	t		
2	$(1 - t)^2$	$2(1 - t)t$	$t^2$	
3	$(1 - t)^3$	$3(1 - t)^2t$	$3(1 - t)t^2$	$t^3$
k	0	1	2	3



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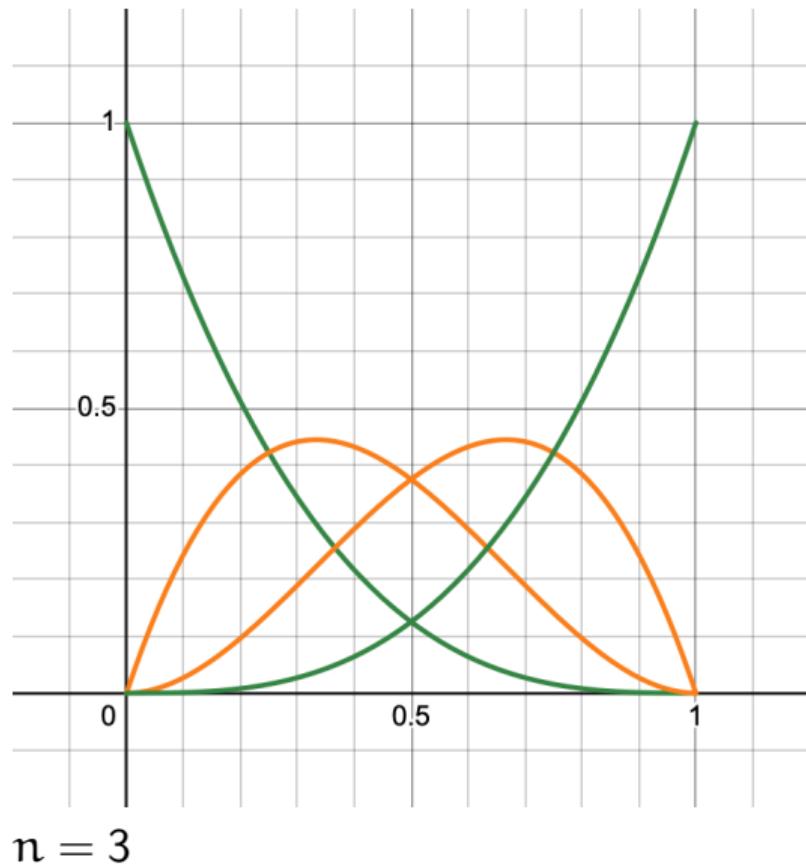
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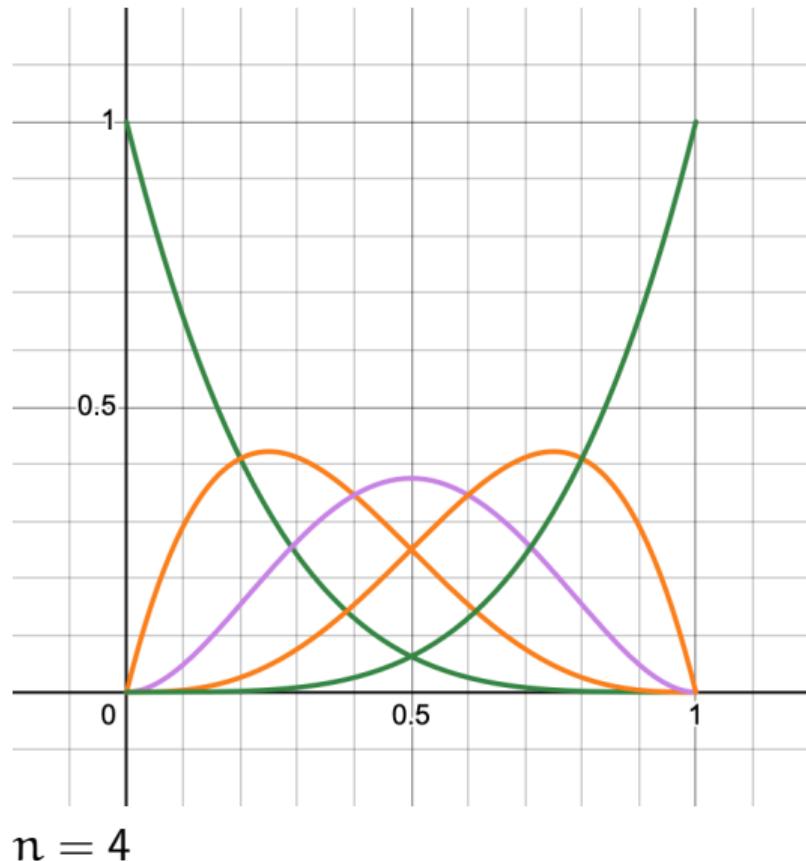
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0				
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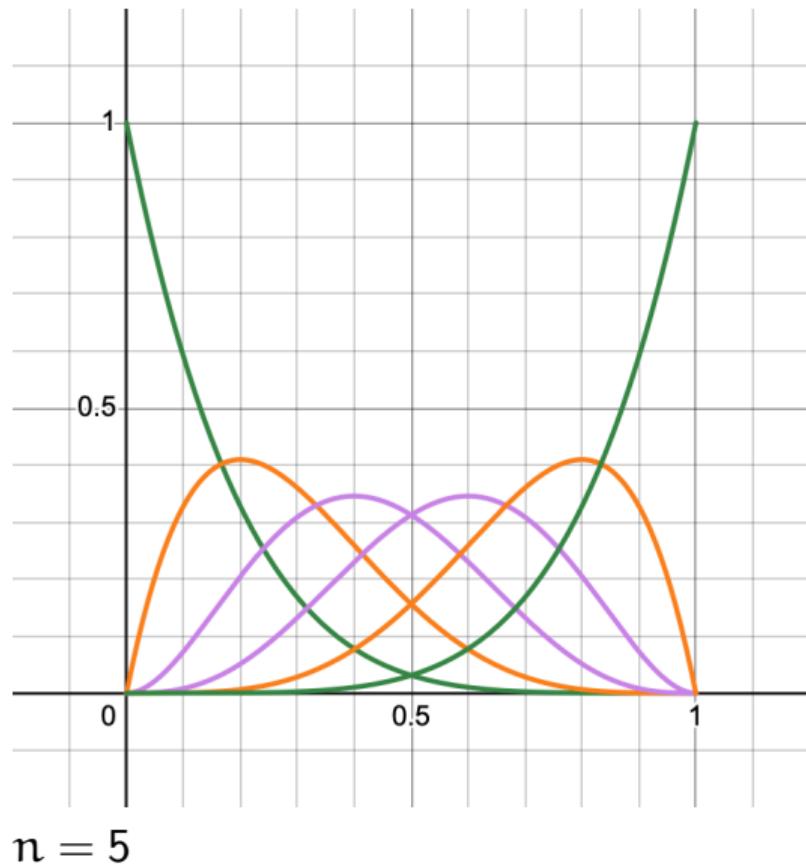
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# Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n	
0	1
1	$1 - t$ $t$
2	$(1 - t)^2$ $2(1 - t)t$ $t^2$
3	$(1 - t)^3$ $3(1 - t)^2t$ $3(1 - t)t^2$ $t^3$
k	0                  1                  2                  3



## Teorema de aproximação

Weierstrass (1885)

Dada  $f \in C[0, 1]$ , para todo  $\varepsilon > 0$  existe um polinômio  $p$  tal que

$$|f(t) - p(t)| < \varepsilon \quad \text{para todo } t \in [0, 1]$$

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$$\lim_{n \rightarrow \infty} p_n = f \quad \text{uniforme}$$

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prova construtiva:

Bernstein (1912)

$$B_n(f)(t) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_k^n(t) \implies \lim_{n \rightarrow \infty} B_n(f) = f \text{ uniforme}$$

$$f \in C^r \implies \lim_{n \rightarrow \infty} B_n(f)^{(r)} = f \text{ uniforme}$$

# Teorema de aproximação de Weierstrass

Bernstein (1912)

## Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités.

Je me propose d'indiquer une démonstration fort simple du théorème suivant de Weierstrass:

Si  $F(x)$  est une fonction continue quelconque dans l'intervalle  $01$ , il est toujours possible, quel que petit que soit  $\varepsilon$ , de déterminer un polynôme  $E_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$  de degré  $n$  assez élevé, tel qu'on ait

$$|F(x) - E_n(x)| < \varepsilon$$

en tout point de l'intervalle considéré.

A cet effet, je considère un événement  $A$ , dont la probabilité est égale à  $x$ . Supposons qu'on effectue  $n$  expériences et que l'on convienne de payer à un joueur la somme  $F\left(\frac{m}{n}\right)$ , si l'événement  $A$  se produit  $m$  fois. Dans ces conditions, l'espérance mathématique  $E_n$  du joueur aura pour valeur

$$E_n = \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}. \quad (1)$$

Or, il résulte de la continuité de la fonction  $F(x)$  qu'il est possible de fixer un nombre  $\delta$ , tel que l'inégalité

$$|x - x_0| \leq \delta$$

entraîne

$$|F(x) - F(x_0)| < \frac{\varepsilon}{2};$$

de sorte que, si  $\bar{F}(x)$  désigne le maximum et  $\underline{F}(x)$  le minimum de  $F(x)$  dans l'intervalle  $(x - \delta, x + \delta)$ , on a

$$\bar{F}(x) - F(x) < \frac{\varepsilon}{2}, \quad F(x) - \underline{F}(x) < \frac{\varepsilon}{2}. \quad (2)$$

— 2 —

Soit de plus  $\eta$  la probabilité de l'inégalité  $|x - \frac{m}{n}| > \delta$ , et  $L$  le maximum de  $|F(x)|$  dans l'intervalle  $01$ .

On aura alors

$$\underline{F}(x) \cdot (1 - \eta) - L \cdot \eta < E_n < \bar{F}(x) \cdot (1 - \eta) + L \cdot \eta. \quad (3)$$

Mais, en vertu du théorème de Bernoulli, on pourra prendre  $n$  assez grand pour avoir

$$\eta < \frac{\varepsilon}{4L}. \quad (4)$$

L'inégalité (3) se mettra donc successivement sous la forme

$$\underline{F}(x) + (\bar{F}(x) - \underline{F}(x)) - \eta(L + \underline{F}(x)) < E_n < \underline{F}(x) + (\bar{F}(x) - \underline{F}(x)) + \eta(L - \bar{F}(x))$$

et ensuite

$$\underline{F}(x) - \frac{\varepsilon}{2} - \frac{2L}{4L} \varepsilon < E_n < \underline{F}(x) + \frac{\varepsilon}{2} + \frac{2L}{4L} \varepsilon;$$

donc

$$|\underline{F}(x) - E_n| < \varepsilon \quad (5)$$

Or  $E_n$  est manifestement un polynôme de degré  $n$ .

Le théorème est donc démontré.

J'ajouterais seulement deux remarques.

Les polynomes approchés  $E_n(x)$  sont surtout commodes, il me semble, lorsqu'on connaît exactement ou approximativement les valeurs de  $F(x)$  pour  $x = \frac{m}{n}$  ( $m = 0, 1, \dots, n$ ).

La formule (1) et l'inégalité (5) montrent que, quelle que soit la fonction continue  $F(x)$ , on a

$$F(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}.$$

S. Bernstein.

## Polinômios de Bernstein – base do espaço de polinômios

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} t^{k+j}$$

base canônica = base monomial:  $1, t, t^2, t^3, \dots, t^n$

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$$\begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3(1-t)^2t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

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$$\begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3(1-t)^2t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$t^k = \frac{1}{\binom{n}{k}} \sum_{j=k}^n \binom{j}{k} B_j^n(t)$$
$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

## Polinômios de Bernstein – propriedades

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

- $B_k^n(t) = B_{n-k}^n(1-t)$  simetria
- $B_k^n(0) = 0, \quad B_k^n(1) = 0, \quad B_0^n(0) = 1, \quad B_n^n(1) = 1$
- $B_k^n(t)$  tem único máximo em  $[0, 1]$ , em  $t = k/n$  unimodal
- $B_k^n(t) \geq 0$  para  $t \in [0, 1]$  não negativo
- $\sum_{k=0}^n B_k^n(t) = ((1-t) + t)^n = 1$  partição da unidade
- $\sum_{k=0}^n \frac{k}{n} B_k^n(t) = t$  precisão linear

## Curvas de Bézier – propriedades

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

- $B(t) = B(1 - t)$  simetria
- $B[p_0, \dots, p_n](t) = B[p_n, \dots, p_0](1 - t)$
- a influência de  $p_k$  é maior perto de  $t = k/n$  unimodal
- $B(t)$  é combinação convexa de  $p_0, \dots, p_n$  não negativo
  - curva contida no fecho convexo + partição da unidade
- $B(t)$  é gráfico de função quando  $x_k = k/n$  precisão linear
  - segmento de reta se pontos de controle uniformemente espaçados

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$B_k^{n+1}(t) = \binom{n+1}{k} (1-t)^{n+1-k} t^k$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \end{aligned}$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \\ &= (1-t) \binom{n}{k} (1-t)^{n-k} t^k + t \binom{n}{k-1} (1-t)^{n-(k-1)} t^{k-1} \end{aligned}$$

## Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

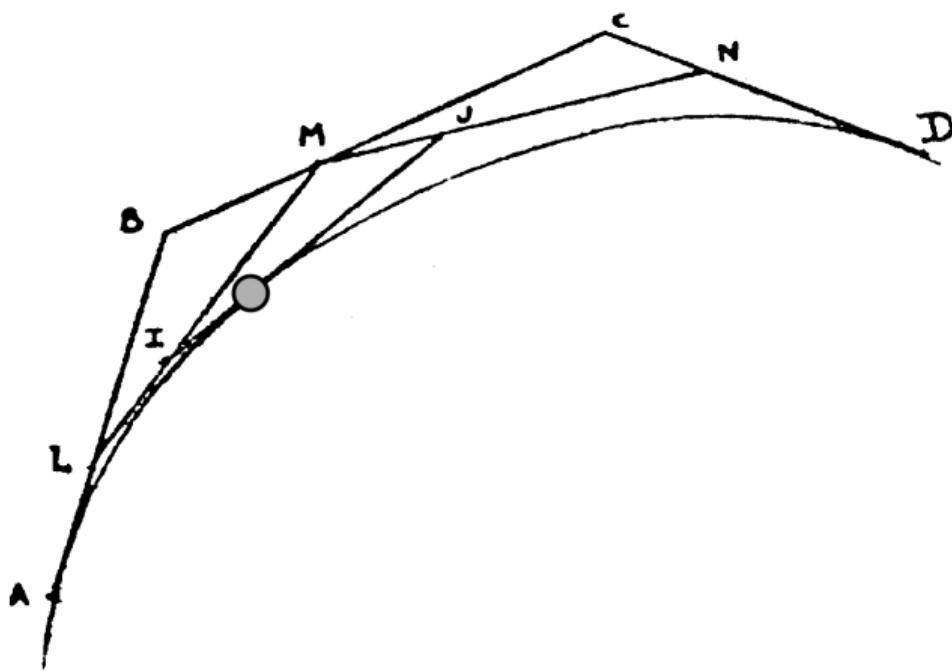
$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \\ &= (1-t) \binom{n}{k} (1-t)^{n-k} t^k + t \binom{n}{k-1} (1-t)^{n-(k-1)} t^{k-1} \end{aligned}$$

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t) \quad \text{interpolação linear repetida}$$

## Curvas de Bézier – algoritmo de de Casteljau

$$B_k^{n+1}(t) = (1 - t) B_k^n(t) + t B_{k-1}^n(t)$$

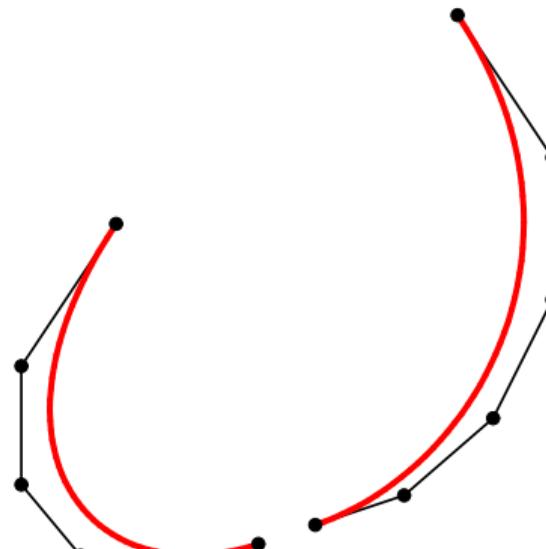
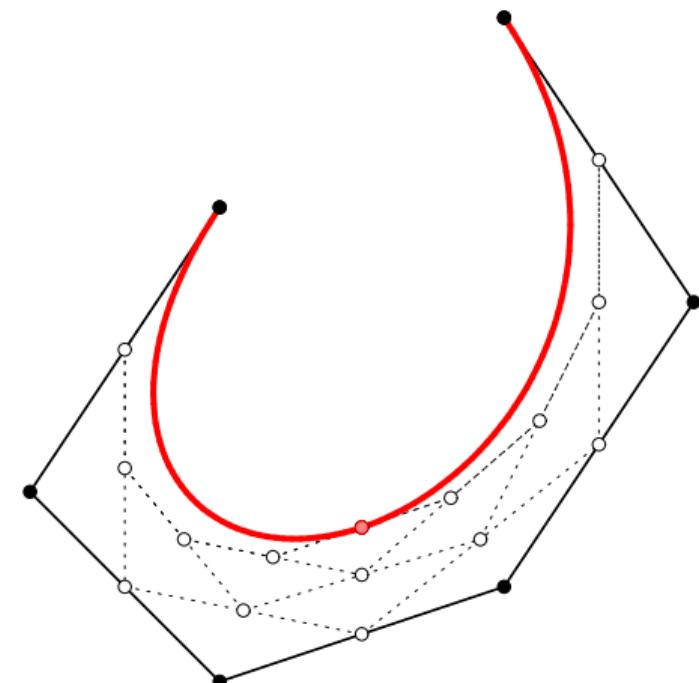
recursão



## Curvas de Bézier – subdivisão adaptativa

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t)$$

recursão



## Curvas de Bézier – presença

grau	PDF	SVG	JavaScript
1	●	●	●
2	●	●	●
3	●	●	●
4+	●	●	●

## Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

### ■ Bézier quadrática

$$\gamma(t) = (1-t)^2 q_0 + 2(1-t)t q_1 + t^2 q_2$$

... multiplicando por  $((1-t) + t) = 1$

### ■ Bézier cúbica

$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

### ■ Bézier de grau n

$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

## Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

### ■ Bézier quadrática

$$\gamma(t) = (1-t)^2 q_0 + 2(1-t)t q_1 + t^2 q_2$$

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### ■ Bézier cúbica

$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

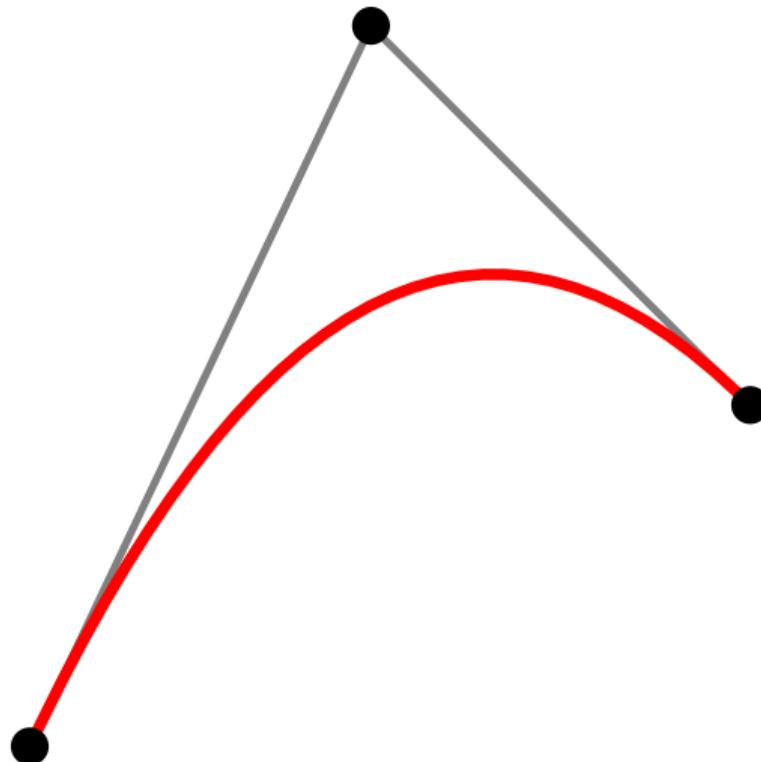
$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

### ■ Bézier de grau n

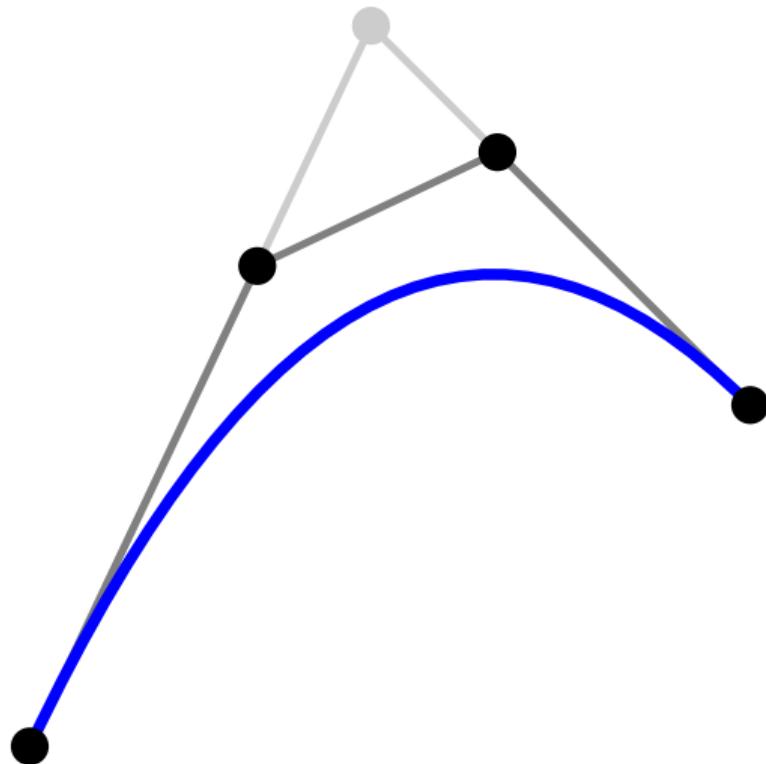
$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

$$P^{n+r} \rightarrow B^n$$

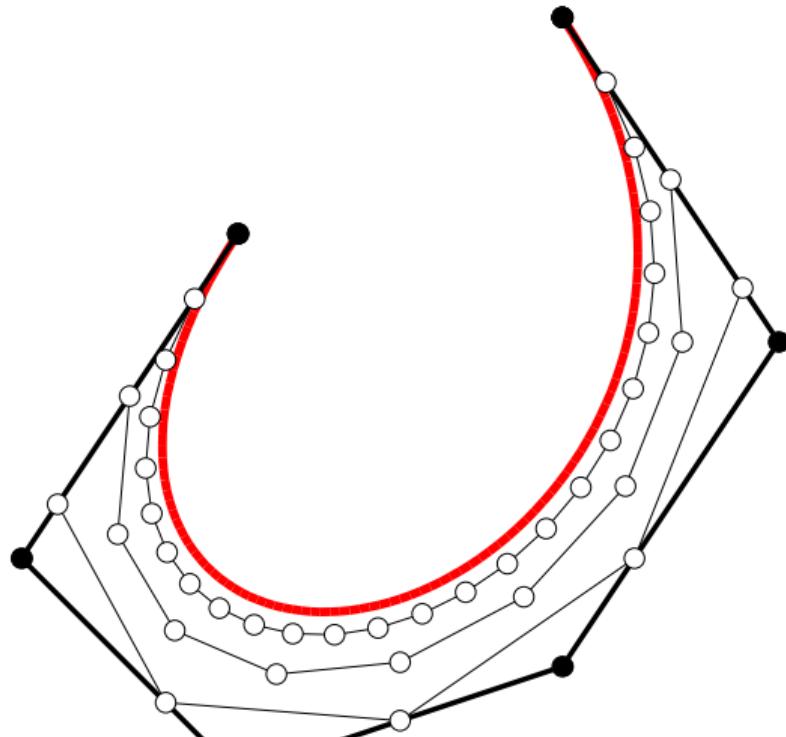
## Curvas de Bézier – elevação de grau



## Curvas de Bézier – elevação de grau



## Curvas de Bézier – elevação de grau



$n = 5 \rightarrow 6, 10, 25$

Farouki

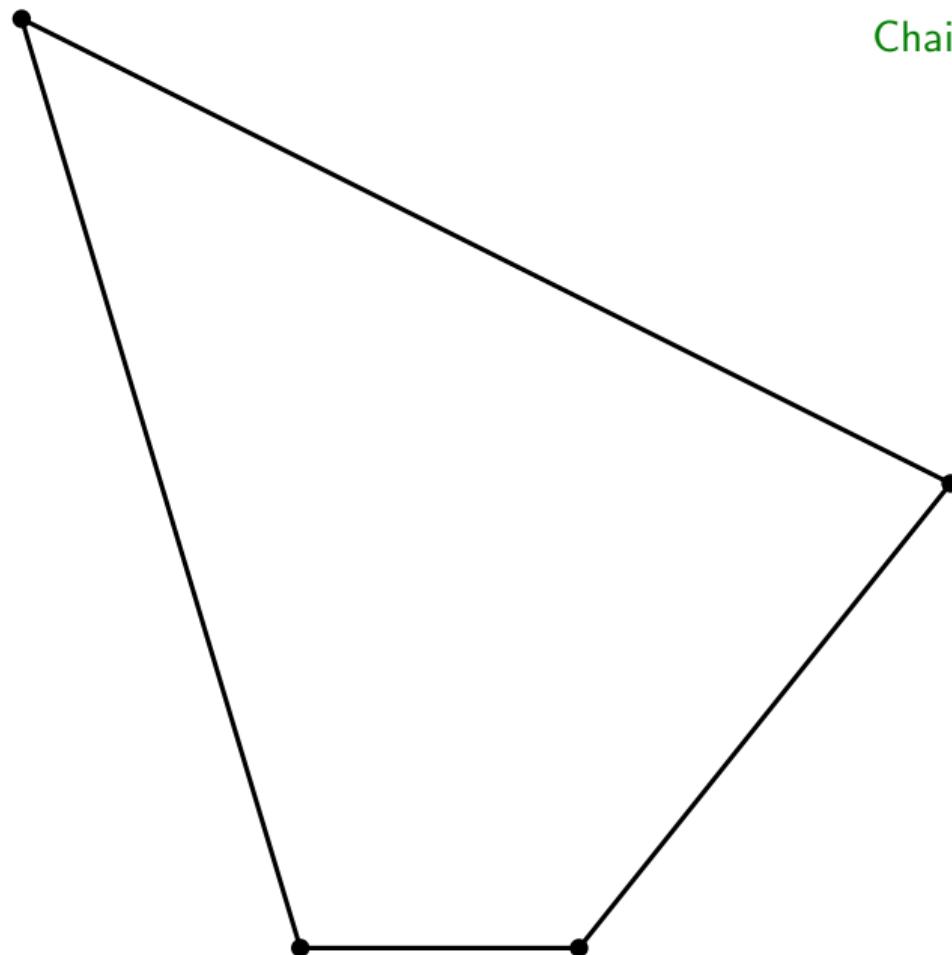
subdivisão

## Curvas de subdivisão

- de Rham (1947, 1956) cantos a  $1/3$
- de Casteljau (1963) Bézier quadrática
- Chaikin (1974) cantos a  $1/4$
- Riesenfeld (1975) spline de Bézier quadráticas
- de Boor (1987) Comer cutting always works

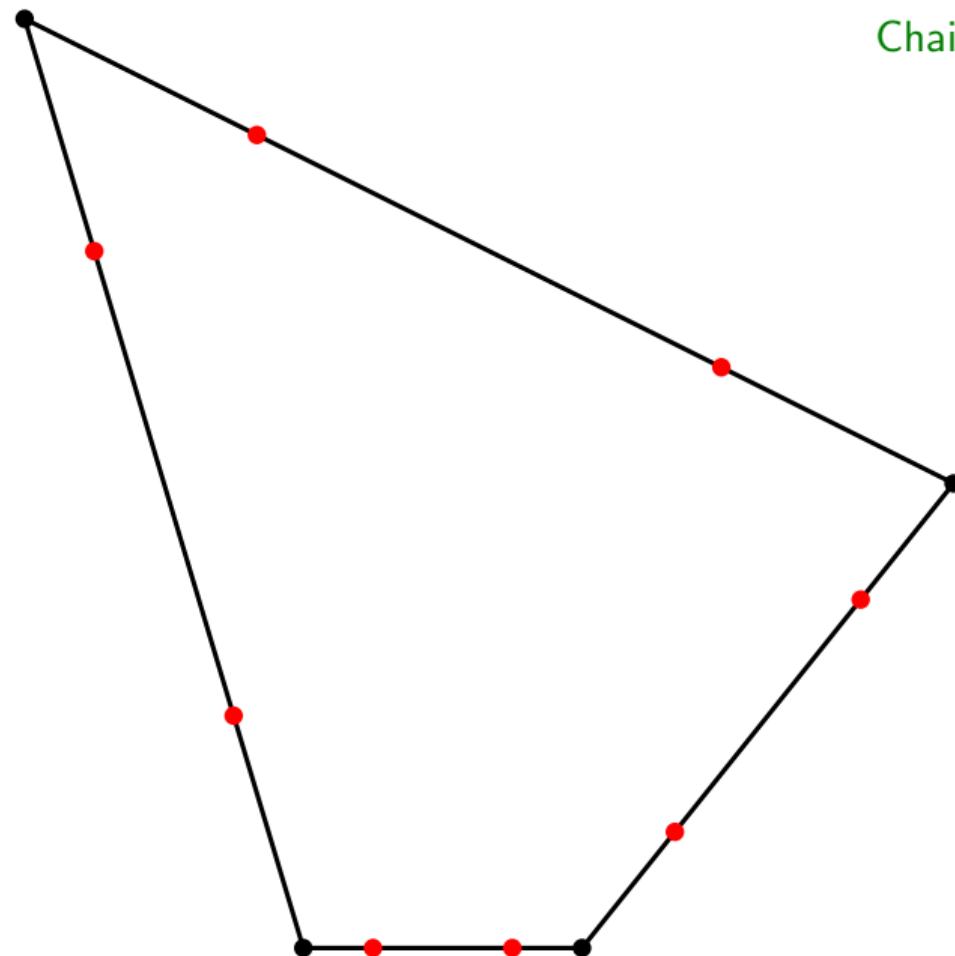
## Curvas de subdivisão

Chaikin (1974)



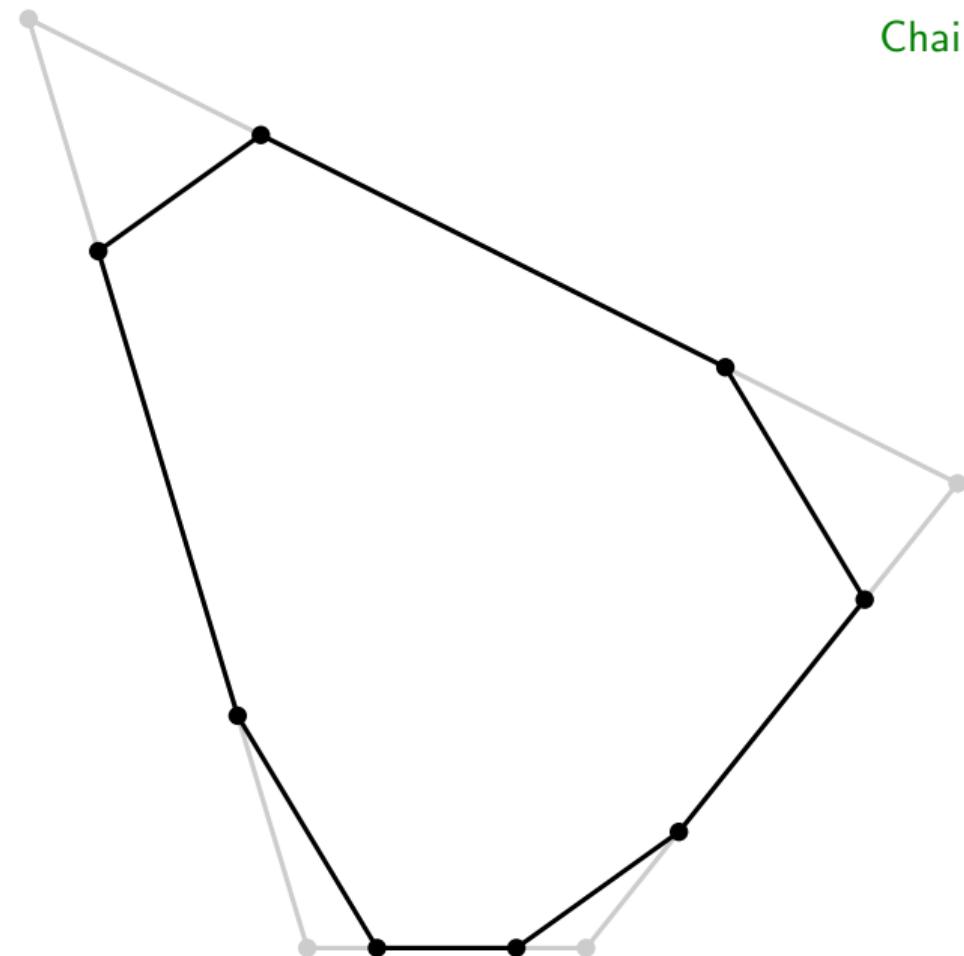
## Curvas de subdivisão

Chaikin (1974)



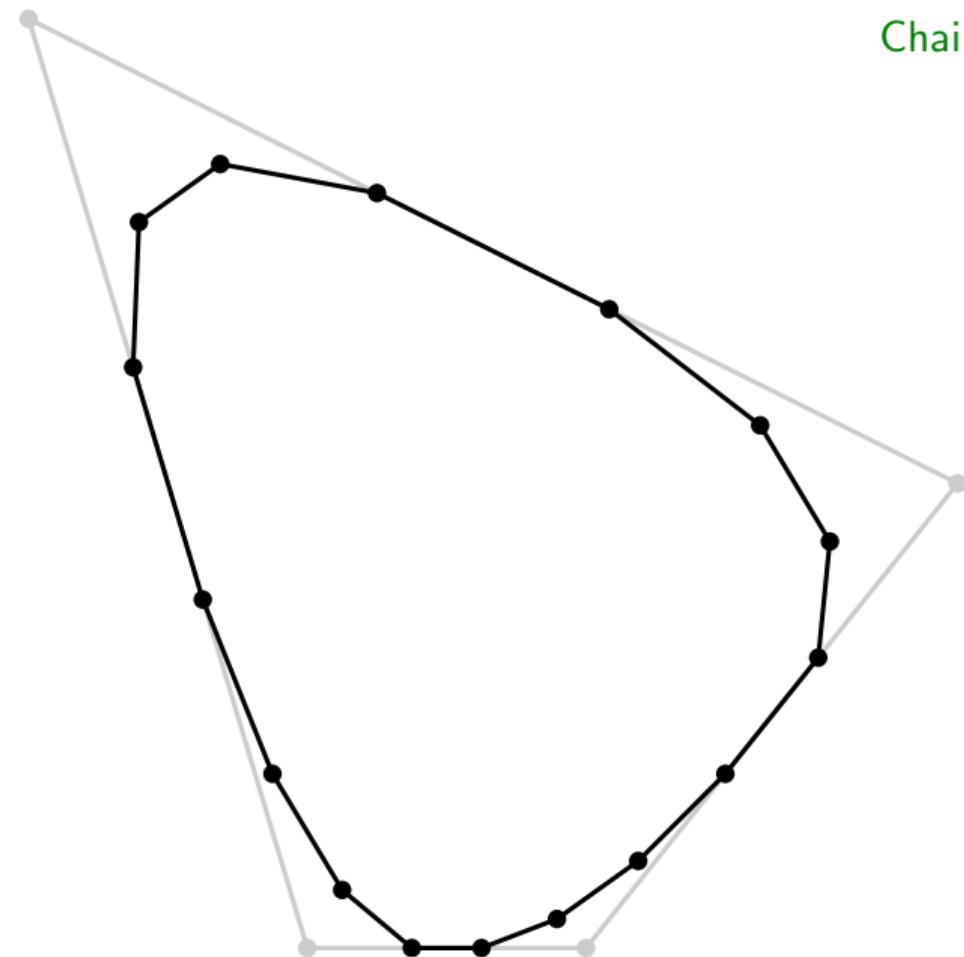
## Curvas de subdivisão

Chaikin (1974)



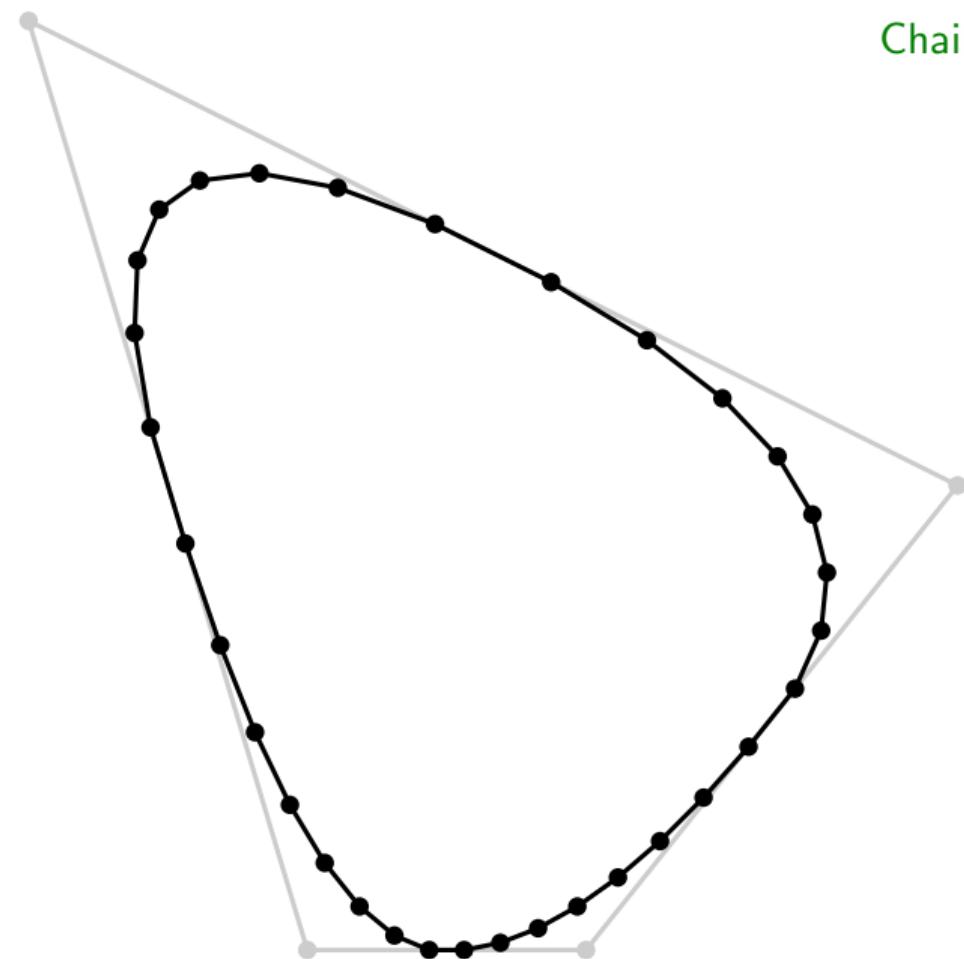
## Curvas de subdivisão

Chaikin (1974)



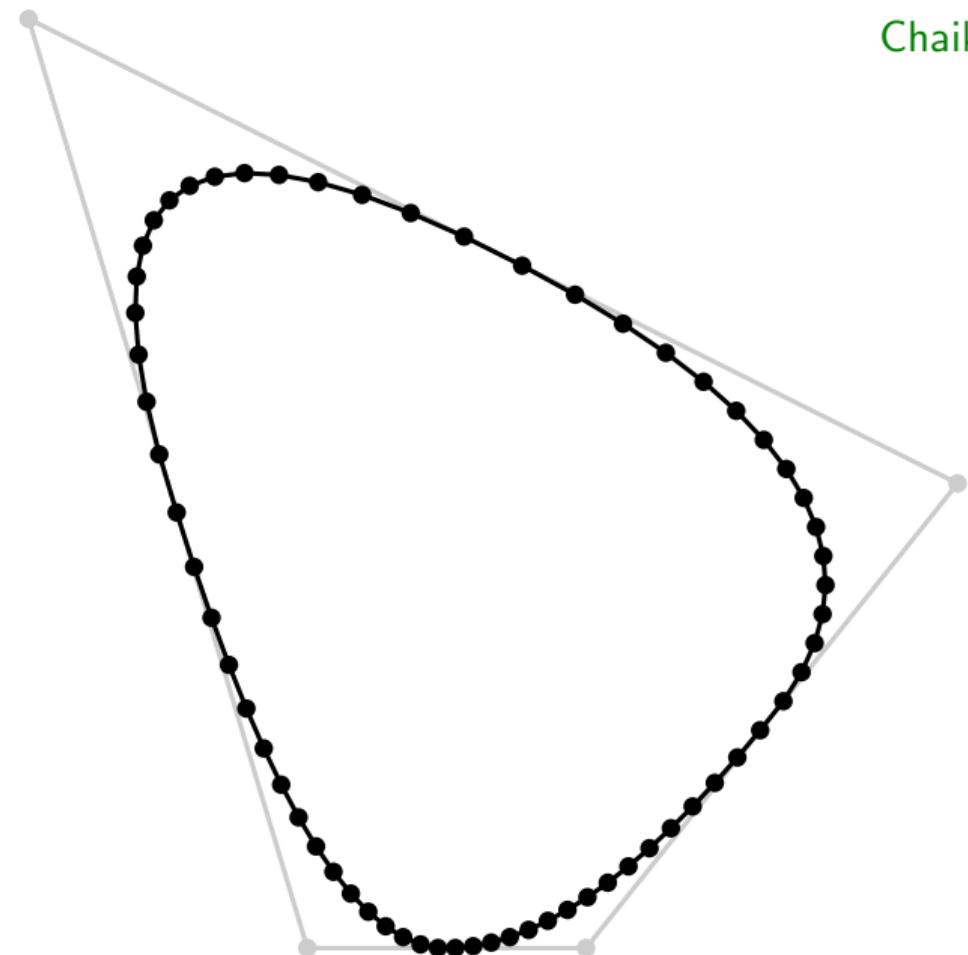
## Curvas de subdivisão

Chaikin (1974)



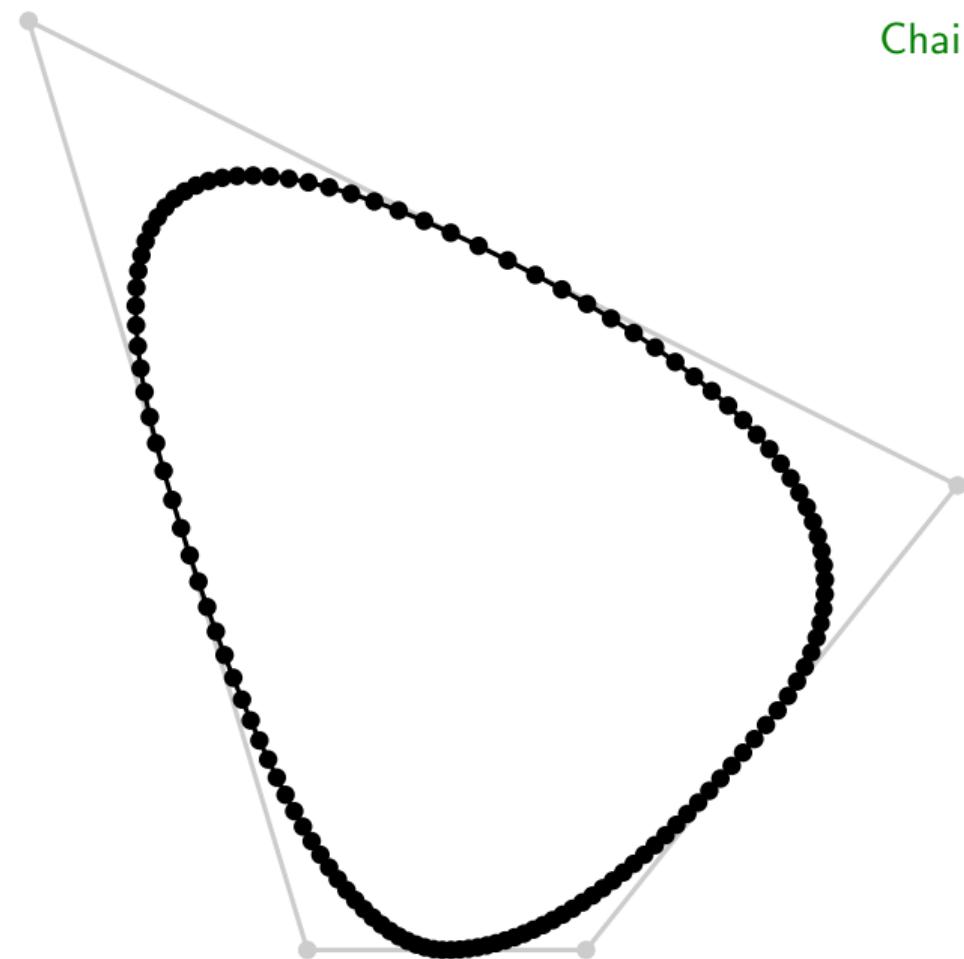
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Chaikin (1974)



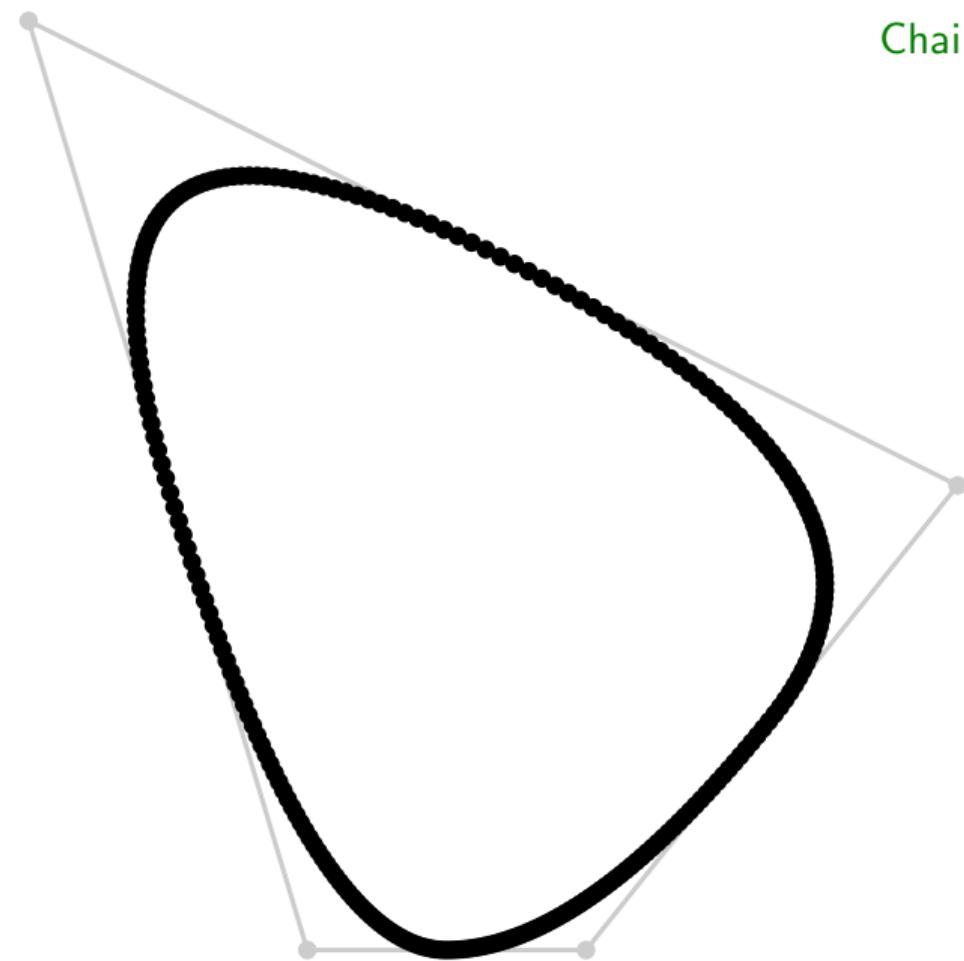
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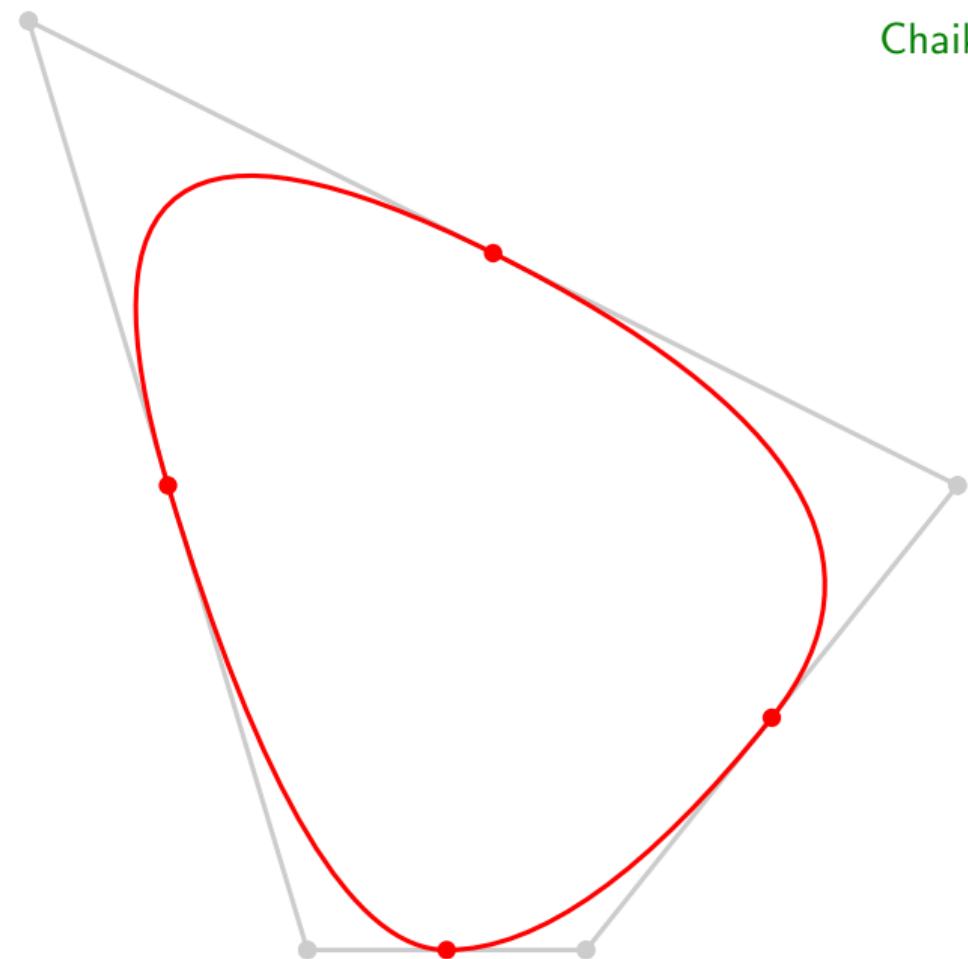
## Curvas de subdivisão

Chaikin (1974)



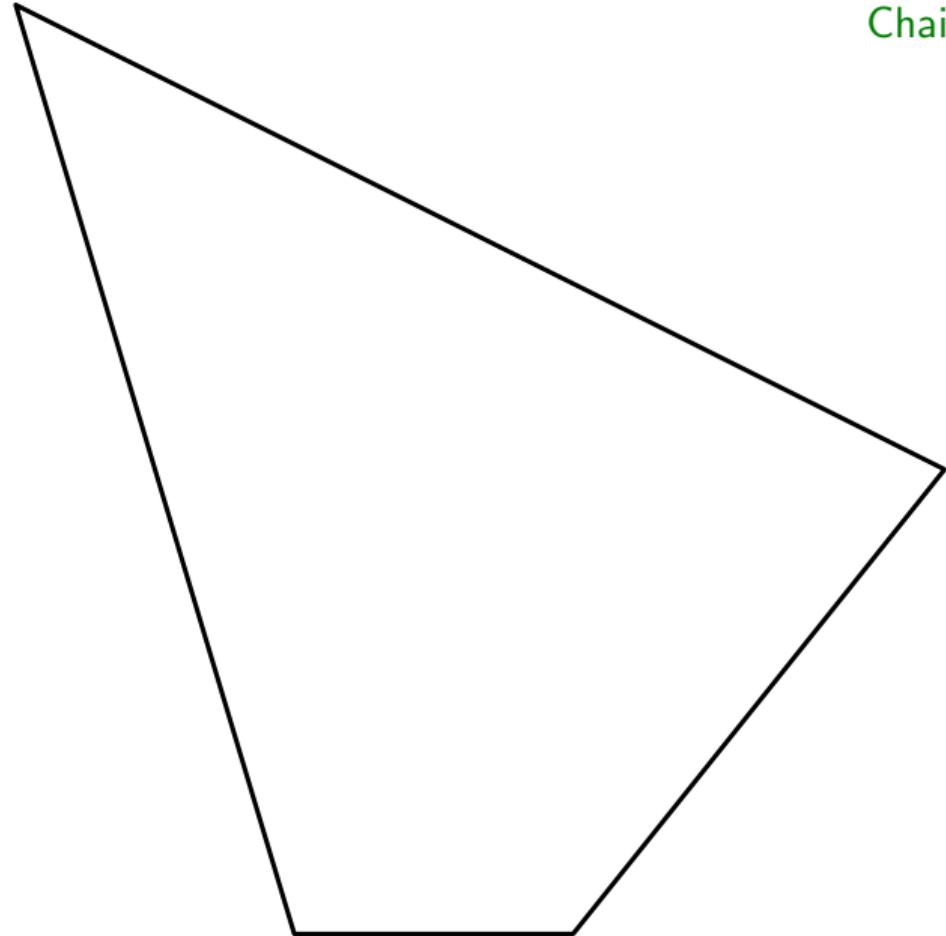
## Curvas de subdivisão

Chaikin (1974)



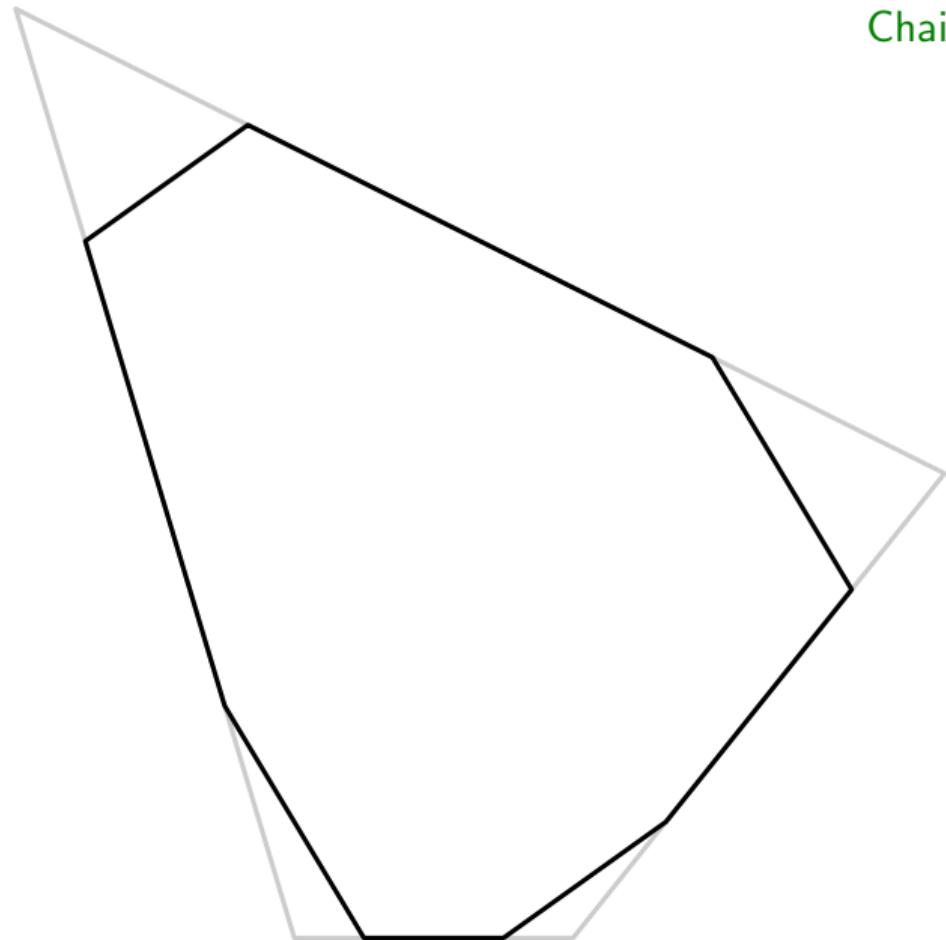
## Curvas de subdivisão

Chaikin (1974)



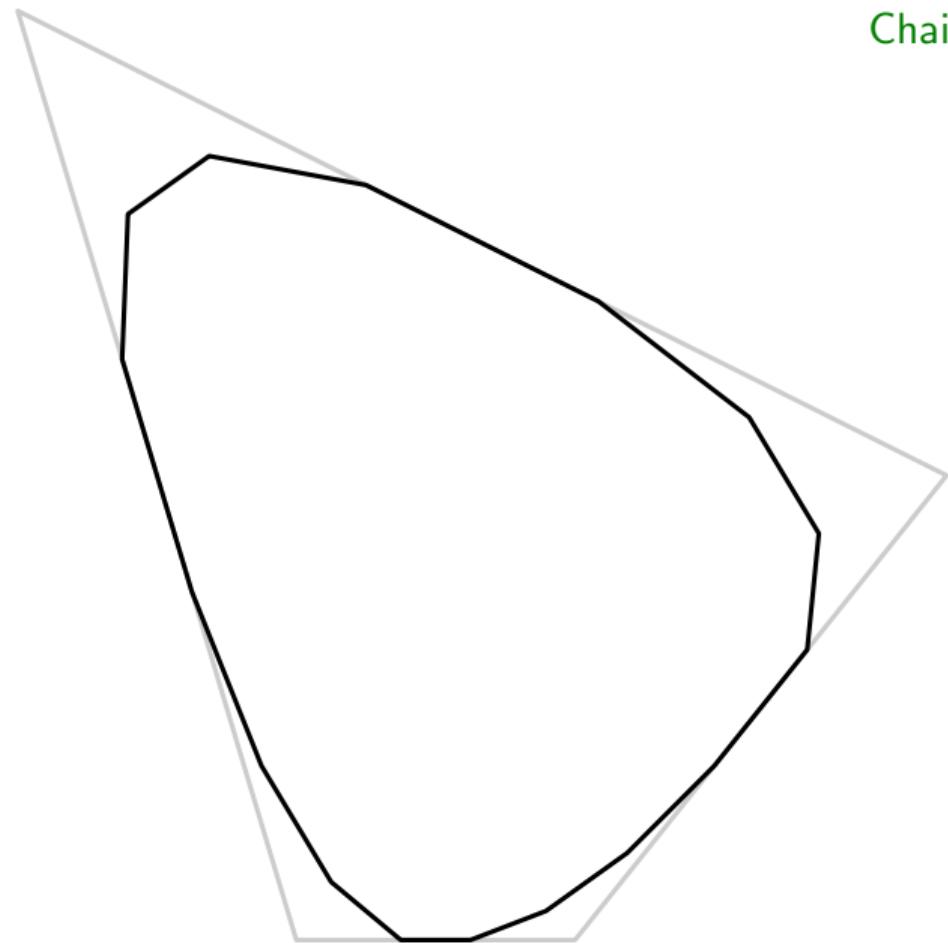
## Curvas de subdivisão

Chaikin (1974)



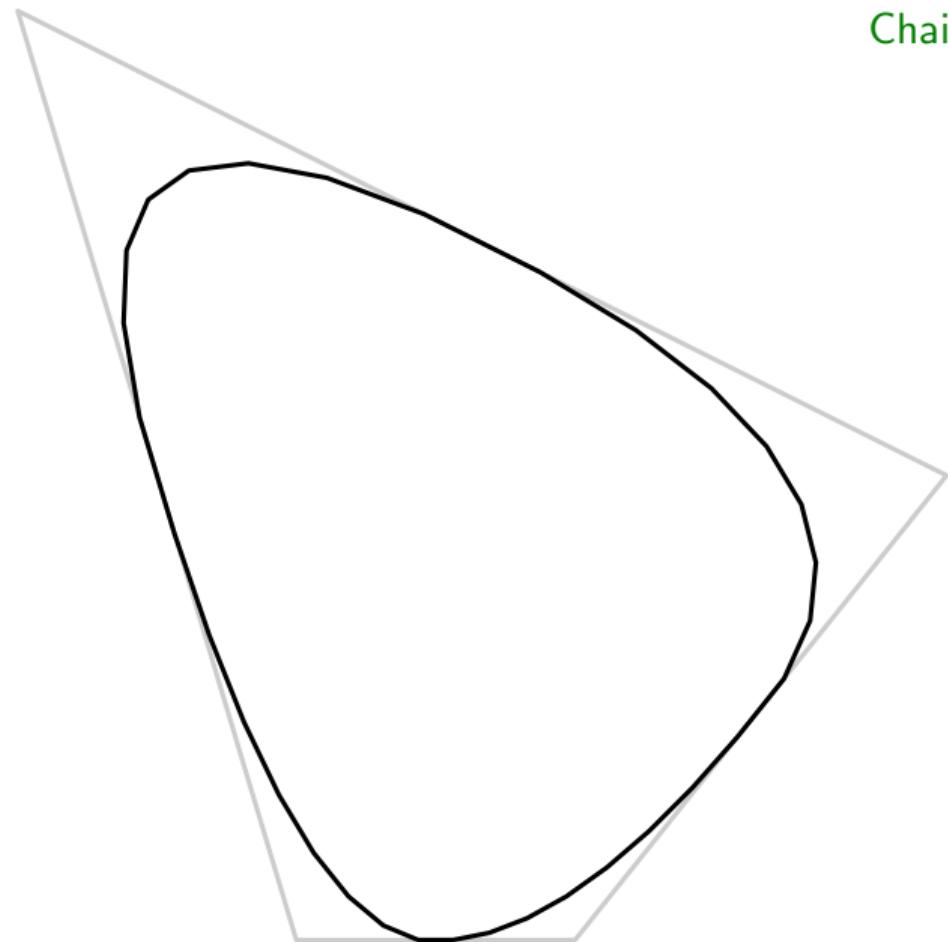
## Curvas de subdivisão

Chaikin (1974)



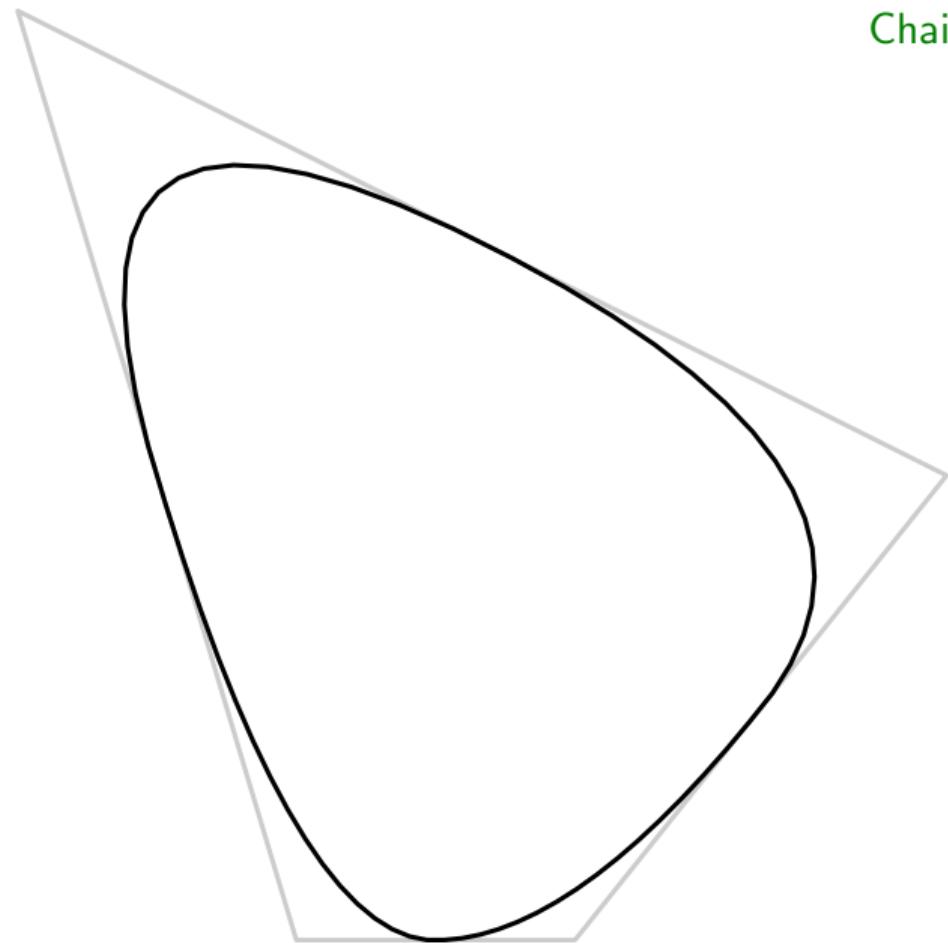
## Curvas de subdivisão

Chaikin (1974)



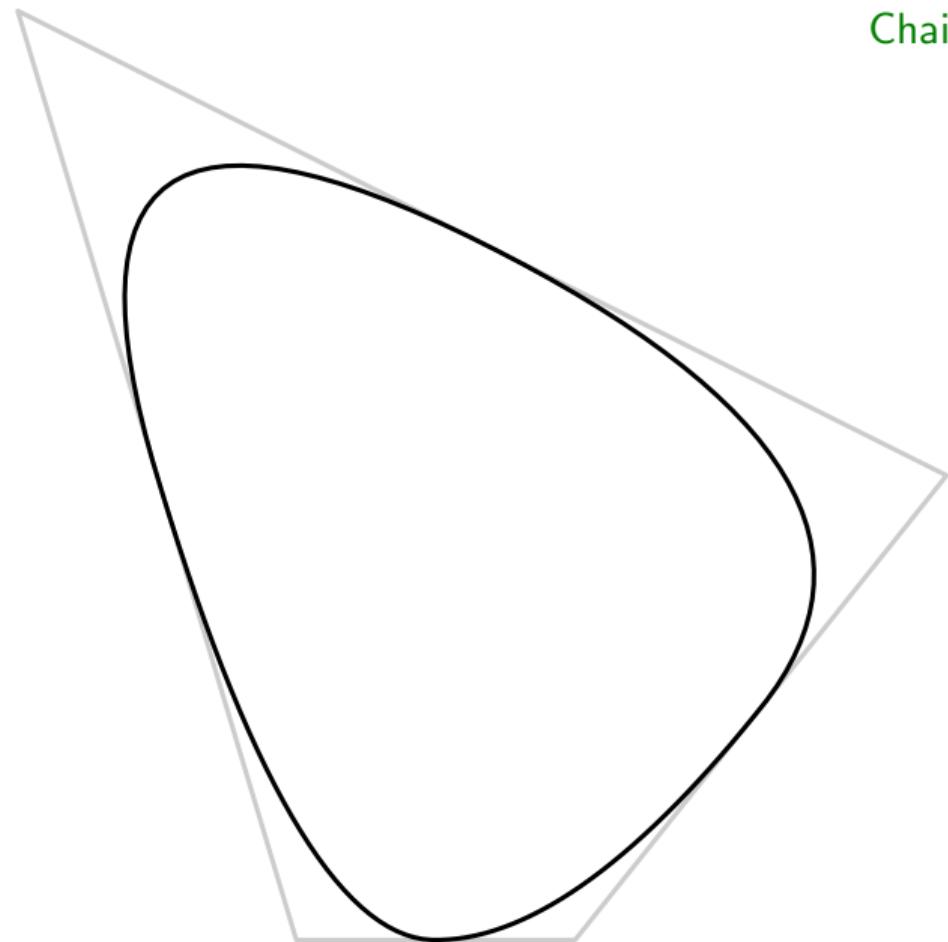
## Curvas de subdivisão

Chaikin (1974)



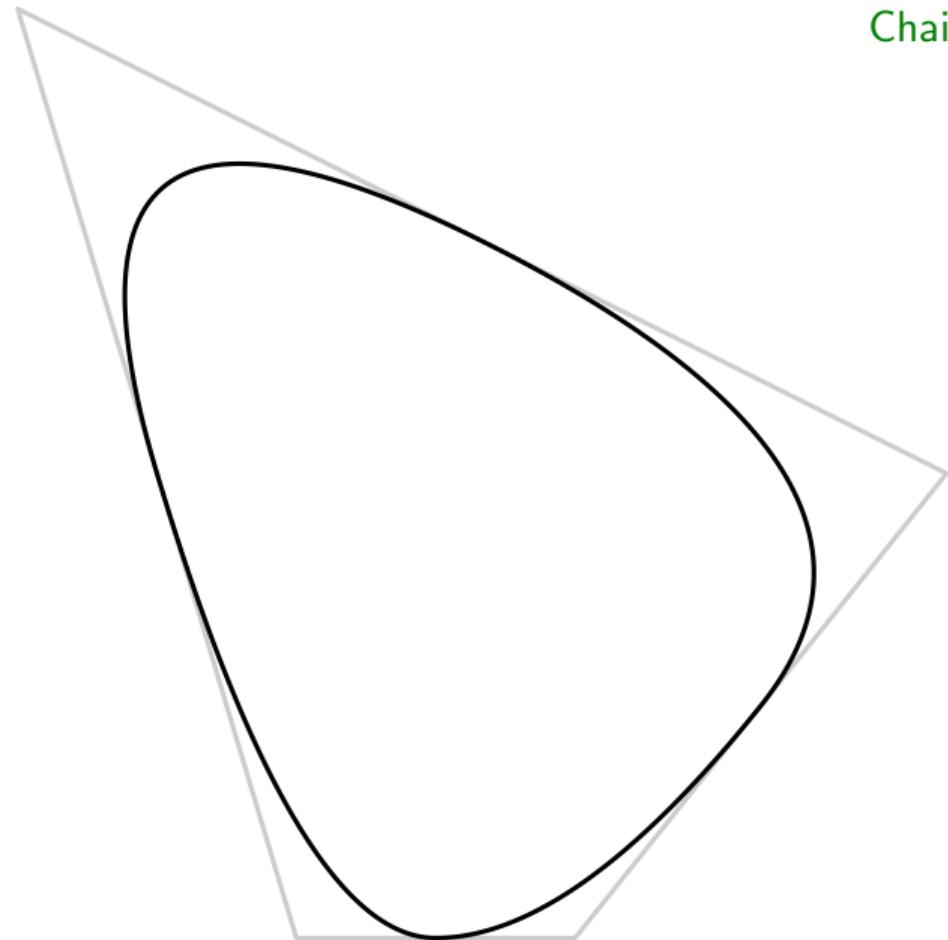
## Curvas de subdivisão

Chaikin (1974)



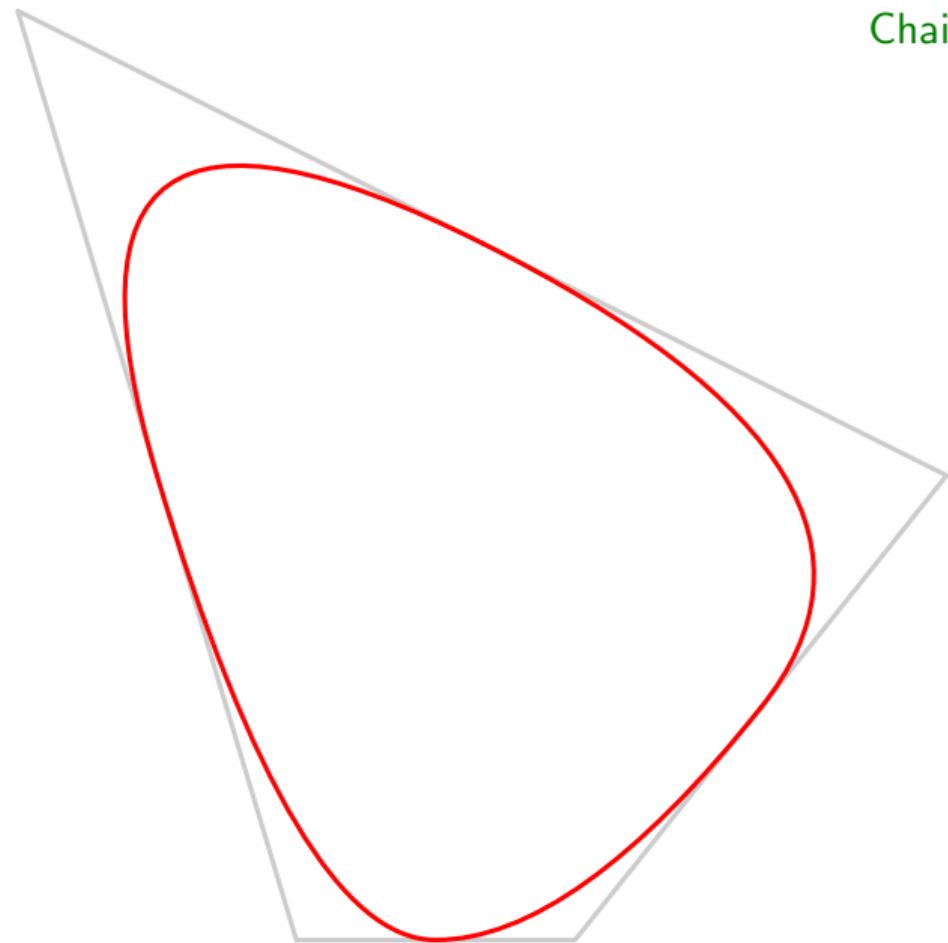
## Curvas de subdivisão

Chaikin (1974)



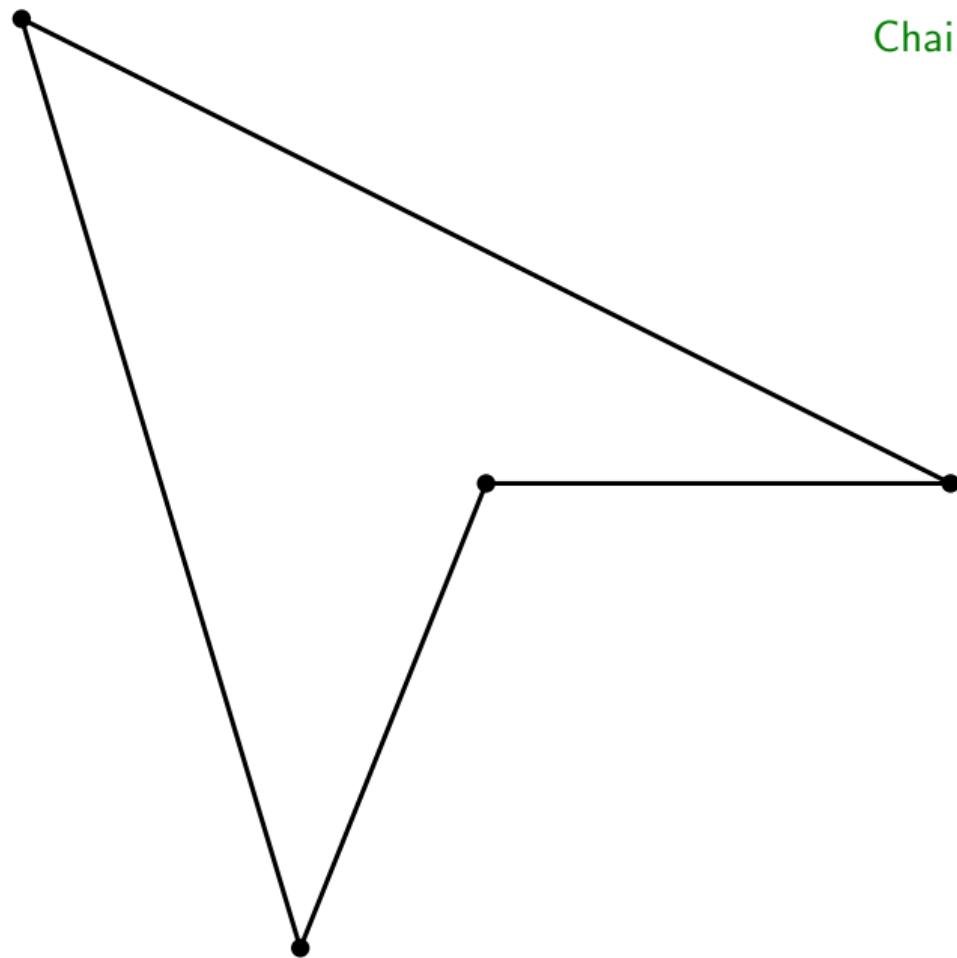
## Curvas de subdivisão

Chaikin (1974)



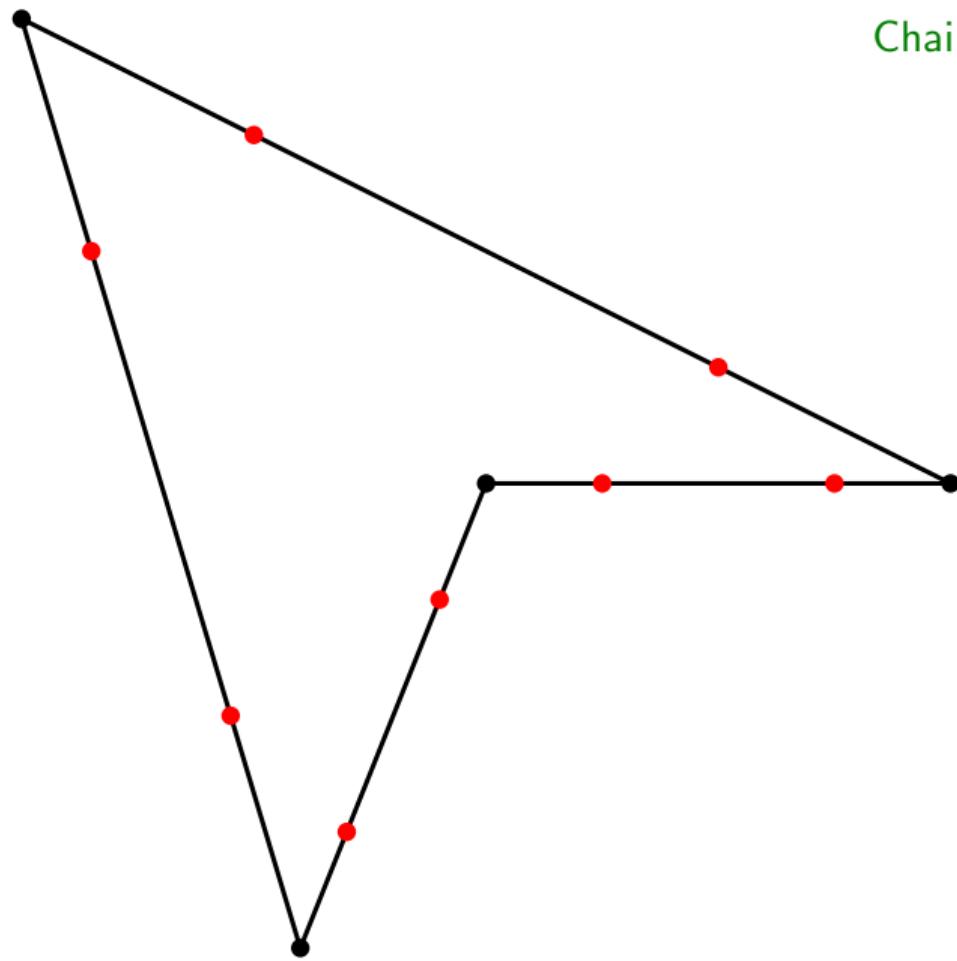
## Curvas de subdivisão

Chaikin (1974)



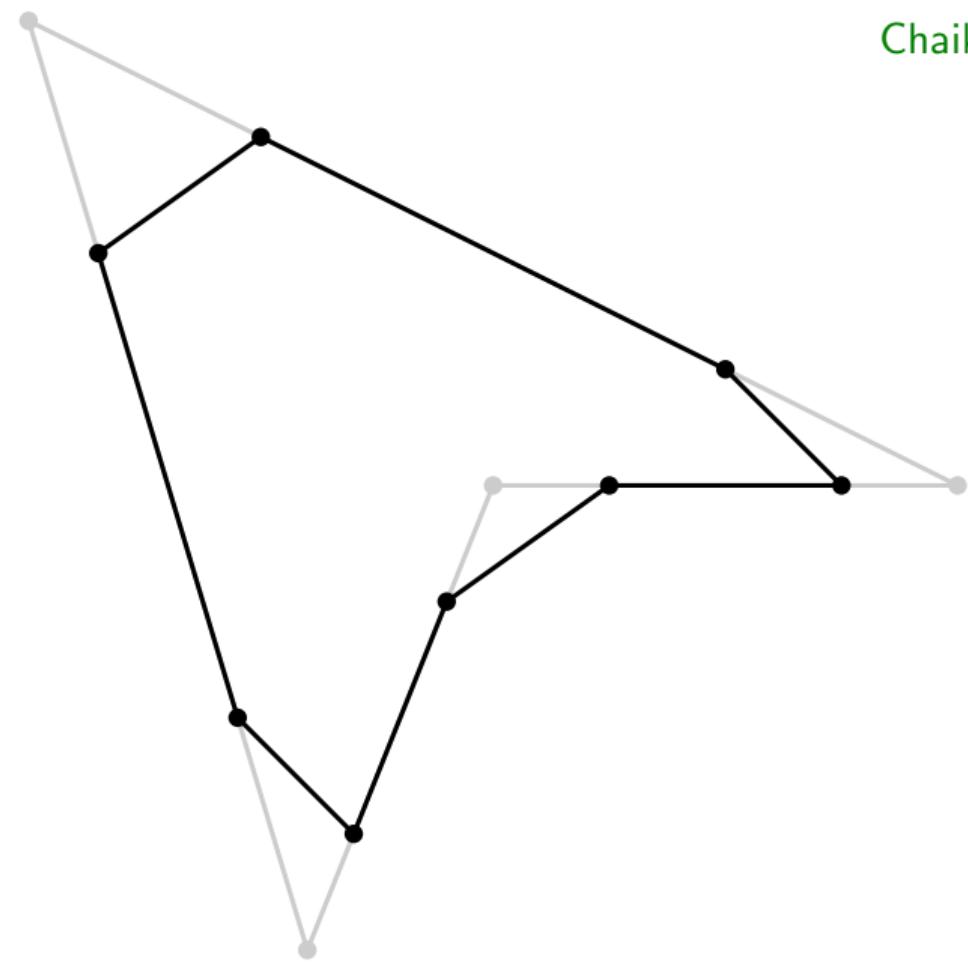
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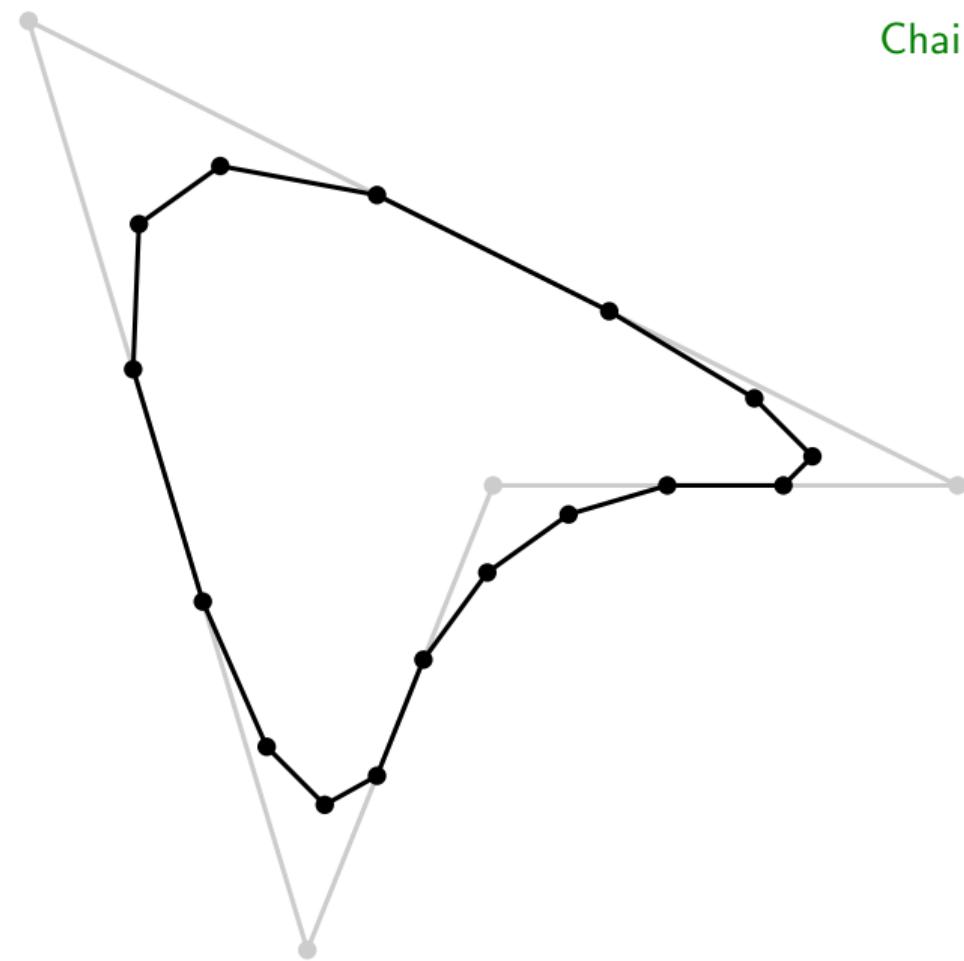
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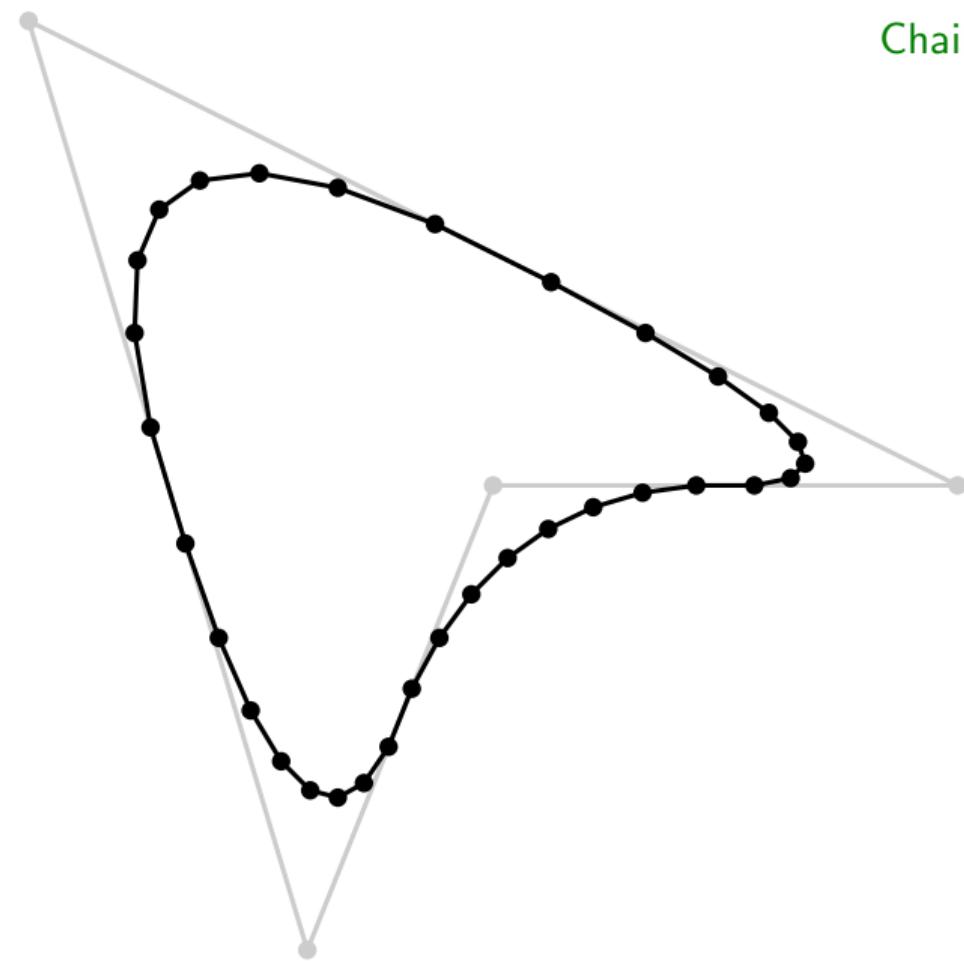
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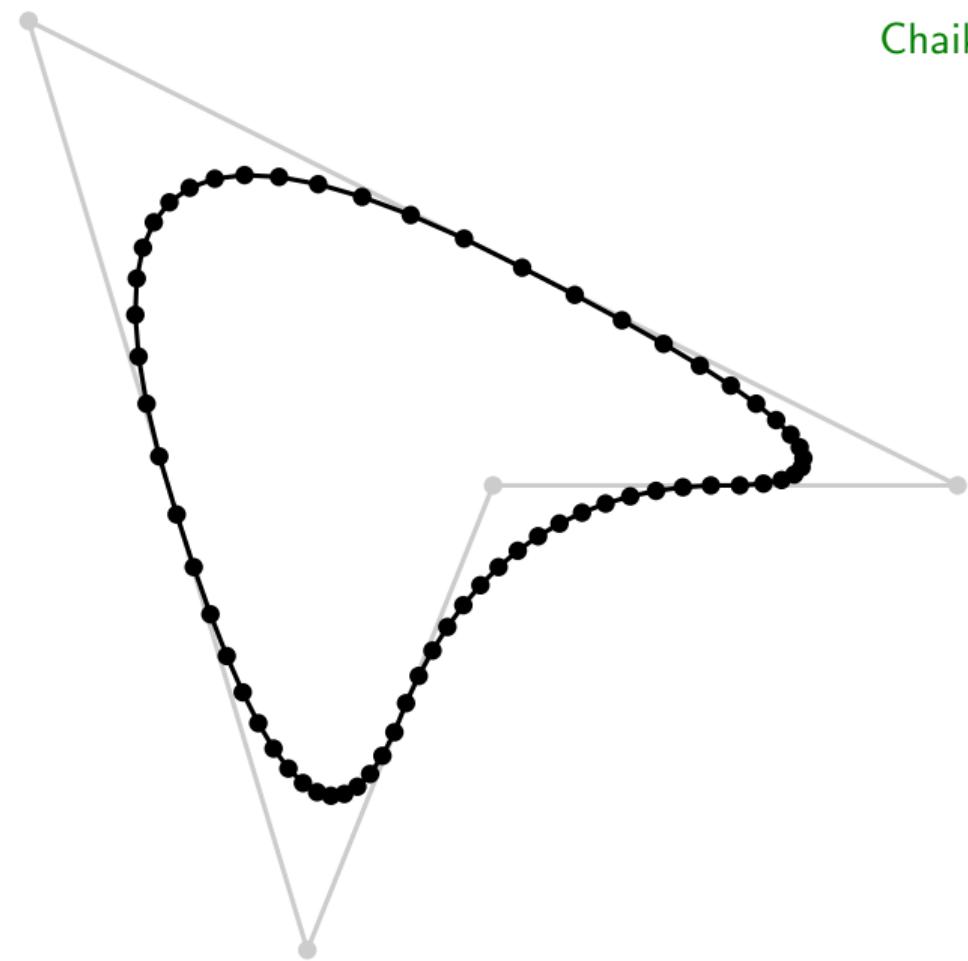
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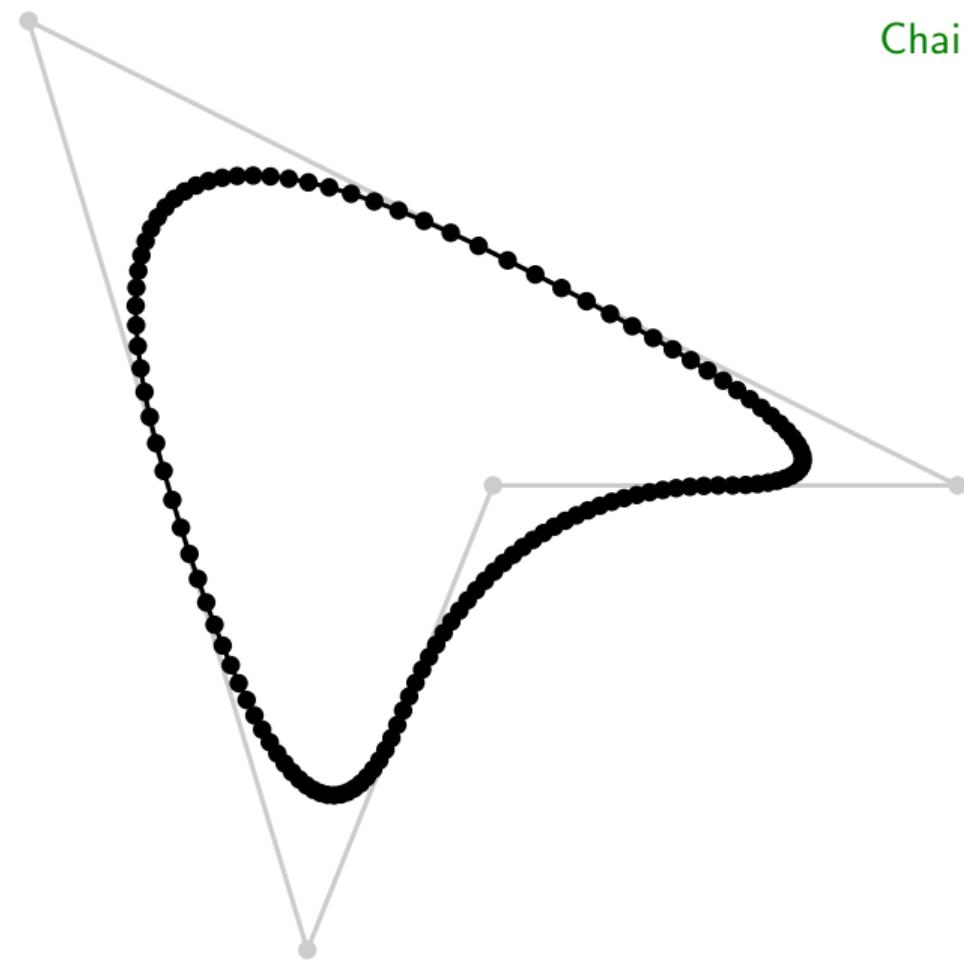
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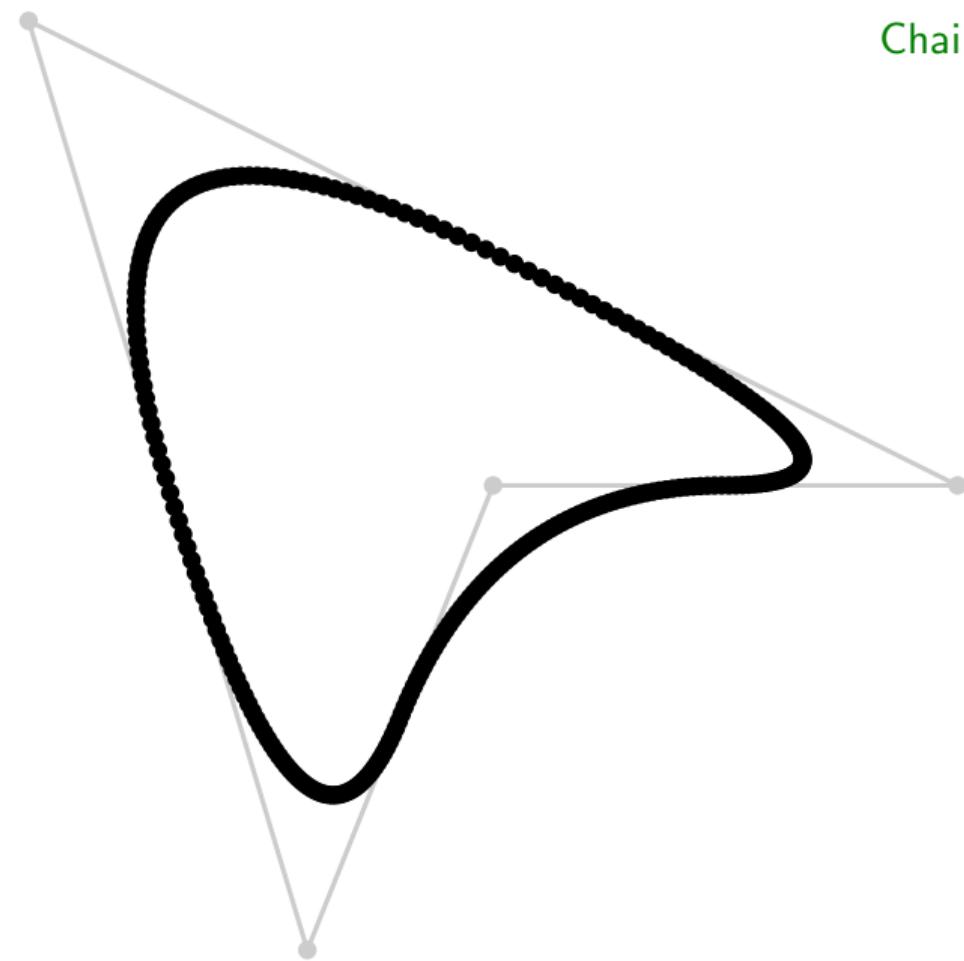
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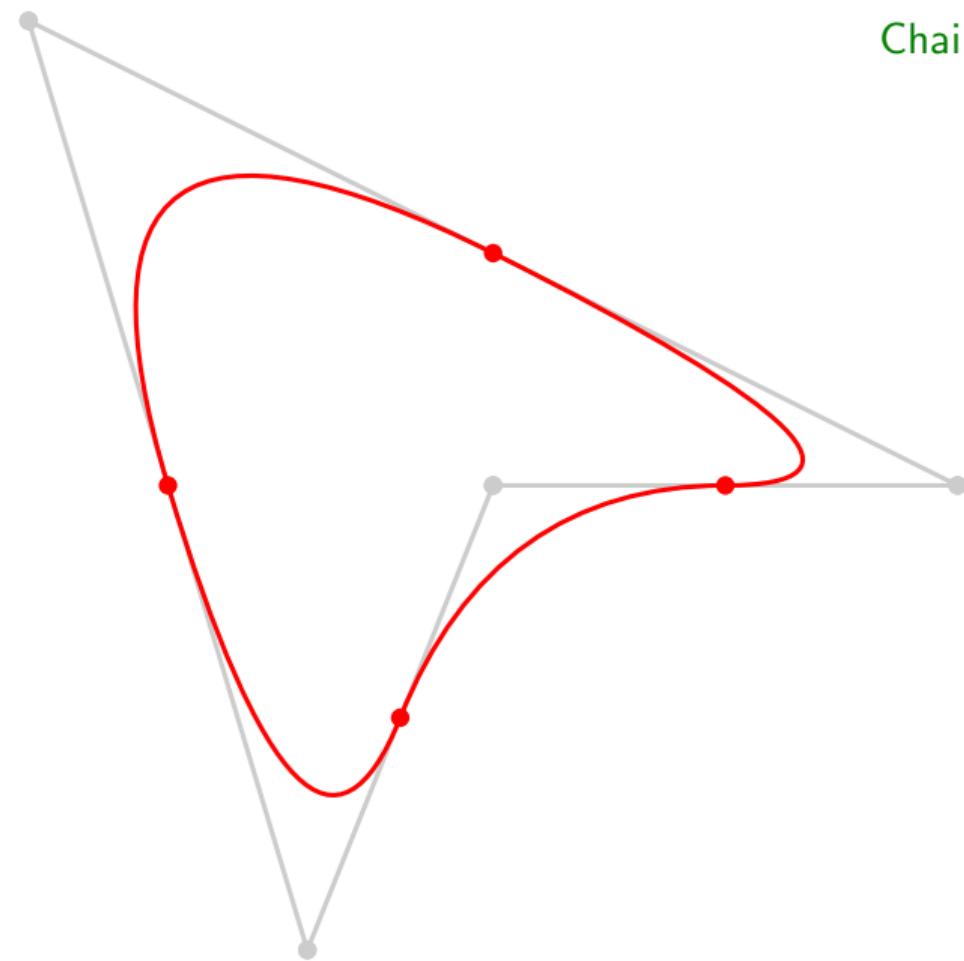
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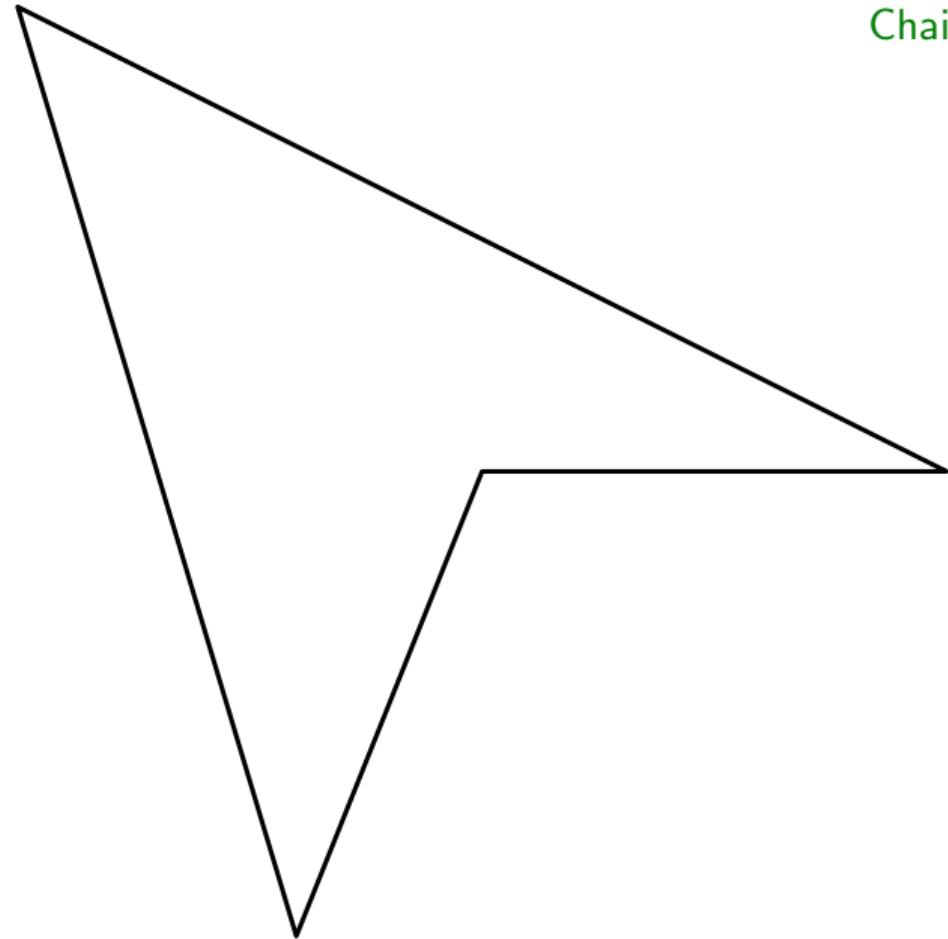
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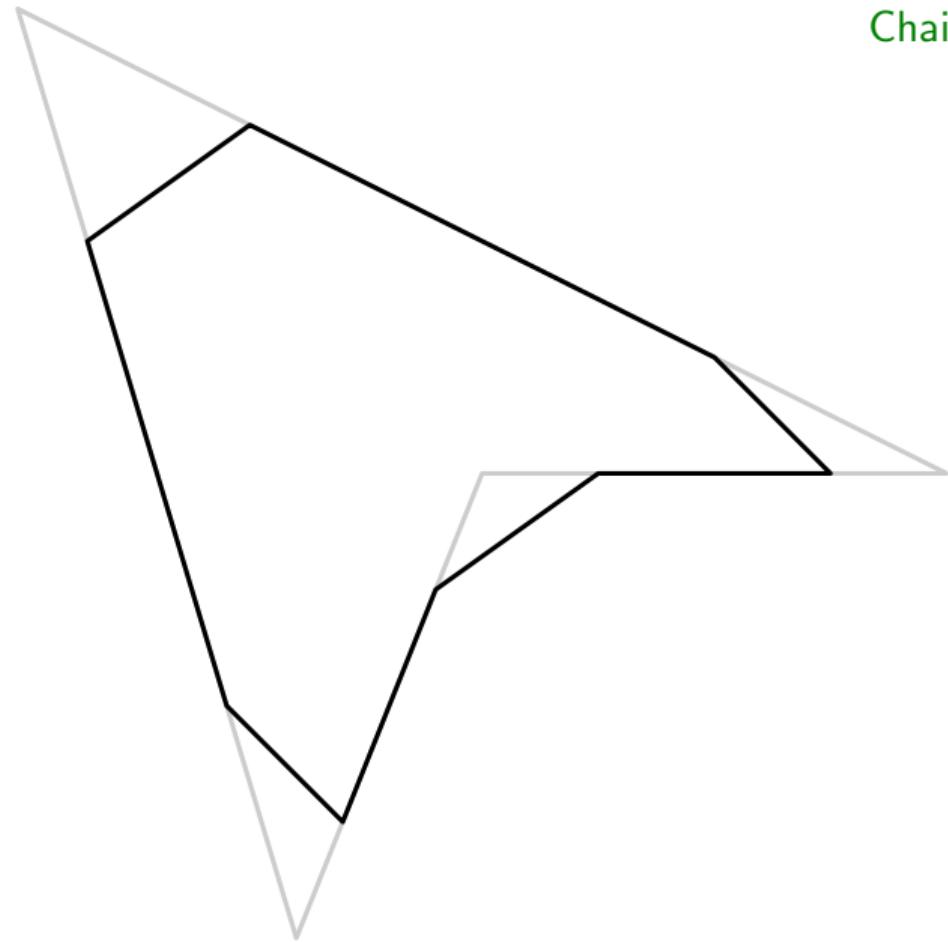
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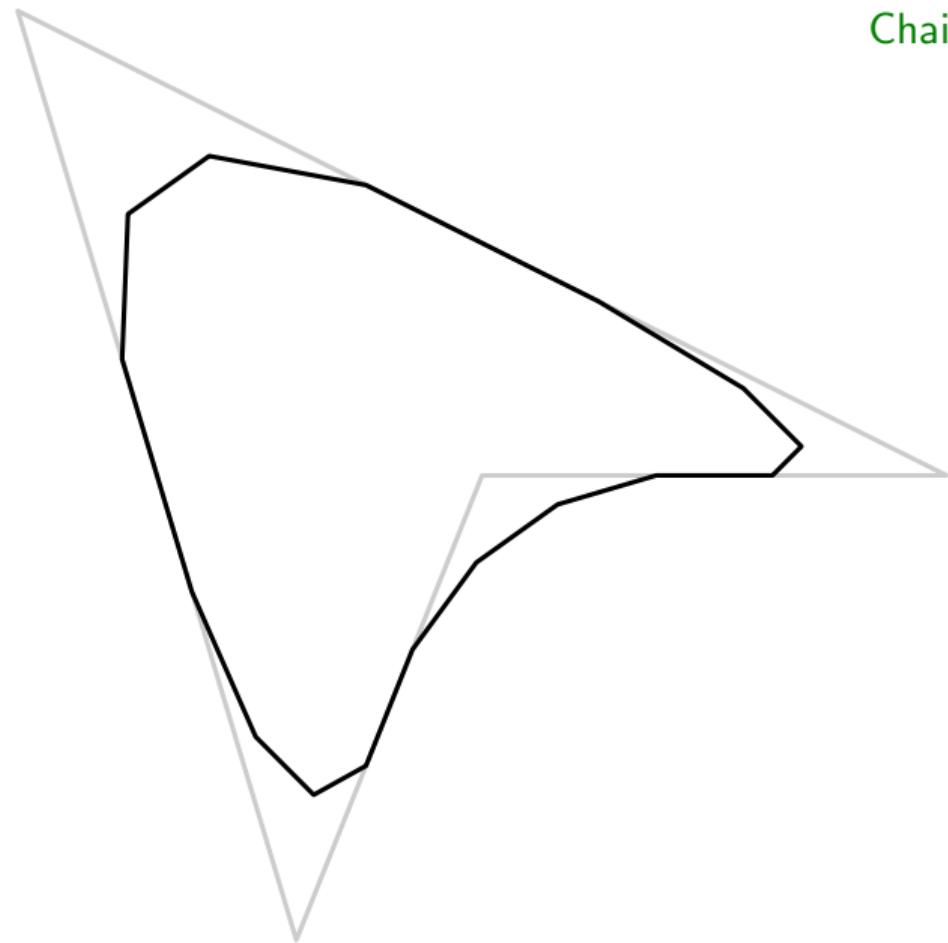
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Chaikin (1974)



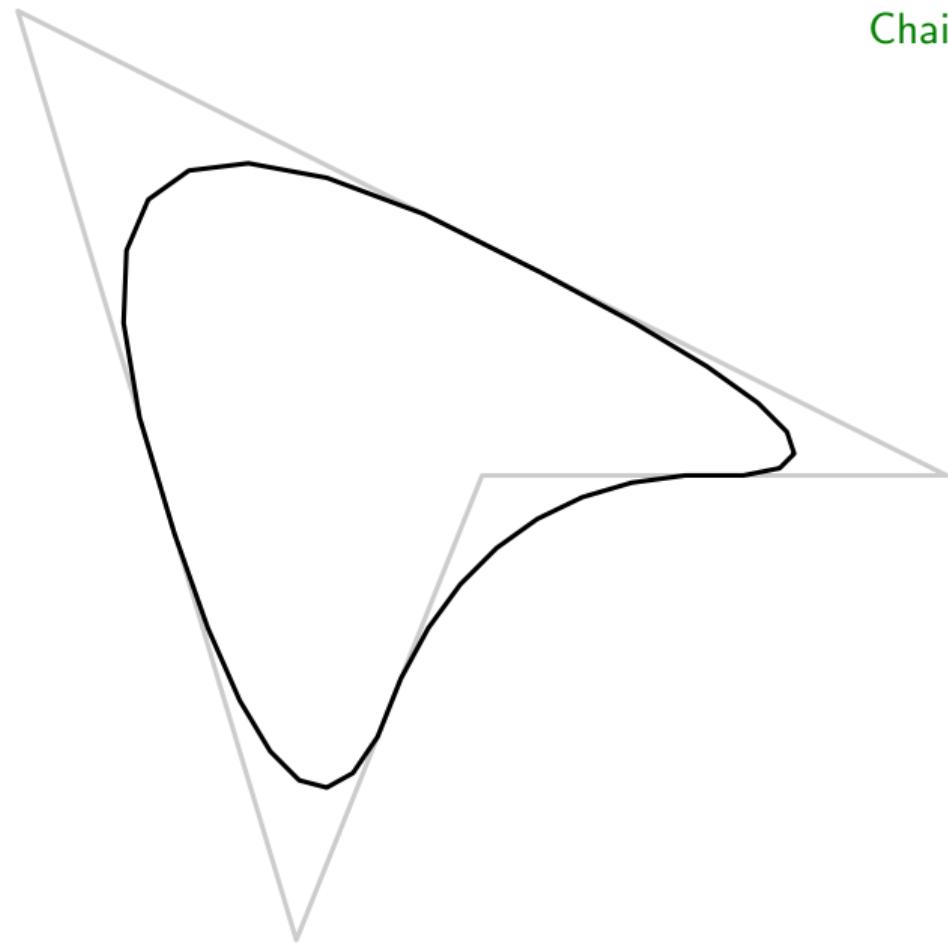
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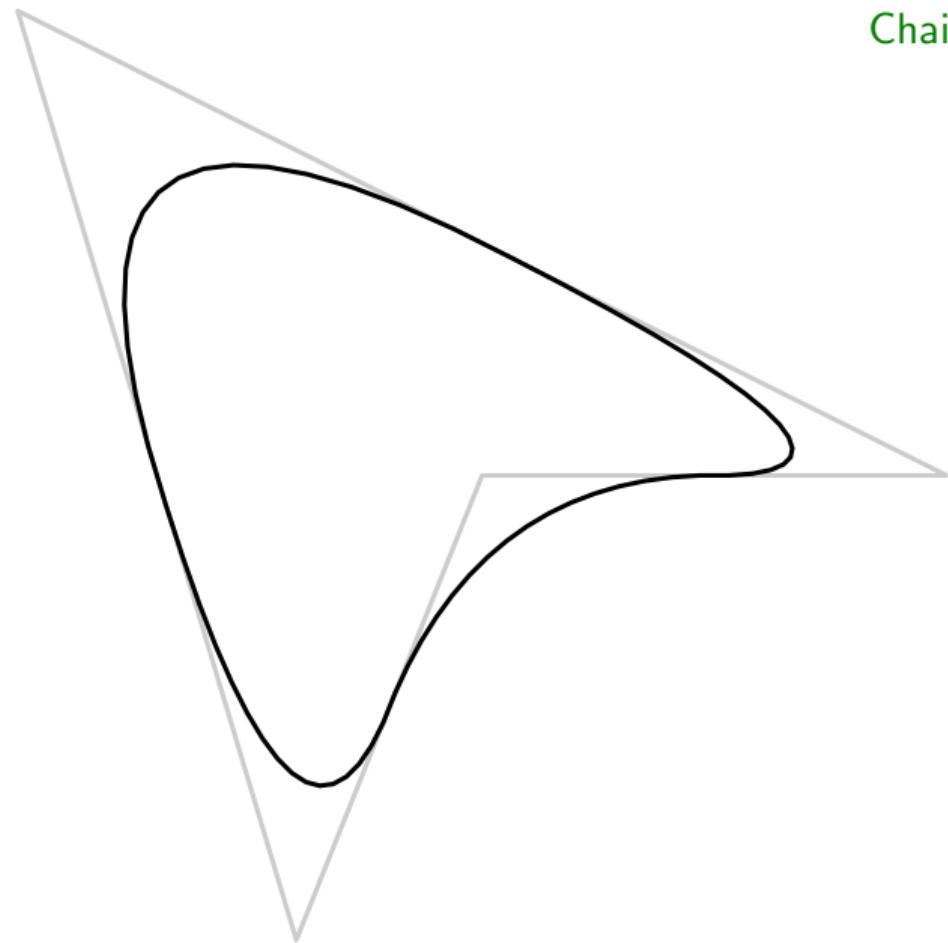
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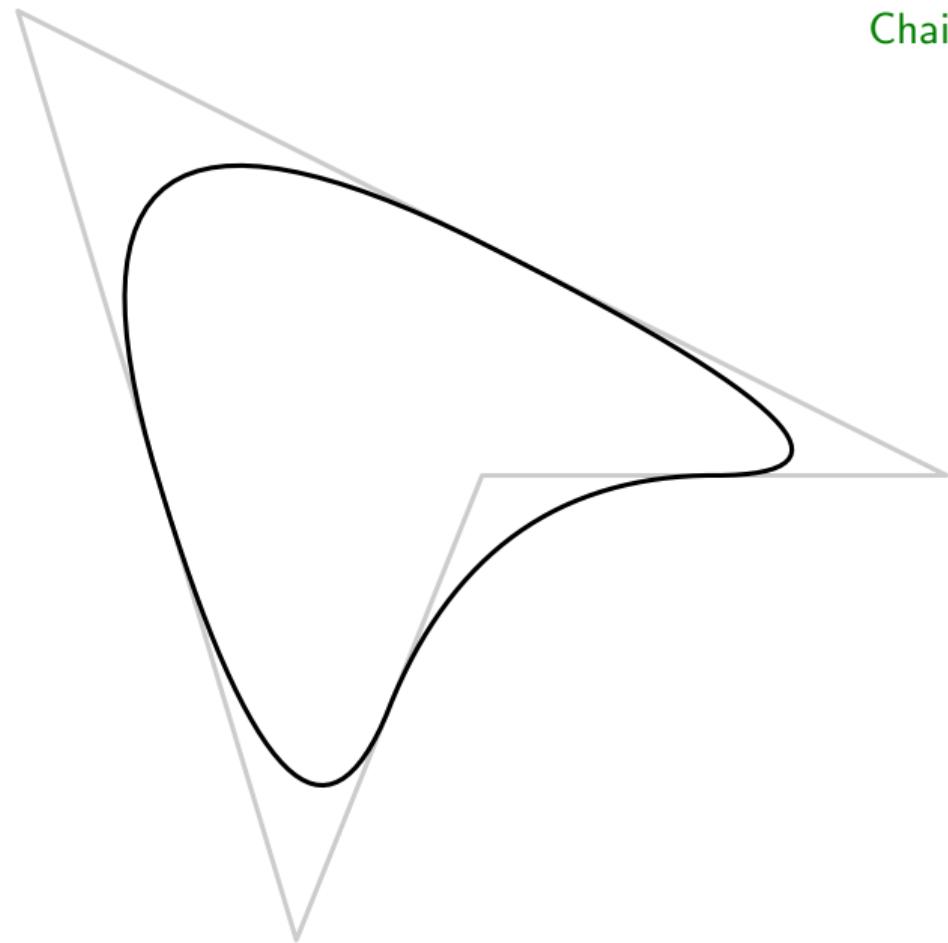
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Chaikin (1974)



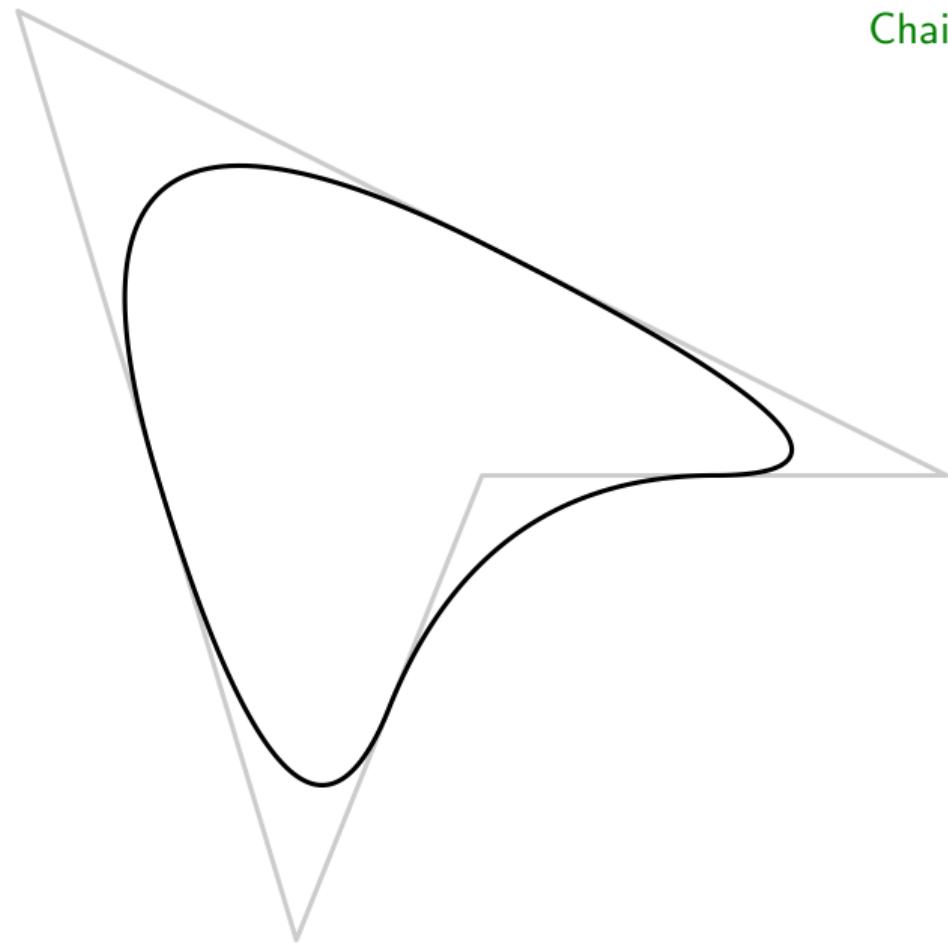
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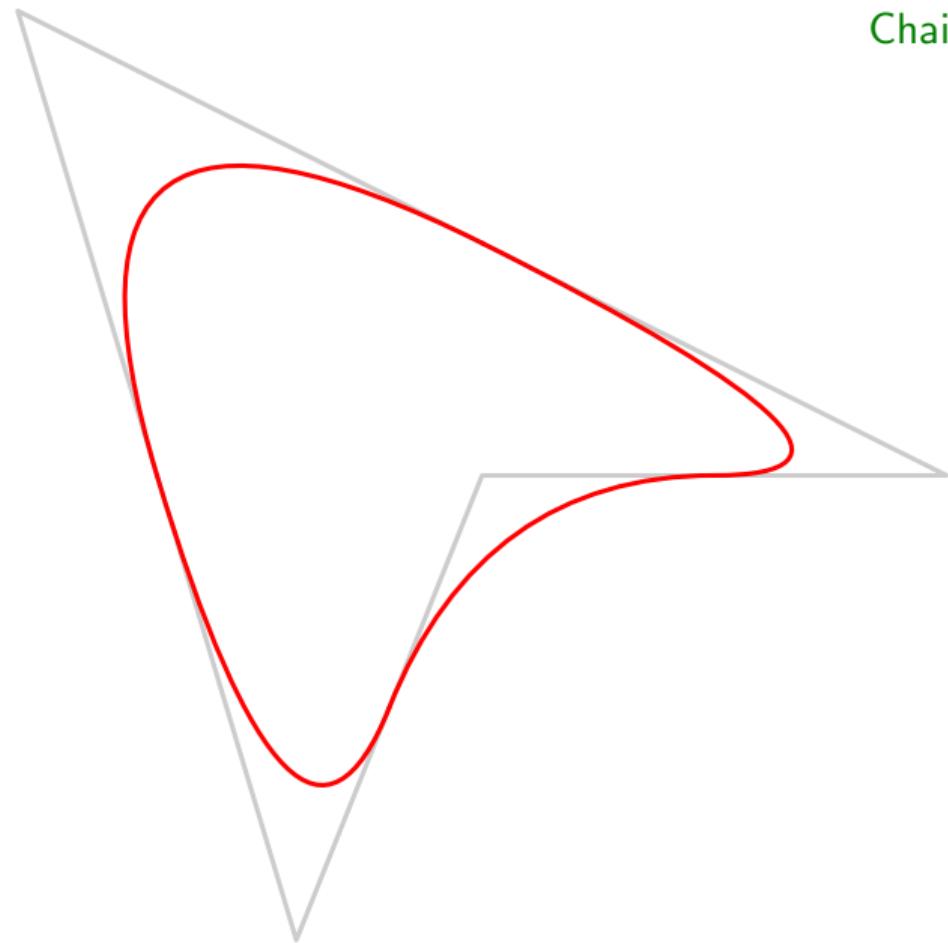
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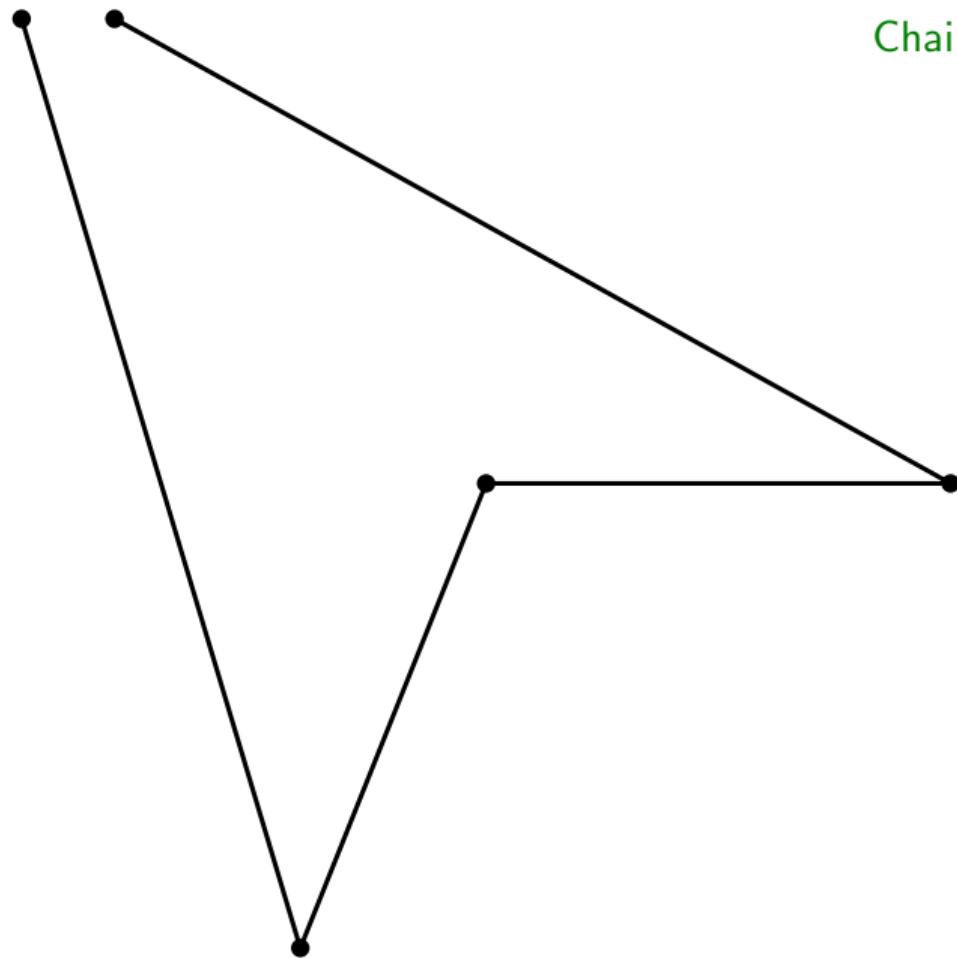
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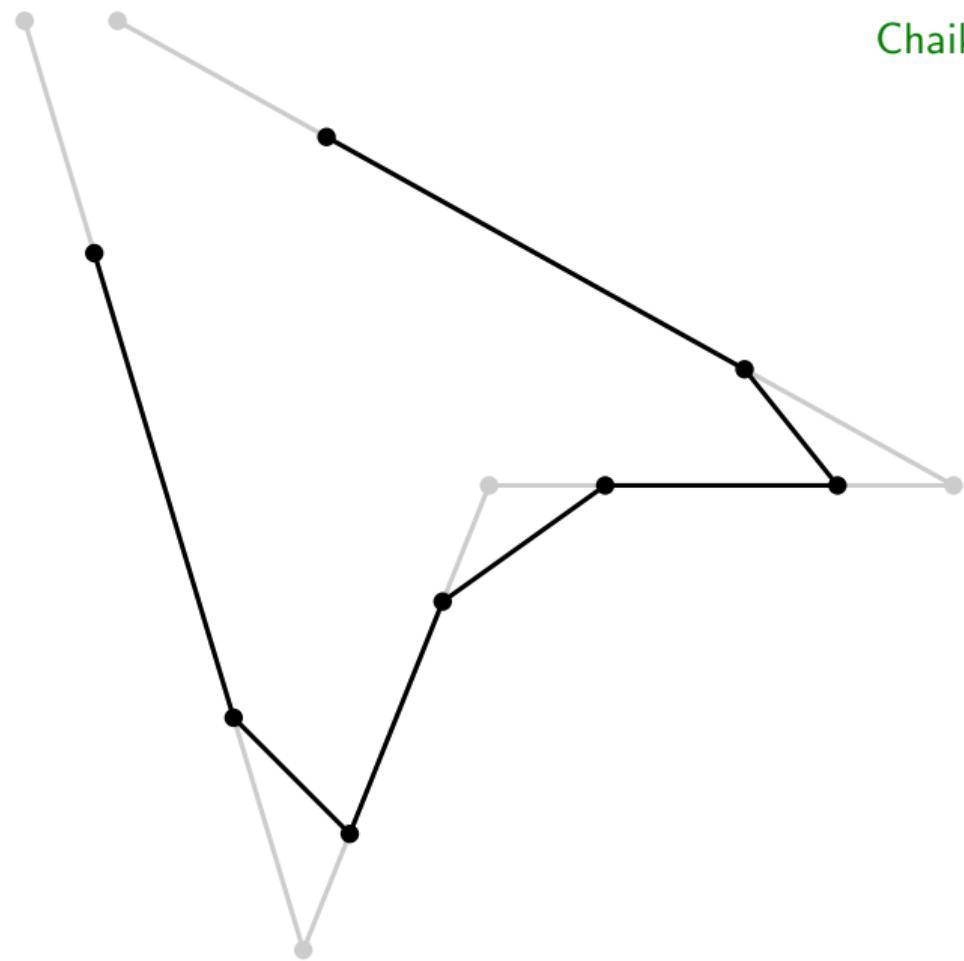
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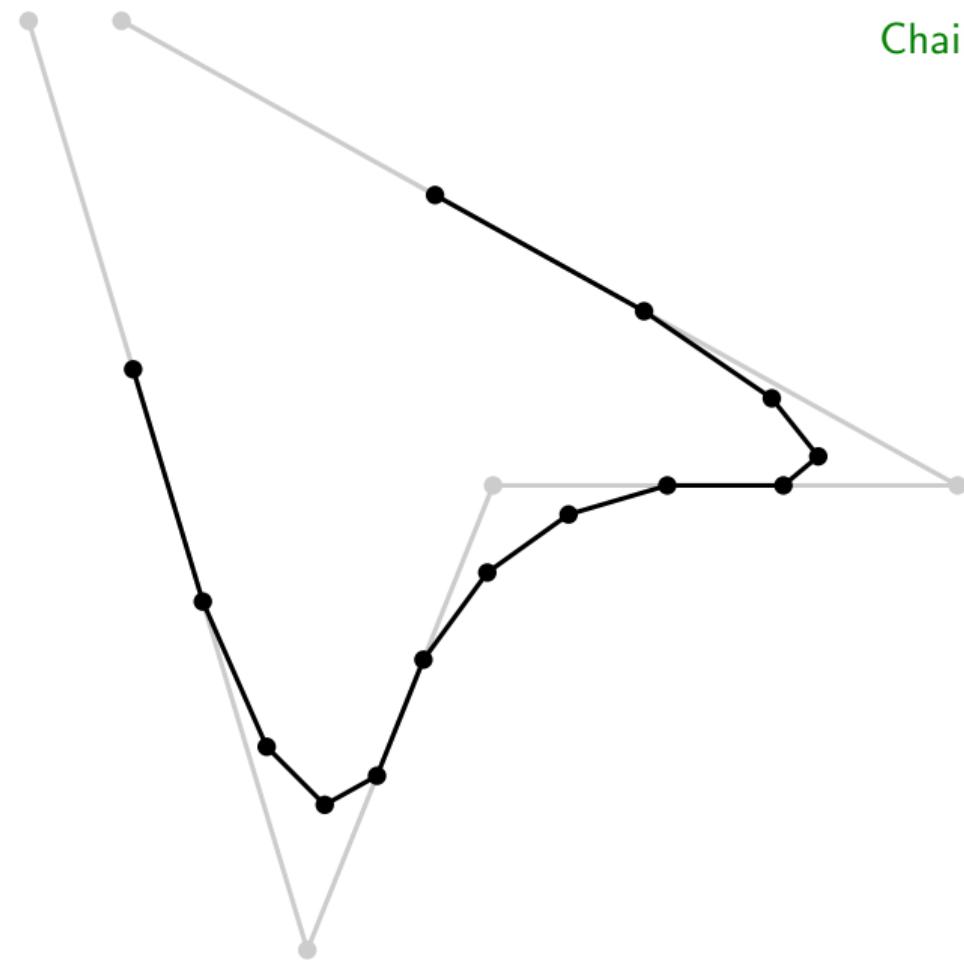
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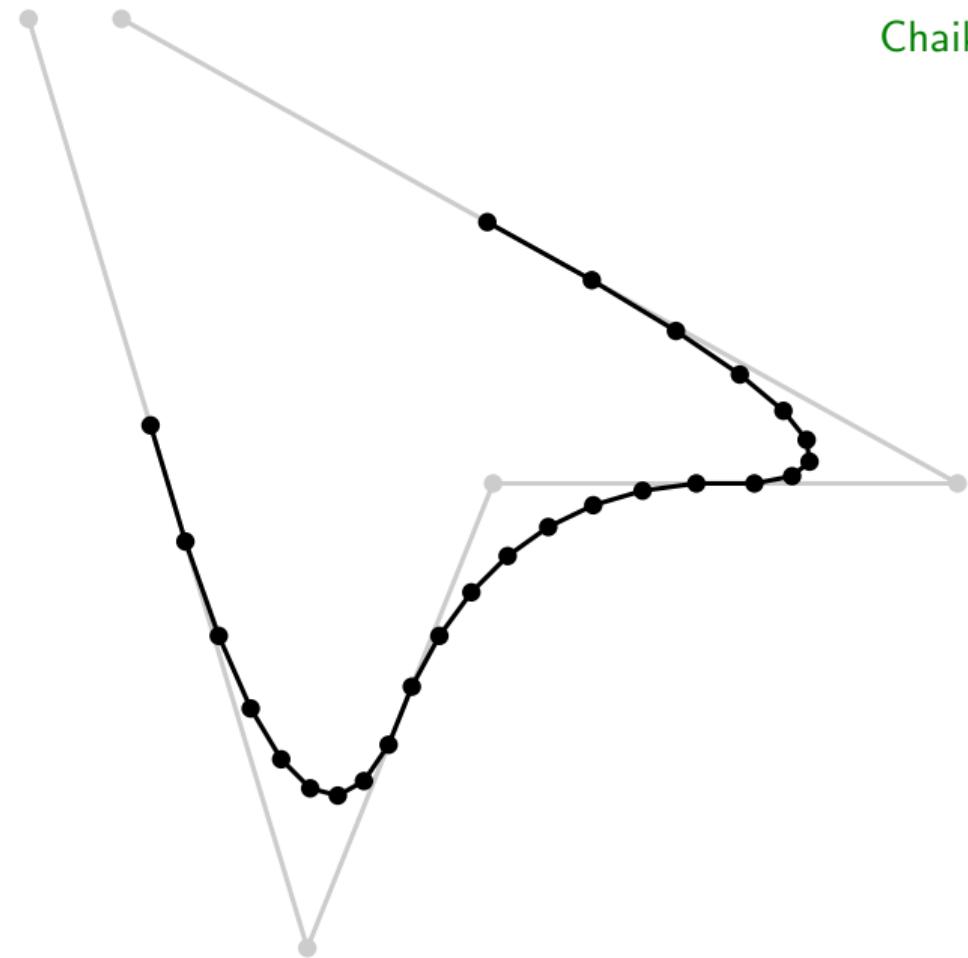
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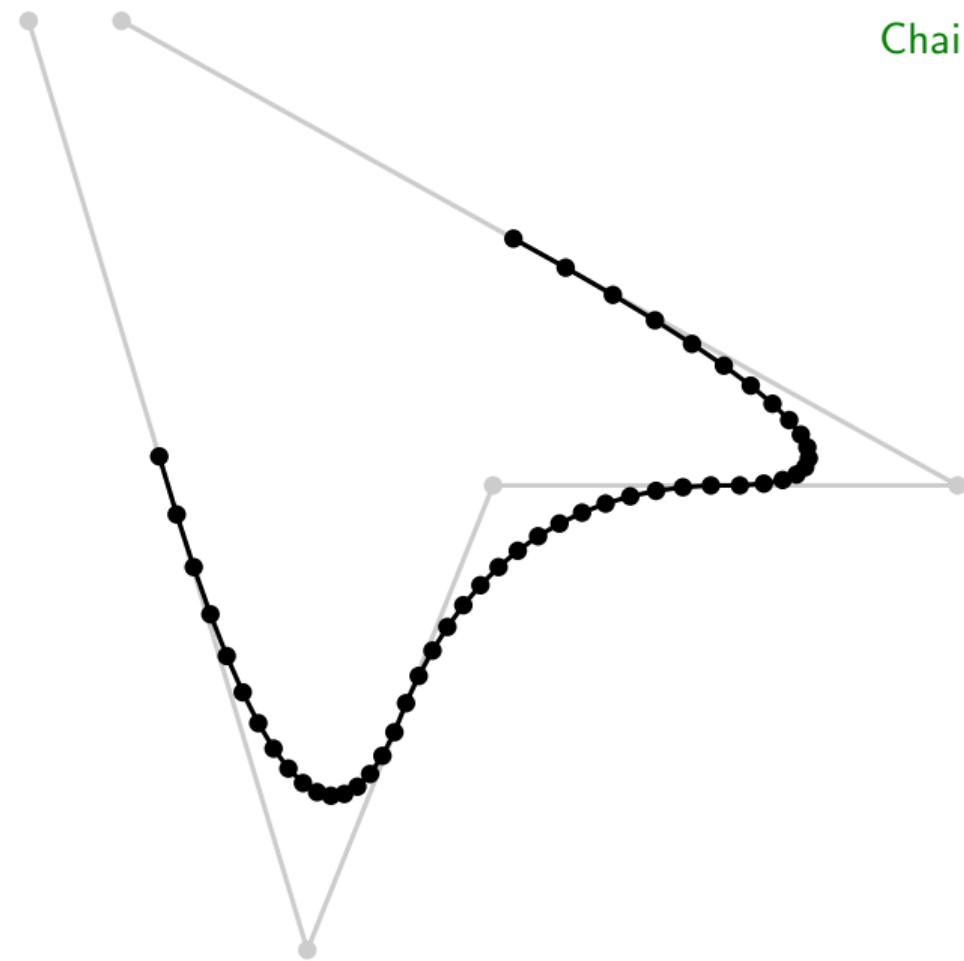
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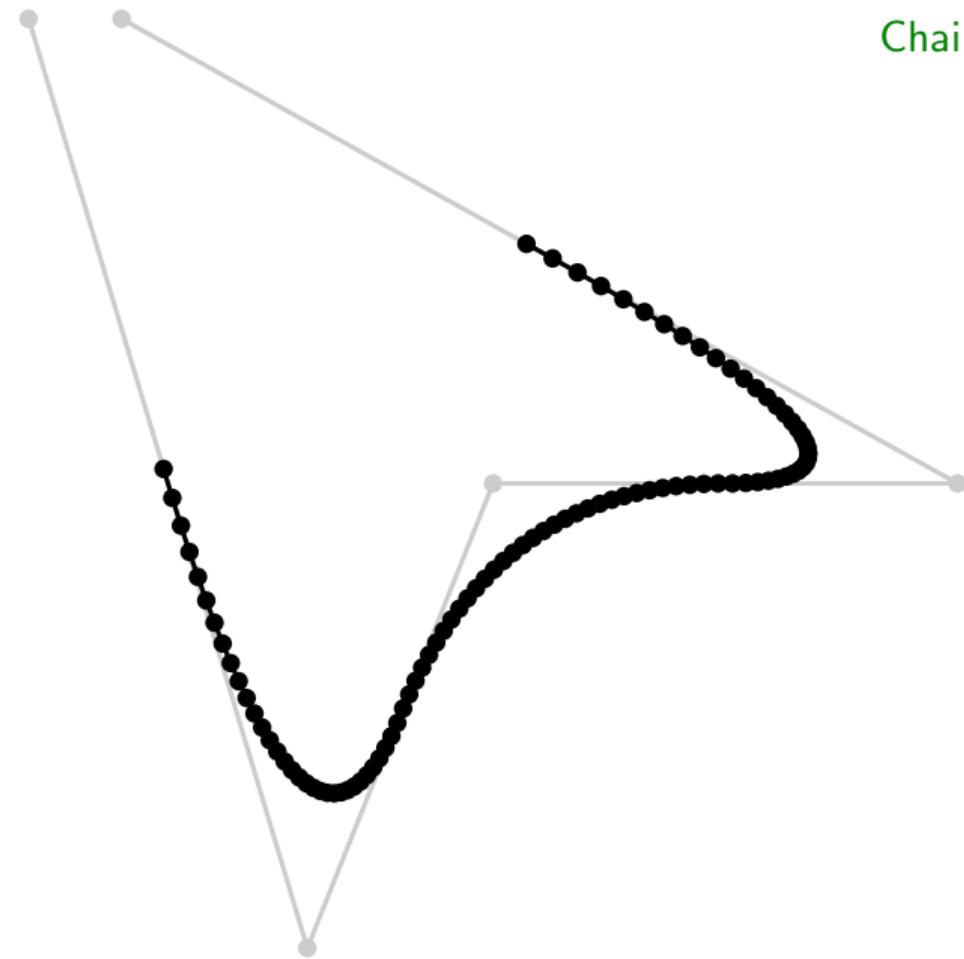
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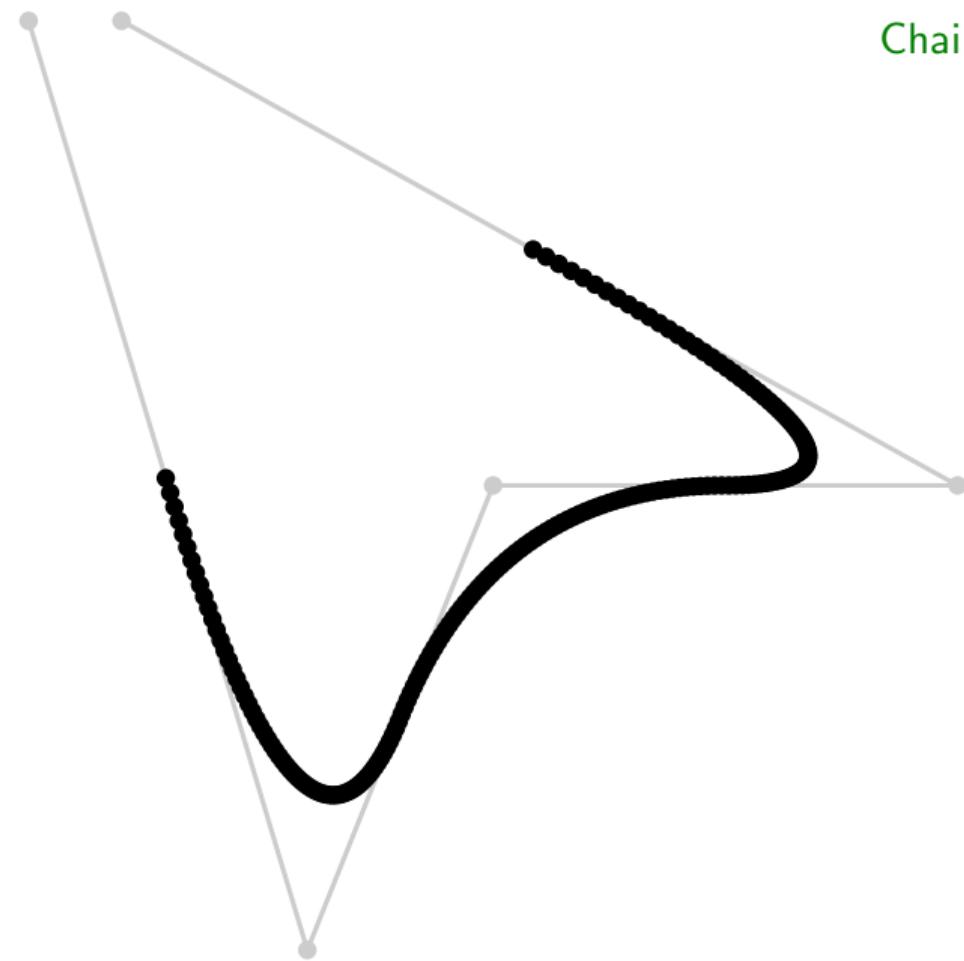
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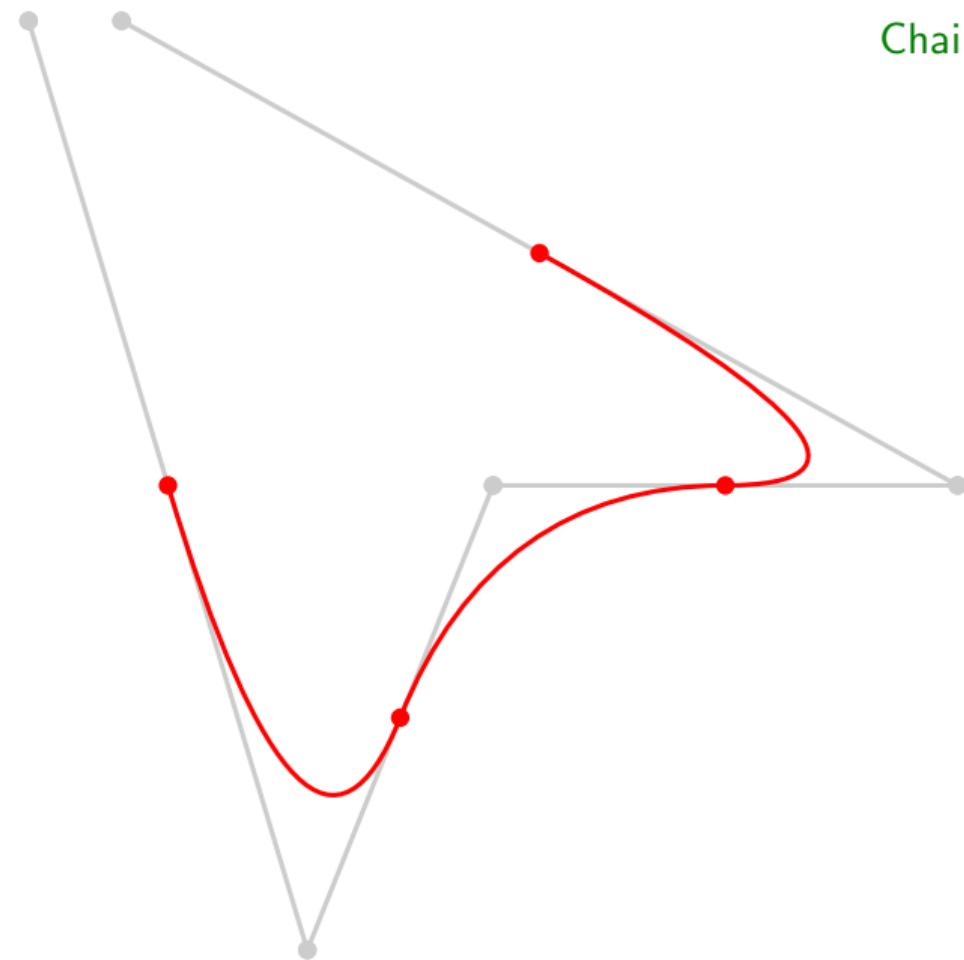
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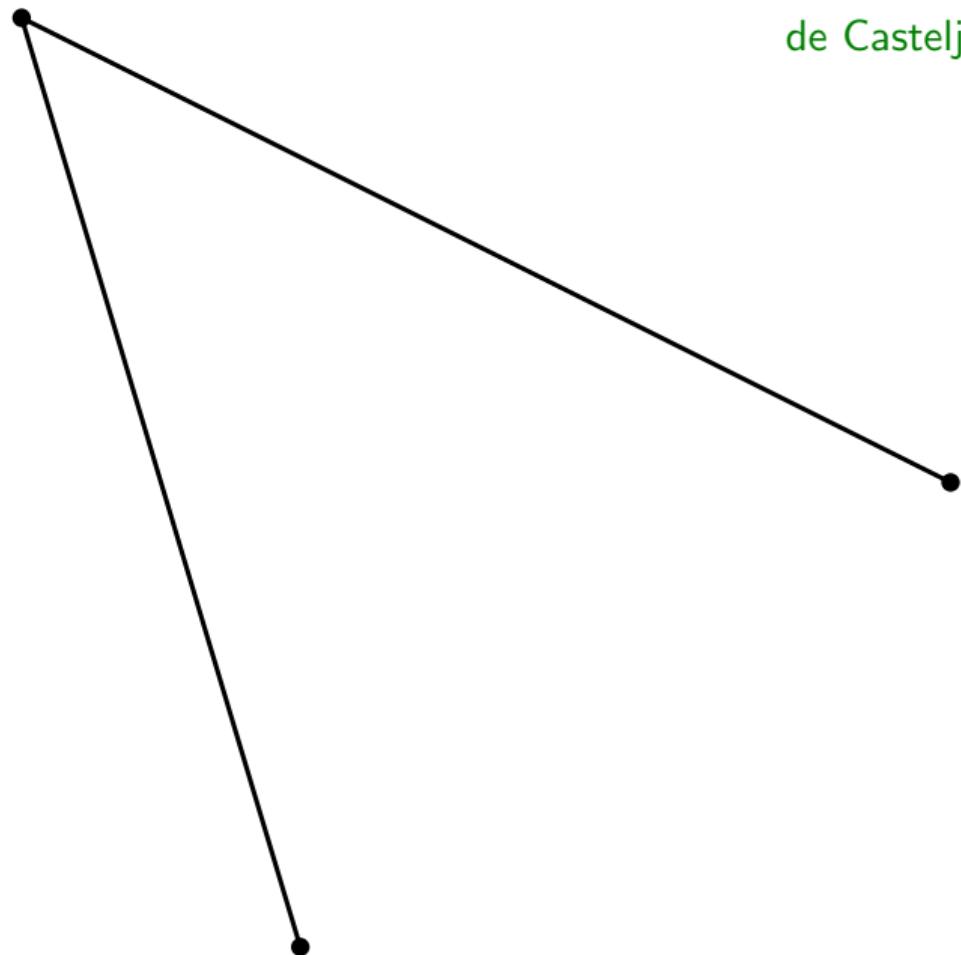
## Curvas de subdivisão

Chaikin (1974)



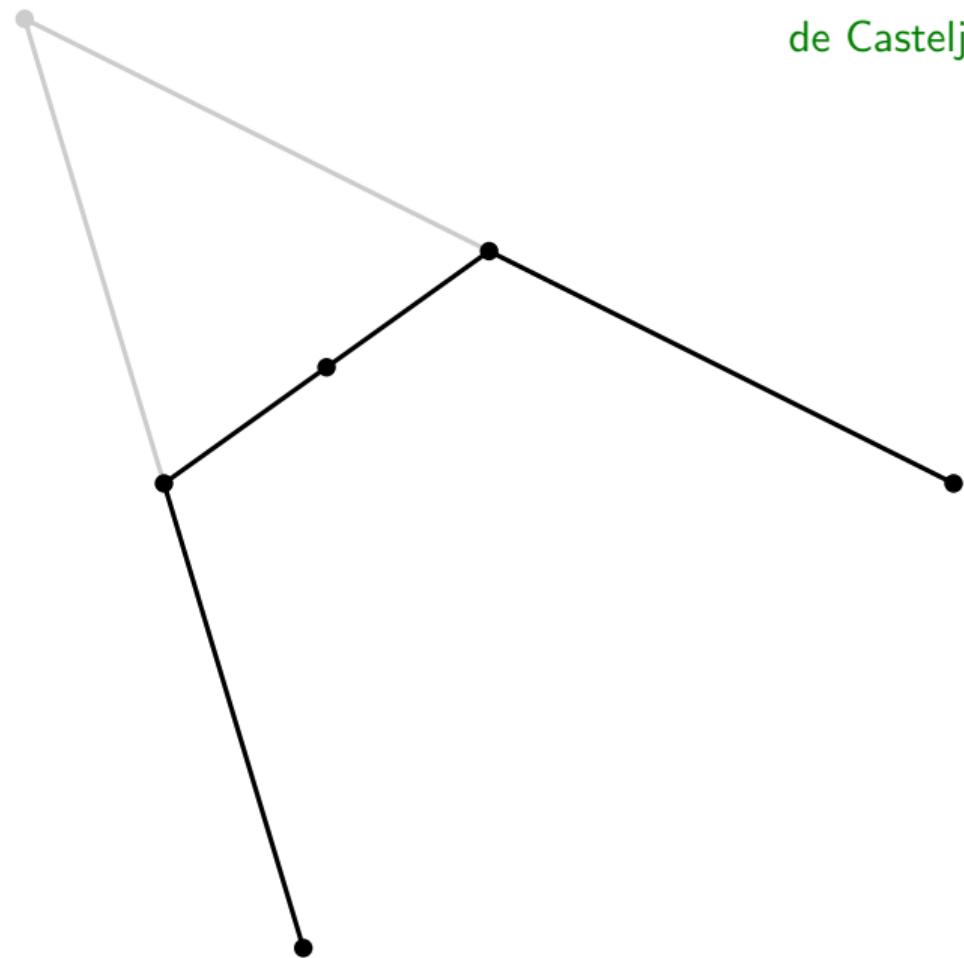
## Curvas de subdivisão

de Casteljau (1963)



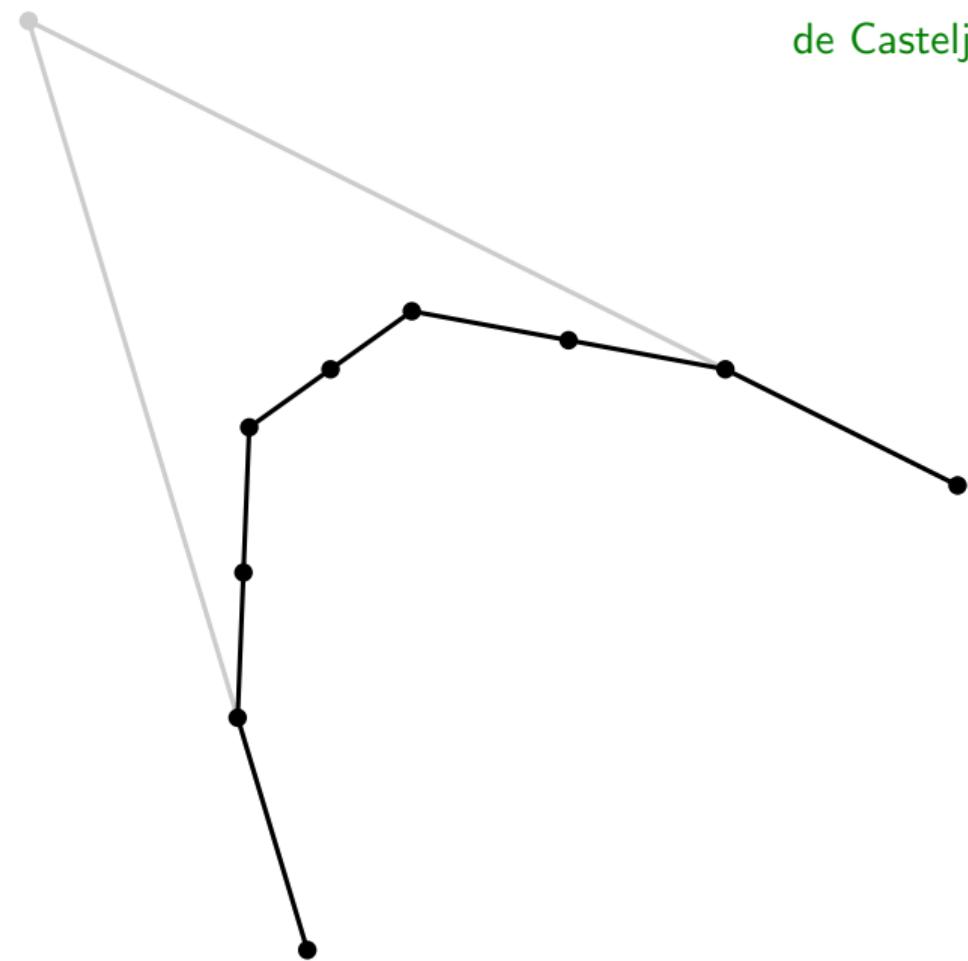
## Curvas de subdivisão

de Casteljau (1963)



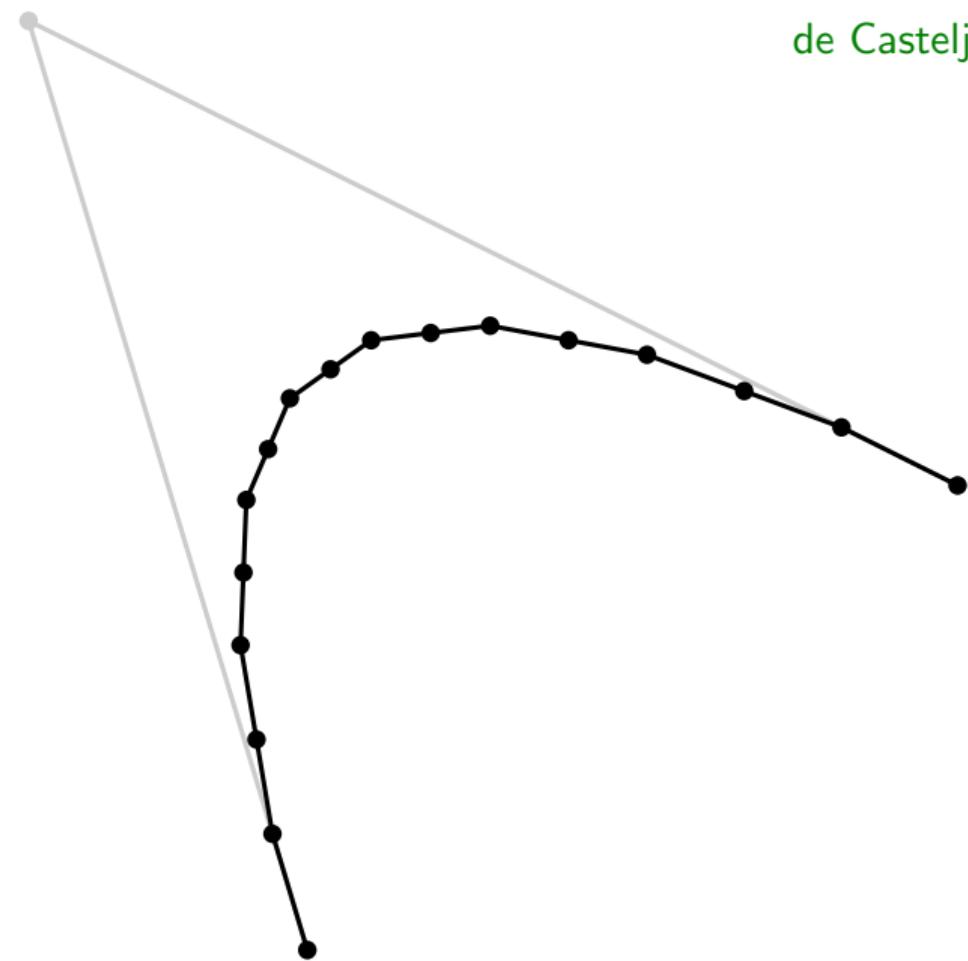
## Curvas de subdivisão

de Casteljau (1963)



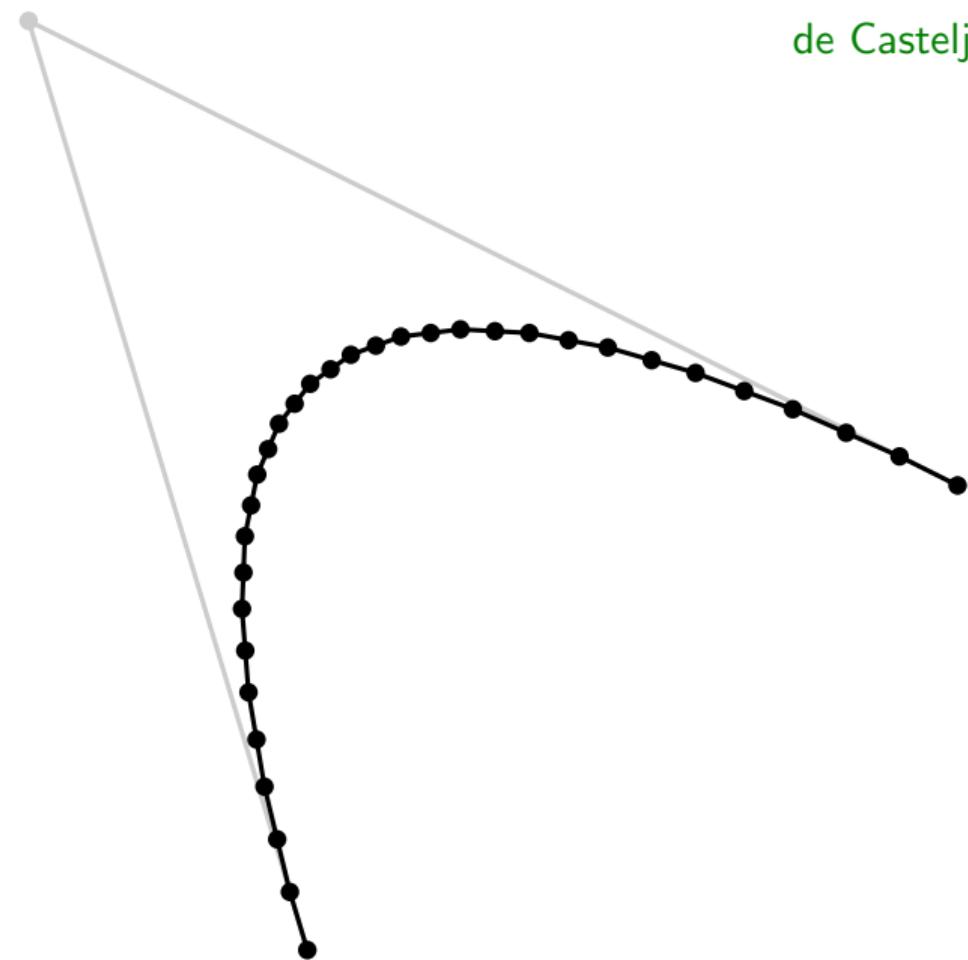
## Curvas de subdivisão

de Casteljau (1963)



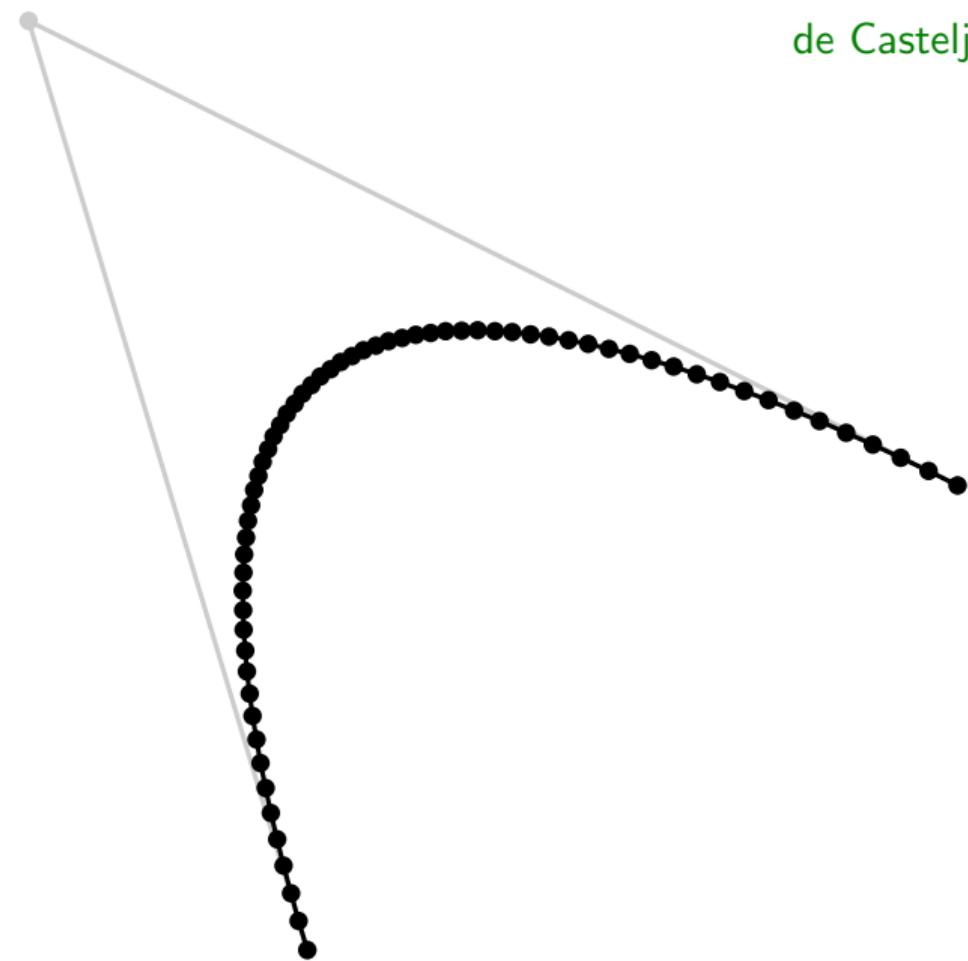
## Curvas de subdivisão

de Casteljau (1963)



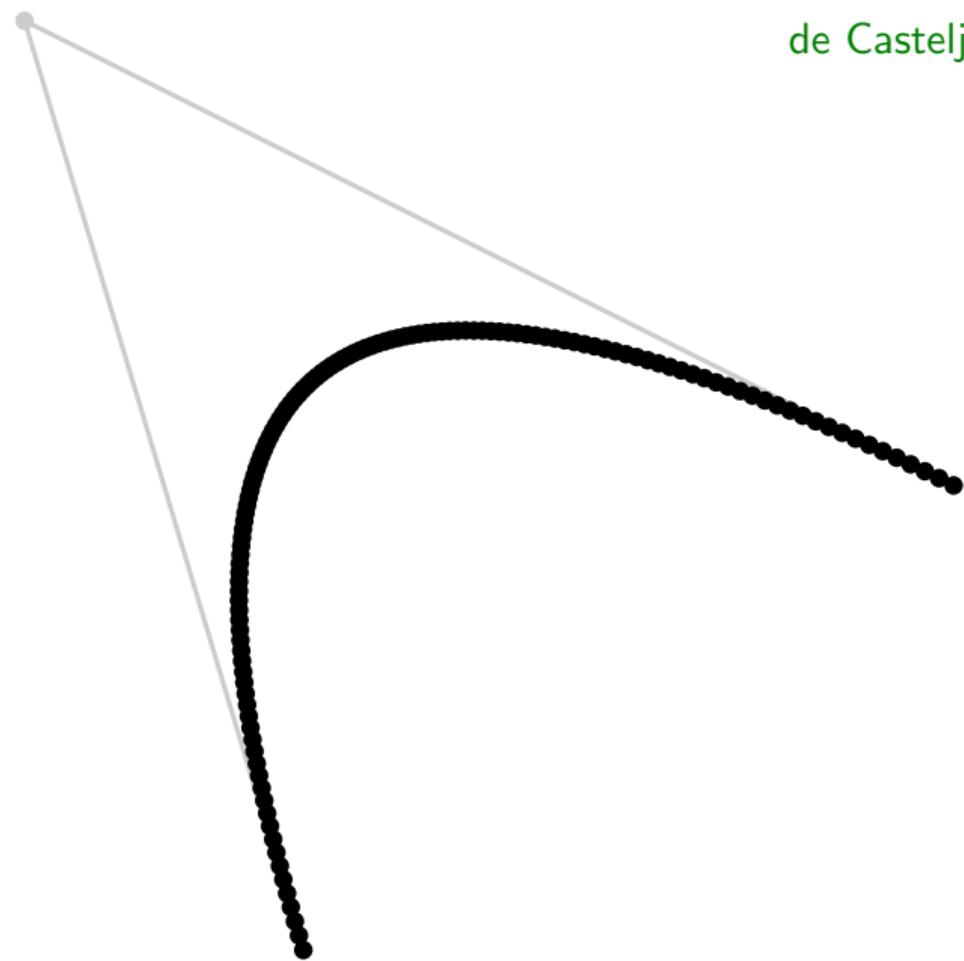
## Curvas de subdivisão

de Casteljau (1963)



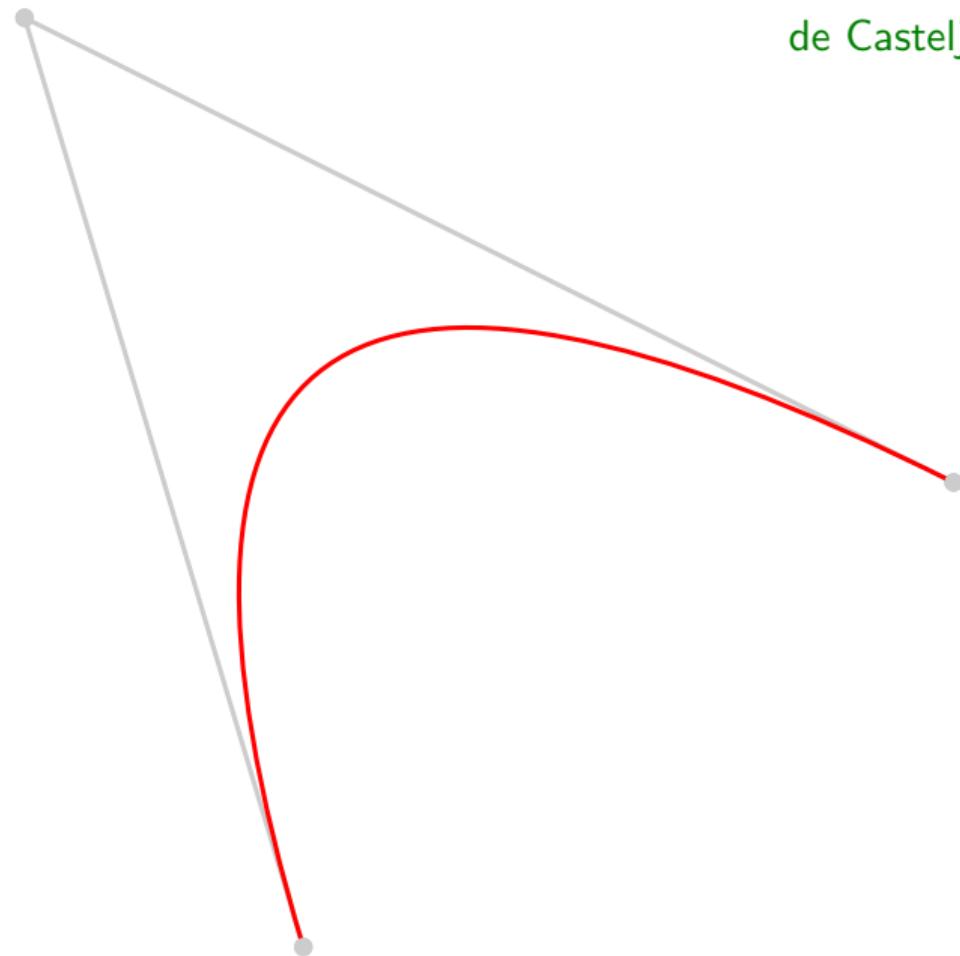
## Curvas de subdivisão

de Casteljau (1963)



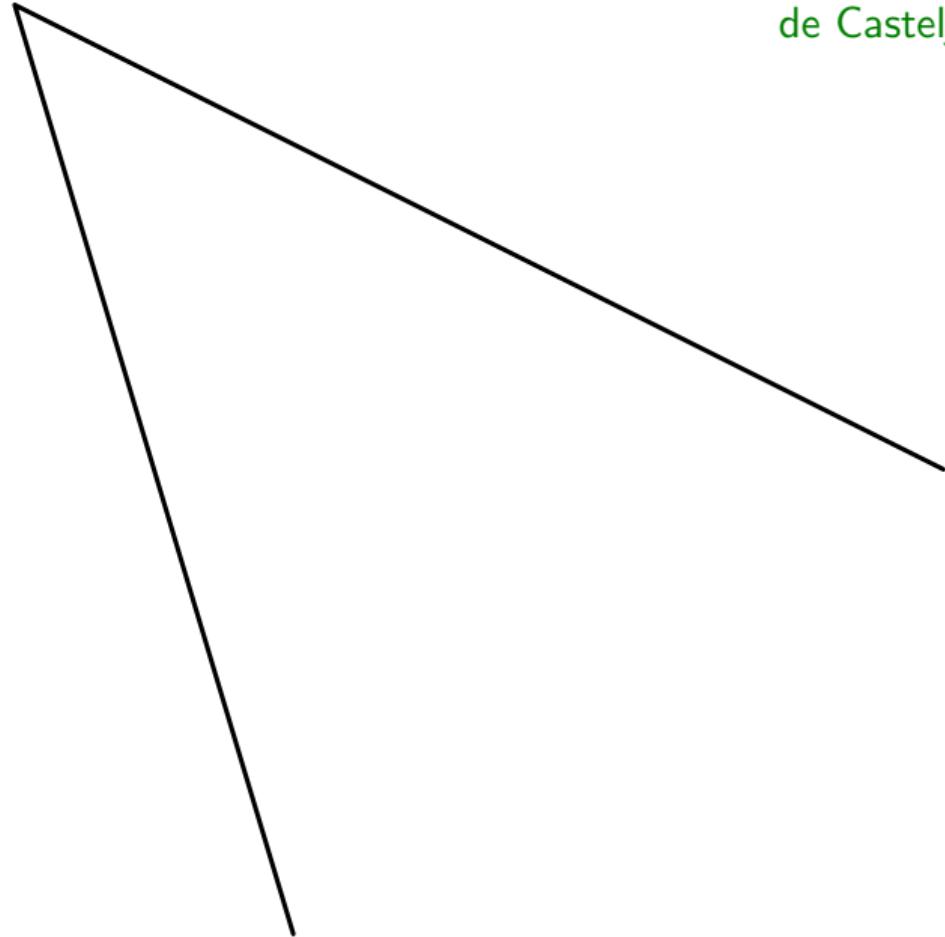
## Curvas de subdivisão

de Casteljau (1963)



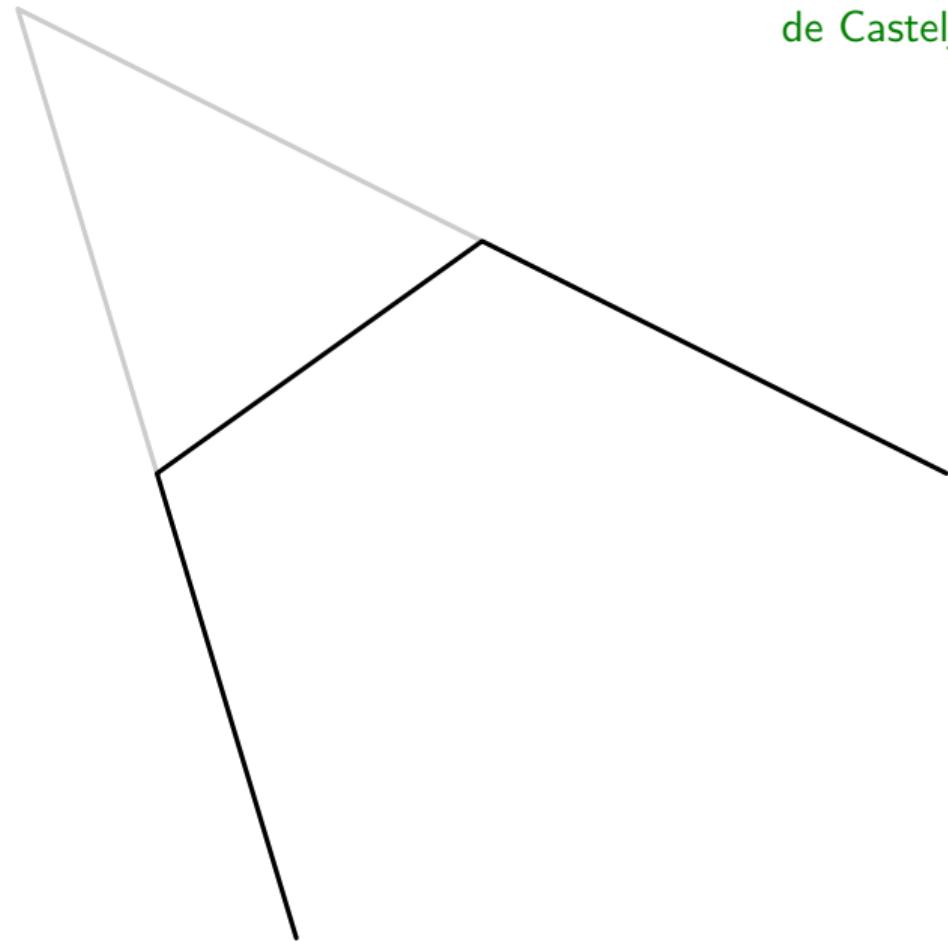
## Curvas de subdivisão

de Casteljau (1963)



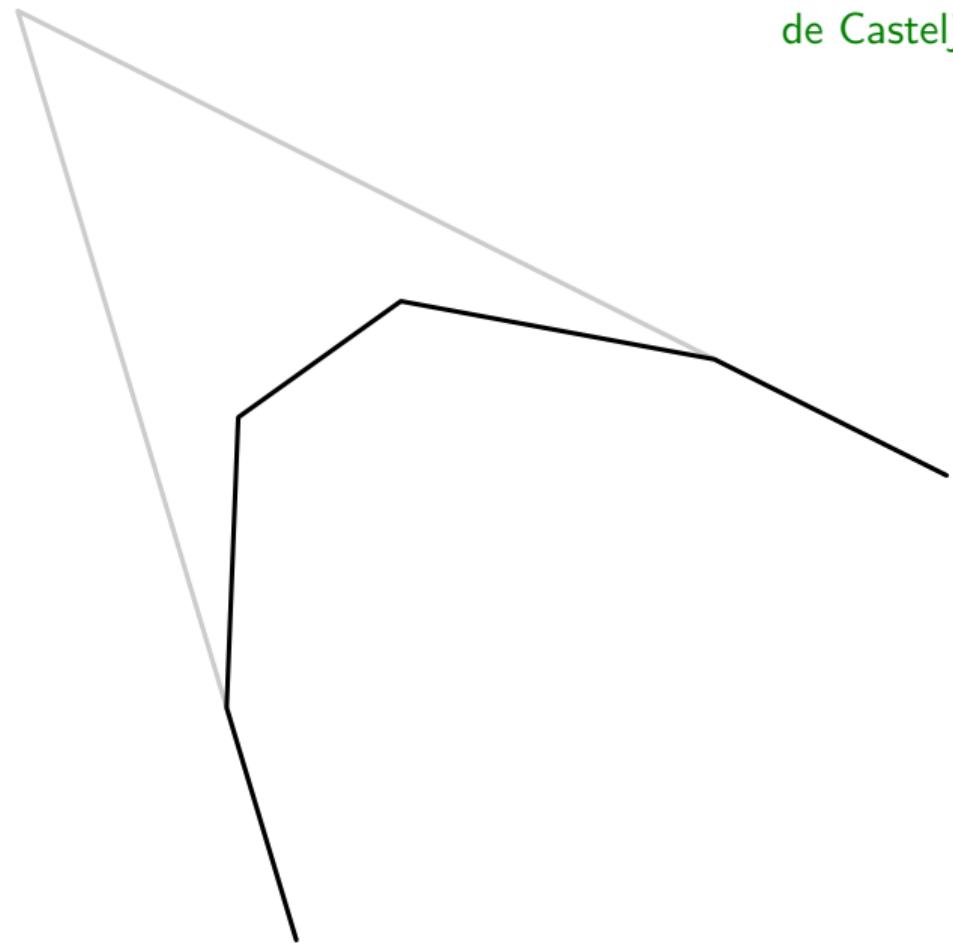
## Curvas de subdivisão

de Casteljau (1963)



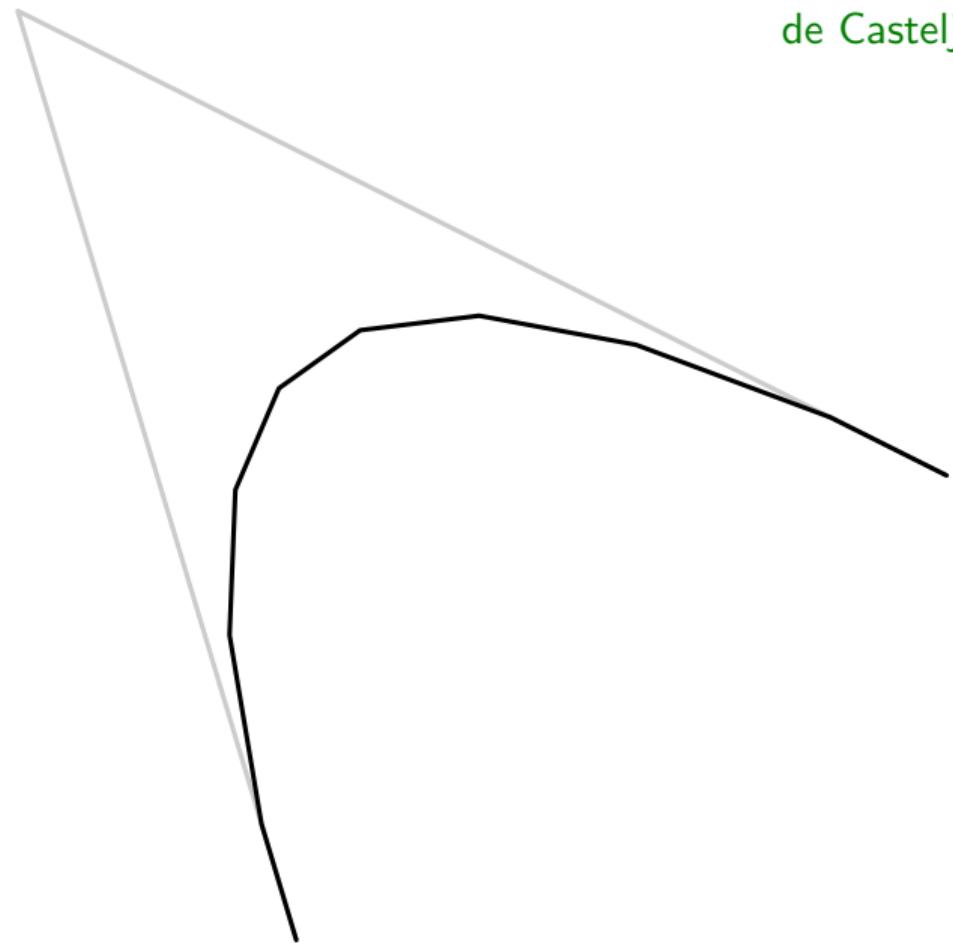
## Curvas de subdivisão

de Casteljau (1963)



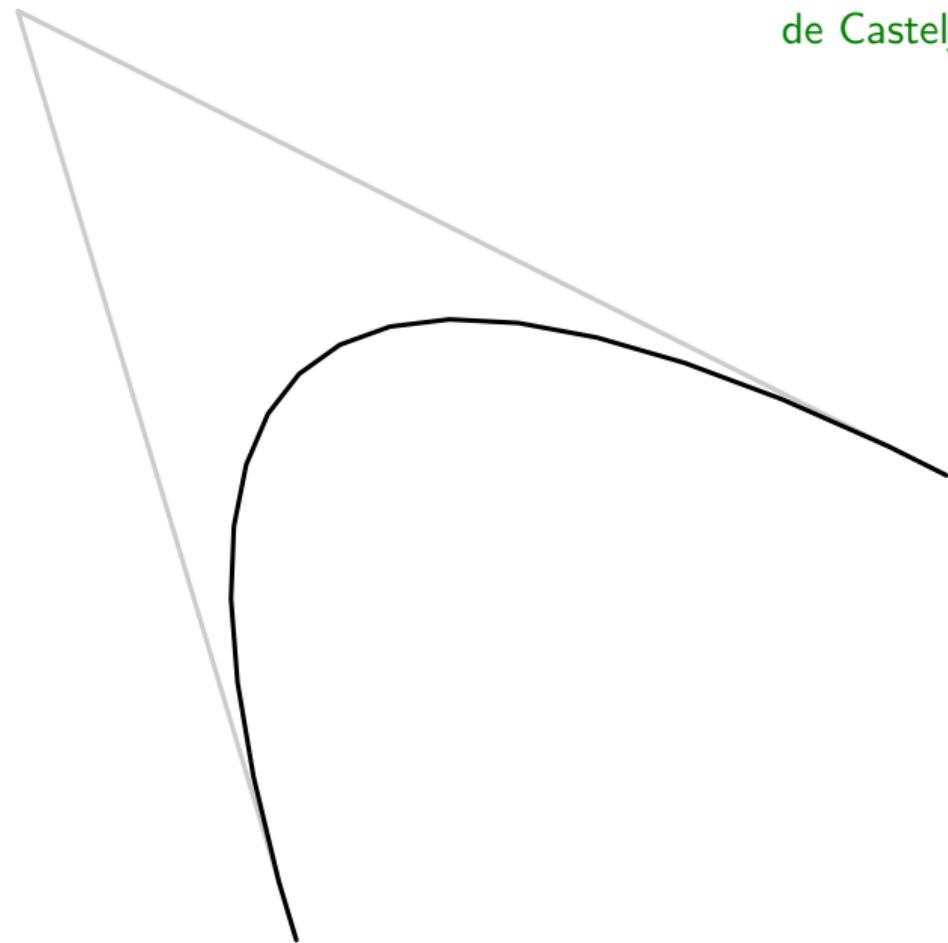
## Curvas de subdivisão

de Casteljau (1963)



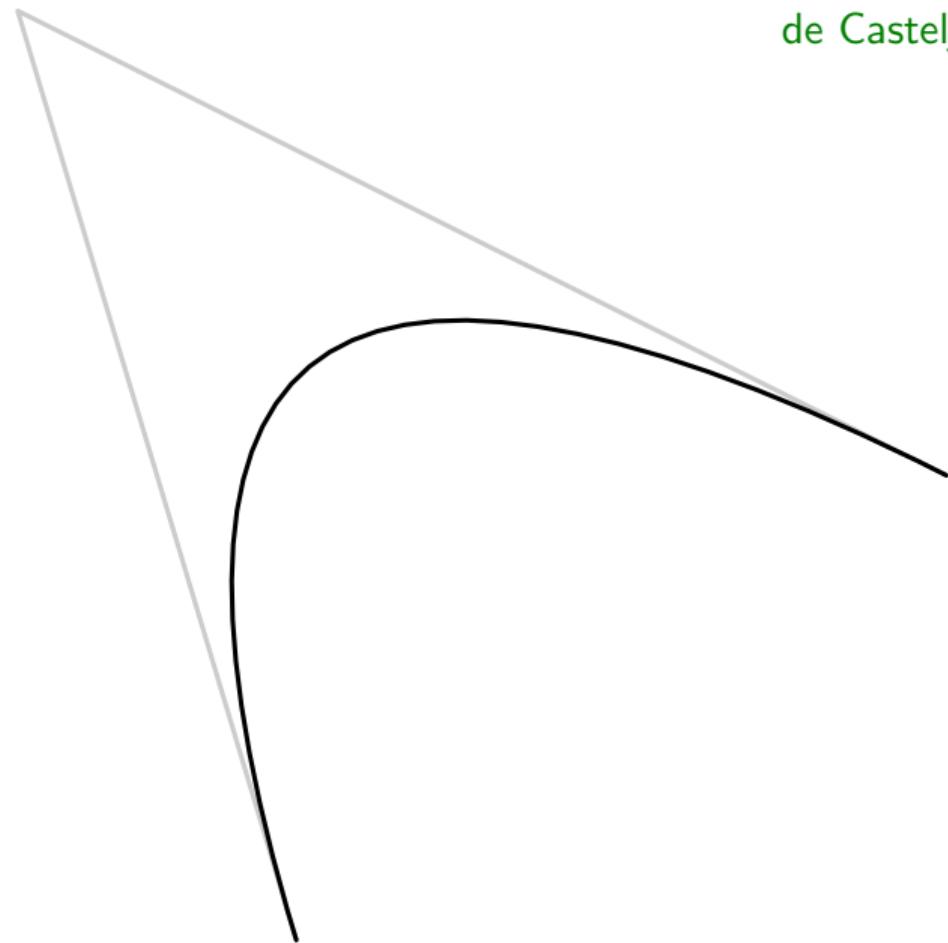
## Curvas de subdivisão

de Casteljau (1963)



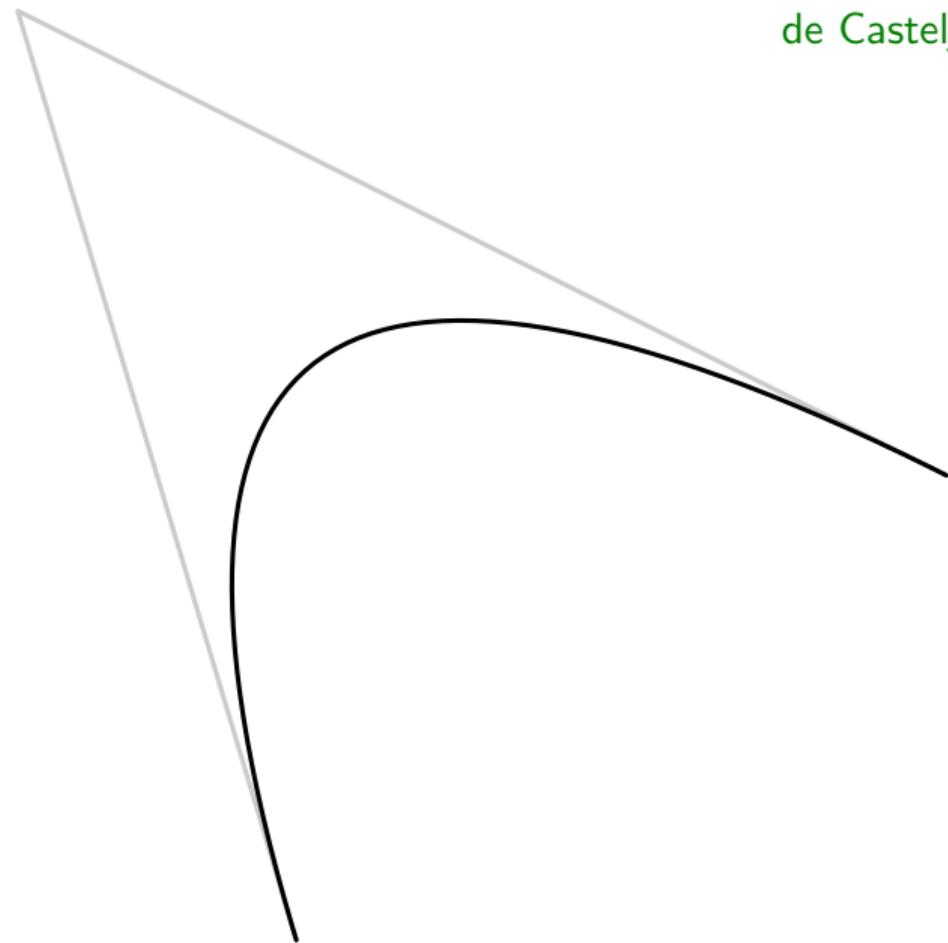
## Curvas de subdivisão

de Casteljau (1963)



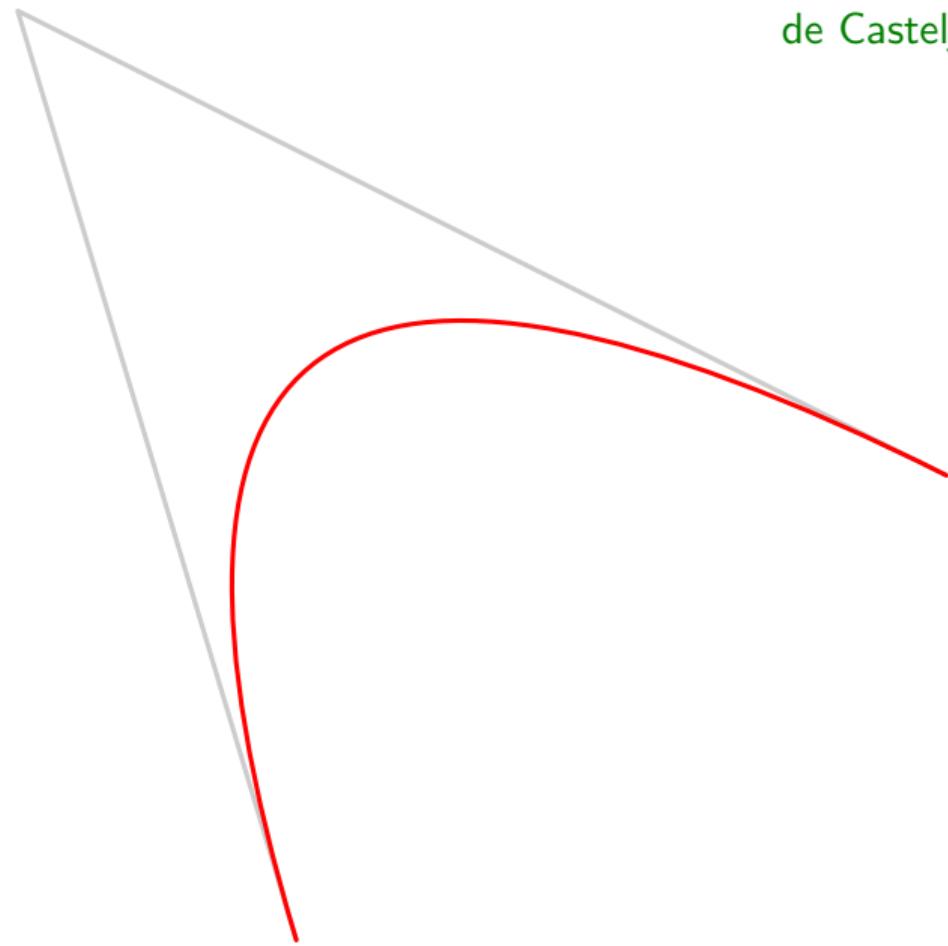
## Curvas de subdivisão

de Casteljau (1963)



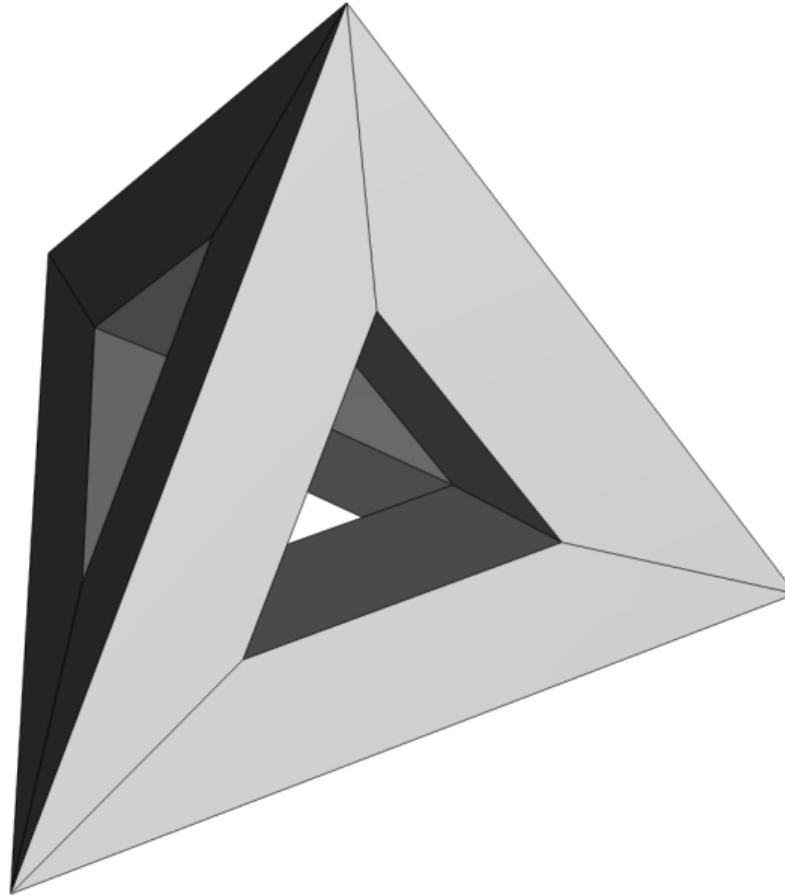
## Curvas de subdivisão

de Casteljau (1963)



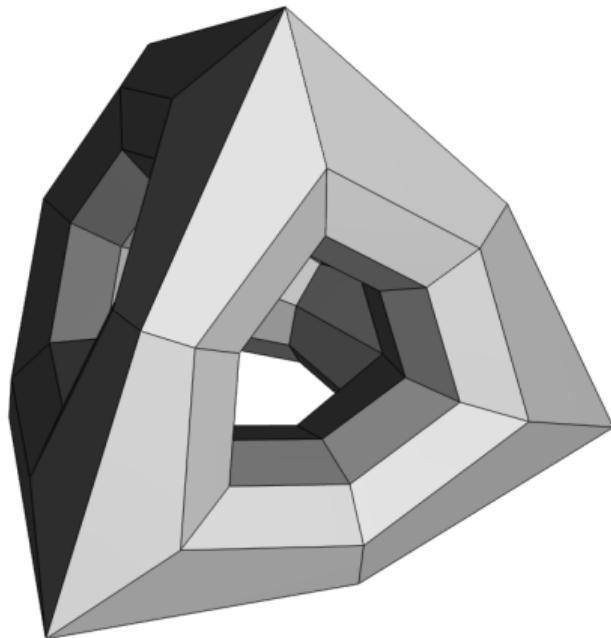
# Superfícies de subdivisão

Catmull–Clark (1978)



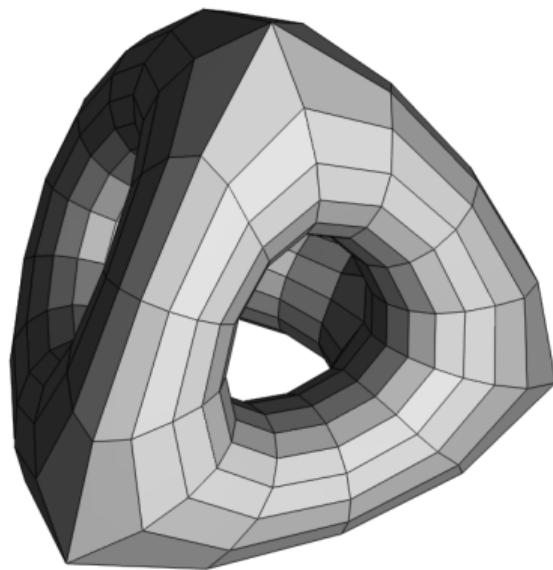
# Superfícies de subdivisão

Catmull–Clark (1978)



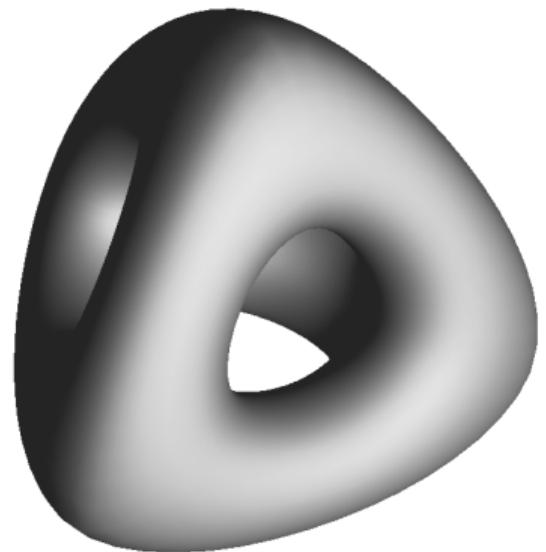
# Superfícies de subdivisão

Catmull–Clark (1978)

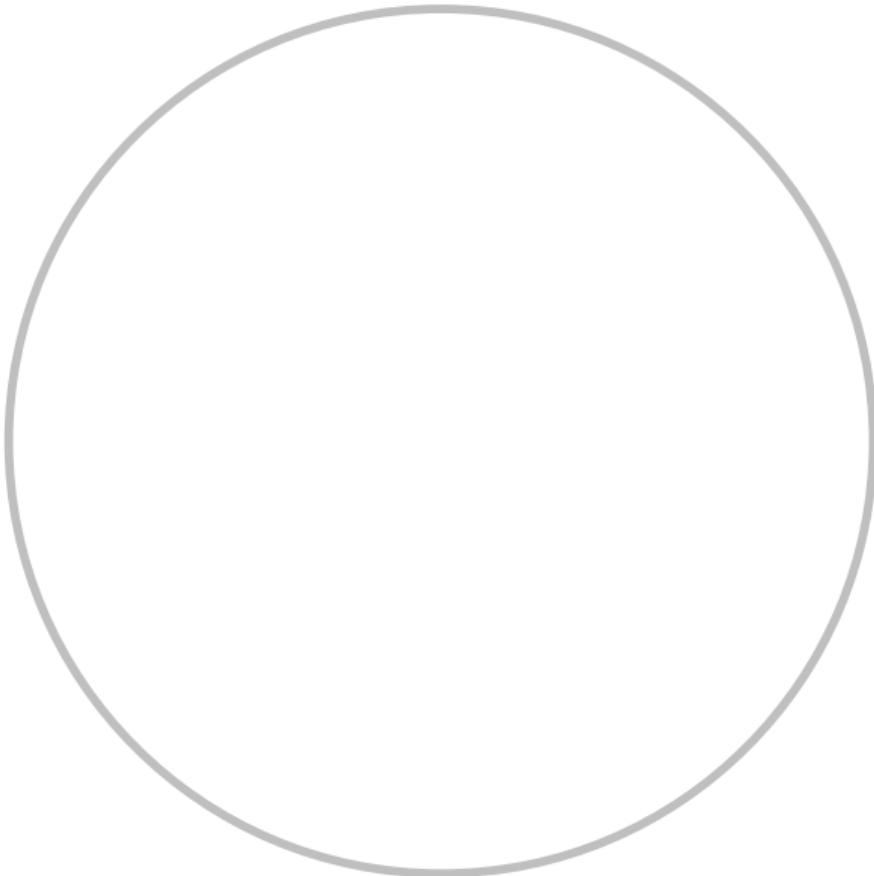


# Superfícies de subdivisão

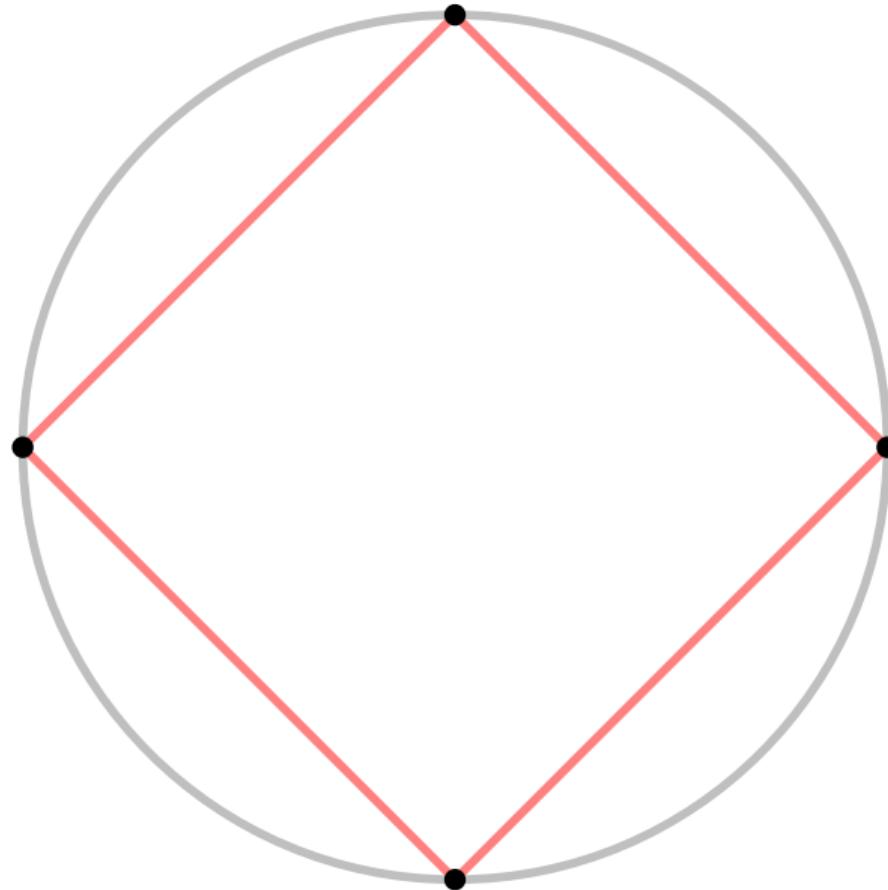
Catmull–Clark (1978)



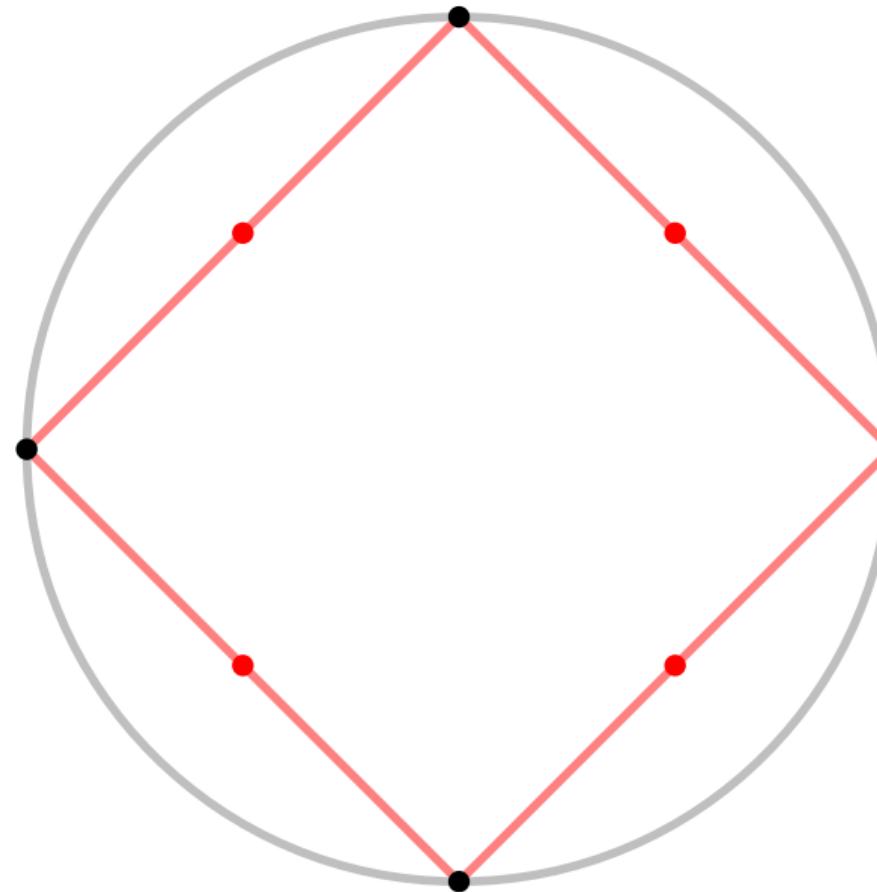
implicit projection



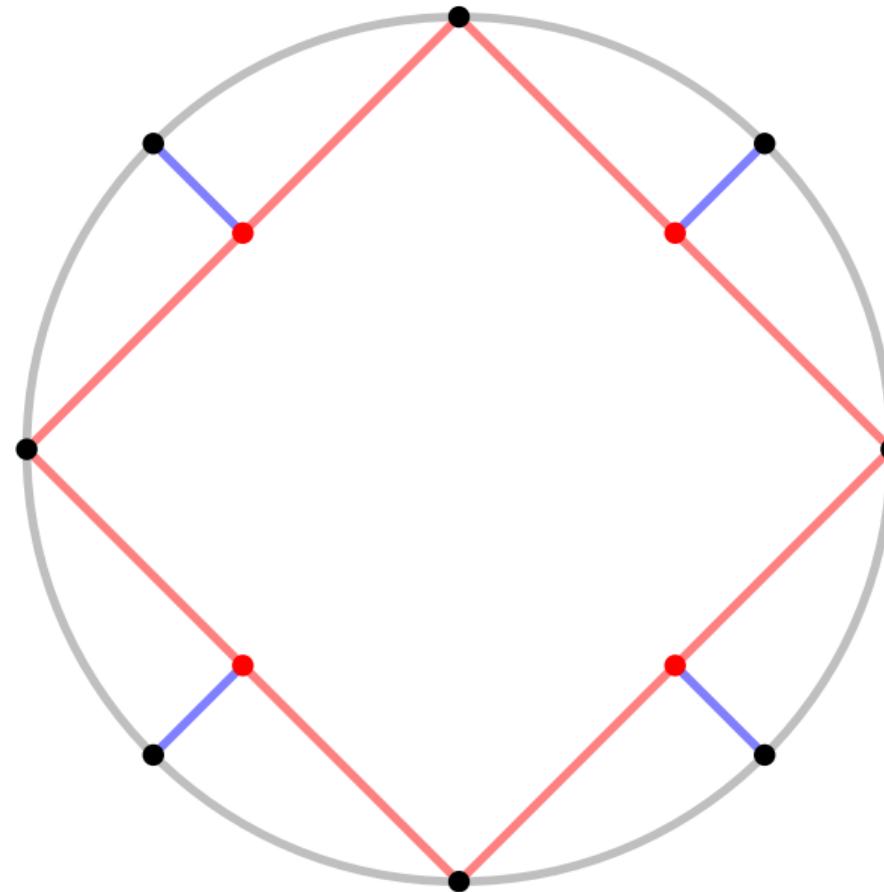
implicit projection



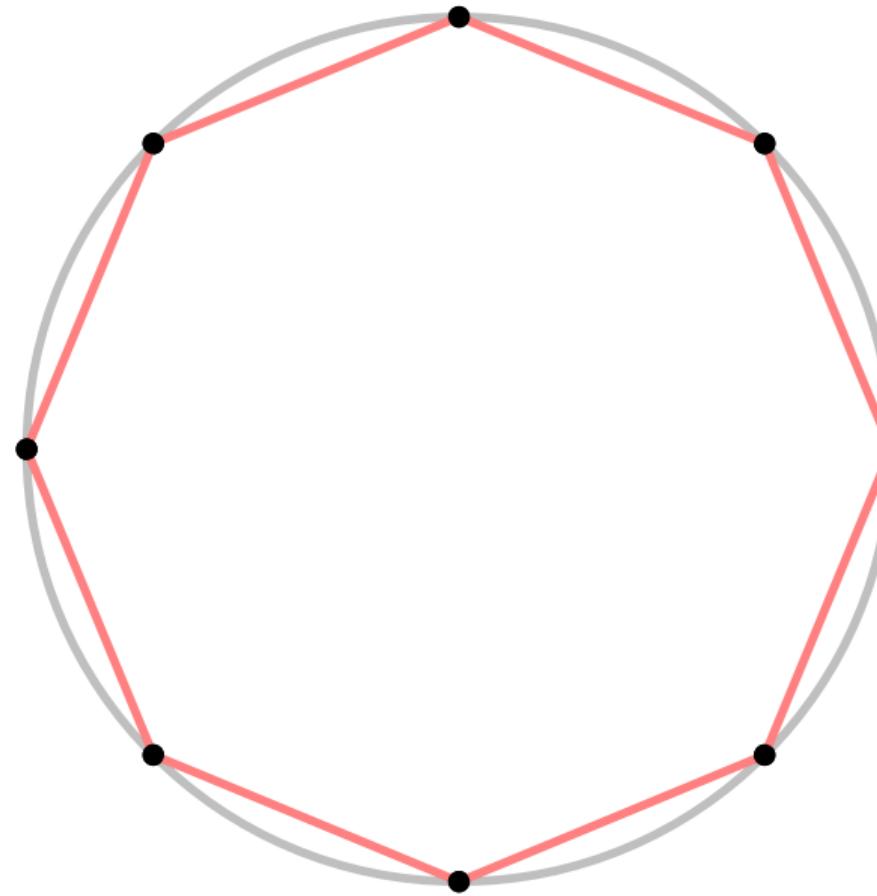
implicit projection



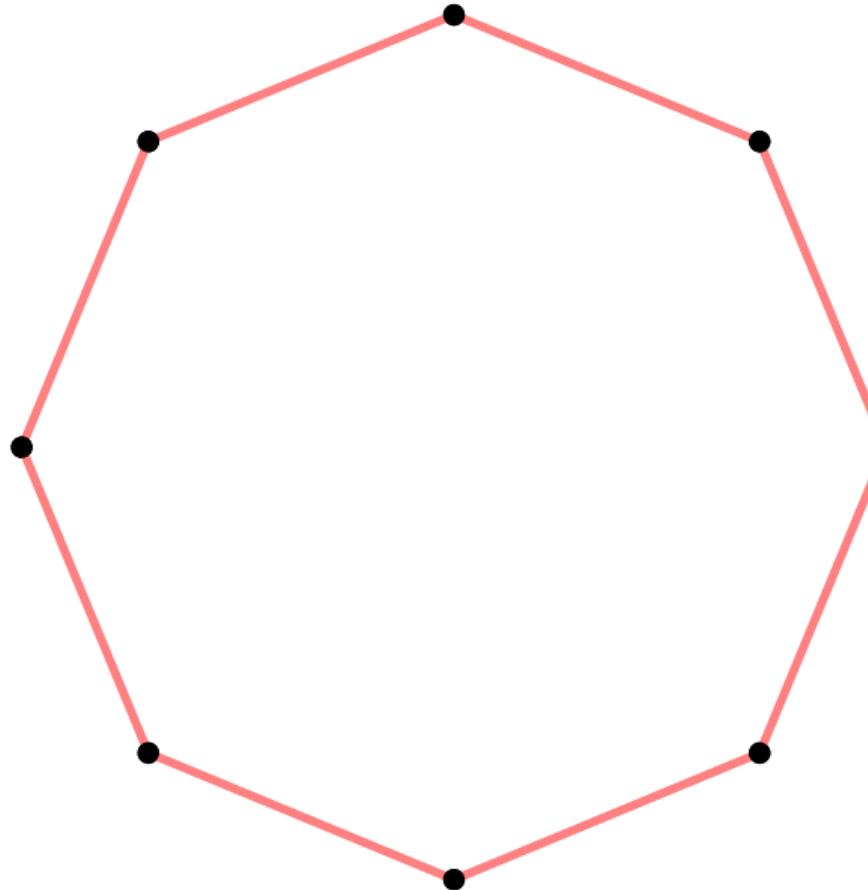
## implicit projection



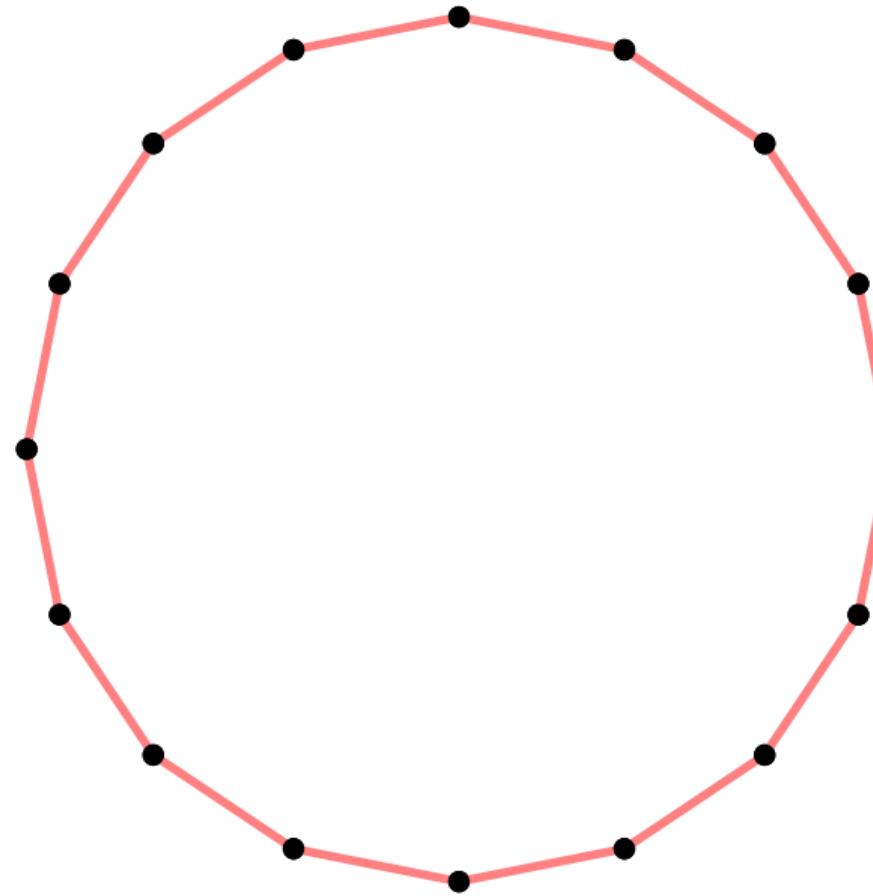
implicit projection



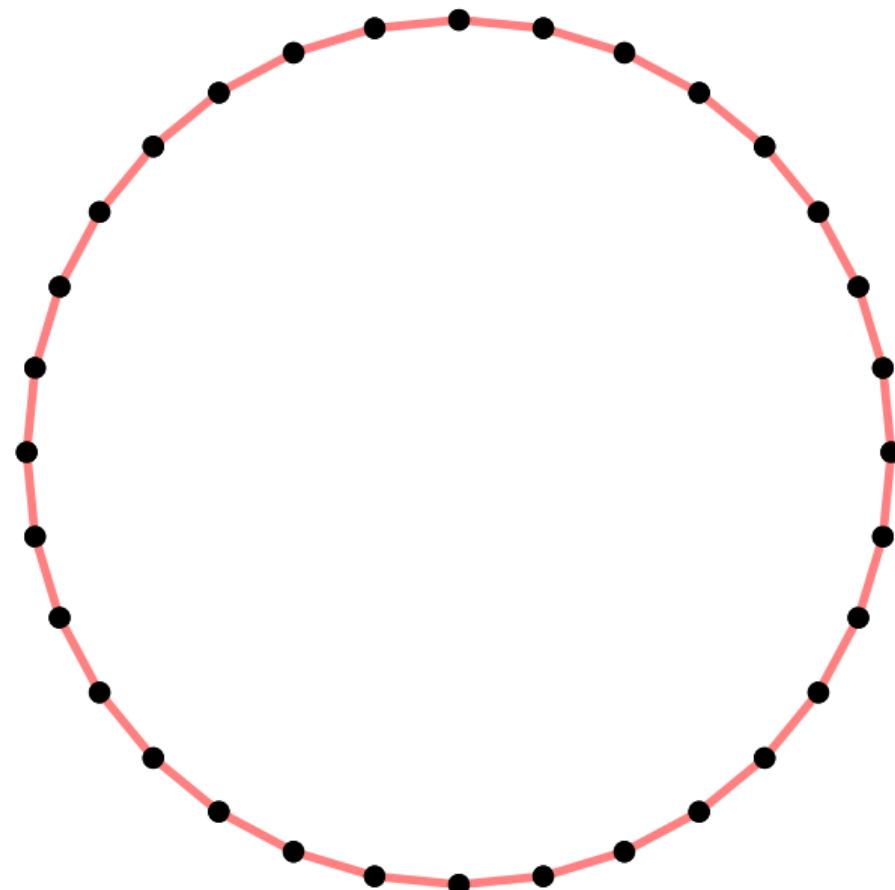
implicit projection



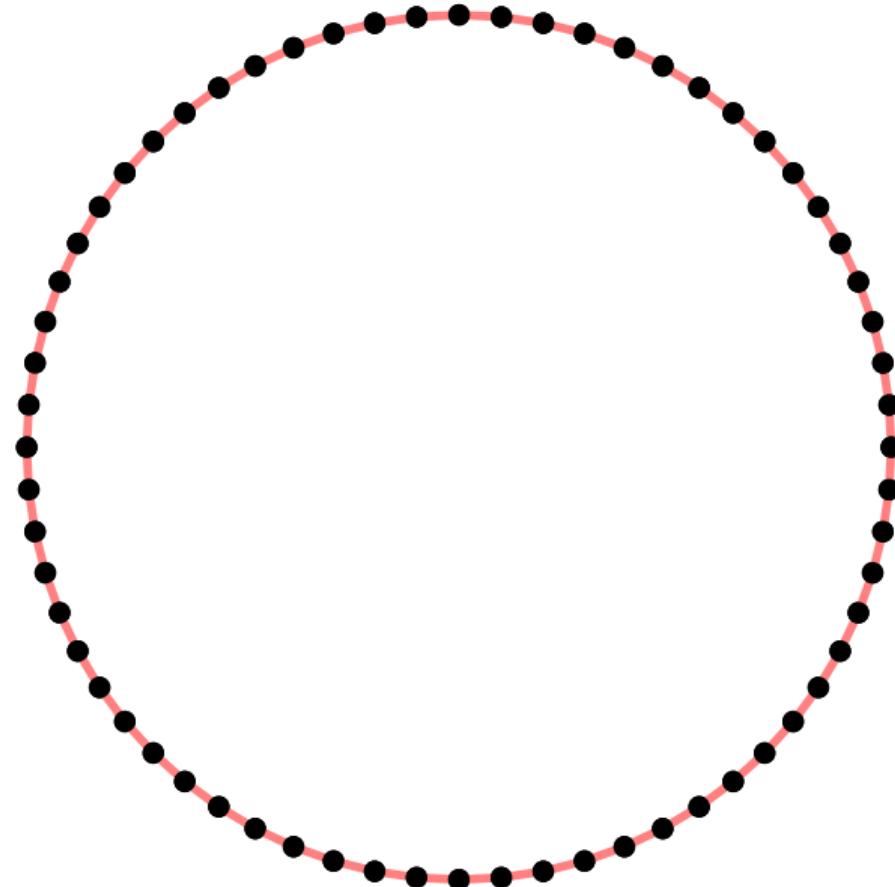
implicit projection



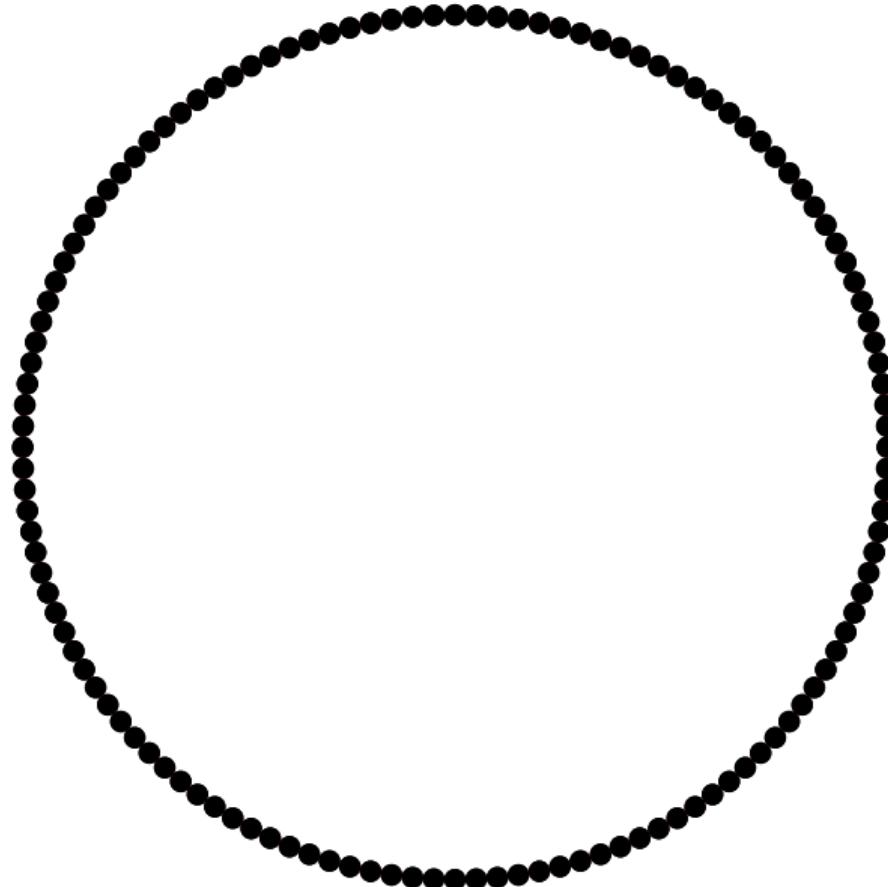
implicit projection



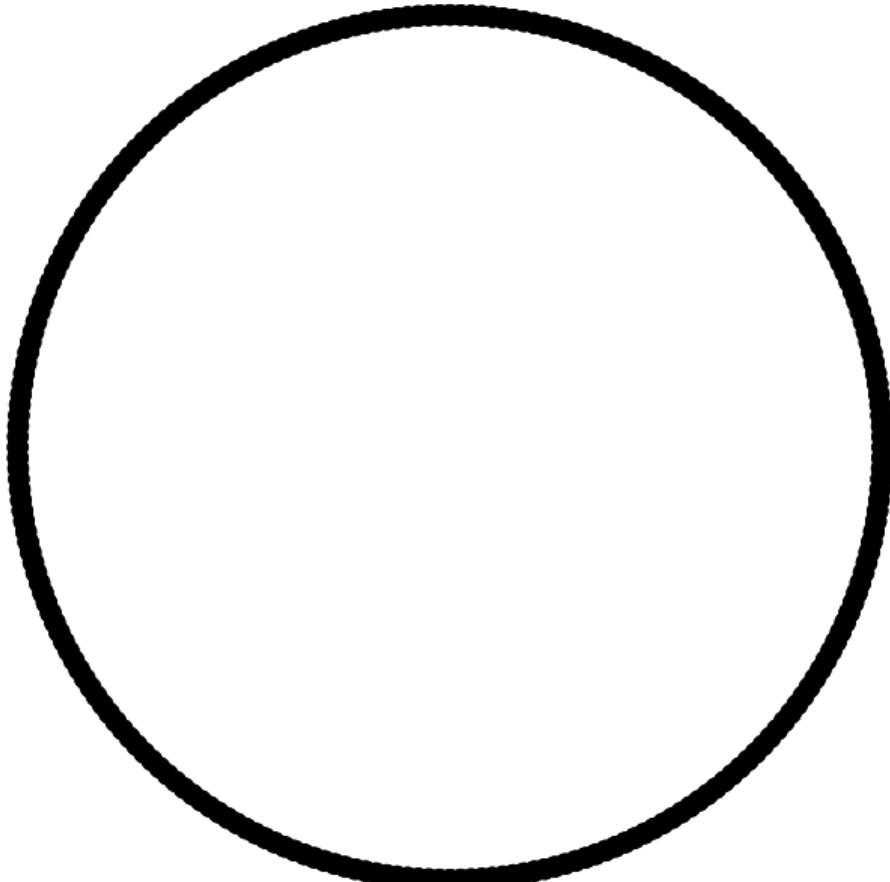
implicit projection



implicit projection



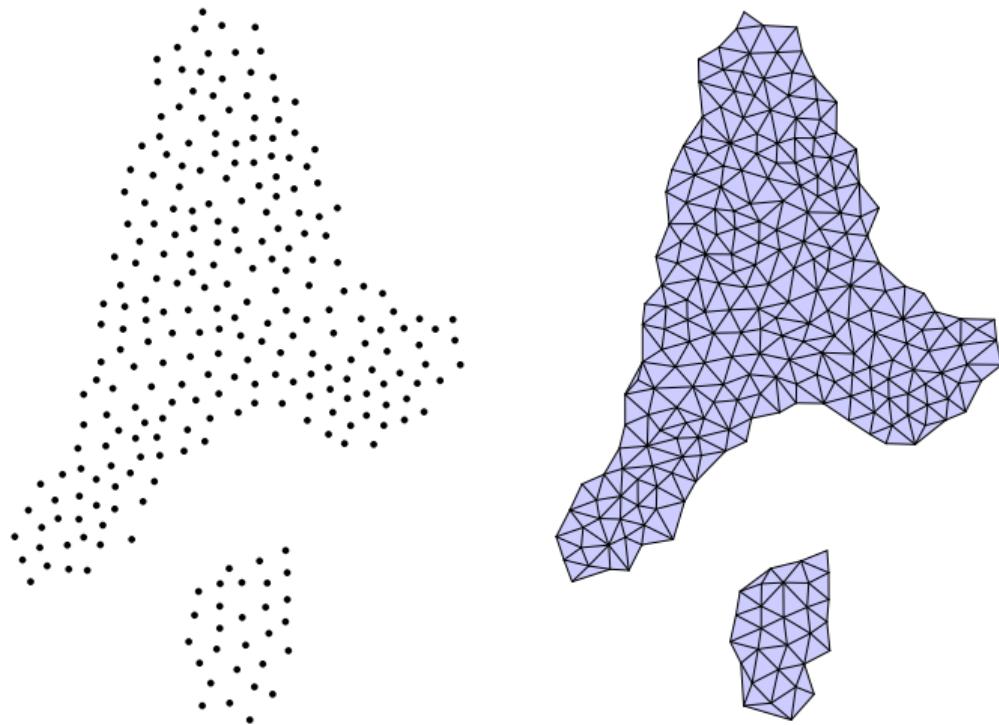
implicit projection



## Aquisição de formas



## Aquisição de formas



## Aquisição de formas

