

Temas de Computação Visual

Modelagem Geométrica

Luiz Henrique de Figueiredo

Formas geométricas em mundos virtuais



Pixar

PIXAR
PRESENTS

Geri's game



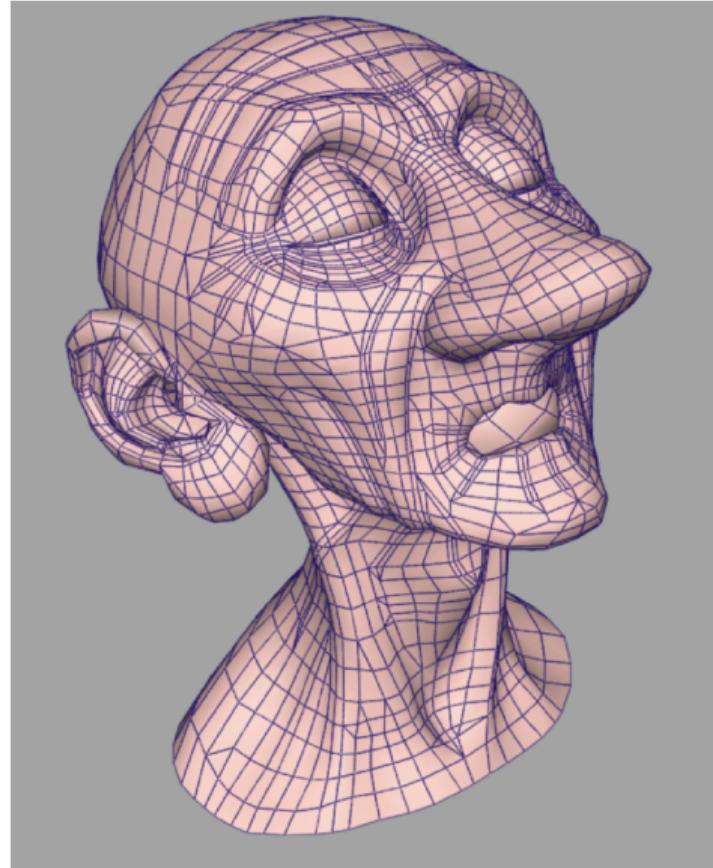
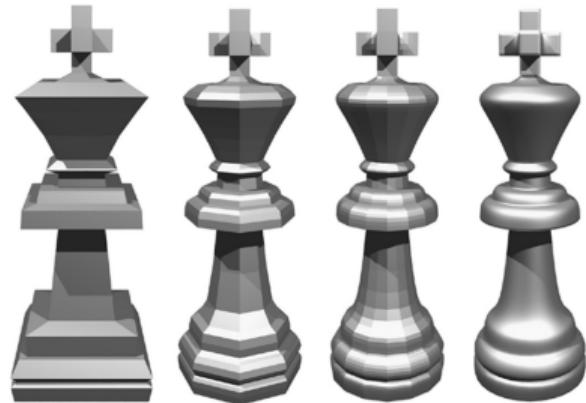
1997

Formas geométricas em mundos virtuais



Pixar

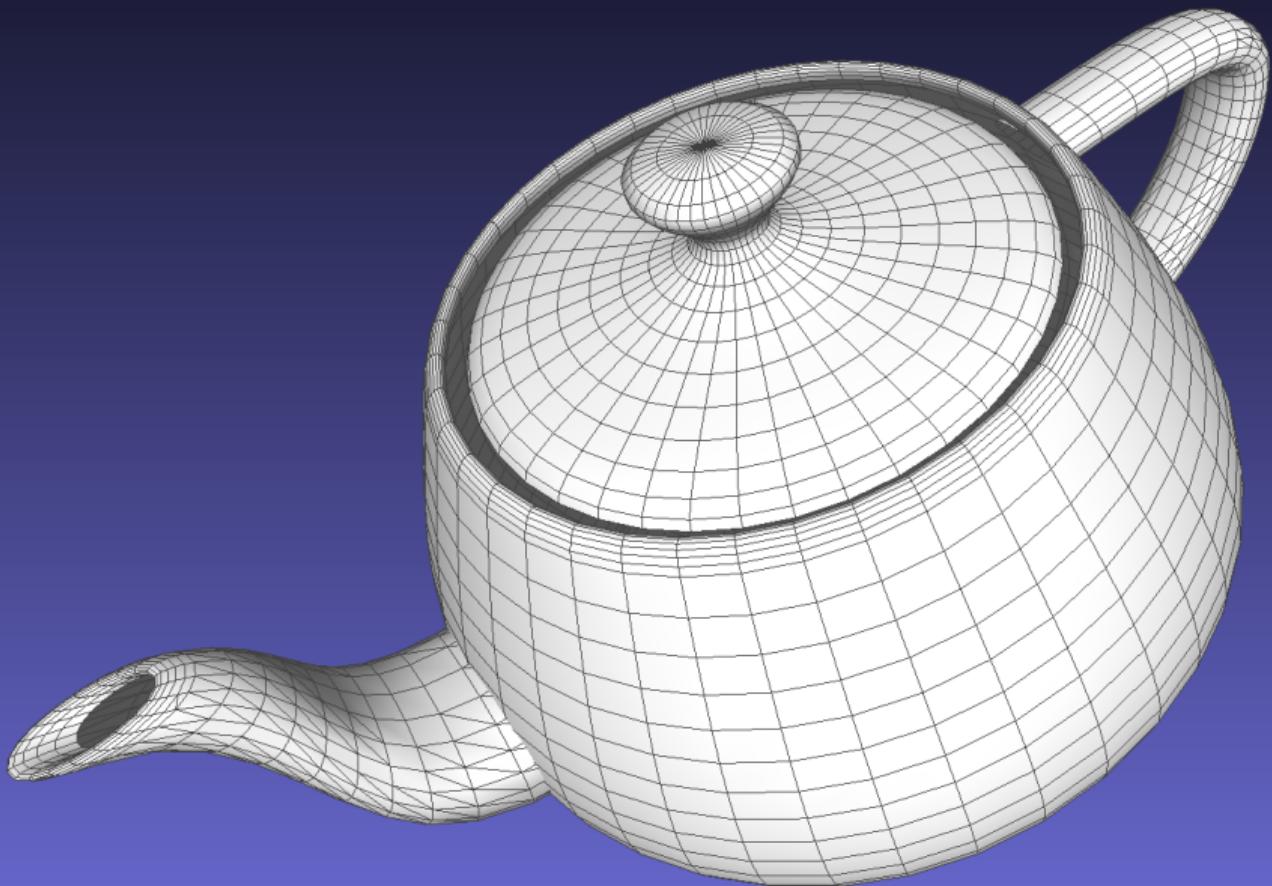
Anatomia das formas virtuais



Pixar



Newell (1975) / Morrical (2019)



Newell (1975) / Morrical (2019)



Gouraud



Gouraud

Anatomia das formas virtuais

a b c 1

Anatomia das formas virtuais

a b c

Anatomia das formas virtuais



fundamentos

Jonas Gomes
Luiz Velho

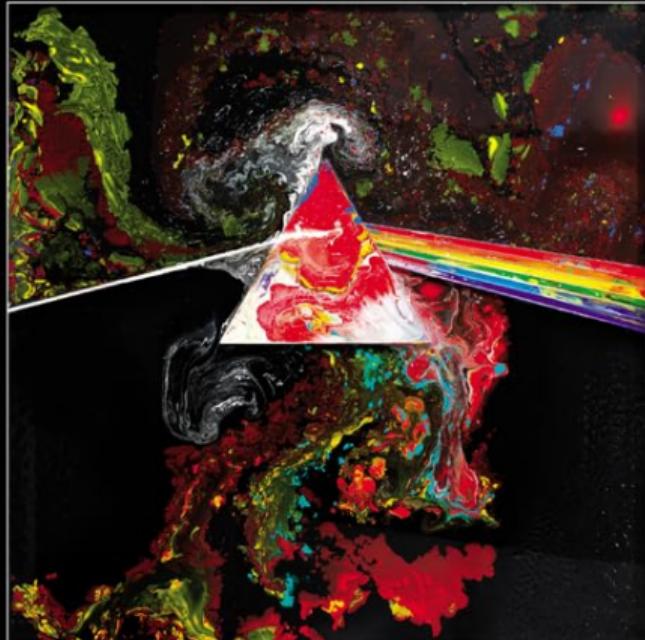
Fundamentos da Computação Gráfica



Instituto Nacional de Matemática Pura e Aplicada - IMPA
Série de Computação e Matemática

2003

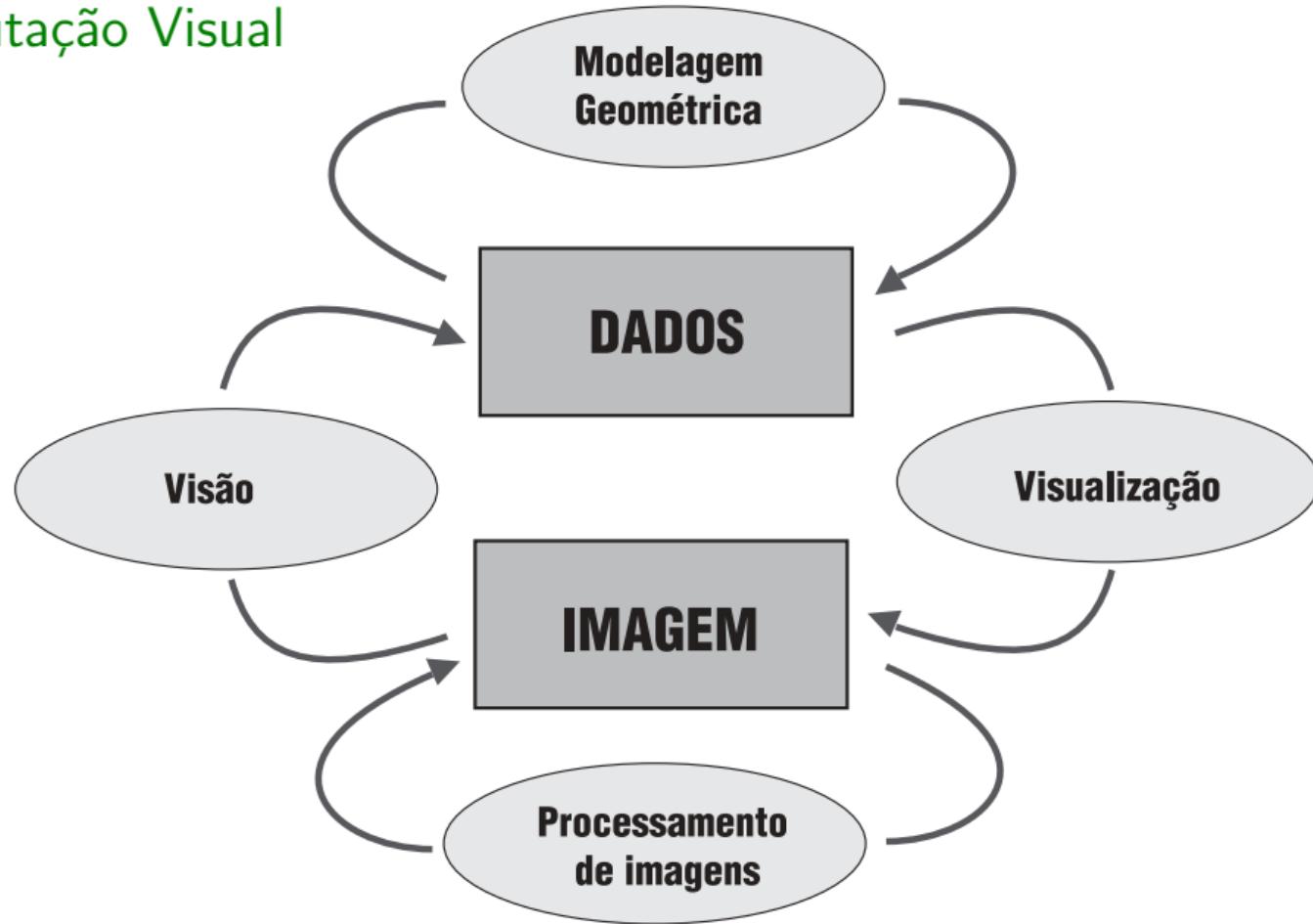
COMPUTER GRAPHICS
THEORY AND PRACTICE

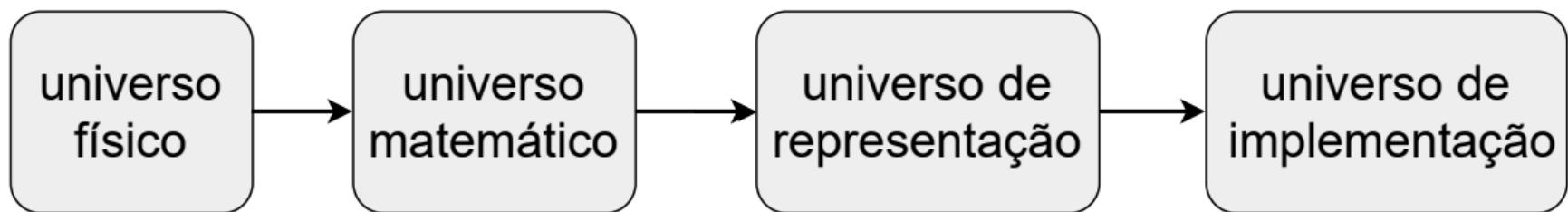


Jonas Gomes • Luiz Velho • Mario Costa Sousa

2012

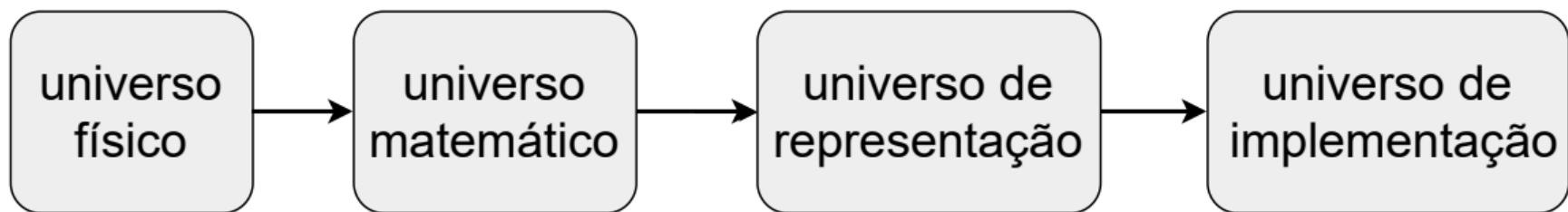
Computação Visual





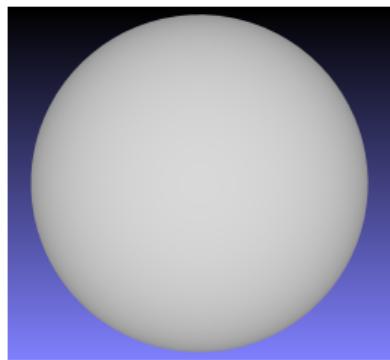
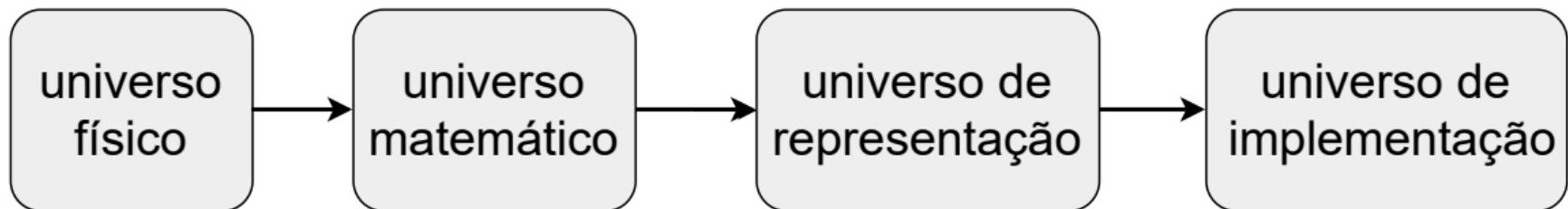
Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)



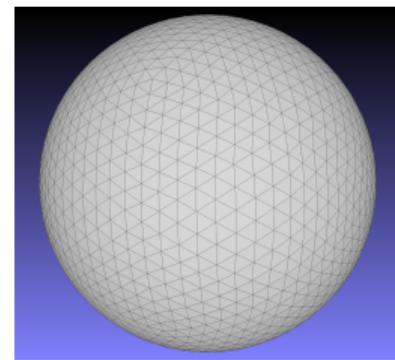
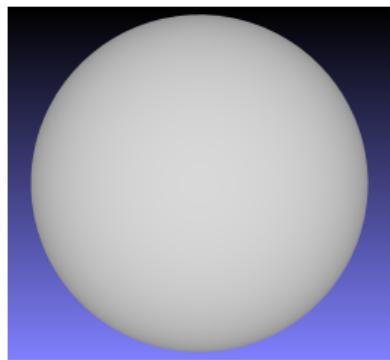
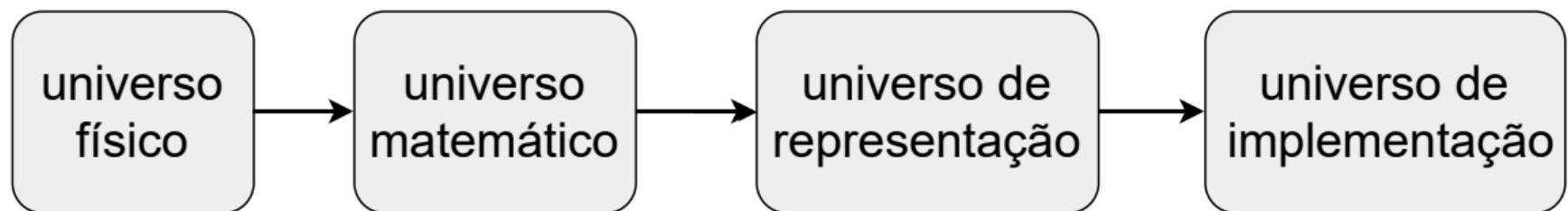
Paradigmas de abstração para computação gráfica

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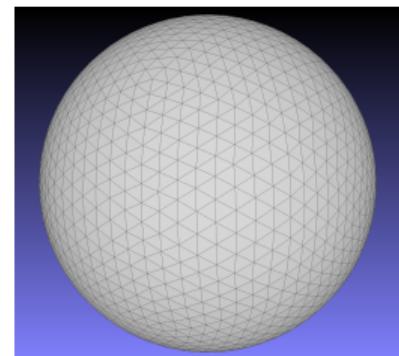
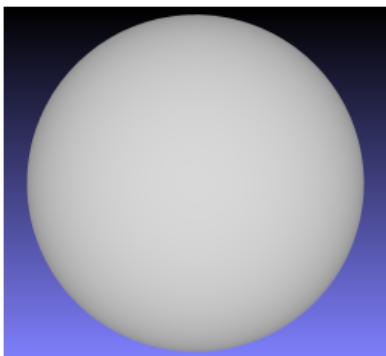
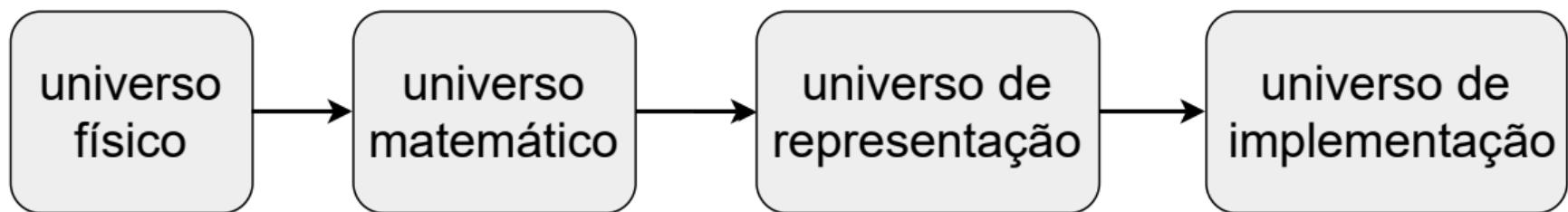
Paradigmas de abstração para computação gráfica

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Paradigmas de abstração para computação gráfica

Gomes–Velho (1995)

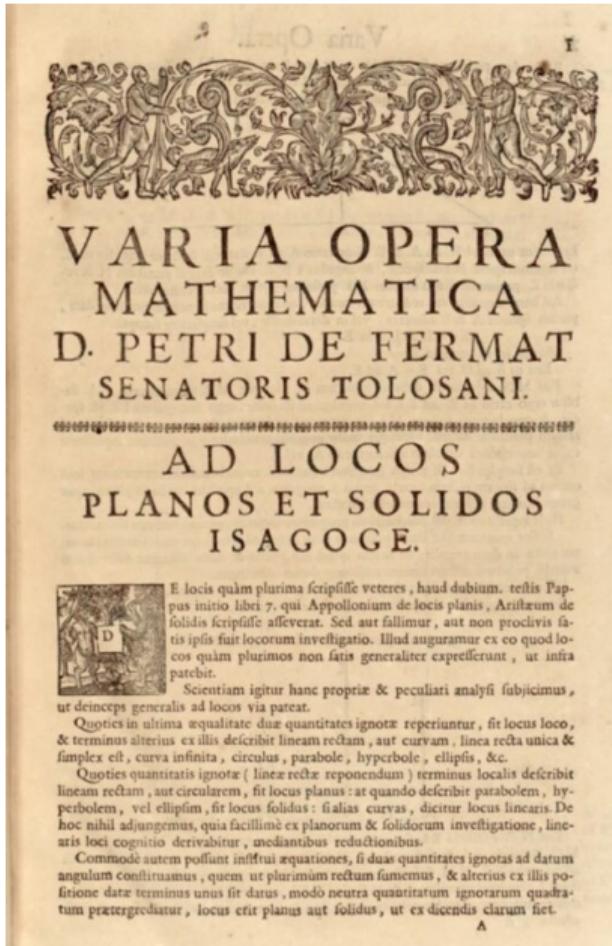


```
v 0.525731 0 -0.850651  
v 0.525731 0 0.850651  
v -0.525731 0 0.850651  
v -0.850651 -0.525731 0  
v -0.850651 0.525731 0  
v 0.850651 0.525731 0  
v 0.850651 -0.525731 0  
v 0 -0.850651 0.525731  
v 0 -0.850651 -0.525731  
v 0 0.850651 -0.525731  
...  
f 23 24 25  
f 25 24 20  
f 25 21 23  
f 22 24 23  
f 37 45 23  
f 17 45 37  
f 23 21 37  
f 23 45 22  
f 31 24 47  
f 28 24 31
```

universo matemático

Geometria analítica

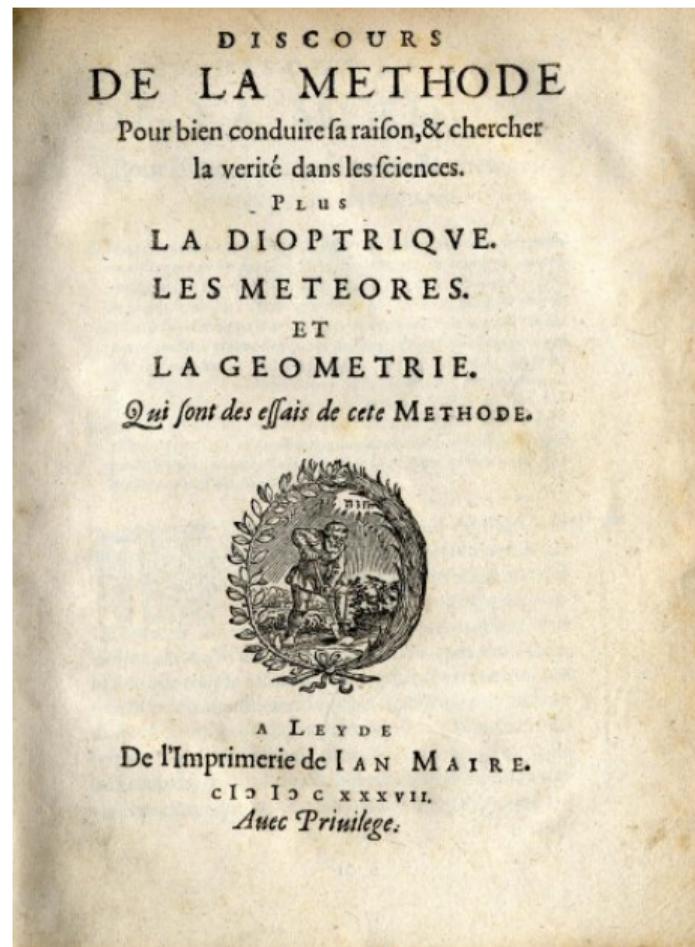
Fermat (1636, 1679)



Geometria analítica

Fermat (1636, 1679)

Descartes (1637)



Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

L A G E O M E T R I E. LIVRE PREMIER.

Des problemes qu'on peut construire sans y employer que des cercles & des lignes droites.



O u s les Problemes de Geometrie se peuvent facilement reduire a tels termes, qu'il n'est besoin par apres que de connoître la longeur de quelques lignes droites, pour les construire.

Et comme toute l'Arithmetique n'est composée, que de quatre ou cinq operations, qui sont l'Addition, la Soustraction, la Multiplication, la Division, & l'Extraction des racines, qu'on peut prendre pour vne espece de Division : Ainsi n'at'on autre chose a faire en Geometrie touchant les lignes qu'on cherche, pour les preparer a estre connues, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnite pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'autre est a l'vnite, ce qui est le mesme que la Multiplication, ou bien en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'vnite

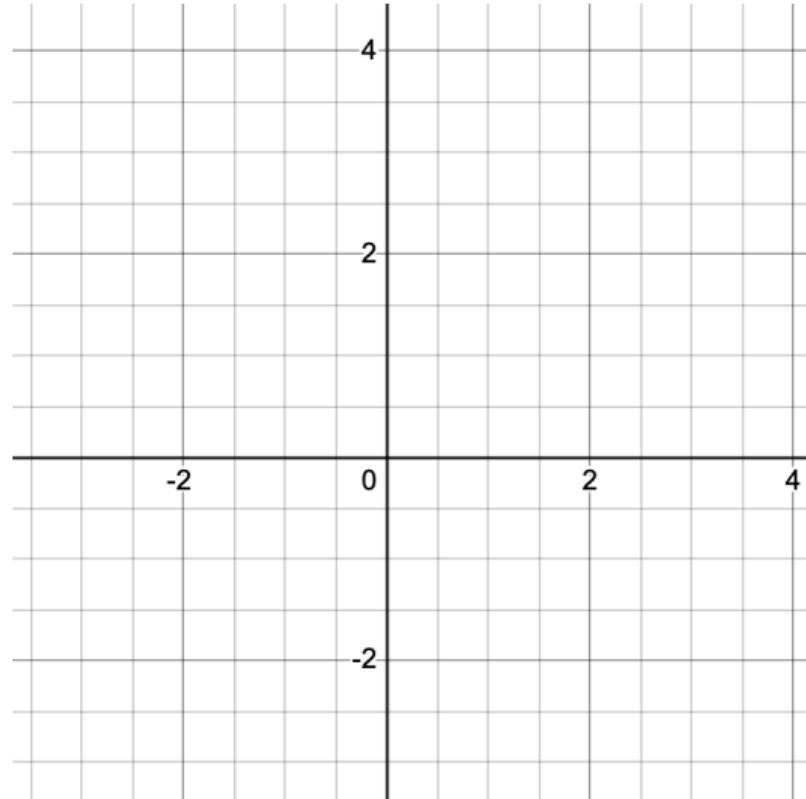
Commencé le calcul d'Addition, la d'At-
Soustraction, la Multiplication, la Division, & l'Extra-
ction des racines, que le rapporte aux opérations de
metrie : Ainsi n'at'on autre chose à faire en Geo-
metrie touchant les lignes qu'on cherche, pour les pre-
parer à estre connues, que leur en adiouster d'autres, ou en oster, Oubien en ayant vne, que ie nommeray l'vnite pour la rapporter d'autant mieux aux nombres, & qui peut ordinairement estre prise a discretion, puis en ayant encore deux autres, en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'autre est a l'vnite, ce qui est le mesme que la Multiplication, ou bien en trouuer vne quatriesme, qui soit a l'vnue de ces deux, comme l'vnite

Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas	álgebra	+	geometria
fórmulas	↔	formas	
finito	↔	infinito	

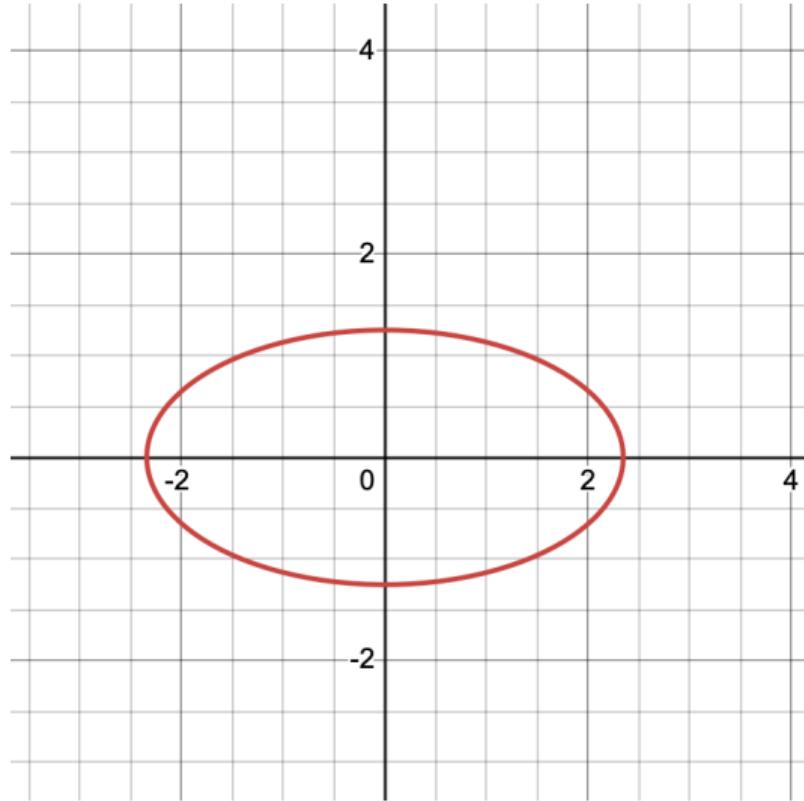


Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas álgebra + geometria
fórmulas \longleftrightarrow formas
finito \longleftrightarrow infinito



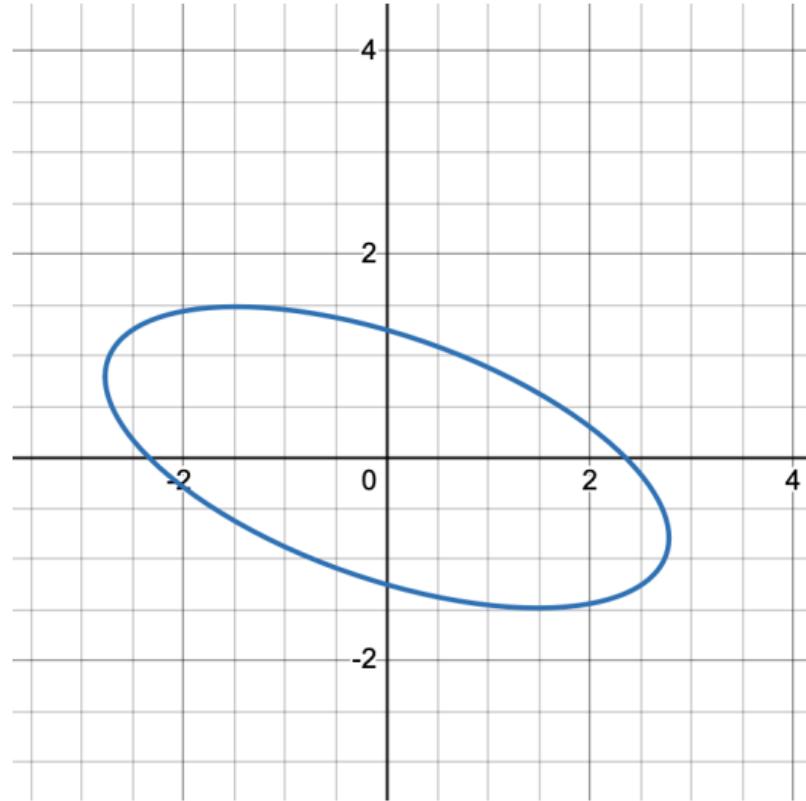
$$2x^2 + 7y^2 = 11$$

Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas álgebra + geometria
fórmulas \longleftrightarrow formas
finito \longleftrightarrow infinito



$$2x^2 + 4xy + 7y^2 = 11$$

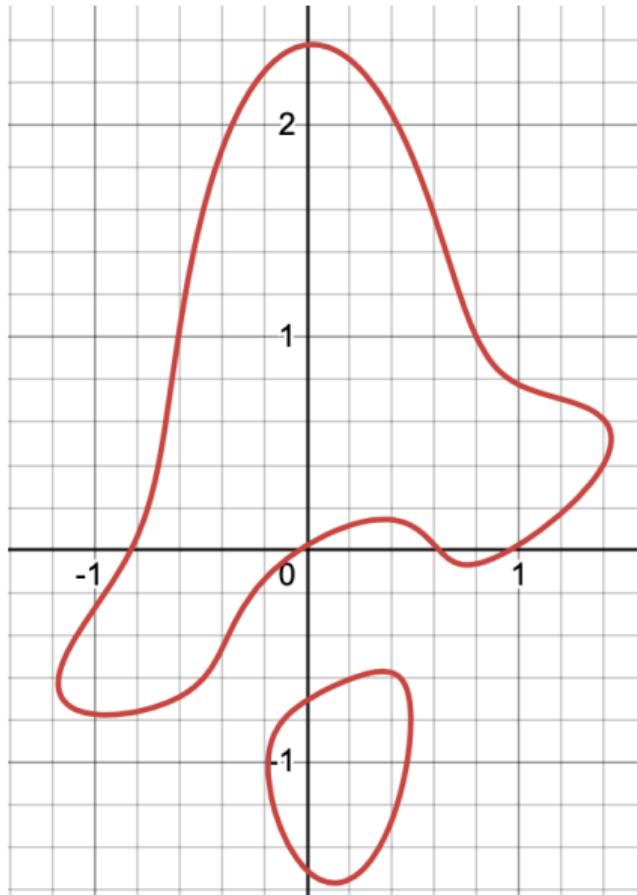
Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas álgebra + geometria
fórmulas \longleftrightarrow formas
finito \longleftrightarrow infinito

$$\begin{aligned} & 0.004 + 0.110x - 0.177y - 0.174x^2 + 0.224xy - 0.303y^2 \\ & - 0.168x^3 + 0.327x^2y - 0.087xy^2 - 0.013y^3 + 0.235x^4 \\ & - 0.667x^3y + 0.745x^2y^2 - 0.029xy^3 + 0.072y^4 = 0 \end{aligned}$$

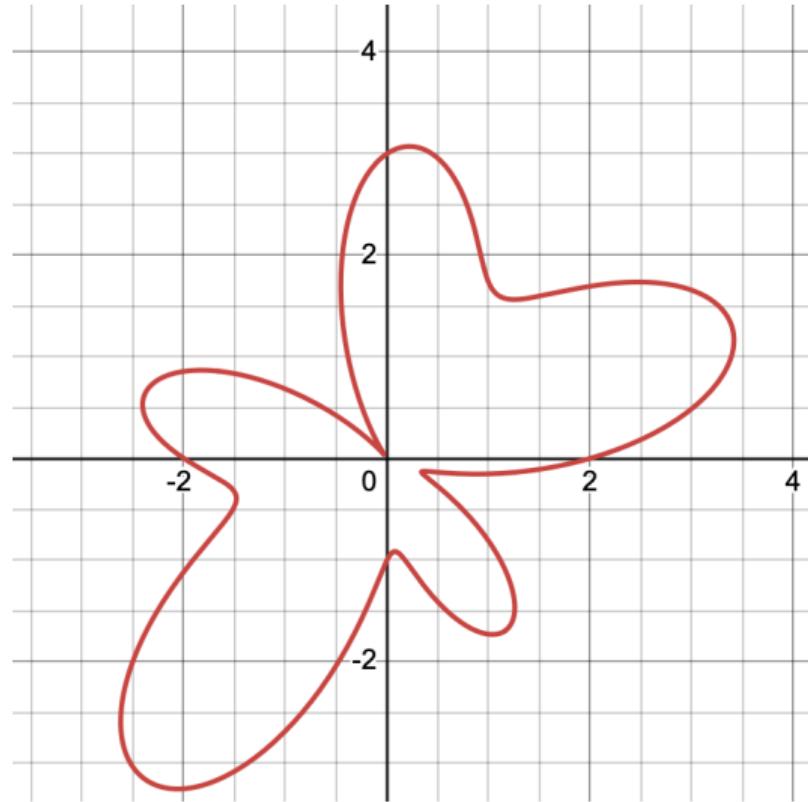


Geometria analítica

Fermat (1636, 1679)

Descartes (1637)

coordenadas	álgebra	+	geometria
fórmulas	↔	formas	
finito	↔	infinito	

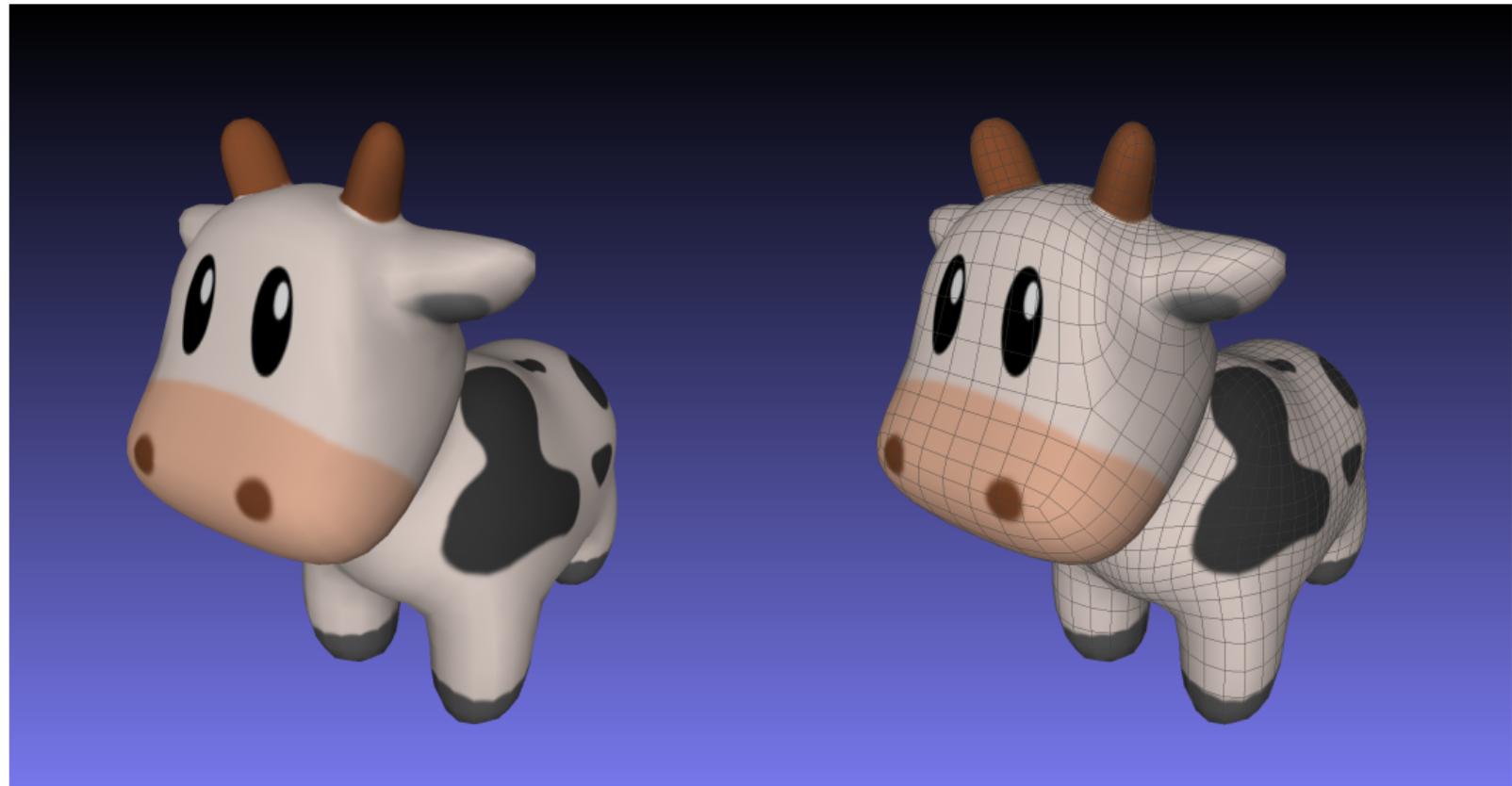


$$r = \sin(2\theta) + \sin(5\theta) + 2$$
$$x = r \cos(\theta)$$
$$y = r \sin(\theta)$$

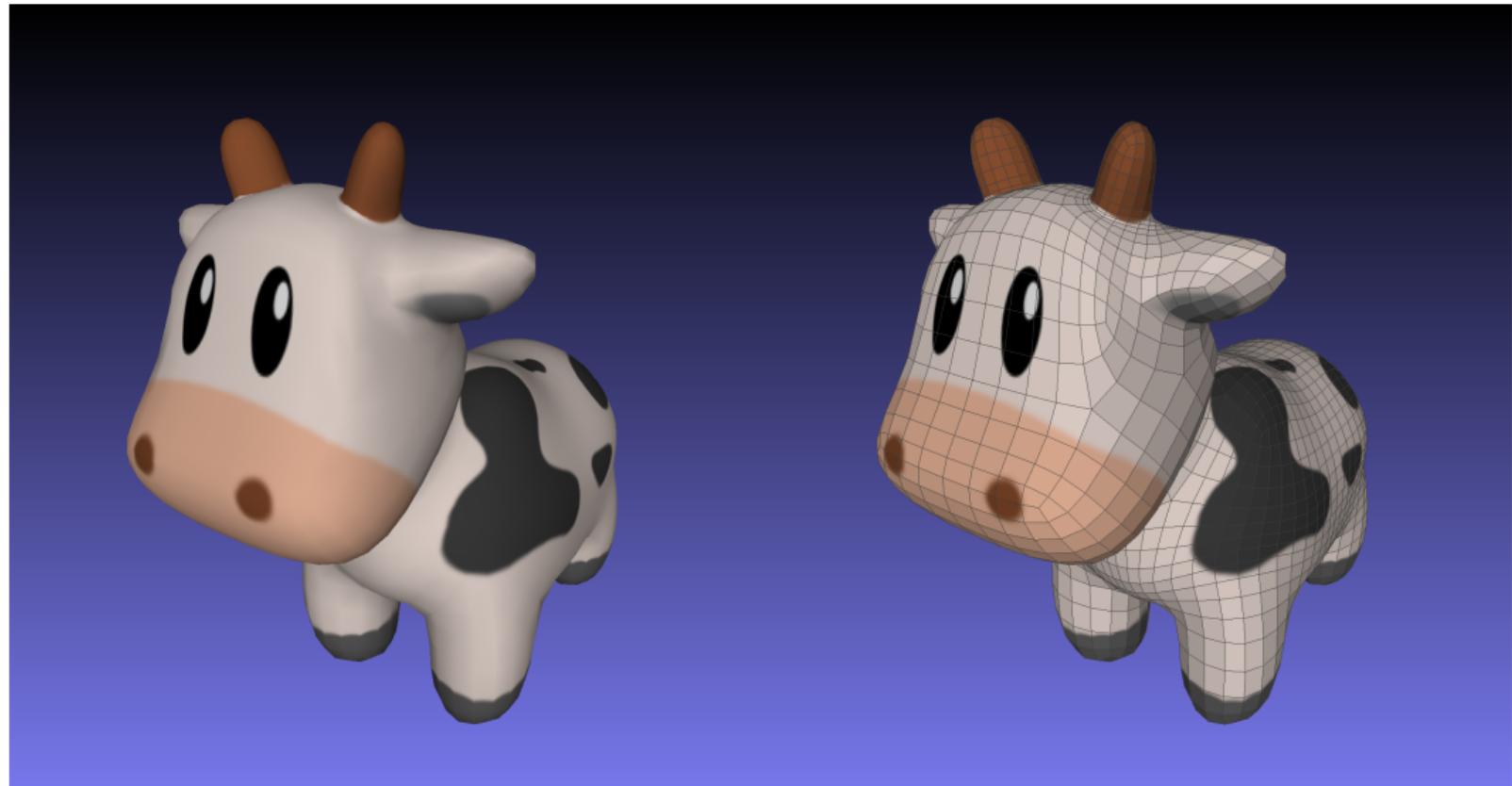
Geometria analítica



Geometria analítica



Geometria analítica



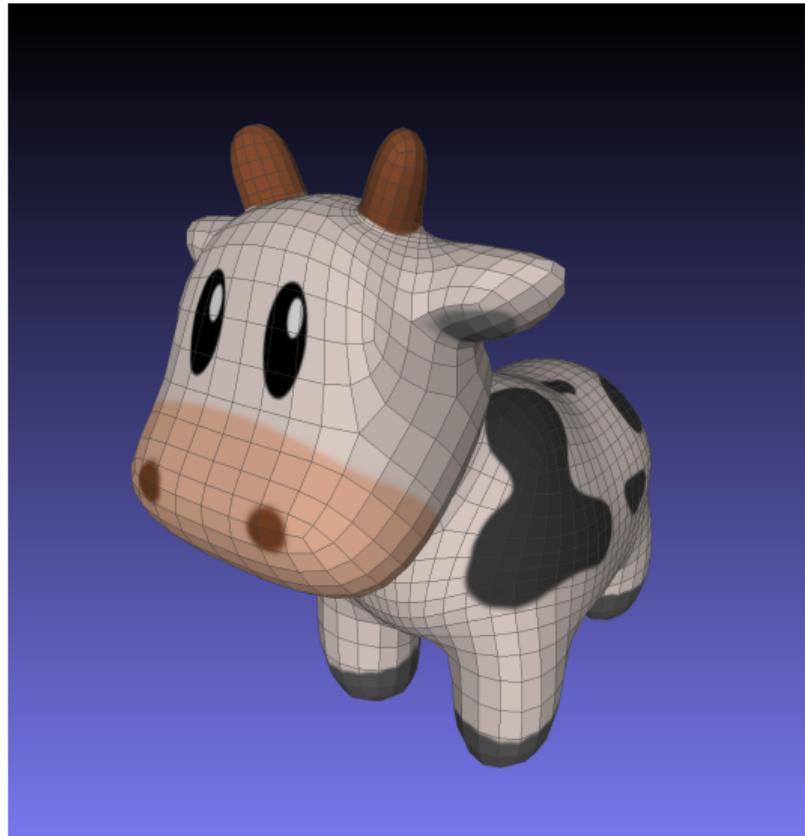
Geometria analítica

```
v 0.348799 -0.334989 -0.0832331
v 0.313132 -0.399051 0.881192
v 0.266758 0.181628 0.122726
v 0.229555 0.0663178 0.828702
v 0.353623 -0.0486456 0.443969
v 0.335878 -0.384495 0.425693
v 0.305362 0.0307983 -0.00655663
v 0.279611 -0.0552387 0.858547
v 0.266719 0.10578 0.46681
v 0.149341 -0.451522 0.166423
v 0.12606 -0.162036 -0.185668
v 0.114009 -0.358339 -0.0759521
v 0.101546 -0.475382 0.663157
v 0.240757 -0.435635 0.421981
v 0.385825 -0.174501 0.180766
```



Geometria analítica

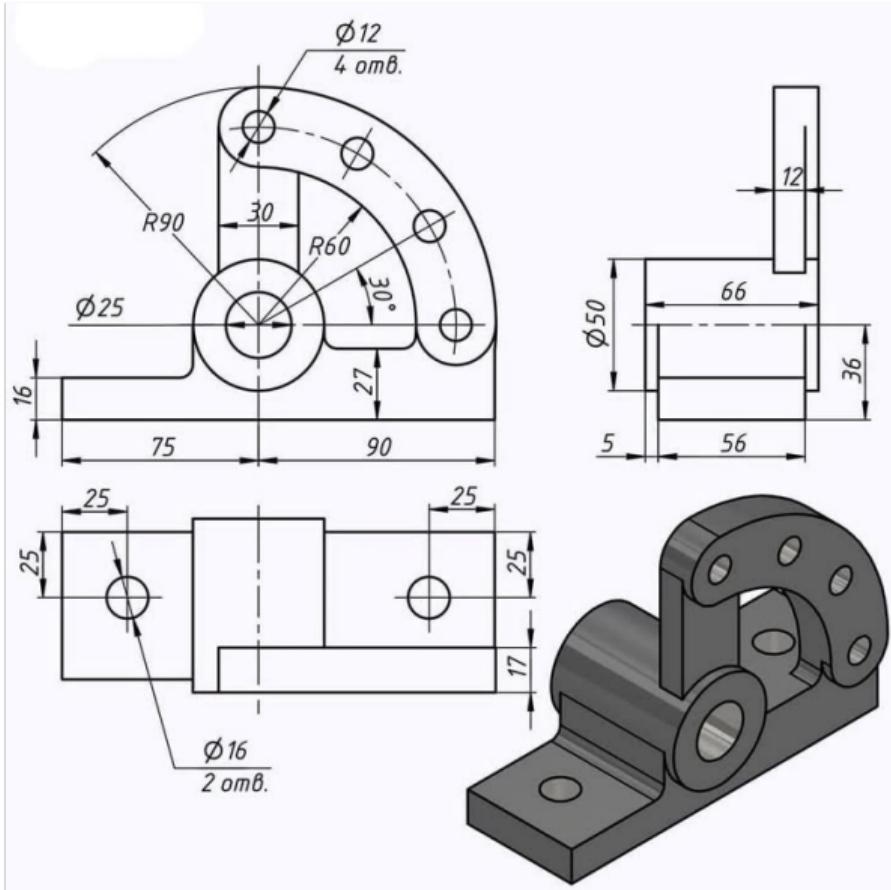
```
f 6 735 739 738
f 189 736 739 735
f 193 737 739 736
f 192 738 739 737
f 14 740 743 742
f 190 741 743 740
f 193 736 743 741
f 189 742 743 736
f 10 744 747 746
f 191 745 747 744
f 193 741 747 745
f 190 746 747 741
f 16 748 750 749
f 192 737 750 748
f 193 745 750 737
```



métodos de modelagem

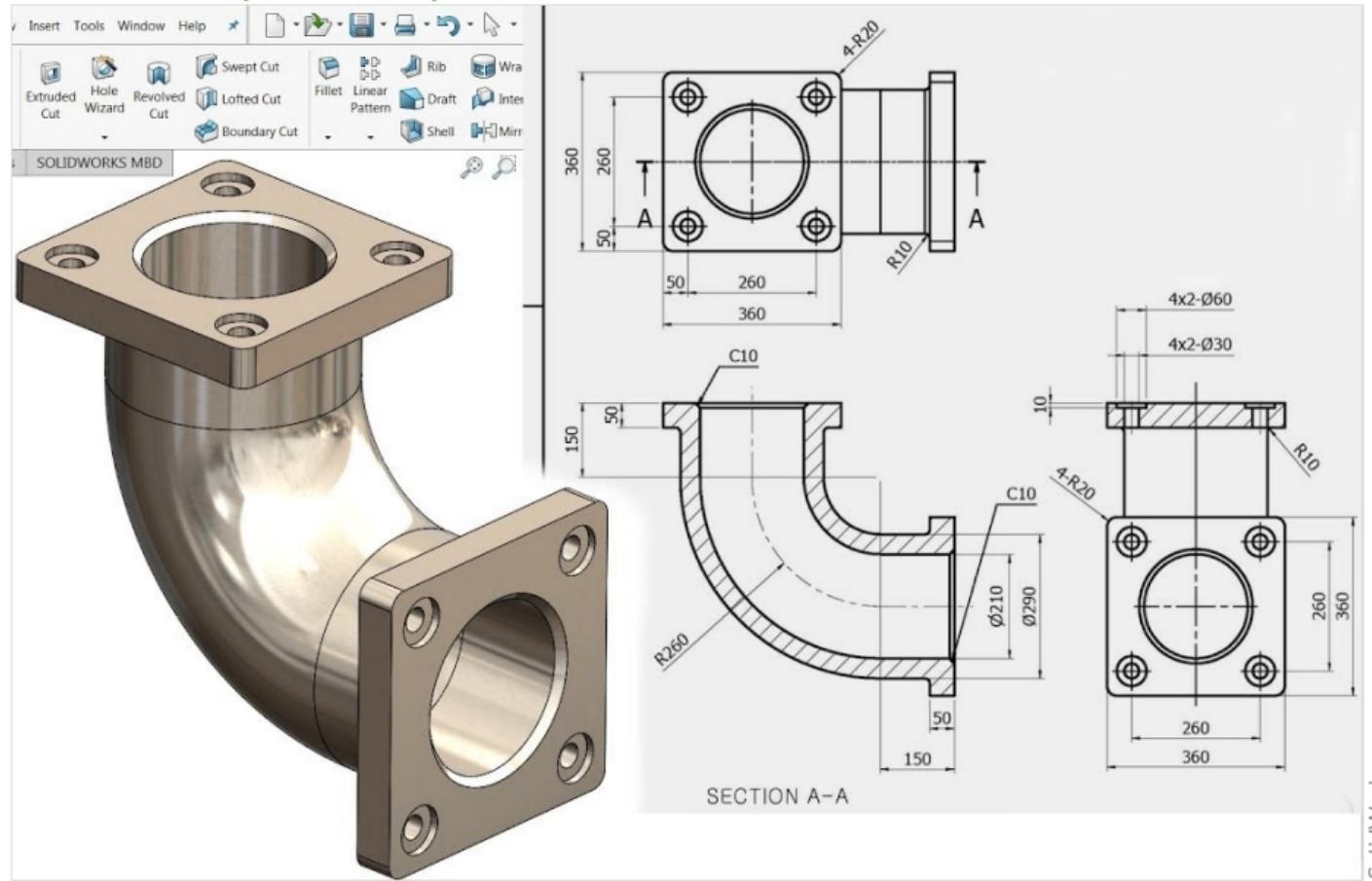
Projeto assistido por computador

CAD



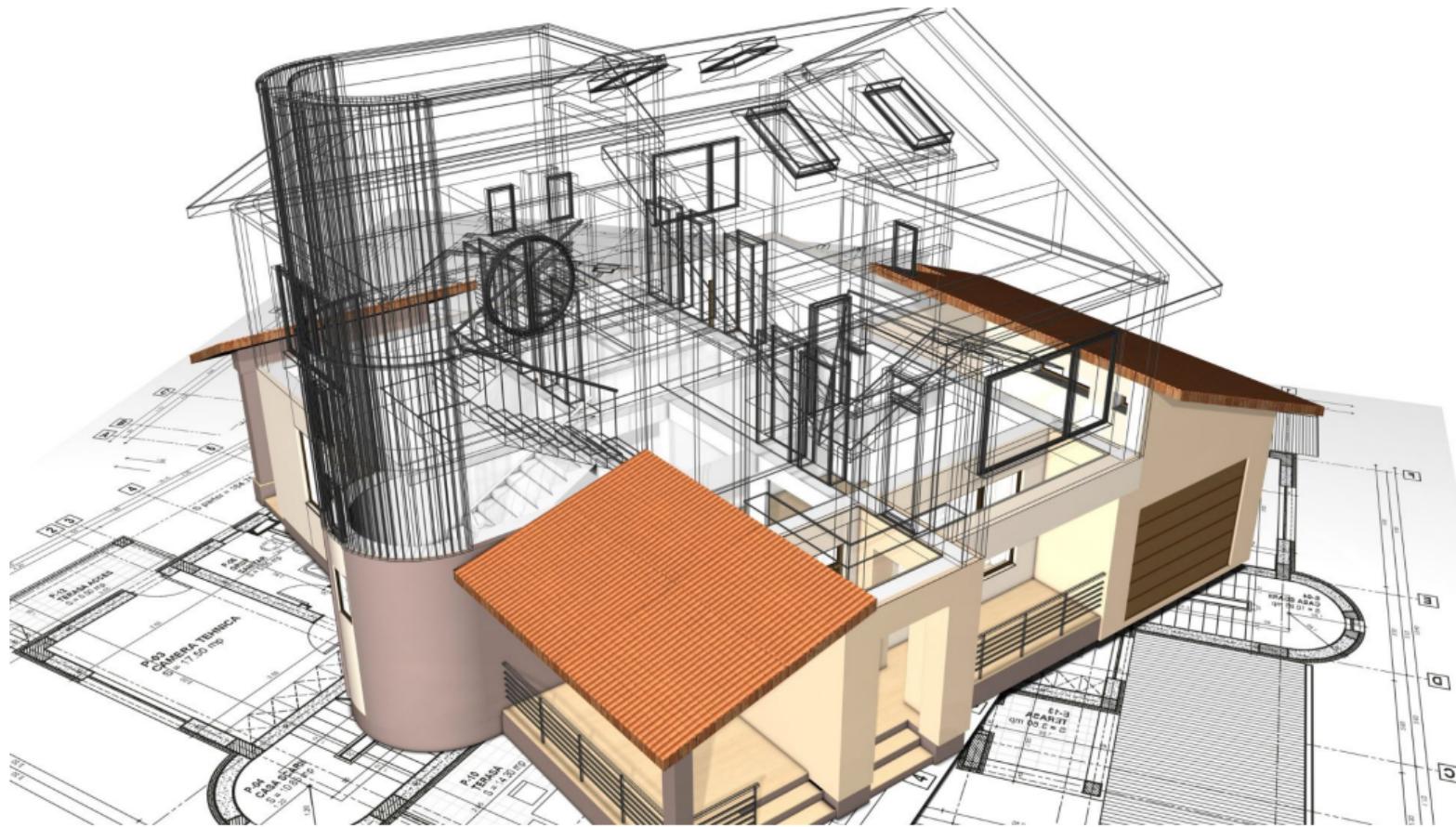
Projeto assistido por computador

CAD



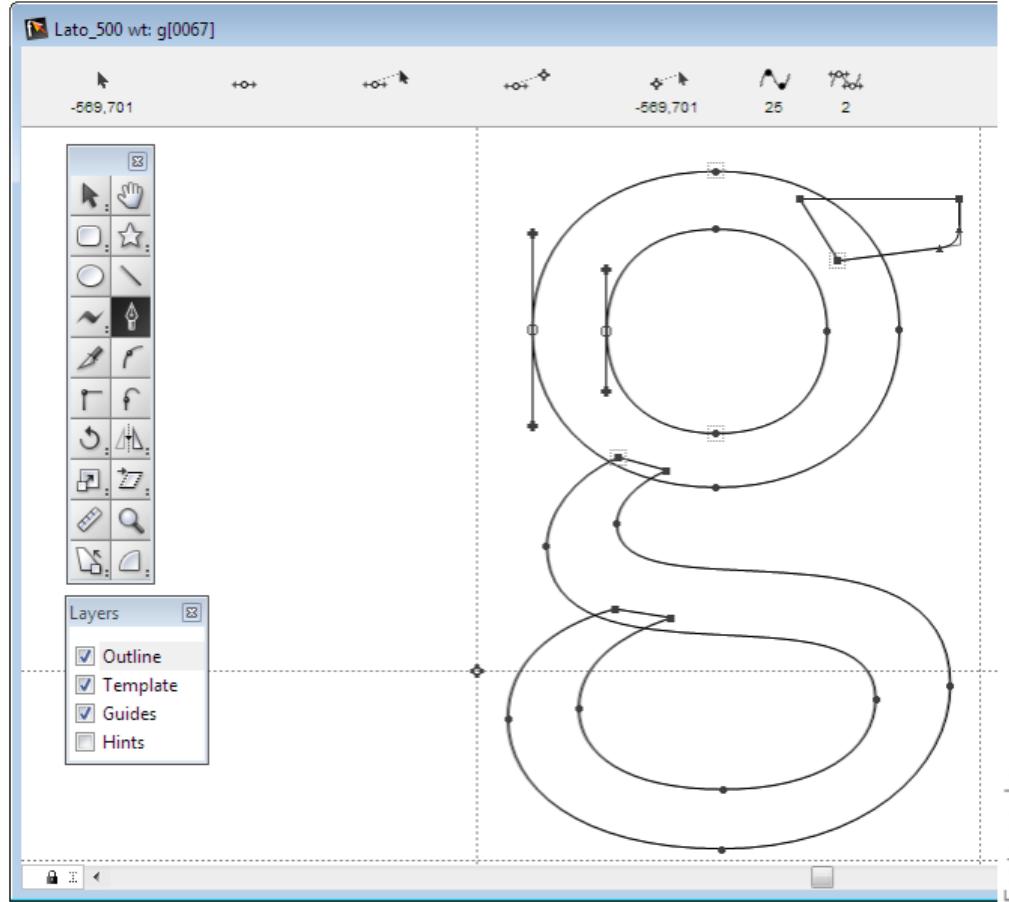
Projeto assistido por computador

CAD



Projeto assistido por computador

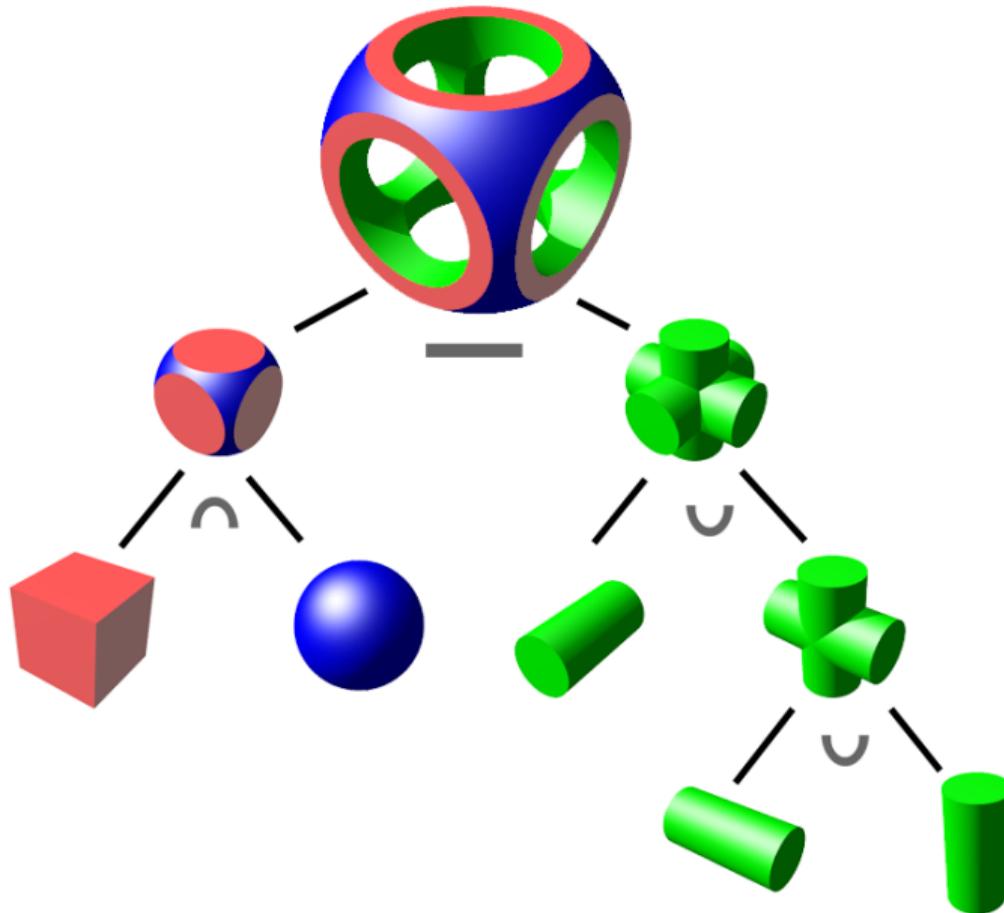
CAD



Fontographer

Geometria sólida construtiva

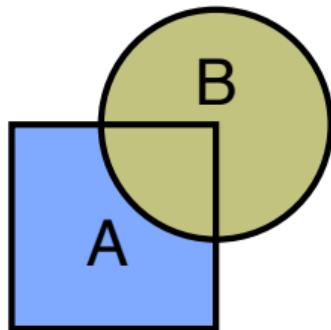
CSG



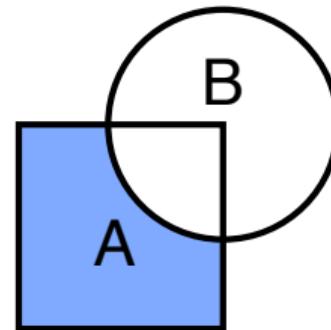
Wikipedia

Operações booleanas

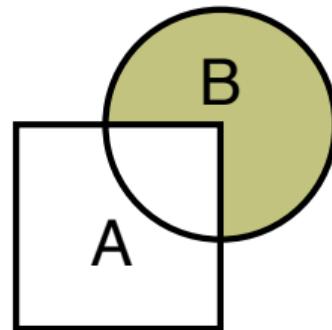
A e B



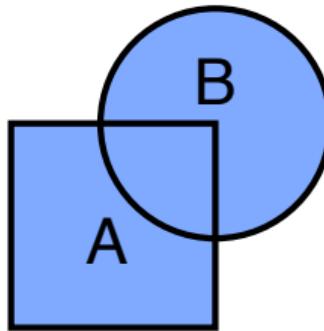
A menos B



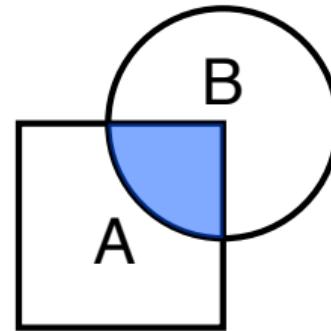
B menos A



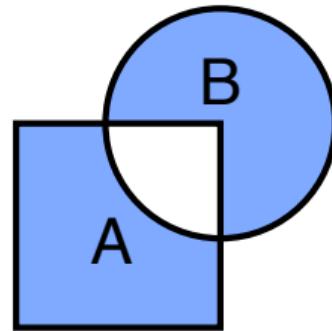
A união B



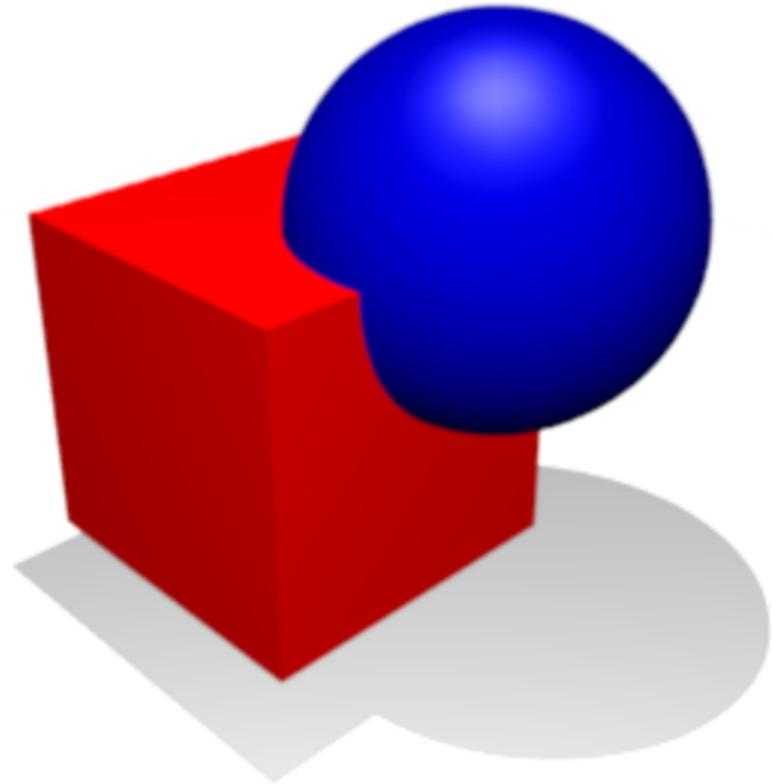
A interseção B



A diff B

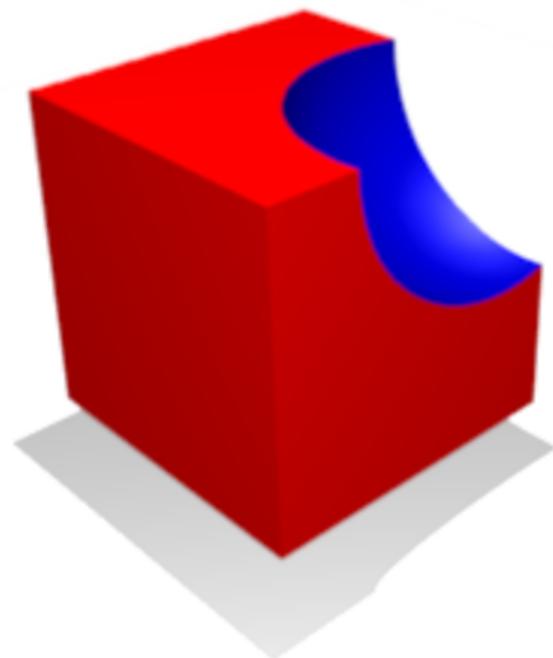


Operações booleanas



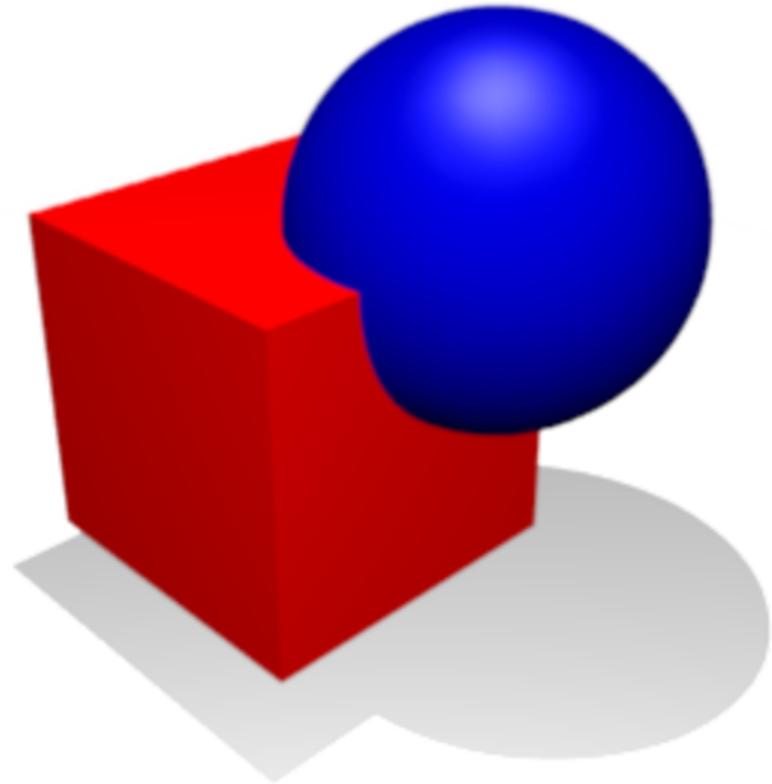
Wikipedia

Operações booleanas



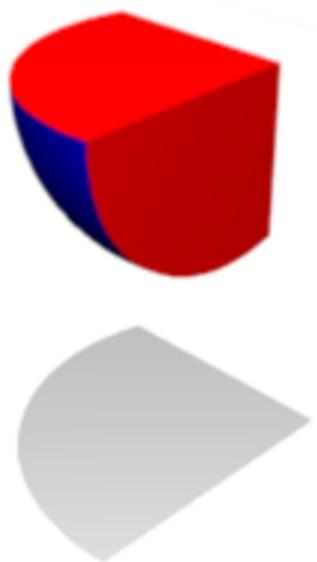
Wikipedia

Operações booleanas



Wikipedia

Operações booleanas



Wikipedia

Operações booleanas

1
d

Operações booleanas



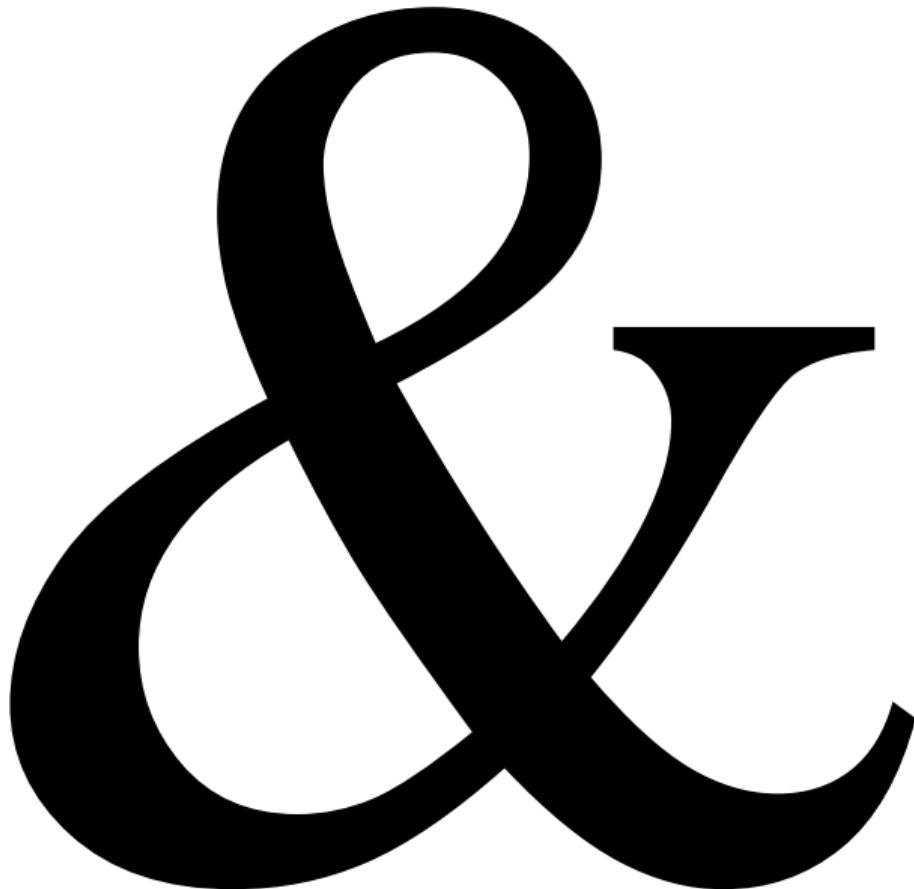
Operações booleanas

d

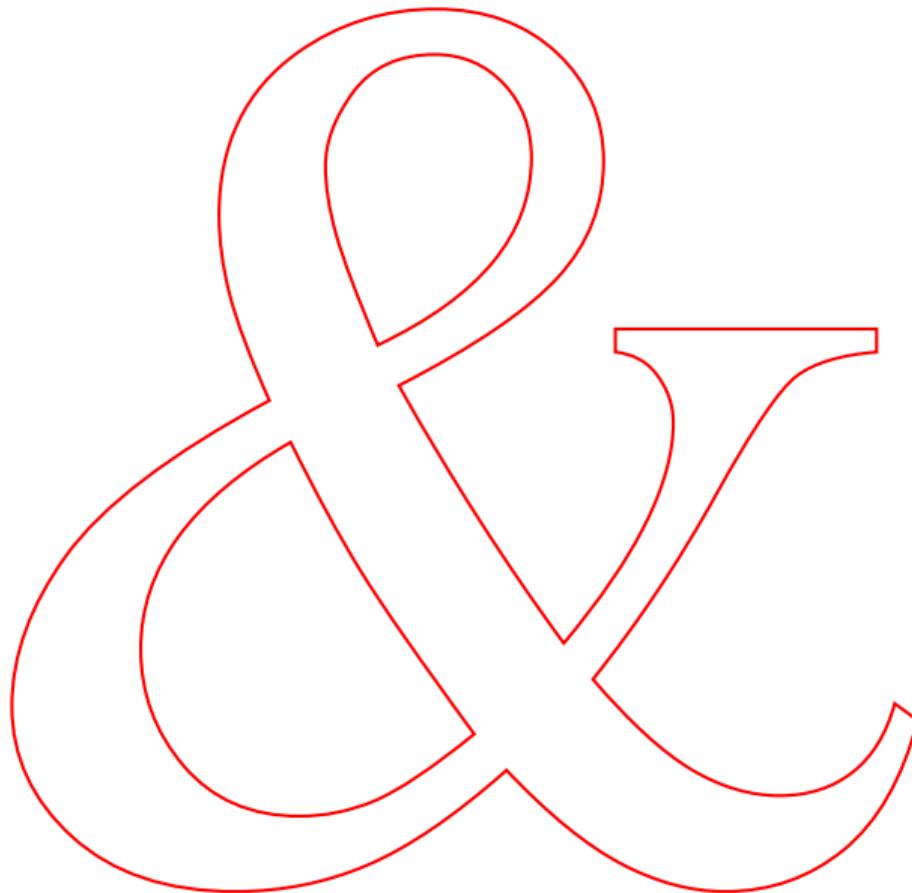
Operações booleanas



Representação por fronteira

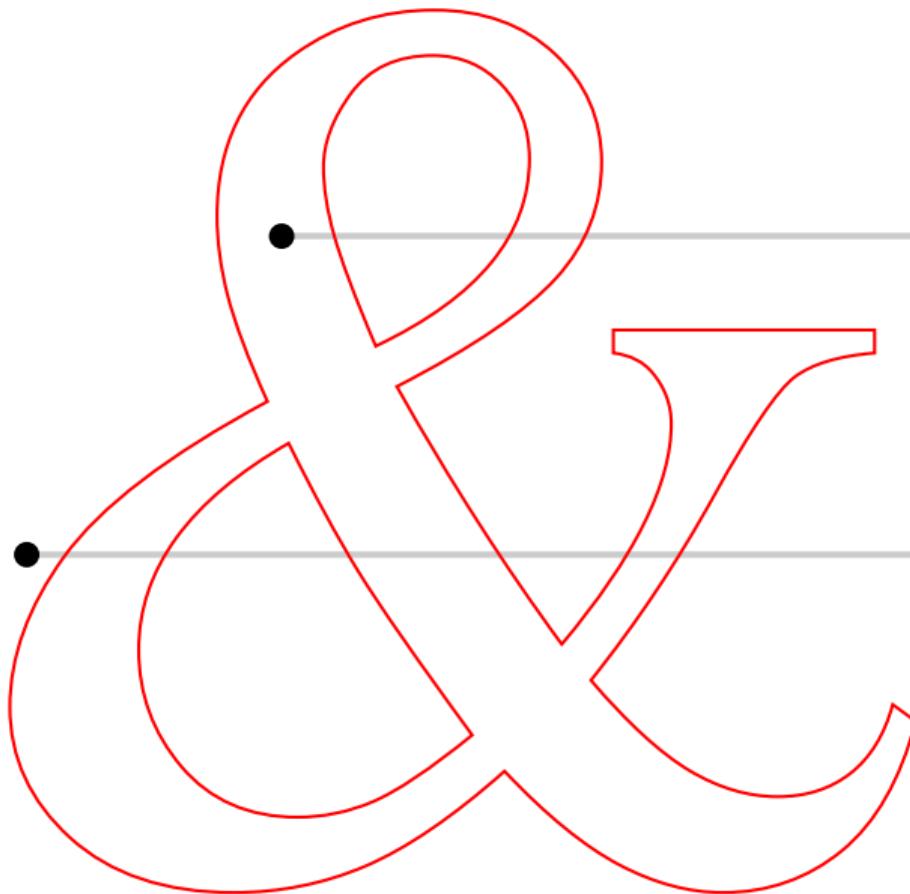


Representação por fronteira



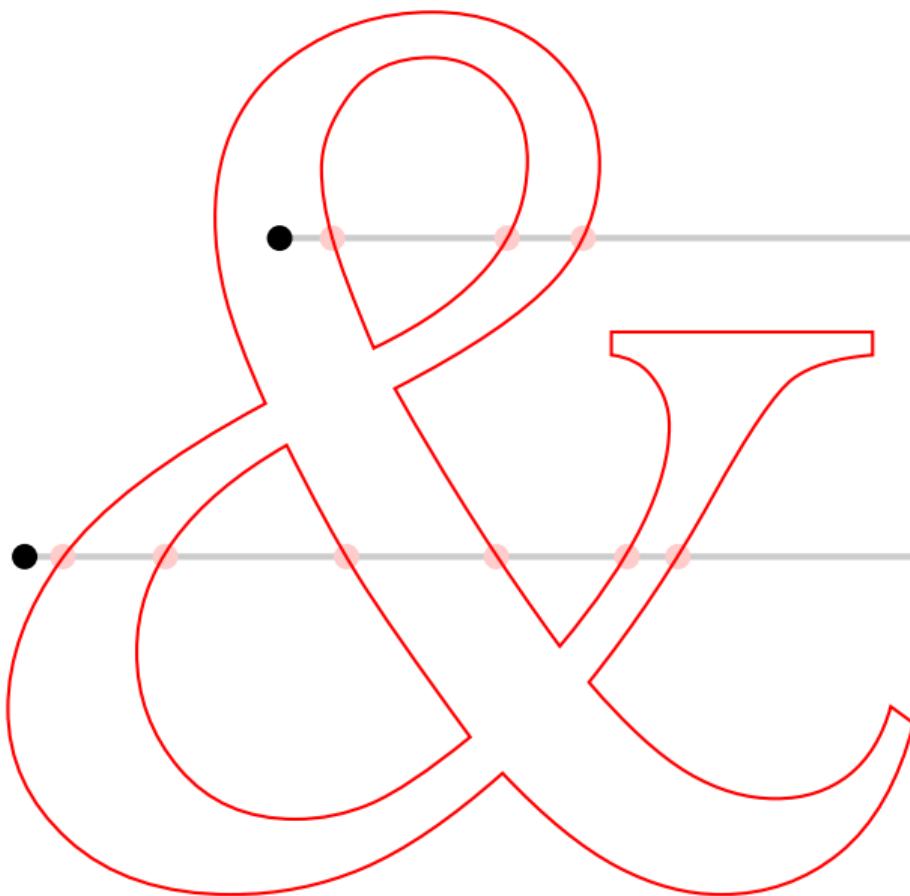
Representação por fronteira

Jordan (1887)

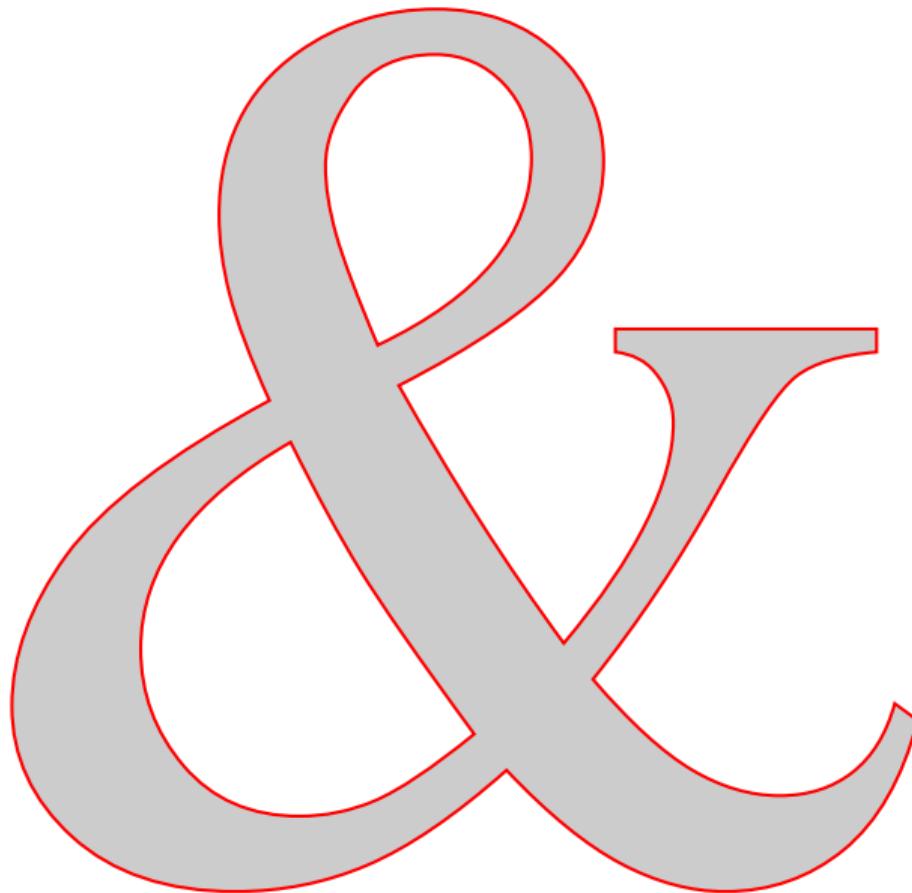


Representação por fronteira

Jordan (1887)



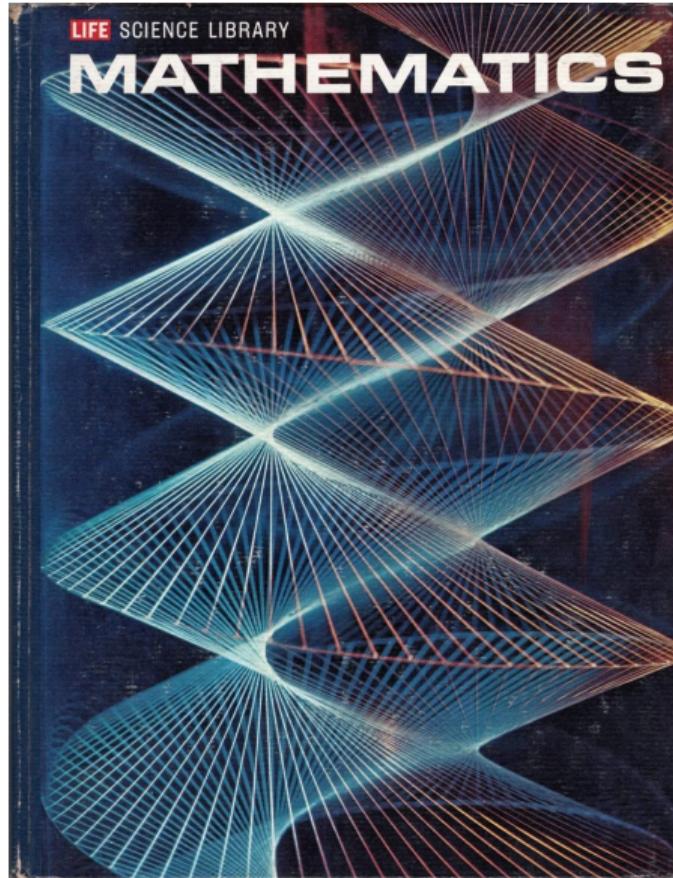
Representação por fronteira



Representação por fronteira

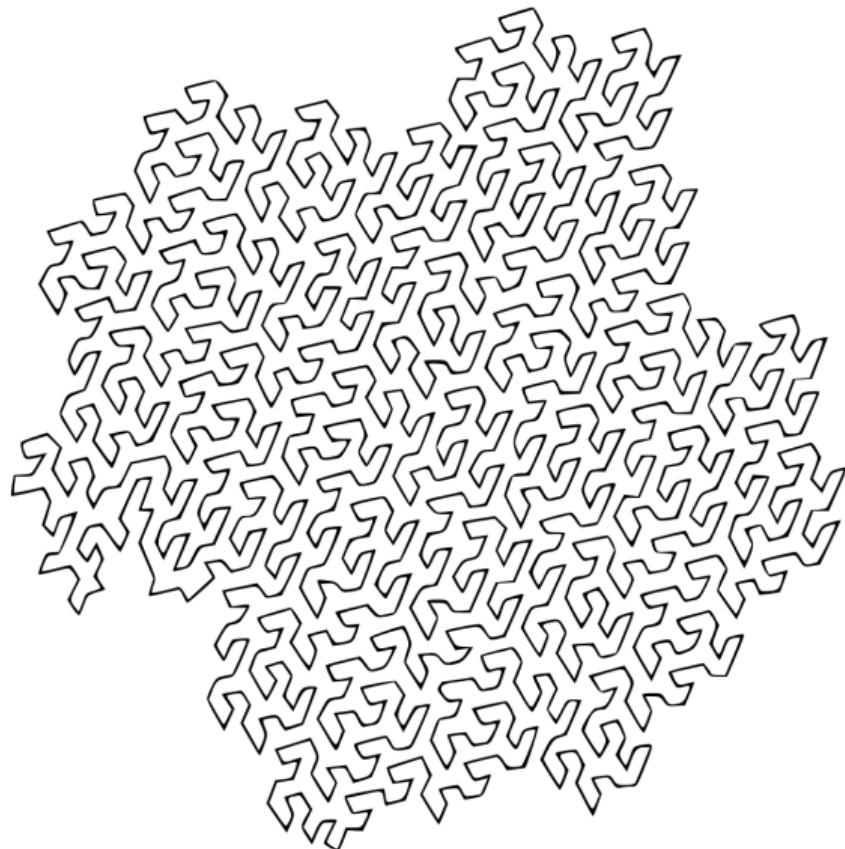


Representação por fronteira

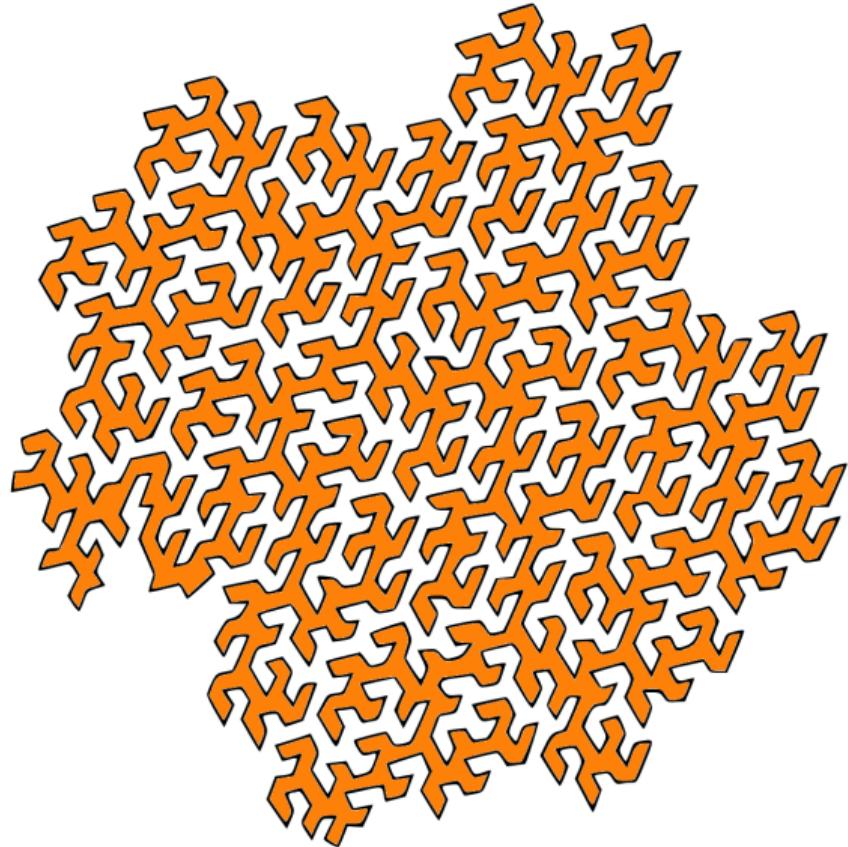


Life Science Library (1963)

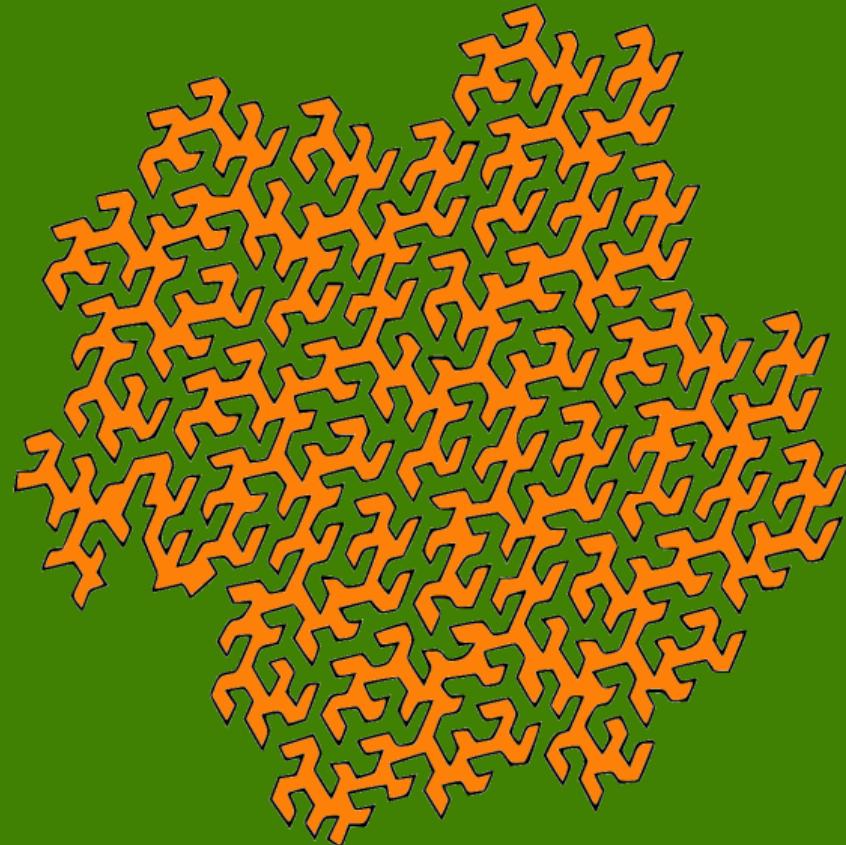
Representação por fronteira



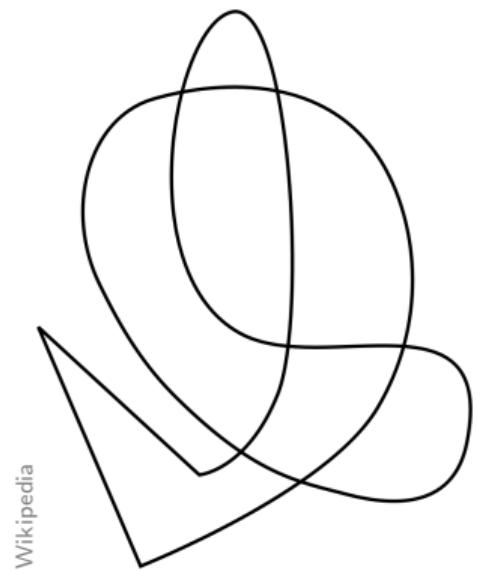
Representação por fronteira



Representação por fronteira



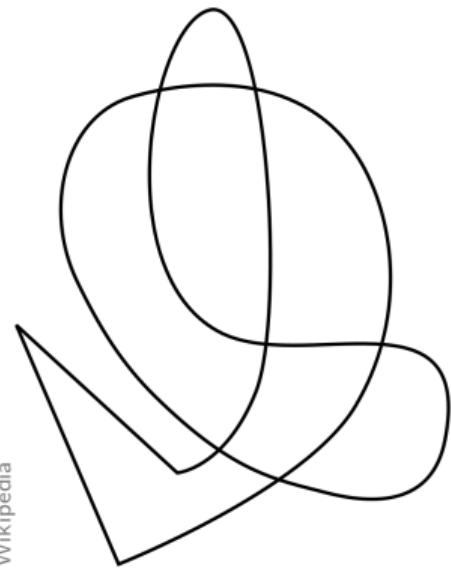
Representação por fronteira



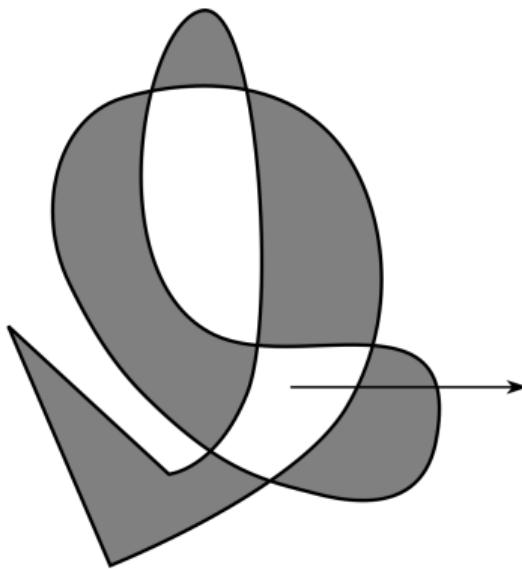
Wikipedia

Representação por fronteira

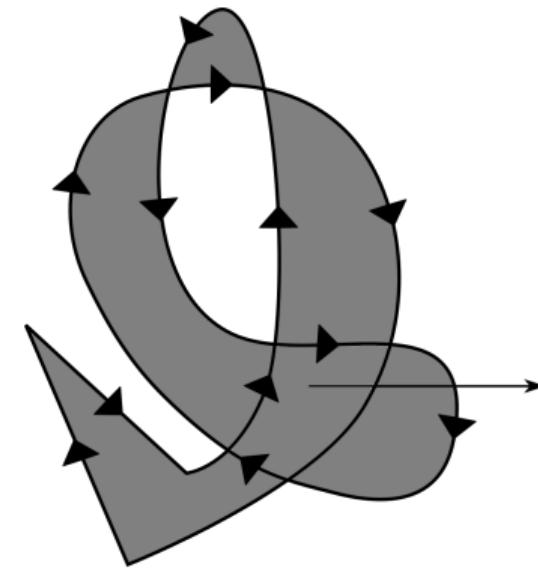
Wikipedia



paridade



número de rotação



Representação por fronteira



Three stylized lowercase letters 'd' in red, each with a red outline and a light red fill. The first letter has a vertical stroke extending upwards.

Representação por fronteira

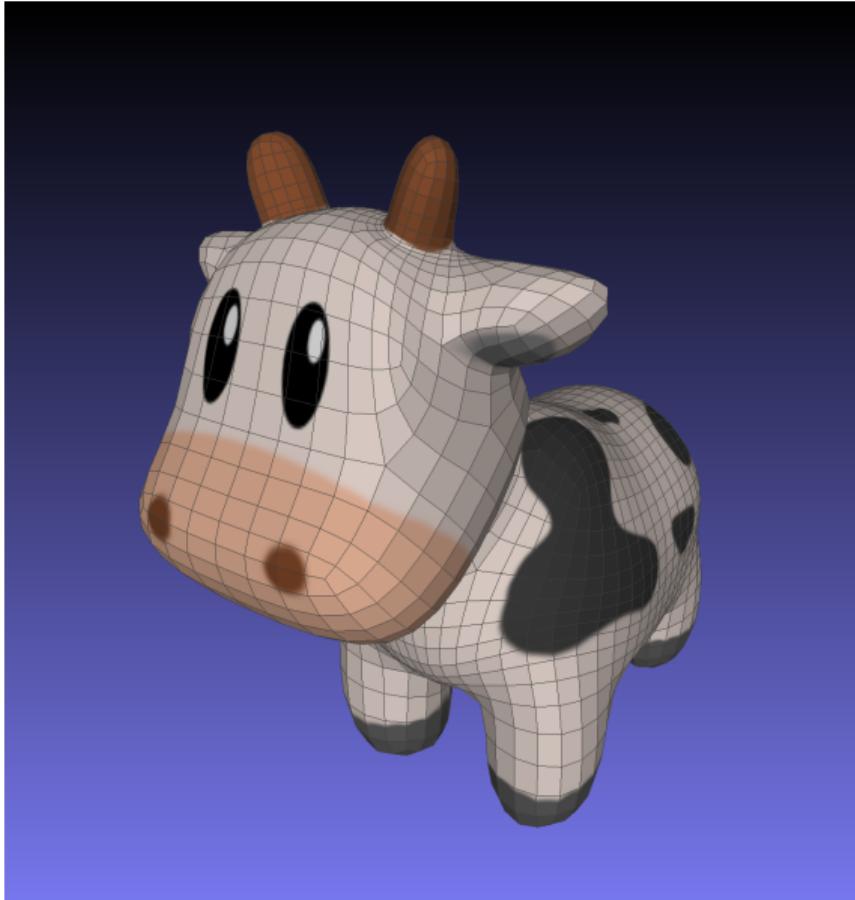
3D



Crane

Representação por fronteira

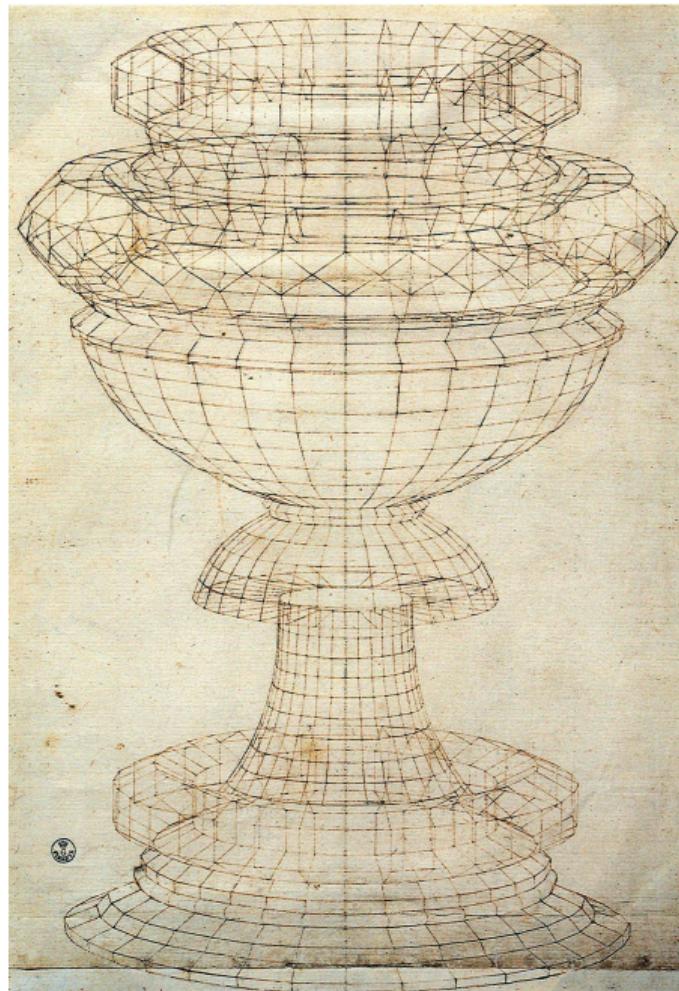
3D



Crane

decomposição

Uccello (1450)



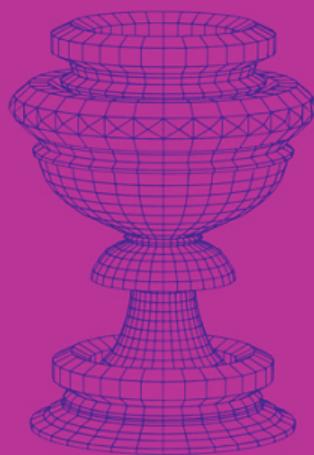


VOLUME 121, SEPTEMBER 2025

ISSN 0167-8396

Uccello (1450)

COMPUTER AIDED GEOMETRIC DESIGN



Métodos de decomposição

- decomposição celular
- decomposição intrínseca
- decomposição extrínseca

Métodos de decomposição

- decomposição celular = discretização
amostragem do objeto
estruturação em células
- decomposição intrínseca
- decomposição extrínseca

Métodos de decomposição

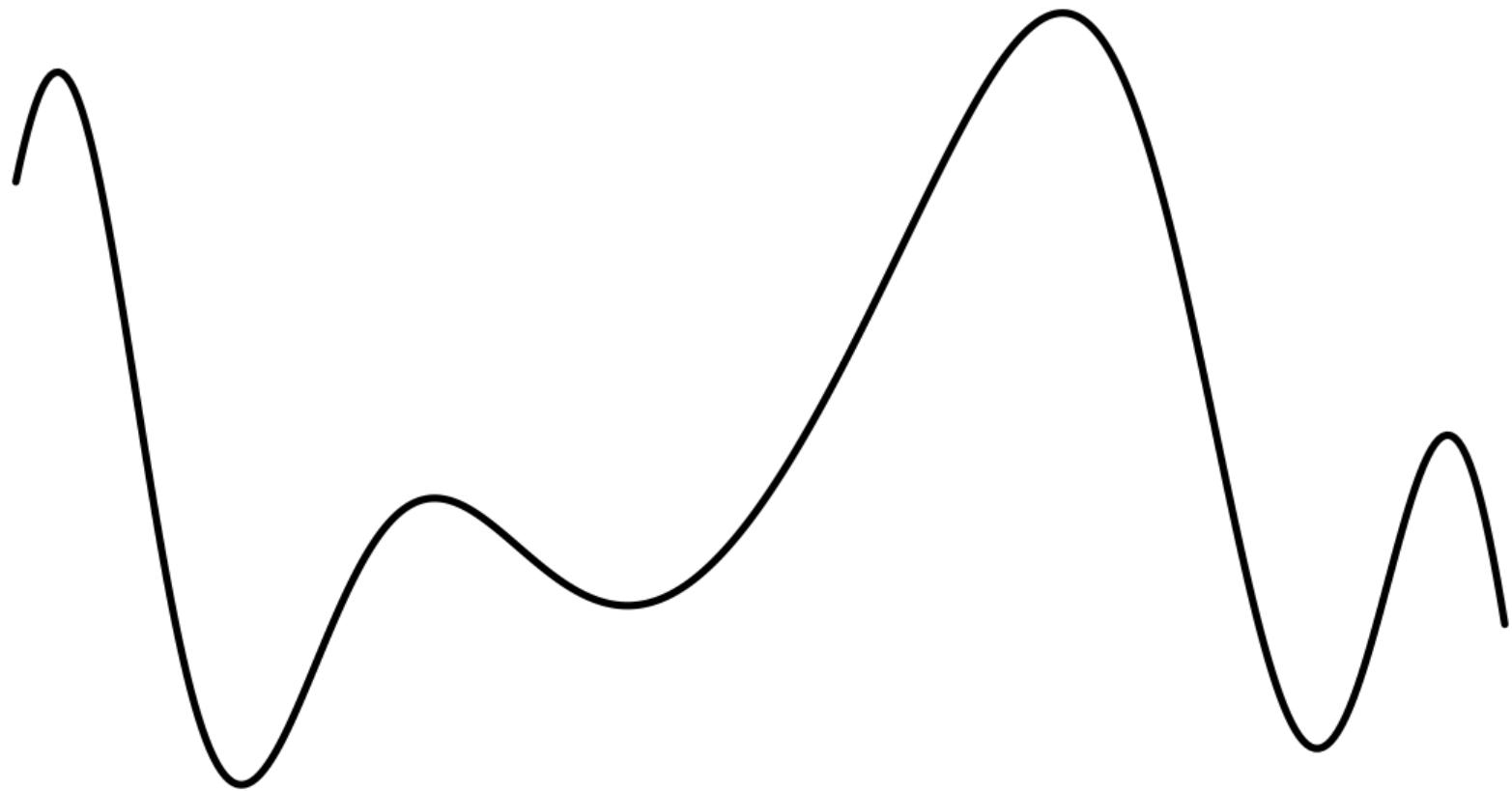
- decomposição celular = discretização
amostragem do objeto
estruturação em células
- decomposição intrínseca
geometria do objeto
poligonização
- decomposição extrínseca
localização do objeto
enumeração espacial

decomposição intrínseca
poligonização

decomposição intrínseca
amostragem + estruturação

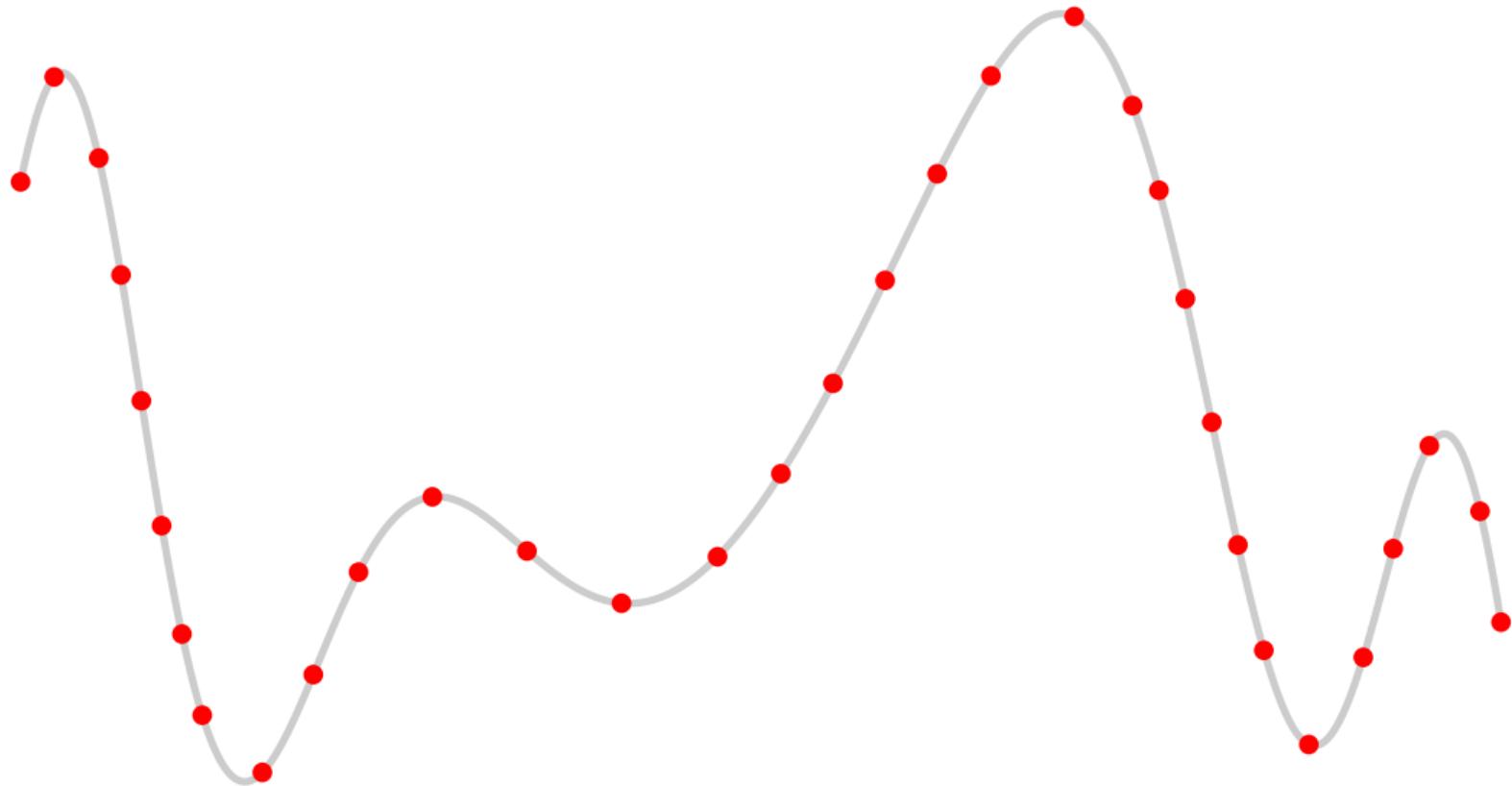
Decomposição celular intrínseca

1D



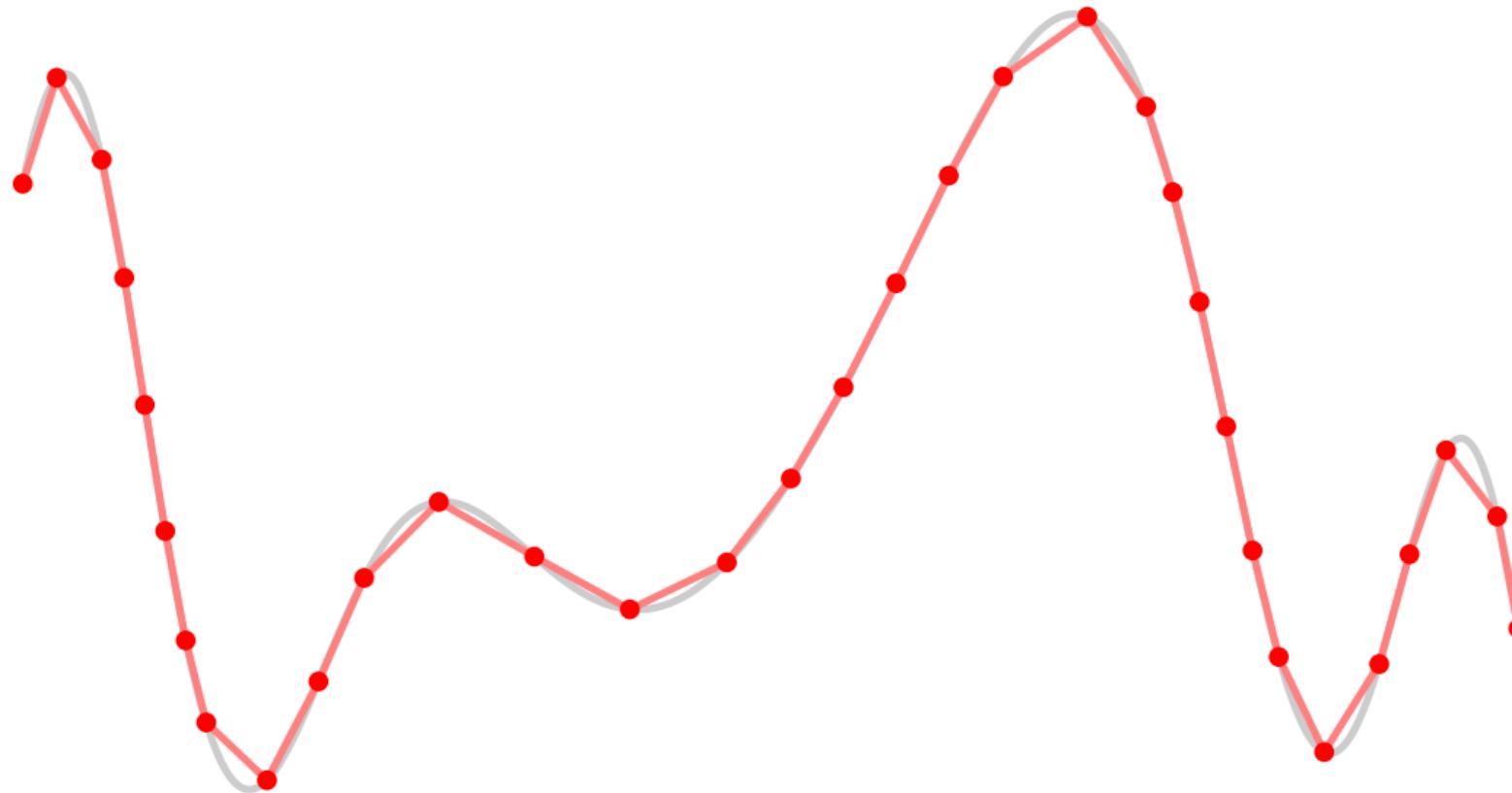
Decomposição celular intrínseca

1D



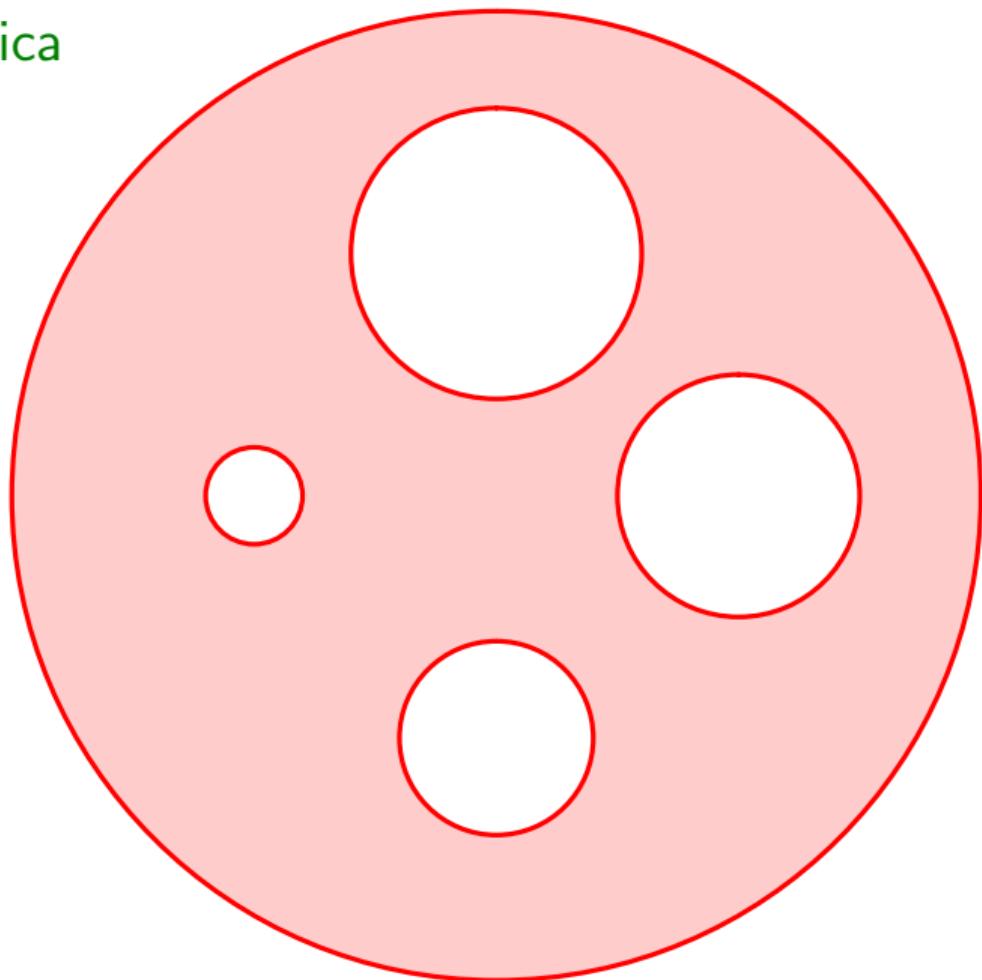
Decomposição celular intrínseca

1D



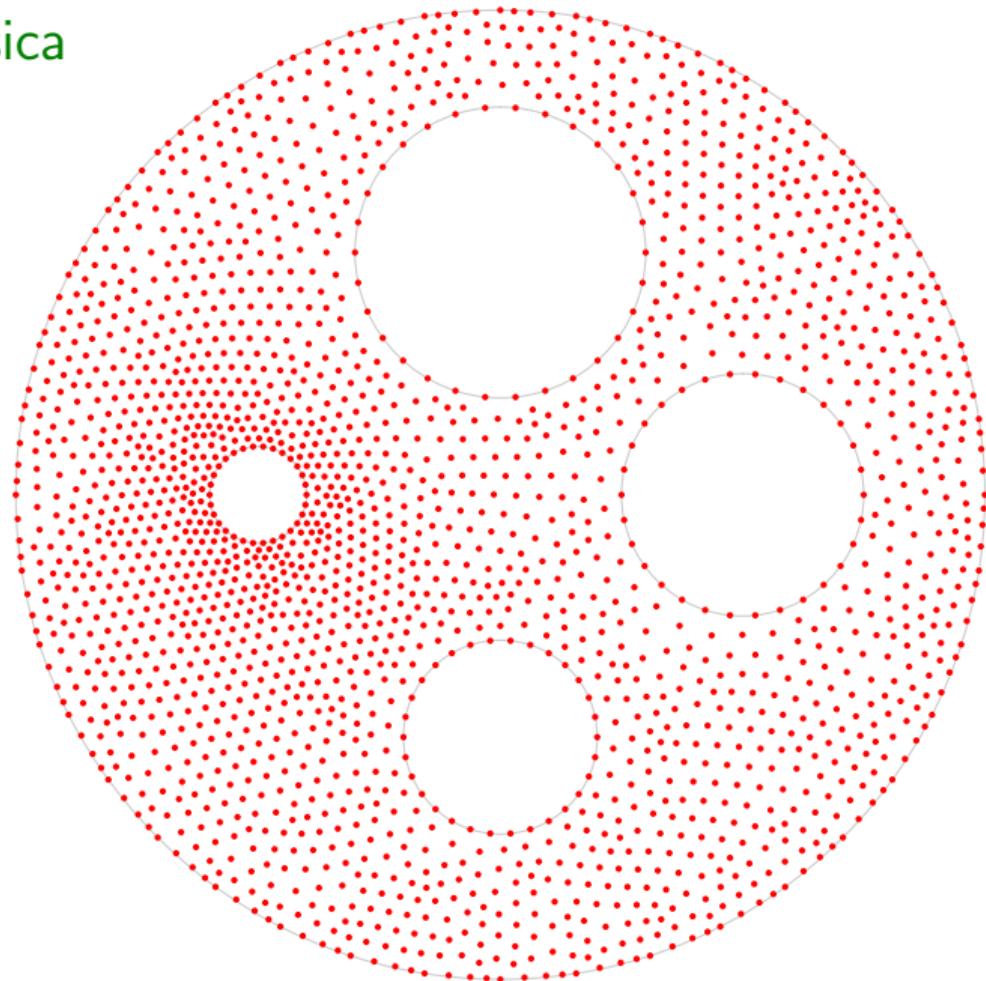
Decomposição celular intrínseca

2D



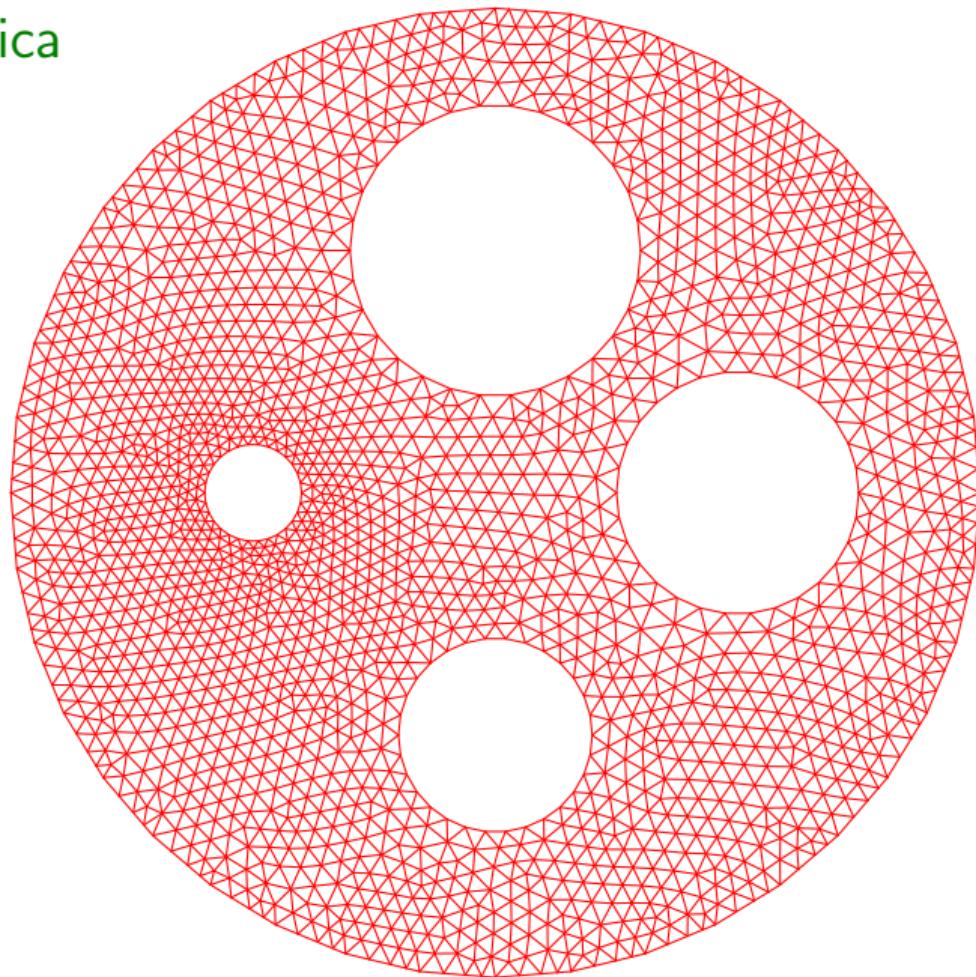
Decomposição celular intrínseca

2D



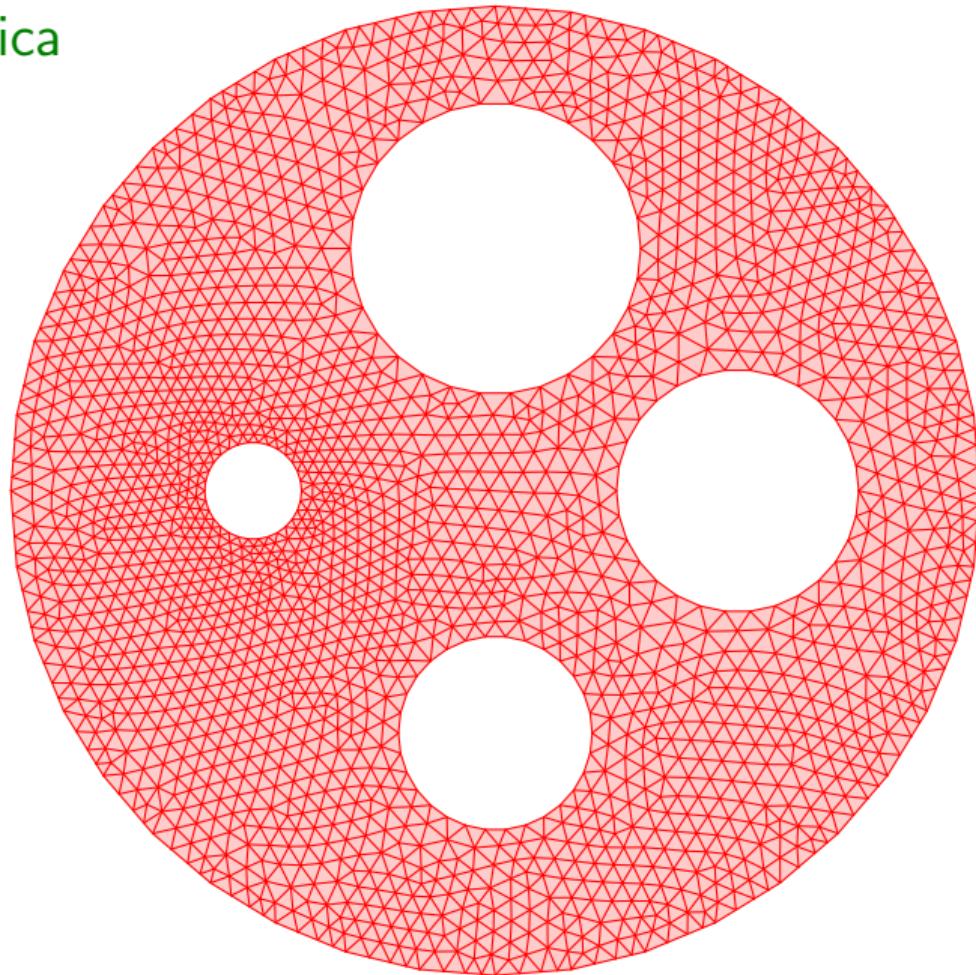
Decomposição celular intrínseca

2D



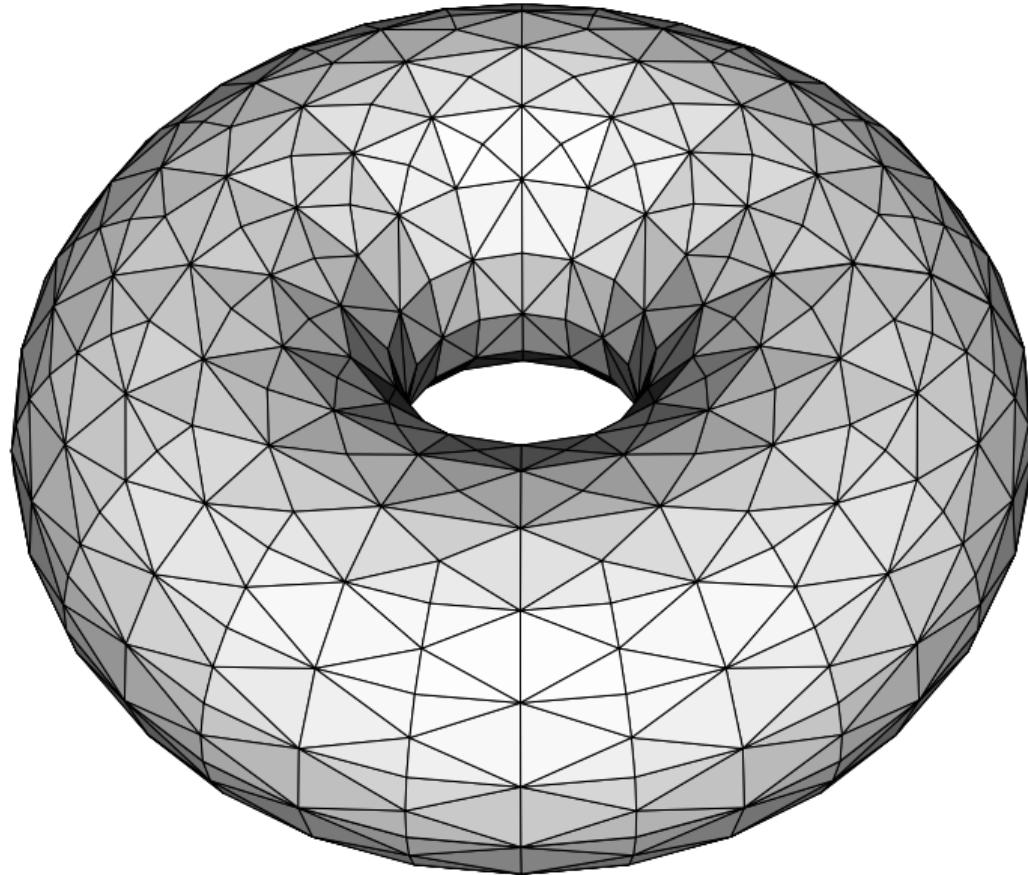
Decomposição celular intrínseca

2D



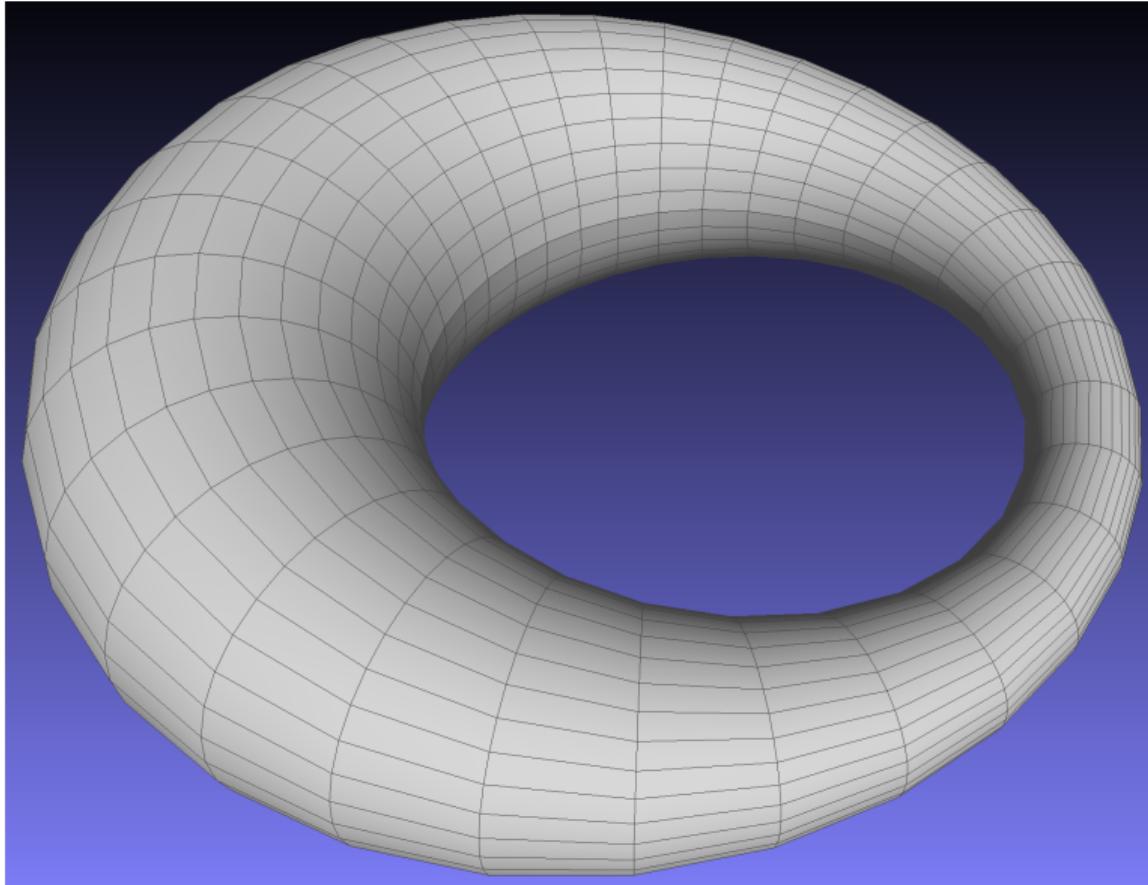
Decomposição celular intrínseca

2D



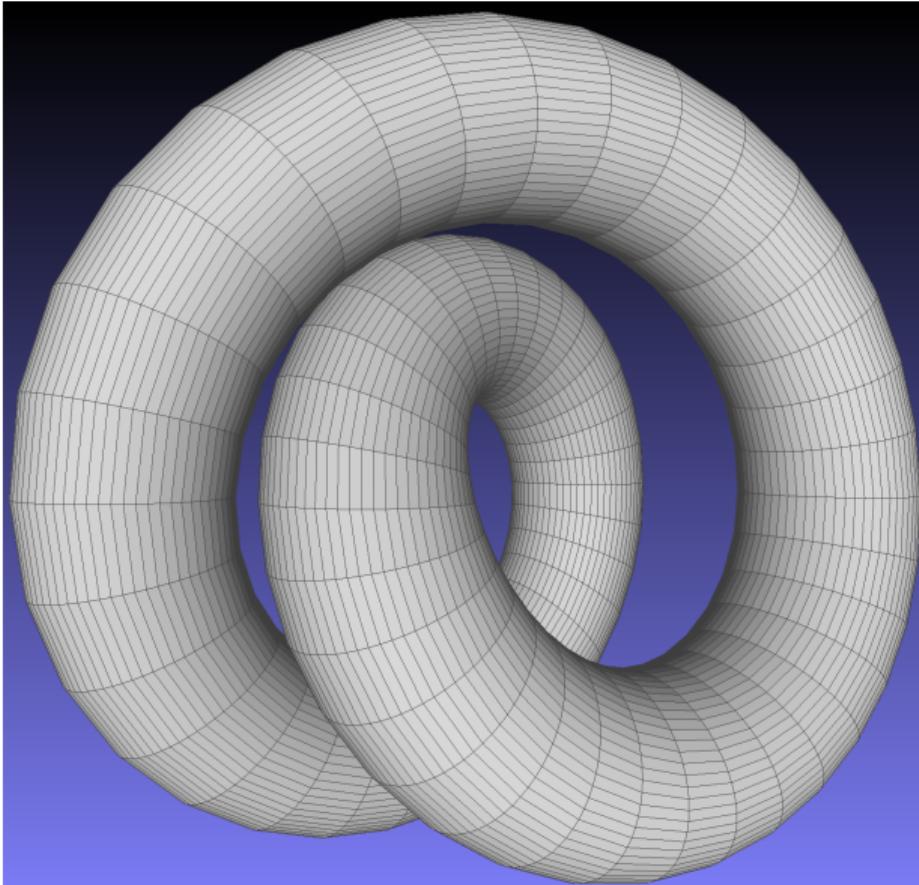
Decomposição celular intrínseca

2D



Decomposição celular intrínseca

2D



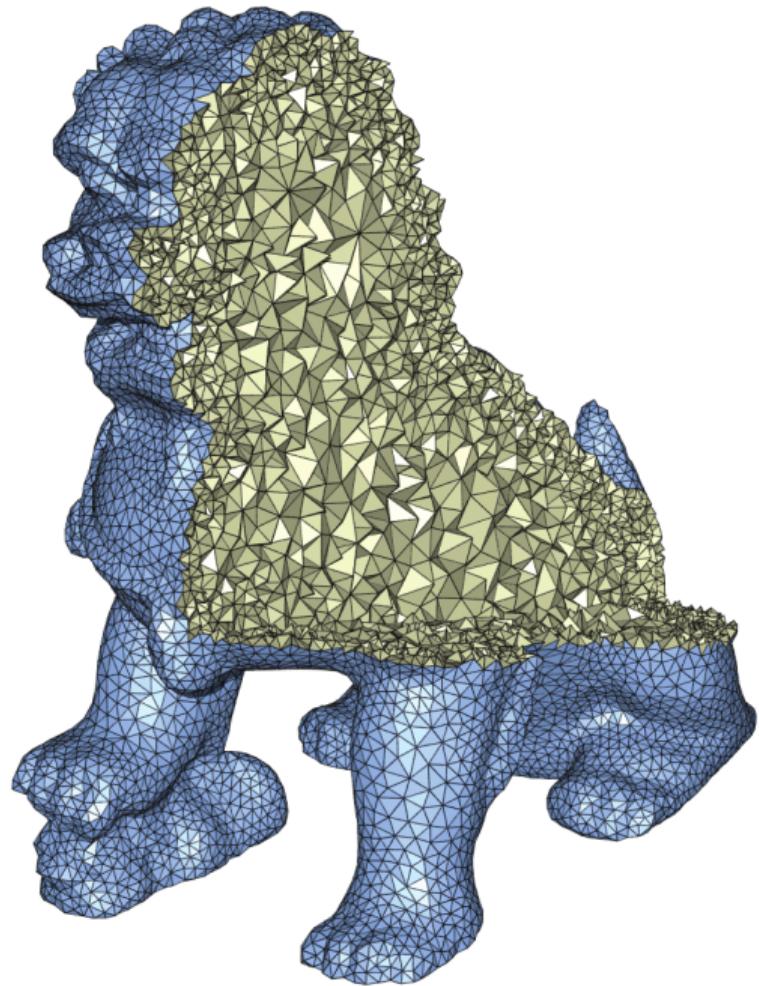
Decomposição celular intrínseca

3D



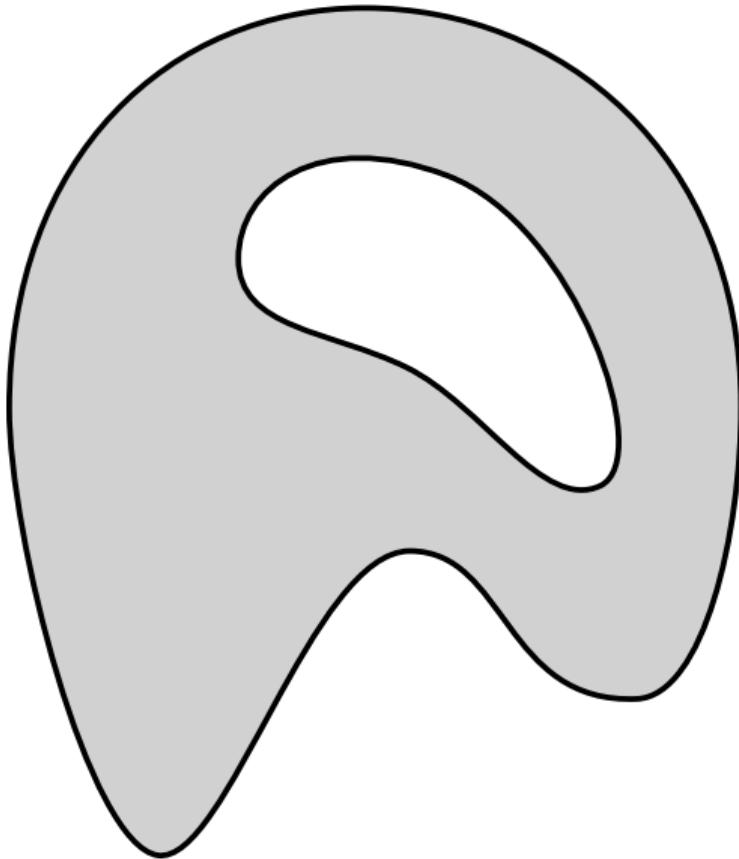
Decomposição celular intrínseca

3D

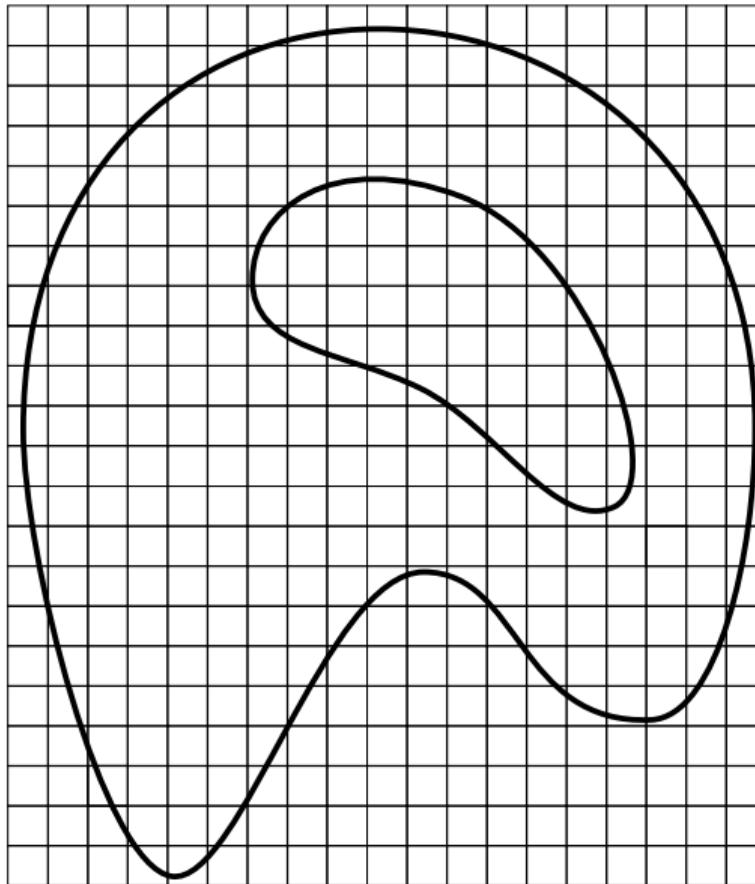


decomposição extrínseca
enumeração espacial

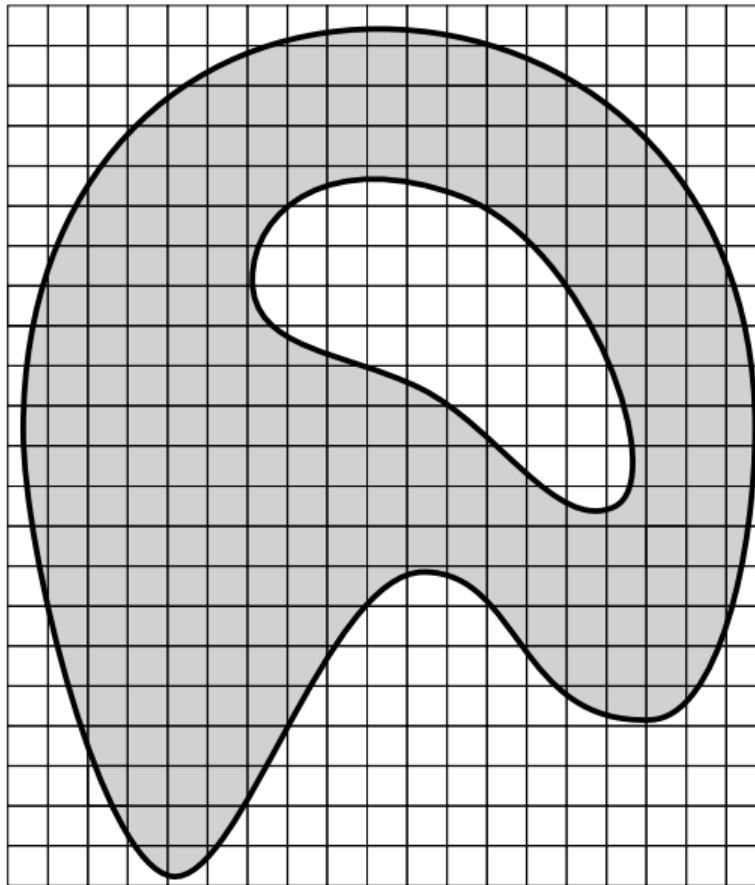
Decomposição celular extrínseca



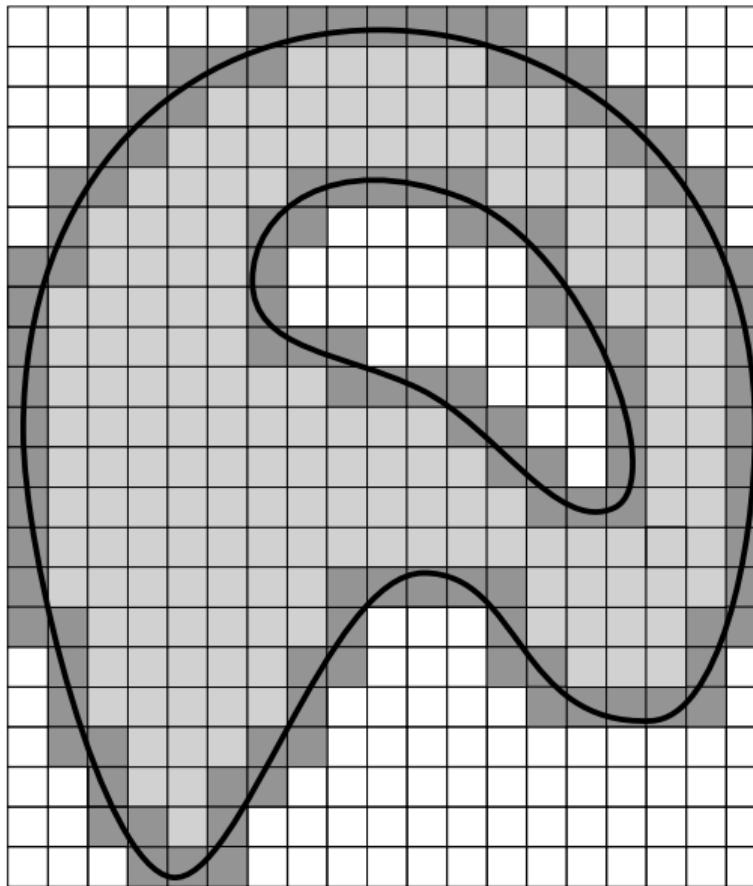
Decomposição celular extrínseca



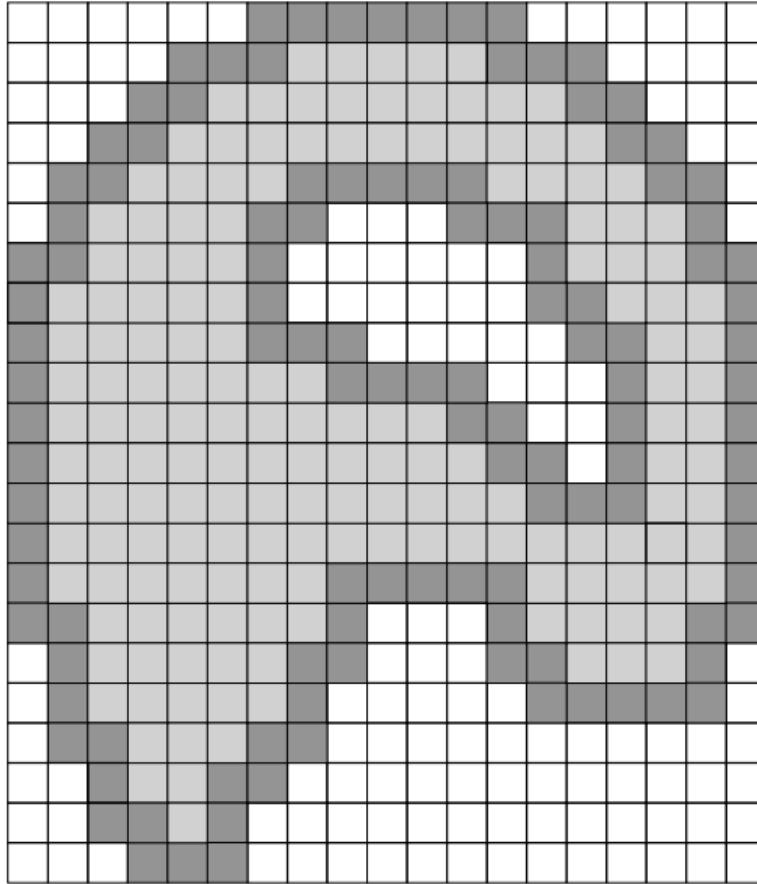
Decomposição celular extrínseca



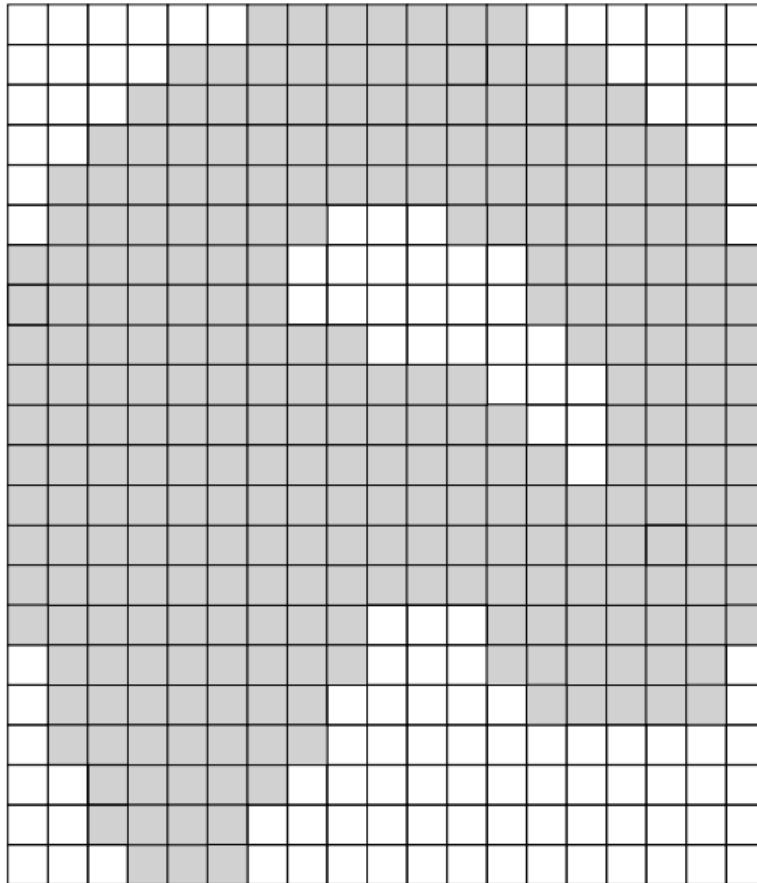
Decomposição celular extrínseca



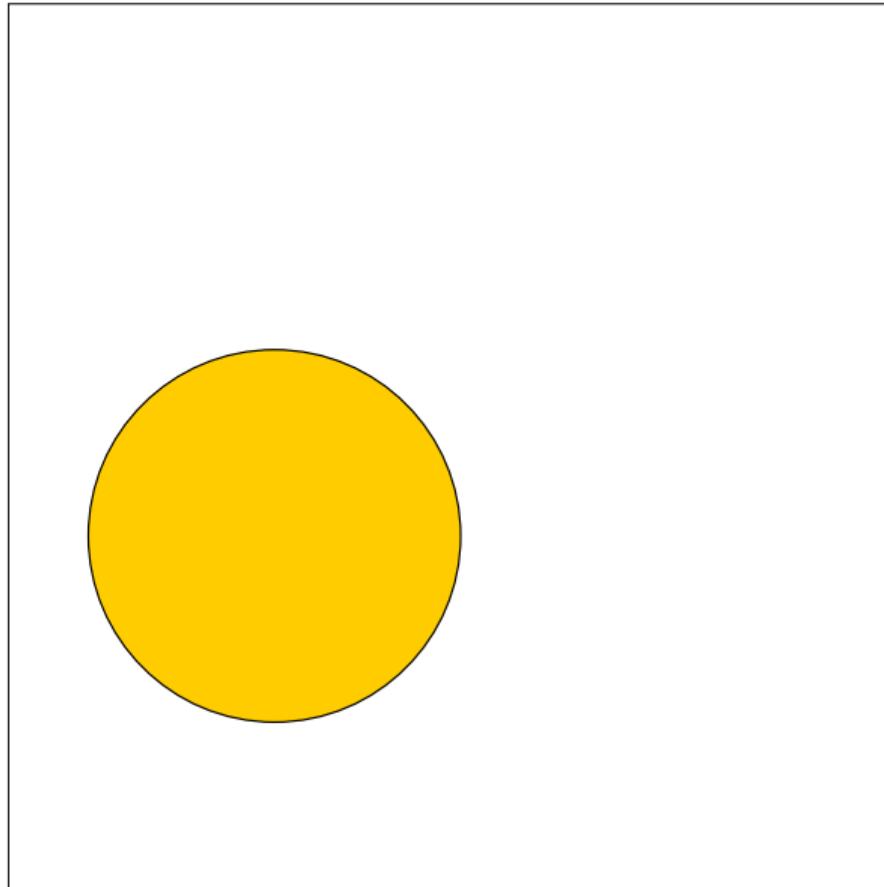
Decomposição celular extrínsica



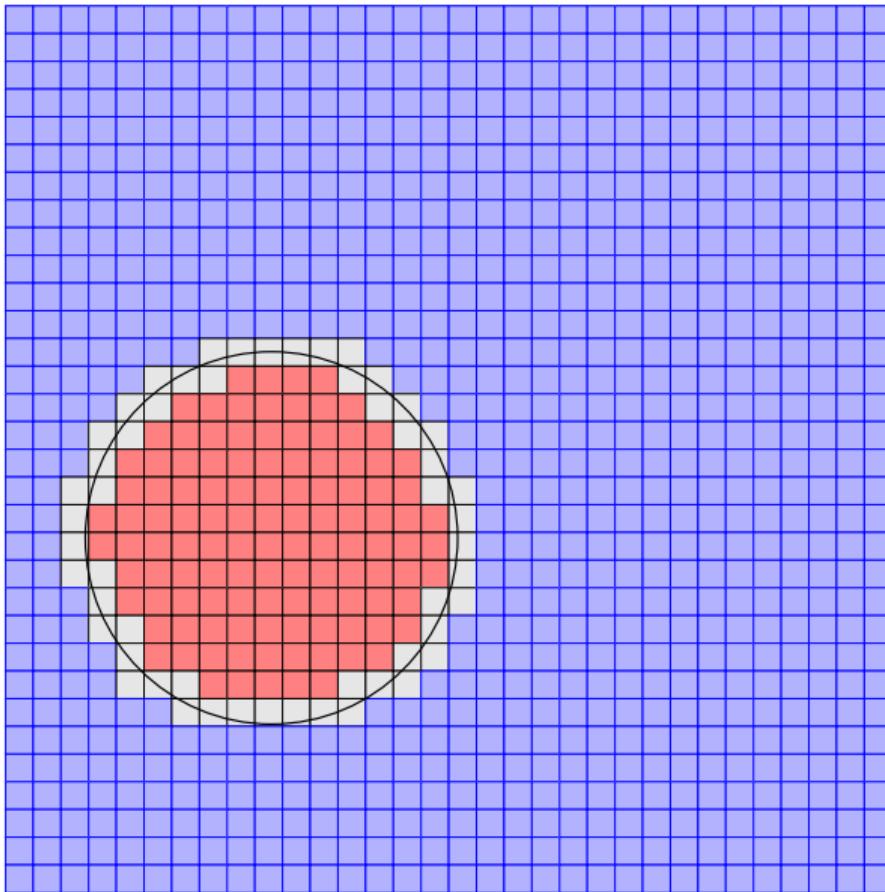
Decomposição celular extrínsica



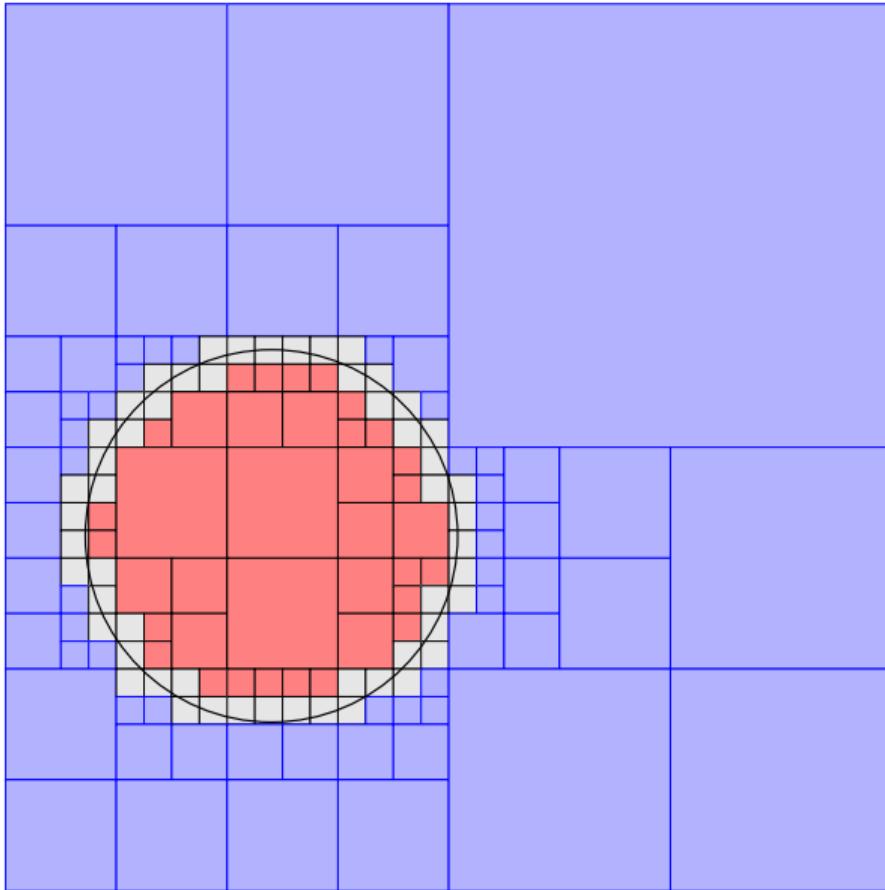
Enumeração espacial



Enumeração espacial uniforme



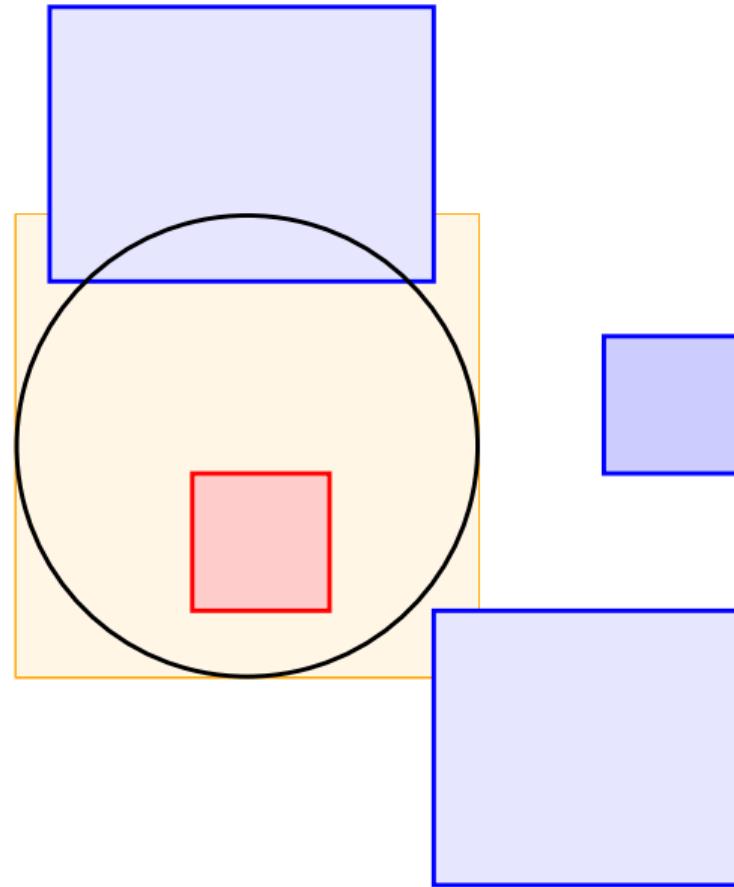
Enumeração espacial adaptativa



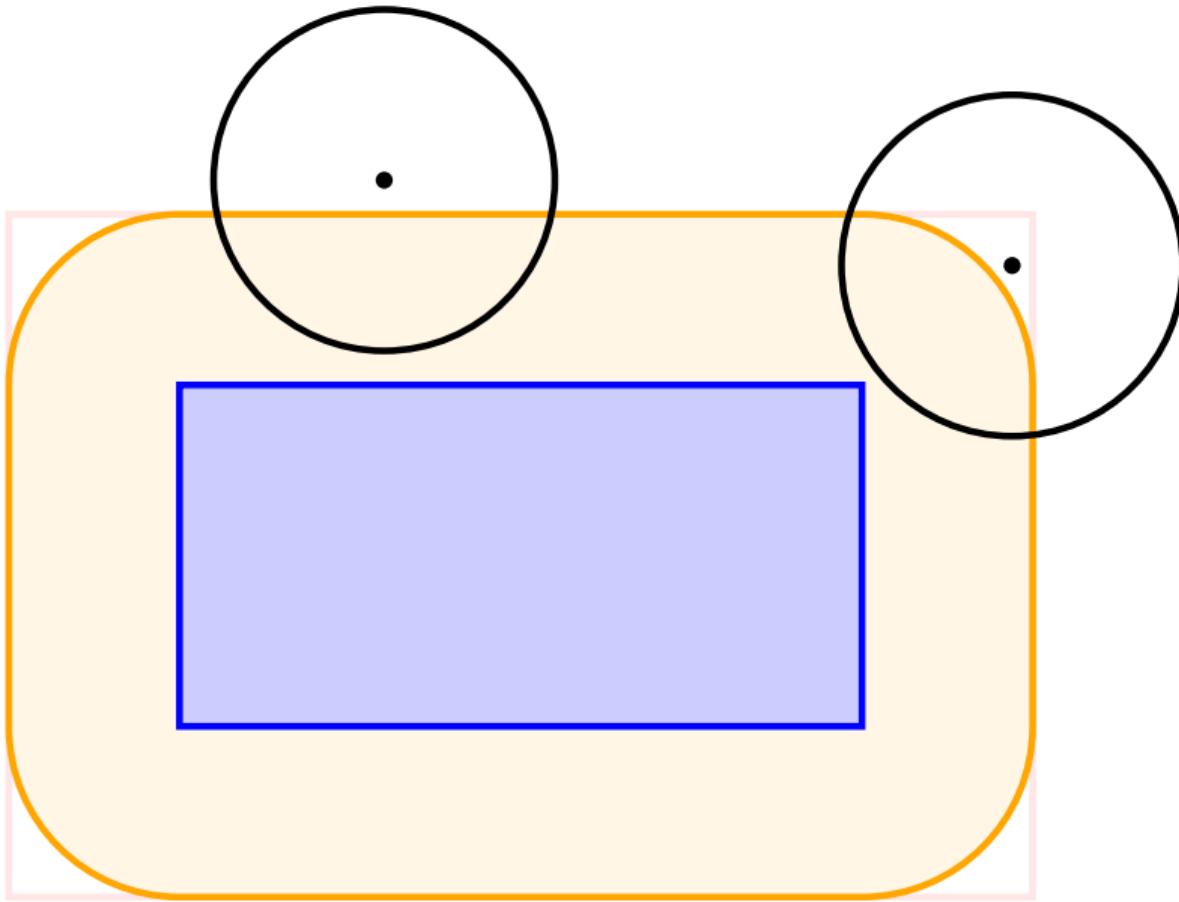
Enumeração espacial adaptativa

```
function explore(xmin,xmax,ymin,ymax,level)
    local c=classify(xmin,xmax,ymin,ymax)
    if c~="none" or level==0 then
        output(xmin,xmax,ymin,ymax,c)
    else
        local xmid=(xmin+xmax)/2
        local ymid=(ymin+ymax)/2
        explore(xmin,xmid,ymid,ymax,level-1)
        explore(xmid,xmax,ymid,ymax,level-1)
        explore(xmin,xmid,ymin,ymid,level-1)
        explore(xmid,xmax,ymin,ymid,level-1)
    end
end
```

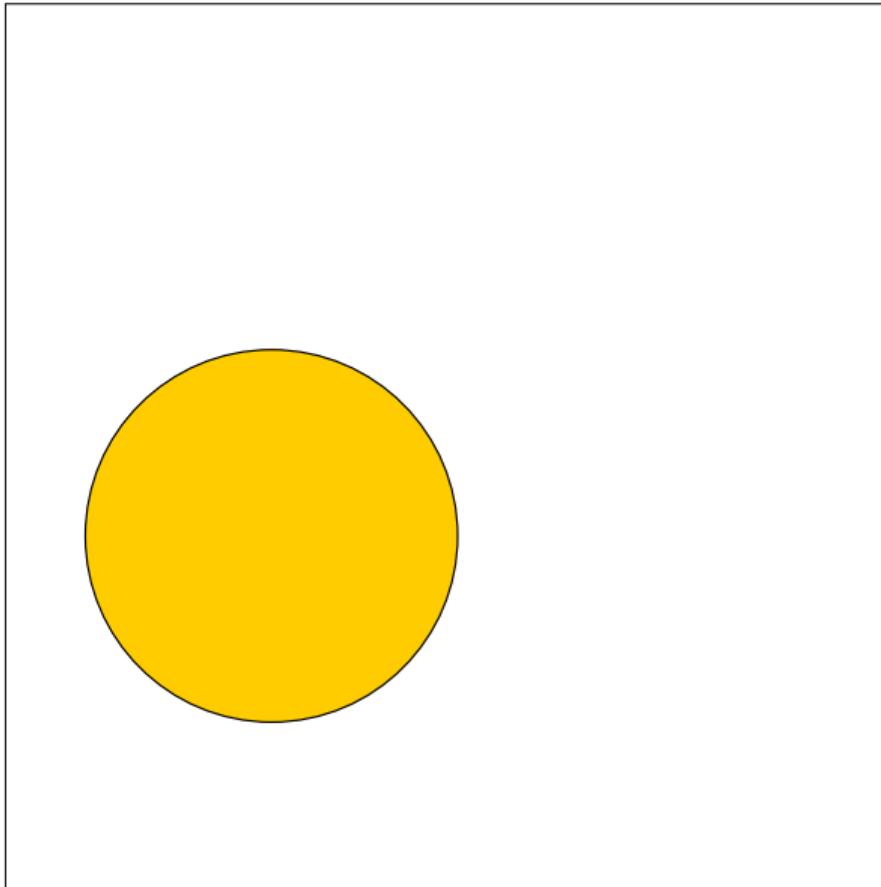
Enumeração espacial



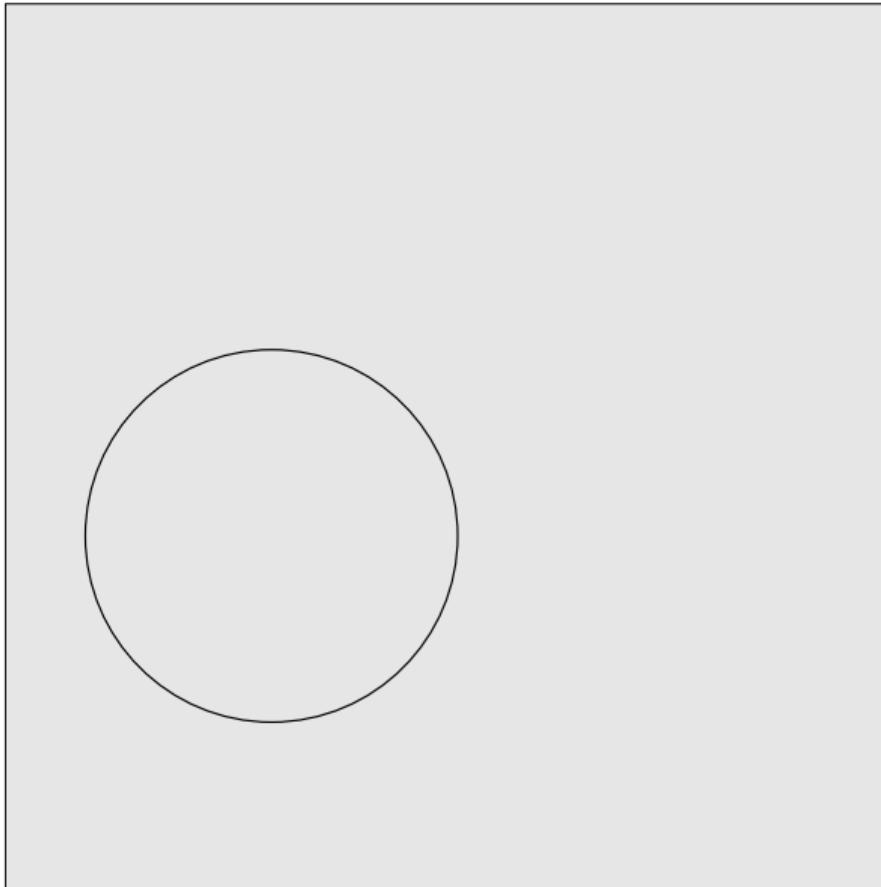
Enumeração espacial



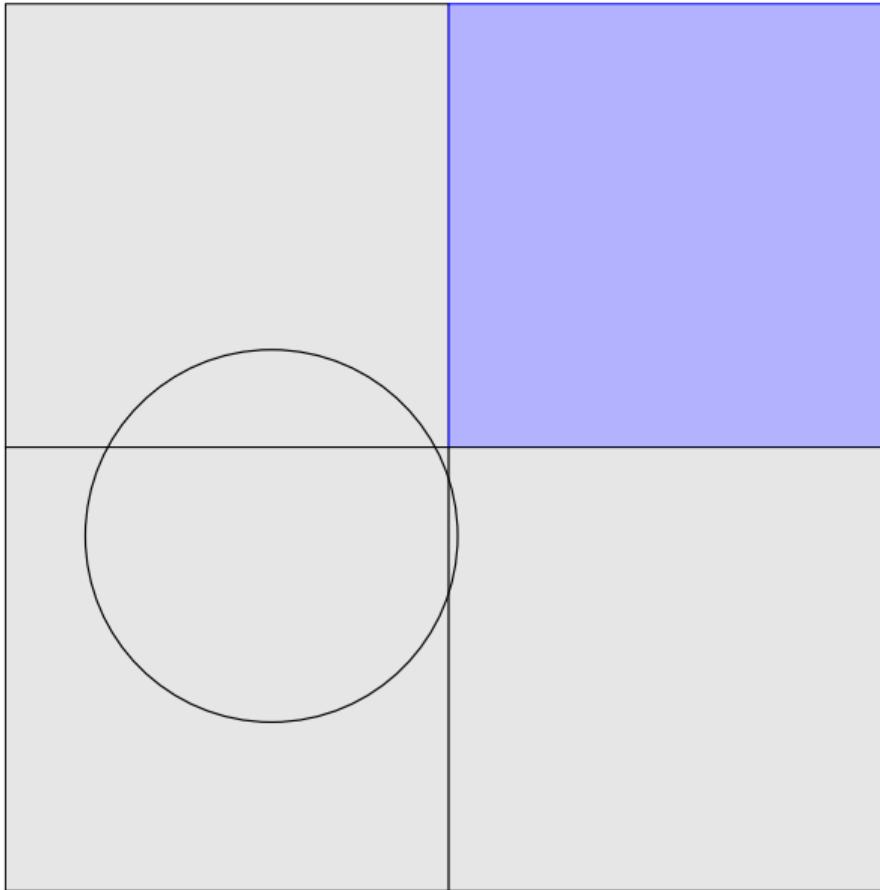
Enumeração espacial adaptativa



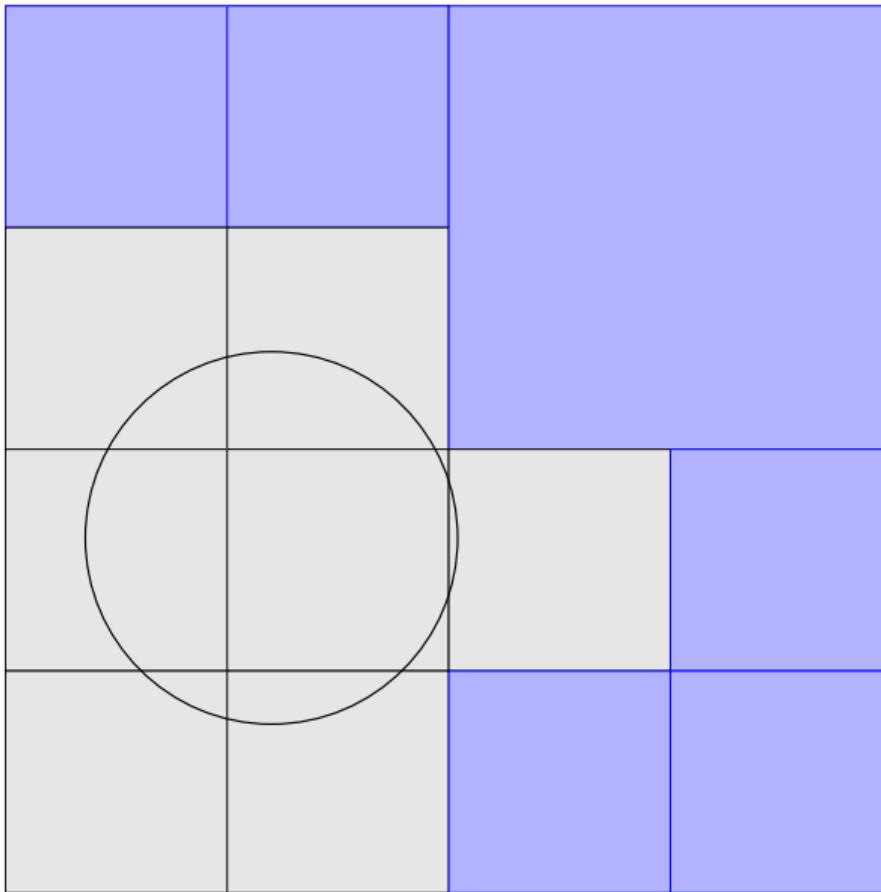
Enumeração espacial adaptativa



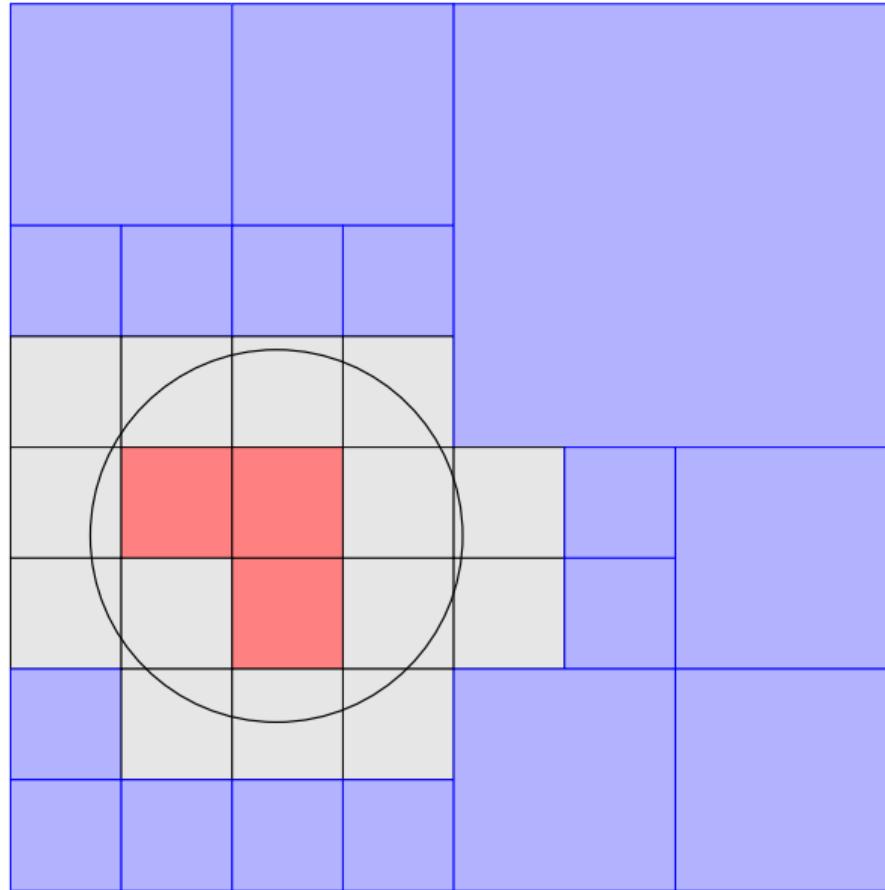
Enumeração espacial adaptativa



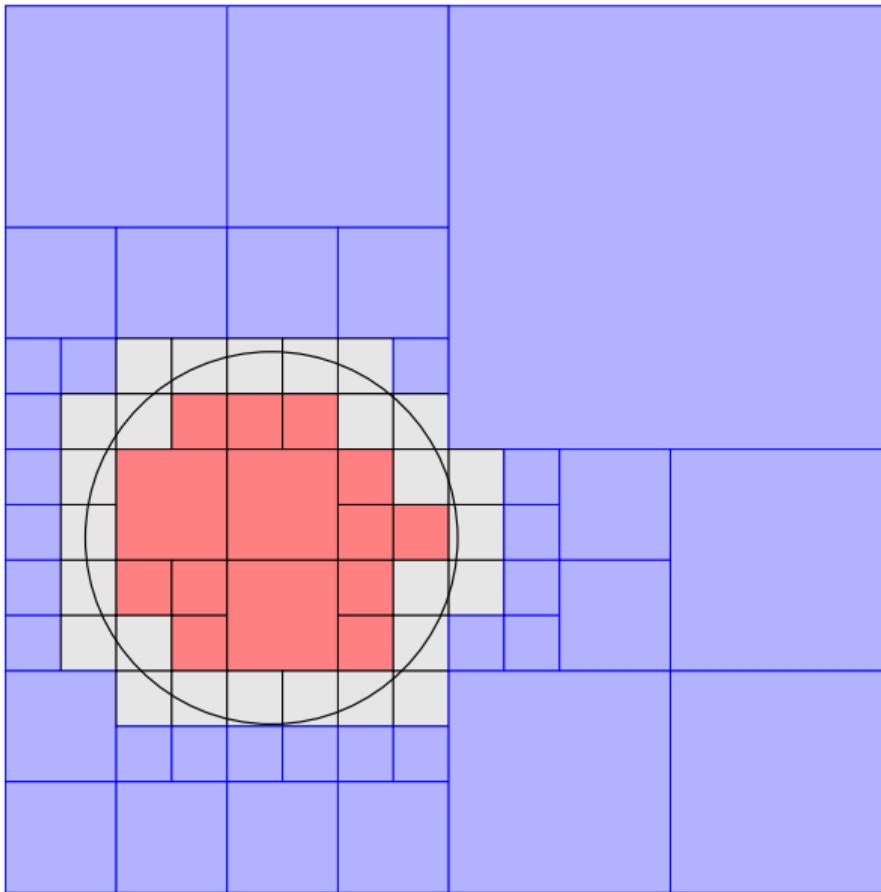
Enumeração espacial adaptativa



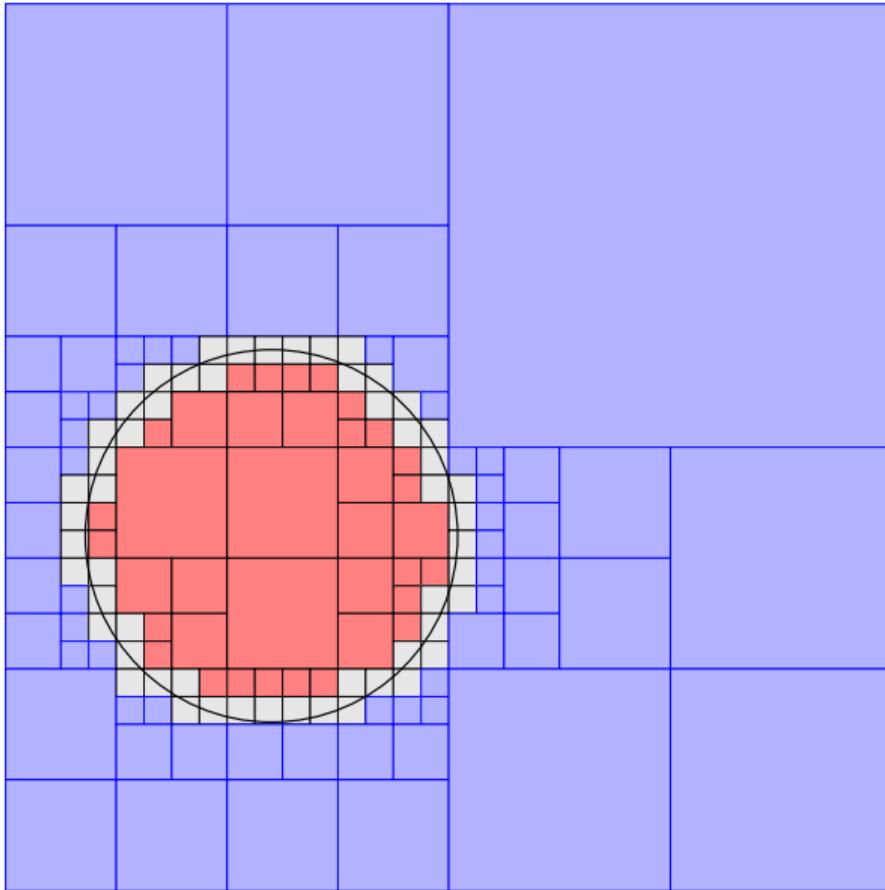
Enumeração espacial adaptativa



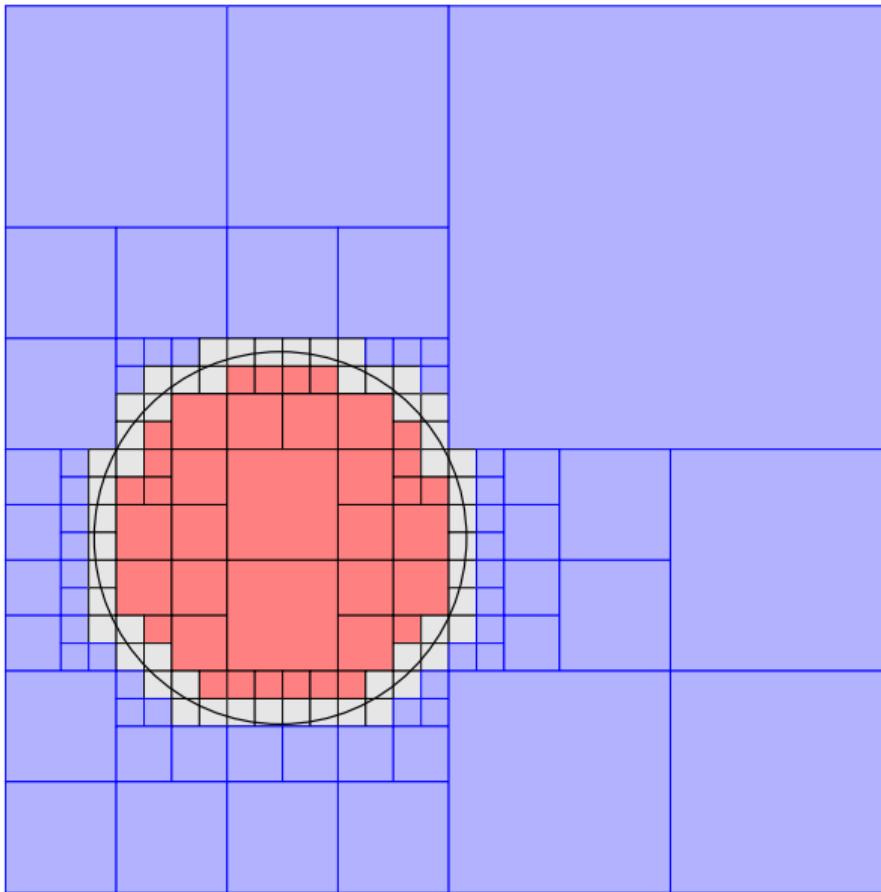
Enumeração espacial adaptativa



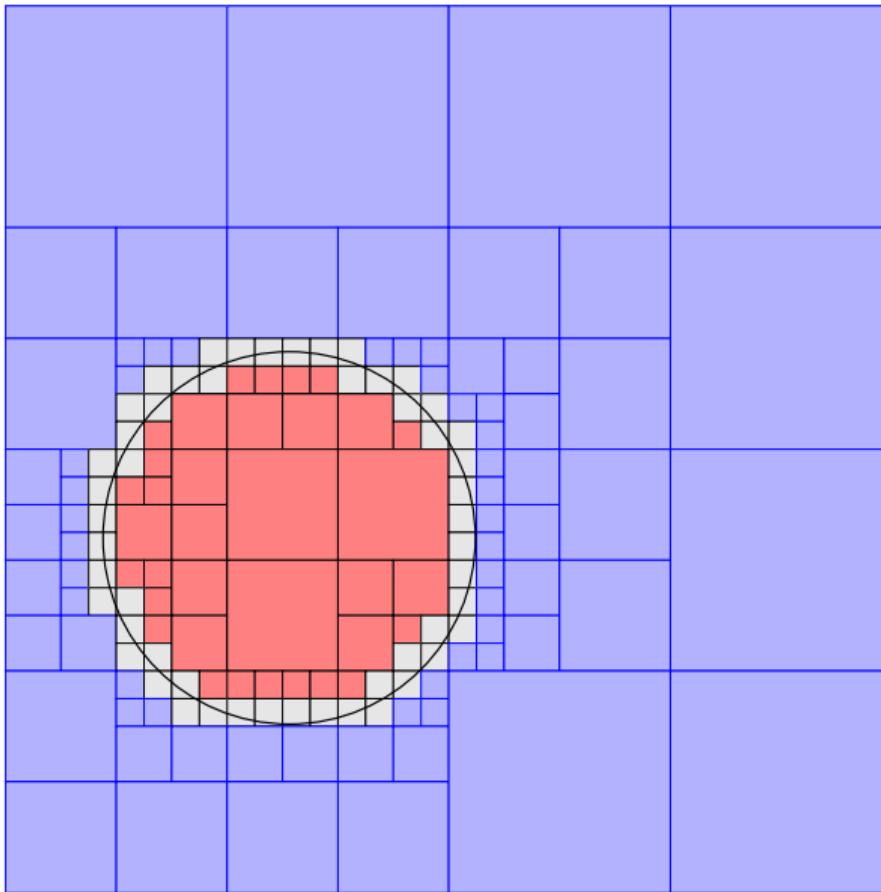
Enumeração espacial adaptativa



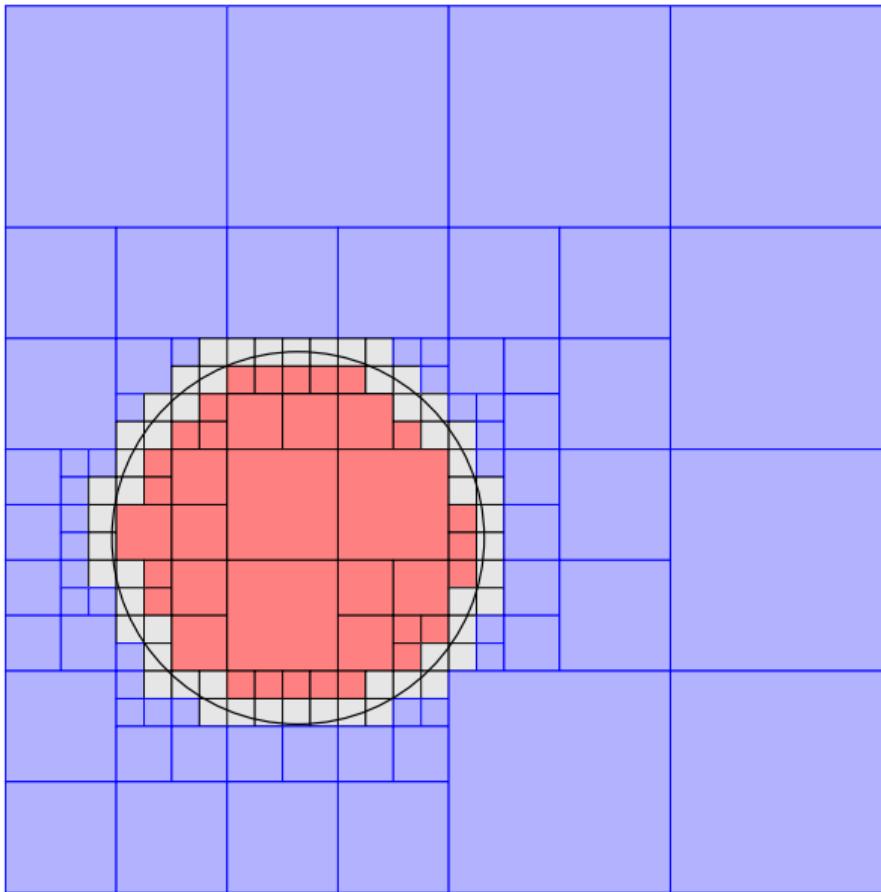
Enumeração espacial adaptativa



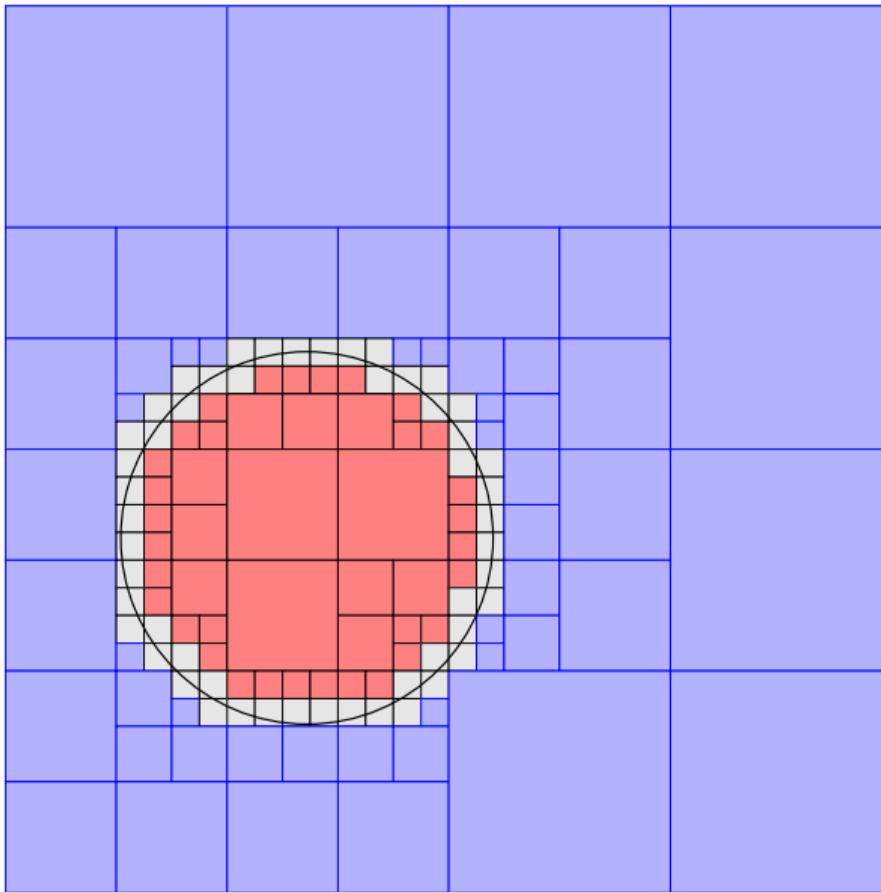
Enumeração espacial adaptativa



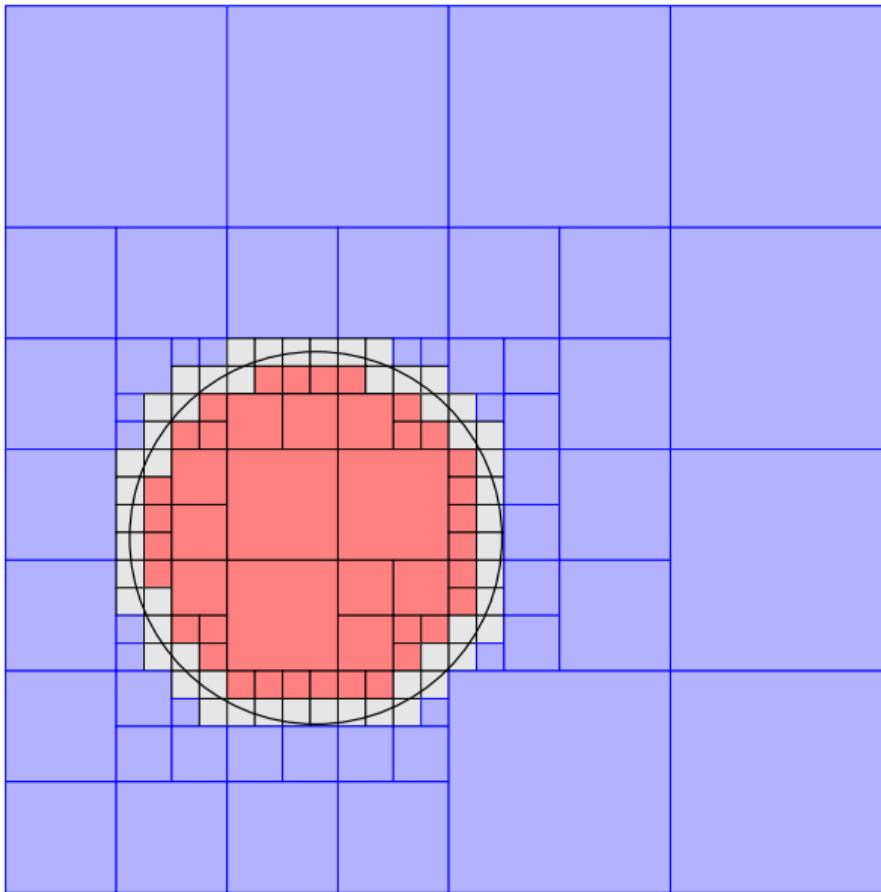
Enumeração espacial adaptativa



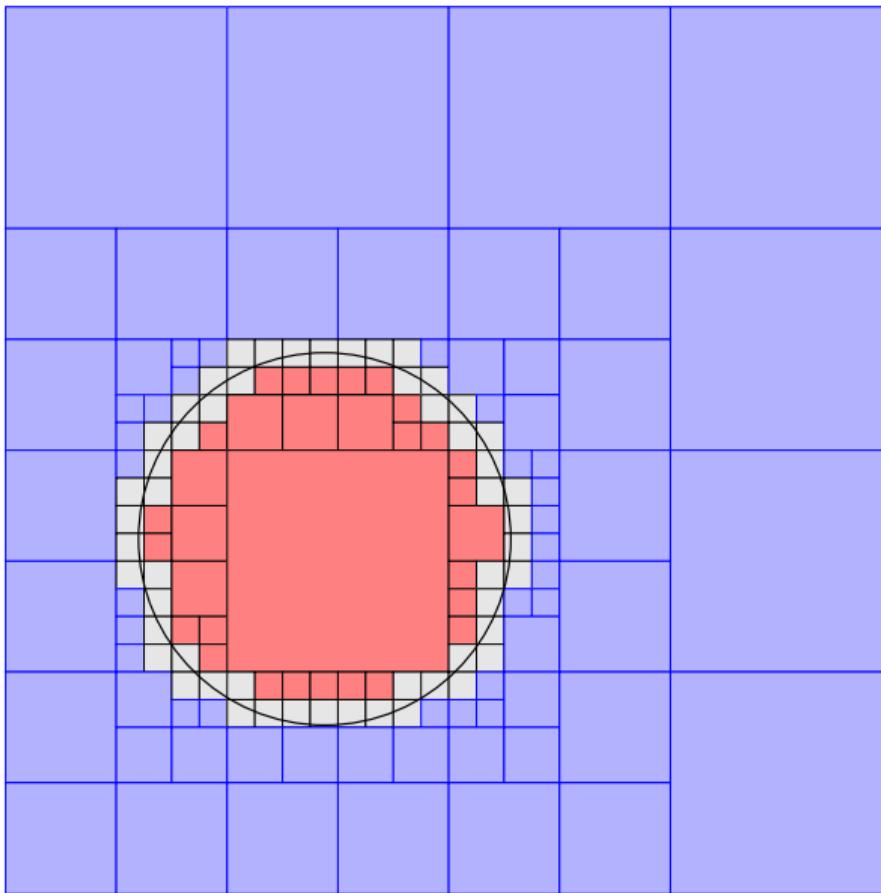
Enumeração espacial adaptativa



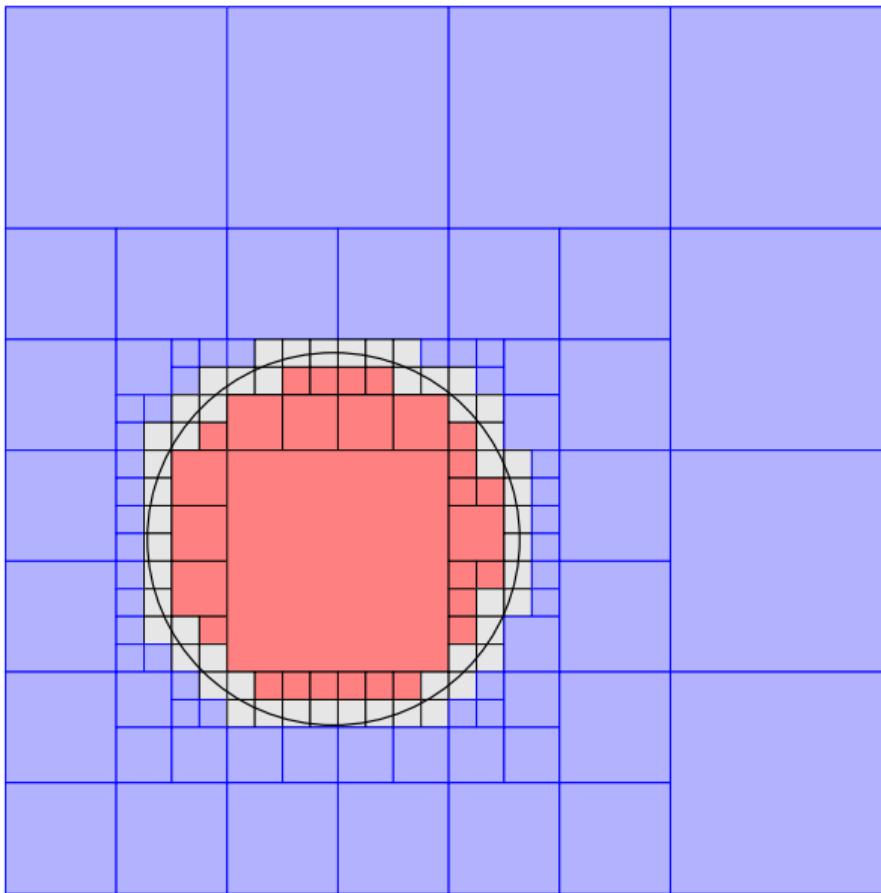
Enumeração espacial adaptativa



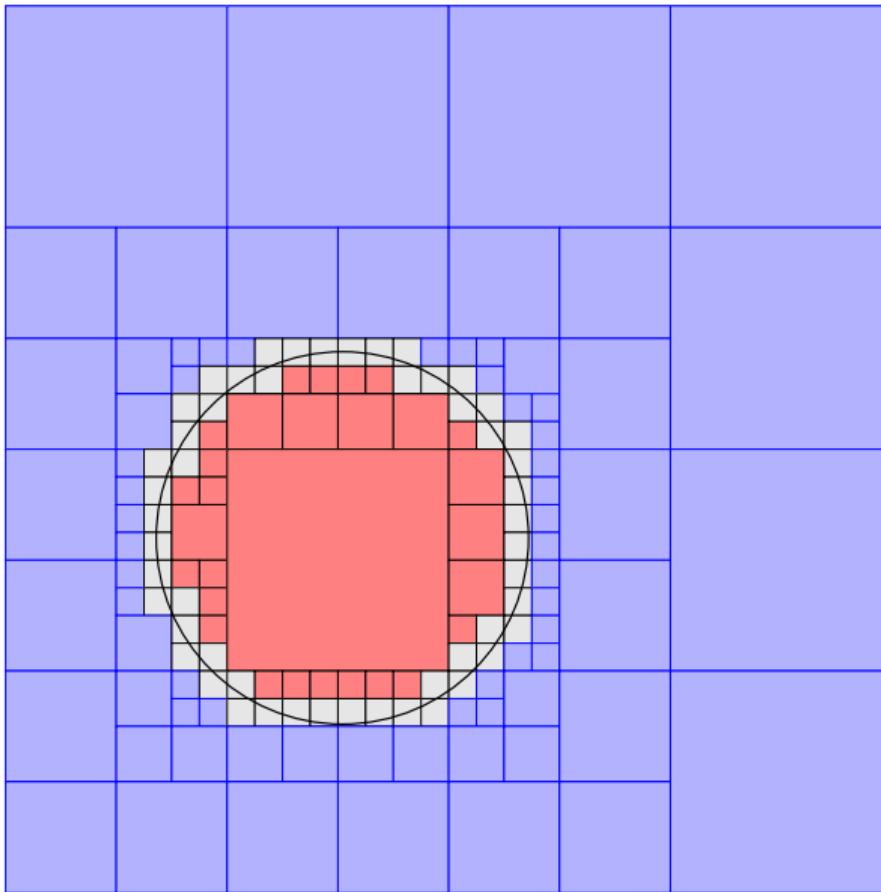
Enumeração espacial adaptativa



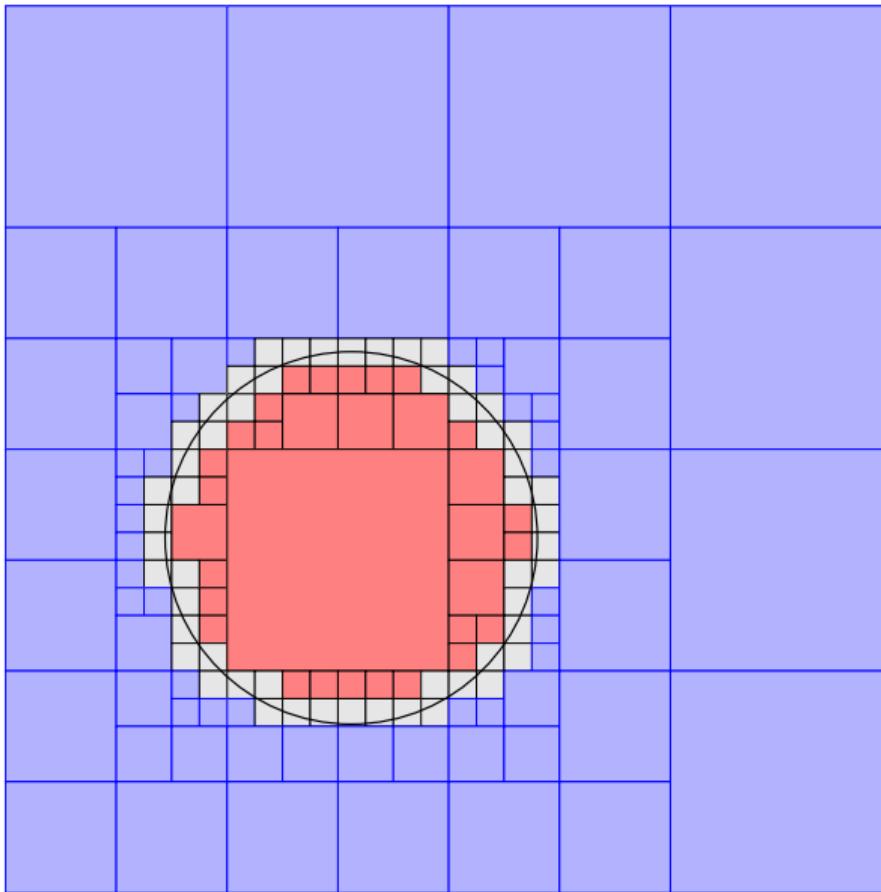
Enumeração espacial adaptativa



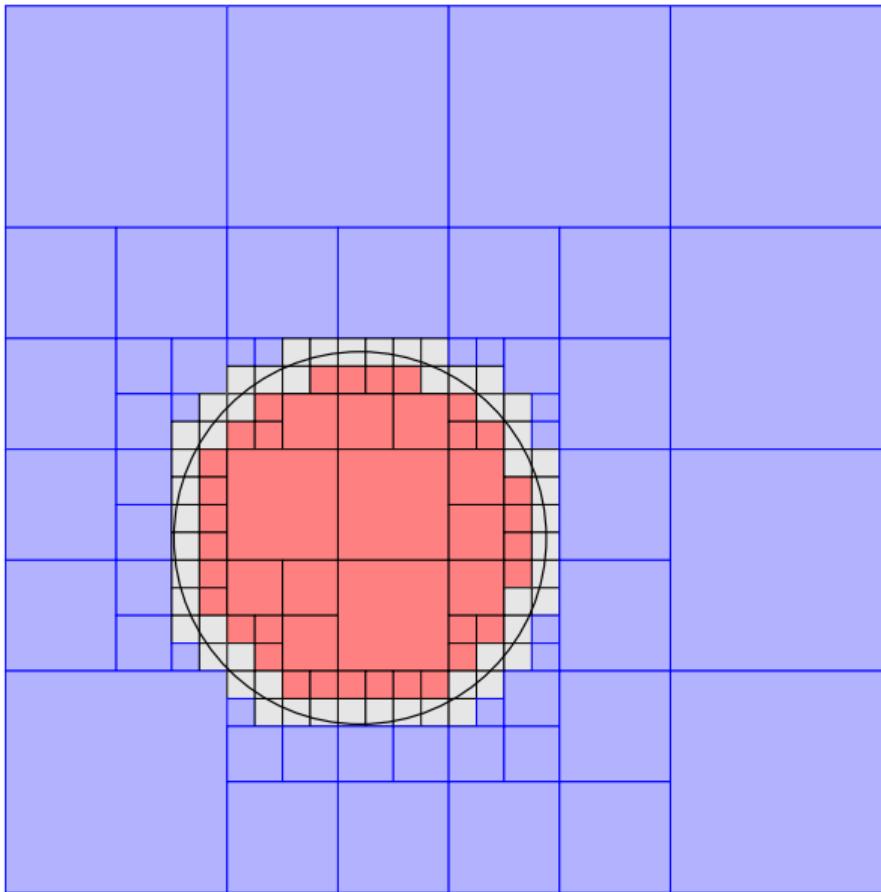
Enumeração espacial adaptativa



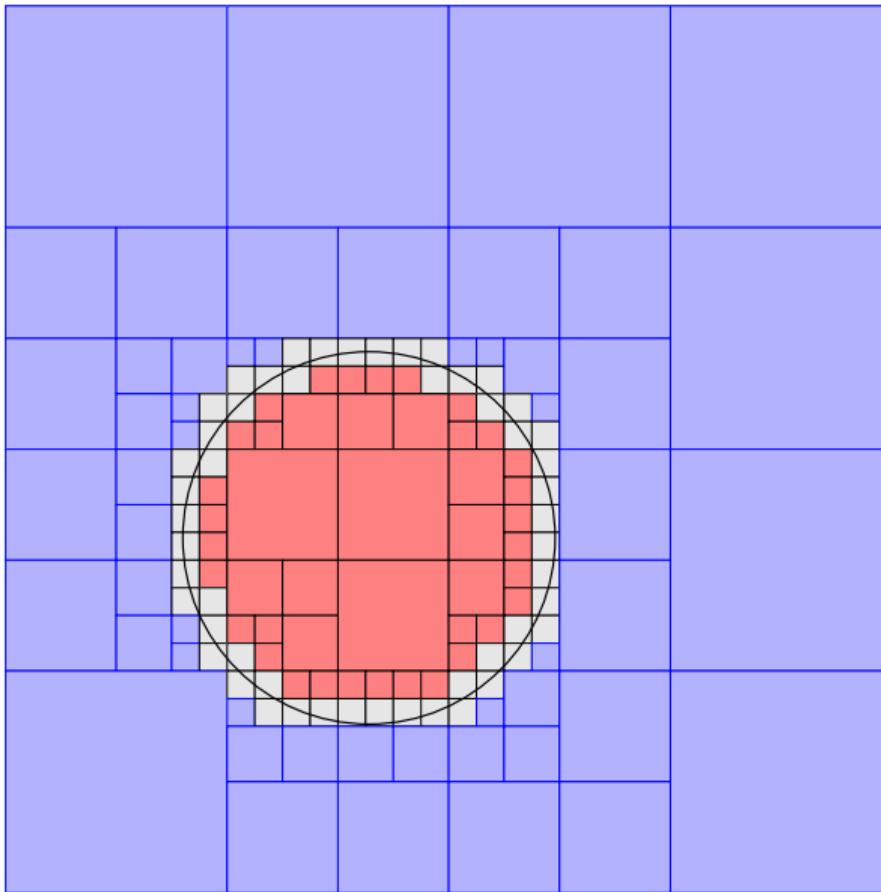
Enumeração espacial adaptativa



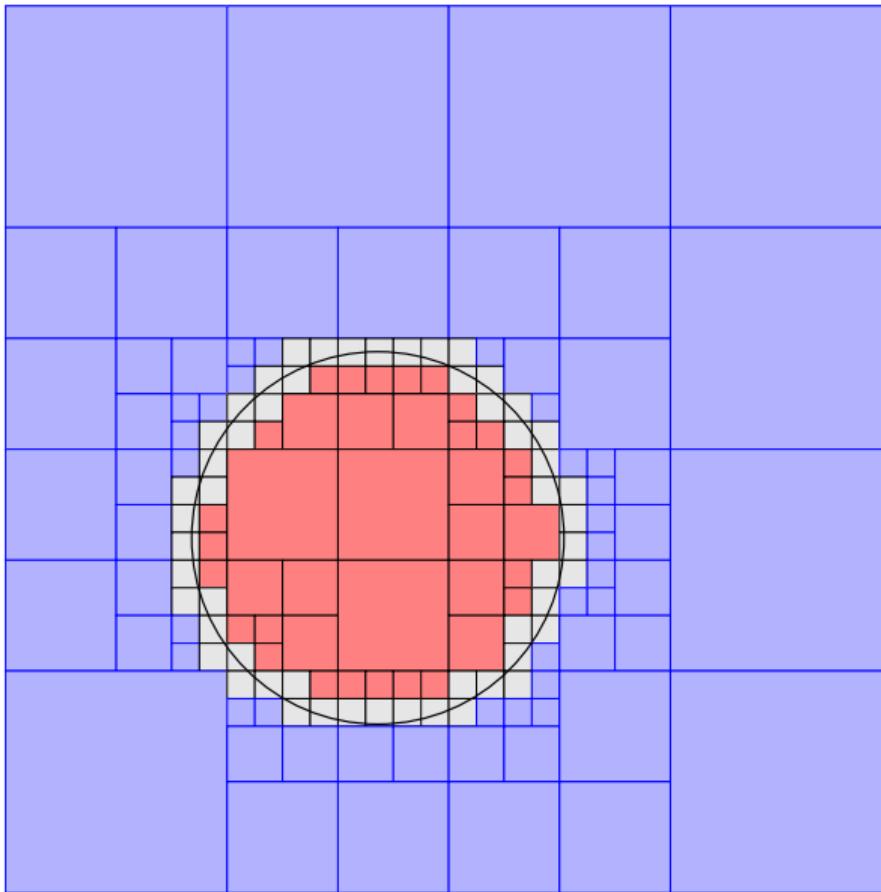
Enumeração espacial adaptativa



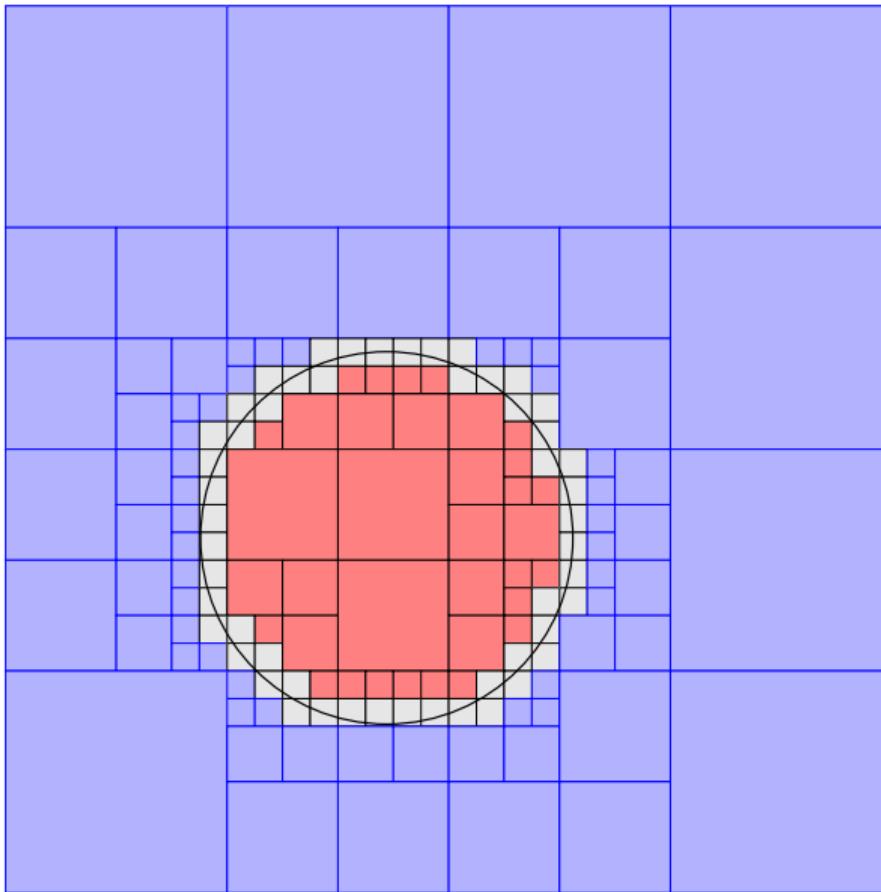
Enumeração espacial adaptativa



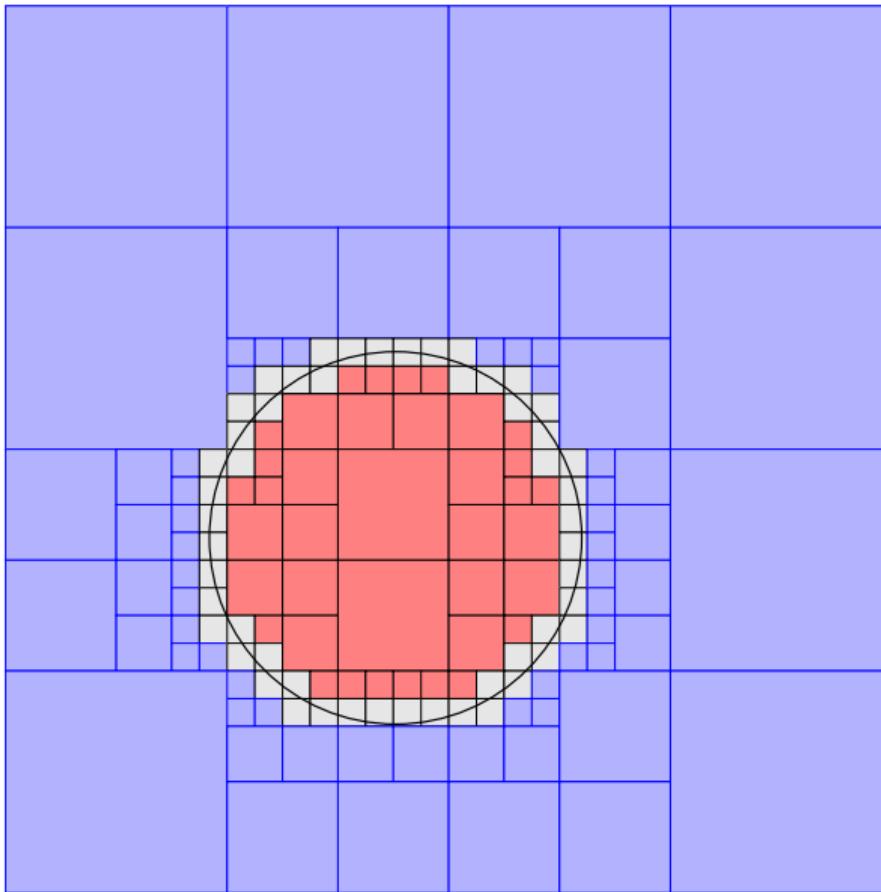
Enumeração espacial adaptativa



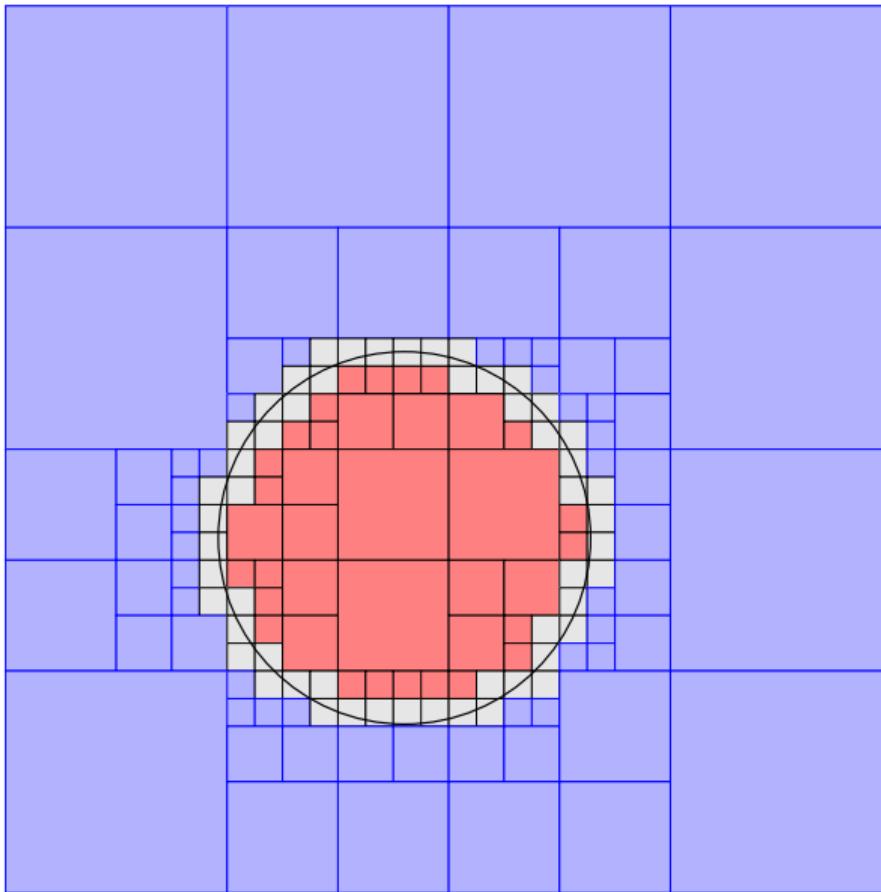
Enumeração espacial adaptativa



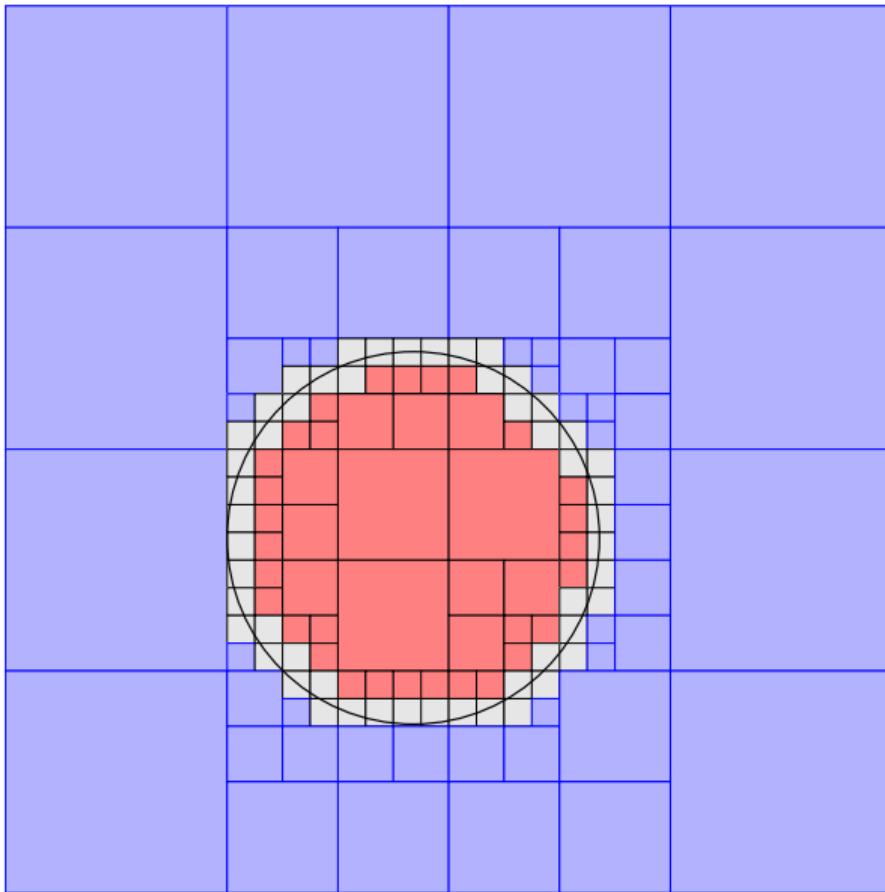
Enumeração espacial adaptativa



Enumeração espacial adaptativa

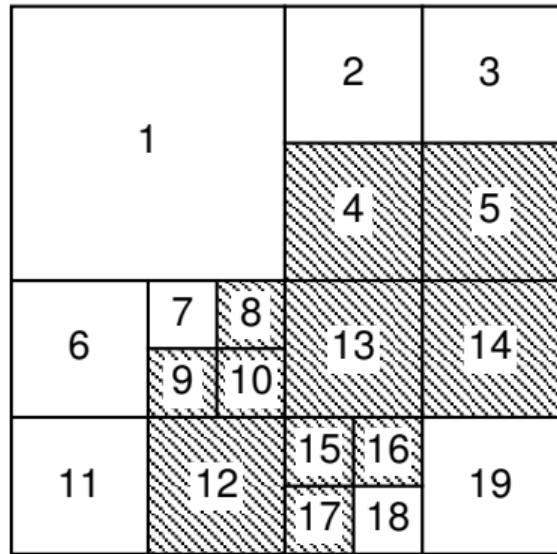


Enumeração espacial adaptativa

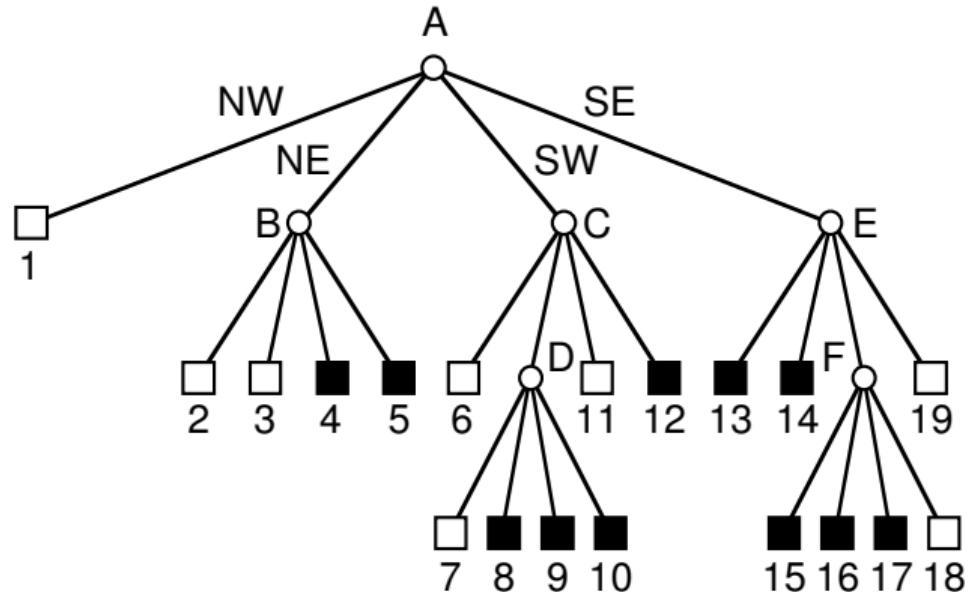


Quadtree

Finkel–Bentley (1974)

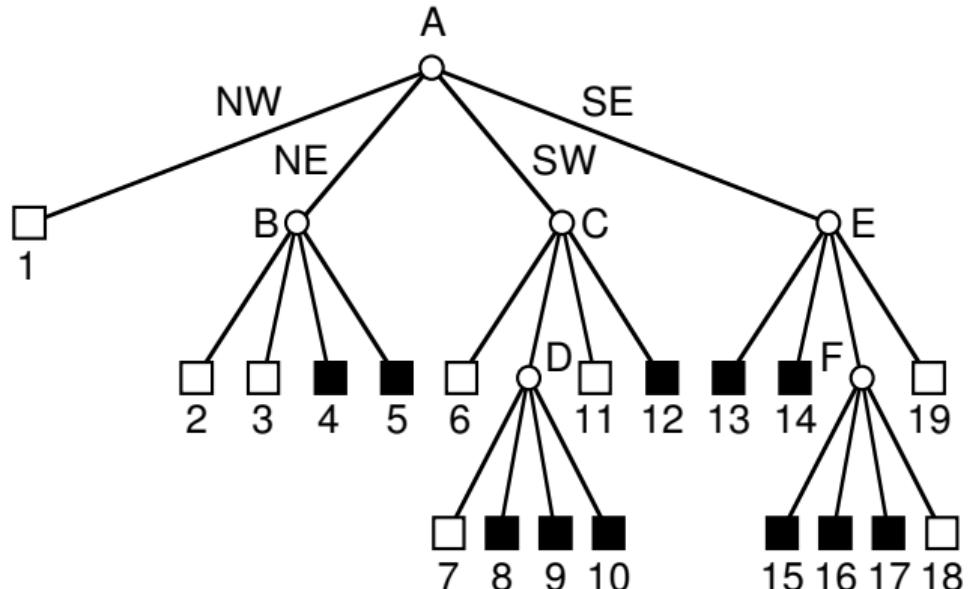
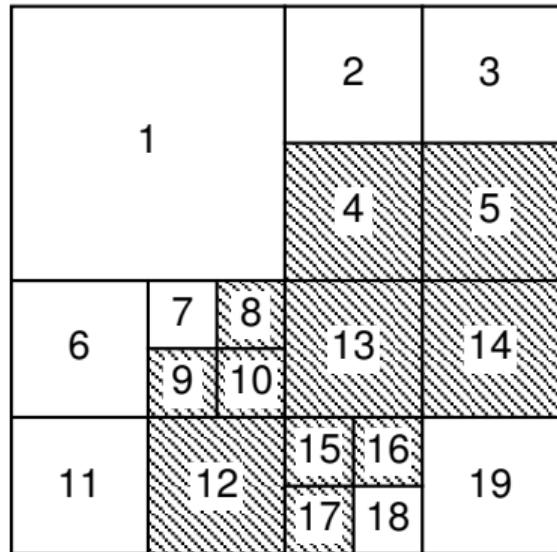


Samet

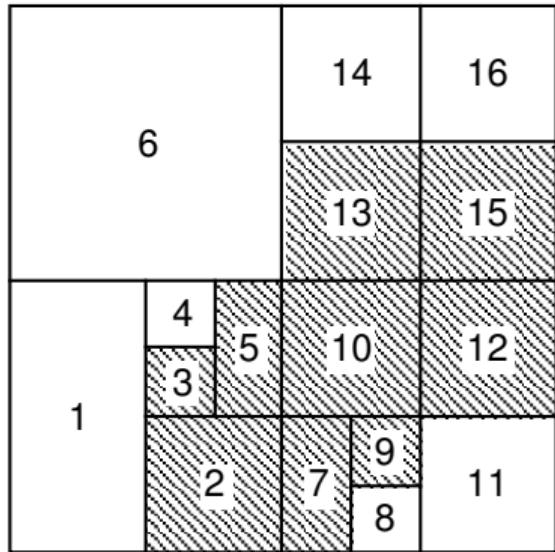


Quadtree

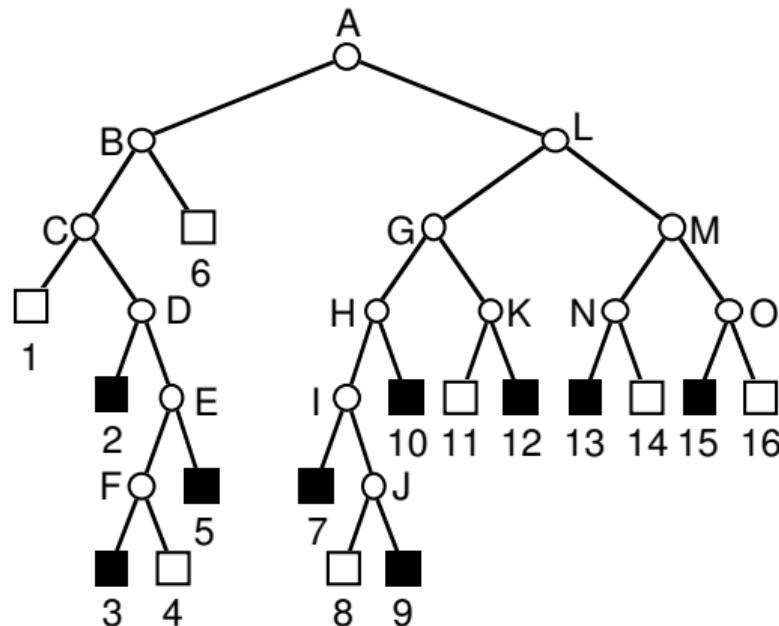
Finkel–Bentley (1974)



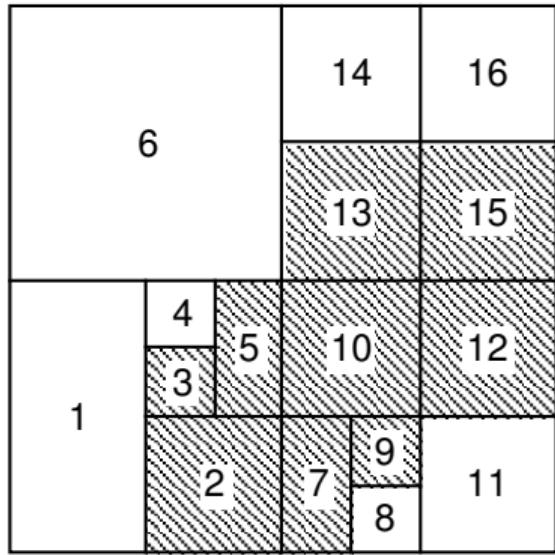
Bintree



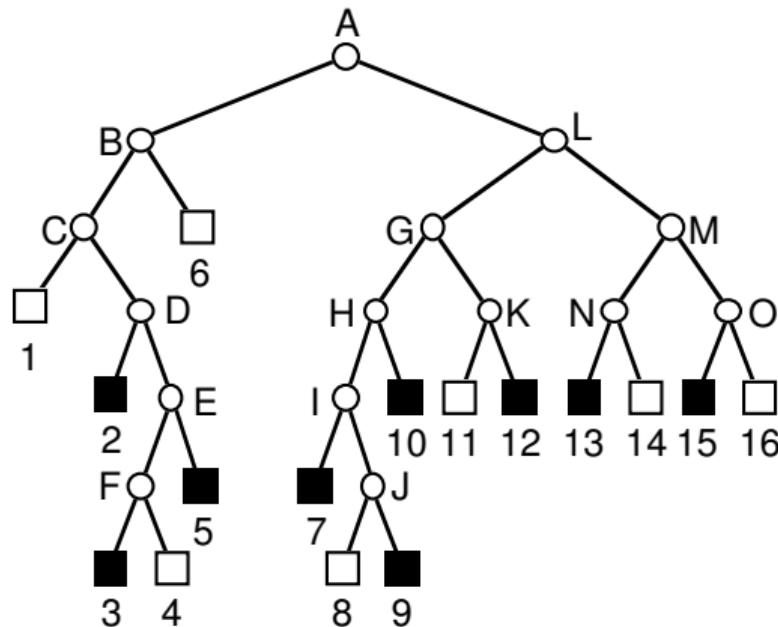
Tamminen et al. (1984)



Bintree

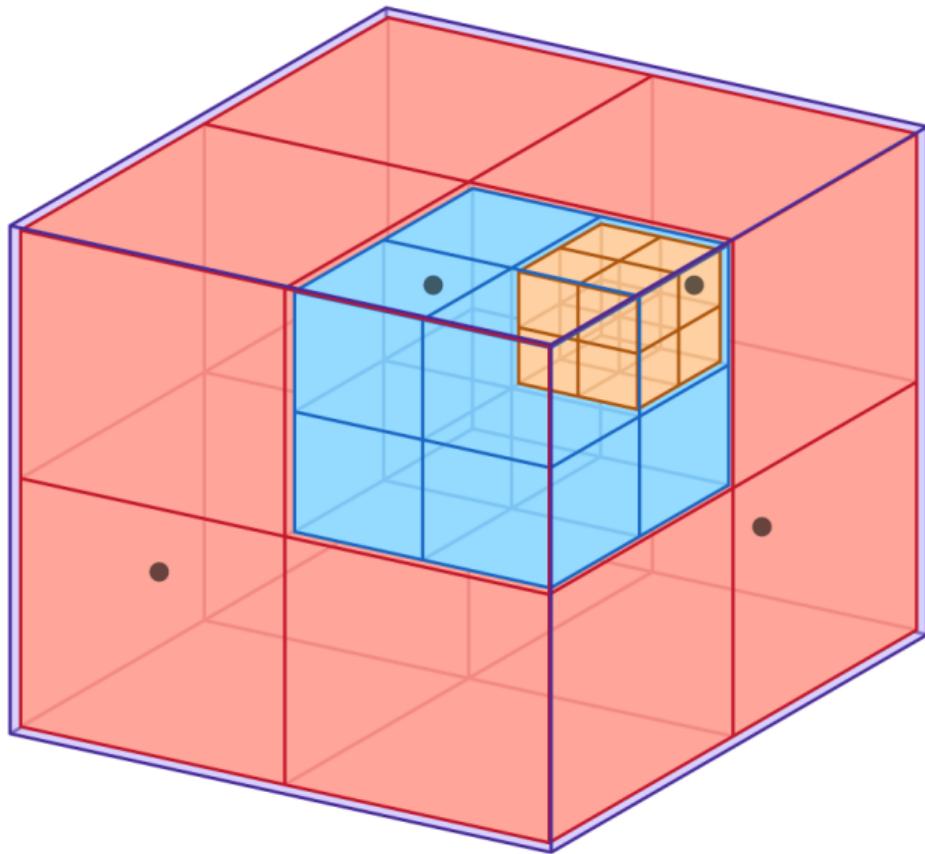


Tamminen et al. (1984)

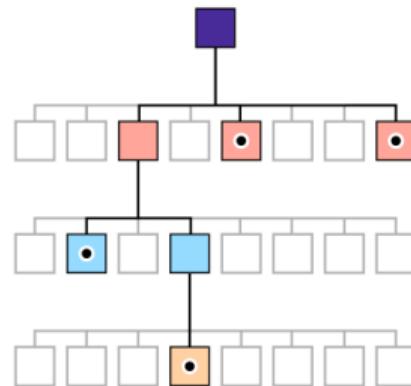


$((((B(P((PB)P)))B))(((P(BP))P)(BP))((PB)(PB))))$

Octree



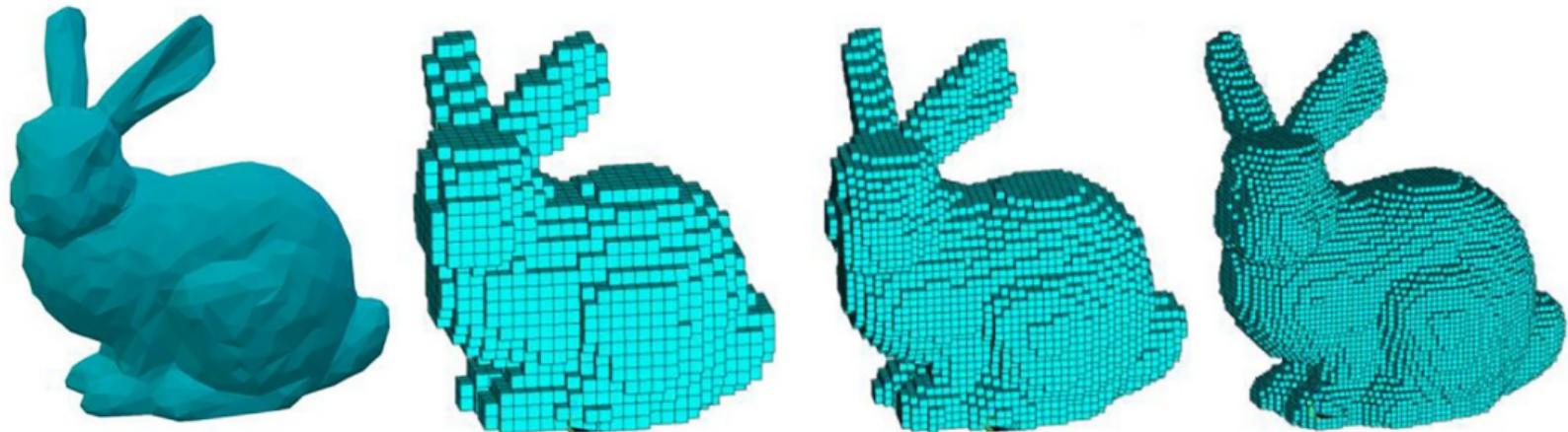
Meagher (1980)



Plasma_Node

Enumeração espacial uniforme

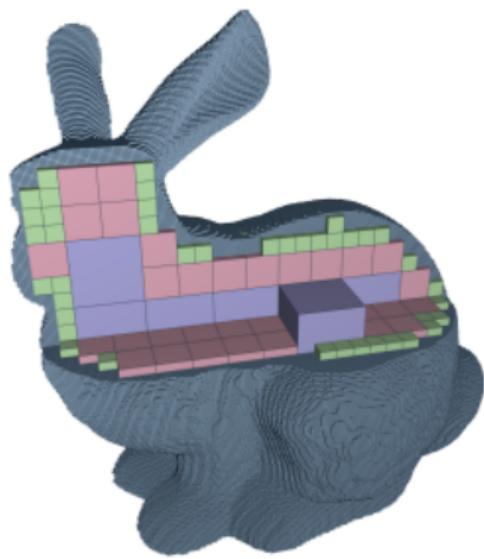
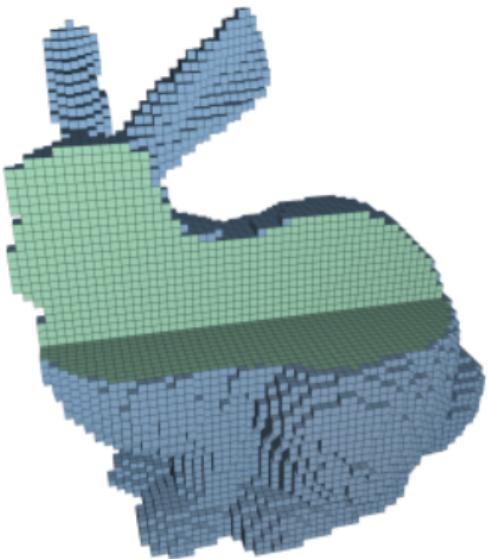
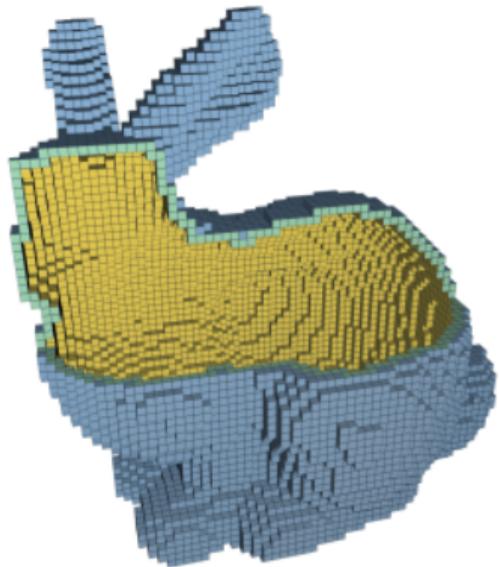
3D



Zhou et al.

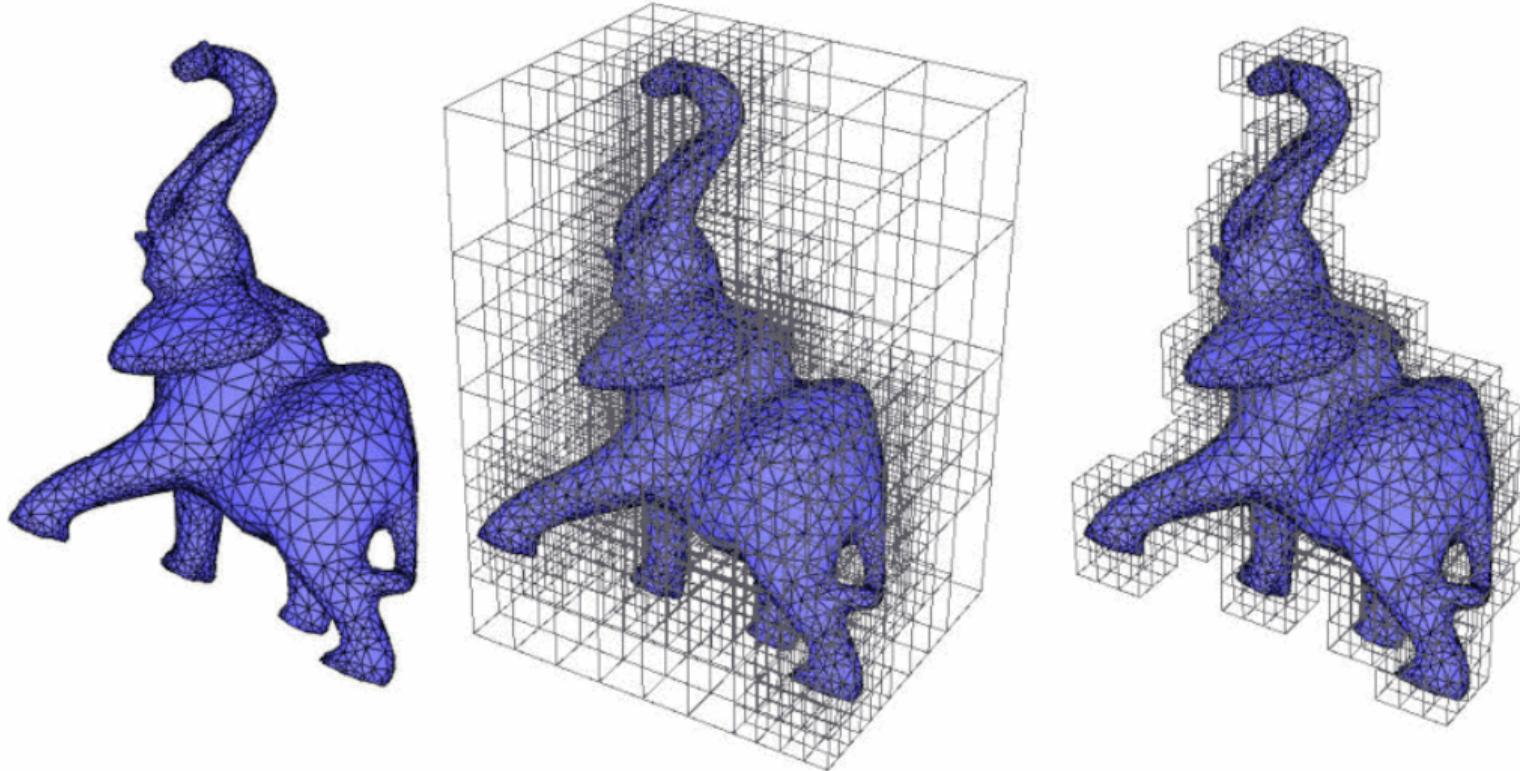
Enumeração espacial adaptativa

3D



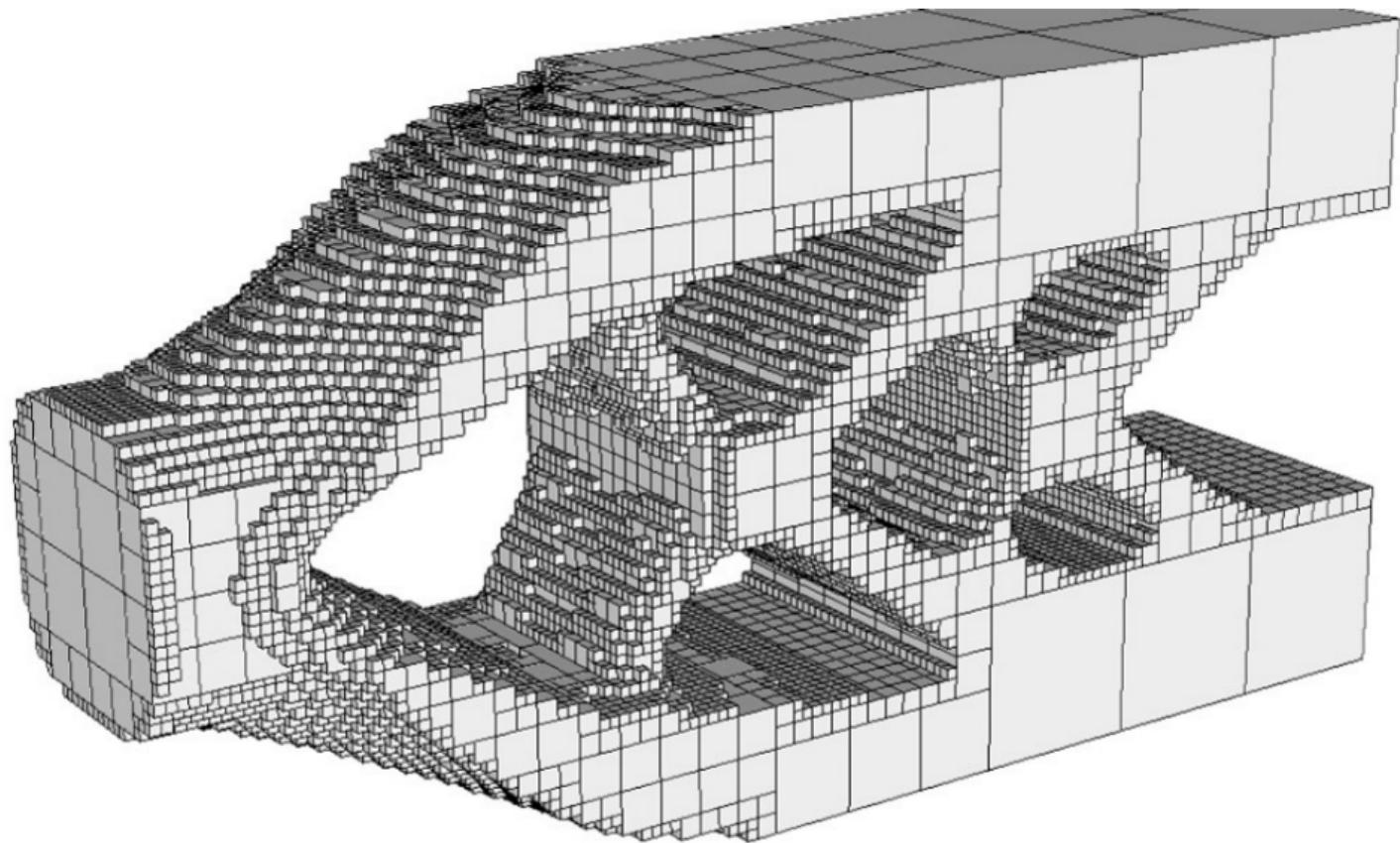
Enumeração espacial adaptativa

3D



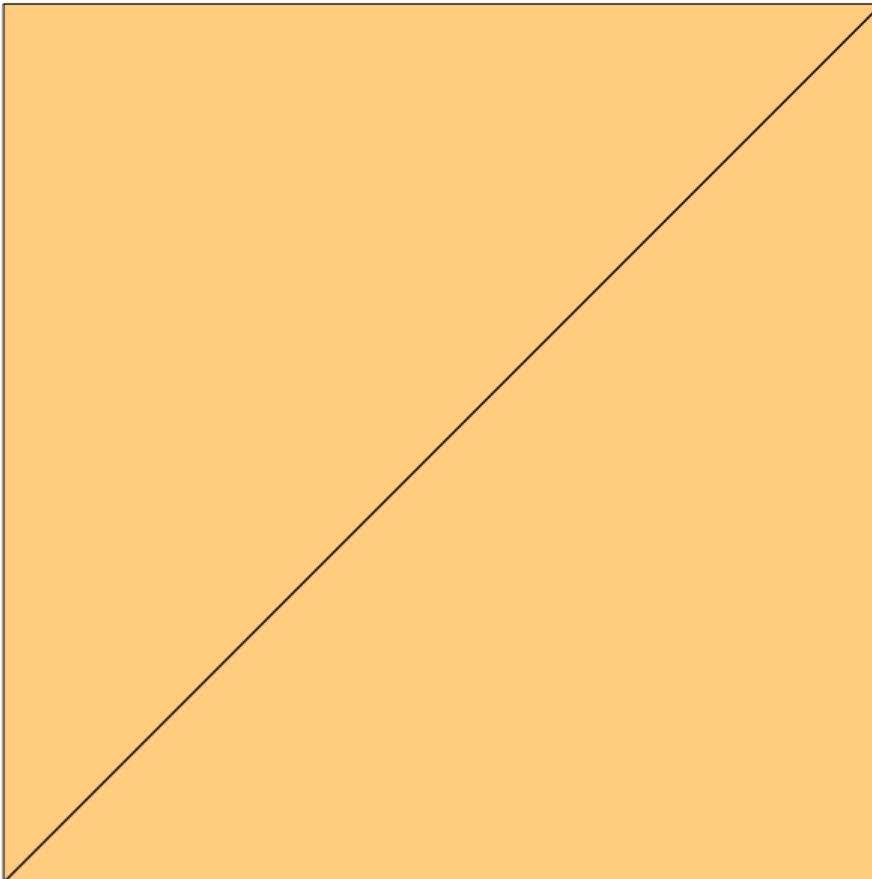
Enumeração espacial adaptativa

3D



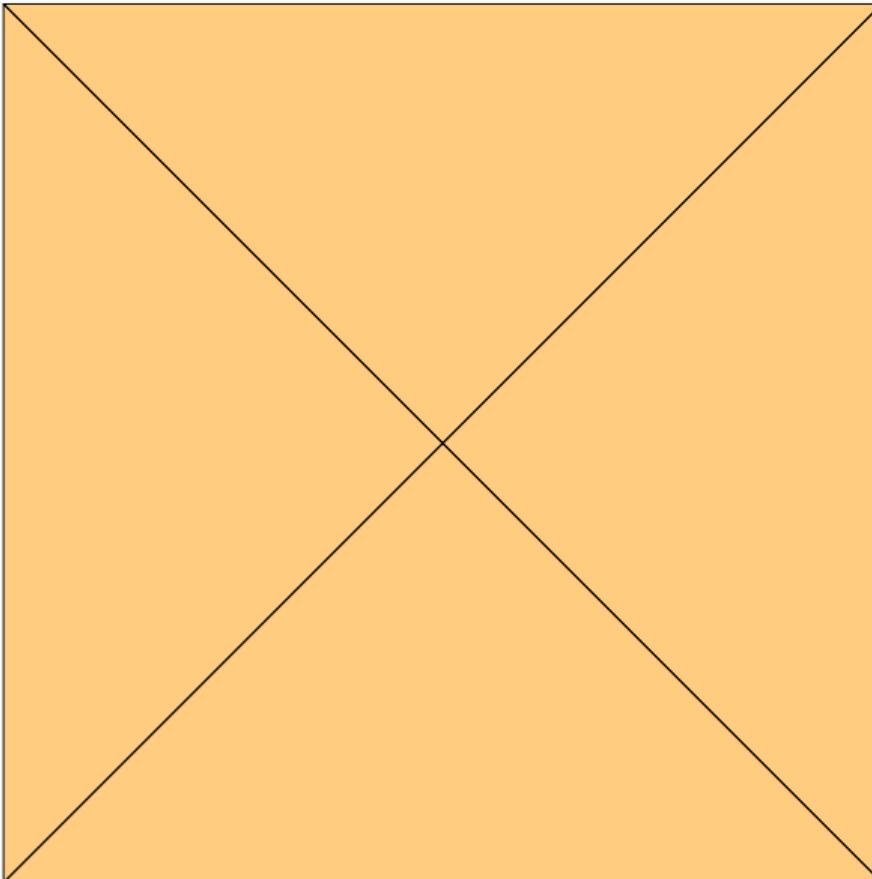
Enumeração espacial adaptativa

4-8



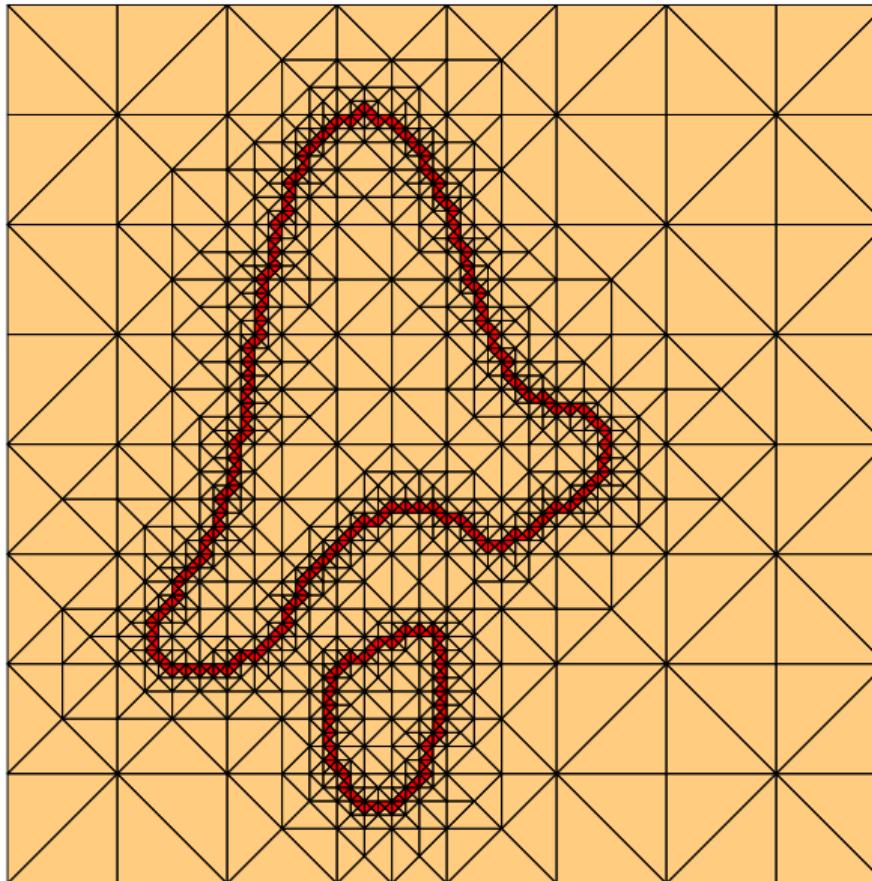
Enumeração espacial adaptativa

4-8



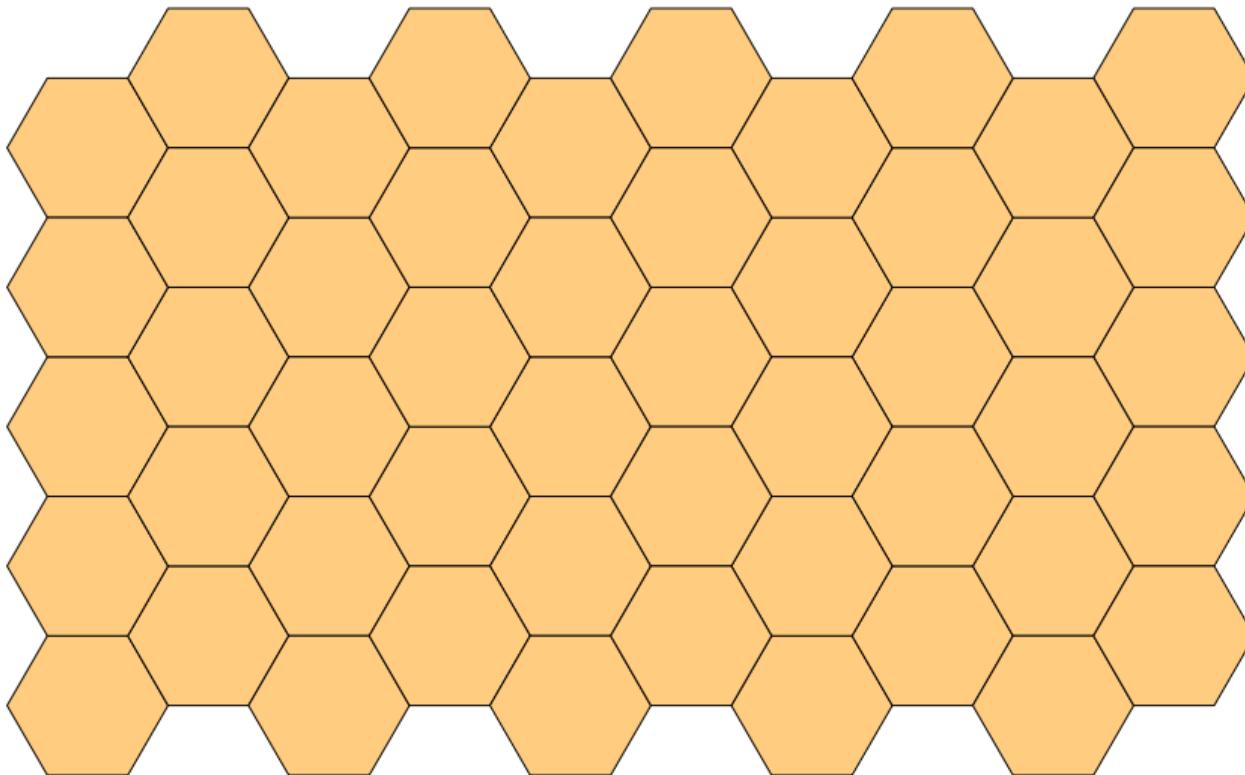
Enumeração espacial adaptativa

4-8



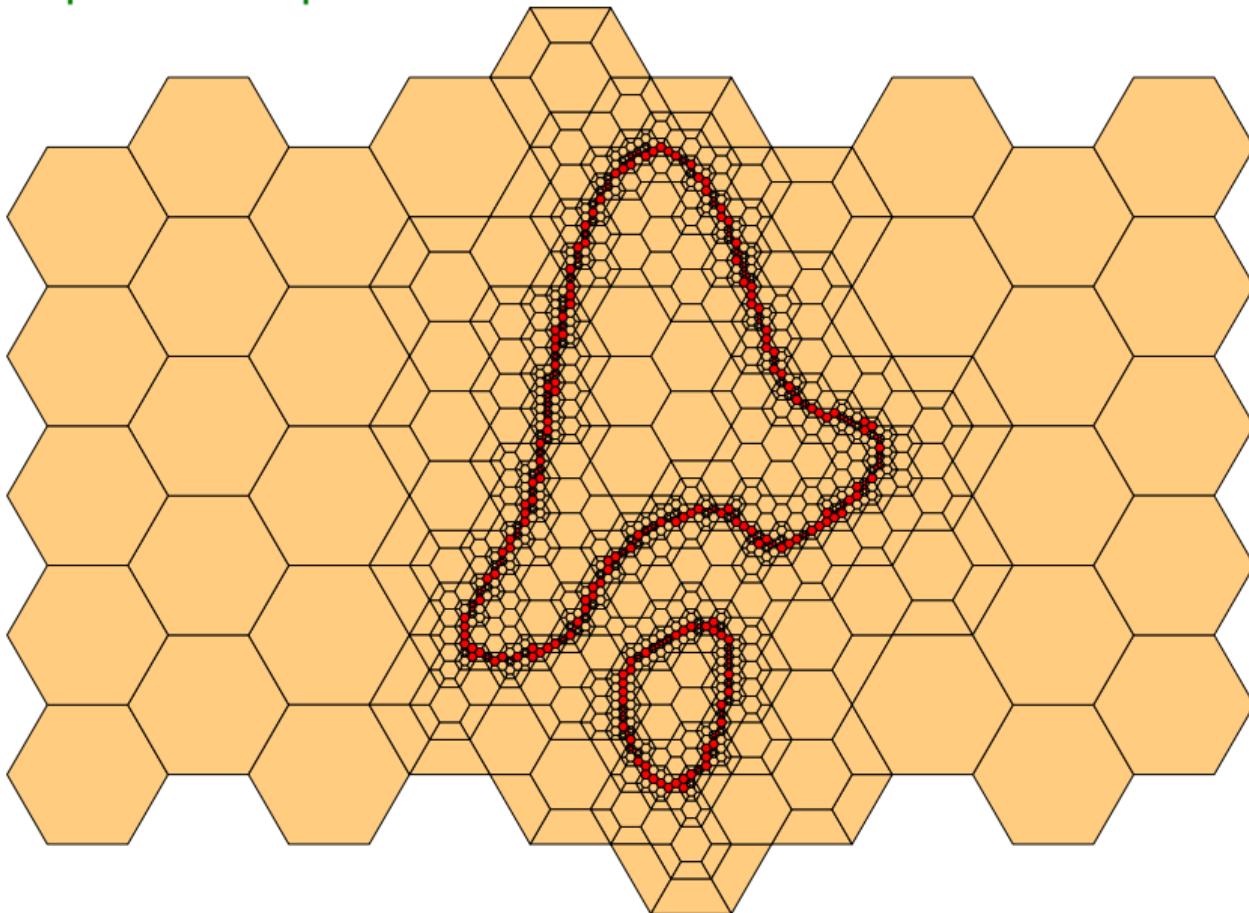
Enumeração espacial adaptativa

hexagonal



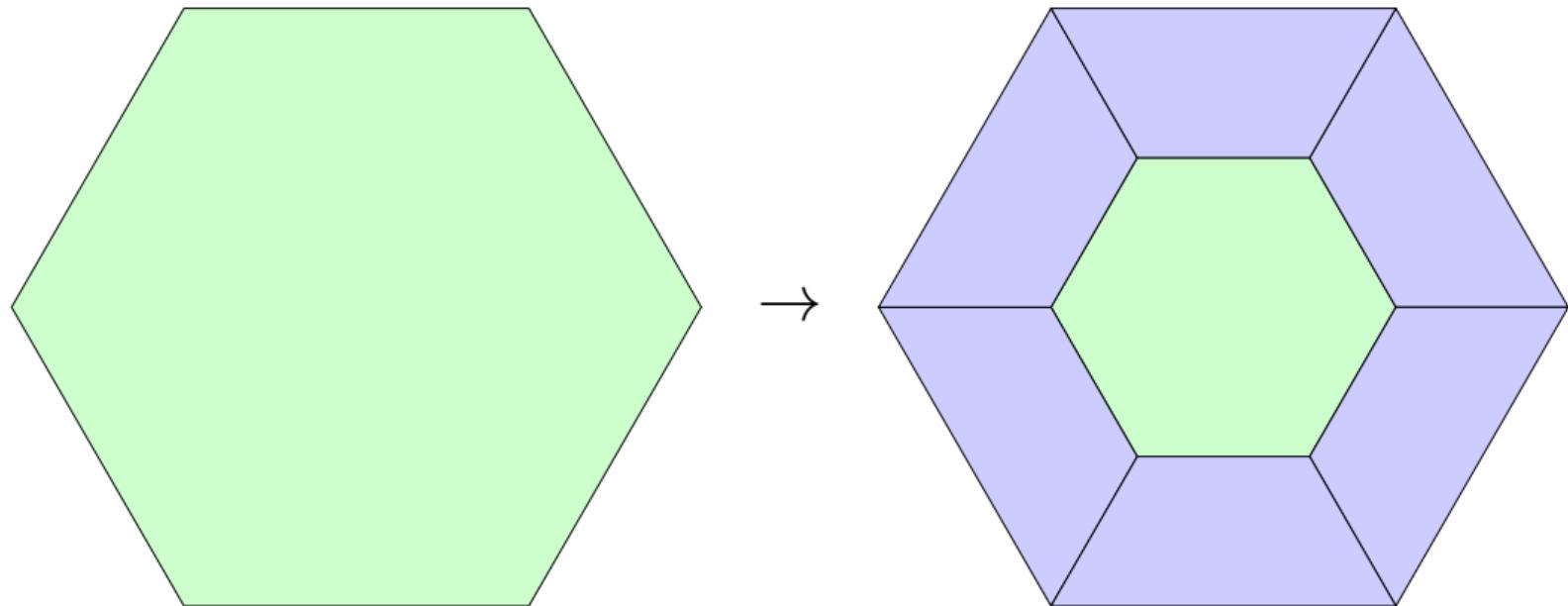
Enumeração espacial adaptativa

hexagonal



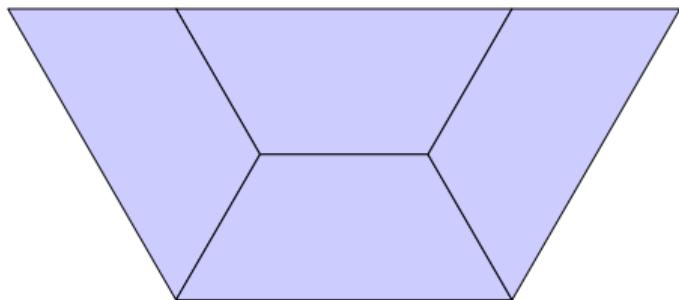
Enumeração espacial adaptativa

hexagonal



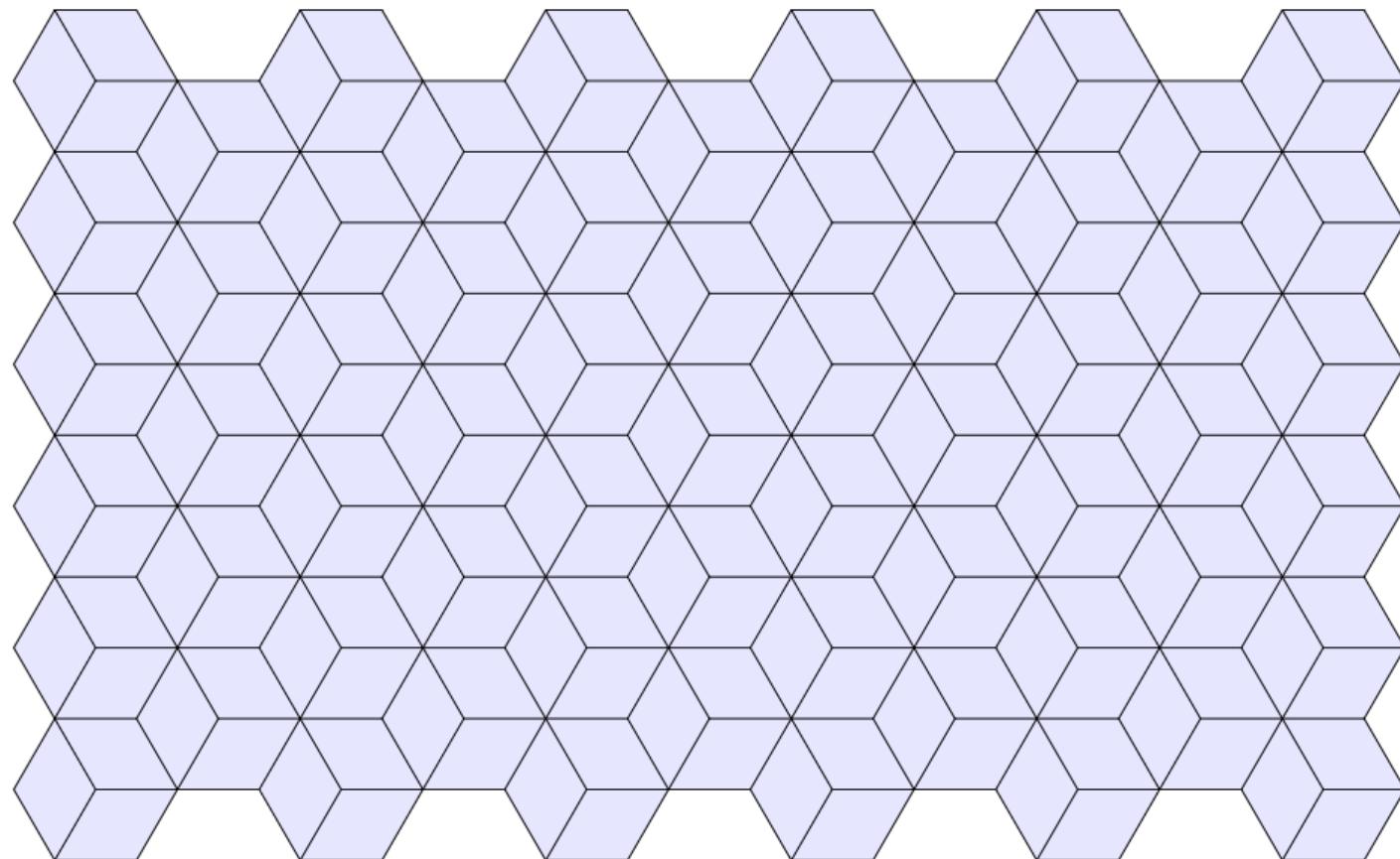
Enumeração espacial adaptativa

hexagonal



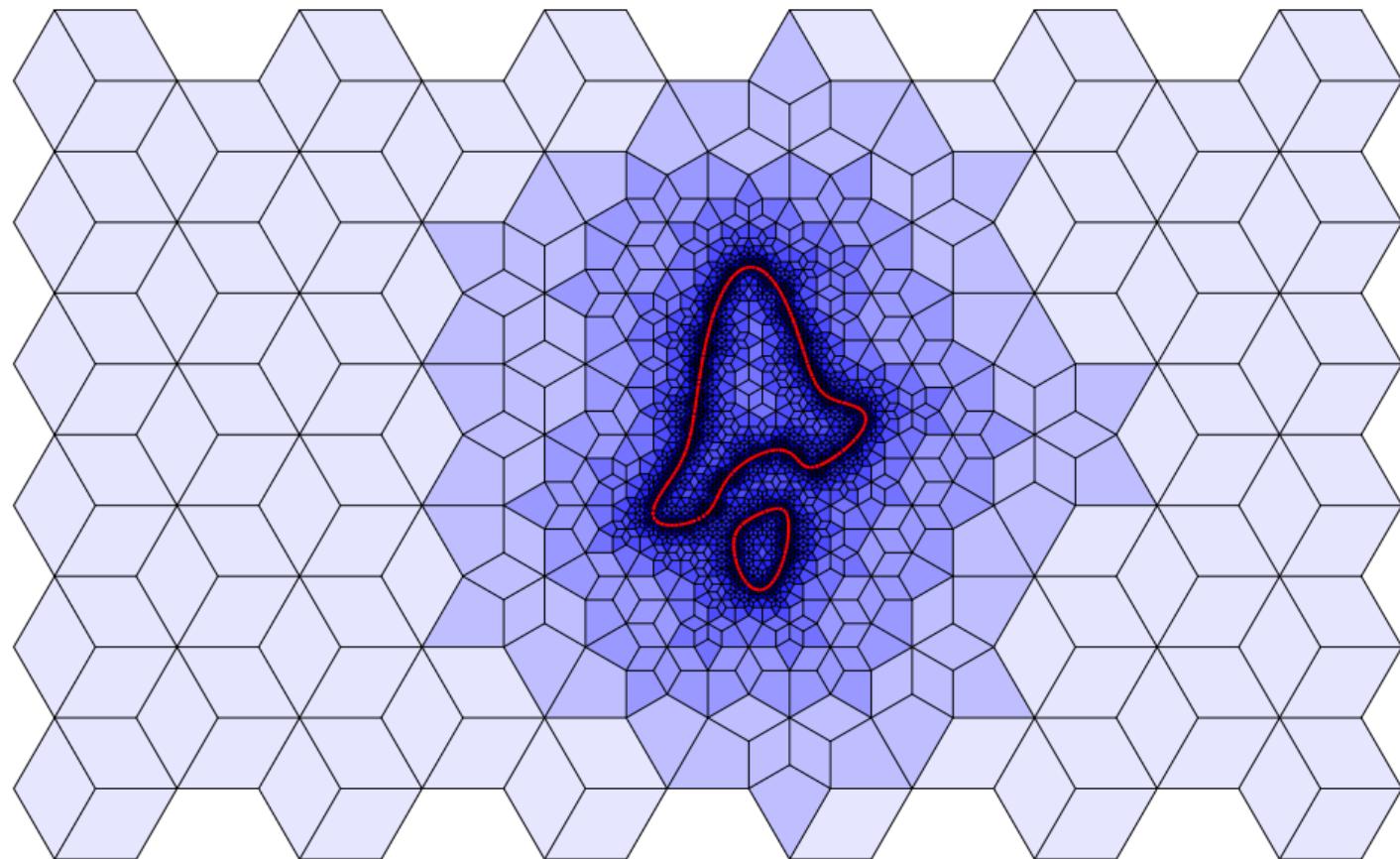
Enumeração espacial adaptativa

dk



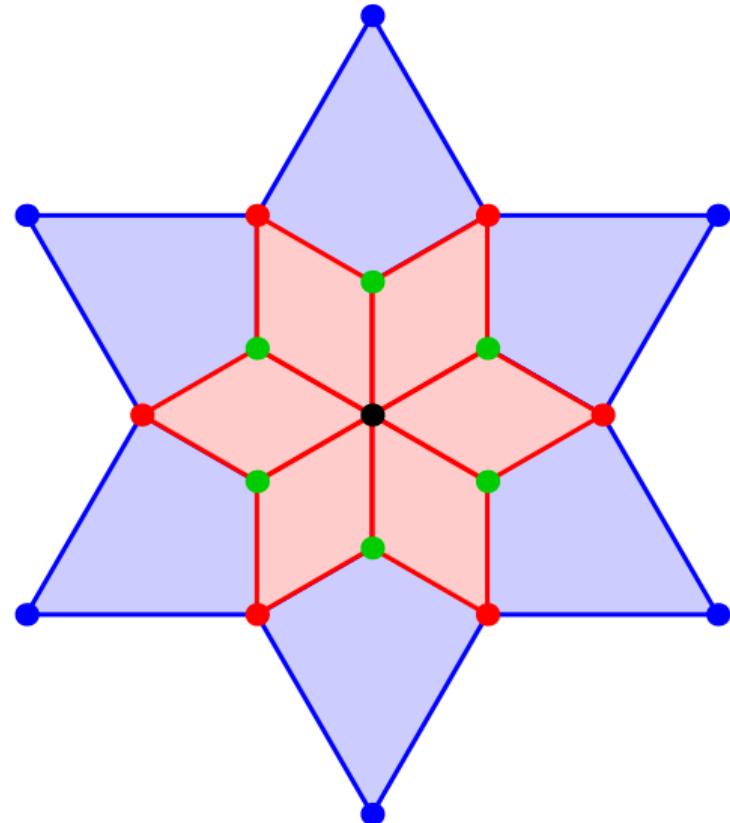
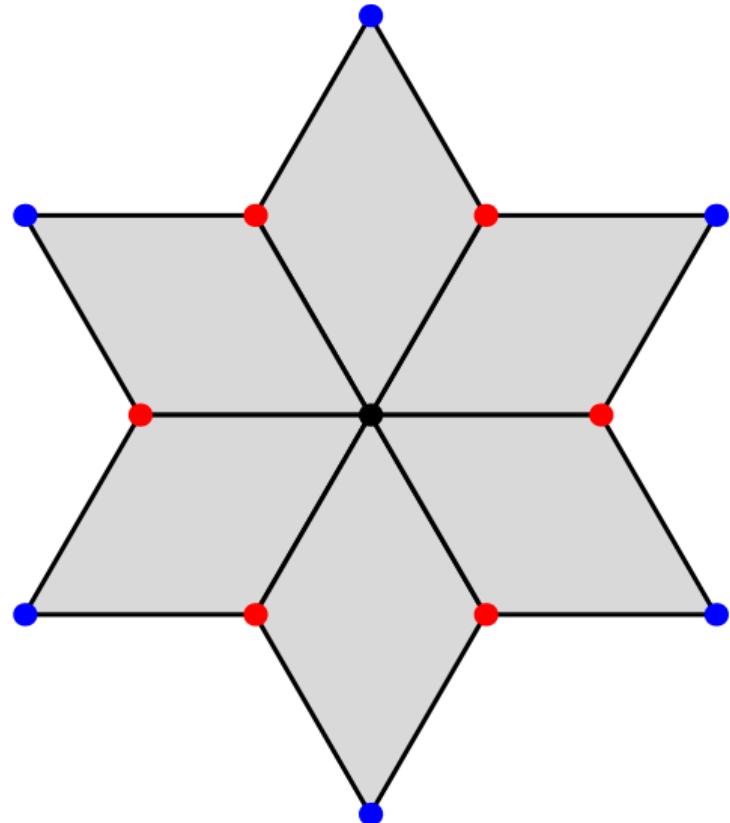
Enumeração espacial adaptativa

dk



Enumeração espacial adaptativa

dk



objetos implícitos

Região implícita

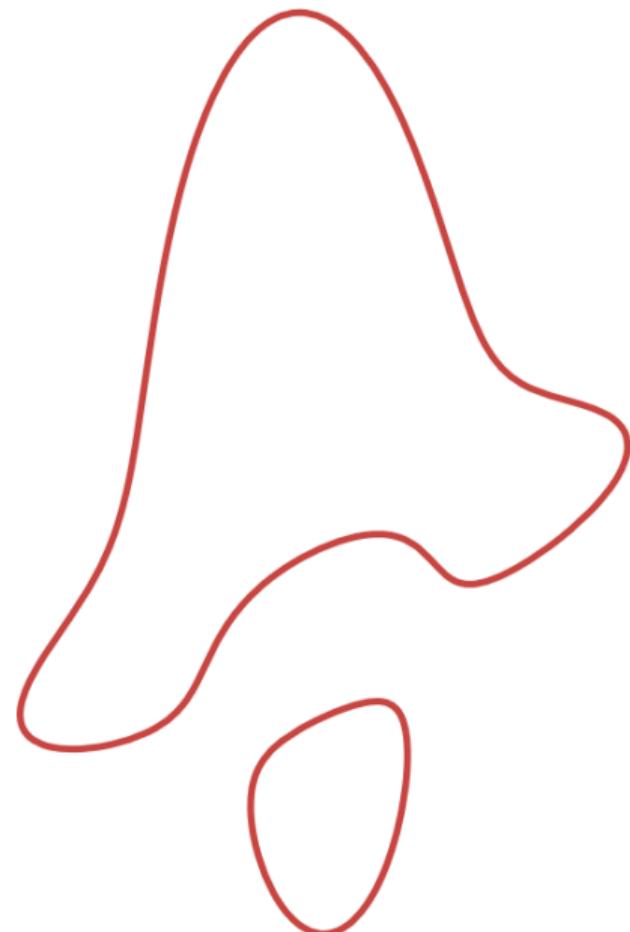
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



Região implícita

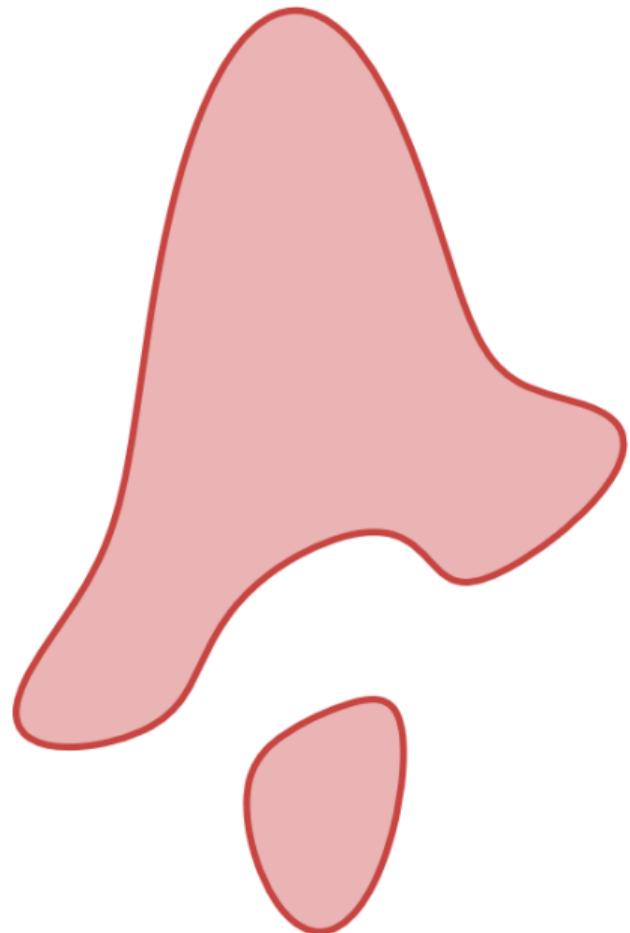
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



Região implícita

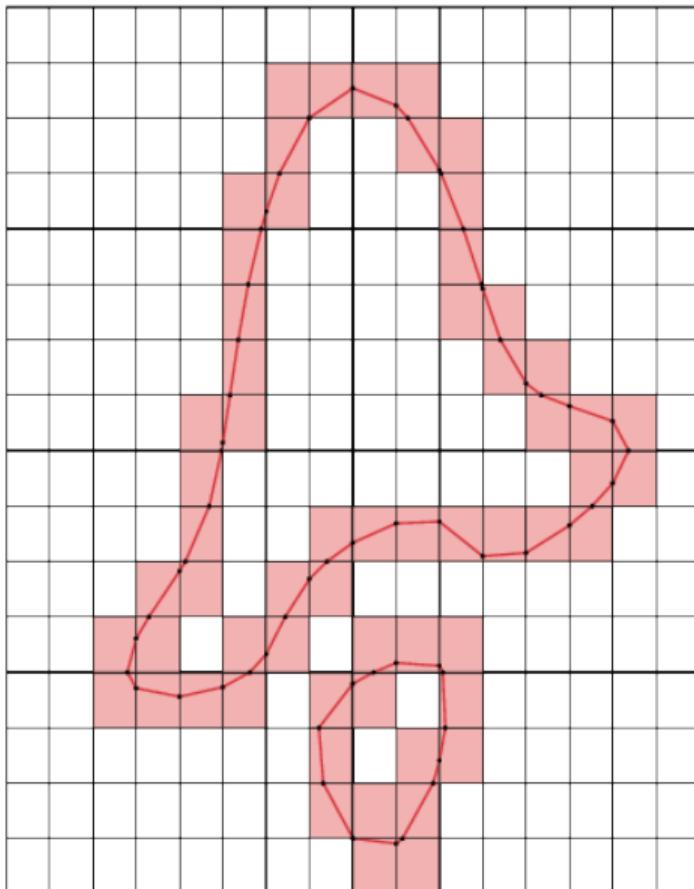
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



Região implícita

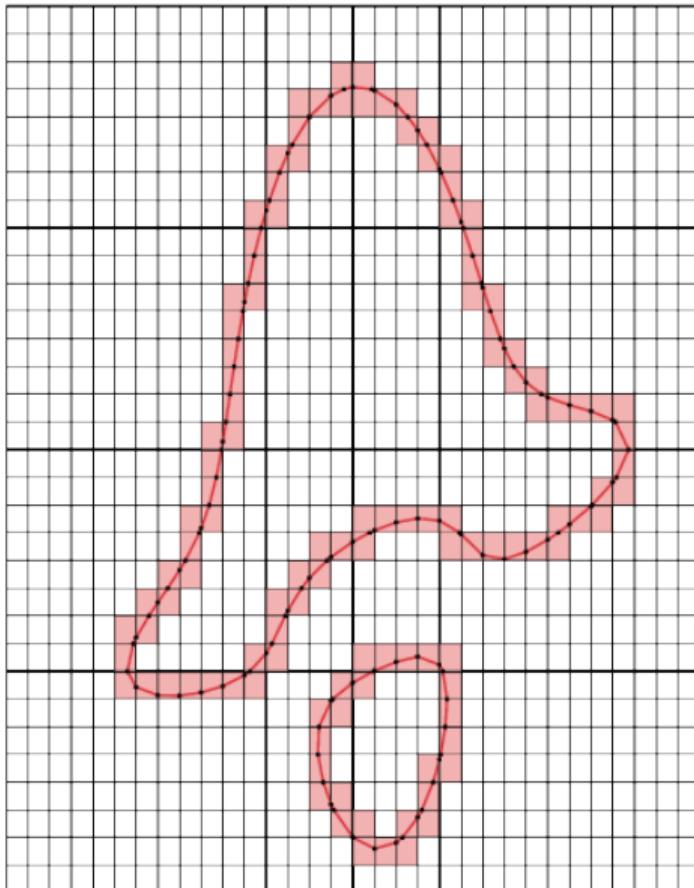
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



Região implícita

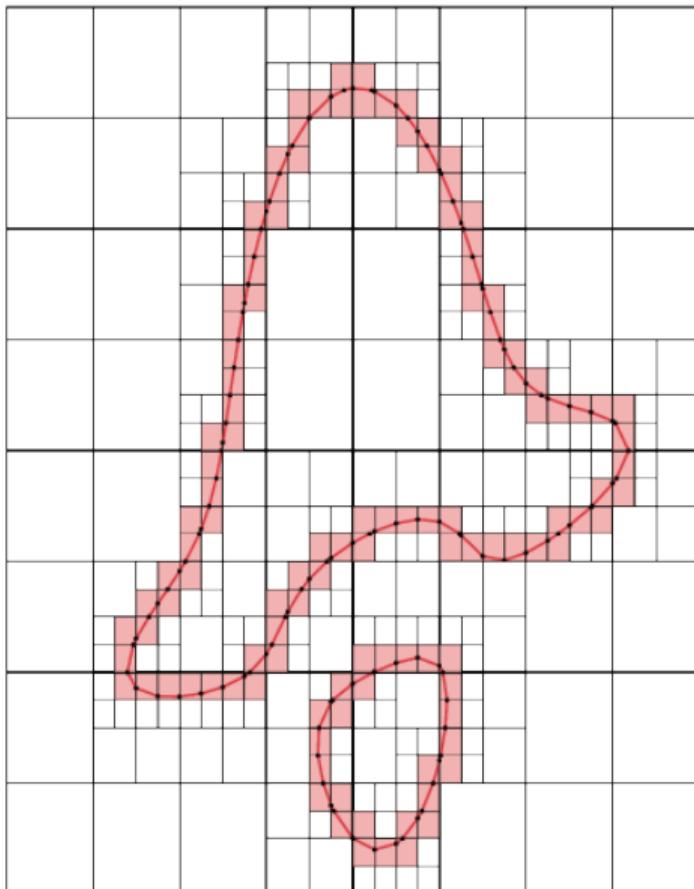
curva = fronteira da região

$$f(x, y) = 0$$

região

$$f(x, y) \leq 0$$

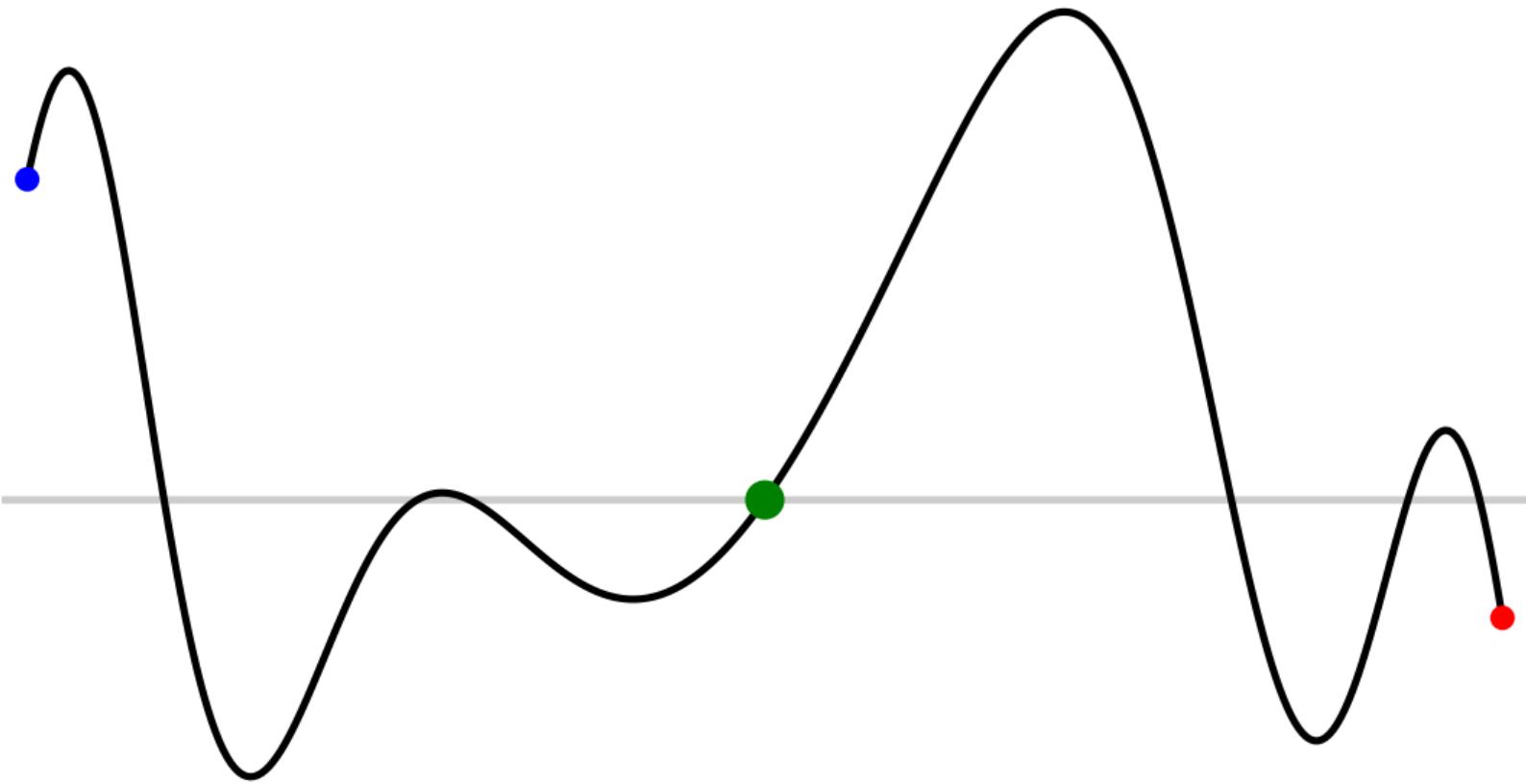
- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$ contínua
 - $[a, b] \subseteq \Omega$
 - $f(a) > 0$
 - $f(b) < 0$
- ⇒ existe $x^* \in (a, b)$ tal que $f(x^*) = 0$

Teorema do valor intermediário

Bolzano (1817)



- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$ contínua
 - $[a, b] \subseteq \Omega$
 - $f(a) > 0$
 - $f(b) < 0$
- ⇒ existe $x^* \in (a, b)$ tal que $f(x^*) = 0$

- $f: \Omega = \mathbf{R} \rightarrow \mathbf{R}$ dada por $f(x) = x^2 - 2$ é contínua
- $[0, 2] \subseteq \Omega$
- $f(0) < 0$
- $f(2) > 0$

\implies existe $r \in (0, 2)$ tal que $f(r) = 0$, isto é, $r^2 = 2$

$$r = \sqrt{2}$$

- $f: \Omega \subseteq \mathbf{R} \rightarrow \mathbf{R}$ contínua
 - $[a, b] \subseteq \Omega$
 - $f(a) > 0$
 - $f(b) < 0$
- ⇒ existe $x^* \in (a, b)$ tal que $f(x^*) = 0$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b$ $f(a) > 0, \quad f(b) < 0$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se $f(m) = 0: \quad x^* = m \quad \text{fim}$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
- $m = \frac{a_n + b_n}{2}$
- se $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
- $f(a_n) > 0, \quad f(b_n) < 0 \quad \text{invariante}$

Teorema do valor intermediário – prova construtiva: método da bisseção

- $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$
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- se $f(m) = 0: \quad x^* = m \quad \text{fim}$
- se $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$
- se $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$
- $f(a_n) > 0, \quad f(b_n) < 0 \quad \text{invariante}$
- $a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1 \leq b_0 = b$

Teorema do valor intermediário – prova construtiva: método da bisseção

■ $a_0 = a, \quad b_0 = b \quad f(a) > 0, \quad f(b) < 0$

■ $m = \frac{a_n + b_n}{2}$

■ se $f(m) = 0: \quad x^* = m \quad \text{fim}$

■ se $f(m) > 0: \quad a_{n+1} = m, \quad b_{n+1} = b_n \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2}$

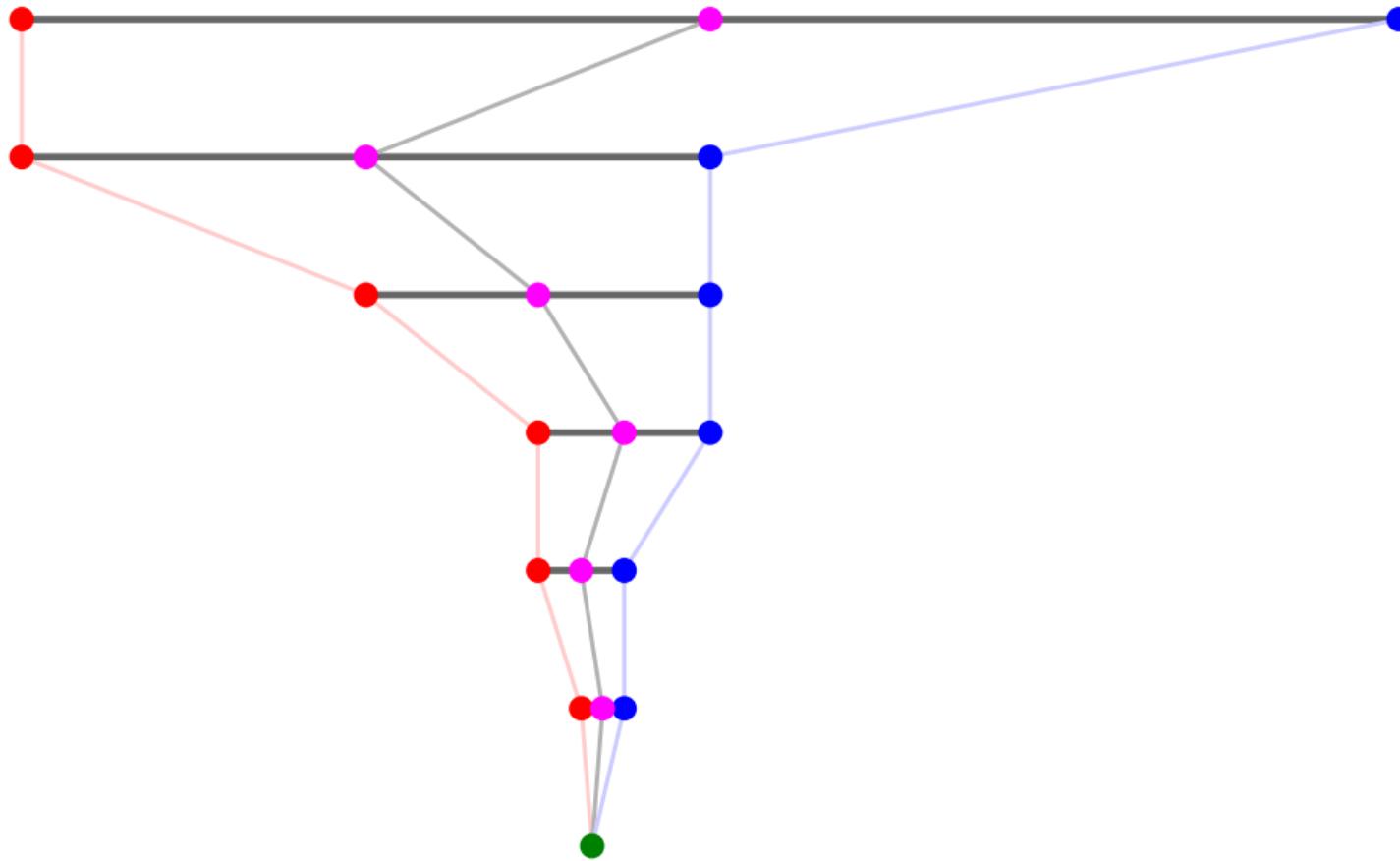
■ se $f(m) < 0: \quad a_{n+1} = a_n, \quad b_{n+1} = m \quad [a_{n+1}, b_{n+1}] \subseteq [a_n, b_n]$

■ $f(a_n) > 0, \quad f(b_n) < 0 \quad \text{invariante}$

■ $a = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1 \leq b_0 = b$

$\Rightarrow x^* = \lim a_n = \lim b_n, \quad f(x^*) = 0 \quad b_n - a_n = \frac{b - a}{2^n} \rightarrow 0$

Teorema do valor intermediário – método da bisseção para $\sqrt{2}$



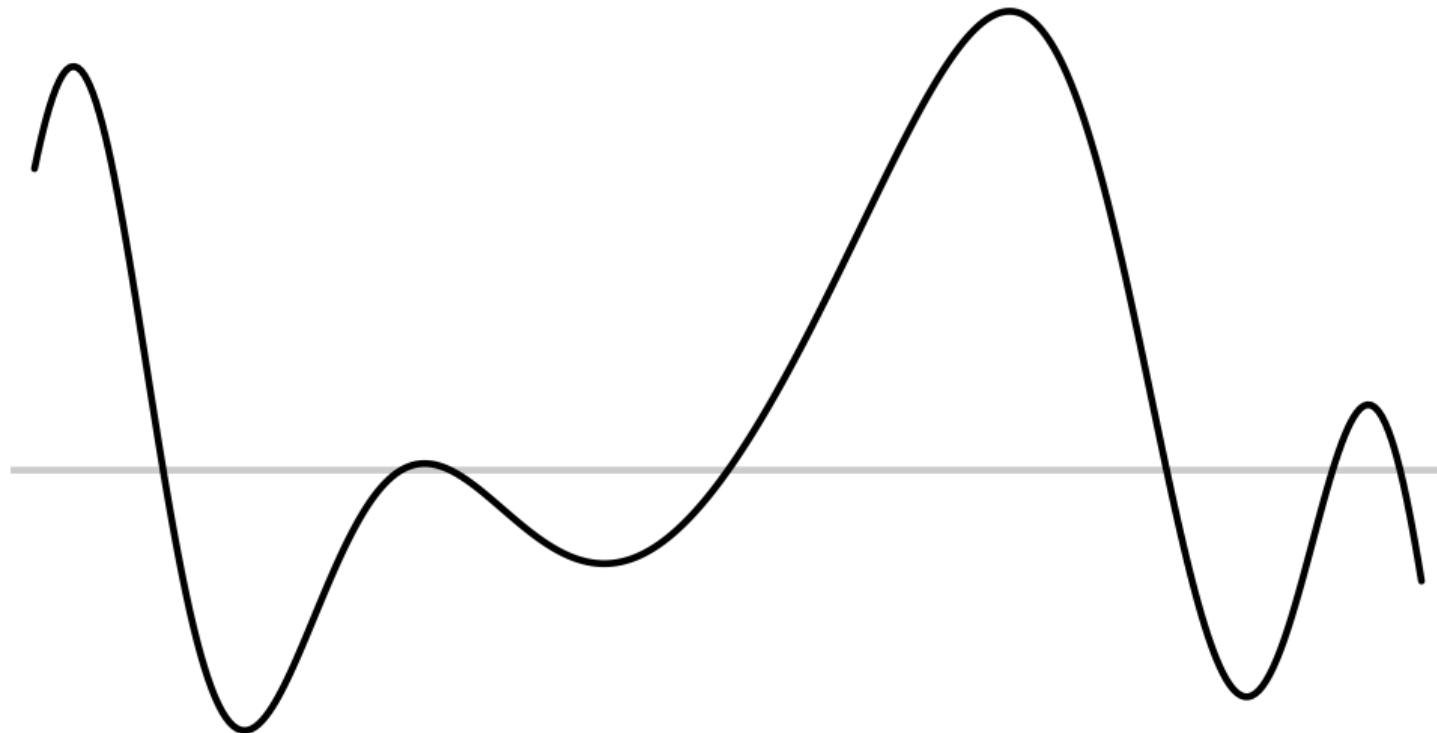
Método da bisseção

```
function bisect(f,a,b)
    local c=(a+b)/2
    if abs(a-b)<delta or c==a or c==b then
        return c
    end
    if f(c)>0 then
        return bisect(f,a,c)
    else
        return bisect(f,c,b)
    end
end
```

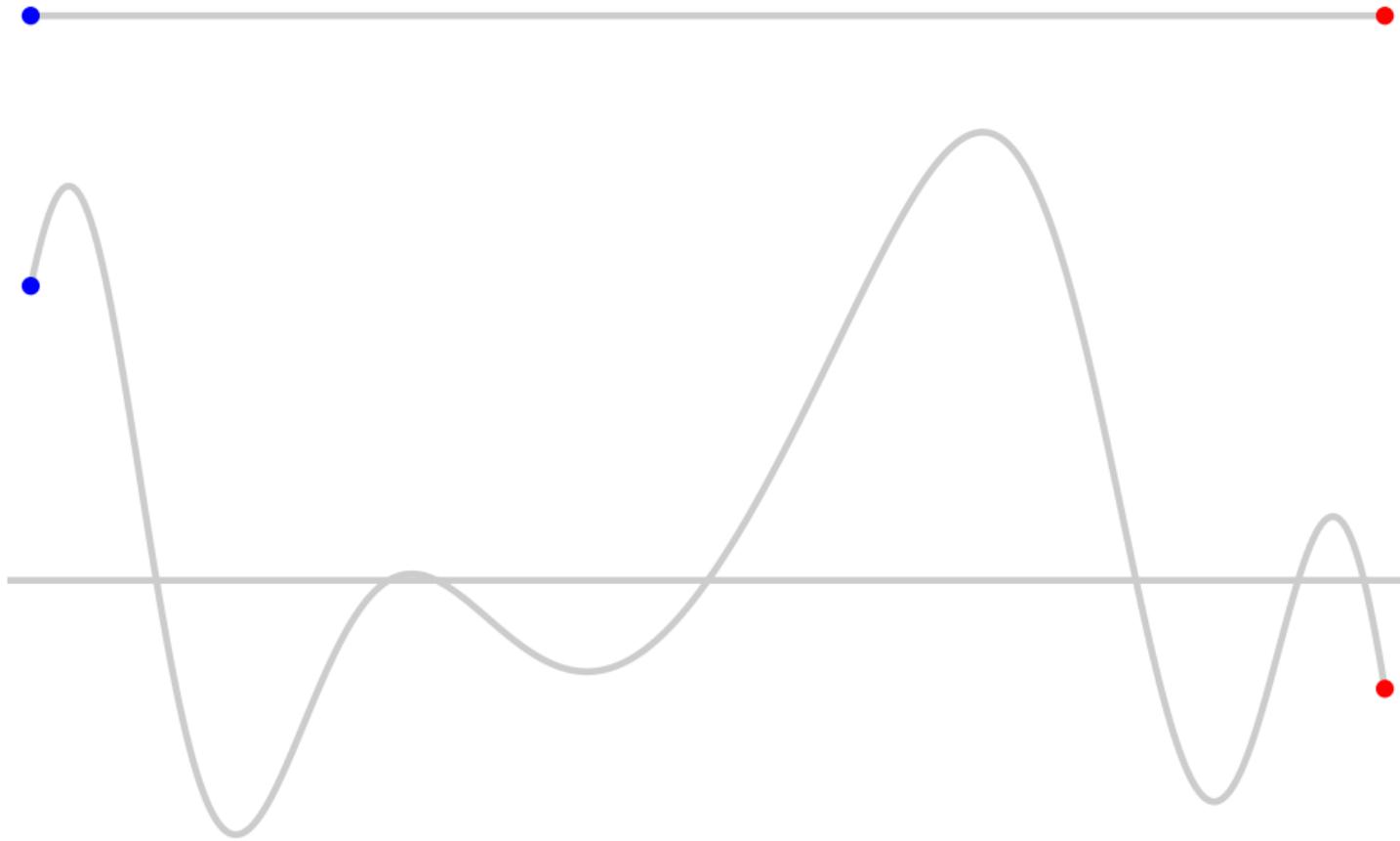
Método da bisseção

```
function bisect(f,a,b)
    while true do
        local c=(a+b)/2
        if abs(a-b)<delta or c==a or c==b then
            return c
        end
        if f(c)>0 then
            b=c
        else
            a=c
        end
    end
end
```

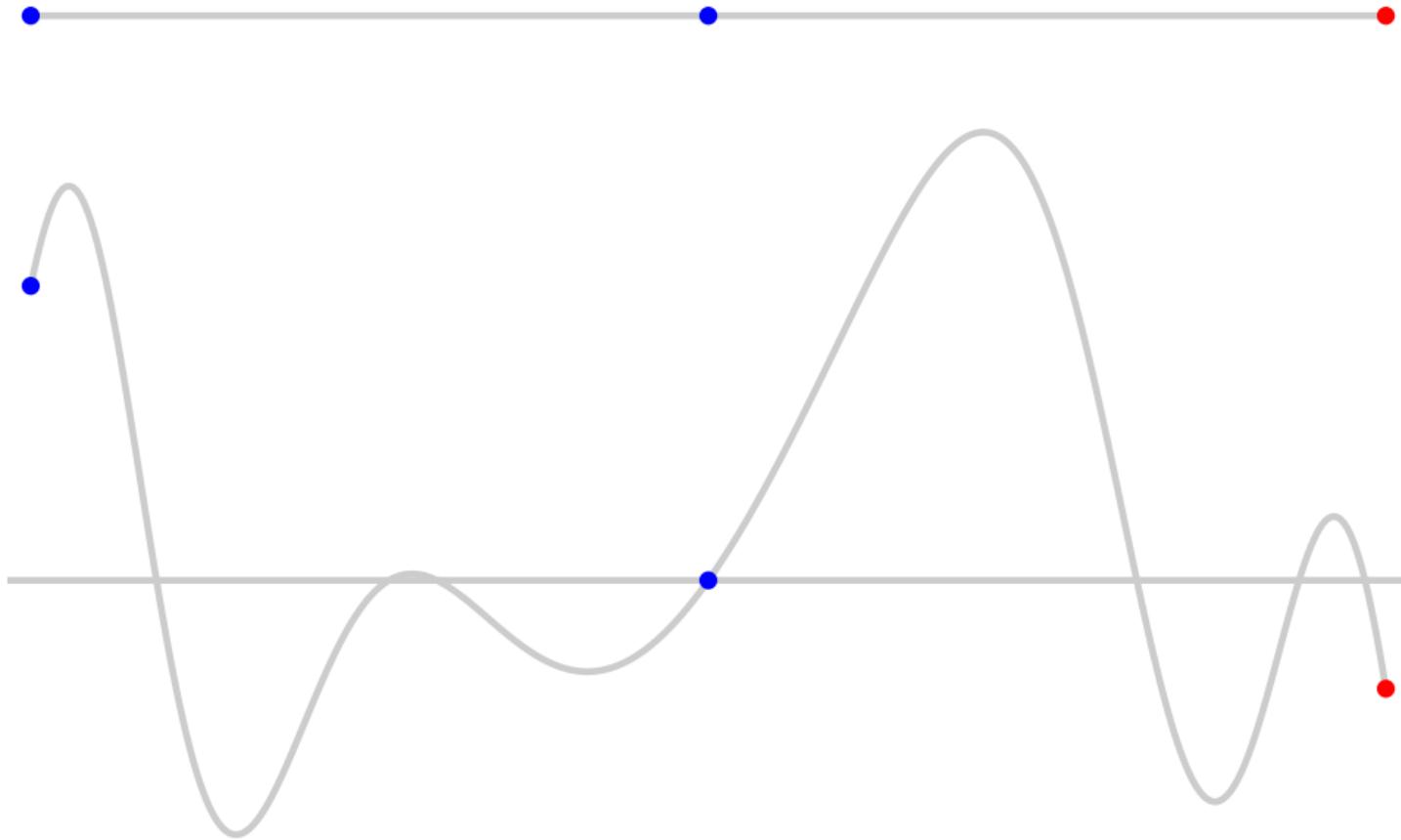
Amostragem em grade



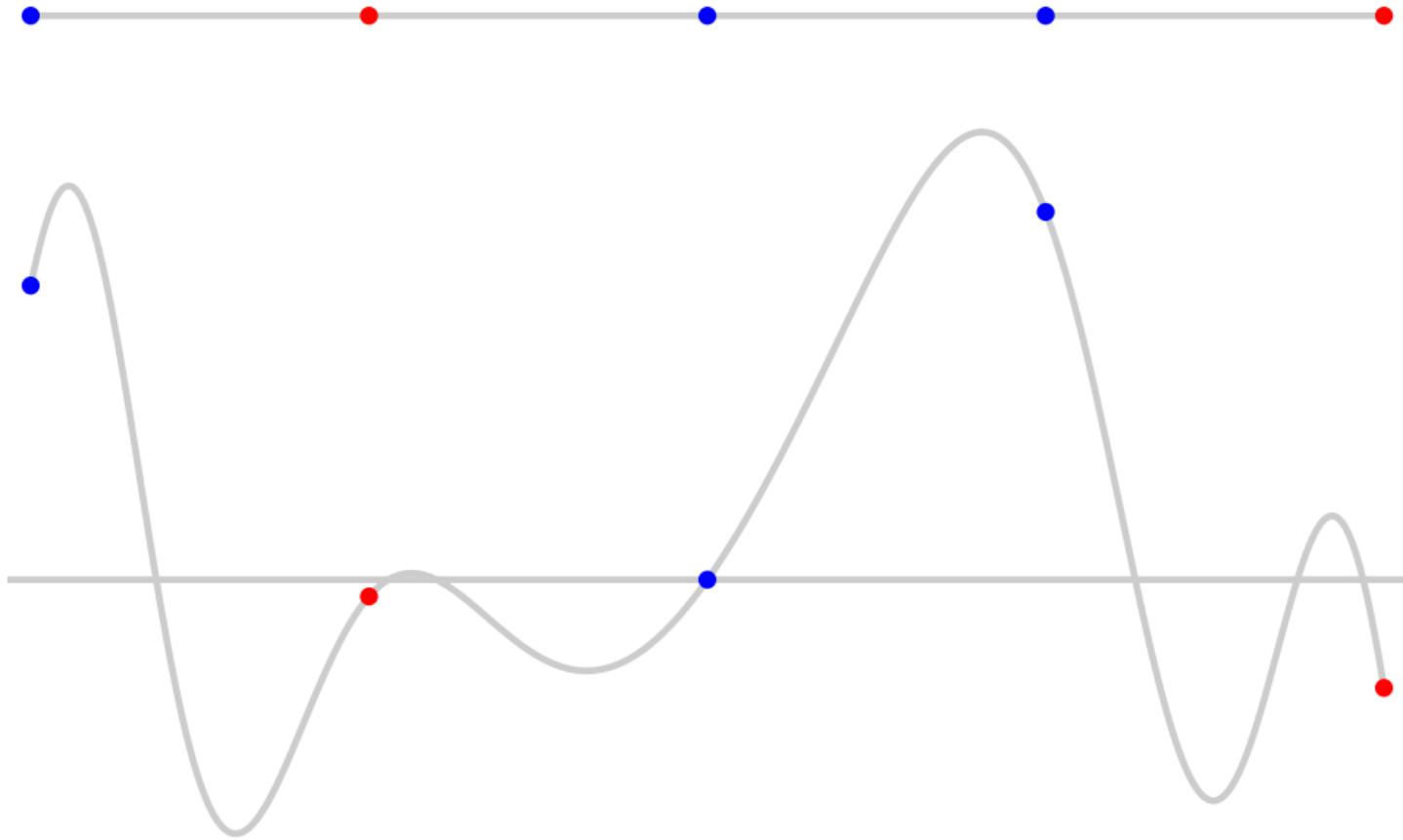
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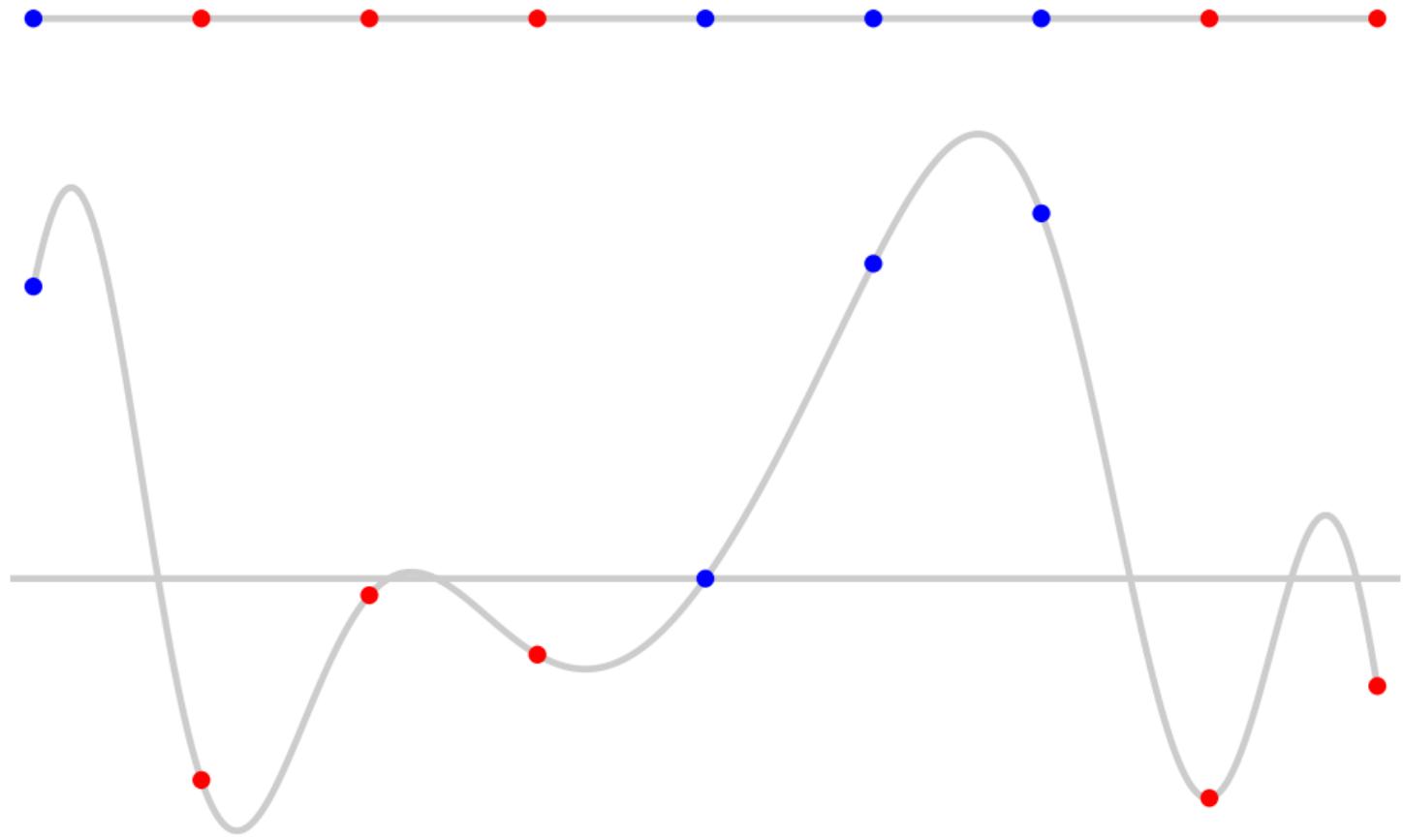
Amostragem em grade



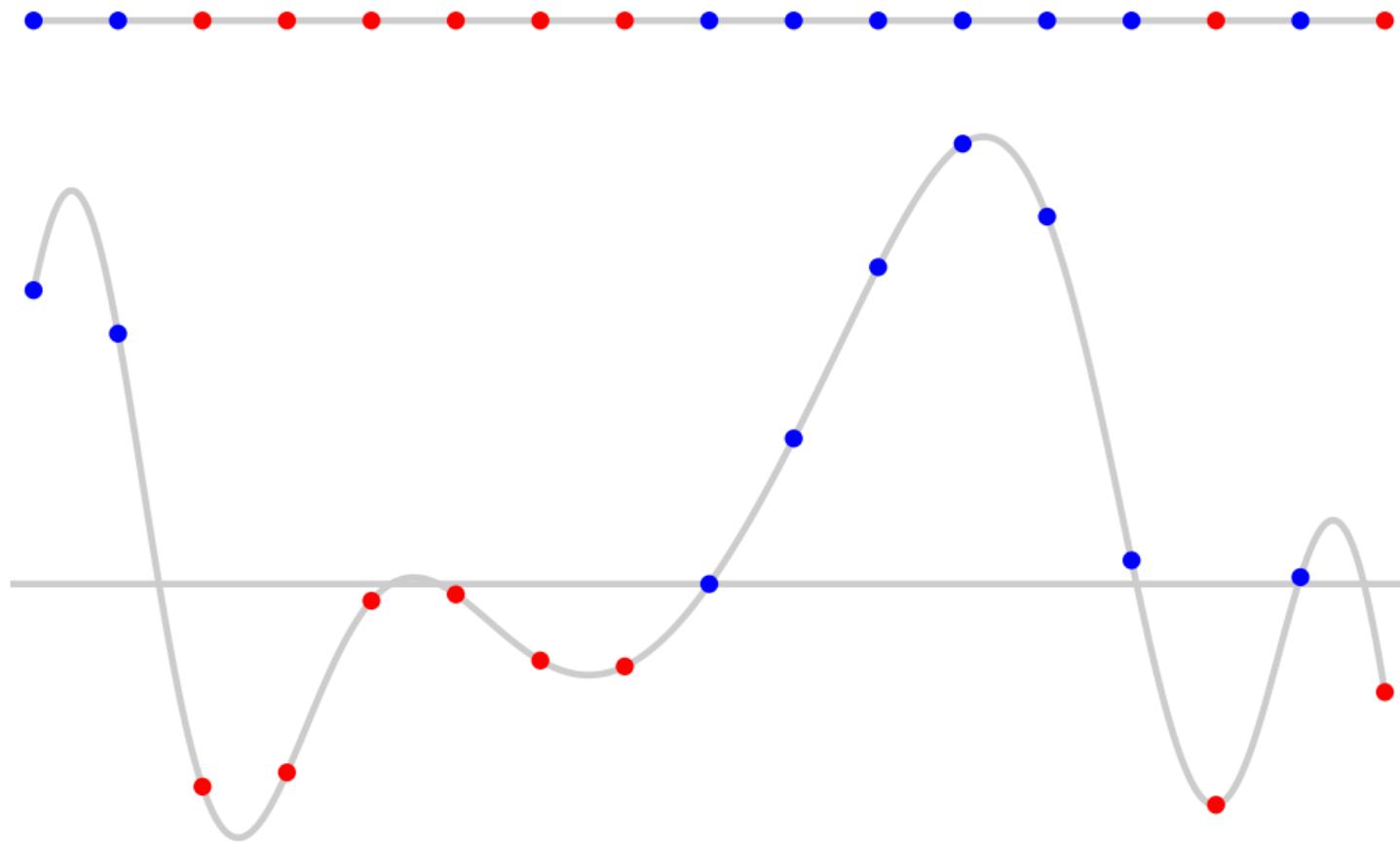
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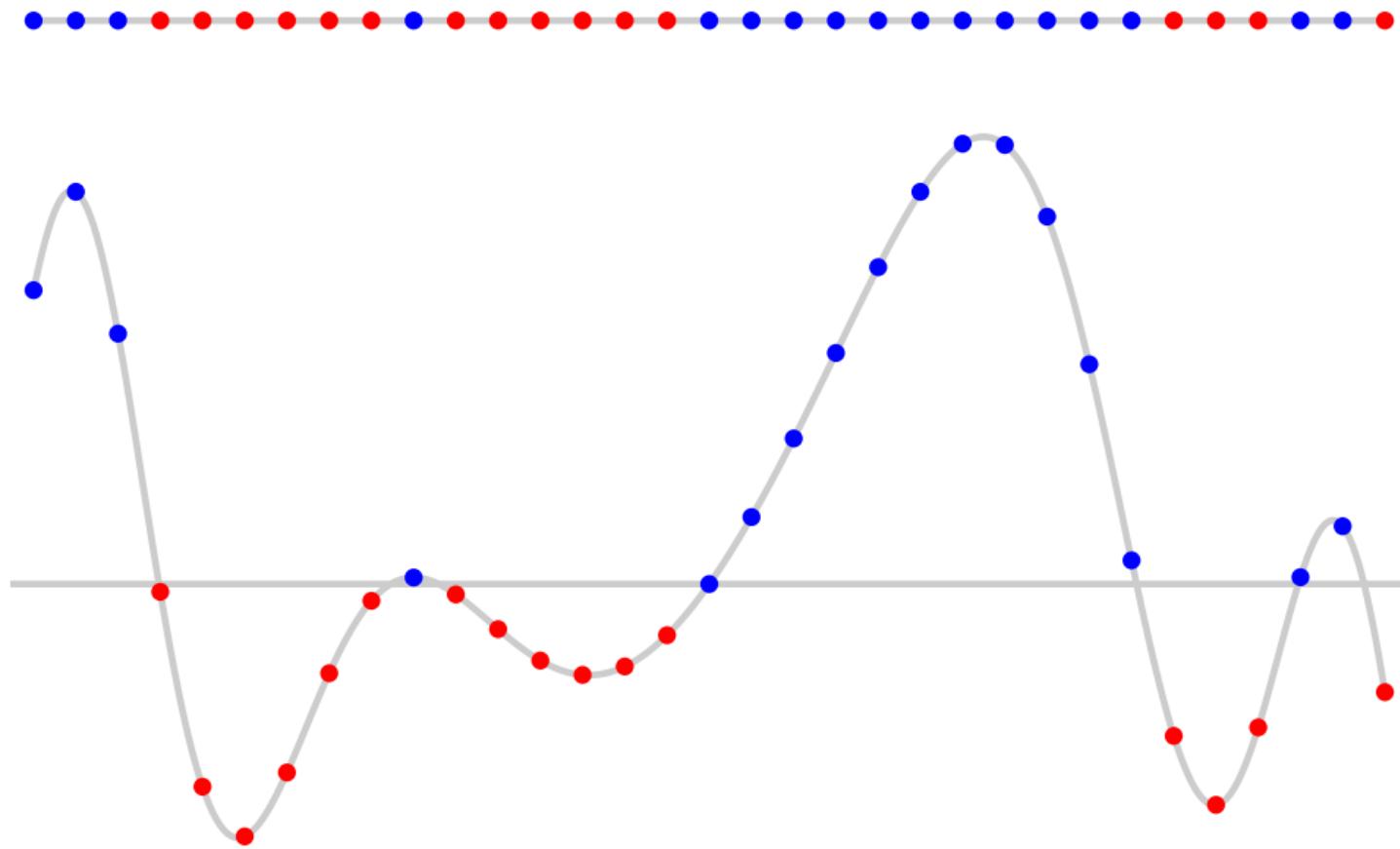
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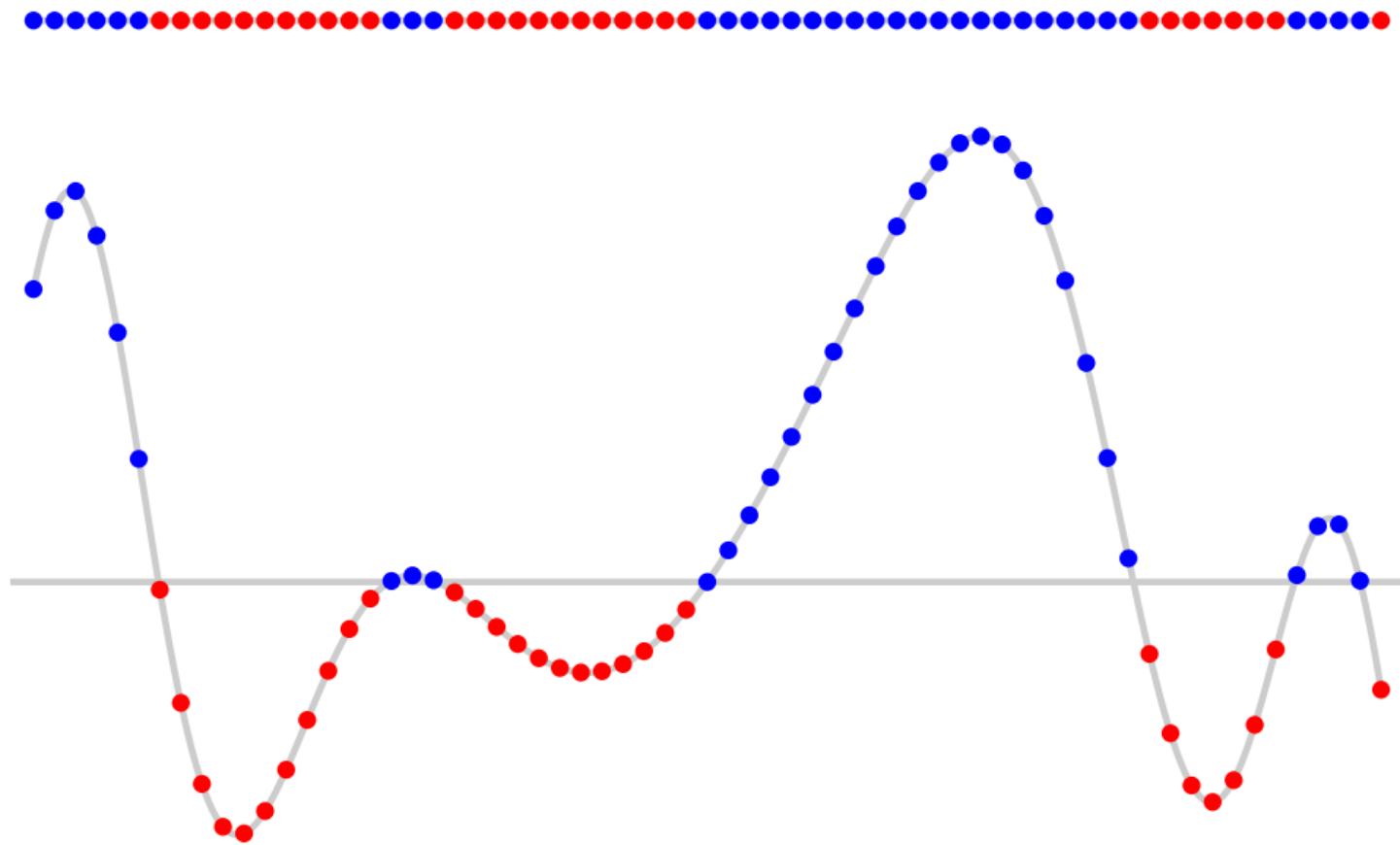
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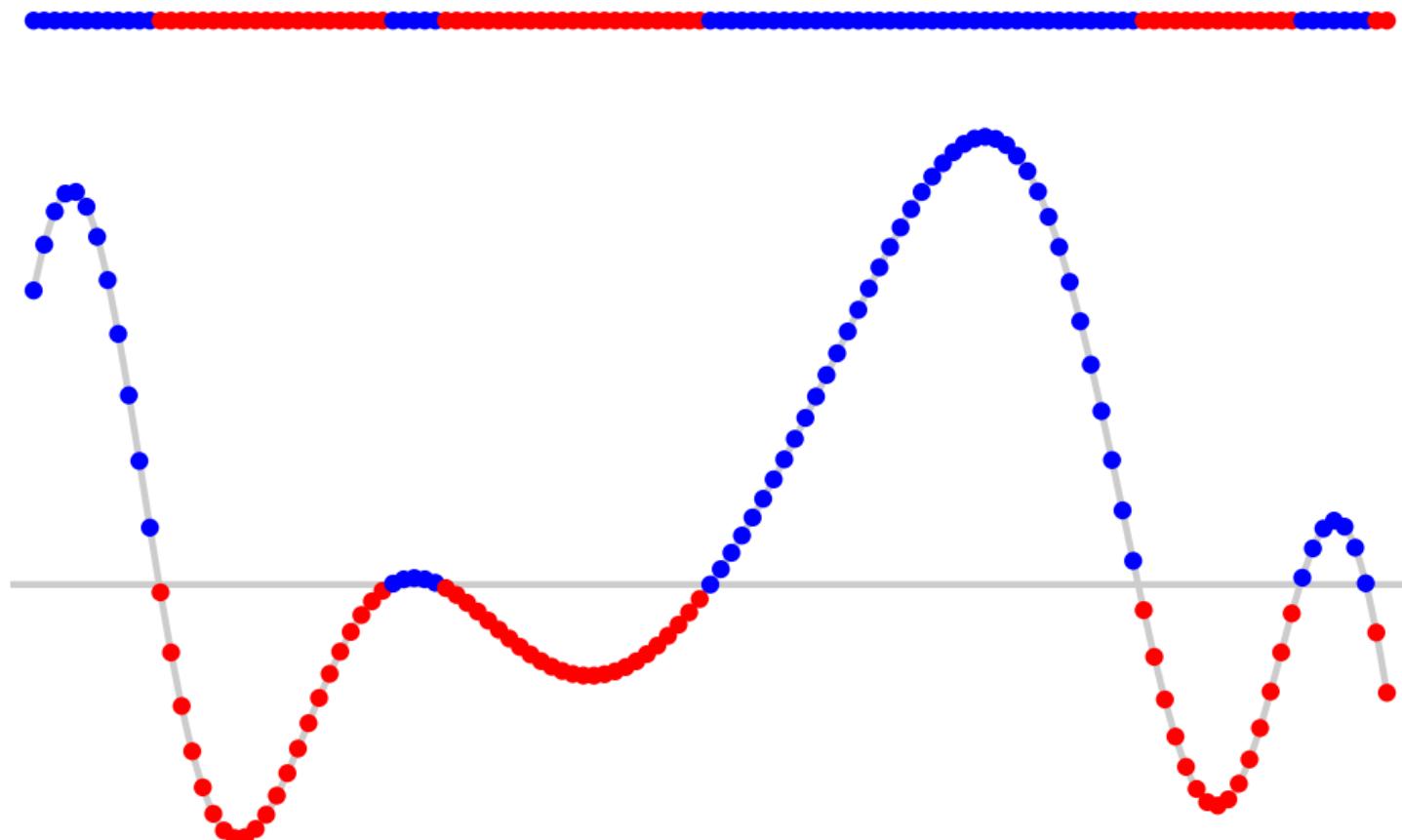
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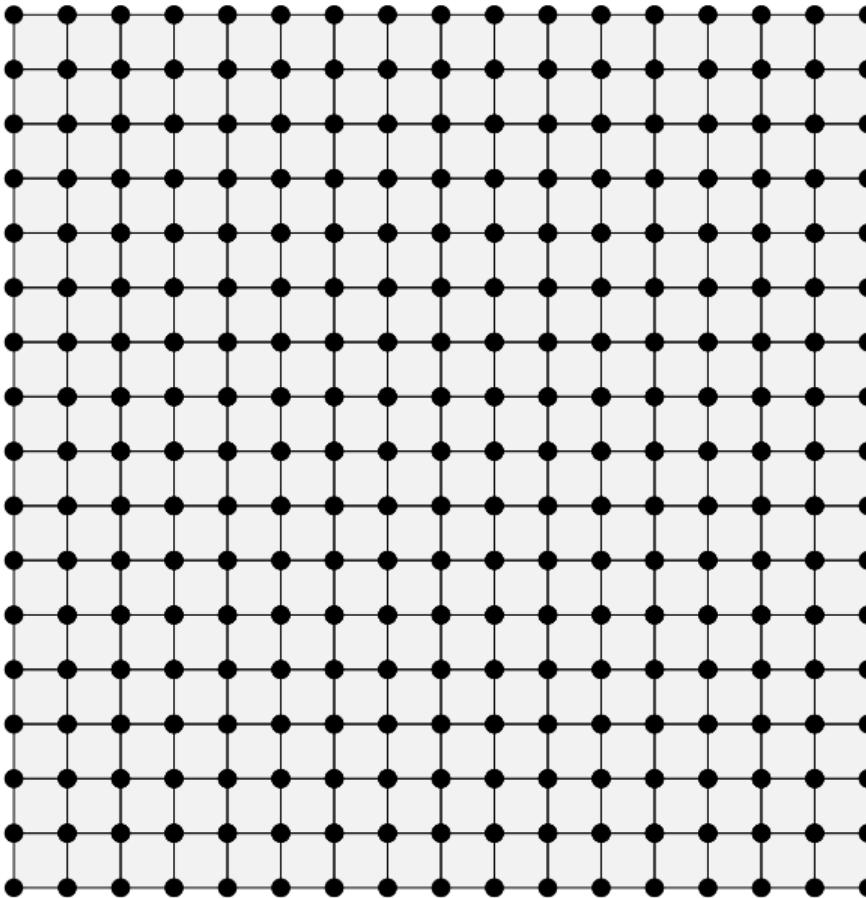
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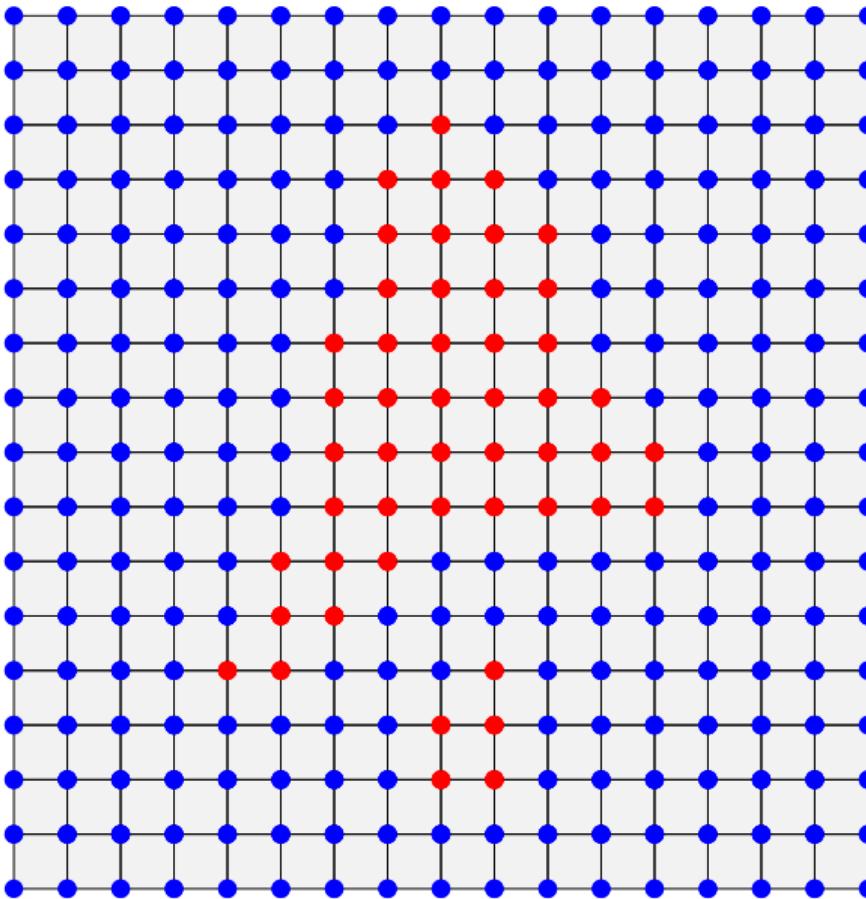
Amostragem em grade



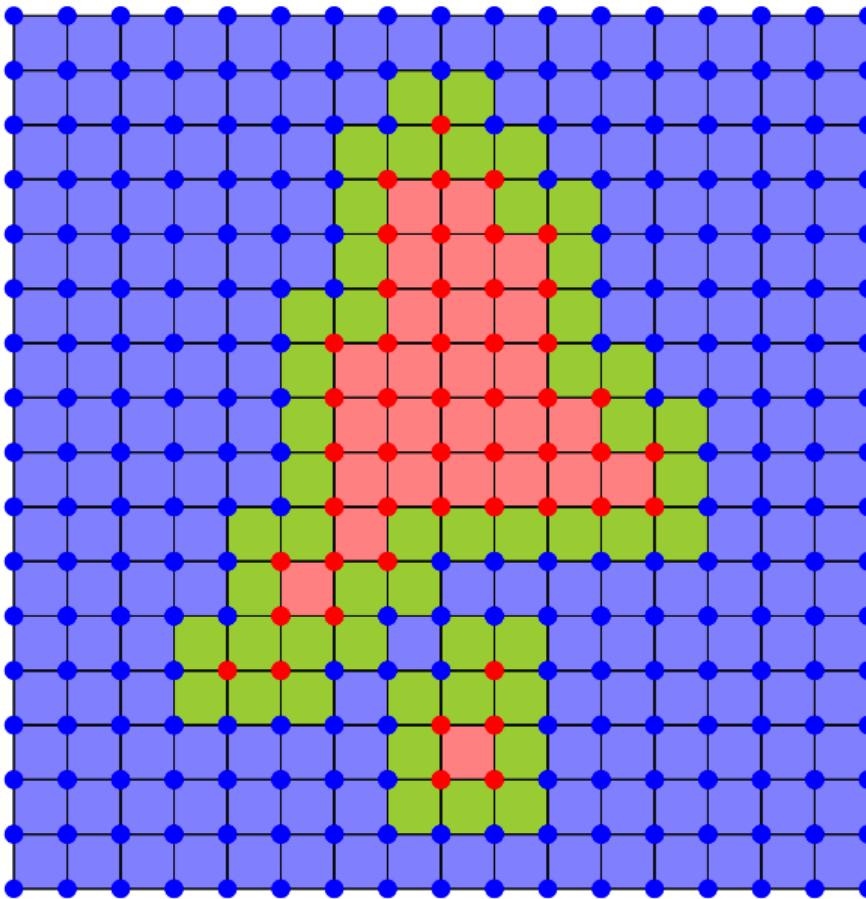
Região implícita – classificação por amostragem



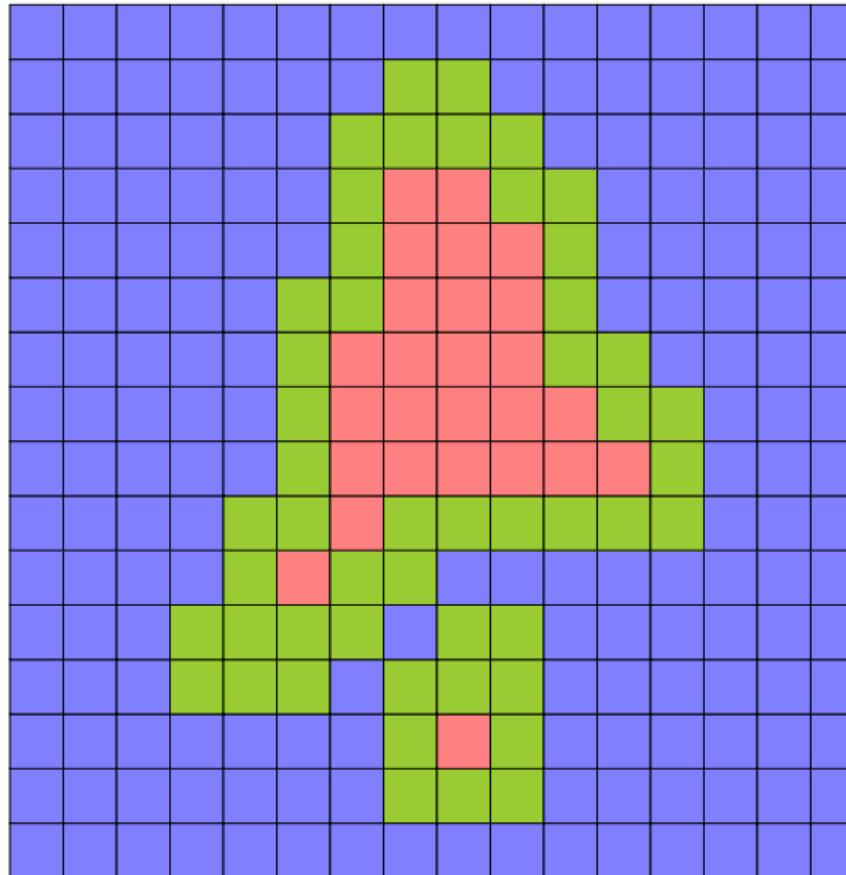
Região implícita – classificação por amostragem



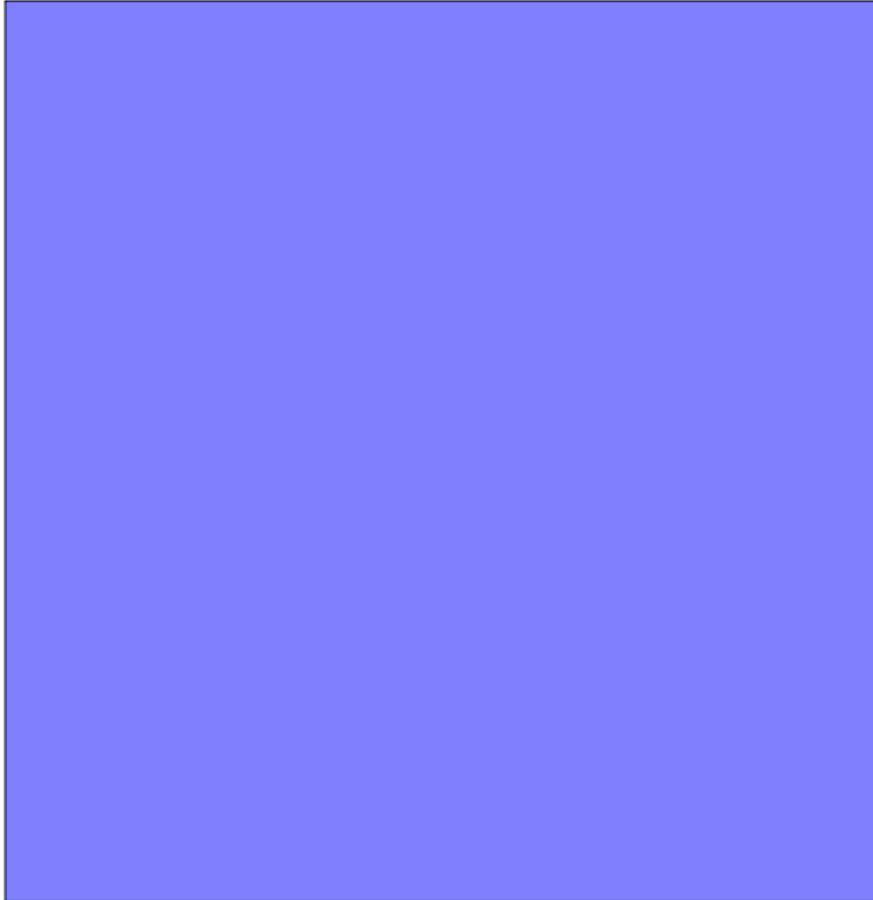
Região implícita – classificação por amostragem



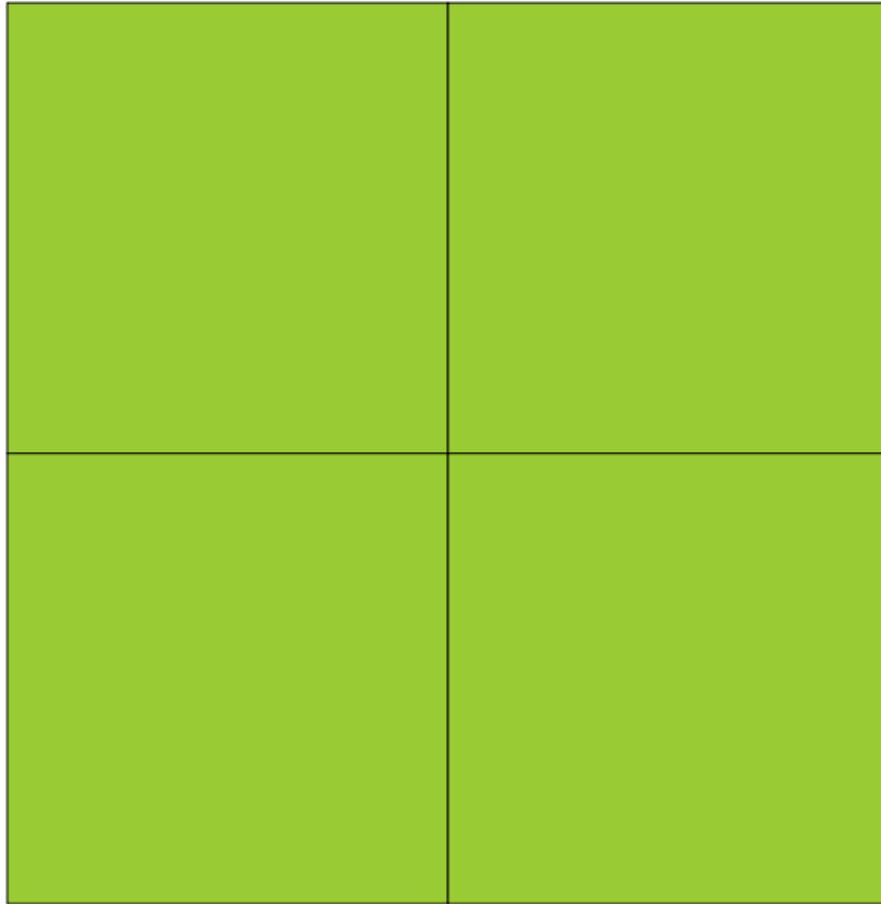
Região implícita – classificação por amostragem



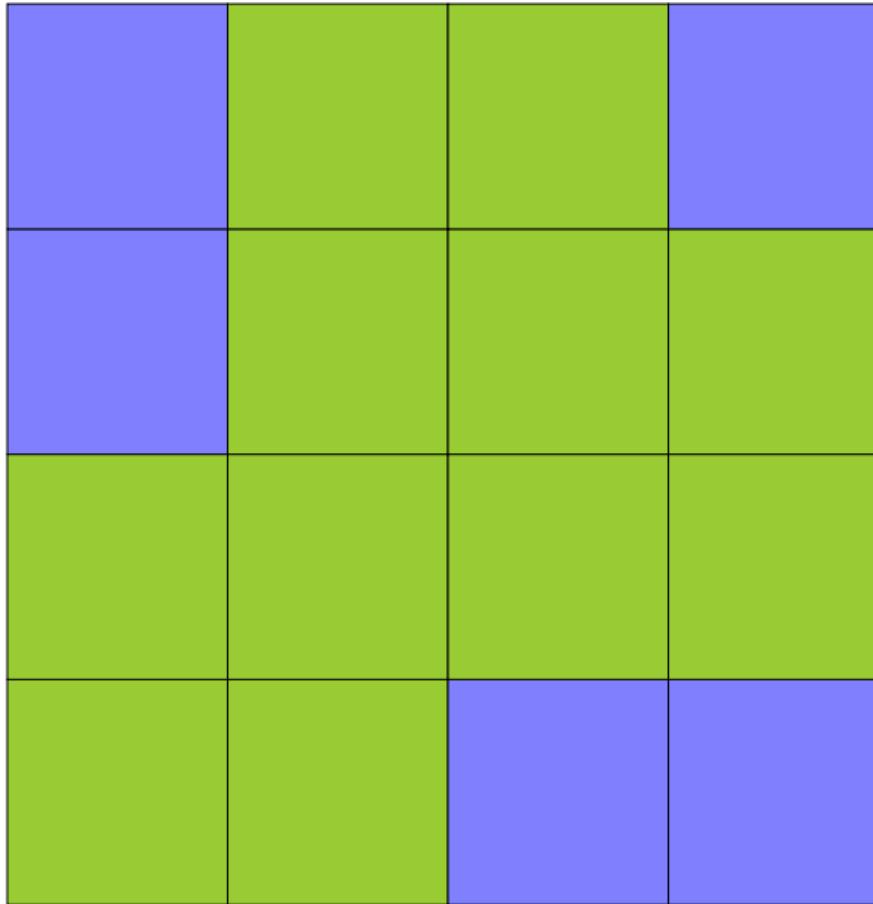
Região implícita – classificação por amostragem



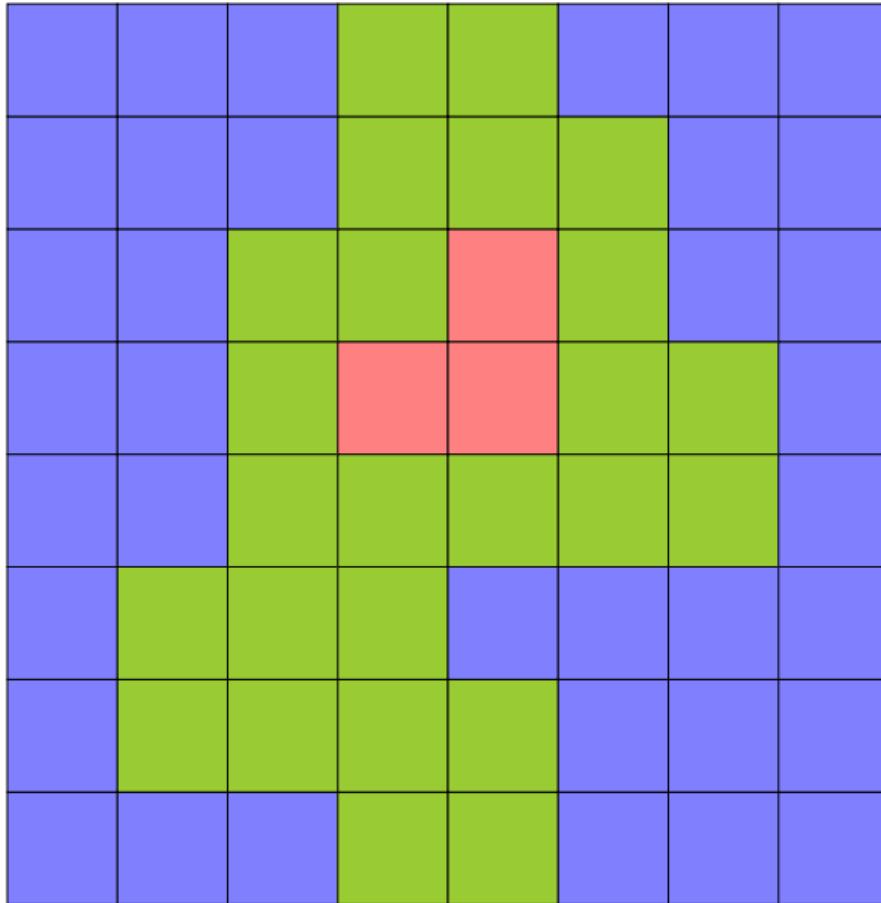
Região implícita – classificação por amostragem



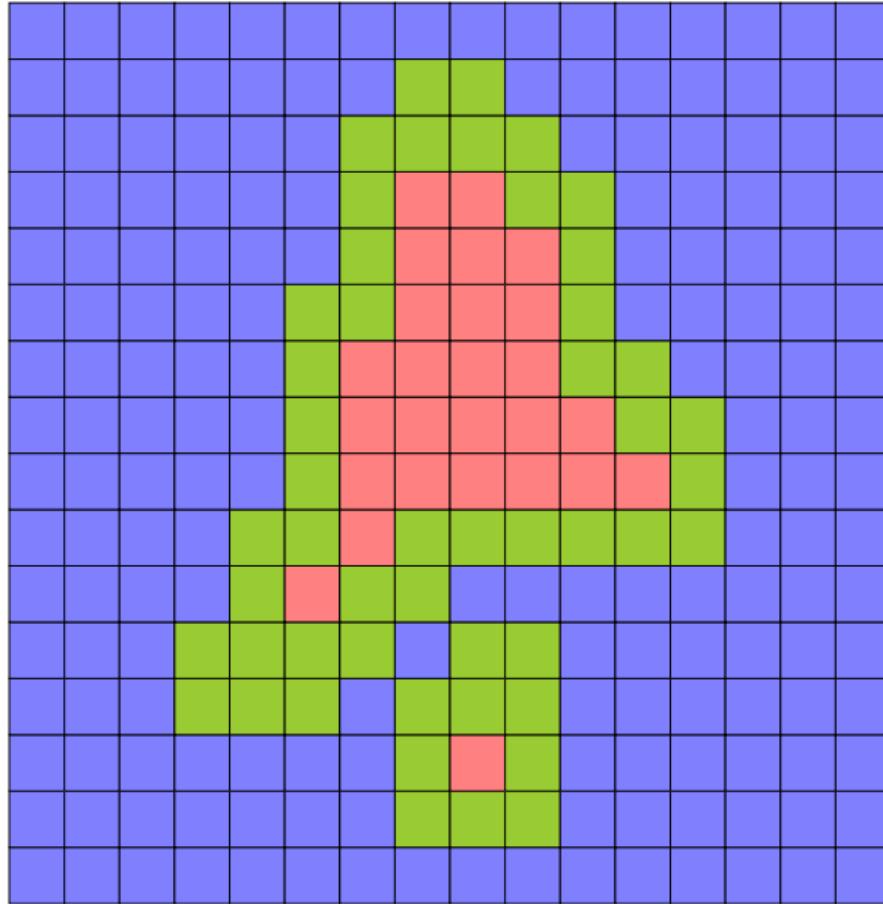
Região implícita – classificação por amostragem



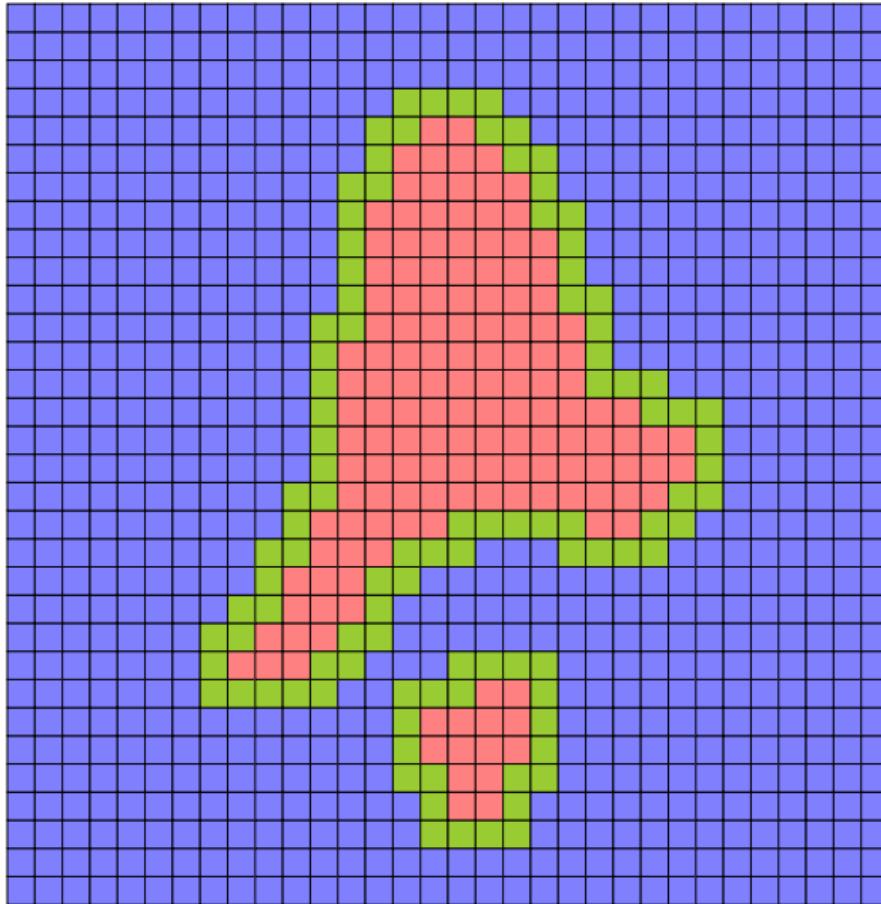
Região implícita – classificação por amostragem



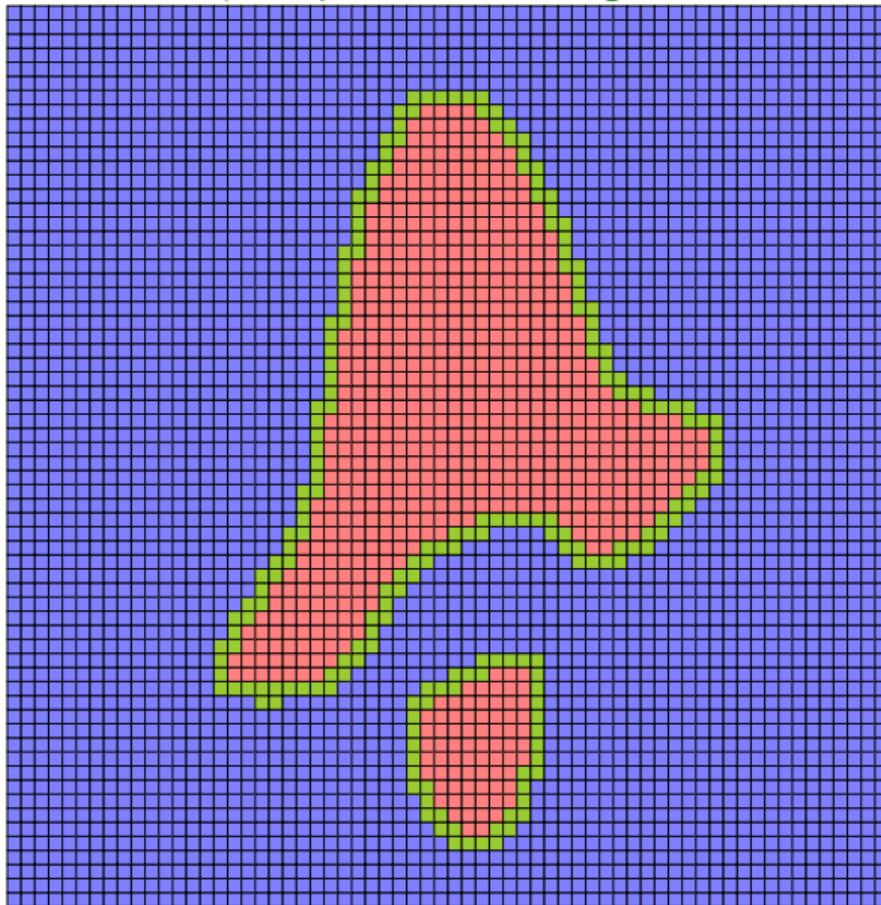
Região implícita – classificação por amostragem



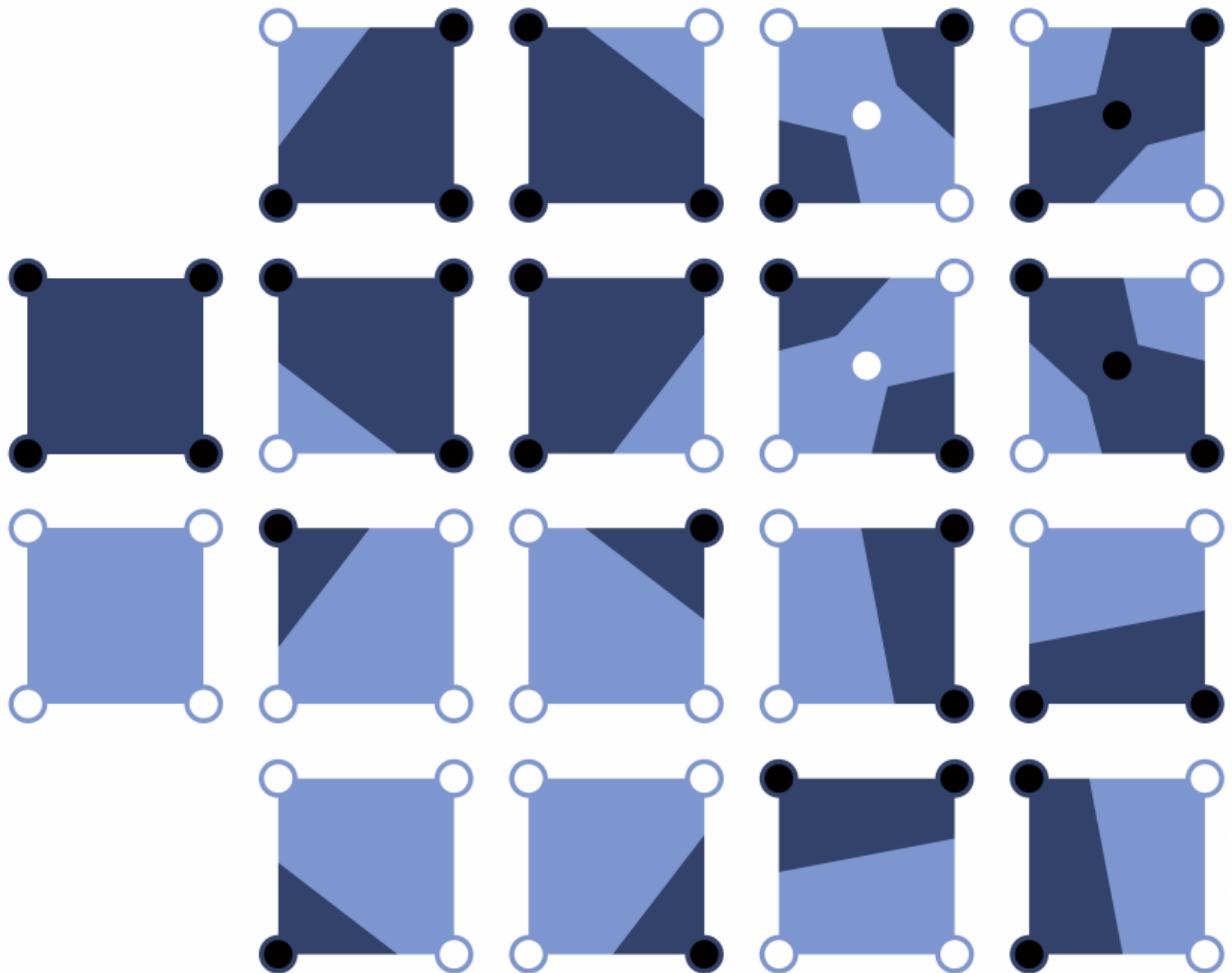
Região implícita – classificação por amostragem



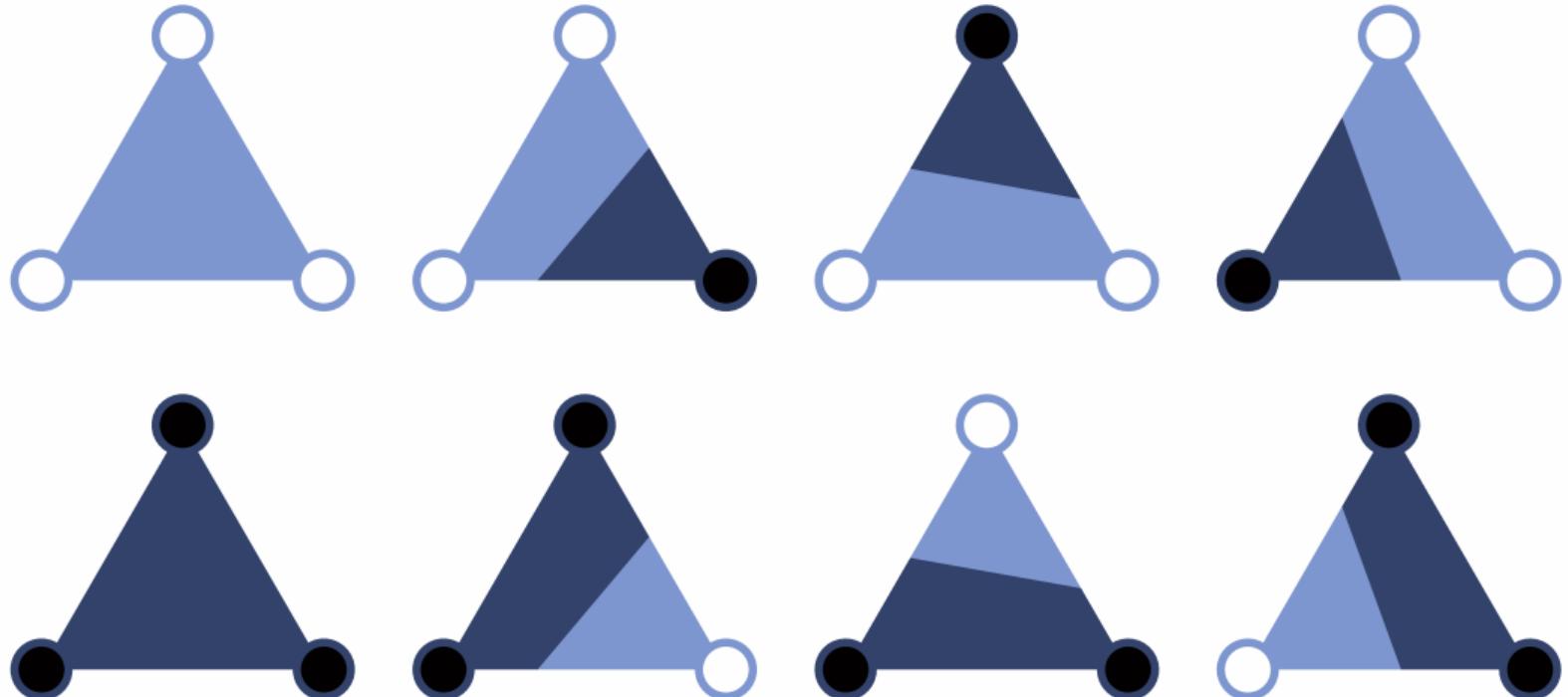
Região implícita – classificação por amostragem



Marching squares

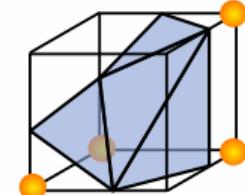
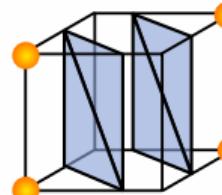
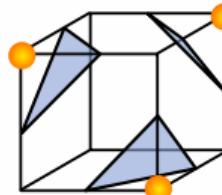
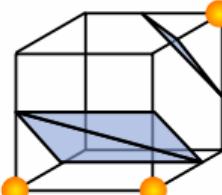
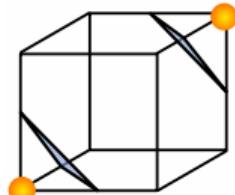
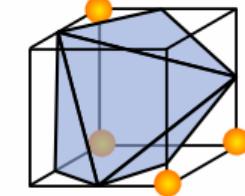
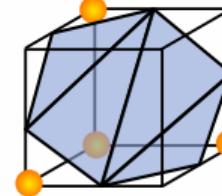
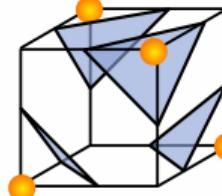
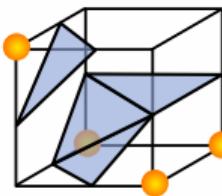
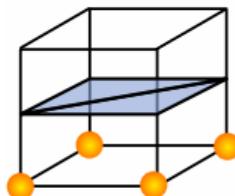
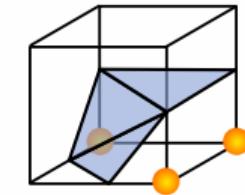
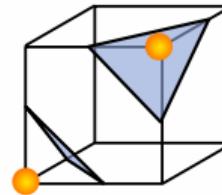
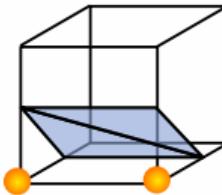
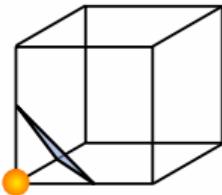
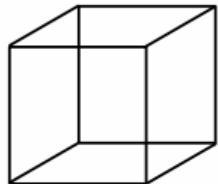


Marching triangles



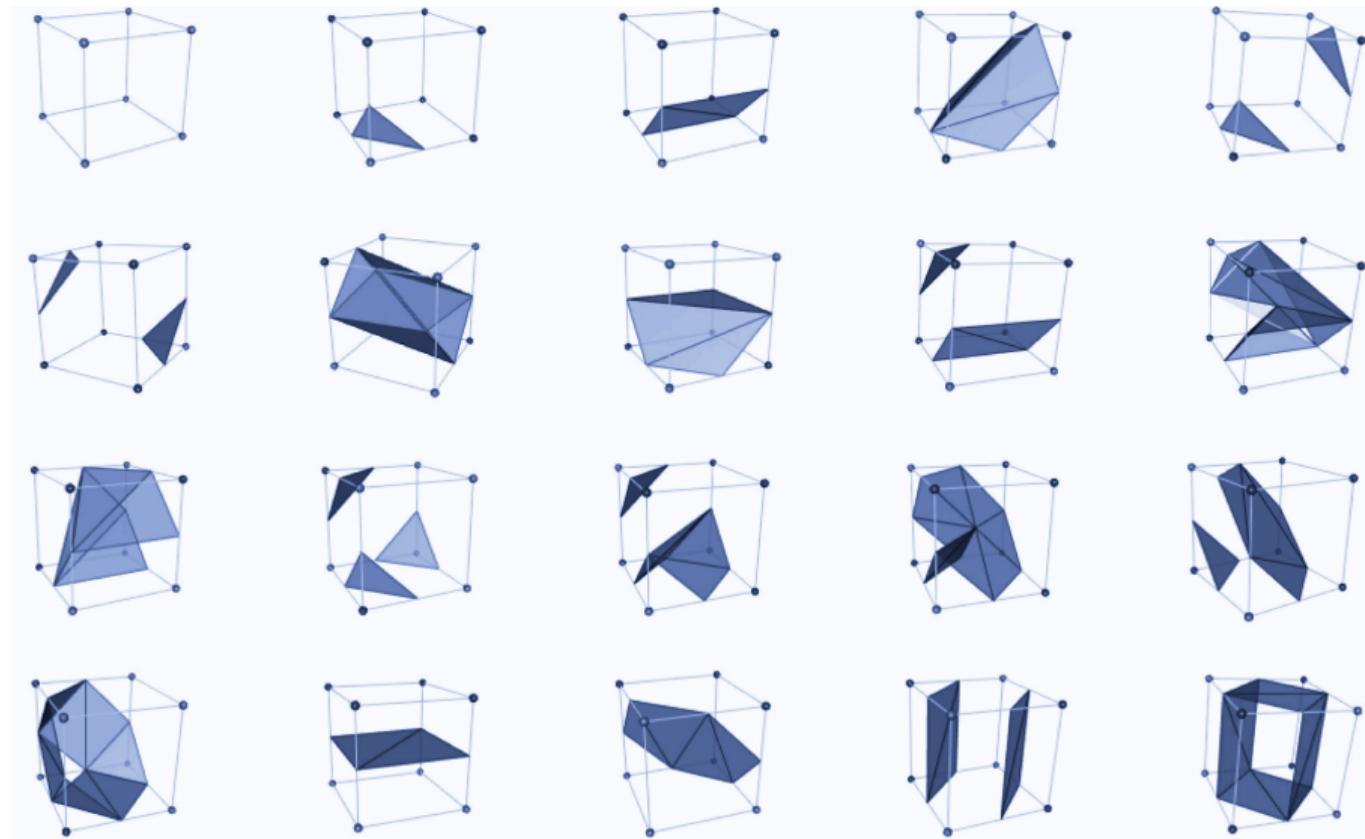
Marching cubes

Lorensen–Cline (1987)



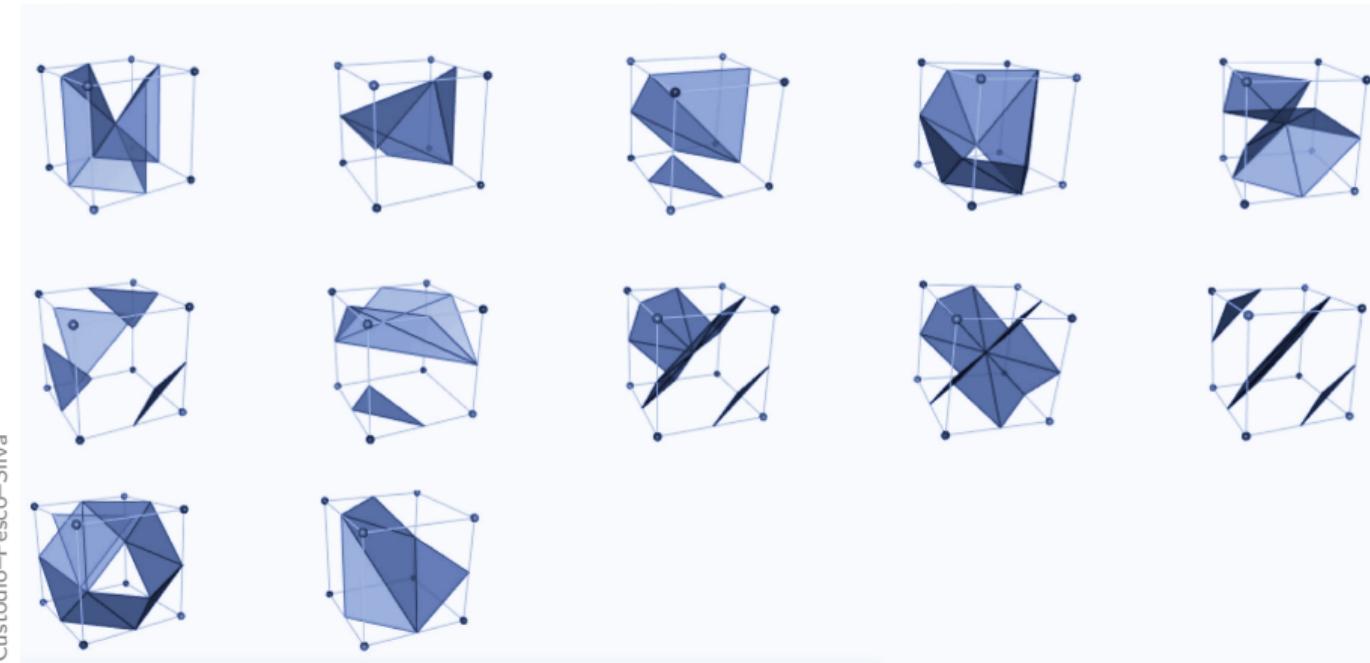
Marching cubes

Chernyaev (1995)

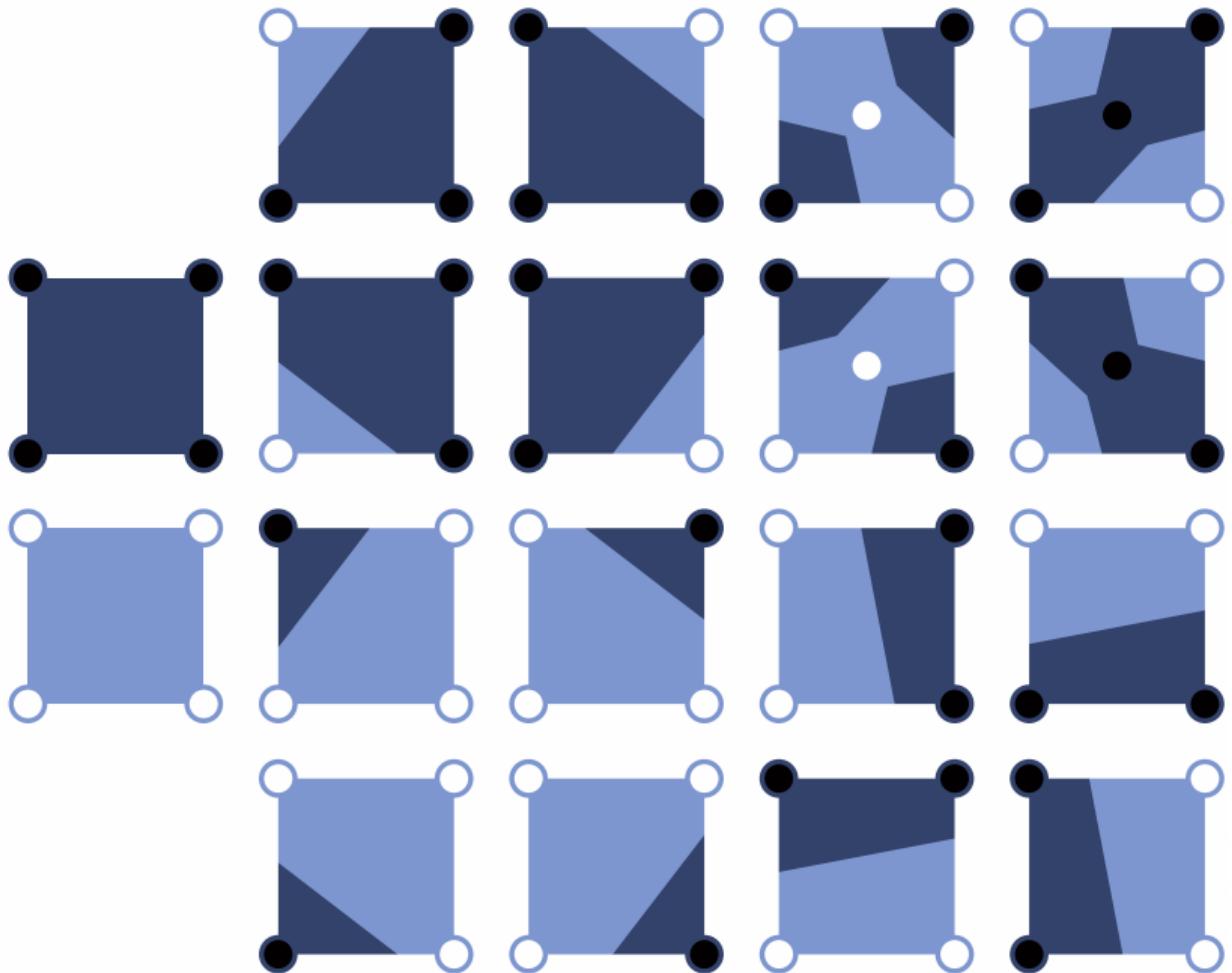


Marching cubes

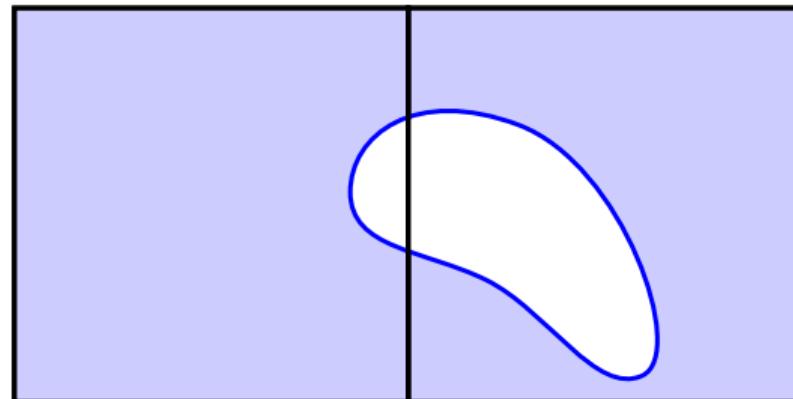
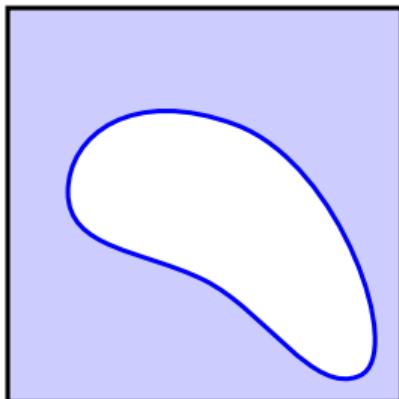
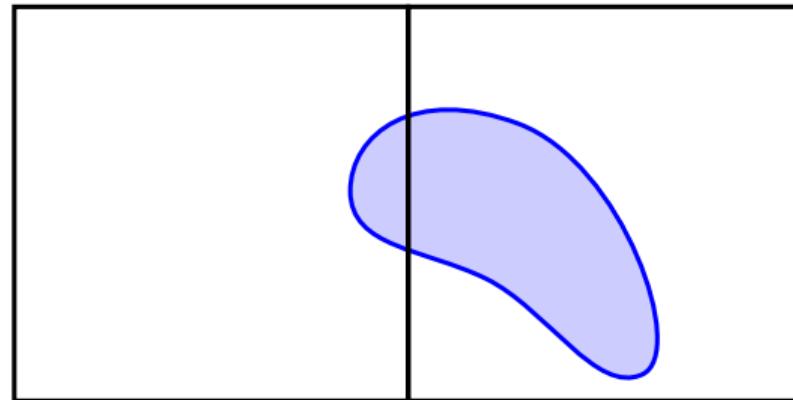
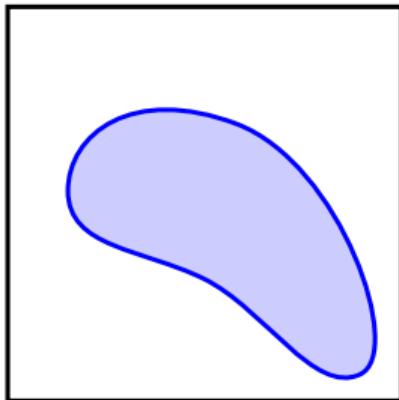
Chernyaev (1995)



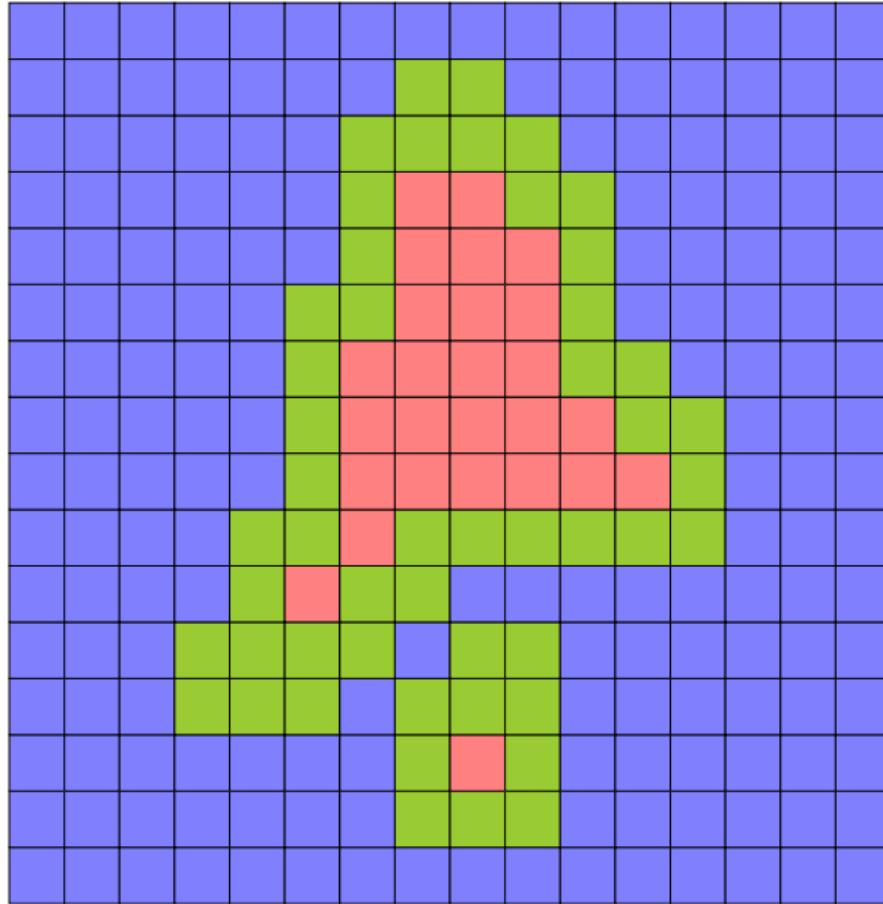
Marching squares



Marching squares – casos omissos

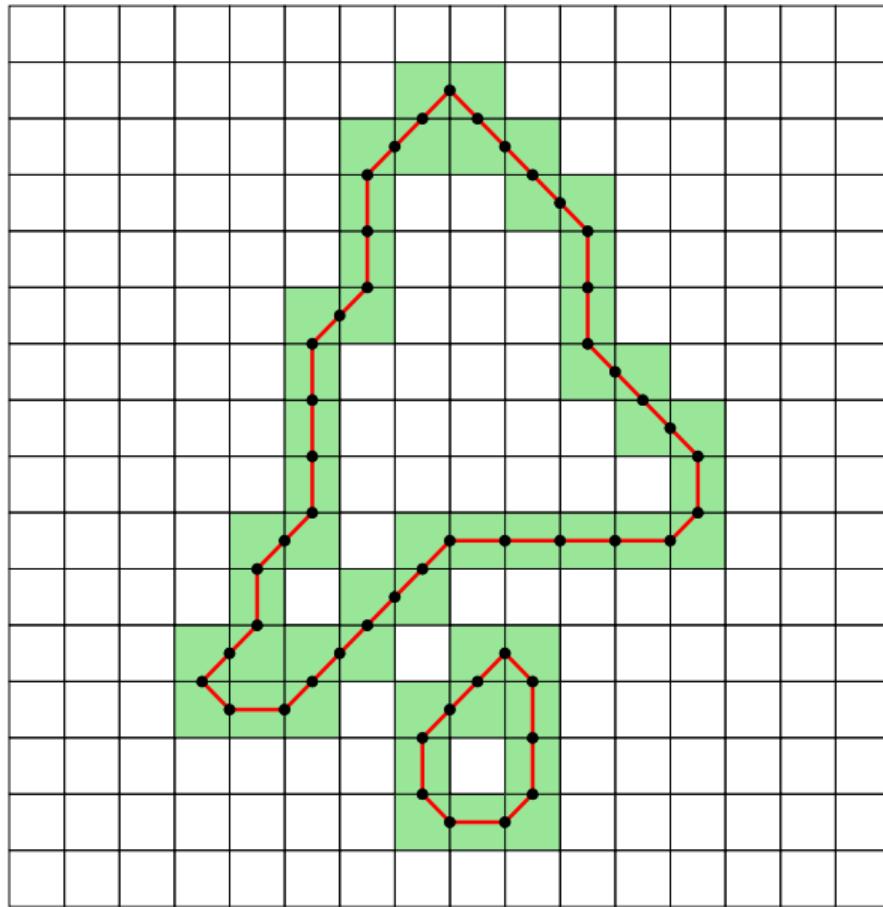


Região implícita – fronteira



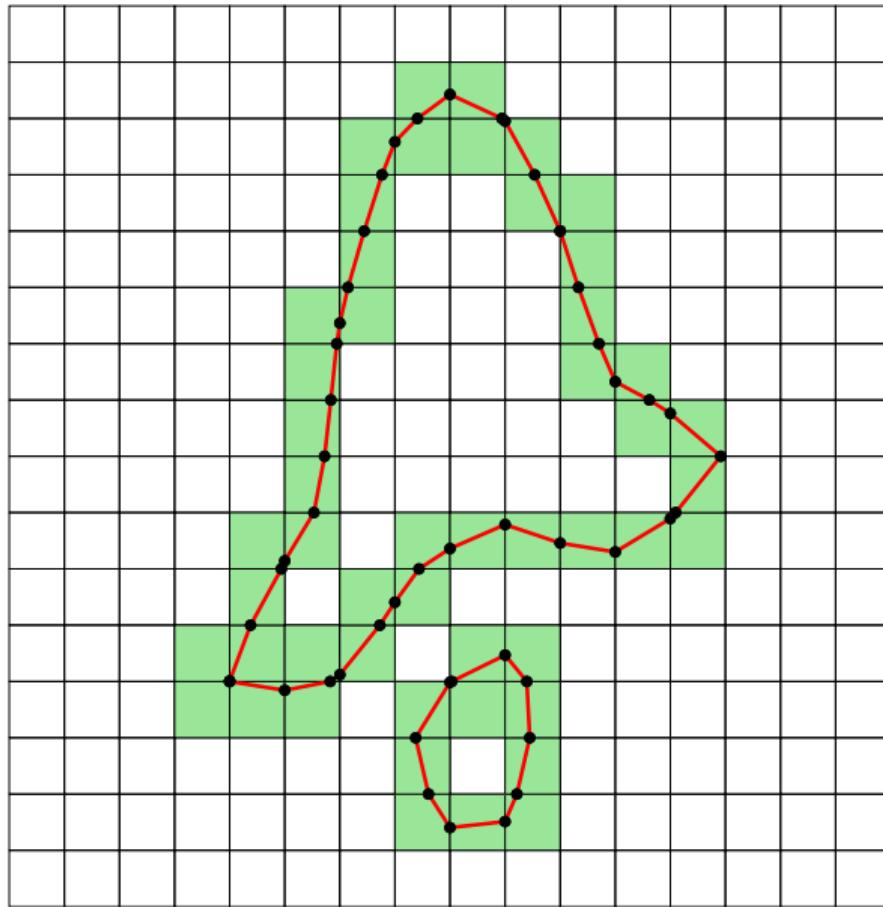
Região implícita – fronteira

ponto médio



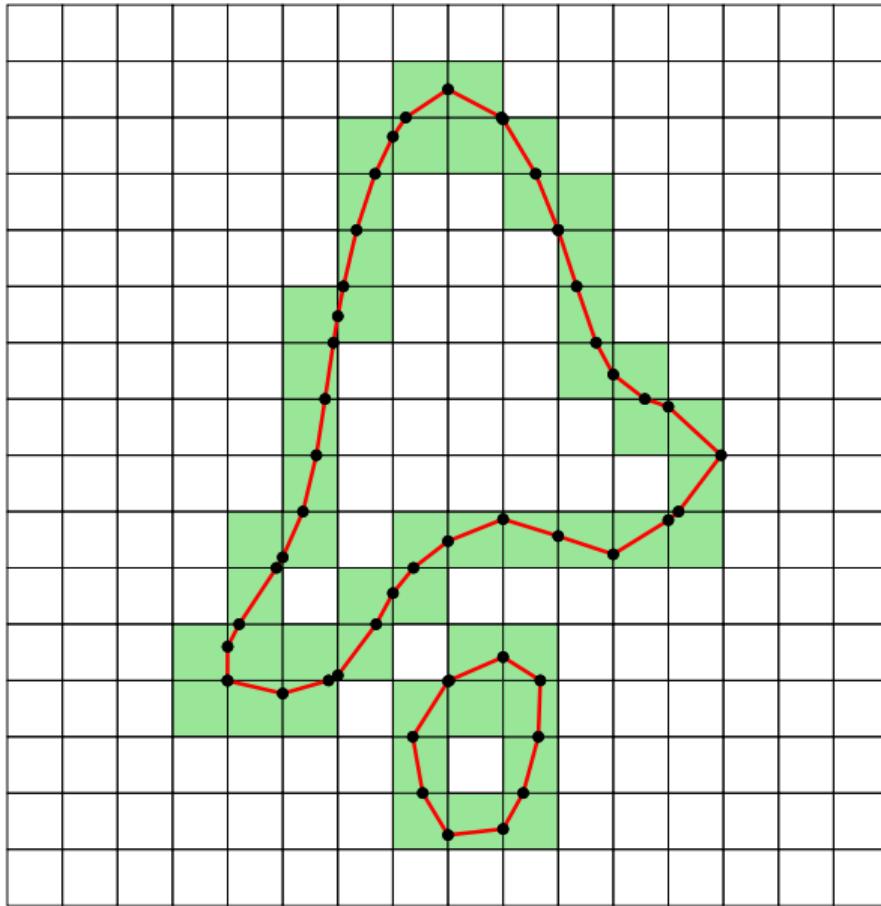
Região implícita – fronteira

interpolação linear

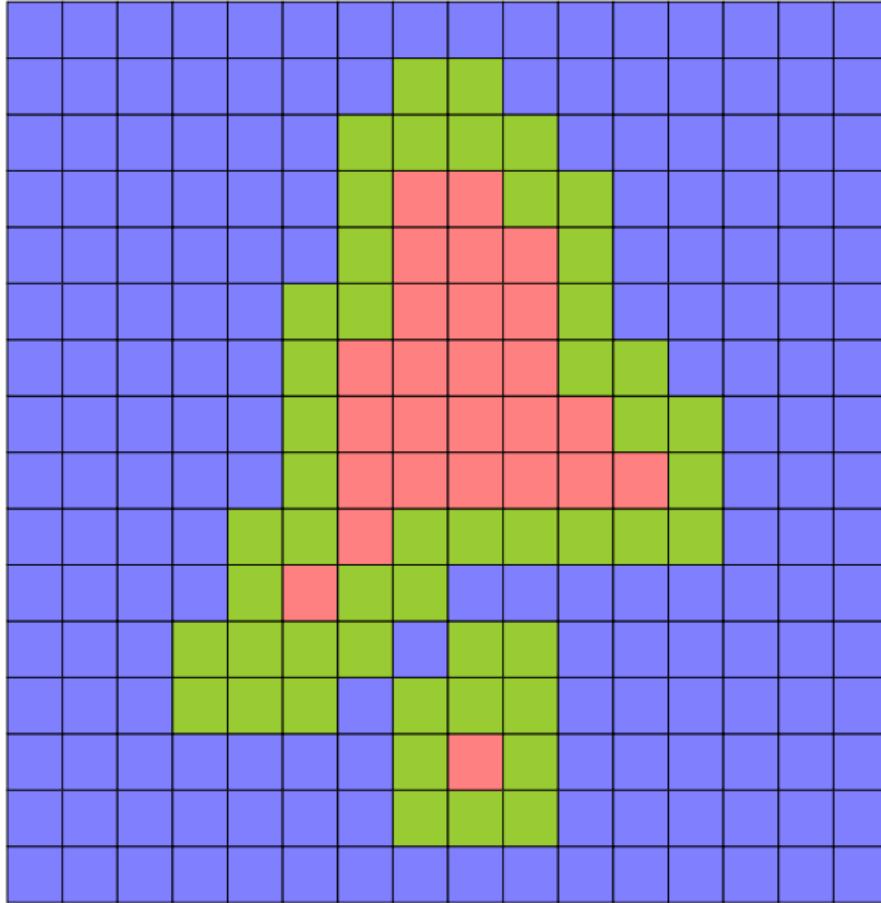


Região implícita – fronteira

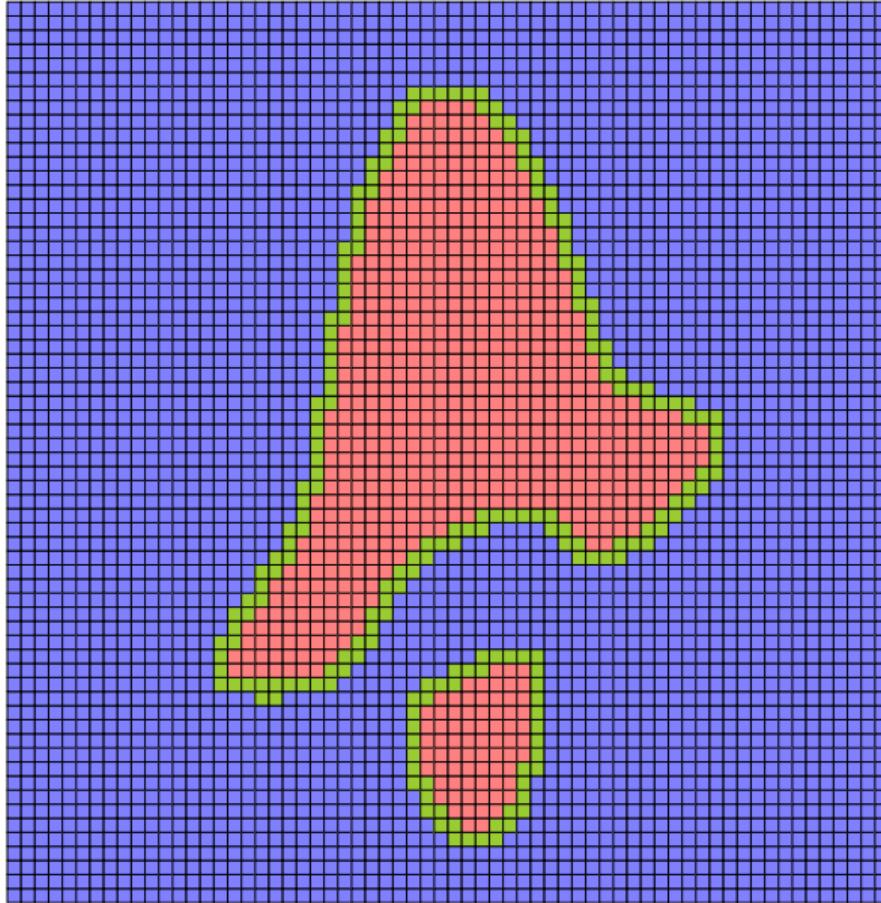
bisseção



Enumeração espacial uniforme

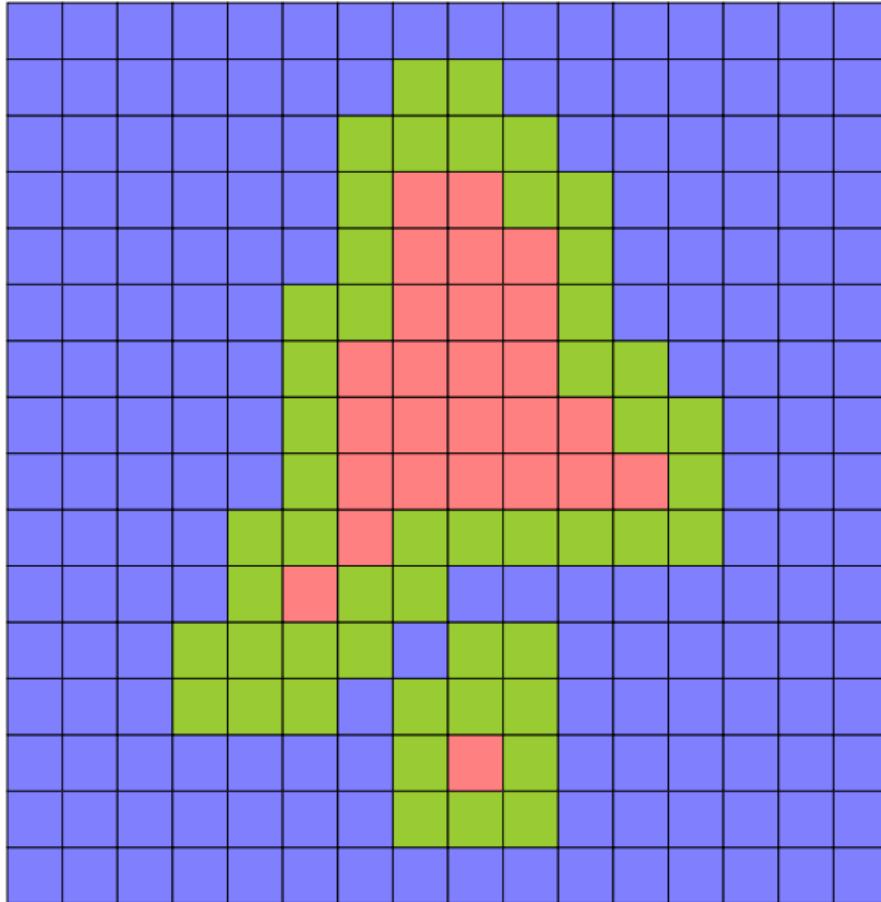


Enumeração espacial uniforme



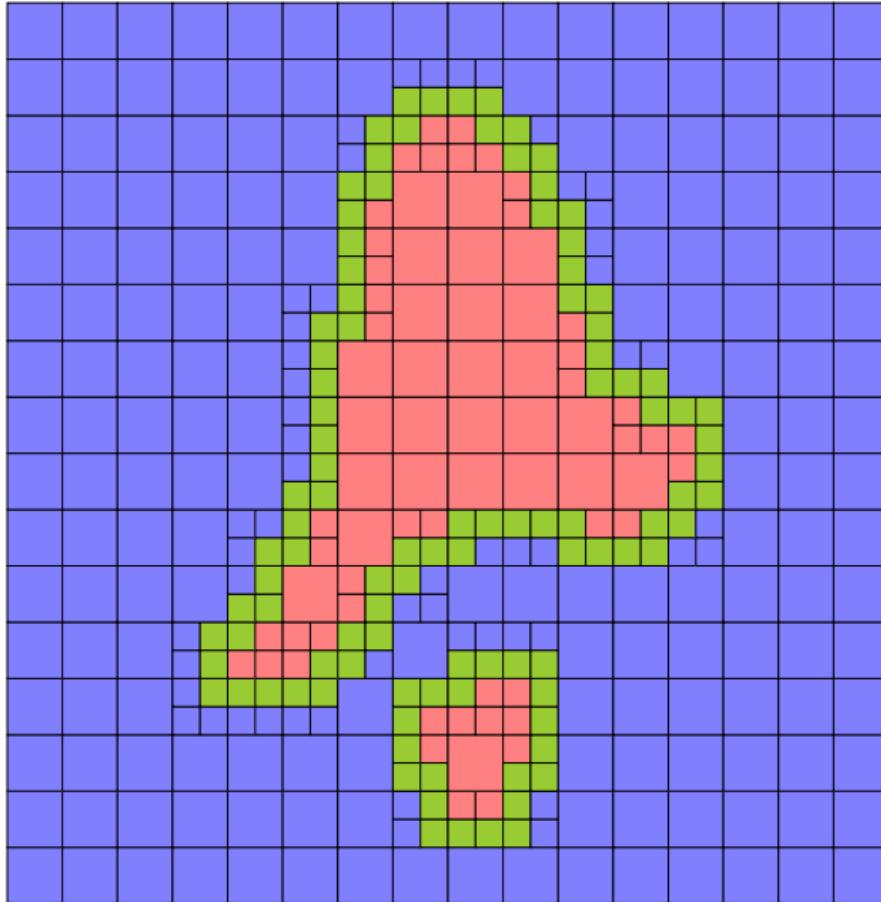
Enumeração espacial semi-adaptativa

Suffern (1990)



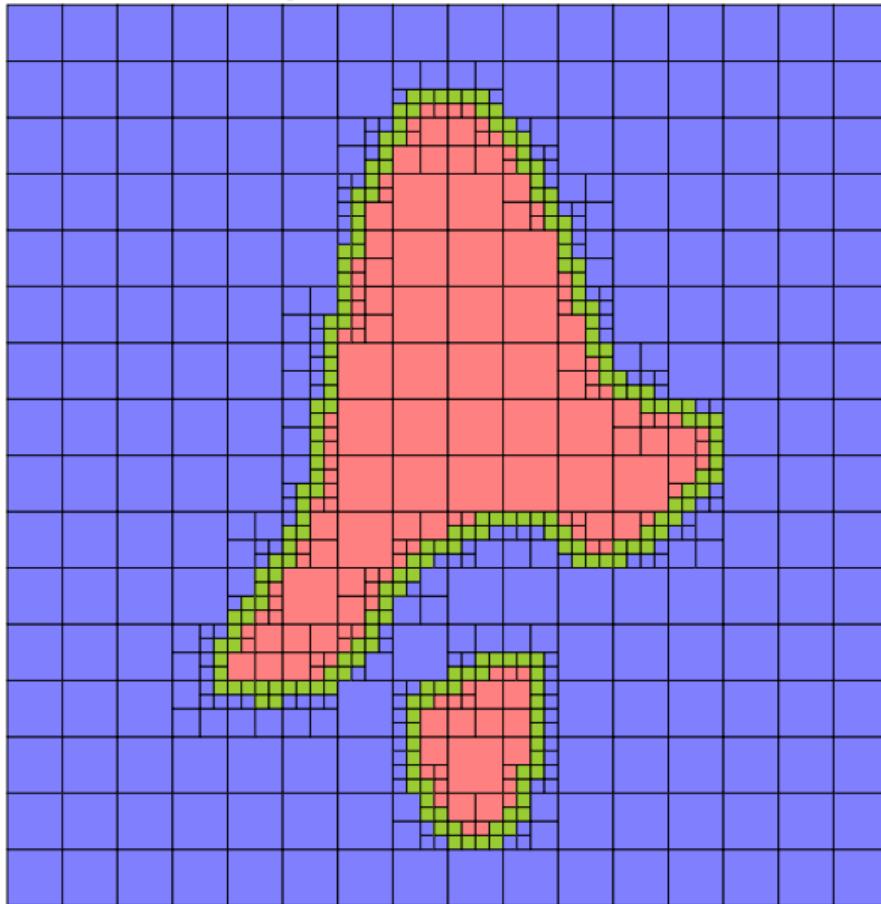
Enumeração espacial semi-adaptativa

Suffern (1990)



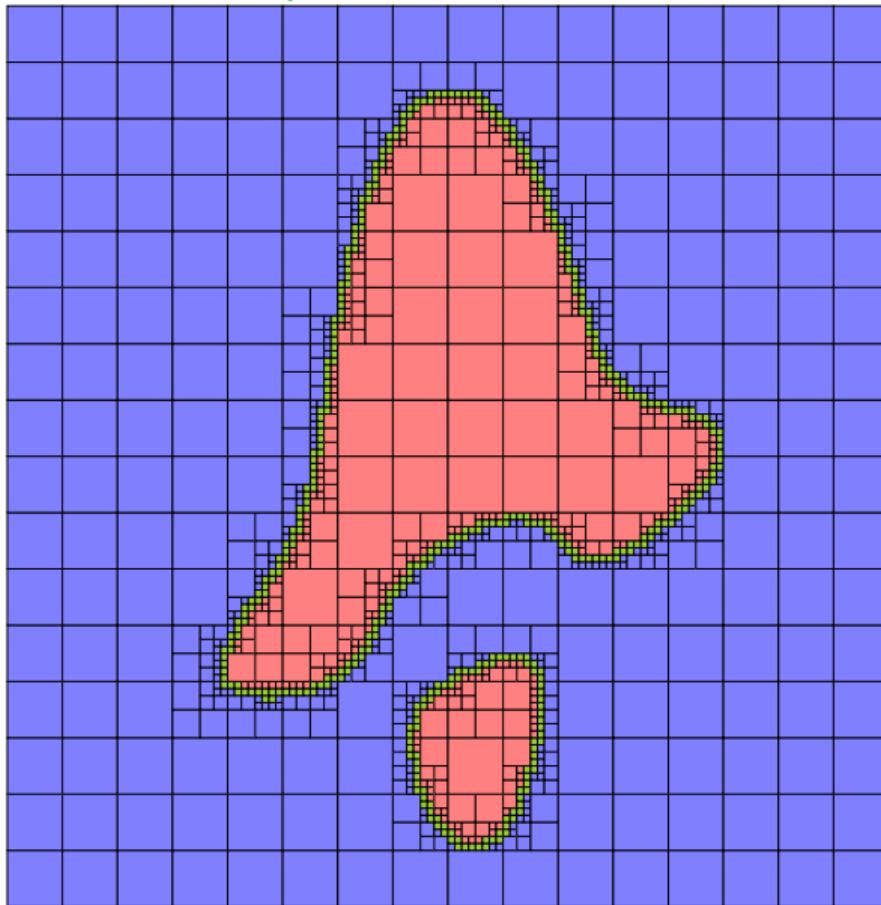
Enumeração espacial semi-adaptativa

Suffern (1990)



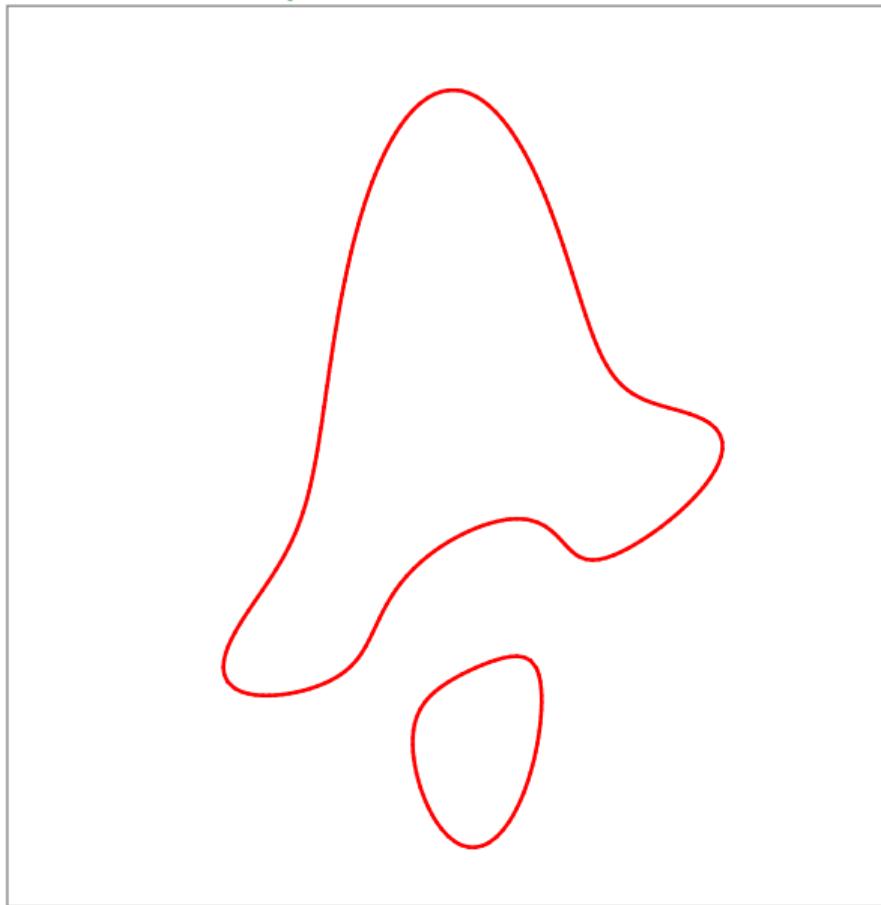
Enumeração espacial semi-adaptativa

Suffern (1990)



Enumeração espacial semi-adaptativa

Suffern (1990)

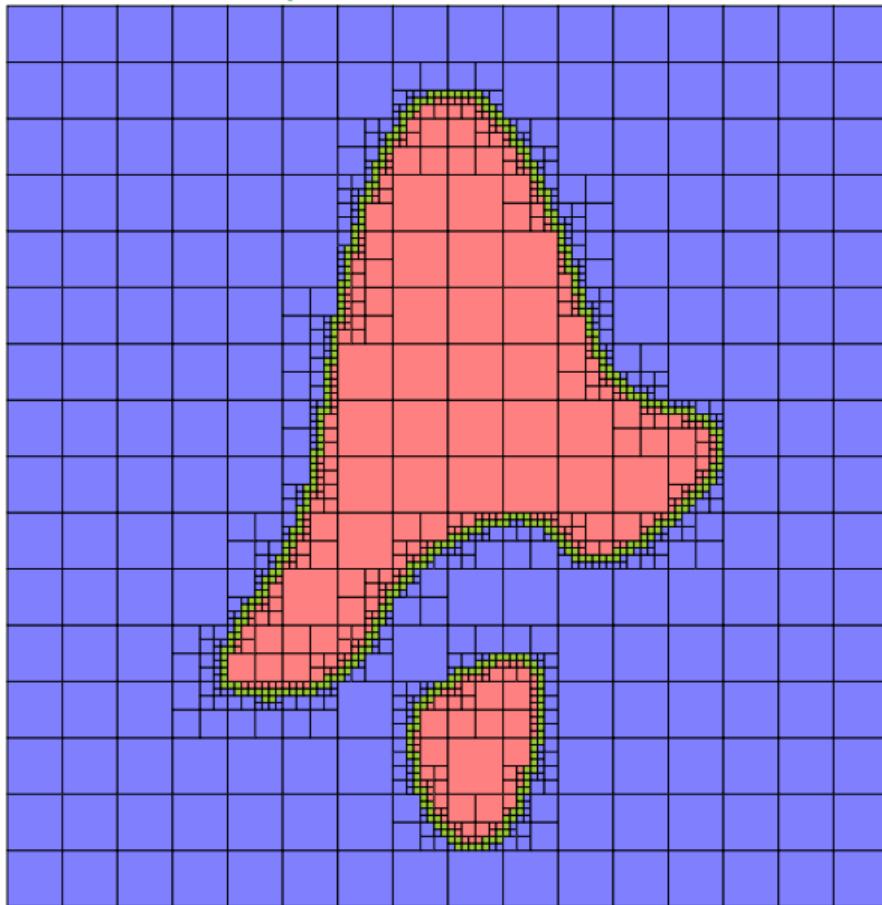


```
function explore(xmin,xmax,ymin,ymax,level)
    if level < searchlevel then
        subdivide(xmin,xmax,ymin,ymax,level)
    elseif contour_present(xmin,xmax,ymin,ymax) then
        if level < plotlevel then
            subdivide(xmin,xmax,ymin,ymax,level)
        else
            draw(xmin,xmax,ymin,ymax)
        end
    else
        draw(xmin,xmax,ymin,ymax)
    end
end
```

```
function subdivide(xmin,xmax,ymin,ymax,level)
    local xmid=(xmin+xmax)/2
    local ymid=(ymin+ymax)/2
    explore(xmin,xmid,ymin,ymid,level+1)
    explore(xmid,xmax,ymin,ymid,level+1)
    explore(xmin,xmid,ymid,ymax,level+1)
    explore(xmid,xmax,ymid,ymax,level+1)
end
```

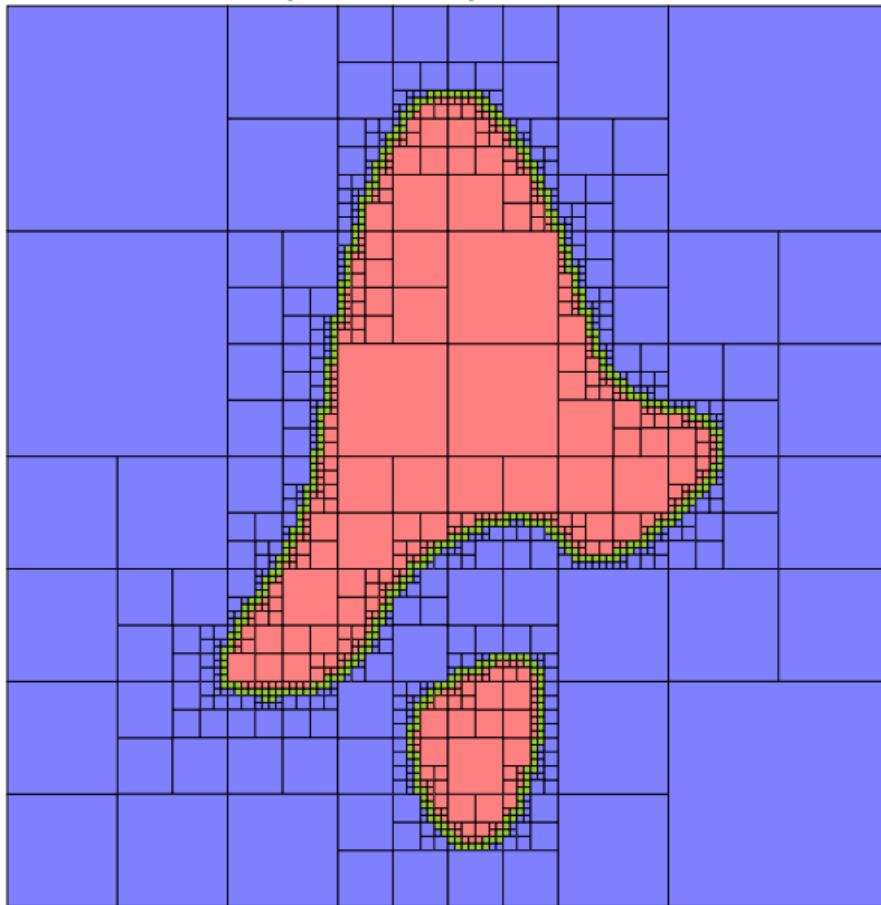
Enumeração espacial semi-adaptativa

Suffern (1990)



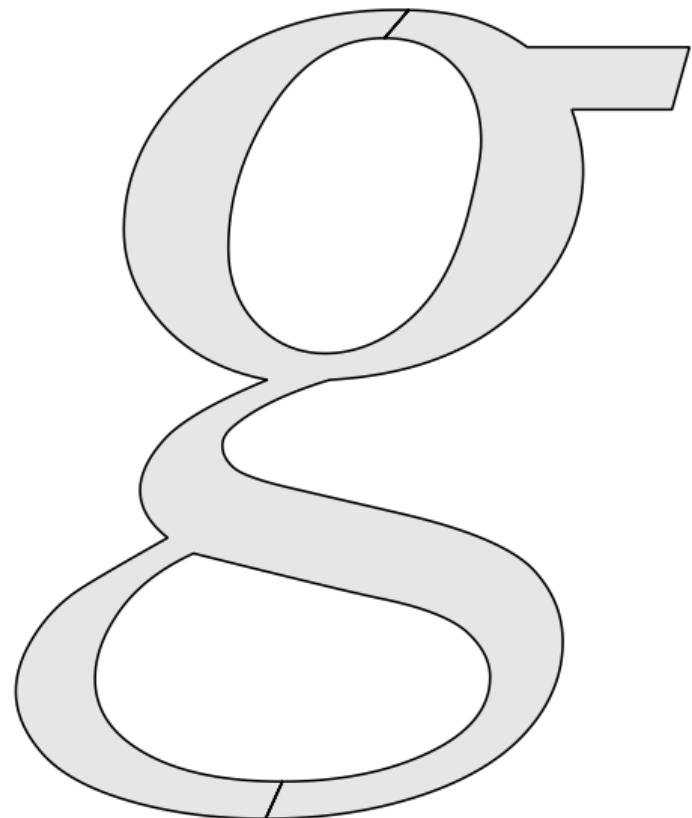
Enumeração espacial semi-adaptativa podada

Suffern (1990)



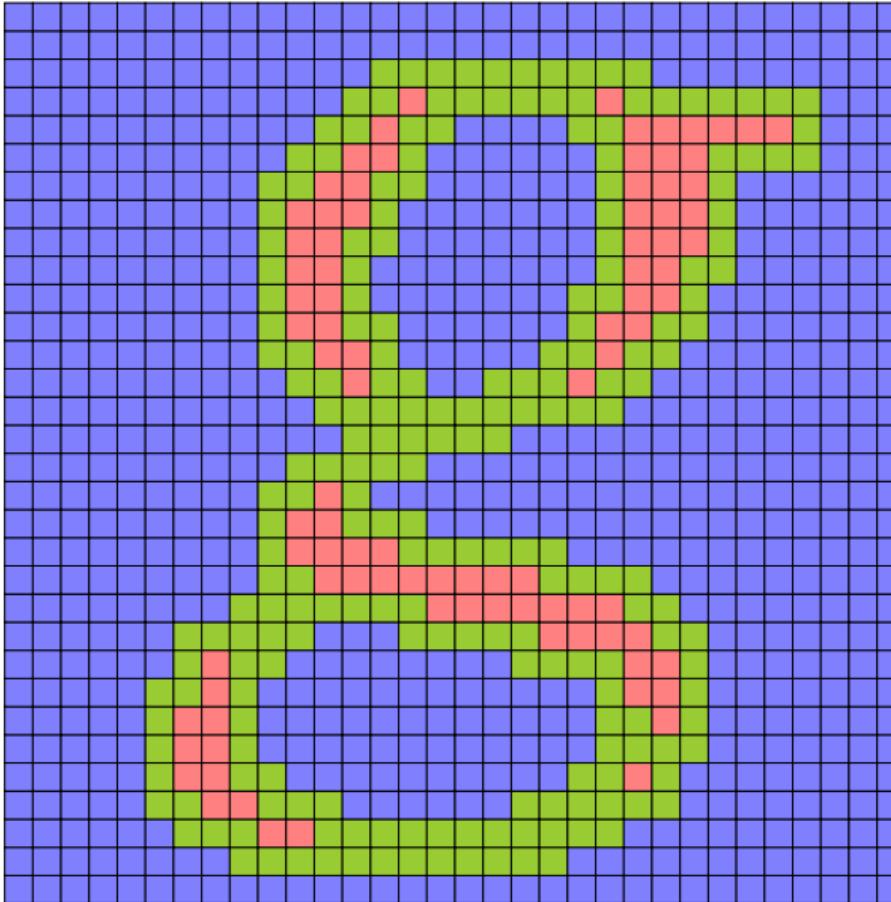
Enumeração espacial semi-adaptativa

Suffern (1990)



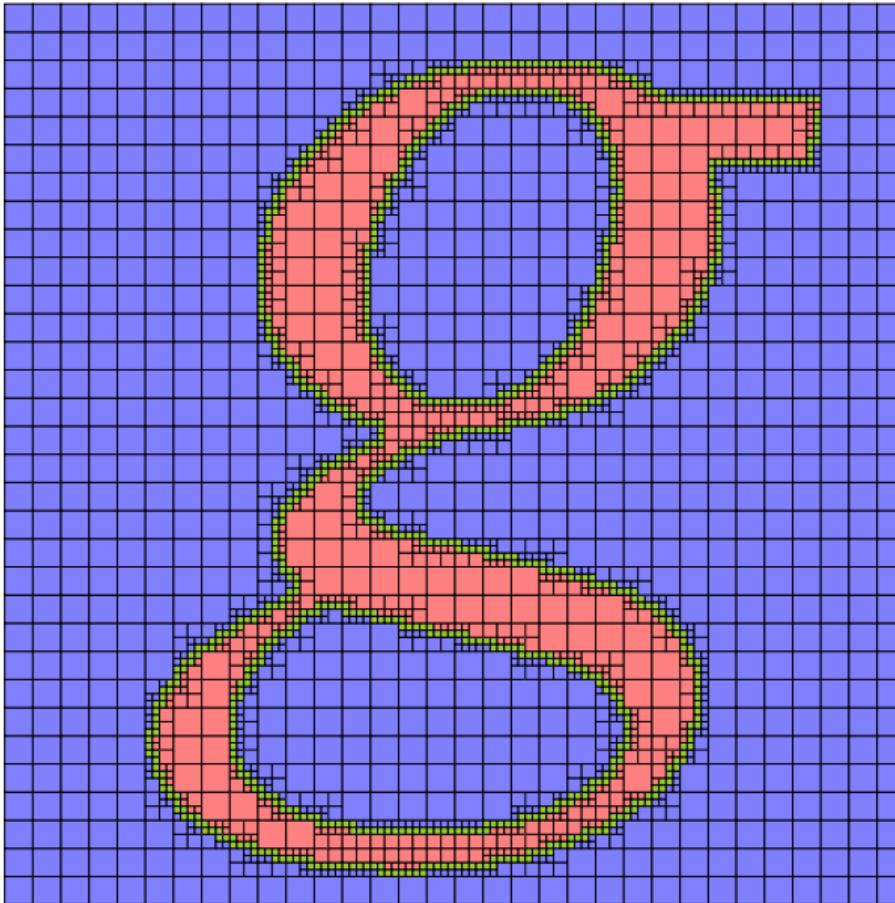
Enumeração espacial semi-adaptativa

Suffern (1990)



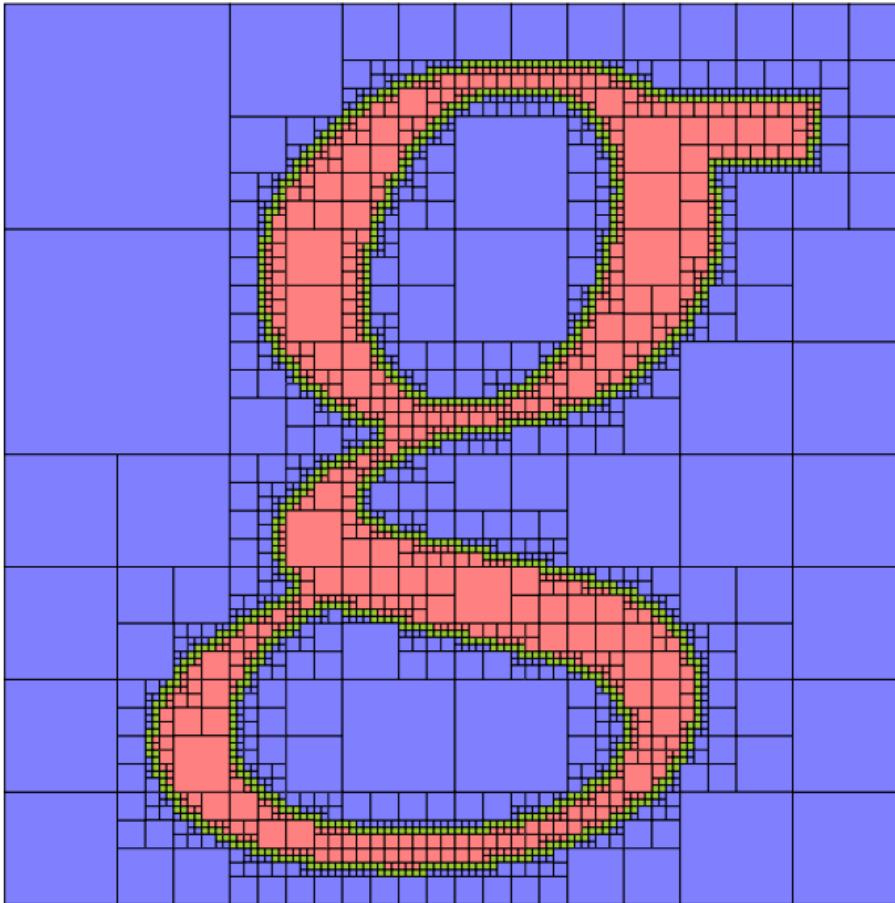
Enumeração espacial semi-adaptativa

Suffern (1990)



Enumeração espacial semi-adaptativa

Suffern (1990)



Enumeração espacial semi-adaptativa

Suffern (1990)



Localizando ponto em relação a polígono

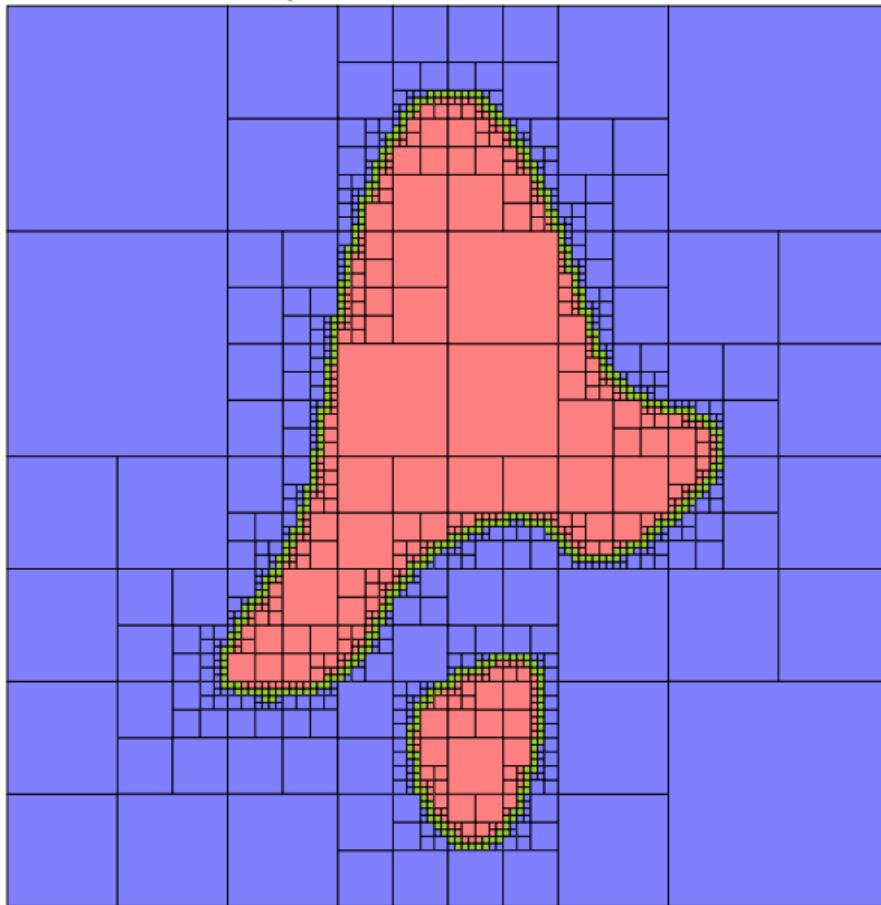
W. Randolph Franklin

```
function f(x,y)
    return inside(x,y) and -1 or 1
end

function inside(x,y)
    local inside=false
    for i=1,N do
        local j=i+1
        if ((Y[i]>y)~=(Y[j]>y)) and
            (x<((X[j]-X[i])*(y-Y[i]))/(Y[j]-Y[i])+X[i])) then
            inside = not inside
        end
    end
    return inside
end
```

Enumeração espacial semi-adaptativa

Suffern (1990)



Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
- $X = X_1 \times \cdots \times X_n \subseteq \Omega$
- oráculo: intervalo $F(X) \subseteq \mathbf{R}$

$$F(X) \supseteq f(X) = \{f(x) : x \in X\}$$

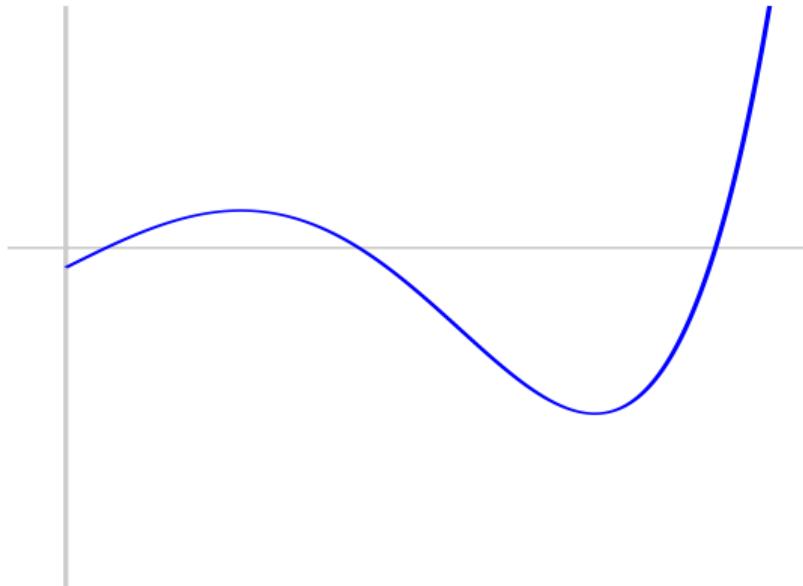
$$x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$$

Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
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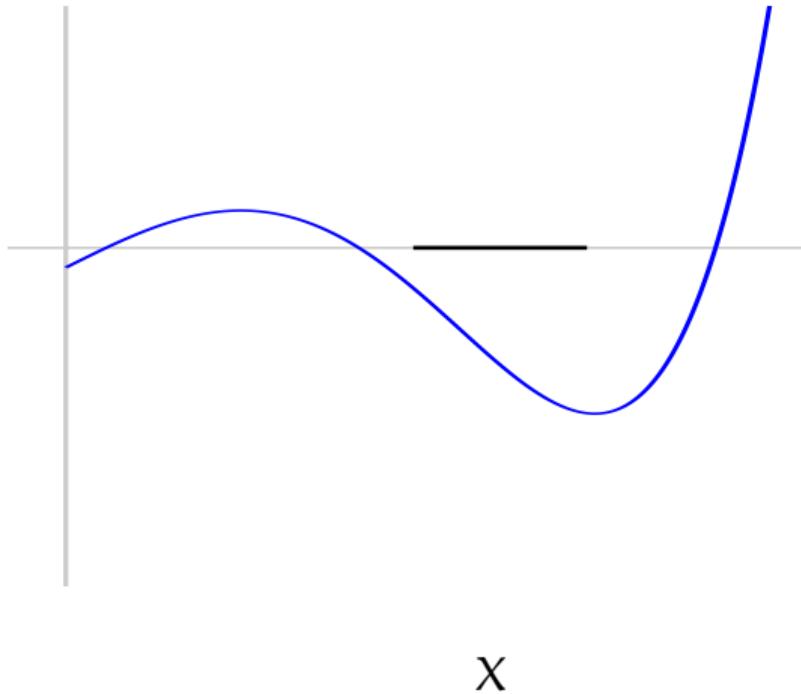
f

Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
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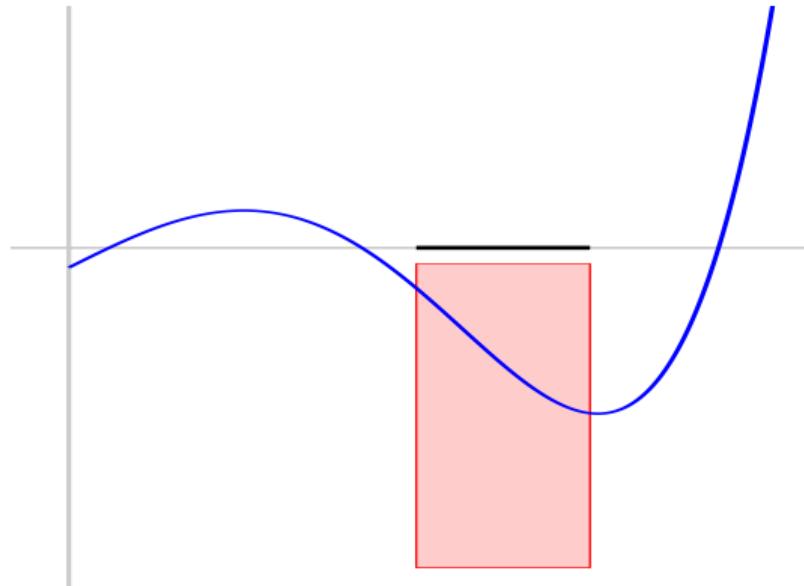


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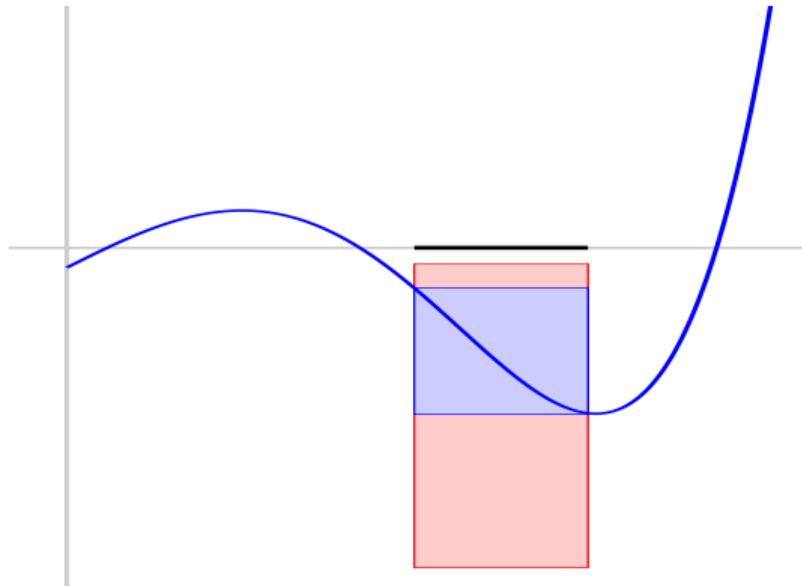
$F(X)$

Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
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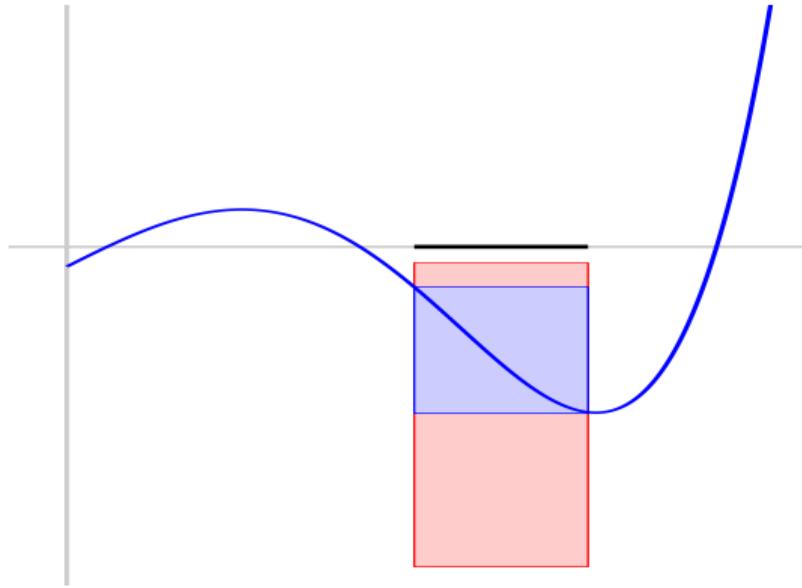


$$F(X) \supsetneq f(X)$$

Oráculos intervalares

- $f: \Omega \subseteq \mathbf{R}^n \rightarrow \mathbf{R}$
 - $X = X_1 \times \cdots \times X_n \subseteq \Omega$
 - oráculo: intervalo $F(X) \subseteq \mathbf{R}$
- $F(X) \supseteq f(X) = \{f(x) : x \in X\}$
- $x_i \in X_i \implies f(x_1, \dots, x_n) \in F(X)$
- $0 \notin F(X) \implies 0 \notin f(X)$
 - prova computational

$f(x) = 0$ não tem solução em X

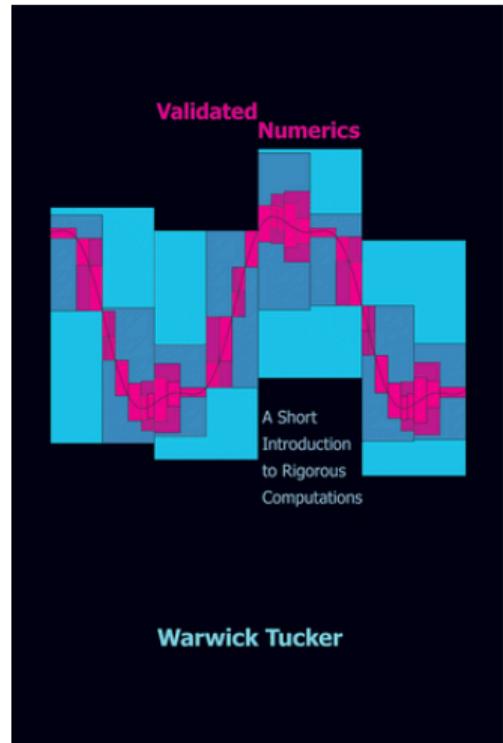
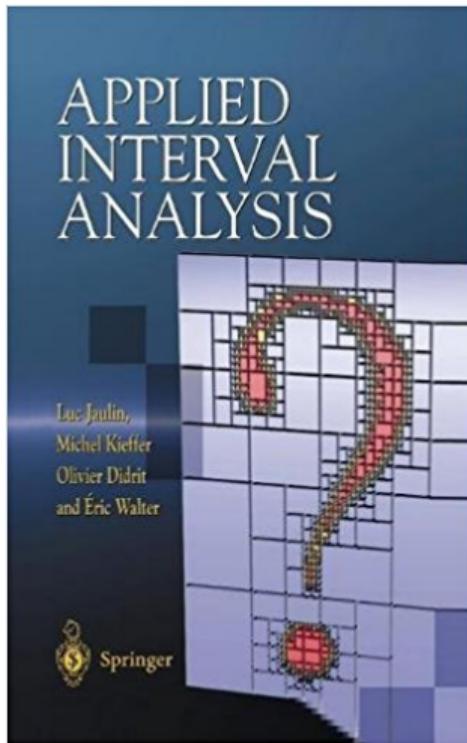
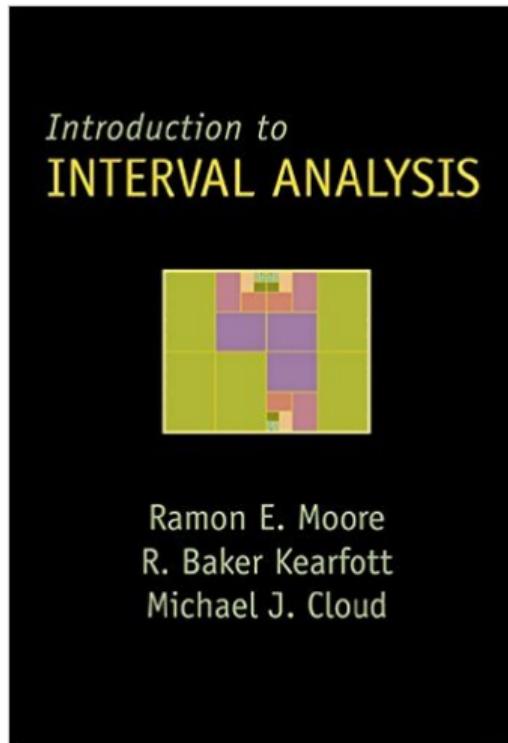


$$F(X) \supsetneq f(X)$$

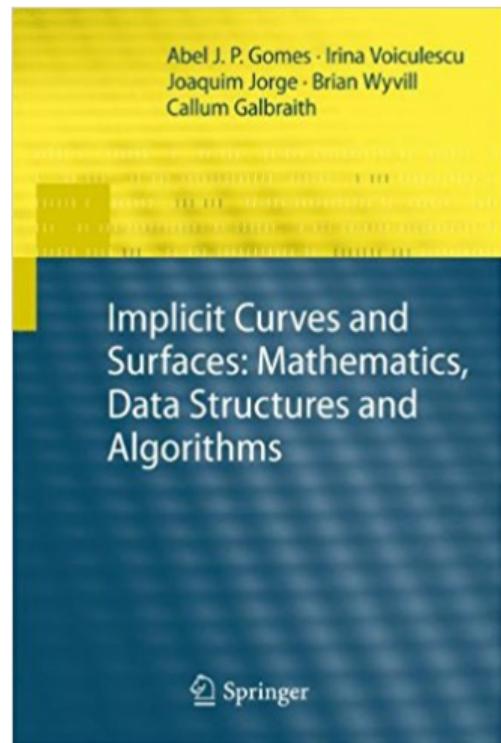
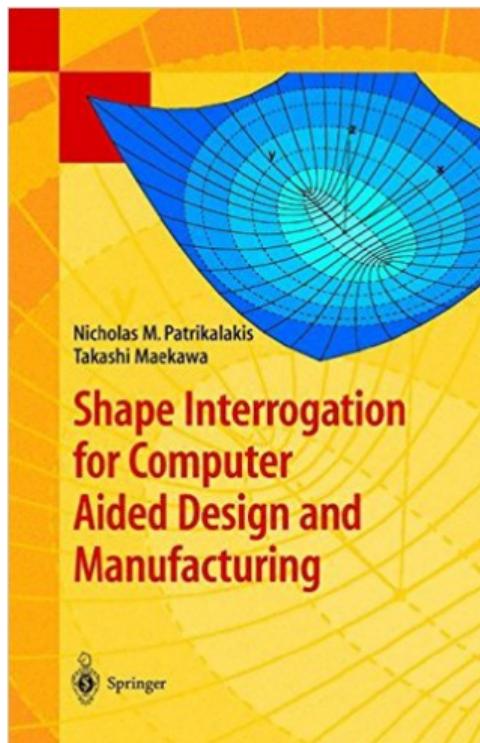
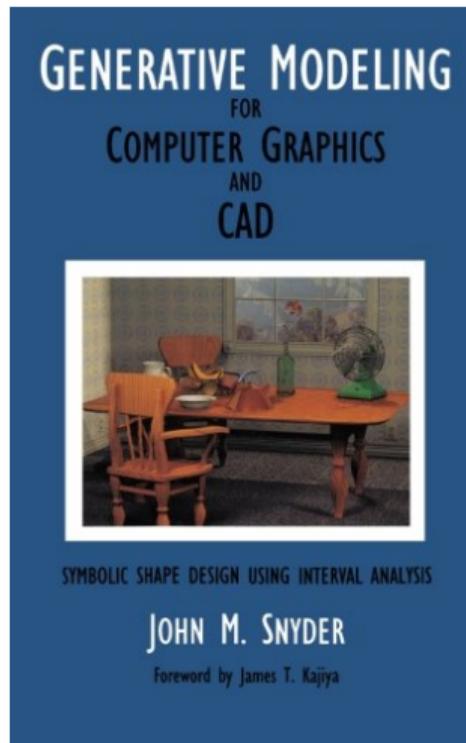
Oráculos intervalares

- aritmética intervalar Moore (1966)
- formas de valor médio Caprani–Madsen (1980)
- aritmética afim Comba–Stolfi (1993)
- formas de Taylor Berz et al. (1996)

Métodos intervalares



Métodos intervalares



Aritmética intervalar

$$\begin{aligned}[a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)]\end{aligned}$$

Aritmética intervalar

$$\begin{aligned}[a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)]\end{aligned}$$

$$[a, b] + [c, d] = [a + c, b + d]$$

$$[a, b] - [c, d] = [a - d, b - c]$$

$$[a, b] \times [c, d] = [\min(ac, ad, bc, bd), \max(ac, ad, bc, bd)]$$

$$[a, b] / [c, d] = [a, b] \times [1/d, 1/c]$$

$$[a, b]^2 = [\min(a^2, b^2), \max(a^2, b^2)] \text{ se } 0 \notin [a, b]$$

$$[a, b]^2 = [0, \max(a^2, b^2)] \text{ se } 0 \in [a, b]$$

$$\exp[a, b] = [\exp(a), \exp(b)]$$

Aritmética intervalar

$$\begin{aligned}[a, b] \star [c, d] &= \{x \star y : x \in [a, b], y \in [c, d]\} \\ &= [\min(x \star y), \max(x \star y)]\end{aligned}$$

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$$[a, b]^2 = [0, \max(a^2, b^2)] \text{ se } 0 \in [a, b]$$

$$\exp[a, b] = [\exp(a), \exp(b)]$$

uso conveniente via sobrecarga de operadores

Região implícita – método intervalar

Suffern–Fackerell (1991), Snyder (1992)

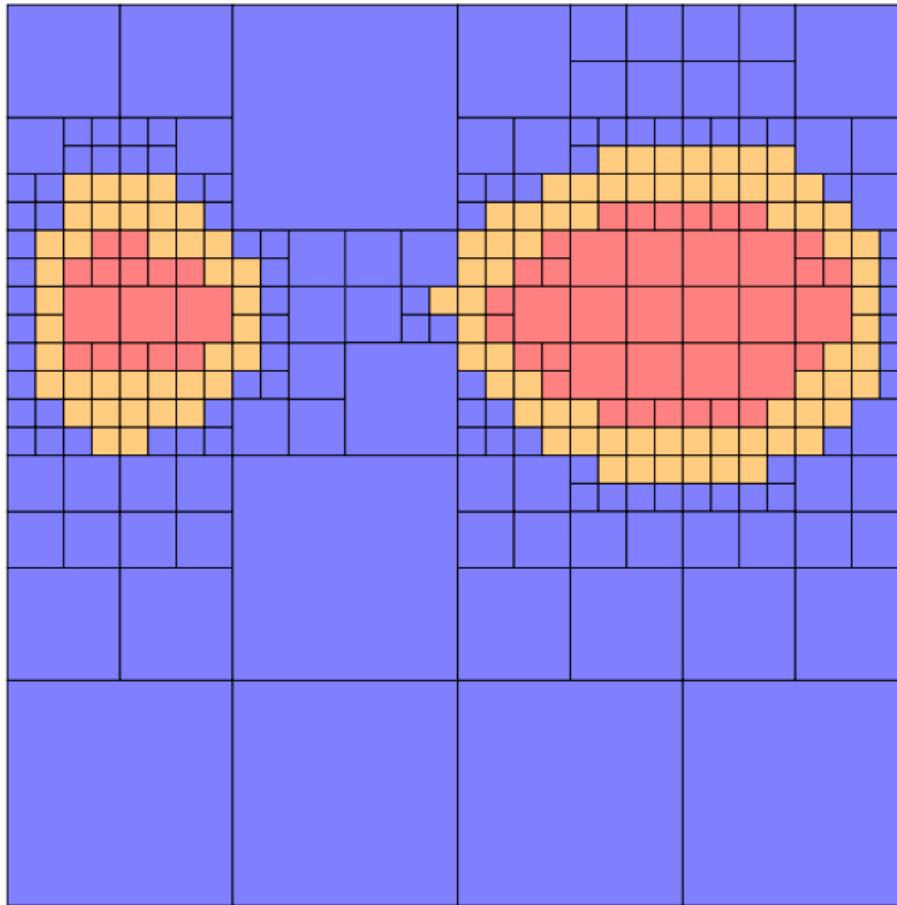
```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return show(x,y,"0") end
    if z<0 then return show(x,y,"I") end
    if level==0 then return show(x,y,"U") end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    explore(xlo,ylo,level-1)
    explore(xhi,ylo,level-1)
    explore(xlo,yhi,level-1)
    explore(xhi,yhi,level-1)
    return nil
end
```

Região implícita – método intervalar



“When you have eliminated the impossible,
whatever remains, however improbable,
must be the truth.”
—Sherlock Holmes in *The Sign of Four*

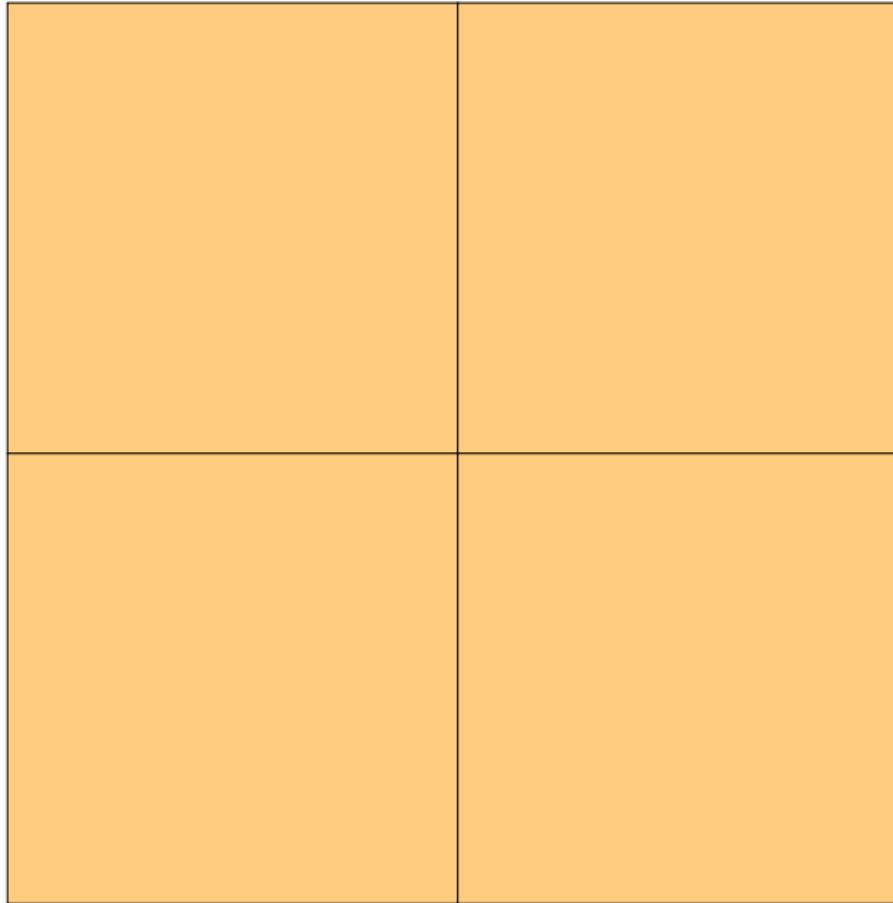
Região implícita – método intervalar



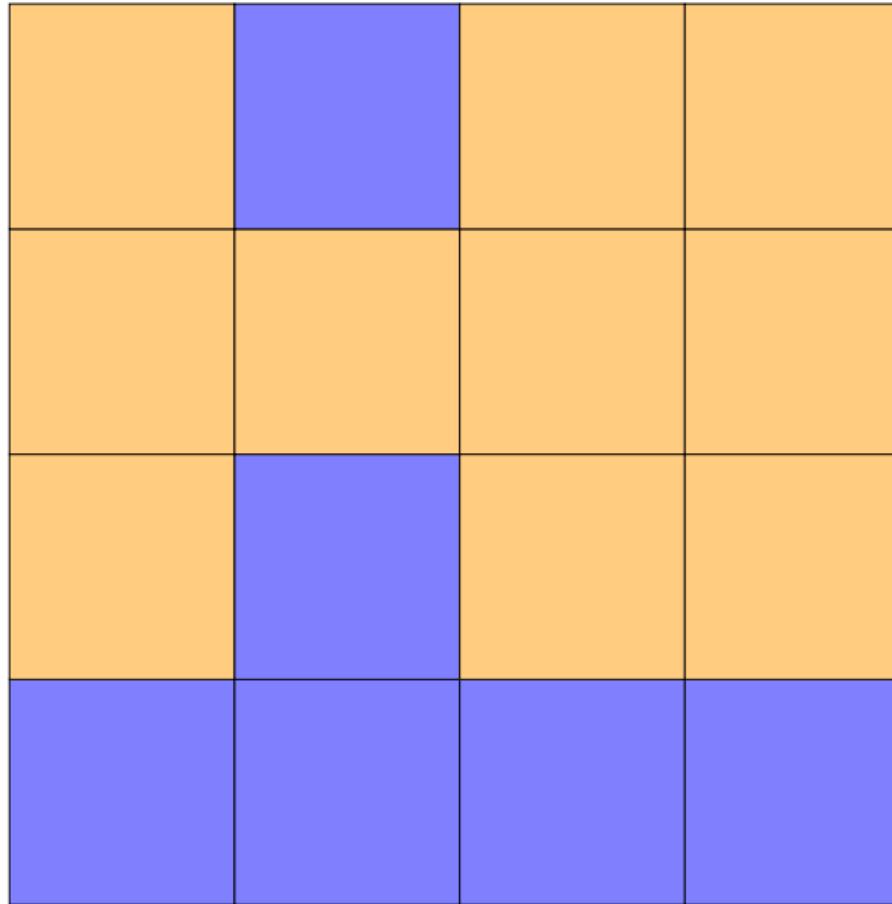
Região implícita – método intervalar



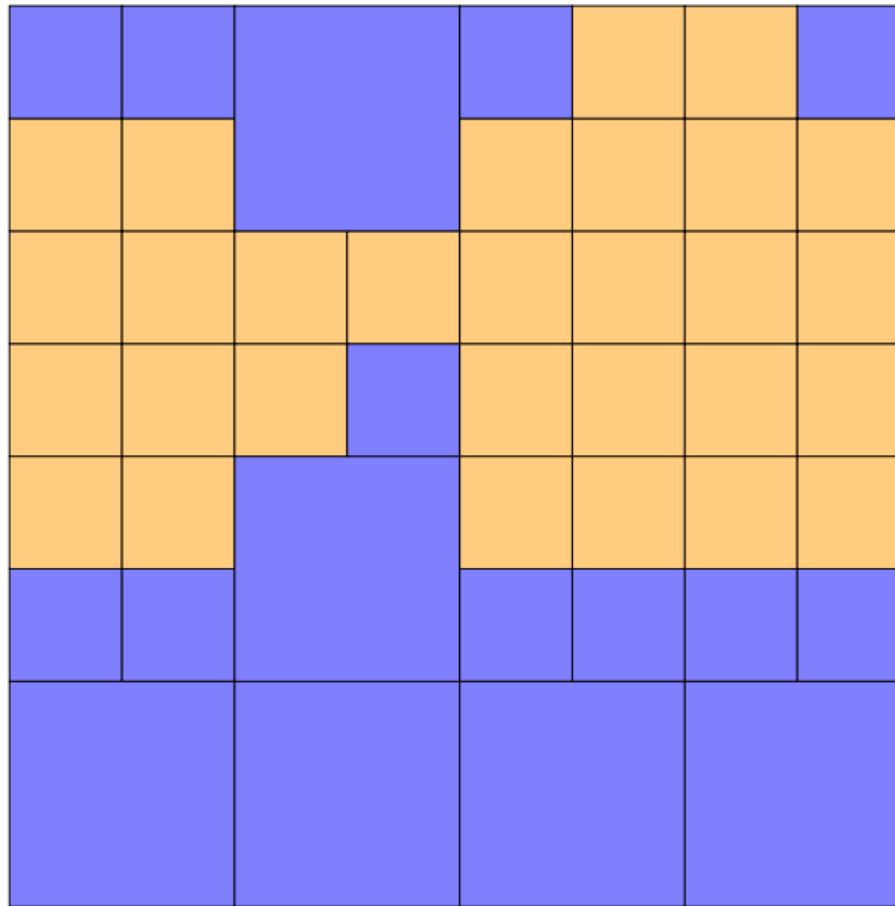
Região implícita – método intervalar



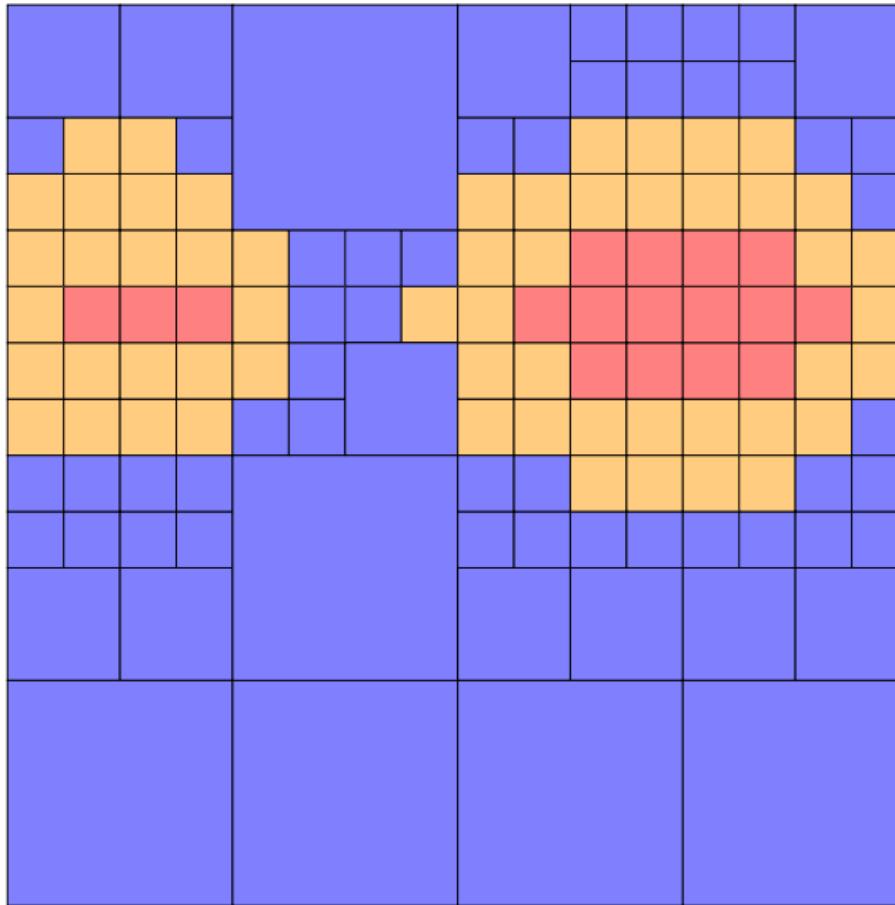
Região implícita – método intervalar



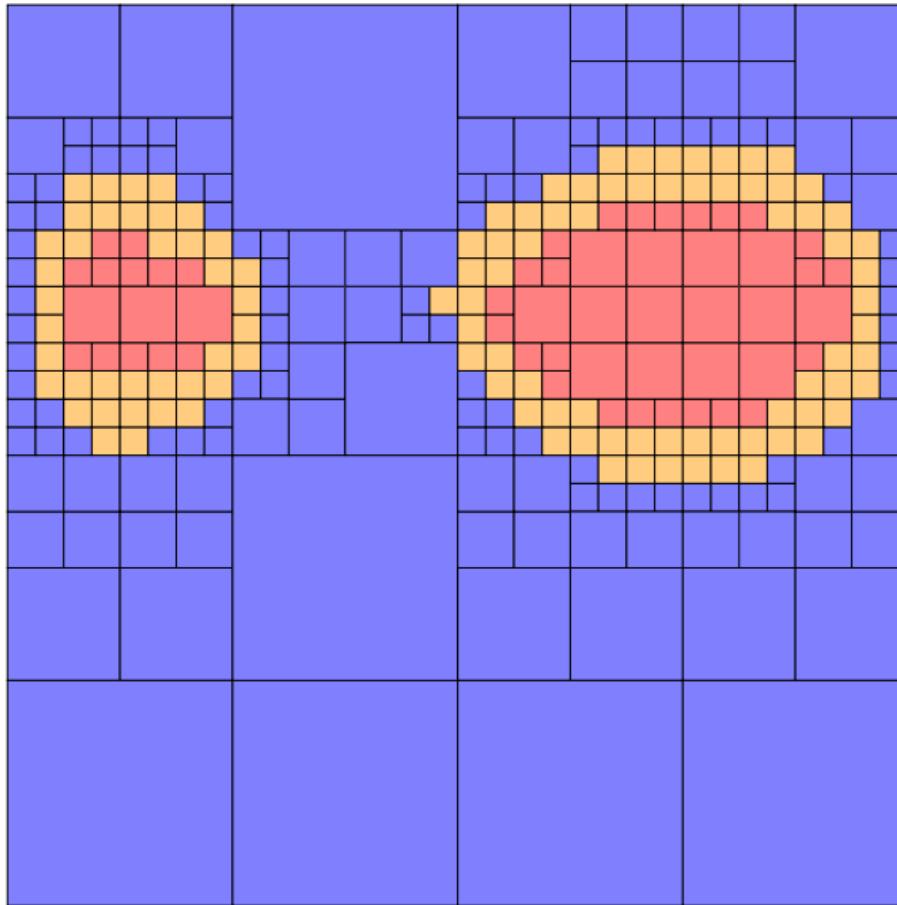
Região implícita – método intervalar



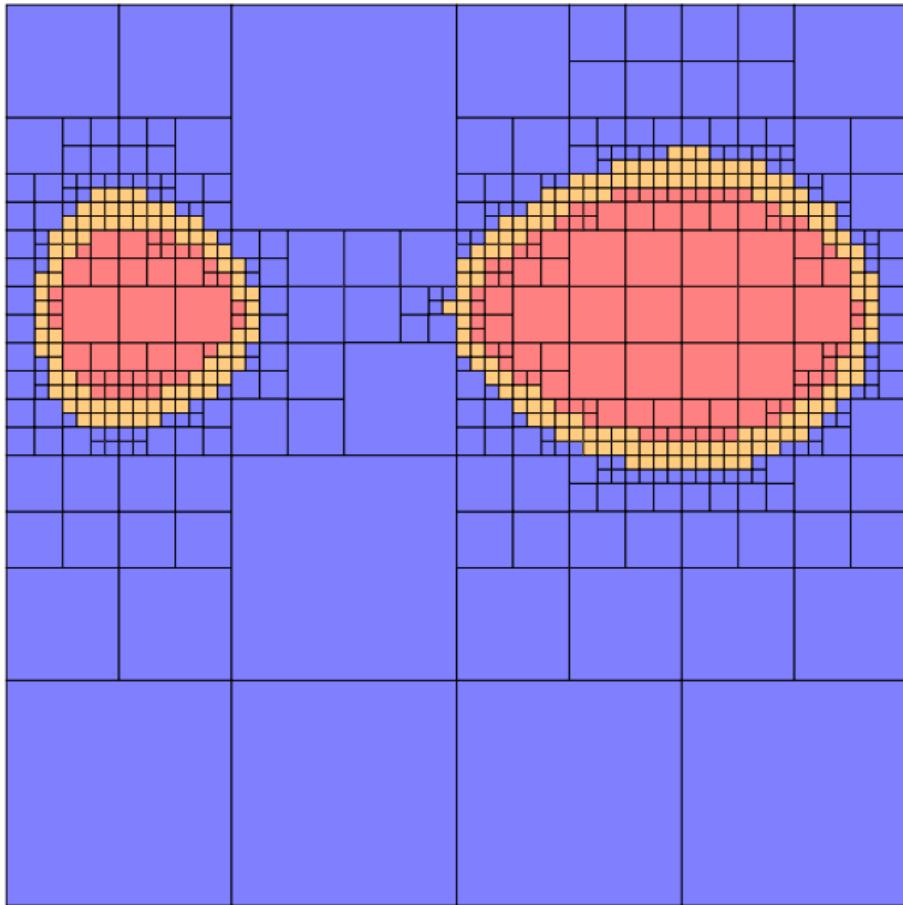
Região implícita – método intervalar



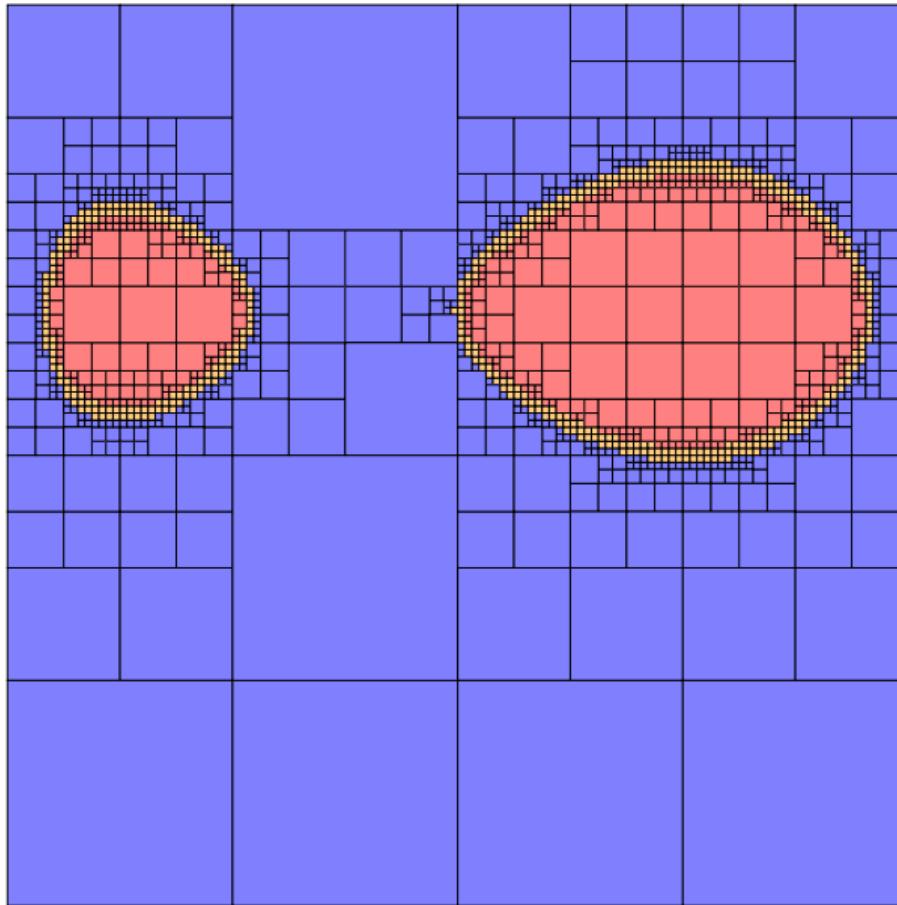
Região implícita – método intervalar



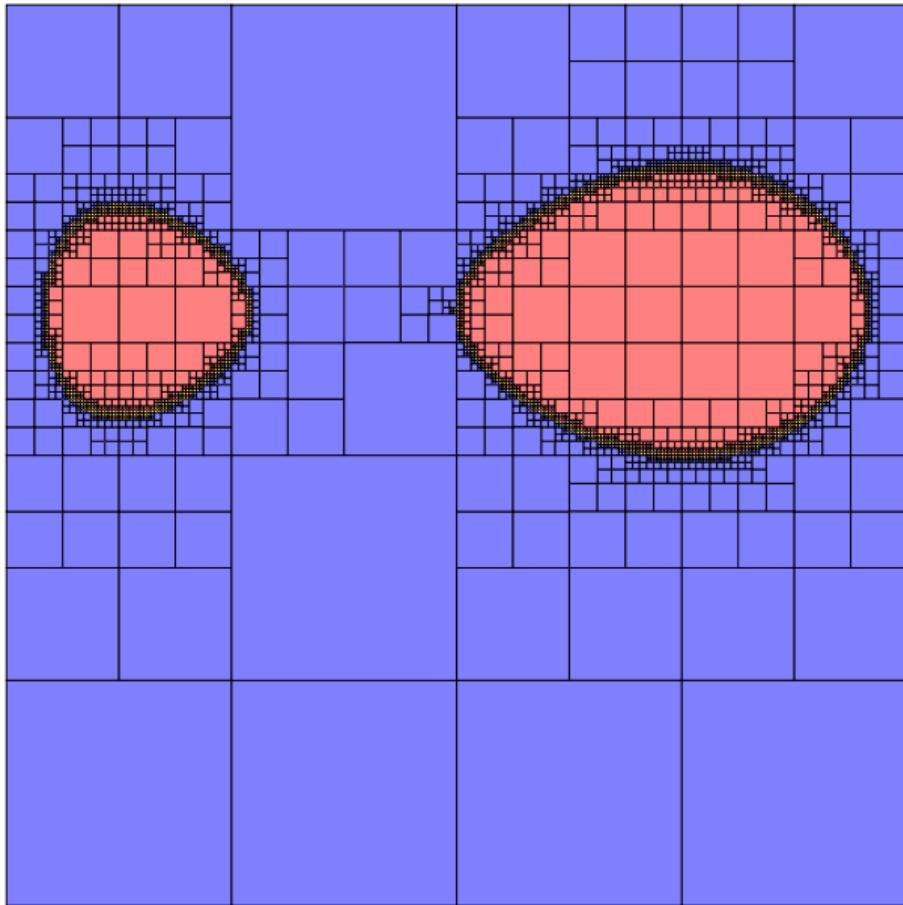
Região implícita – método intervalar



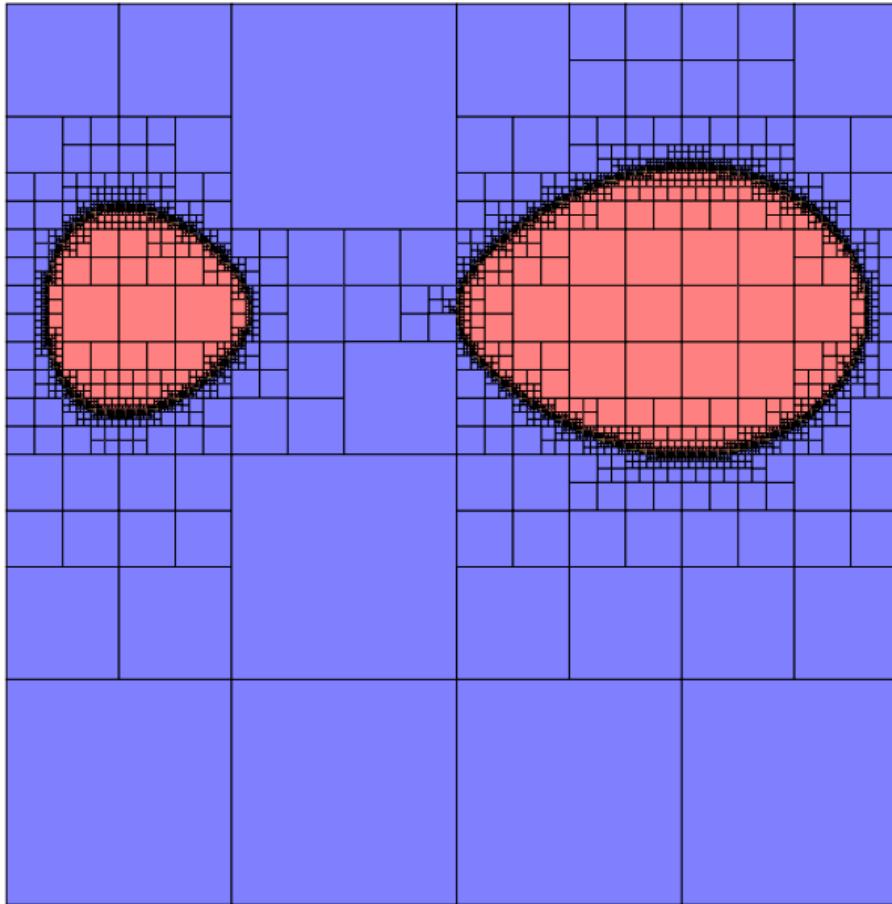
Região implícita – método intervalar



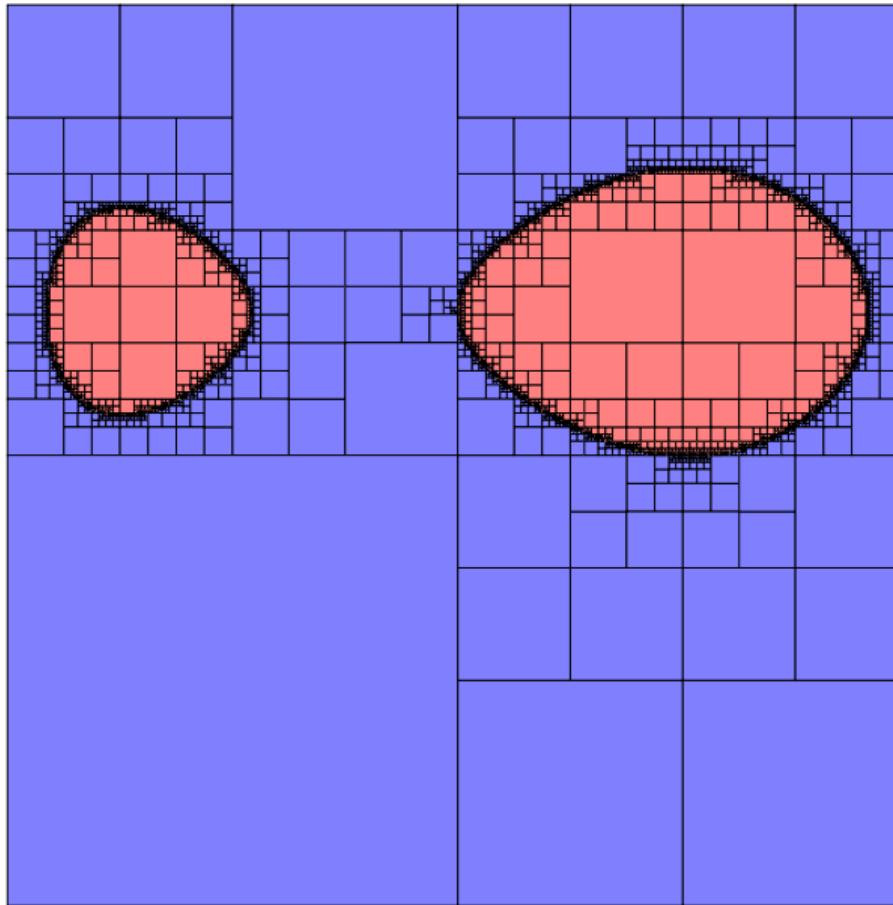
Região implícita – método intervalar



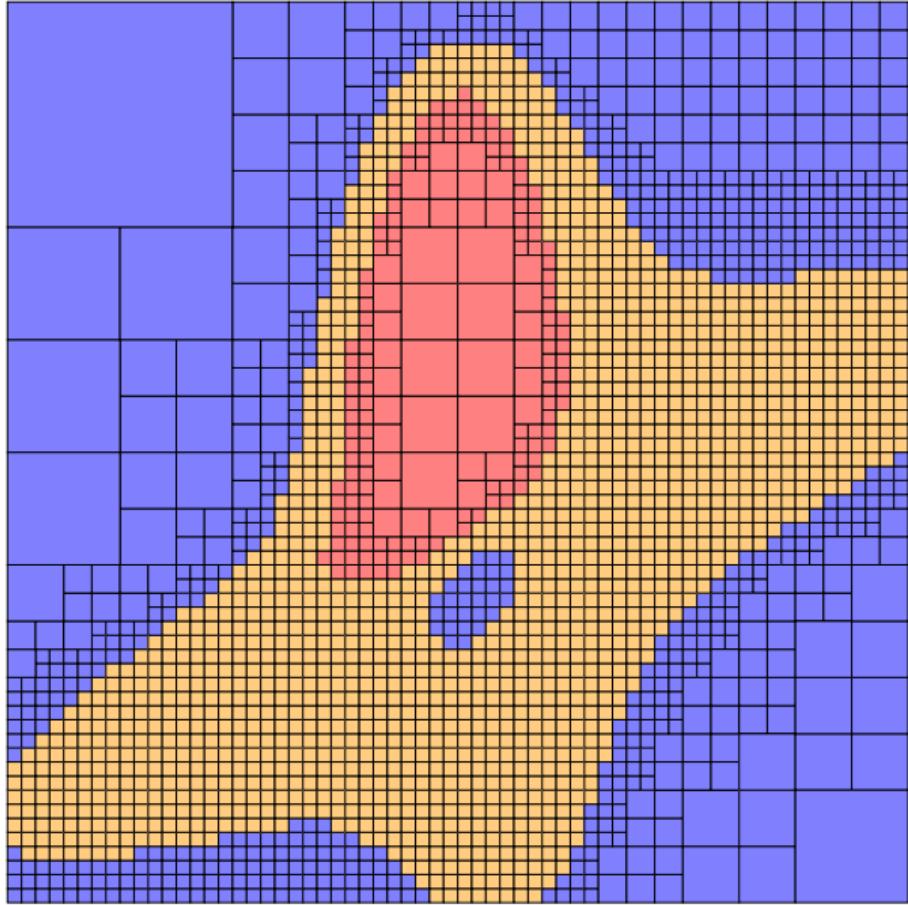
Região implícita – método intervalar



Região implícita – método intervalar

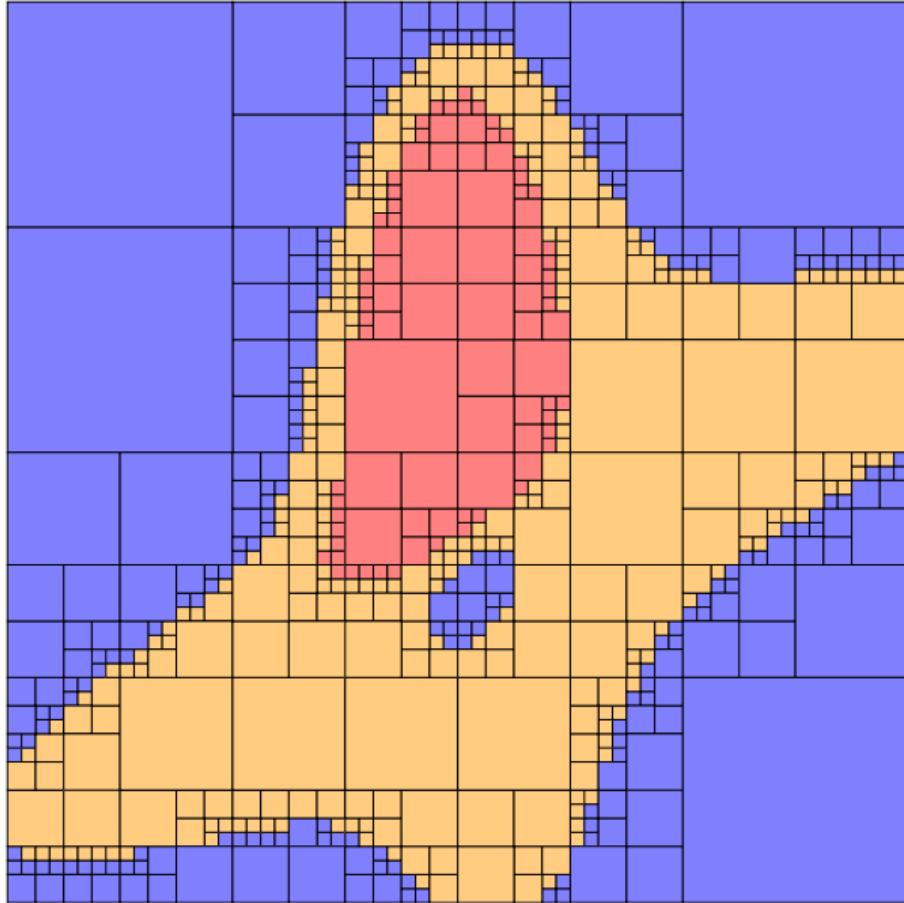


Região implícita – método intervalar



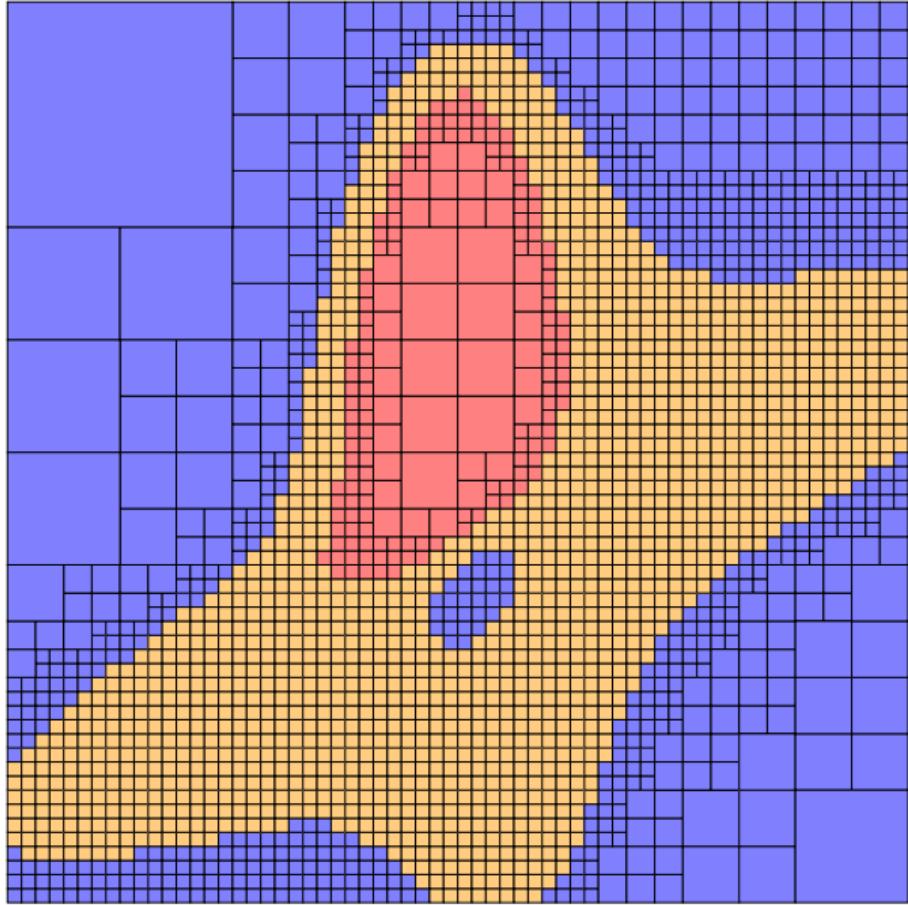
2423 leaves

Região implícita – método intervalar

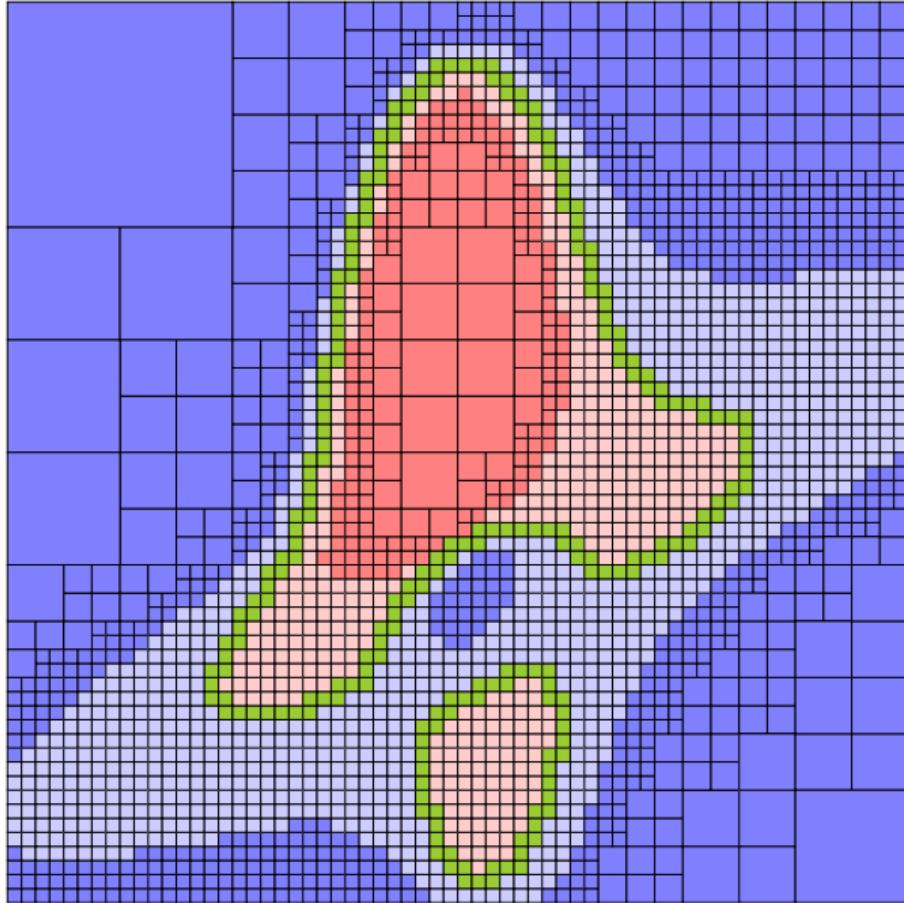


635 leaves

Região implícita – método intervalar

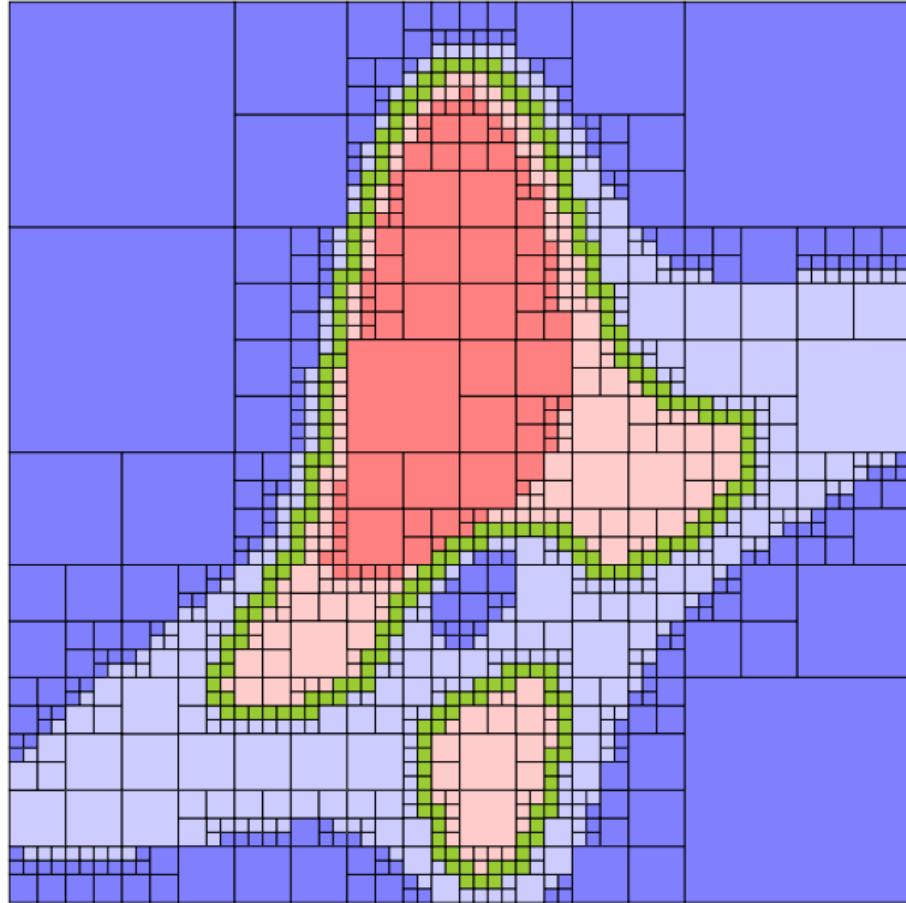


Região implícita – método intervalar híbrido

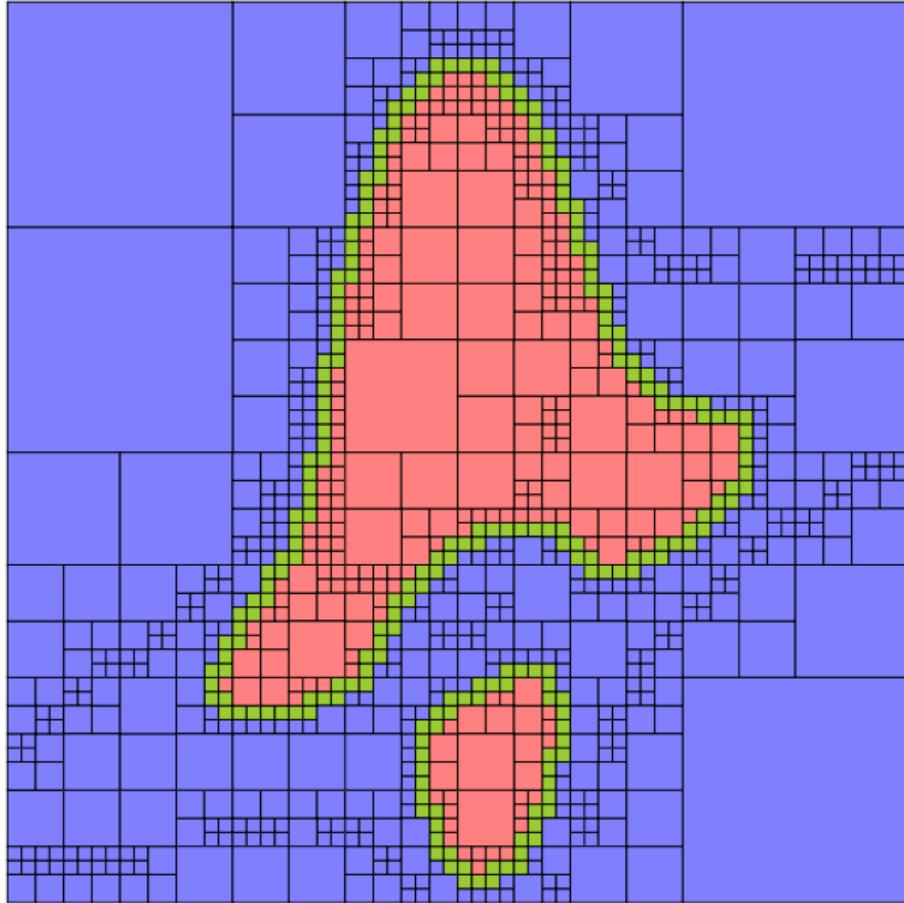


2423 leaves

Região implícita – método intervalar híbrido

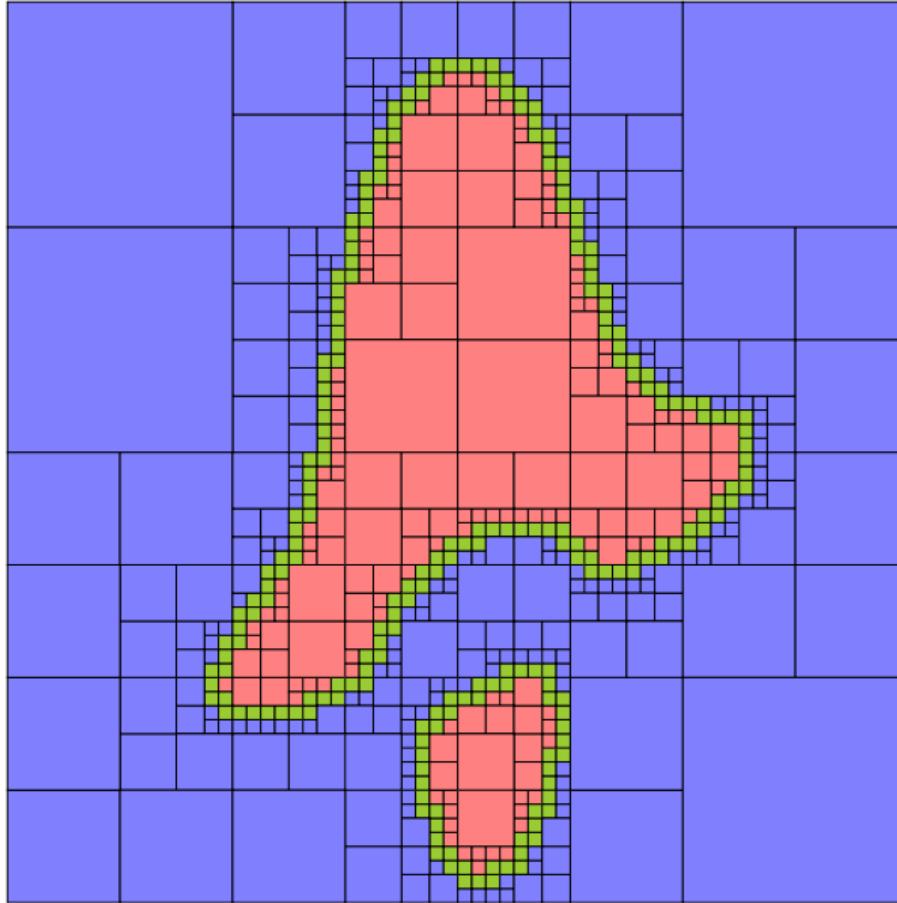


Região implícita – método intervalar híbrido otimista



1061 leaves

Região implícita – método intervalar híbrido otimista



644 leaves

Região implícita – método intervalar

```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return show(x,y,"O") end
    if z<0 then return show(x,y,"I") end
    if level==0 then return show(x,y,"U") end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    explore(xlo,ylo,level-1)
    explore(xhi,ylo,level-1)
    explore(xlo,yhi,level-1)
    explore(xhi,yhi,level-1)
    return nil
end
```

Região implícita – método intervalar híbrido otimista

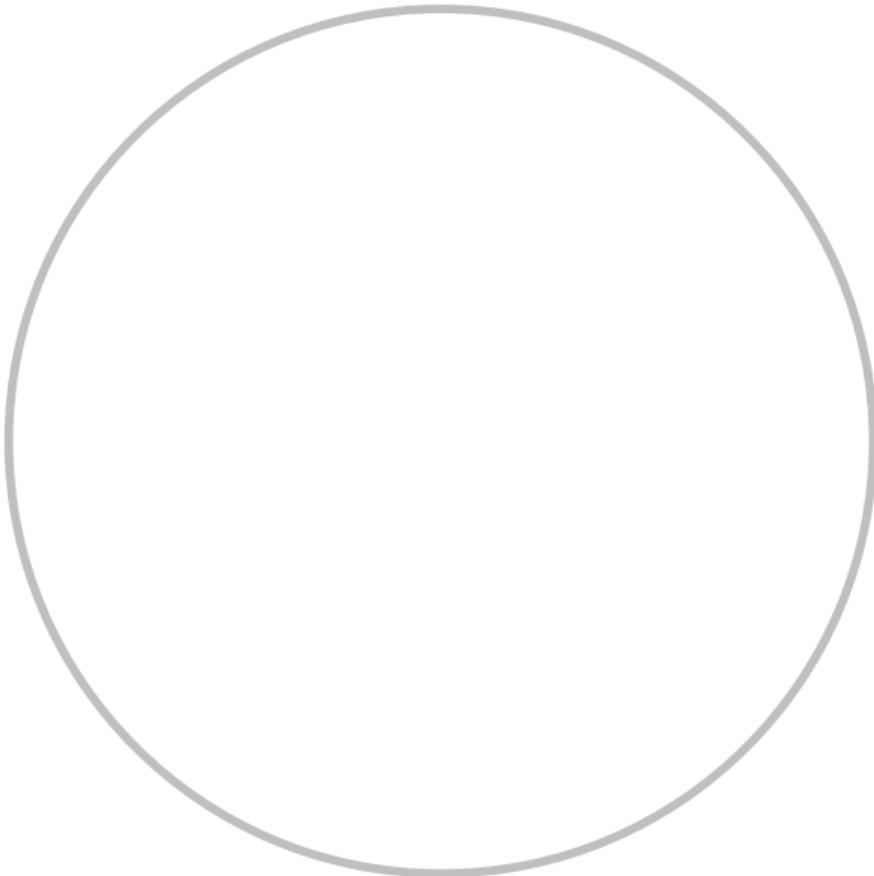
```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explode(xlo,ylo,level-1)
    local c2=explode(xhi,ylo,level-1)
    local c3=explode(xlo,yhi,level-1)
    local c4=explode(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
```

Região implícita – método intervalar híbrido otimista

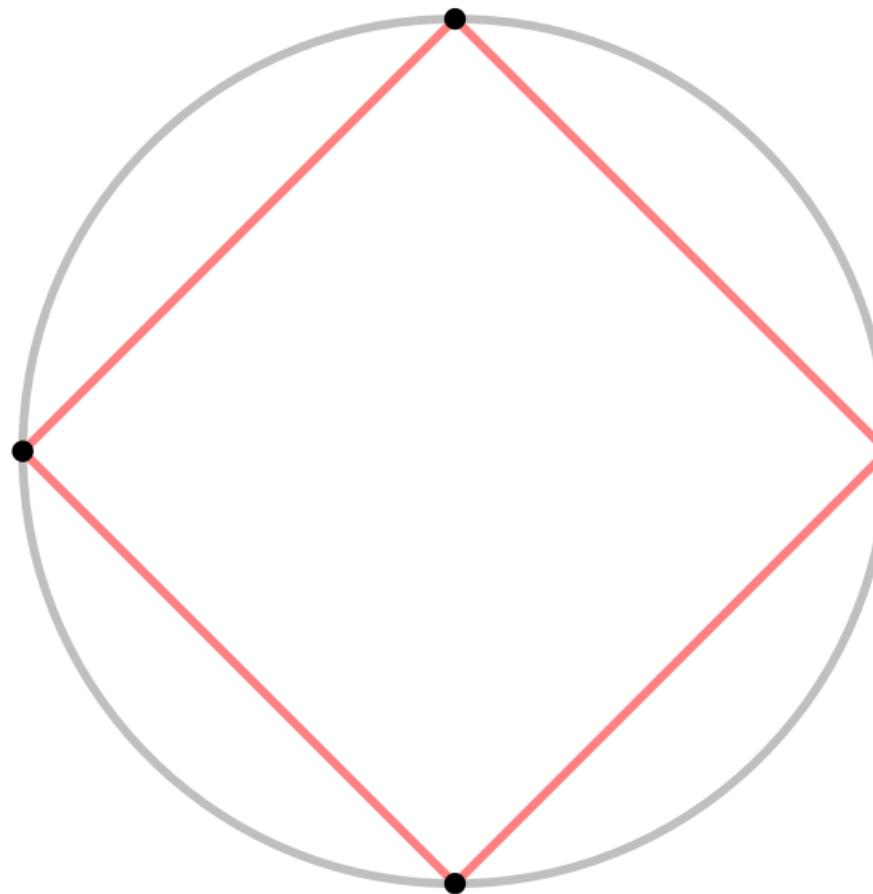
```
function explore(x,y,level)
    local z=f(x,y)
    if z>0 then return "O" end
    if z<0 then return "I" end
    if level==0 then return classify(x,y) end
    local xlo,xhi=split(x)
    local ylo,yhi=split(y)
    local c1=explode(xlo,ylo,level-1)
    local c2=explode(xhi,ylo,level-1)
    local c3=explode(xlo,yhi,level-1)
    local c4=explode(xhi,yhi,level-1)
    if c1==c2 and c1==c3 and c1==c4 then
        return c1
    else
        show(xlo,ylo,c1)
        show(xhi,ylo,c2)
        show(xlo,yhi,c3)
        show(xhi,yhi,c4)
        return nil
    end
end
```

```
function classify(x,y)
    local xmin,xmax=x:extremes()
    local ymin,ymax=y:extremes()
    local f1=f(xmin,ymin)
    local f2=f(xmax,ymin)
    local f3=f(xmax,ymax)
    local f4=f(xmin,ymax)
    if (f1>0 and f2>0 and f3>0 and f4>0) then
        return optimistic and "O" or "PO"
    end
    if (f1<0 and f2<0 and f3<0 and f4<0) then
        return optimistic and "I" or "PI"
    end
    return "V"
end
```

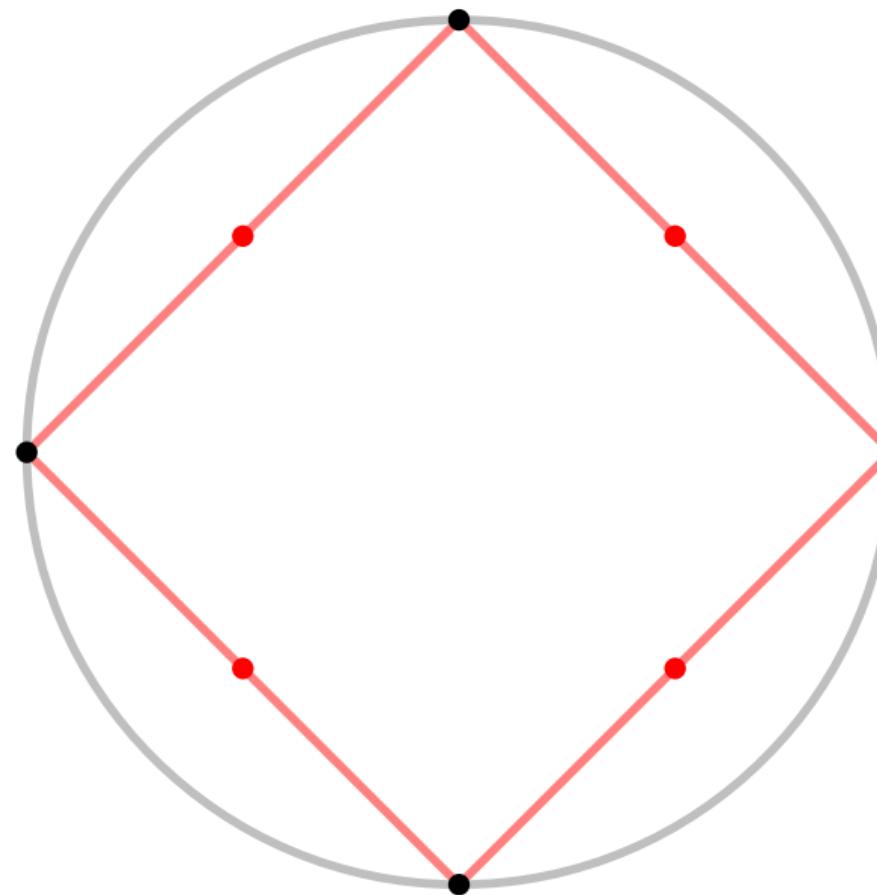
Curva implícita – refinamento intrínseco



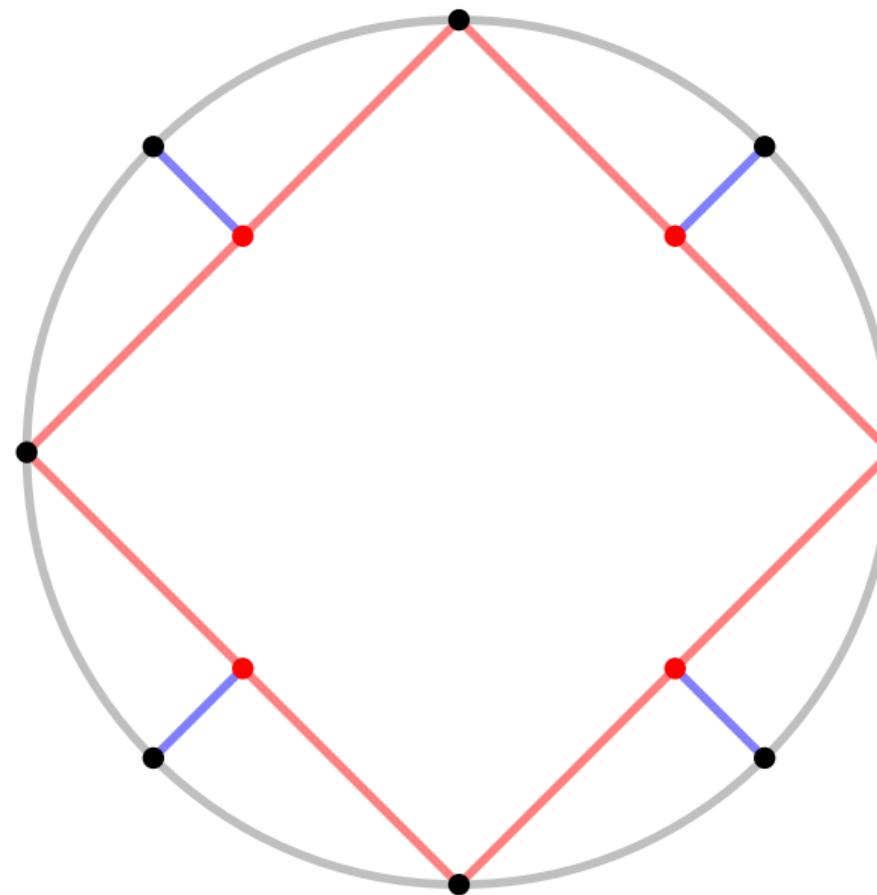
Curva implícita – refinamento intrínseco



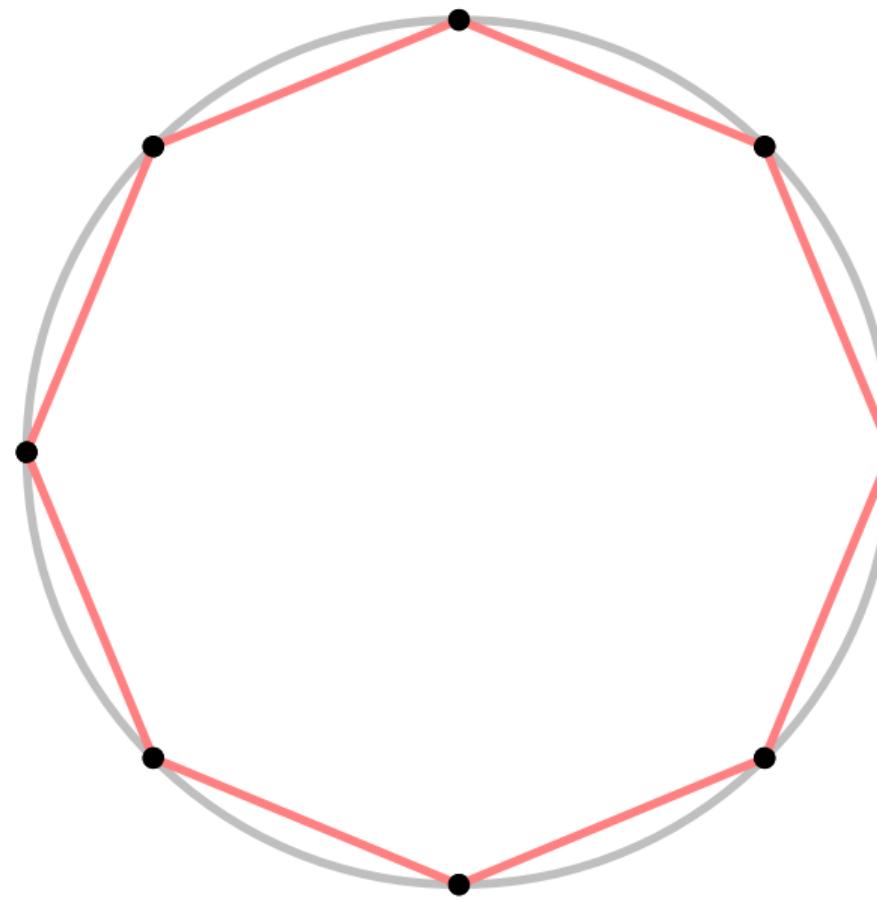
Curva implícita – refinamento intrínseco



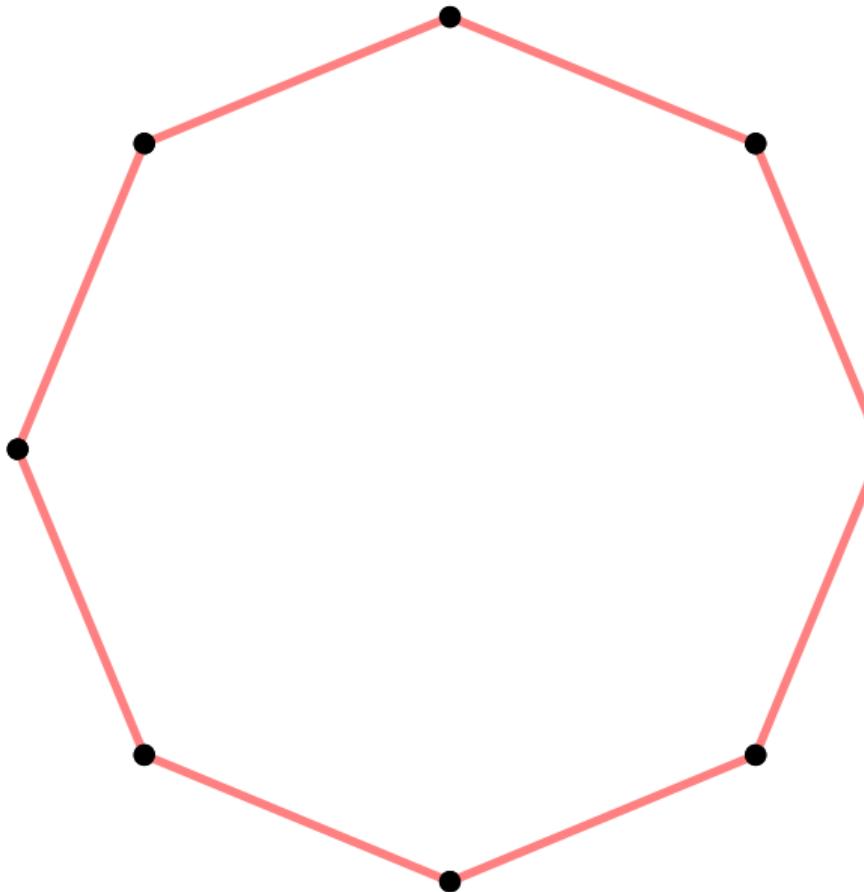
Curva implícita – refinamento intrínseco



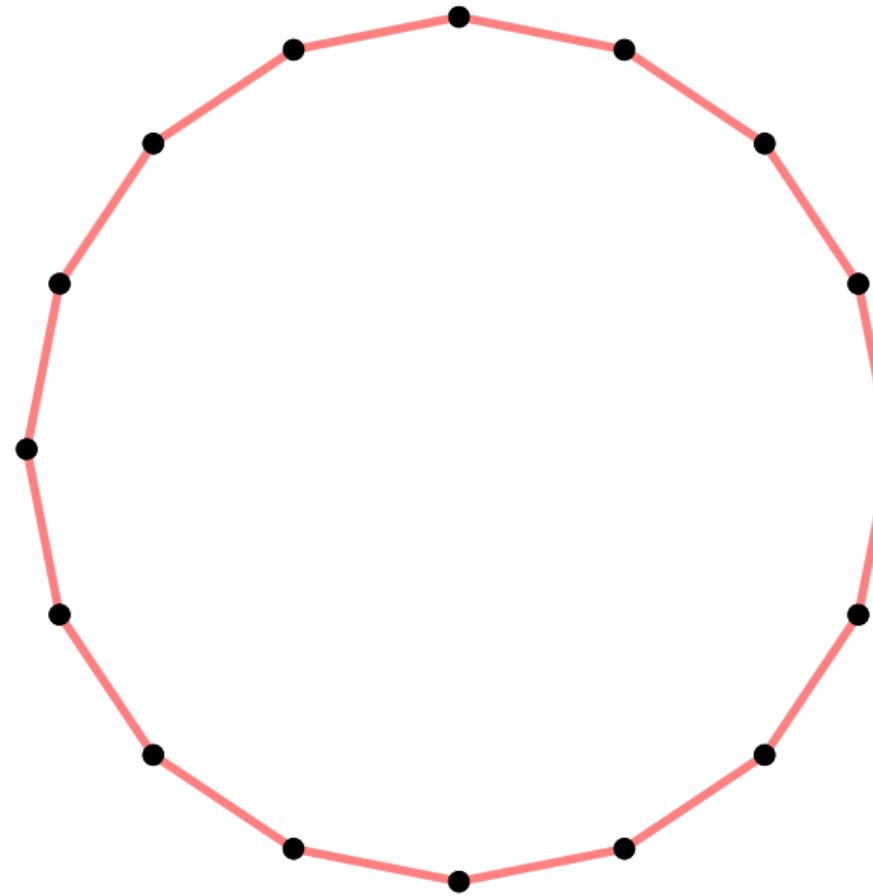
Curva implícita – refinamento intrínseco



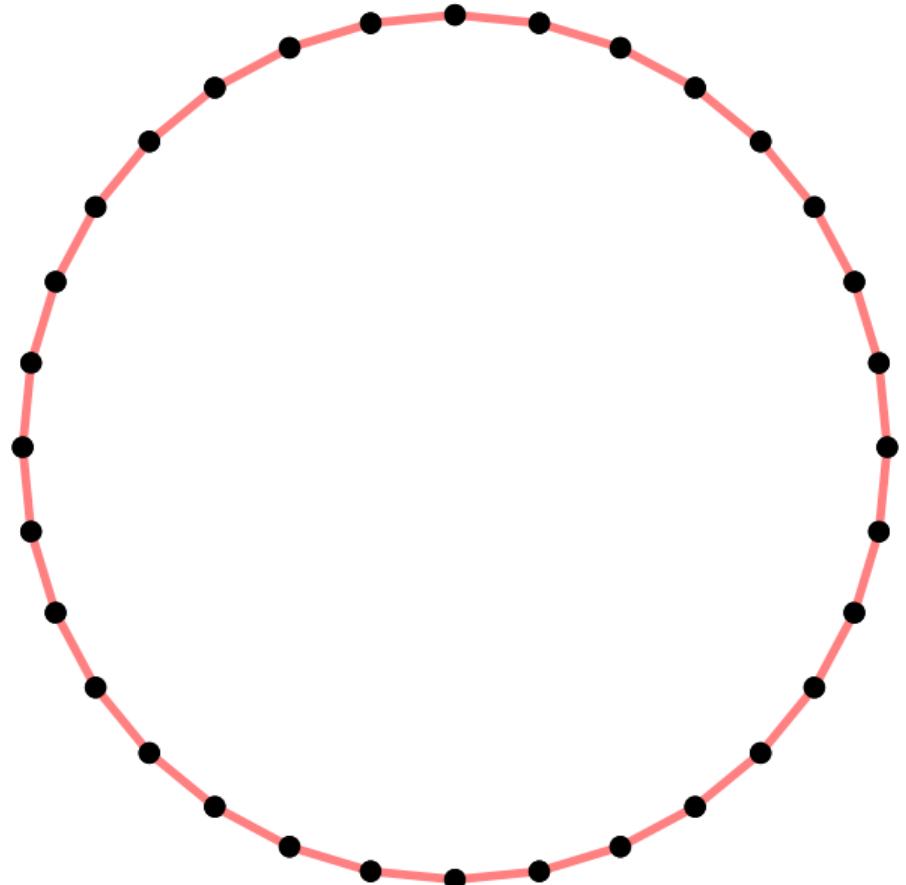
Curva implícita – refinamento intrínseco



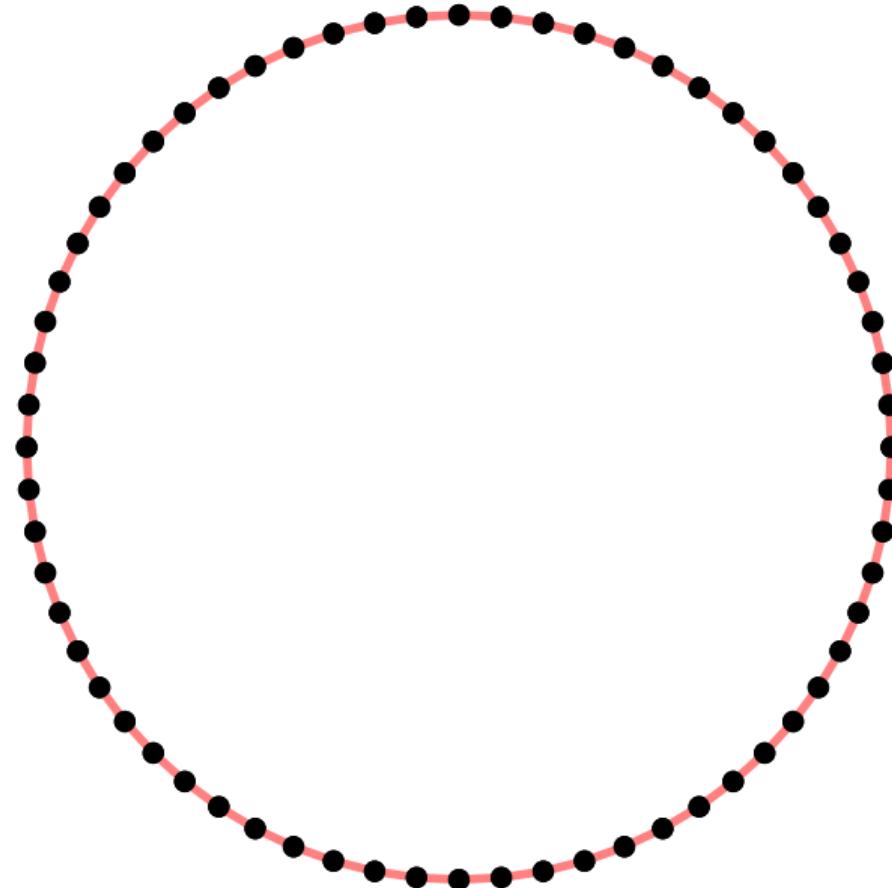
Curva implícita – refinamento intrínseco



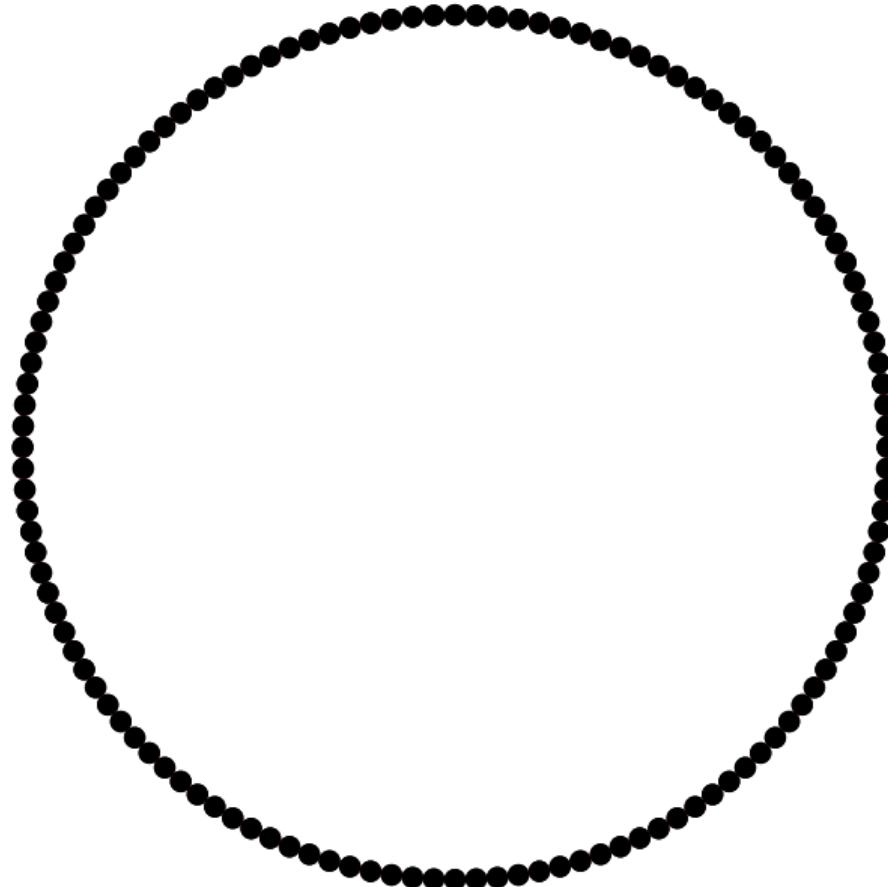
Curva implícita – refinamento intrínseco



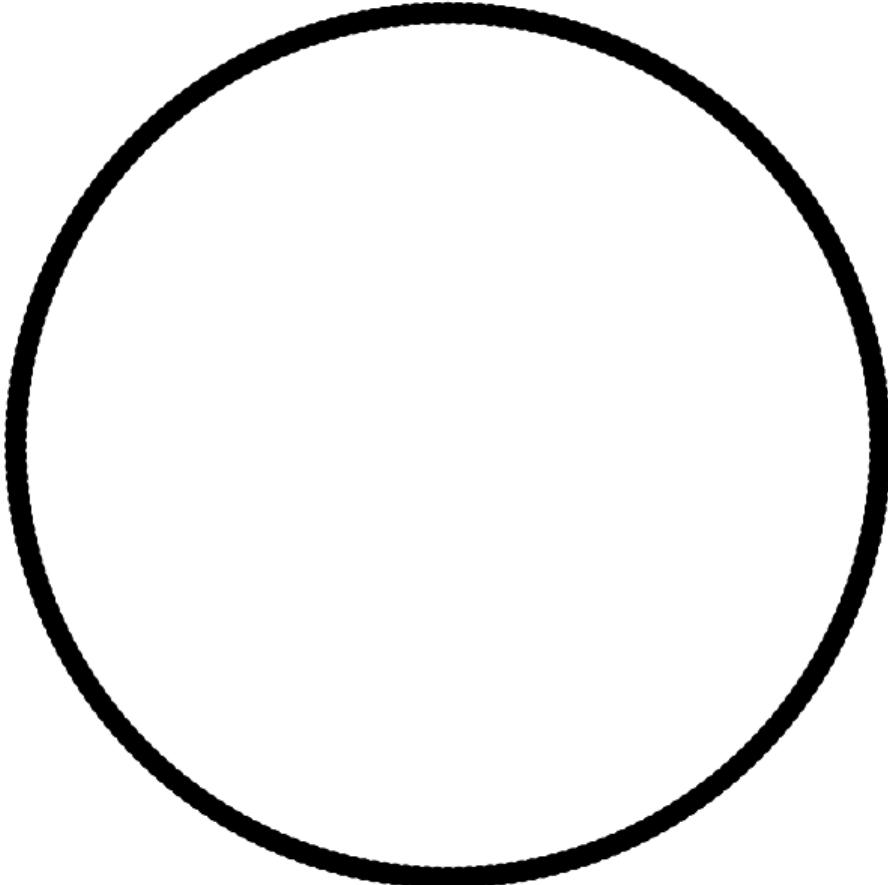
Curva implícita – refinamento intrínseco



Curva implícita – refinamento intrínseco



Curva implícita – refinamento intrínseco



objetos paramétricos

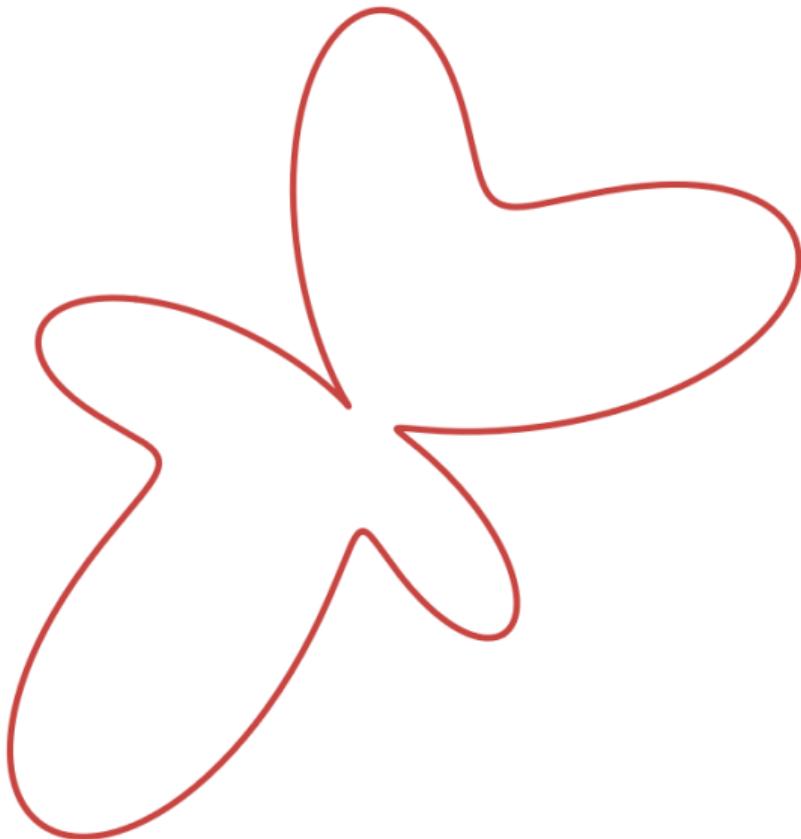
Curva paramétrica

curva = fronteira da região

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



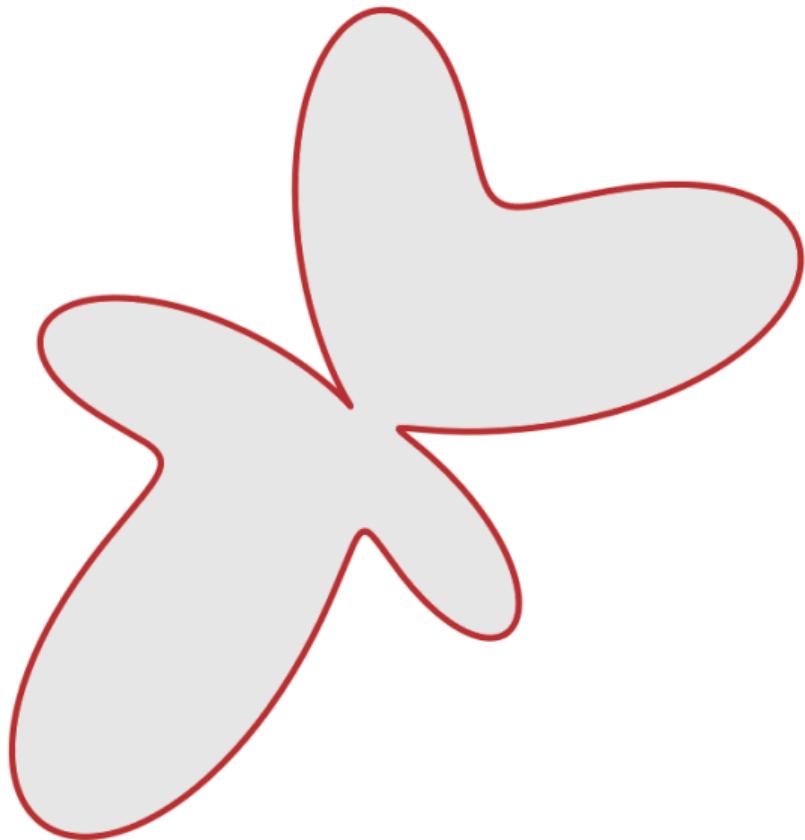
Curva paramétrica

curva = fronteira da região

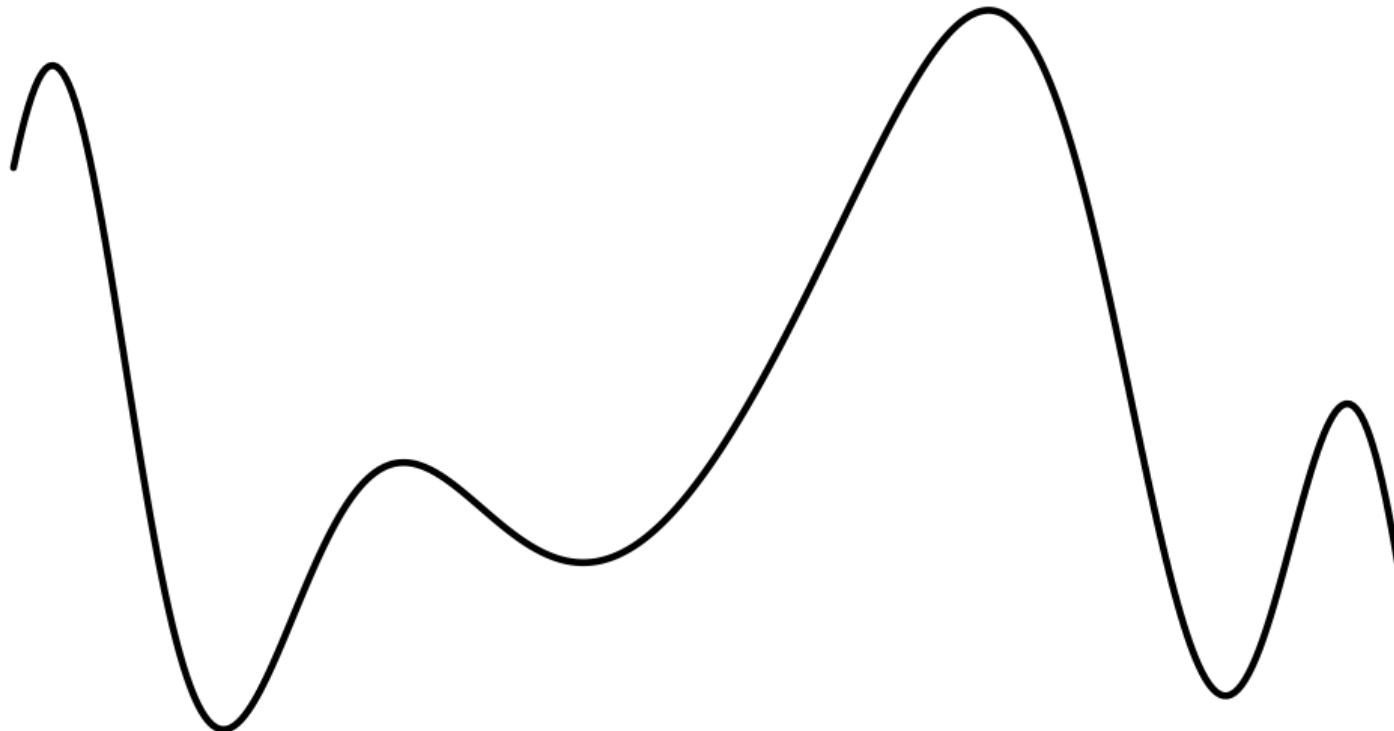
$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

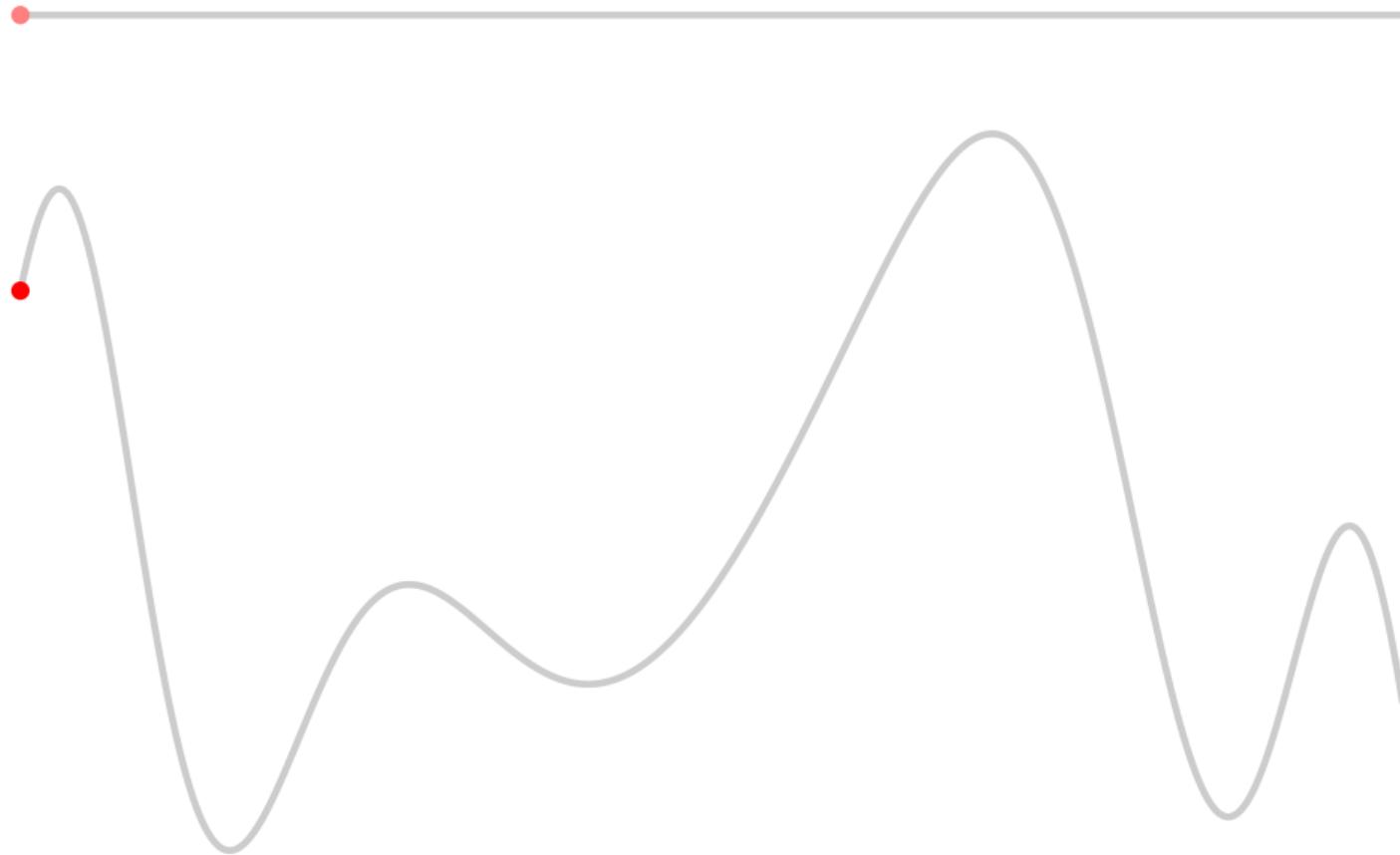
- amostrar fronteira
- amostrar interior
- classificação de pontos
- classificação de caixas



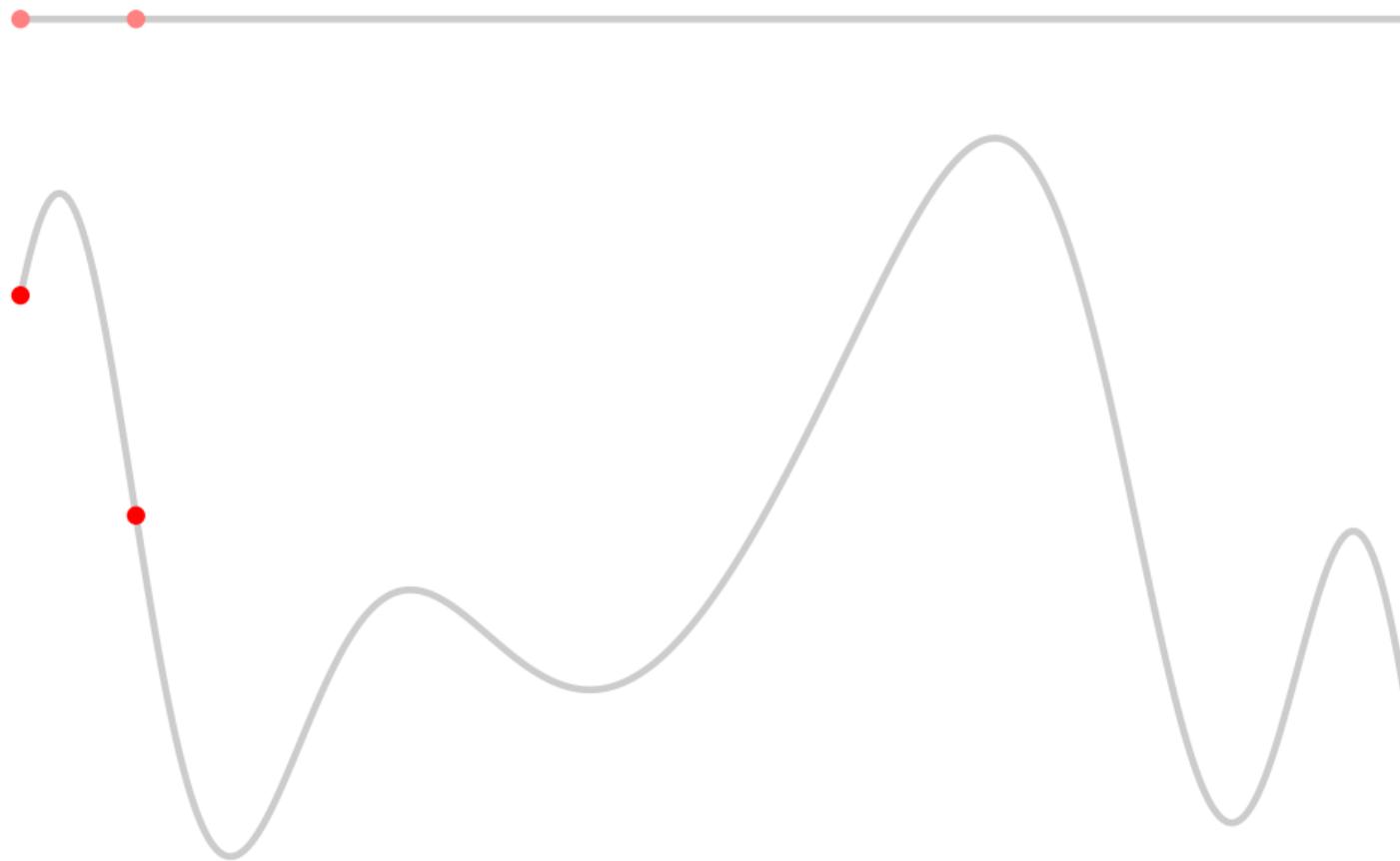
Curva paramétrica – amostragem uniforme no domínio



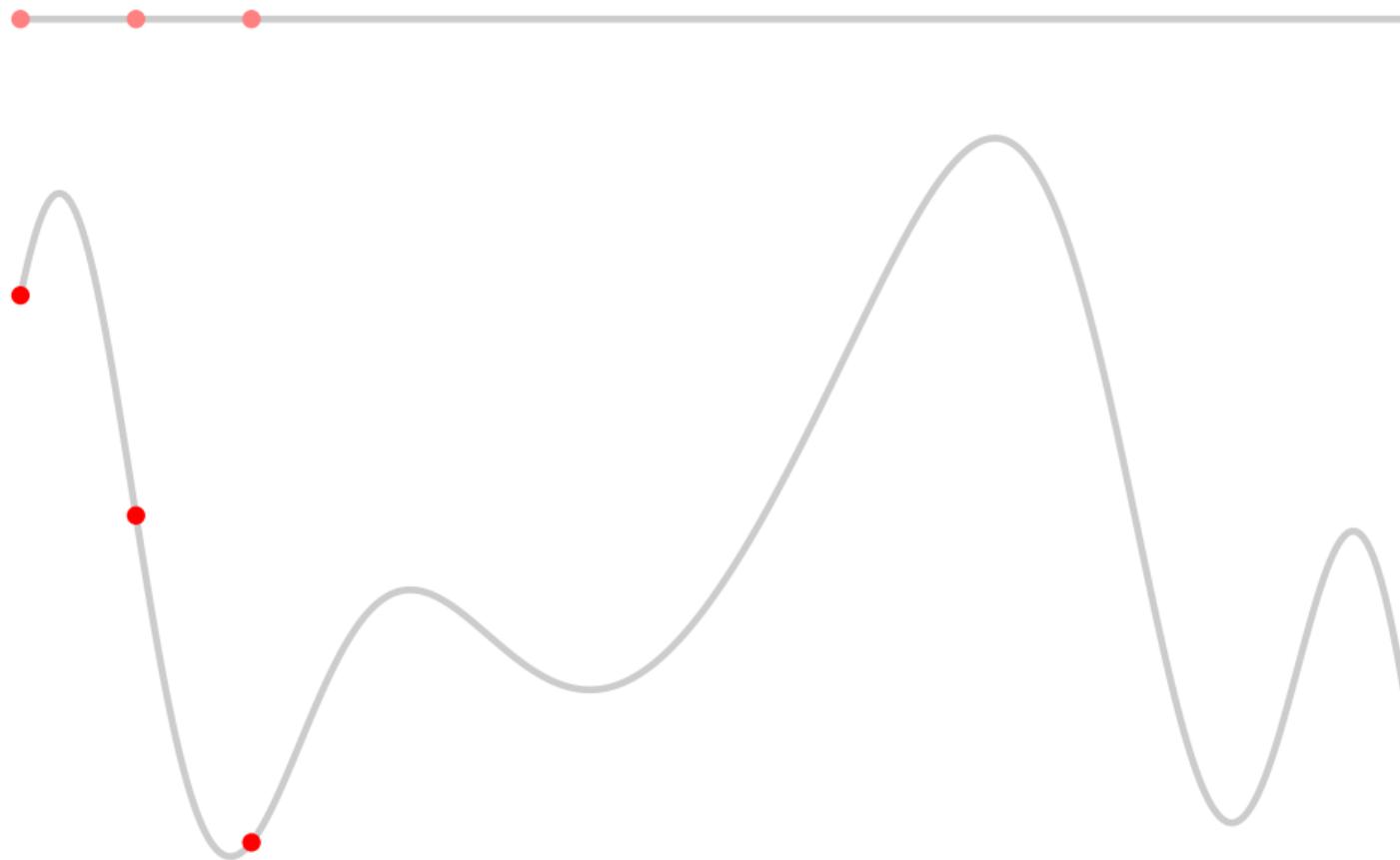
Curva paramétrica – amostragem uniforme no domínio



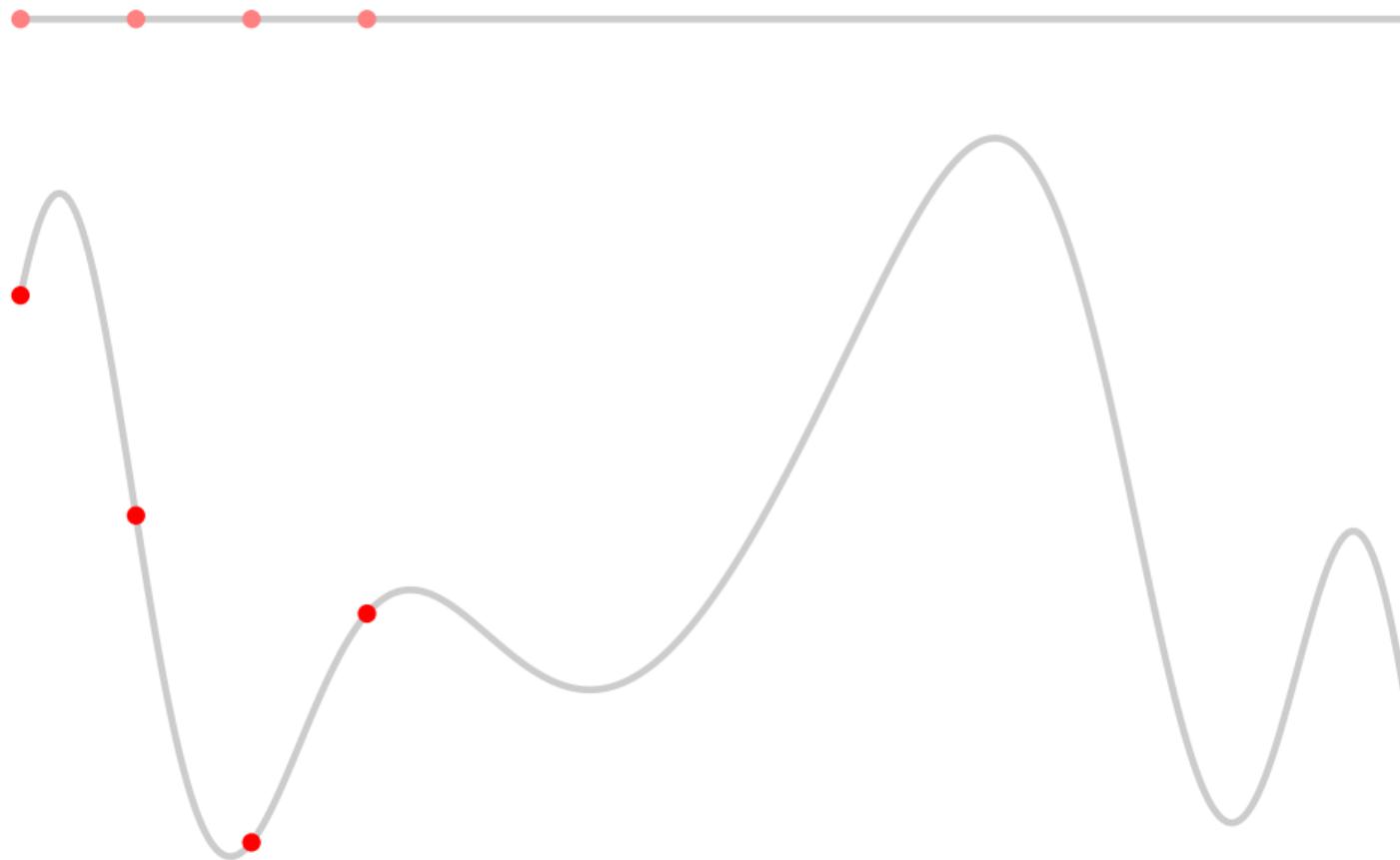
Curva paramétrica – amostragem uniforme no domínio



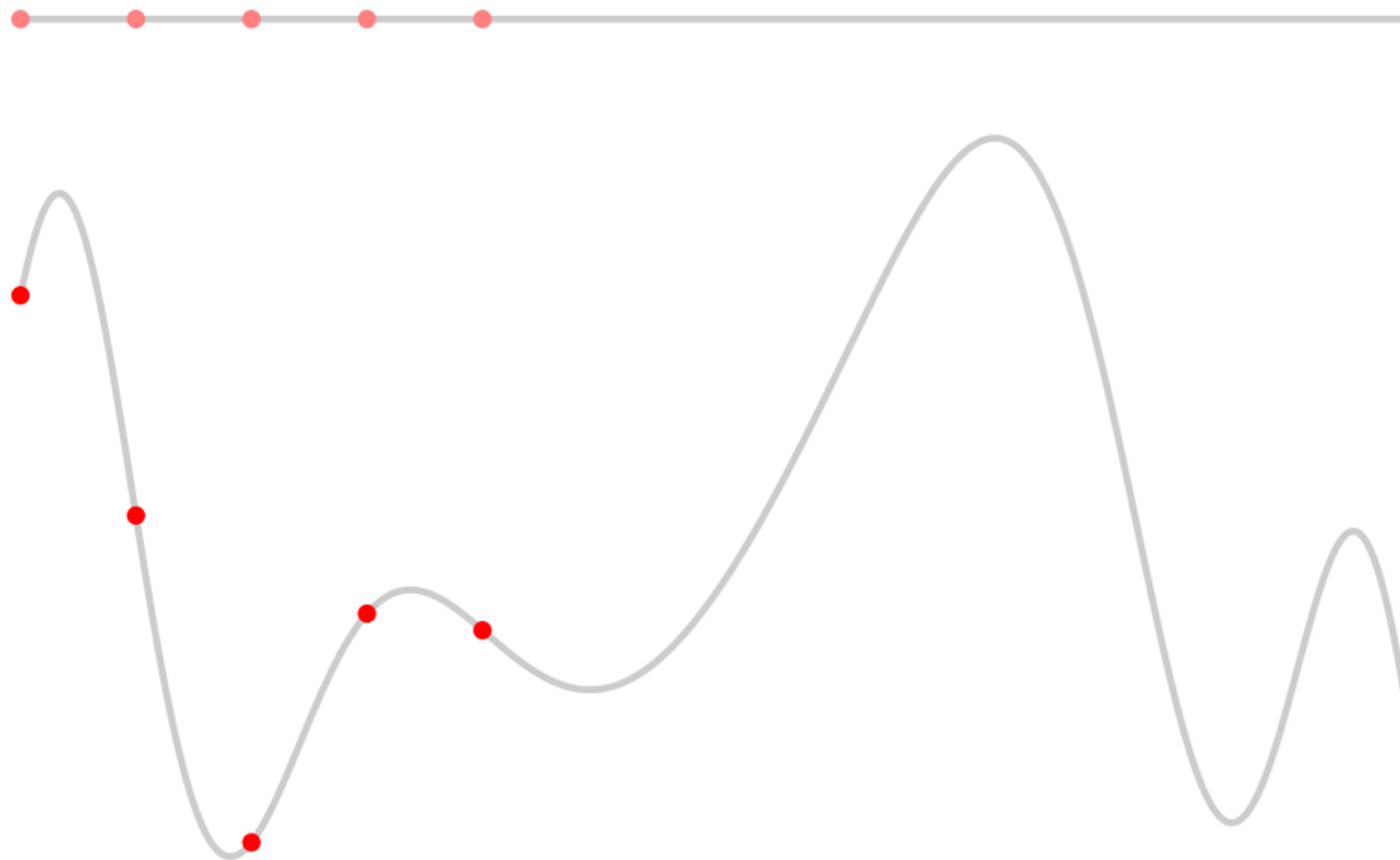
Curva paramétrica – amostragem uniforme no domínio



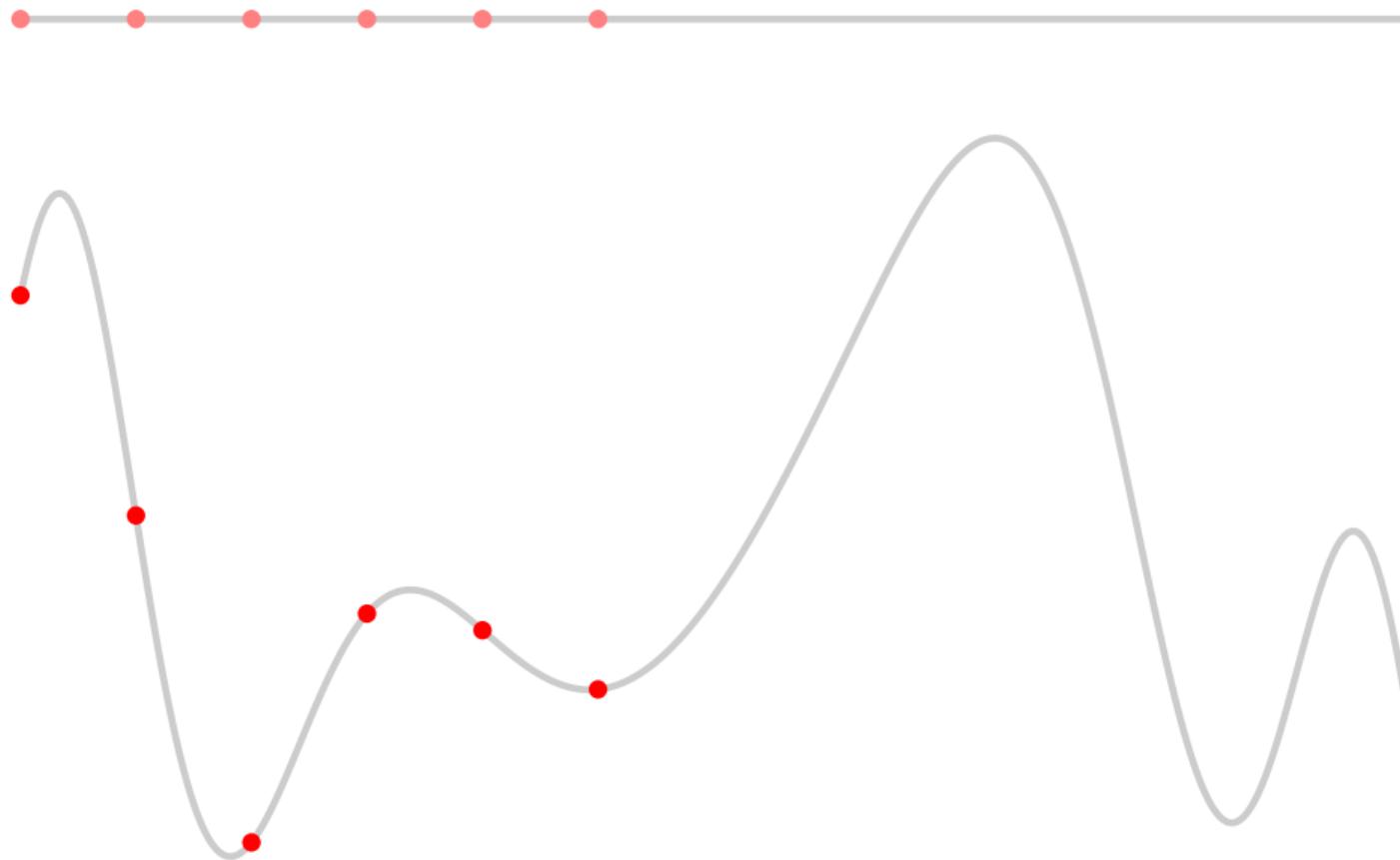
Curva paramétrica – amostragem uniforme no domínio



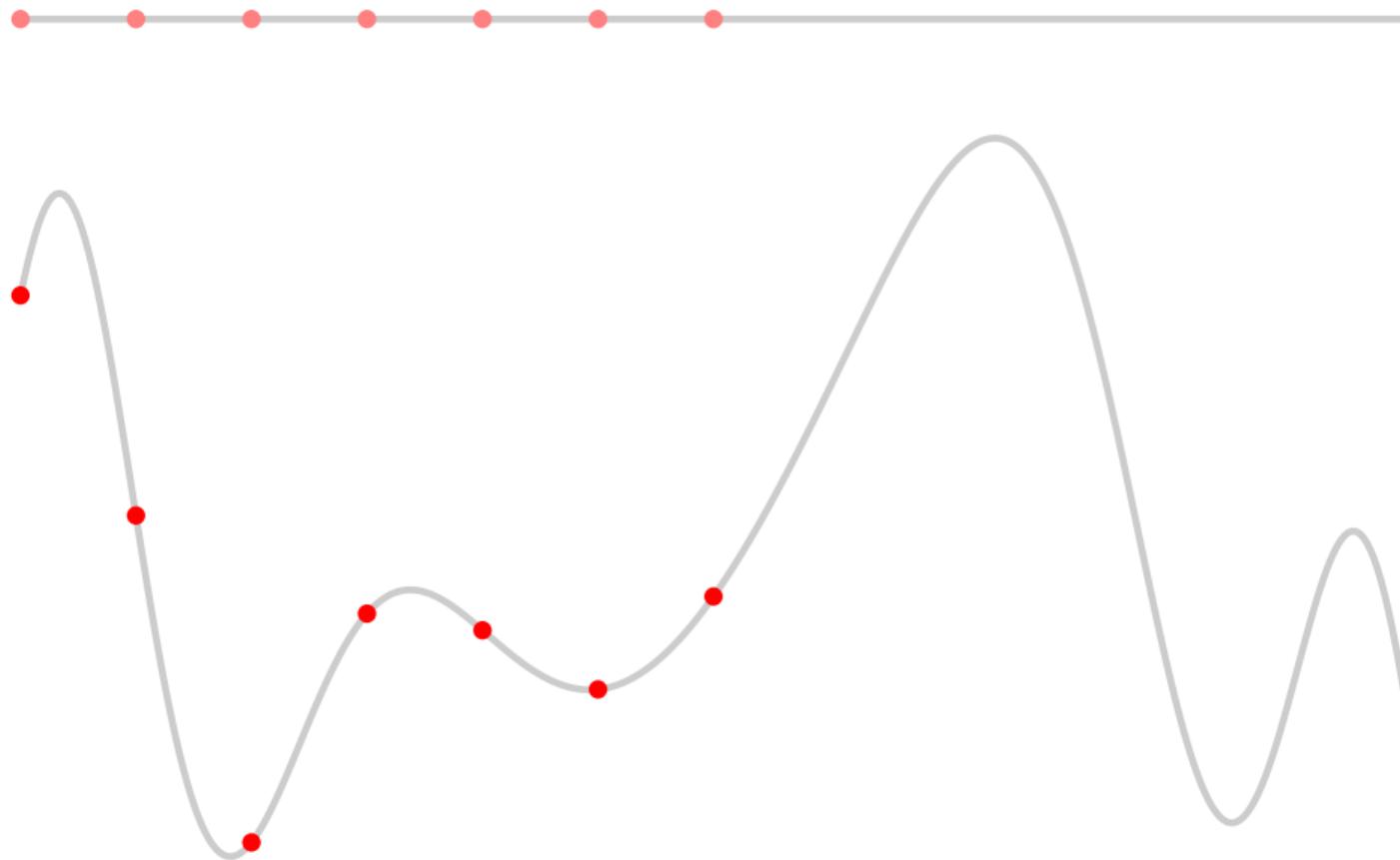
Curva paramétrica – amostragem uniforme no domínio



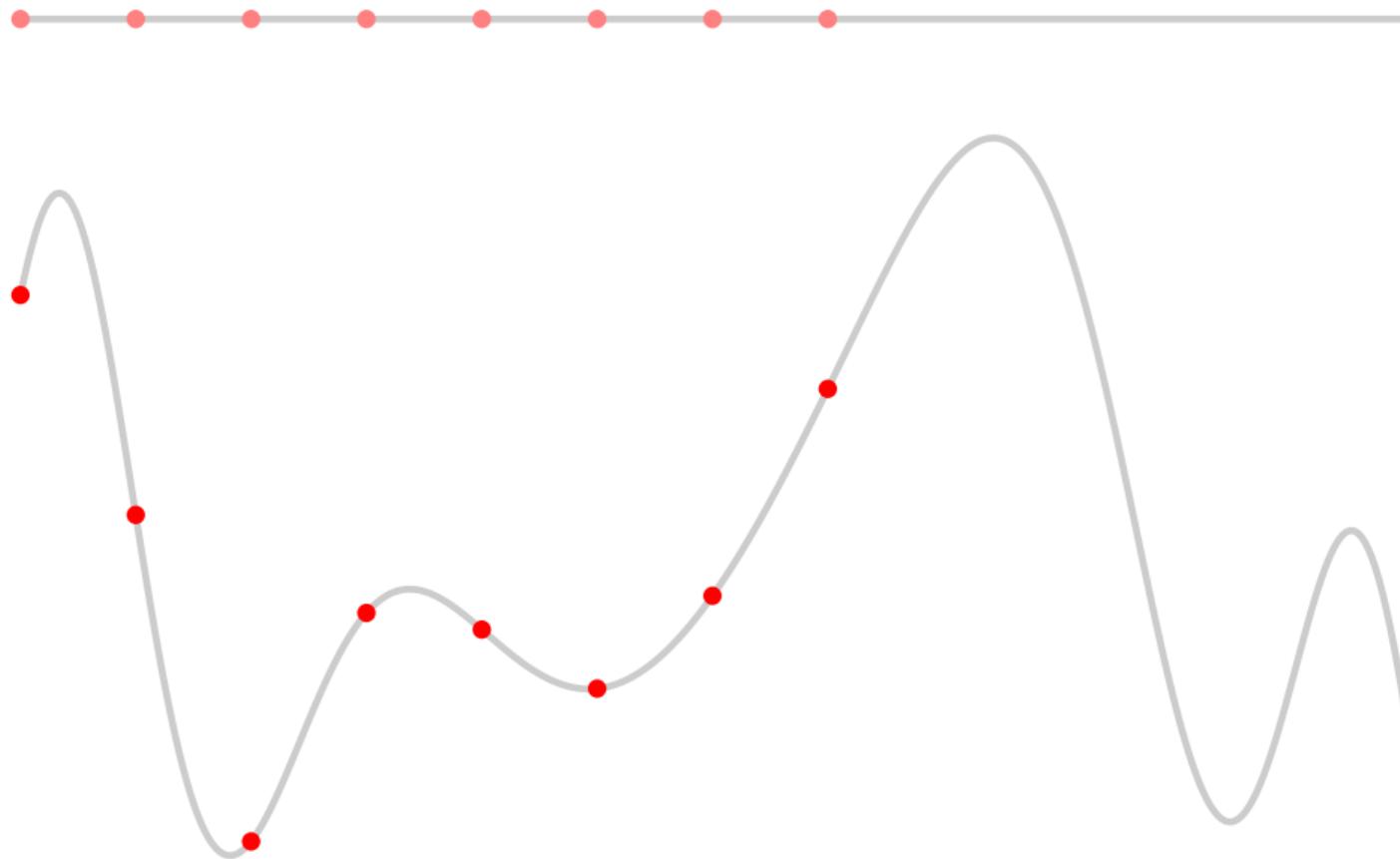
Curva paramétrica – amostragem uniforme no domínio



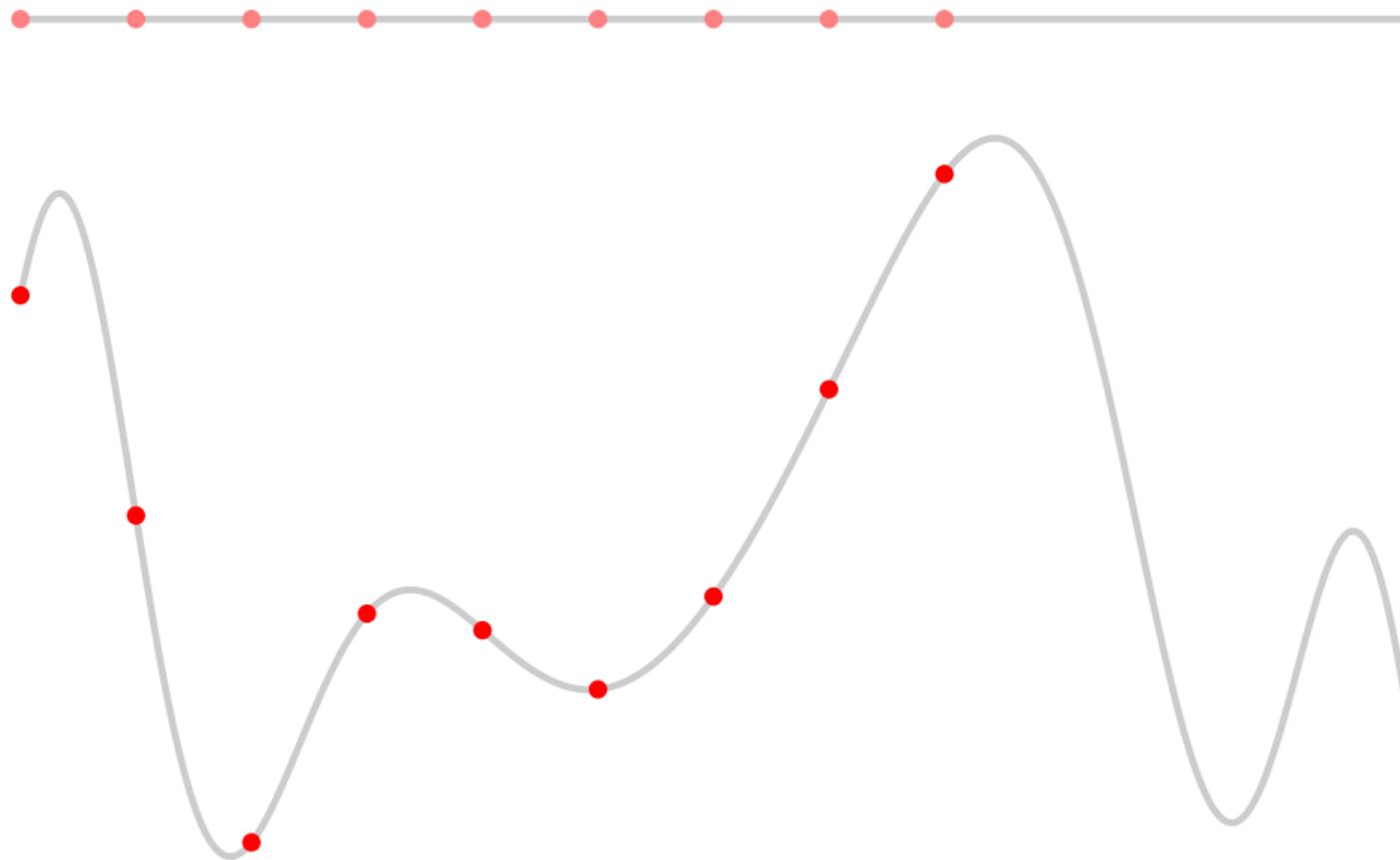
Curva paramétrica – amostragem uniforme no domínio



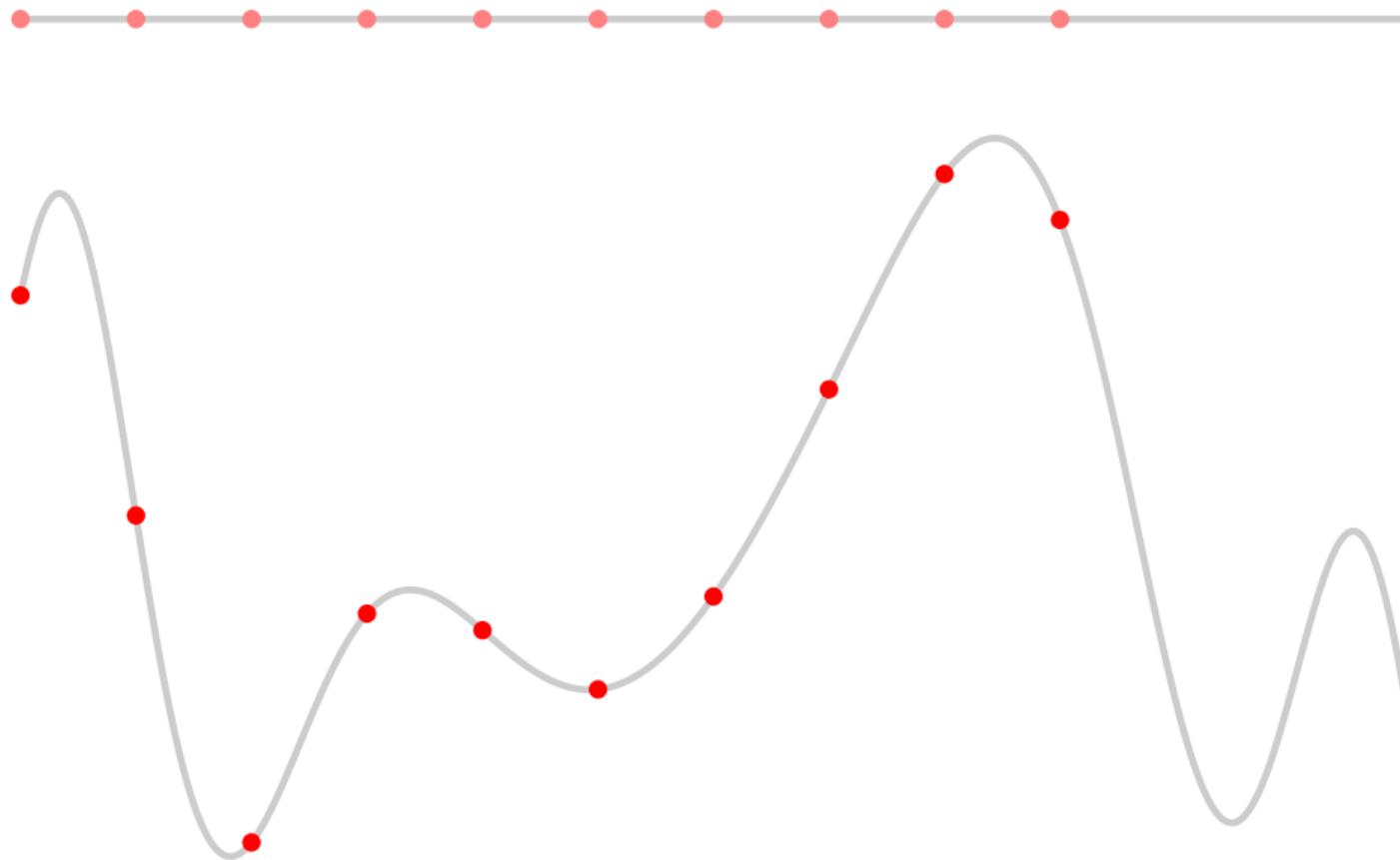
Curva paramétrica – amostragem uniforme no domínio



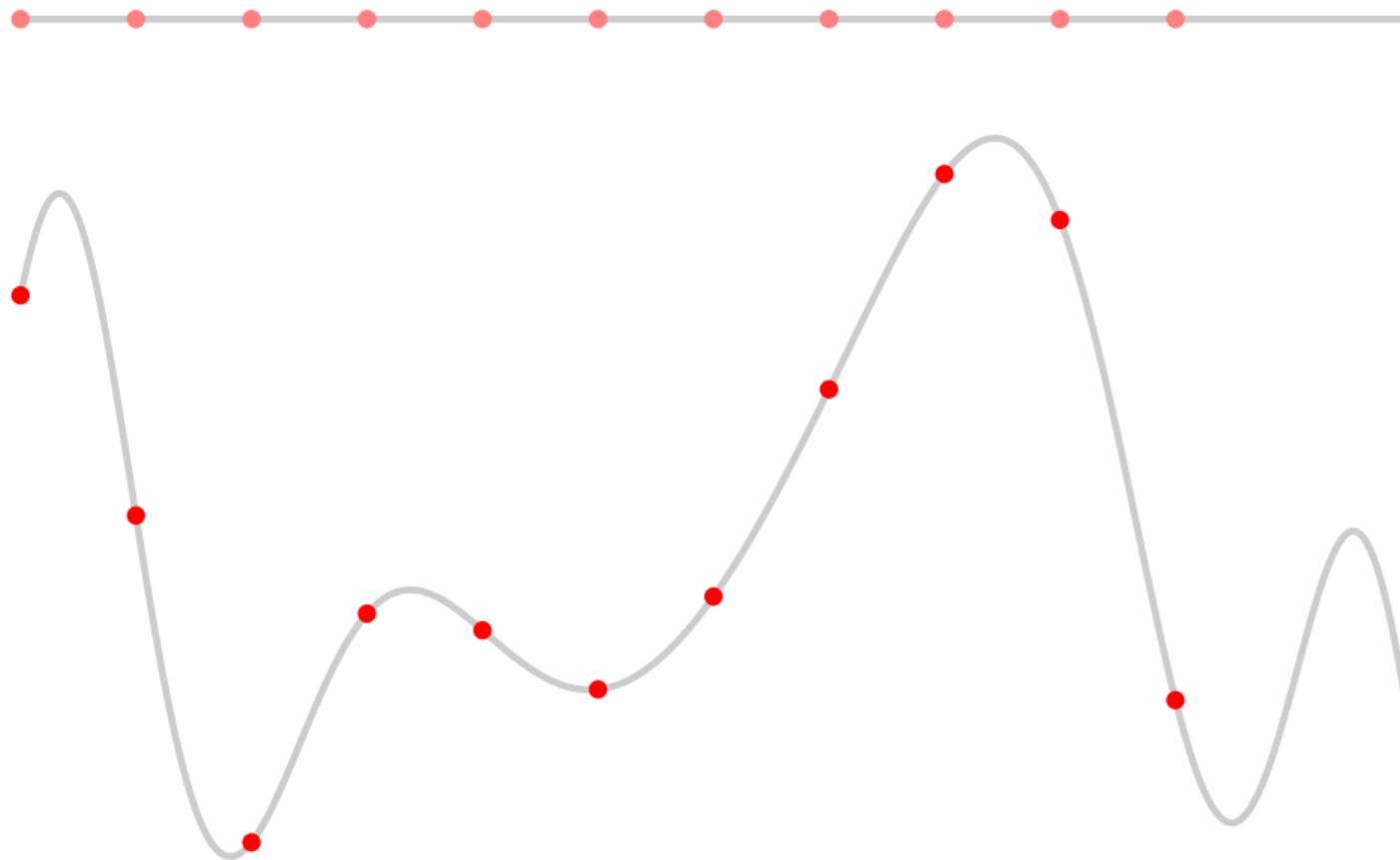
Curva paramétrica – amostragem uniforme no domínio



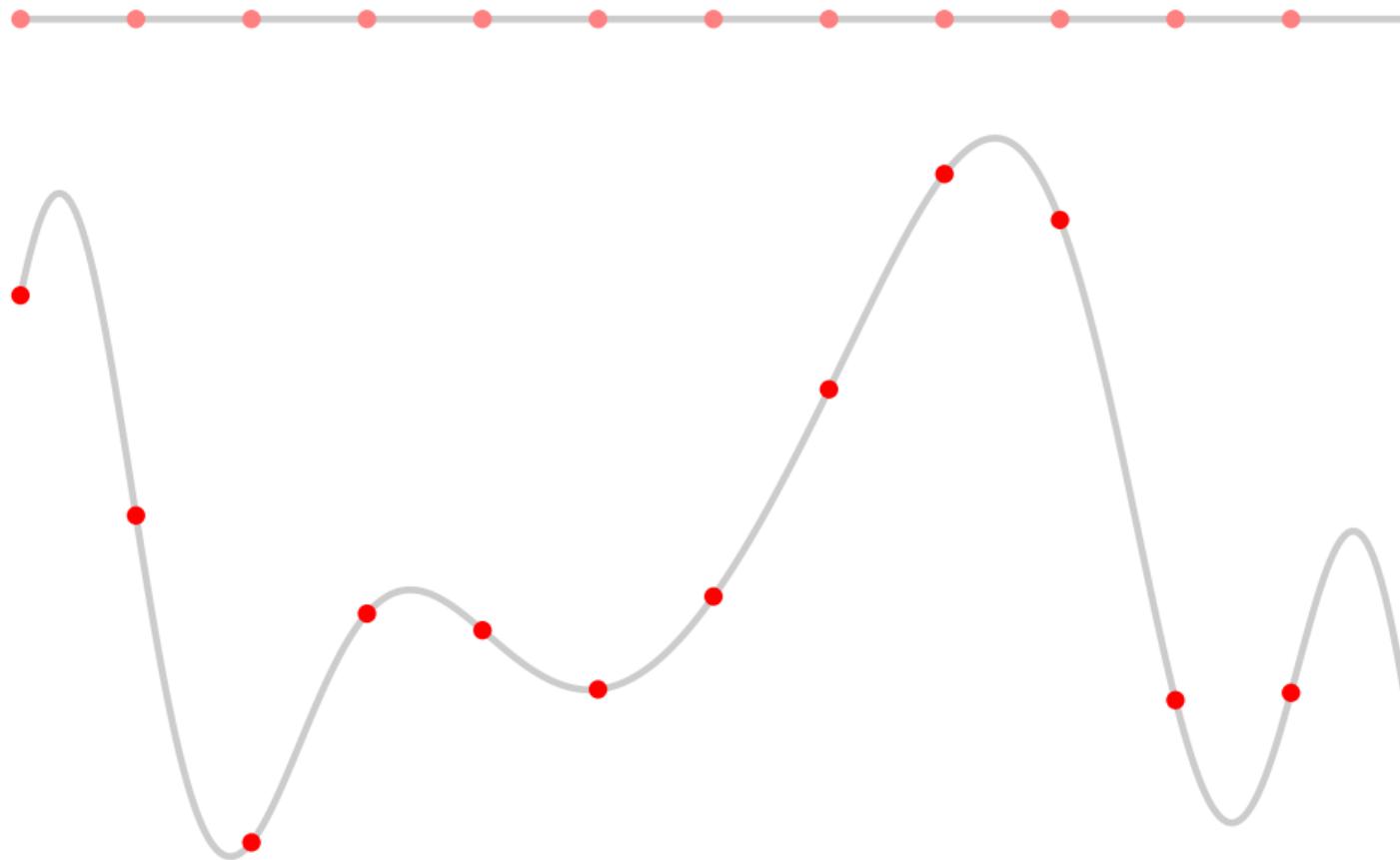
Curva paramétrica – amostragem uniforme no domínio



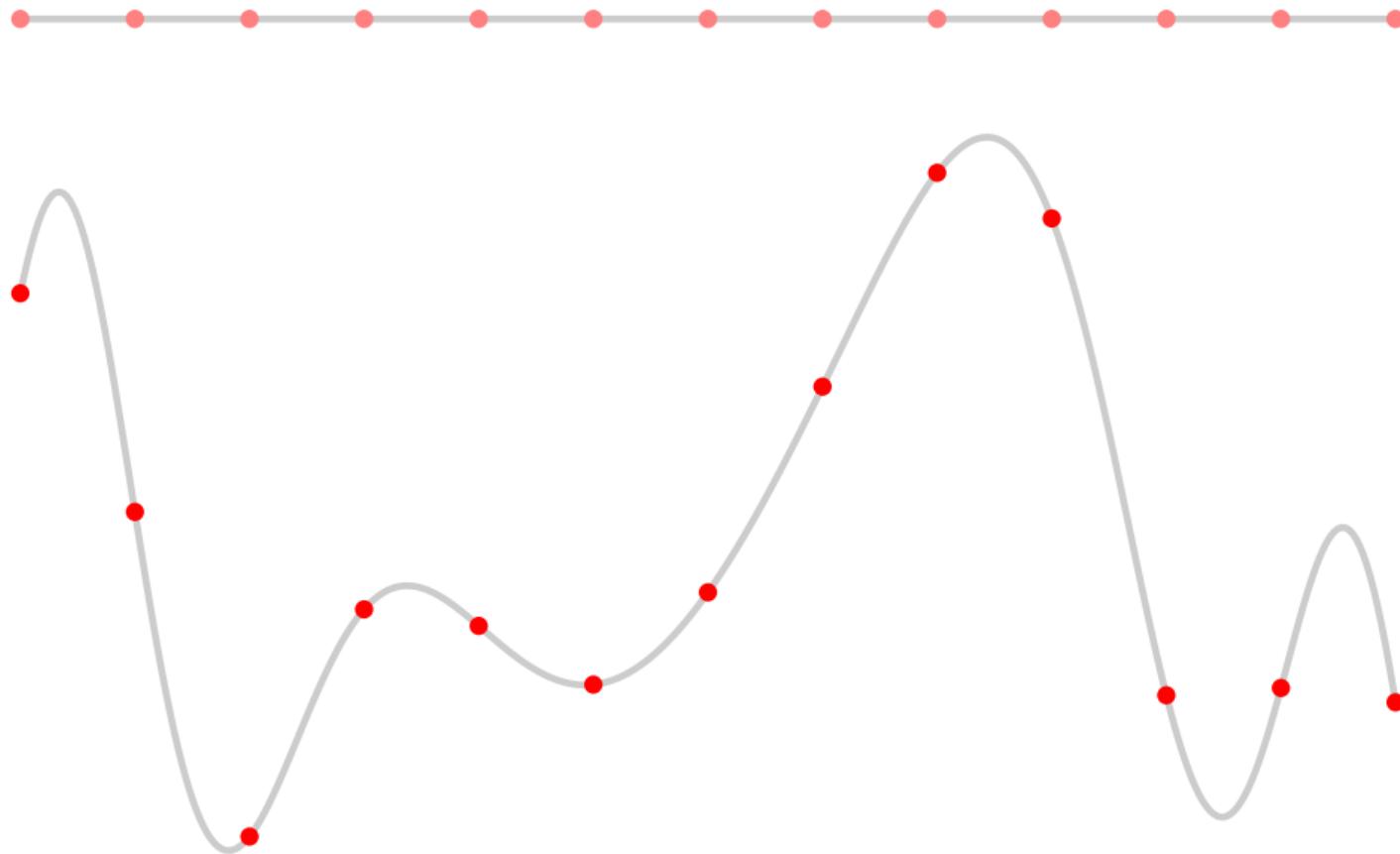
Curva paramétrica – amostragem uniforme no domínio



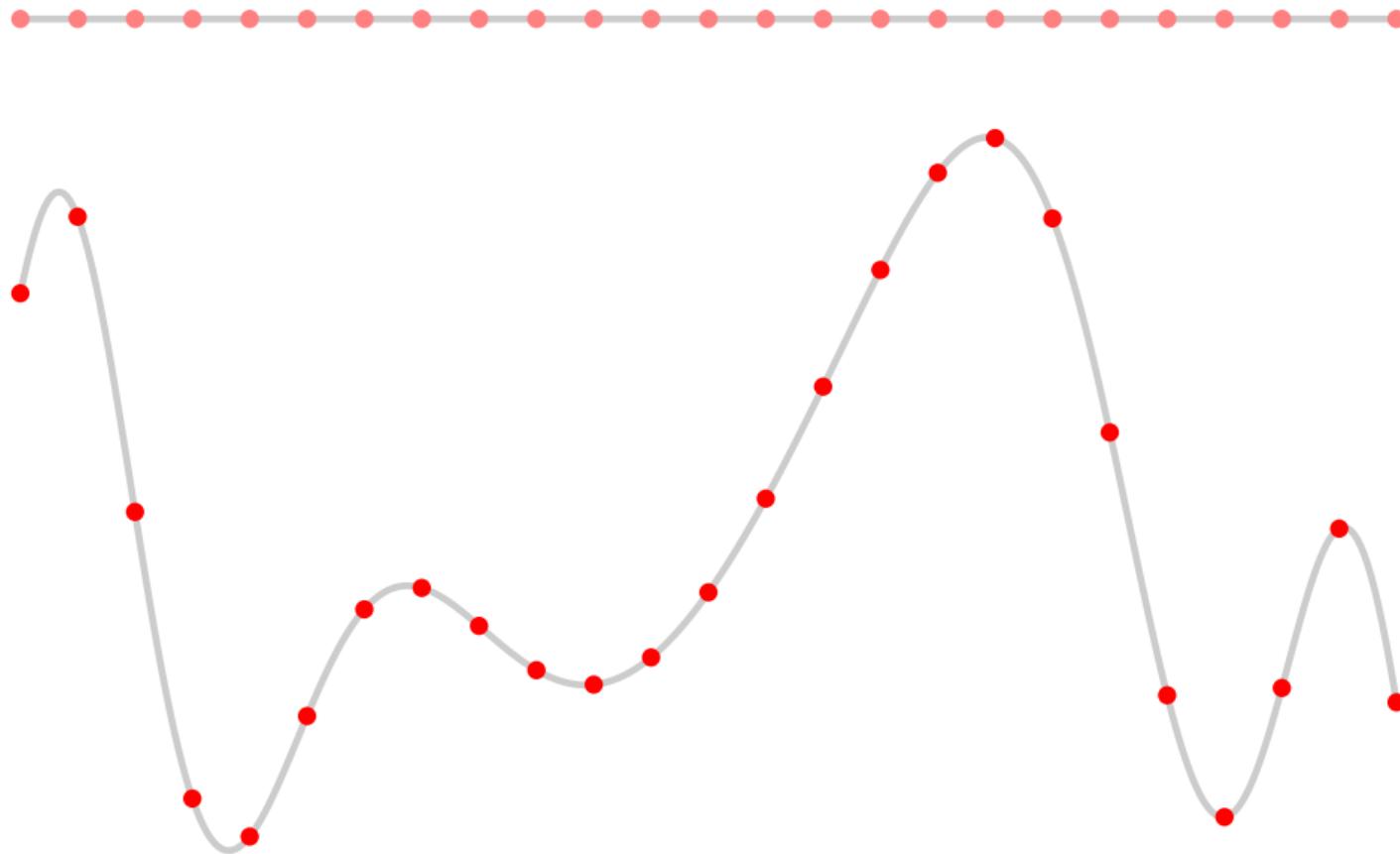
Curva paramétrica – amostragem uniforme no domínio



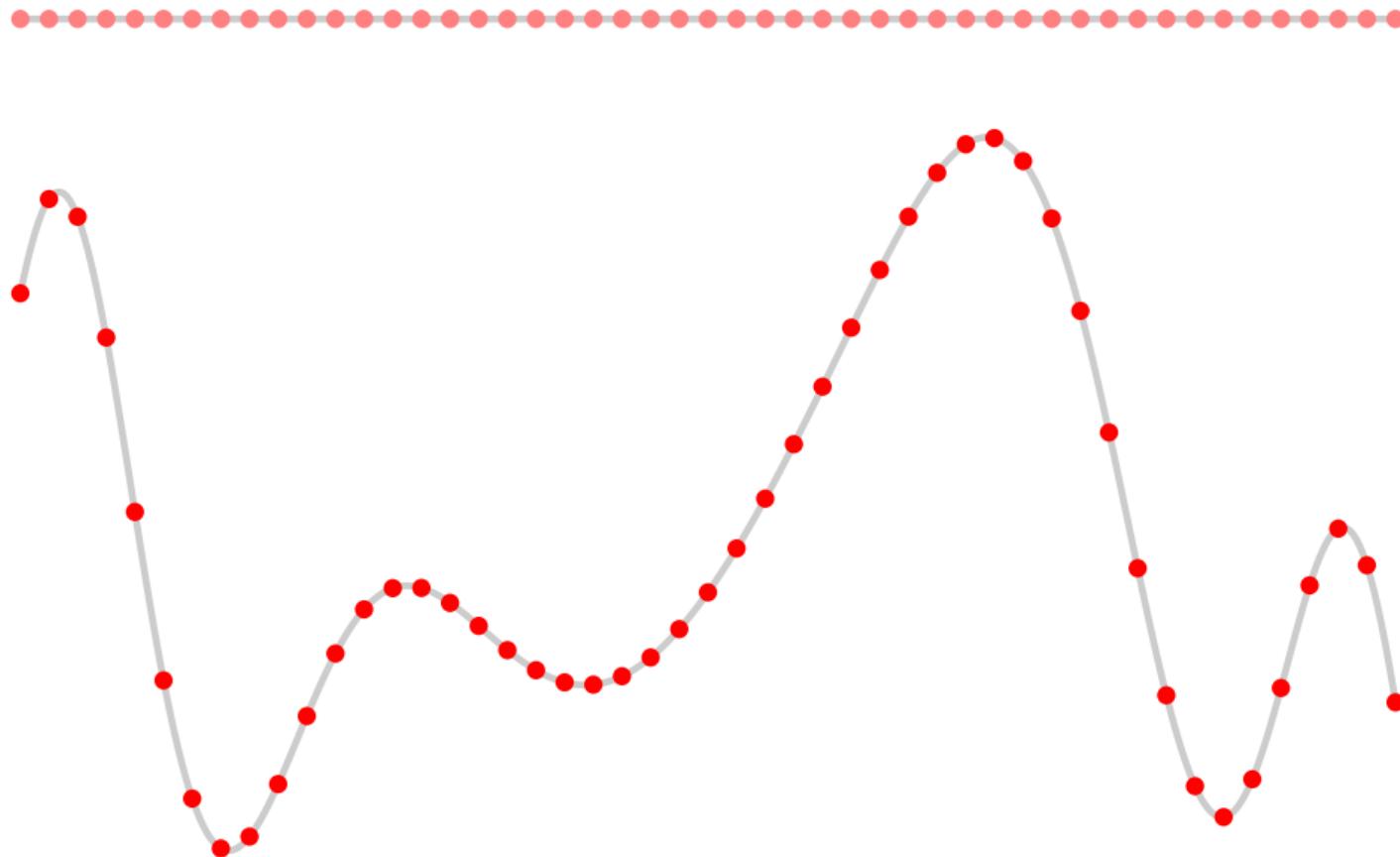
Curva paramétrica – amostragem uniforme no domínio



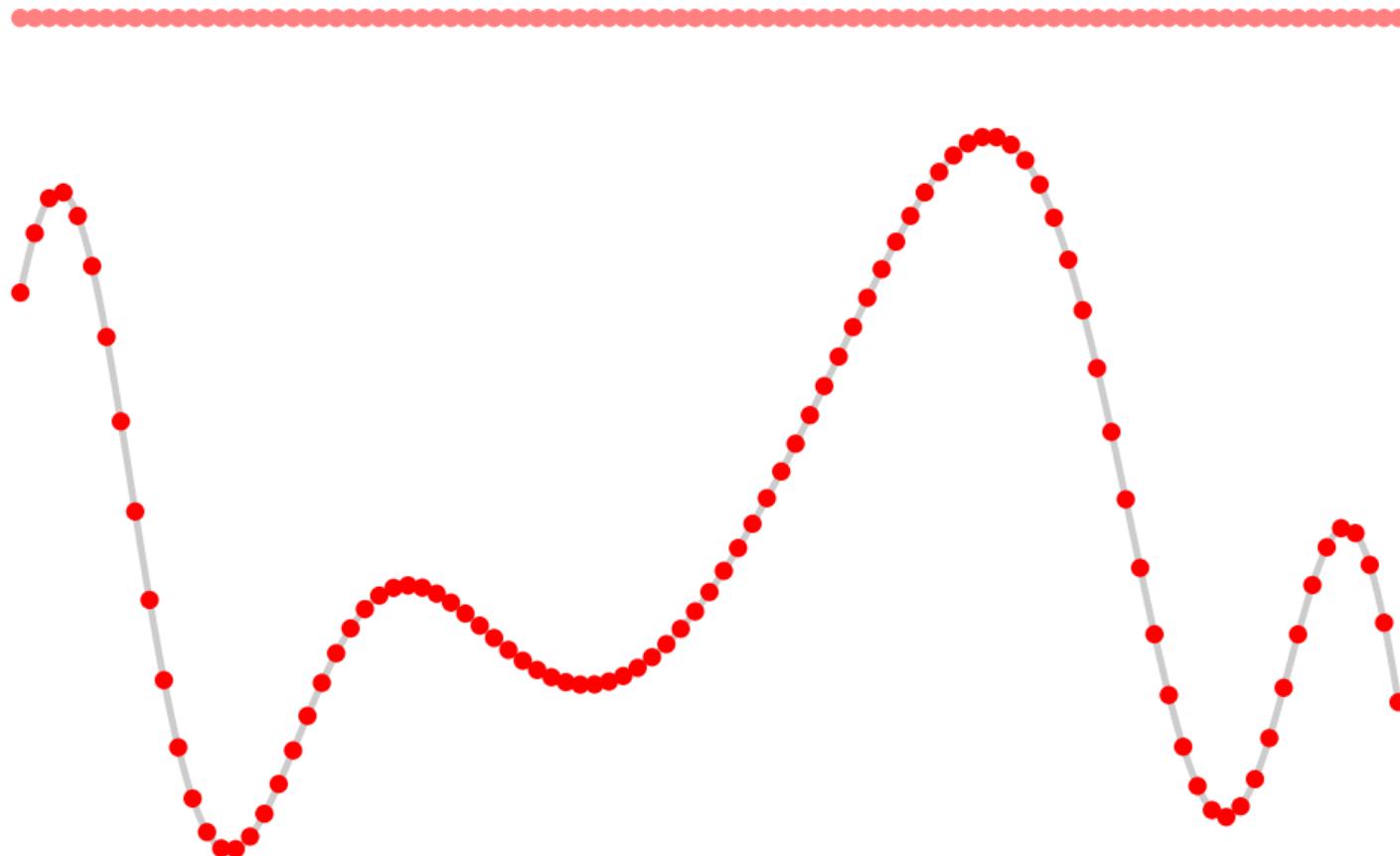
Curva paramétrica – amostragem uniforme no domínio



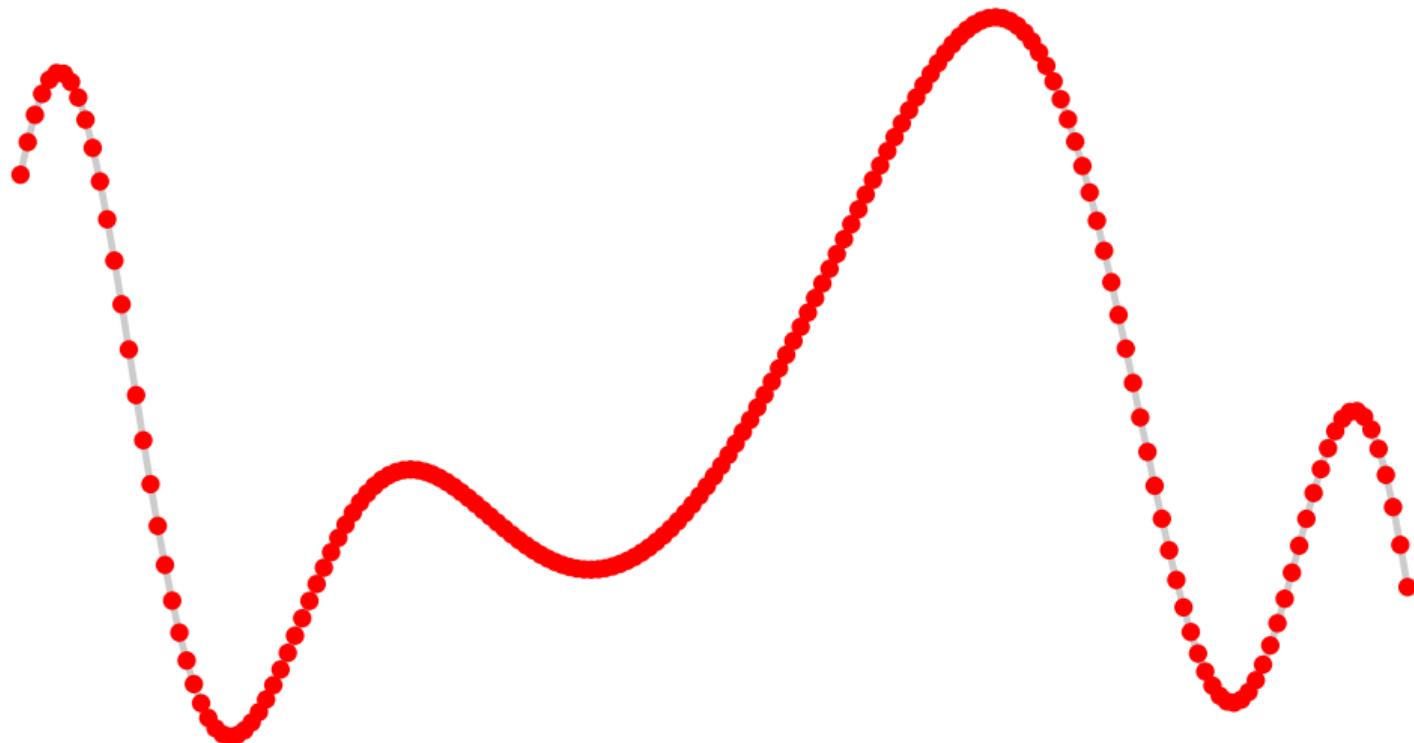
Curva paramétrica – amostragem uniforme no domínio



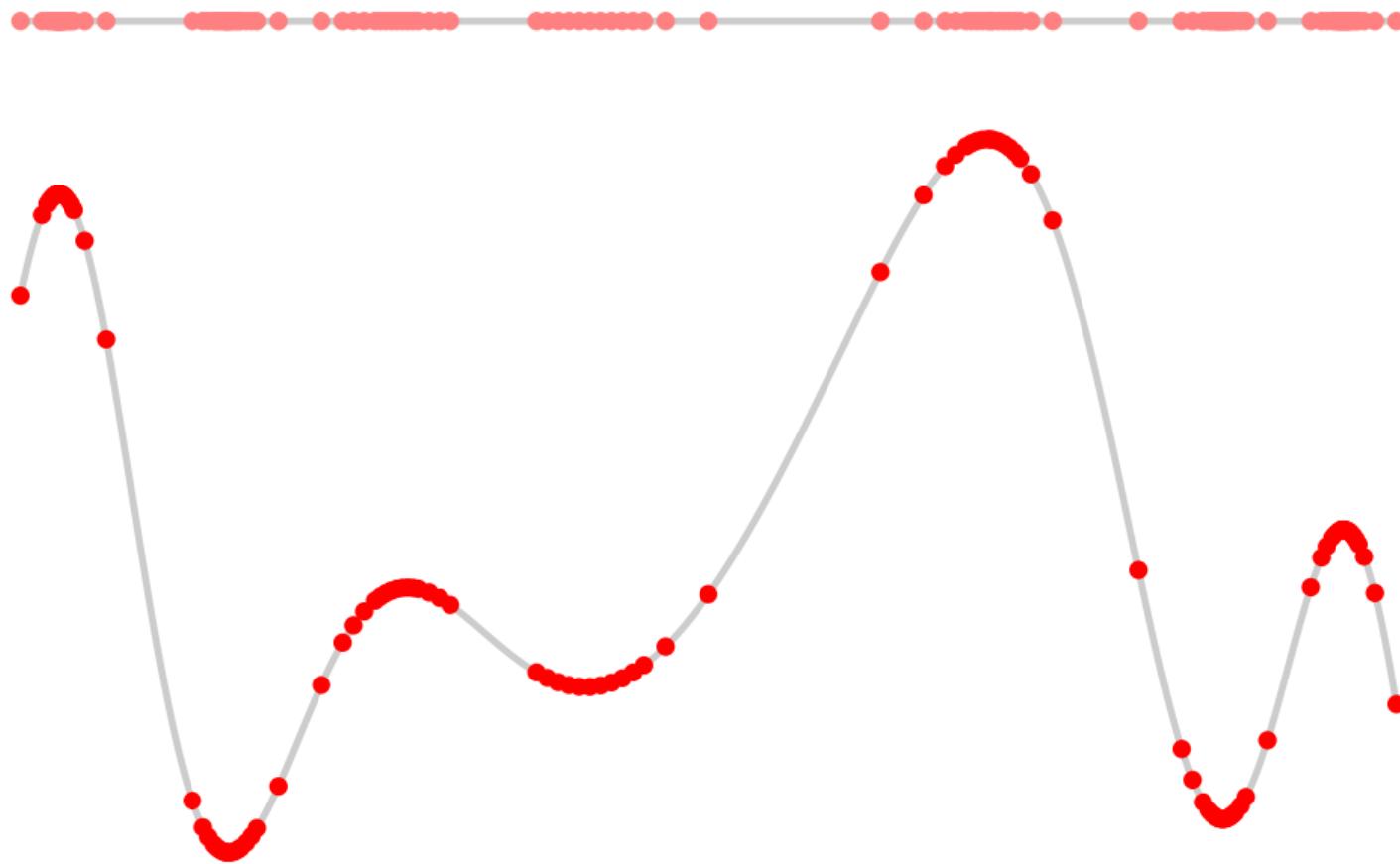
Curva paramétrica – amostragem uniforme no domínio



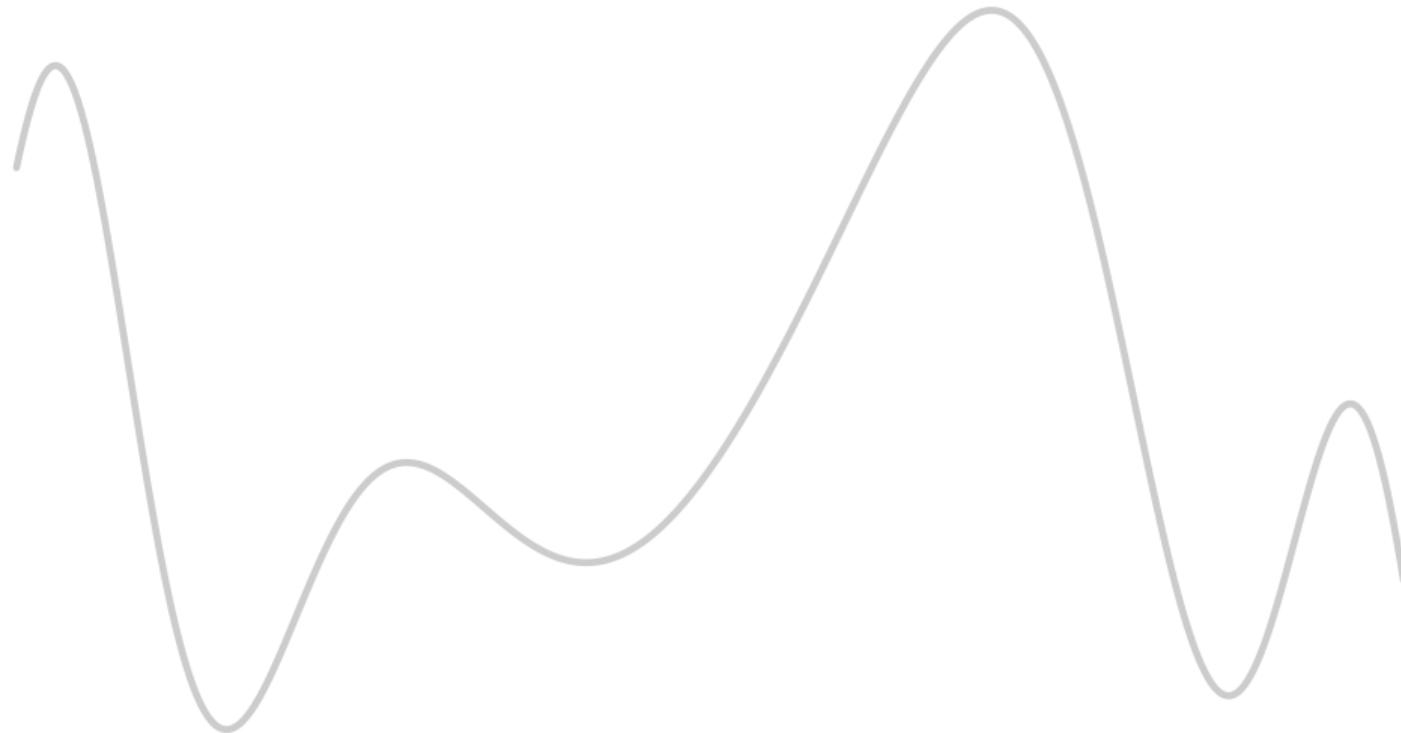
Curva paramétrica – amostragem uniforme no domínio



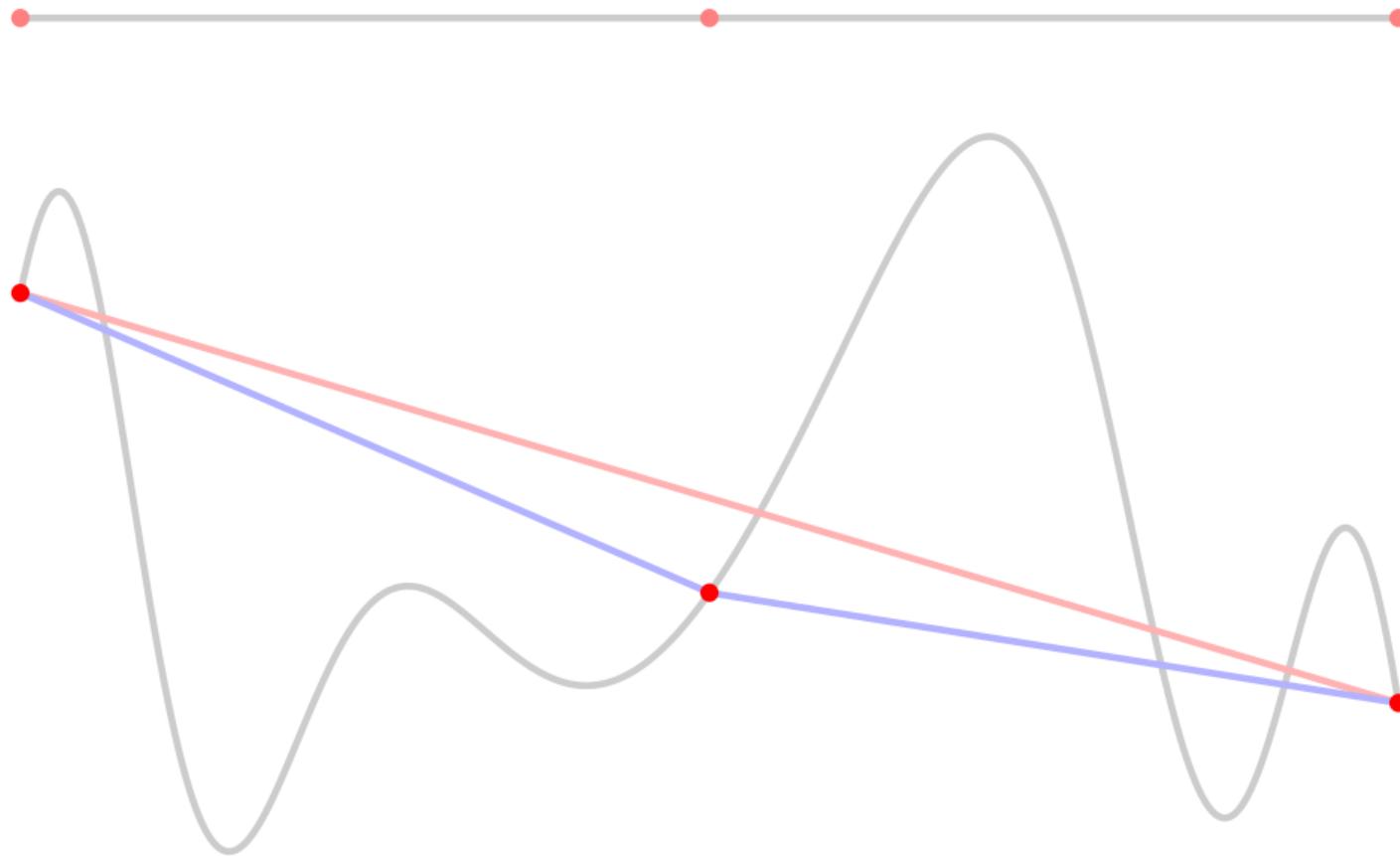
Curva paramétrica – amostragem adaptativa



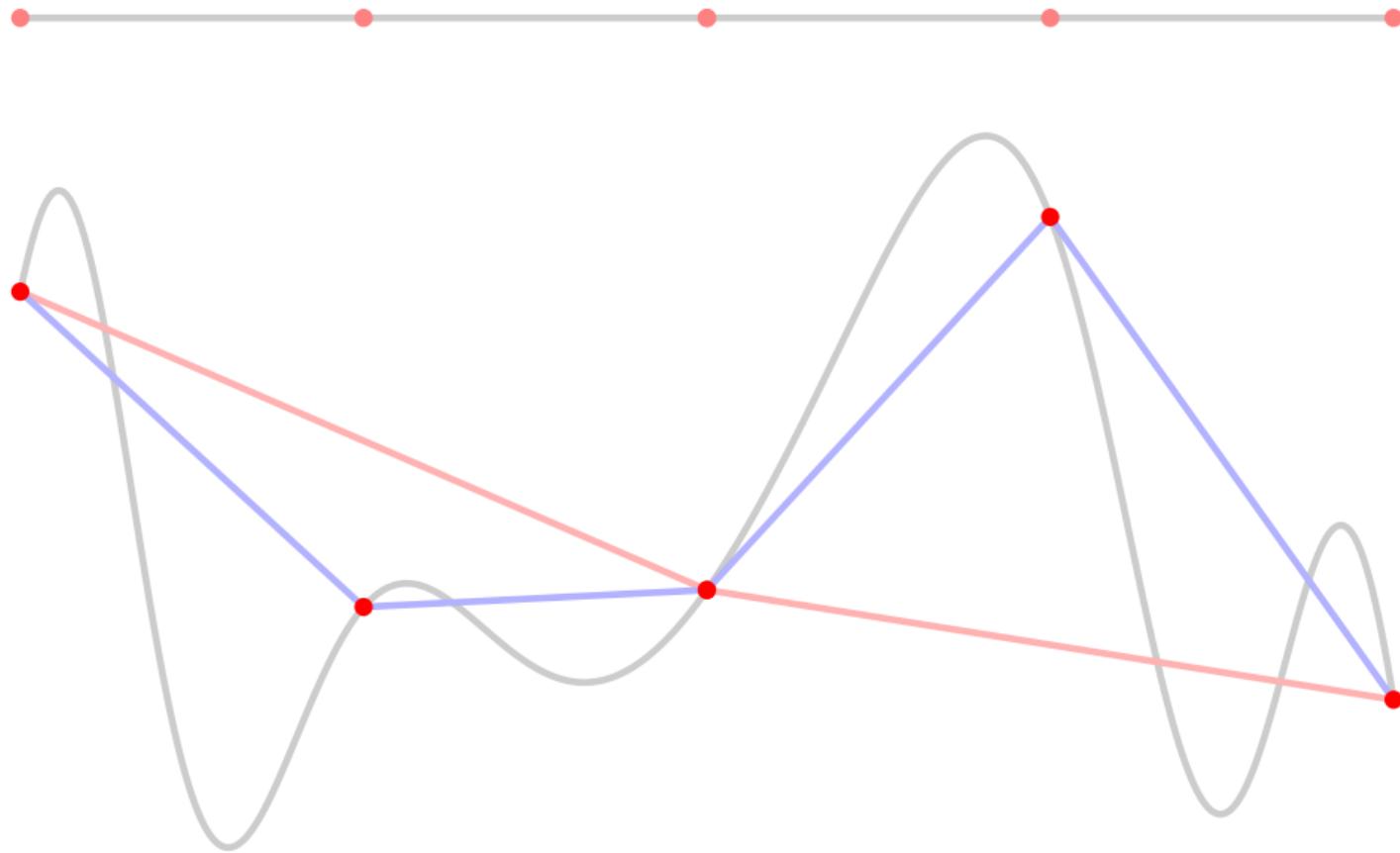
Curva paramétrica – amostragem adaptativa



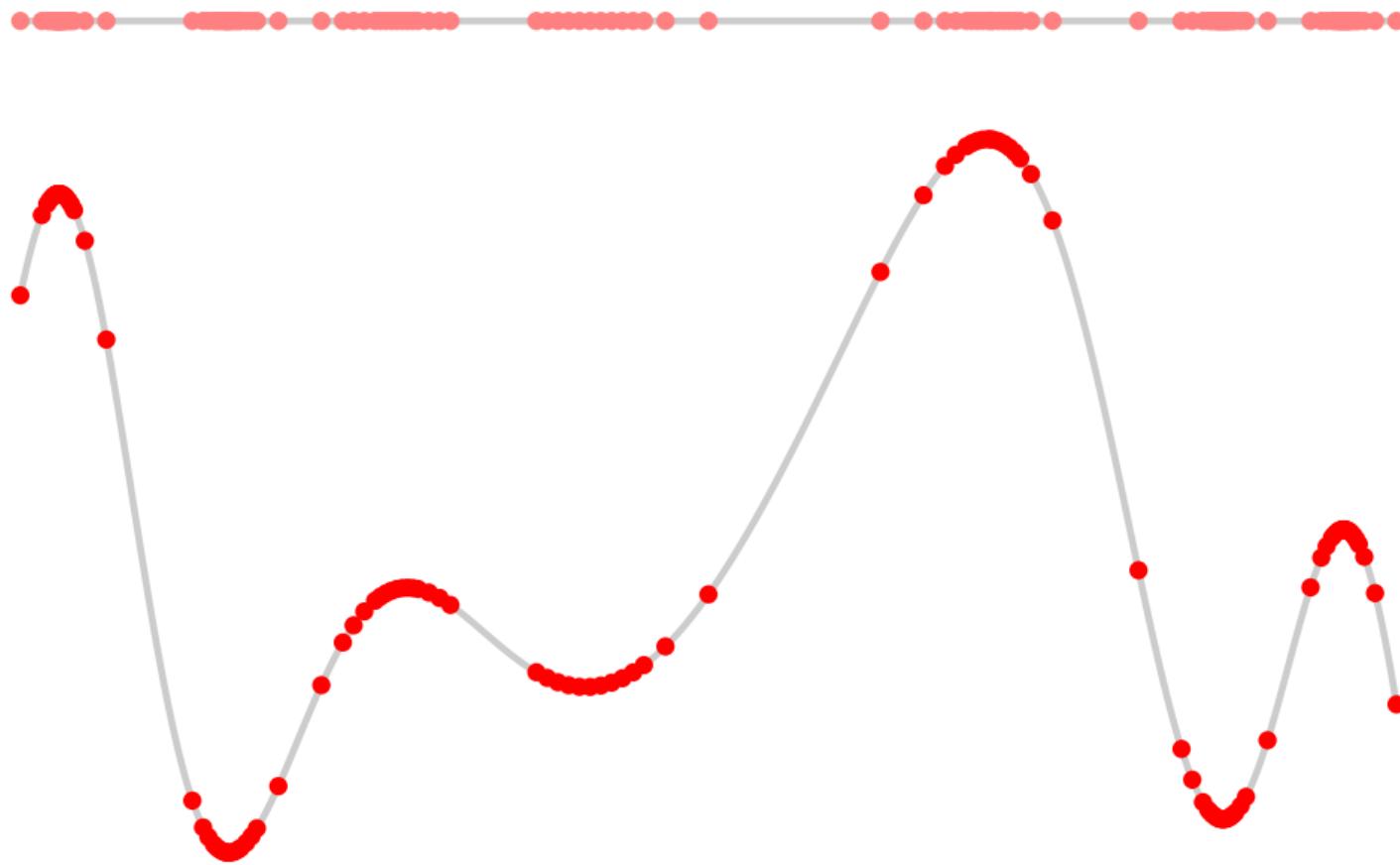
Curva paramétrica – amostragem adaptativa



Curva paramétrica – amostragem adaptativa



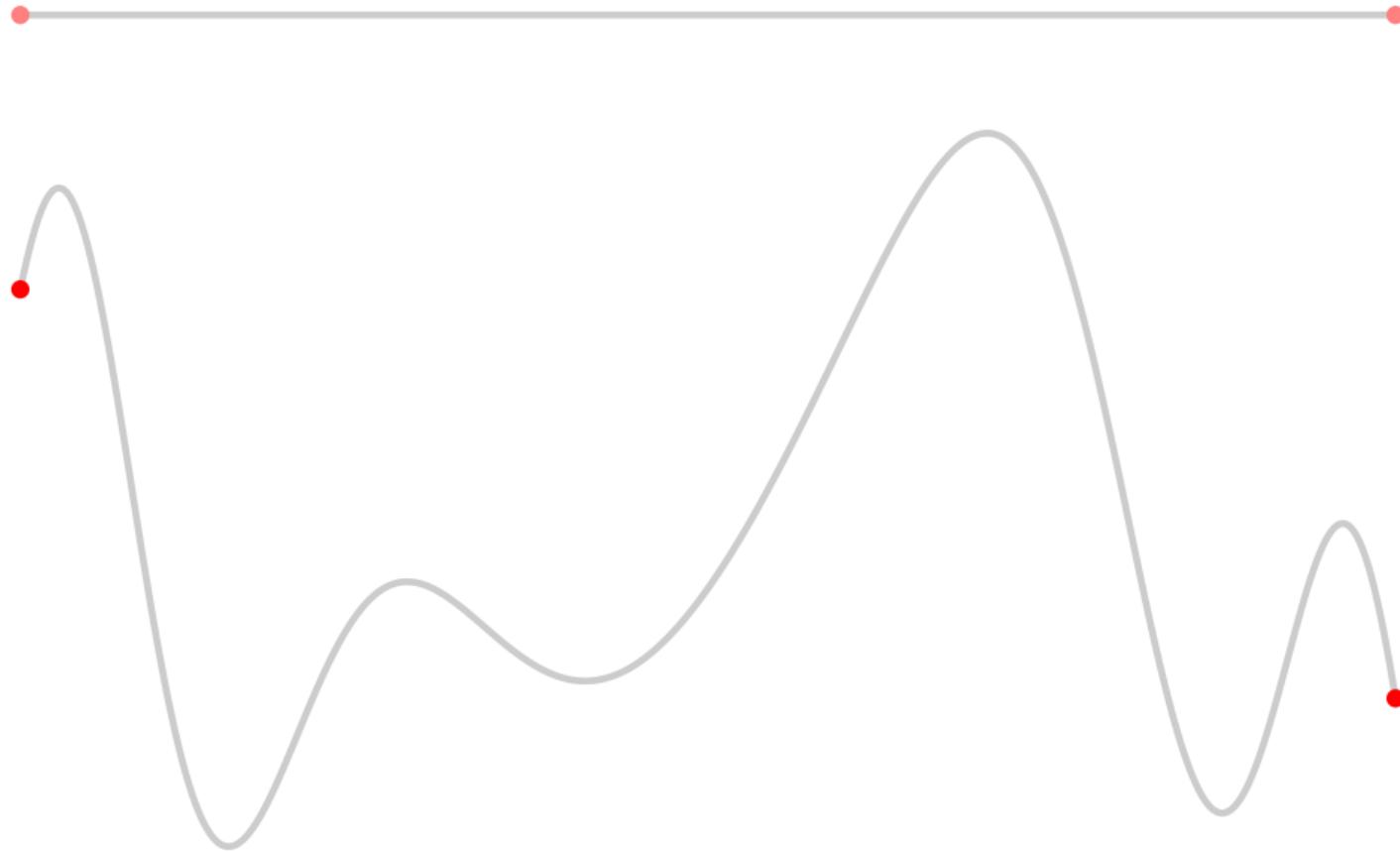
Curva paramétrica – amostragem adaptativa



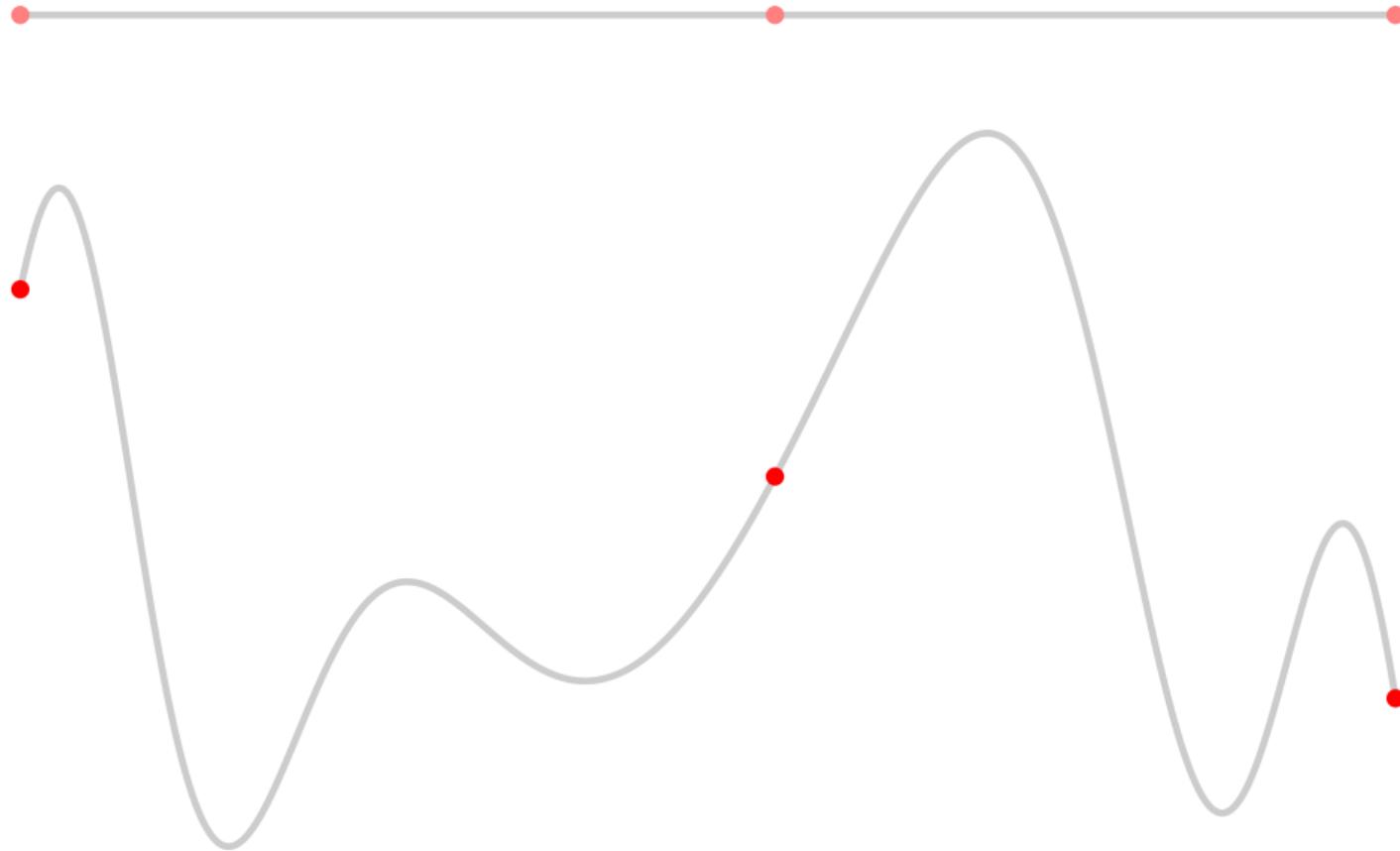
Curva paramétrica – amostragem adaptativa

```
function sample(t0,t1)
    local tm=(t0+t1)/2
    local x0,y0=gamma(t0)
    local x1,y1=gamma(t1)
    local xm,ym=gamma(tm)
    if flat(x0,y0,x1,y1,xm,ym) then
        save(t0,x0,y0)
    else
        sample(t0,tm)
        sample(tm,t1)
    end
end
```

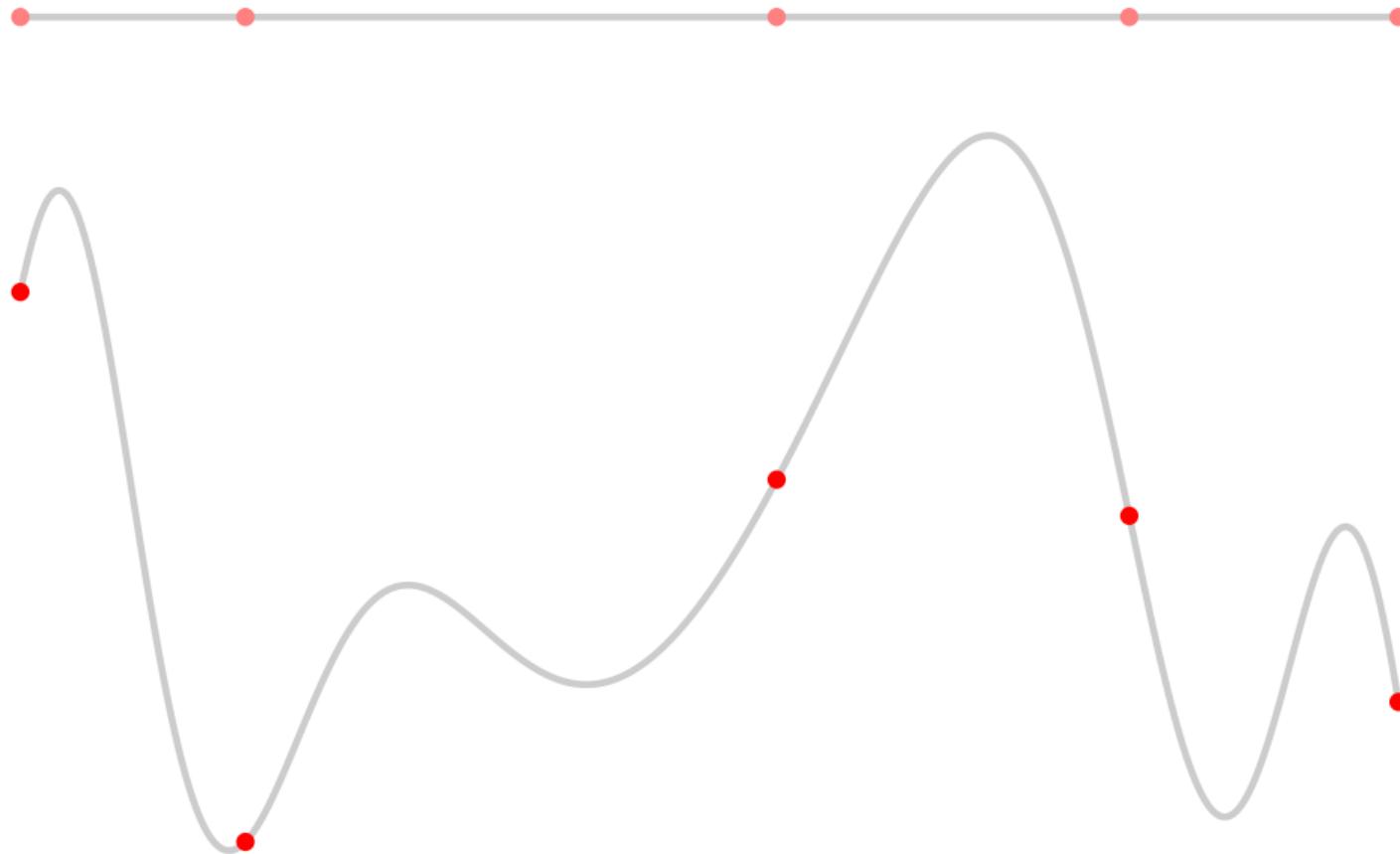
Curva paramétrica – amostragem uniforme na curva



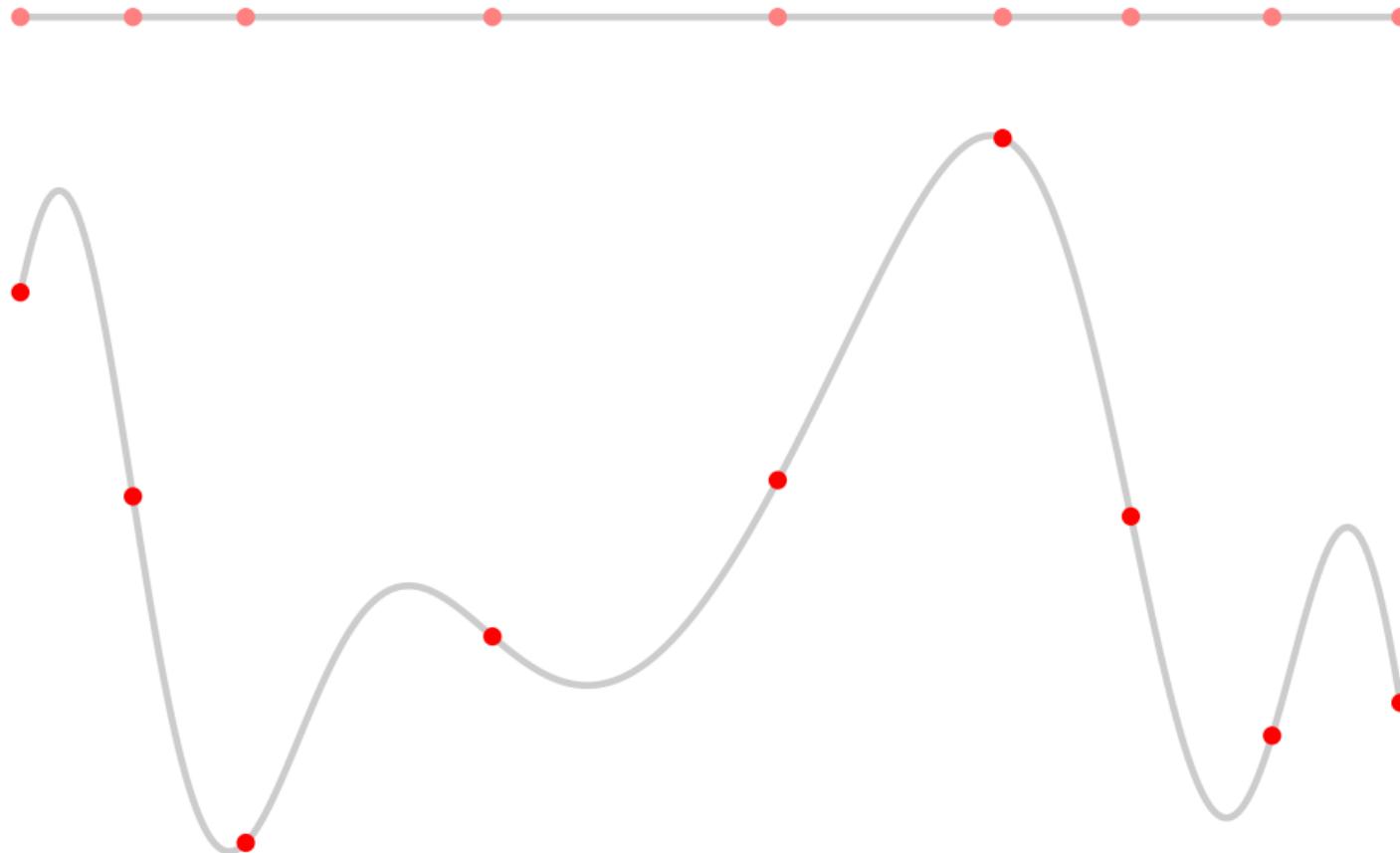
Curva paramétrica – amostragem uniforme na curva



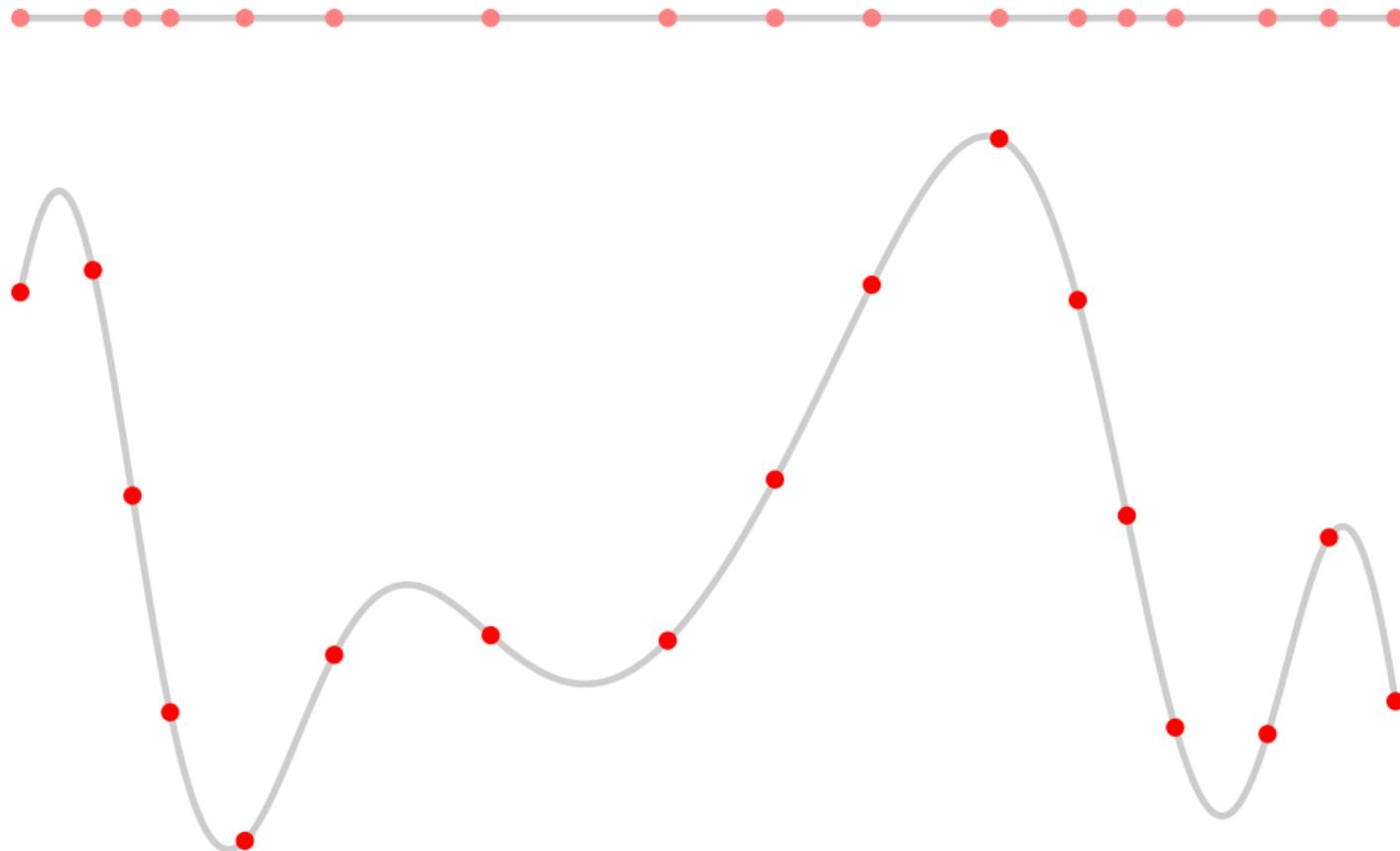
Curva paramétrica – amostragem uniforme na curva



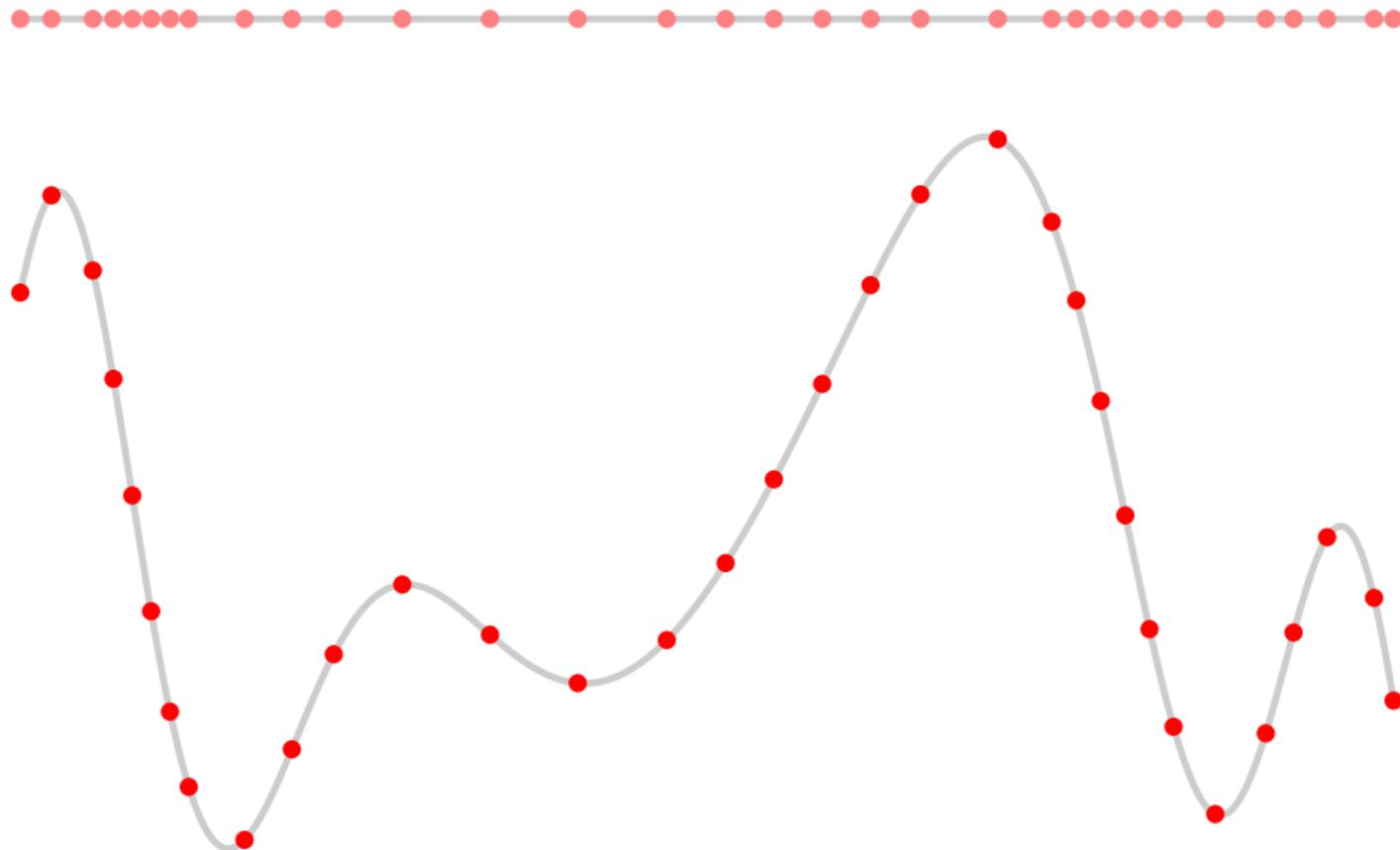
Curva paramétrica – amostragem uniforme na curva



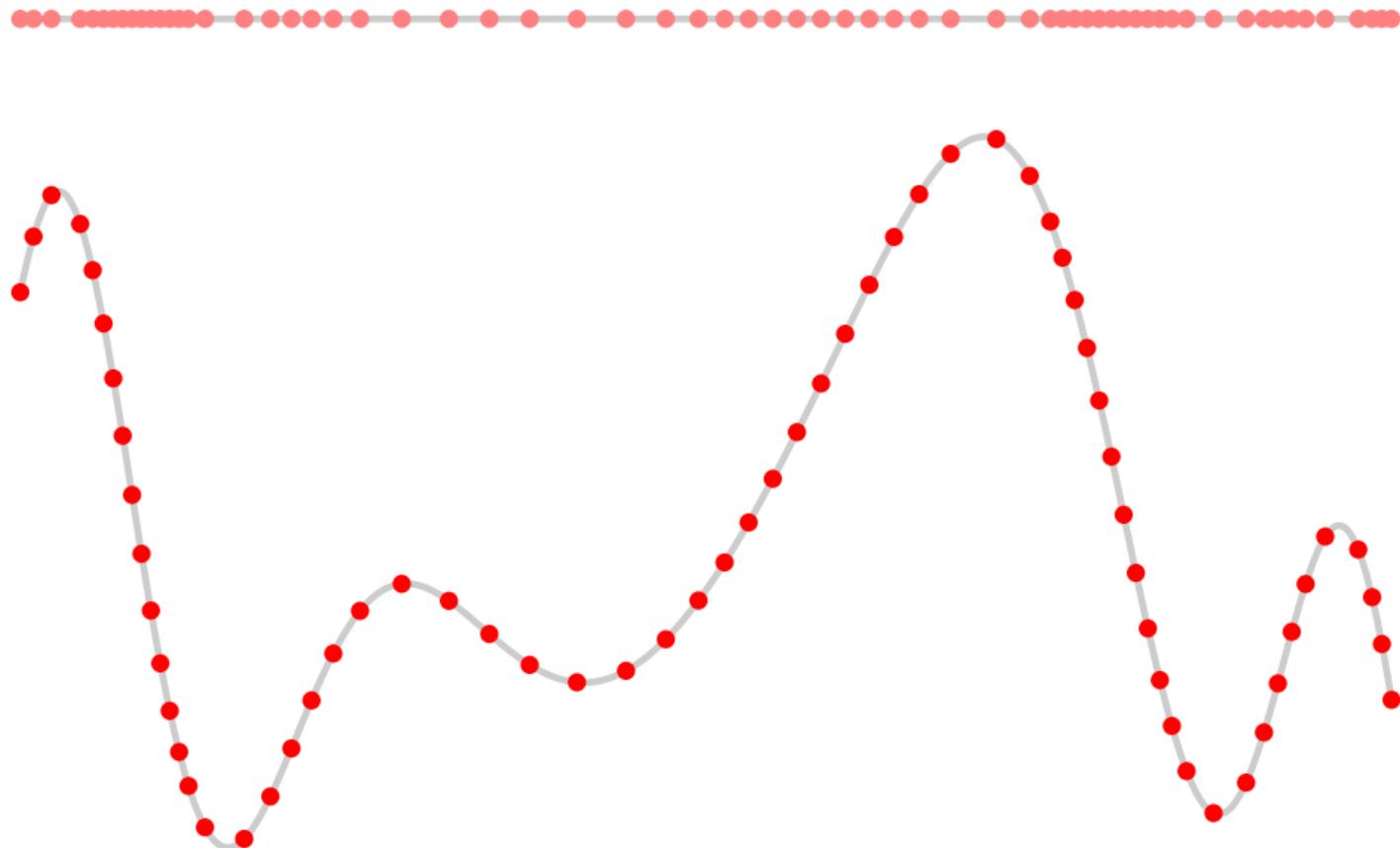
Curva paramétrica – amostragem uniforme na curva



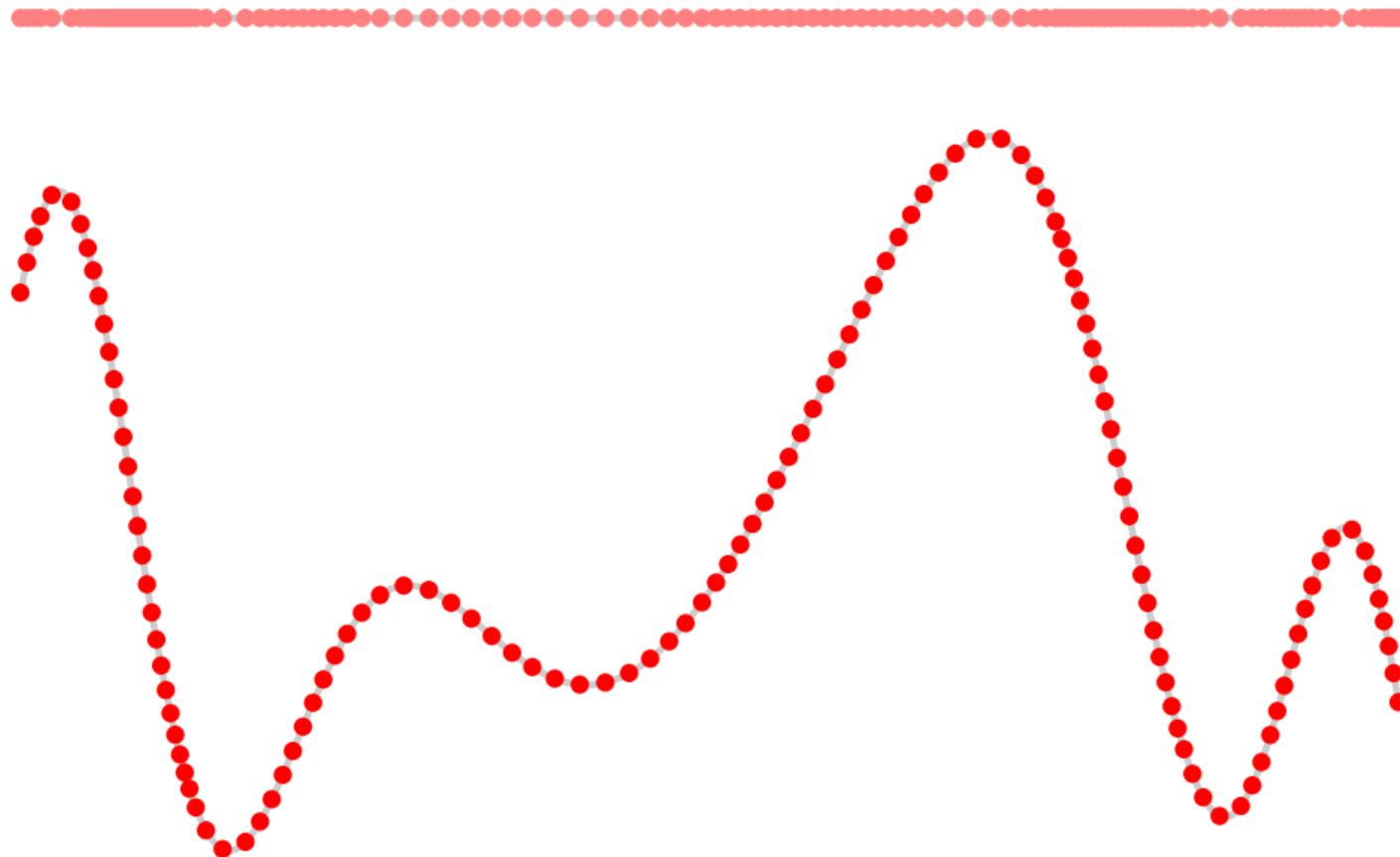
Curva paramétrica – amostragem uniforme na curva



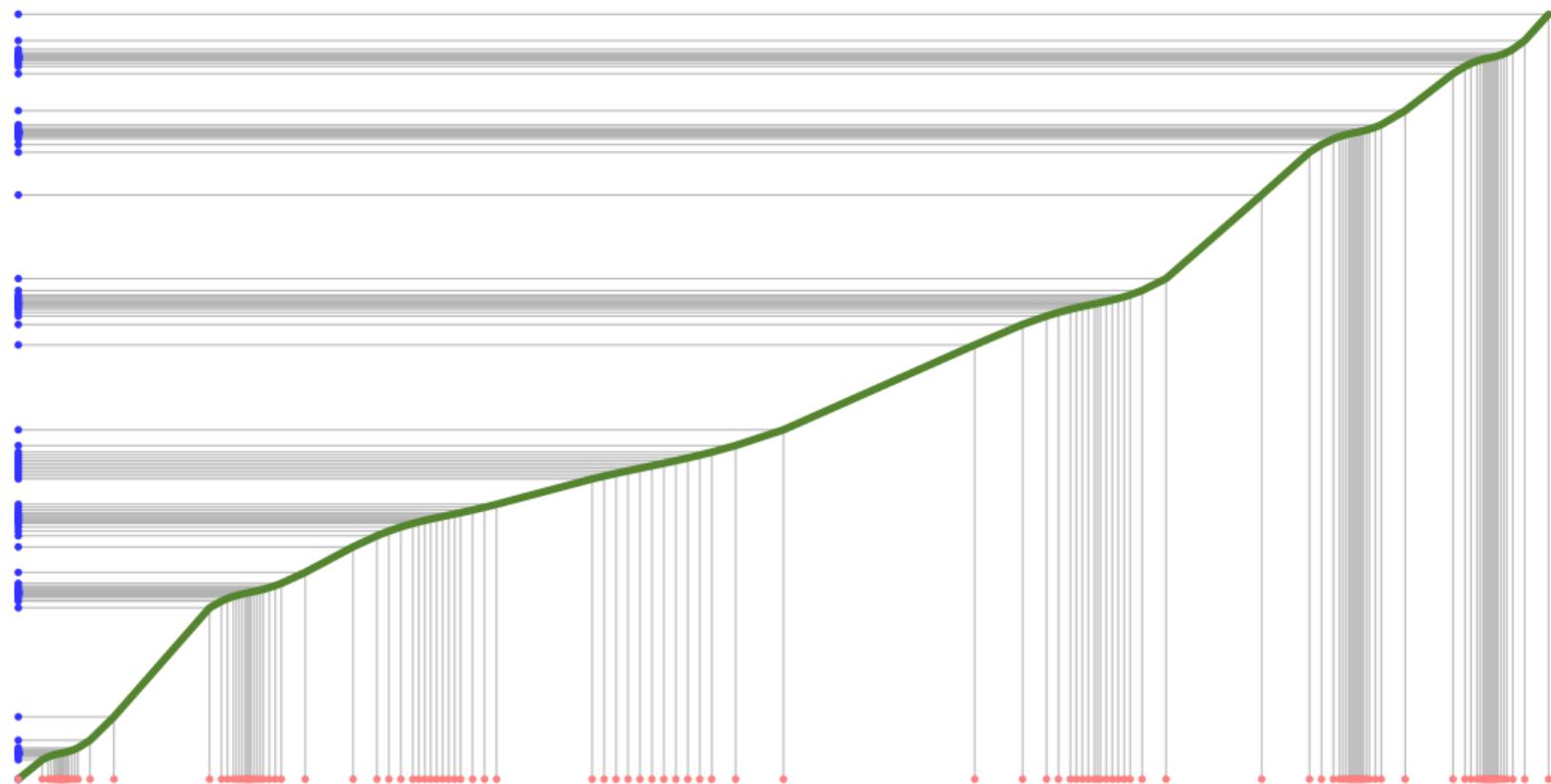
Curva paramétrica – amostragem uniforme na curva



Curva paramétrica – amostragem uniforme na curva



Curva paramétrica – amostragem uniforme na curva

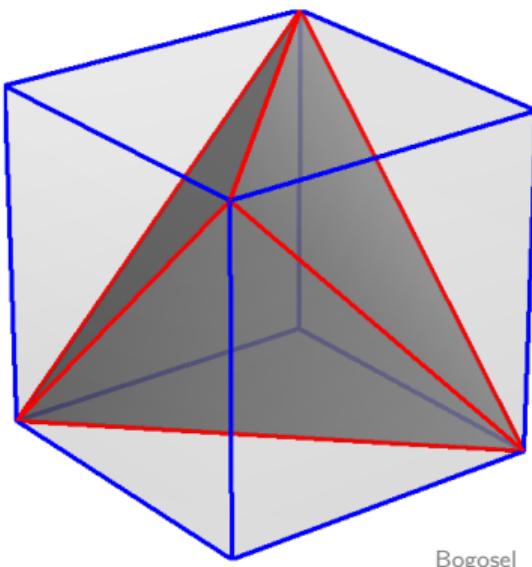


representação de malhas

Representação de malhas: sopa de polígonos

STL

- conjunto de faces sem organização
- vértices não compartilhados
- difícil de editar
- geometria das faces

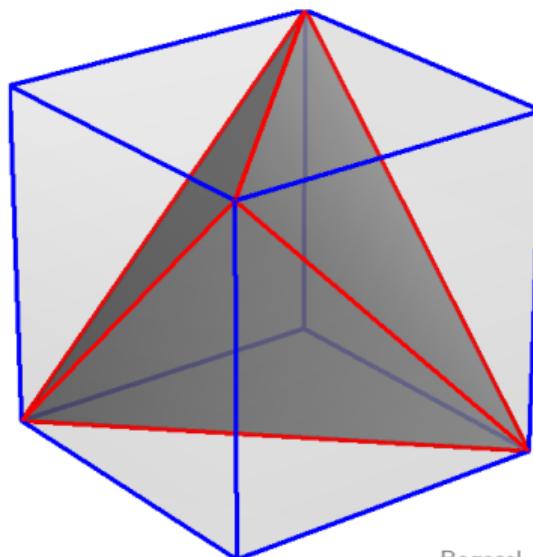


f			
-1	1	-1	
1	1	1	
1	-1	-1	
f			
-1	-1	1	
-1	1	-1	
1	-1	-1	
f			
1	-1	-1	
1	1	1	
-1	-1	1	
f			
-1	-1	1	
1	1	1	
-1	1	-1	

Representação de malhas: geometria + topologia

OBJ

- conjunto de faces sem organização
- vértices compartilhados
- mais fácil de editar
- geometria dos vértices
- topologia das faces



geometria

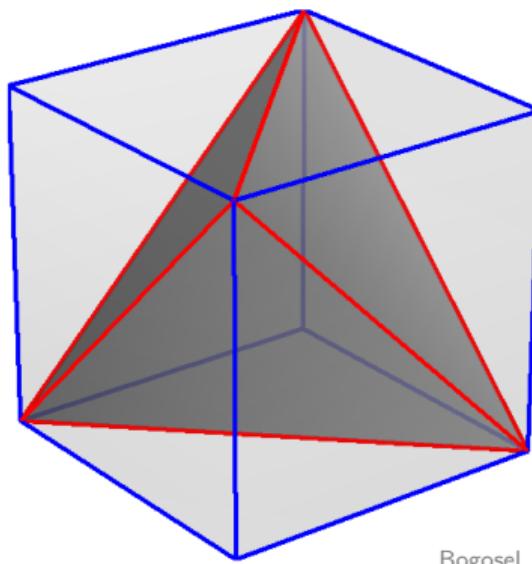
v	1	1	1
v	-1	1	-1
v	1	-1	-1
v	-1	-1	1

topologia

f	1	2	3
f	2	4	3
f	1	3	4
f	1	4	2

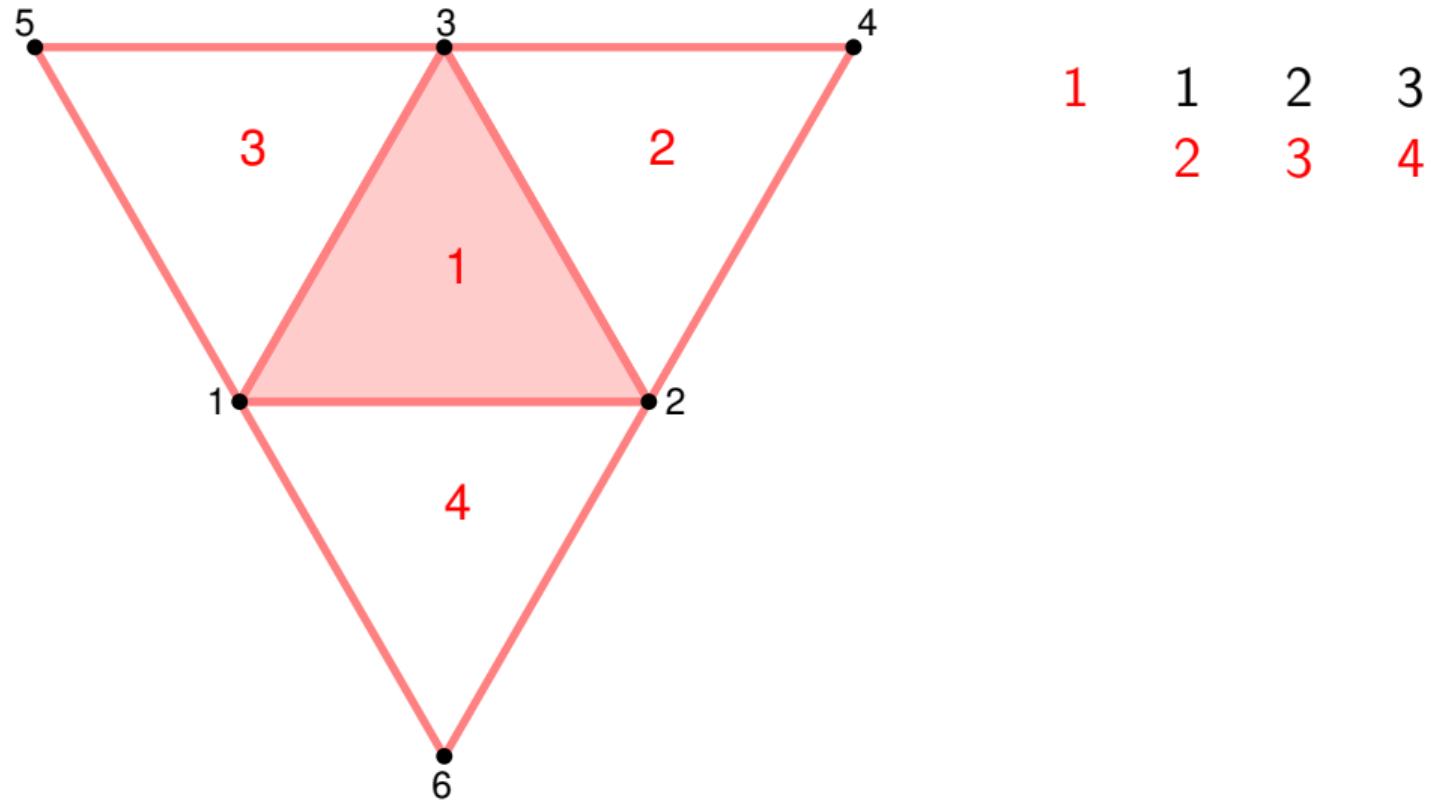
Representação de malhas: geometria + topologia + adjacências

- conjunto de faces com organização
- vértices compartilhados
- menos fácil de editar
- geometria dos vértices
- topologia das faces

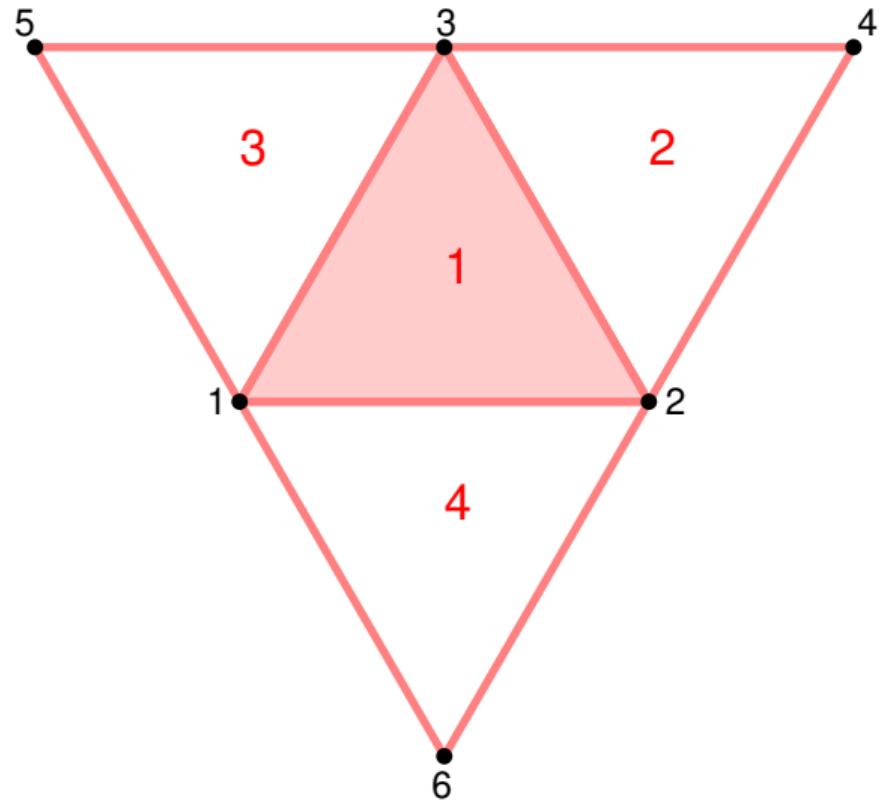


	geometria			
v	1	1	1	1
v	-1	1	-1	
v	1	-1	-1	
v	-1	-1	1	
	topologia			
f	1	2	3	
	2	3	4	
f	2	4	3	
	3	1	4	
f	1	3	4	
	2	4	1	
f	1	4	2	
	2	1	3	

Representação de malhas de triângulos



Representação de malhas de triângulos



1	1	2	3
2	3	4	
2	2	4	3
	1		
3	1	3	5
	1		1
4	1	6	2
	1		

superfícies paramétricas

Superfície paramétrica

superfície = fronteira da região

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

Superfície paramétrica

superfície = fronteira da região (retalho)

$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

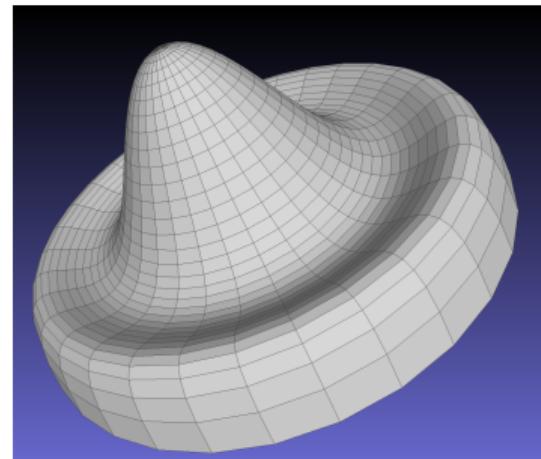
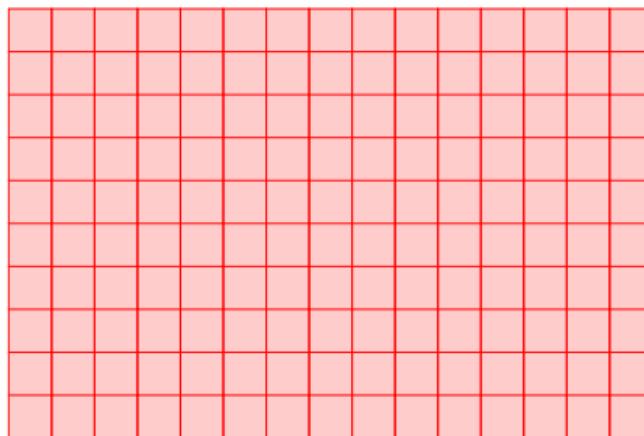
Superfície paramétrica

superfície = fronteira da região (retalho)

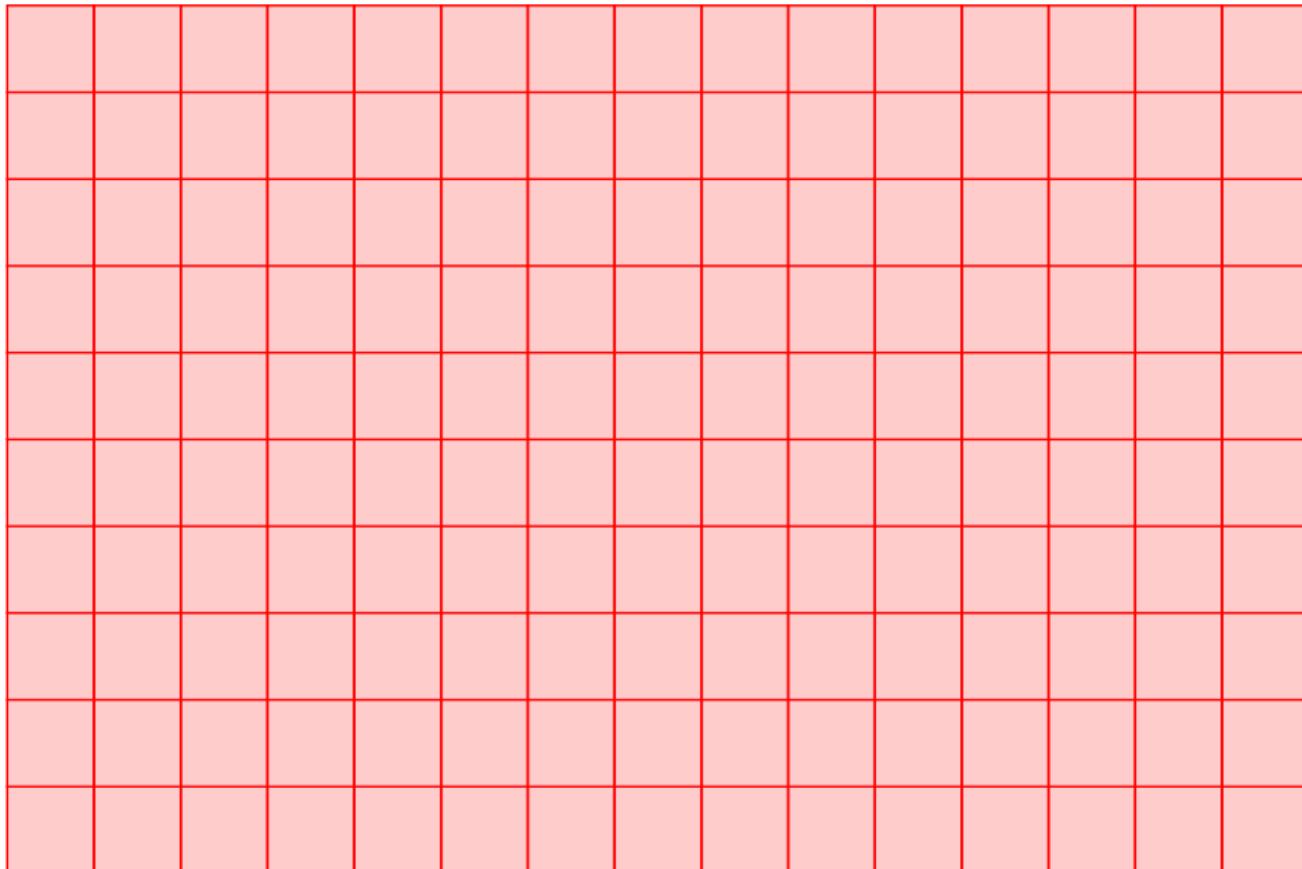
$$f: \Omega \subseteq \mathbf{R}^2 \rightarrow \mathbf{R}^3 \quad \Omega = [a, b] \times [c, d]$$

$$f(u, v) = (x(u, v), y(u, v), z(u, v))$$

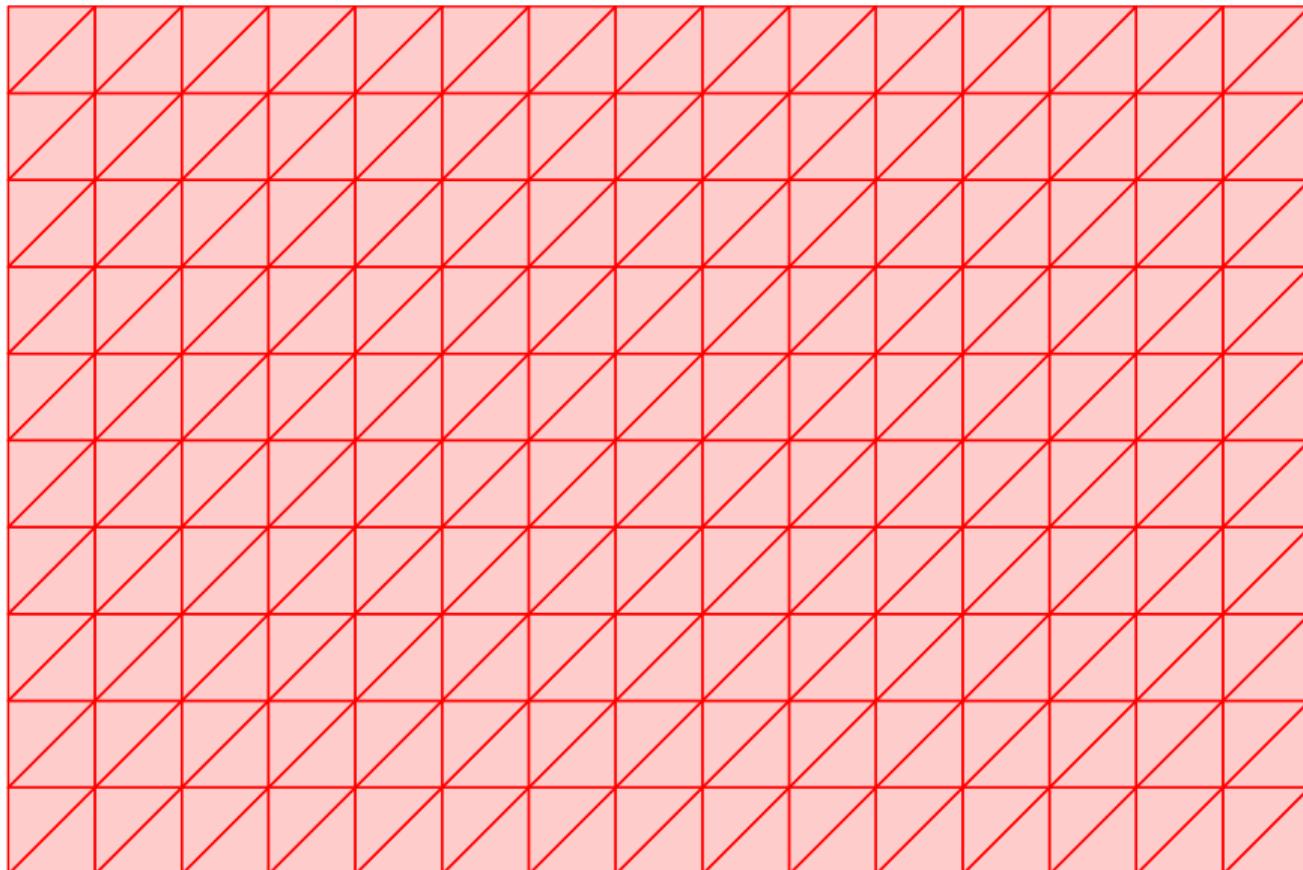
discretização da superfície = discretização do domínio



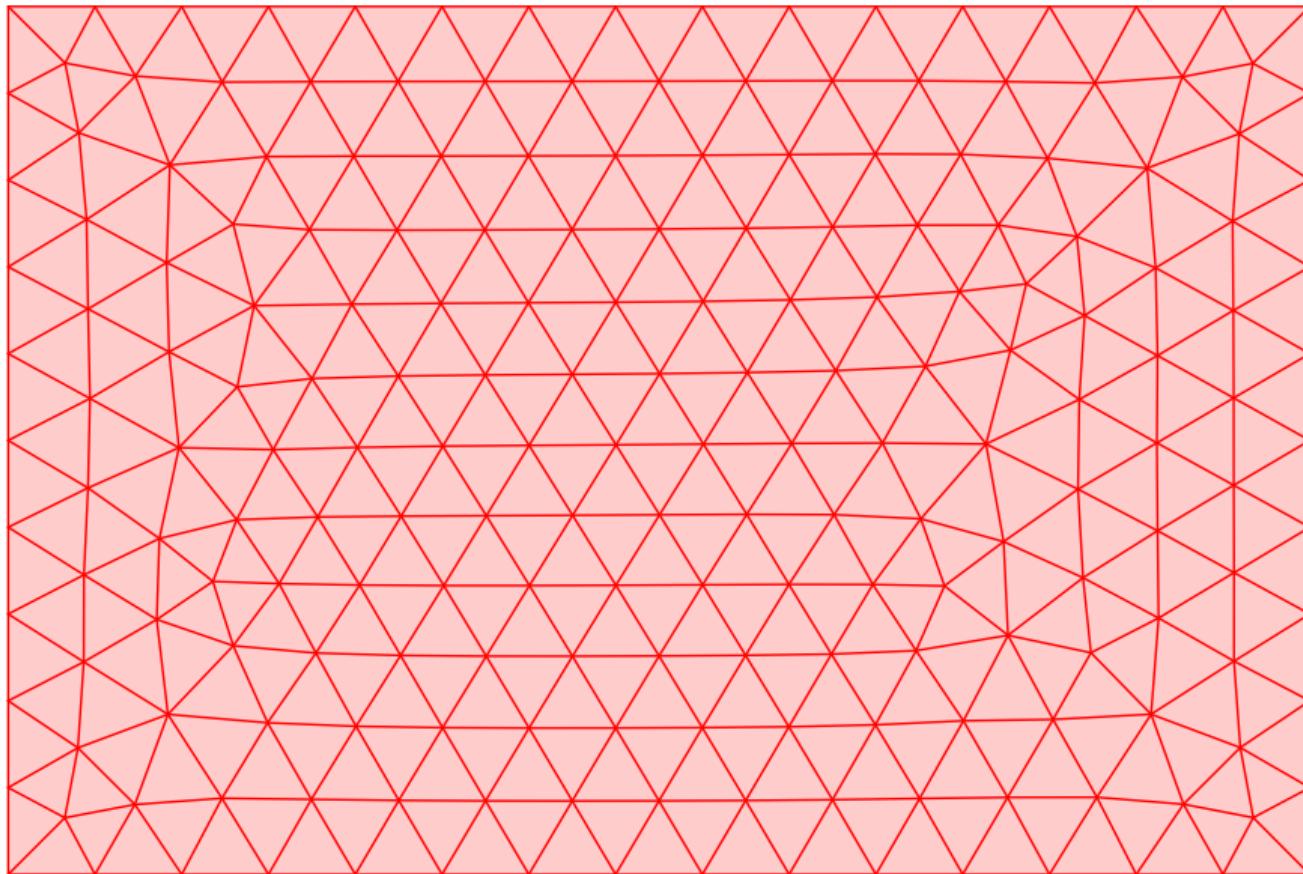
Superfície paramétrica – discretização do domínio



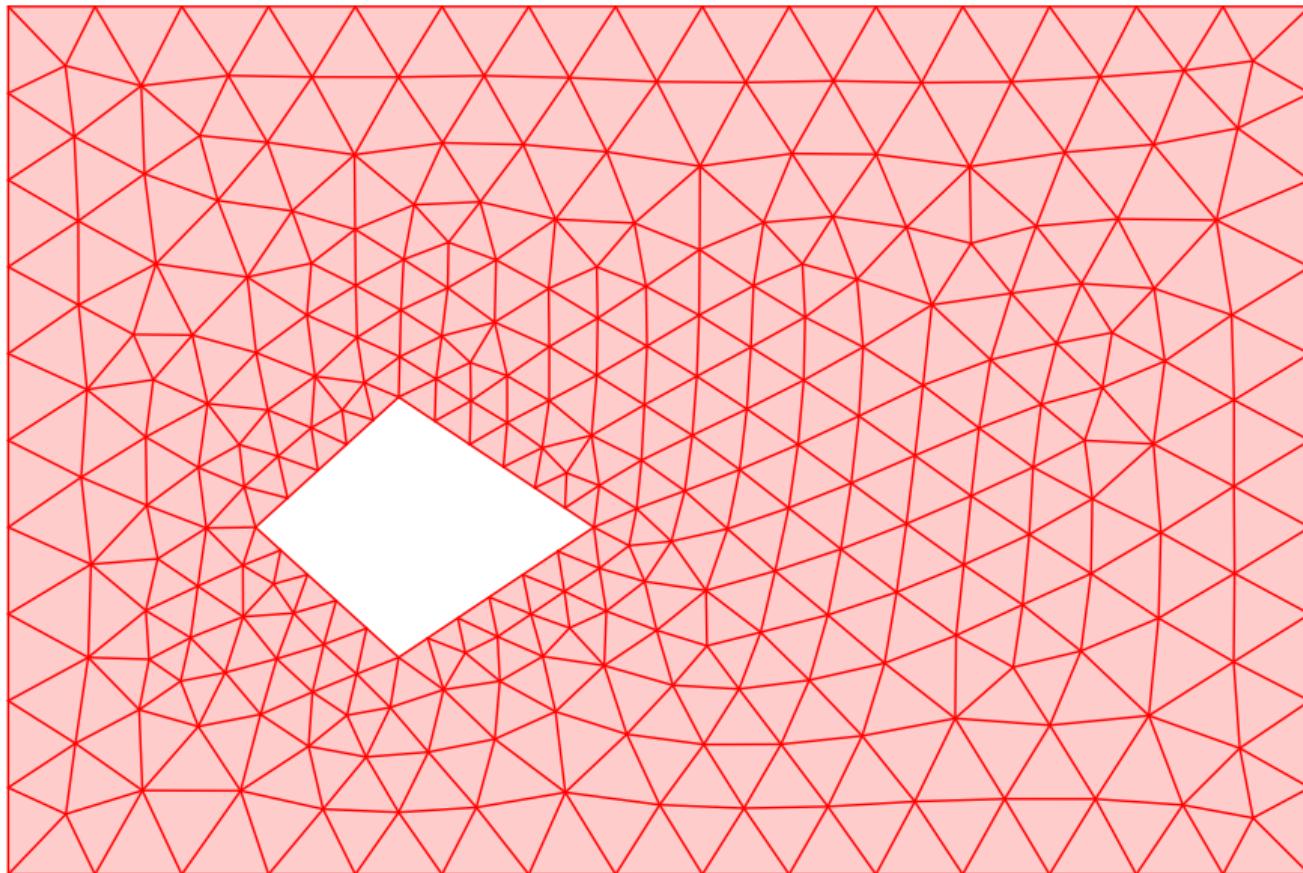
Superfície paramétrica – discretização do domínio



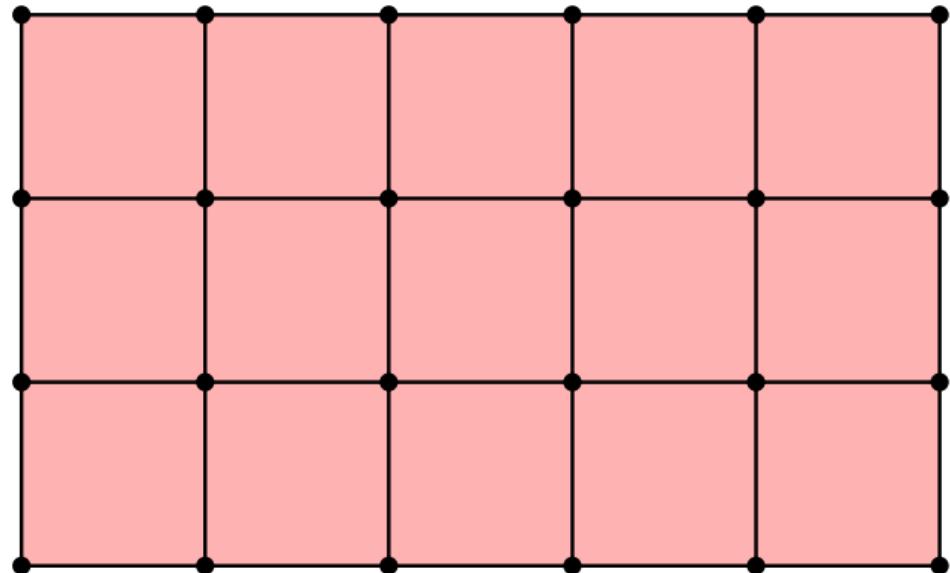
Superfície paramétrica – discretização do domínio



Superfície paramétrica – discretização do domínio



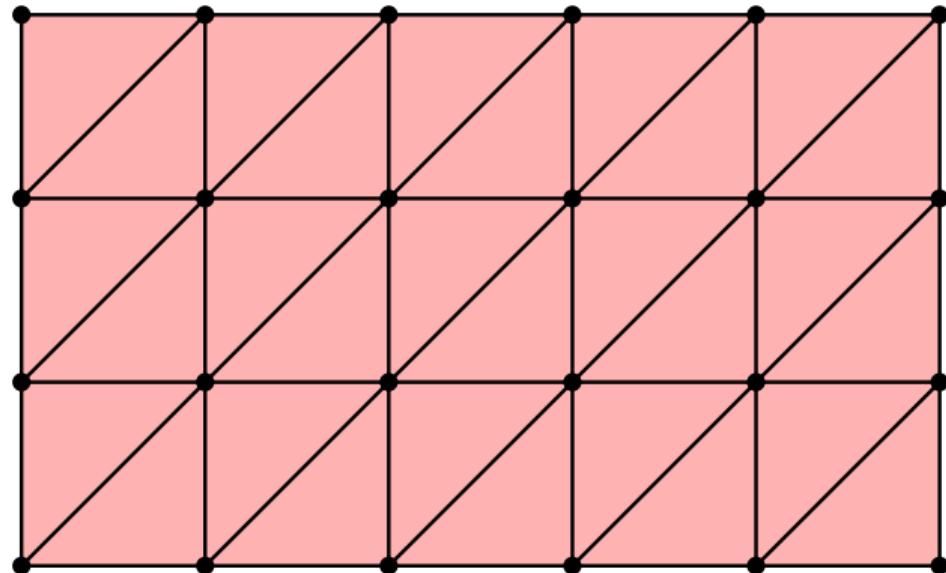
Superfície paramétrica – representação



19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

f	1	2	8	7
f	2	3	9	8
f	3	4	10	9
f	4	5	11	10
f	5	6	12	11
f	7	8	14	13
f	8	9	15	14
f	9	10	16	15
f	10	11	17	16
f	11	12	18	17
f	13	14	20	19
f	14	15	21	20
f	15	16	22	21
f	16	17	23	22
f	17	18	24	23

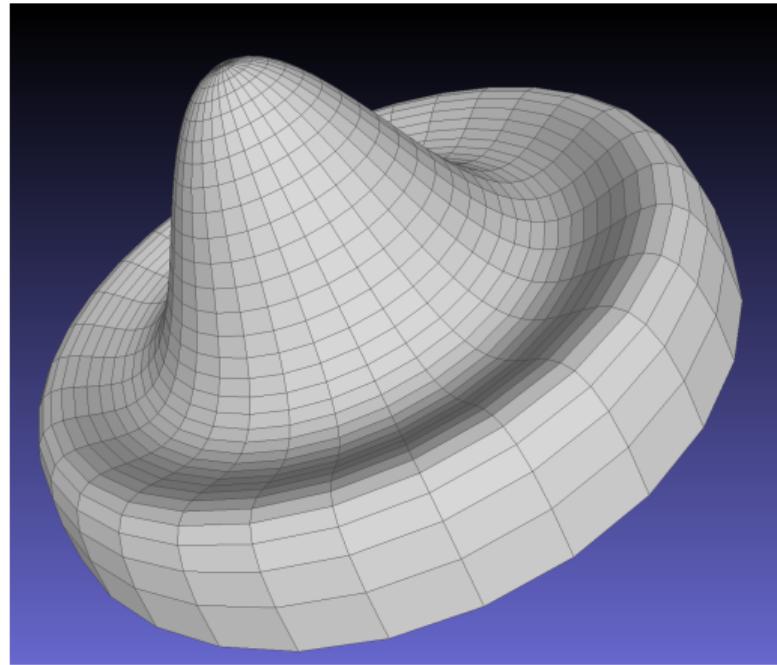
Superfície paramétrica – representação



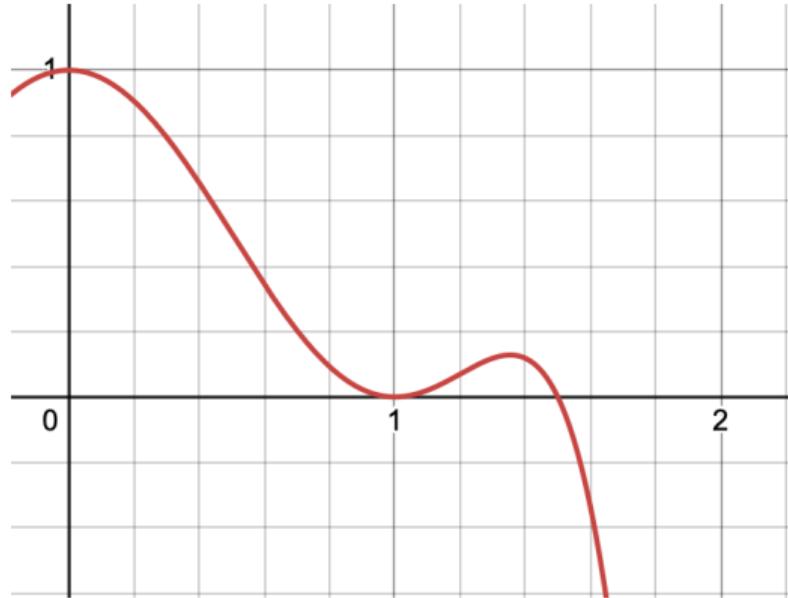
19	20	21	22	23	24
13	14	15	16	17	18
7	8	9	10	11	12
1	2	3	4	5	6

f	1	2	8
f	1	8	7
f	2	3	9
f	2	9	8
f	3	4	10
f	3	10	9
f	4	5	11
f	4	11	10
f	5	6	12
f	5	12	11
f	7	8	14
f	7	14	13
f	8	9	15
f	8	15	14
f	...		

Superfície de revolução

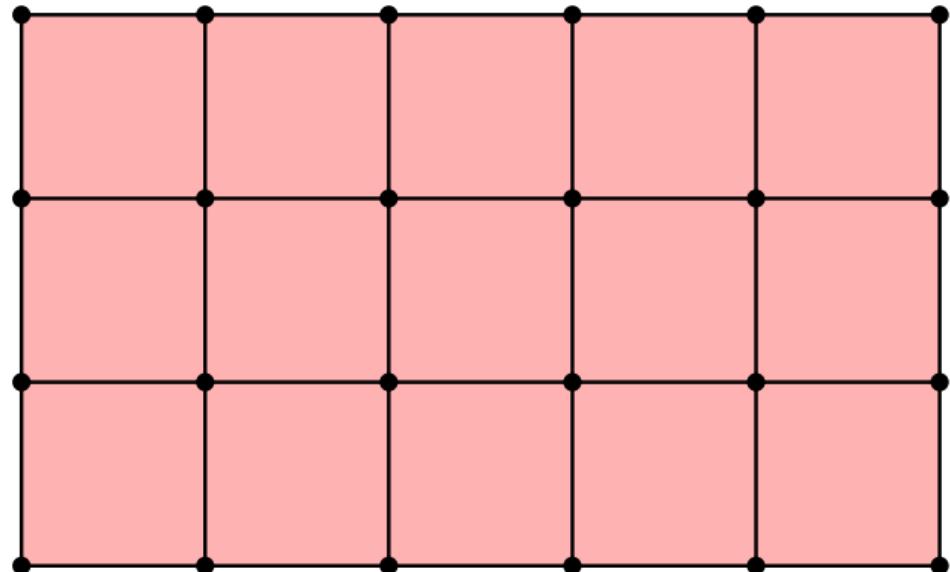


$$f(u, v) = (v \cos(u), v \sin(u), g(v))$$
$$u \in [0, 2\pi], v \in [0, 1.6]$$



$$g(x) = -\frac{4}{9}x^6 + \frac{17}{9}x^4 - \frac{22}{9}x^2 + 1$$

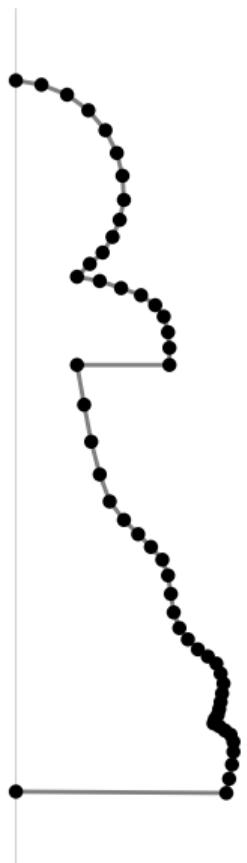
Superfície de revolução – representação



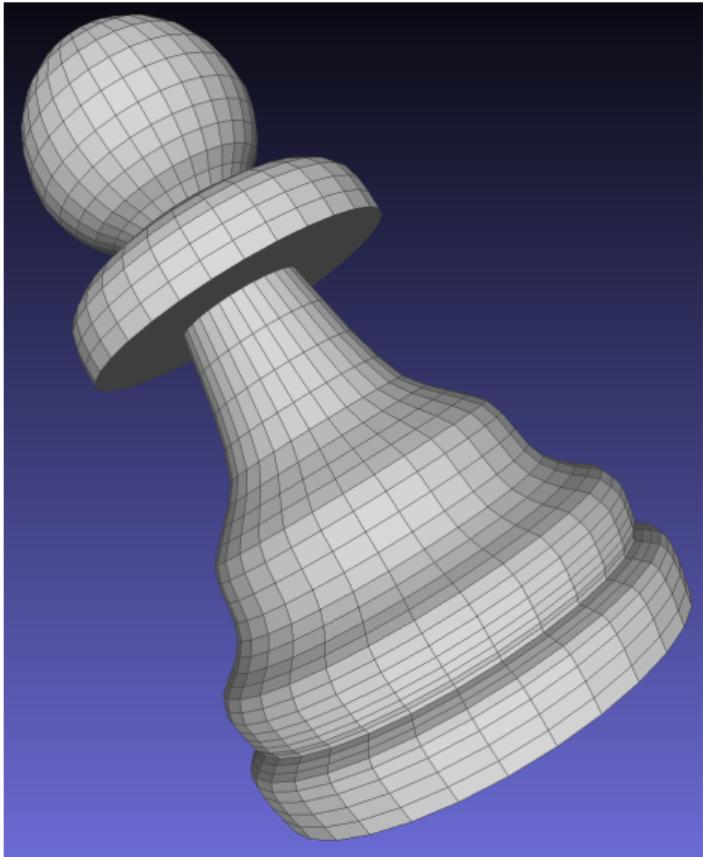
16	17	18	19	20	16
11	12	13	14	15	11
6	7	8	9	10	6
1	2	3	4	5	1

f	1	2	7	6
f	2	3	8	7
f	3	4	9	8
f	4	5	10	9
f	5	1	6	10
f	6	7	12	11
f	7	8	13	12
f	8	9	14	13
f	9	10	15	14
f	10	6	11	15
f	11	12	17	16
f	12	13	18	17
f	13	14	19	18
f	14	15	20	19
f	15	11	16	20

Superfície de revolução



Warren–Weimer



Retalho bilinear

- vértices do bordo

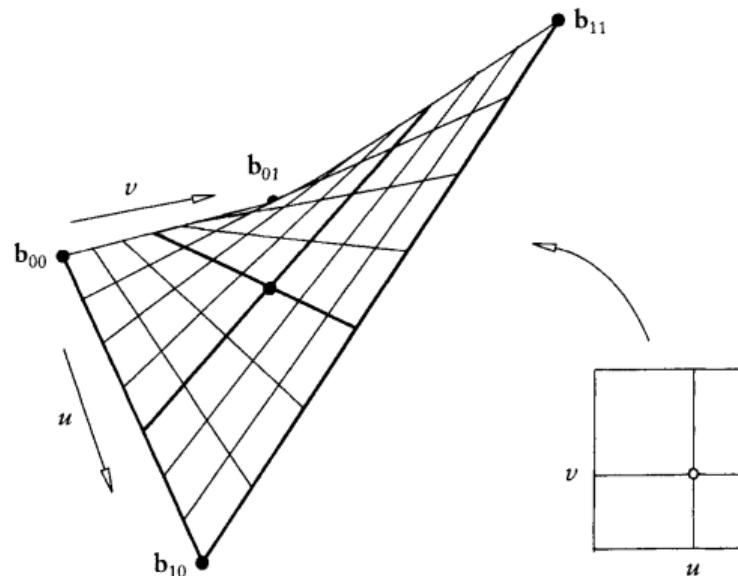
$p_{00}, p_{10}, p_{01}, p_{11}$

- interpolação bilinear

$$f(u, v) = (1 - u)(1 - v) p_{00} + u(1 - v) p_{10} + (1 - u)v p_{01} + uv p_{11}$$

$$u, v \in [0, 1]$$

- paraboloide hiperbólico



Retalho de Coons

Coons (1967)

- curvas de bordo

$$\alpha_0, \alpha_1, \beta_0, \beta_1: [0, 1] \rightarrow \mathbf{R}^3$$

- vértices do bordo

$$p_{00}, p_{10}, p_{01}, p_{11}$$

- superfícies regradas

$$A(u, v) = (1 - v)\alpha_0(u) + v\alpha_1(u)$$

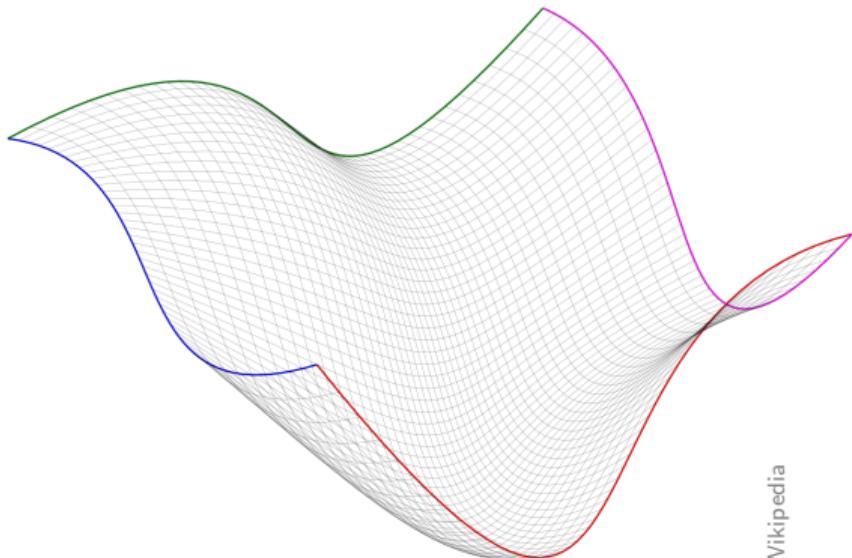
$$B(u, v) = (1 - u)\beta_0(v) + u\beta_1(v)$$

- interpolação bilinear

$$C(u, v) = (1 - u)(1 - v)p_{00} + u(1 - v)p_{10} + (1 - u)v p_{01} + uv p_{11}$$

- $f(u, v) = A(u, v) + B(u, v) - C(u, v)$

$$u, v \in [0, 1]$$

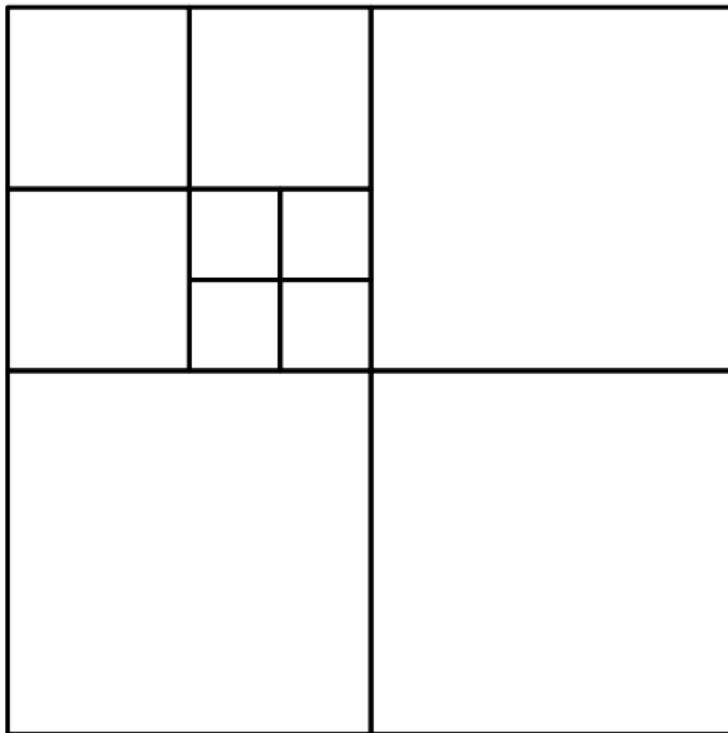


Wikipedia

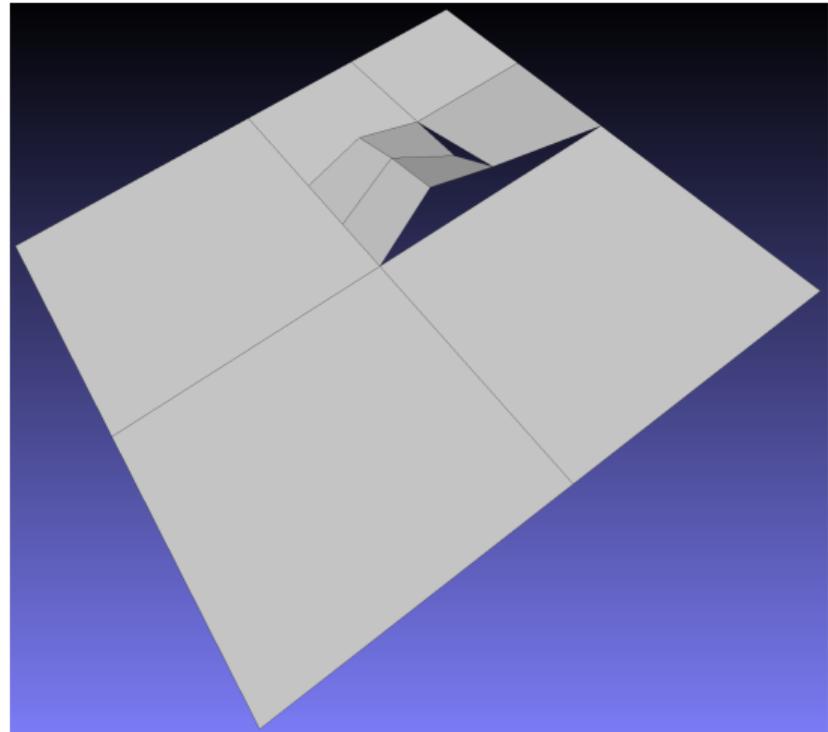
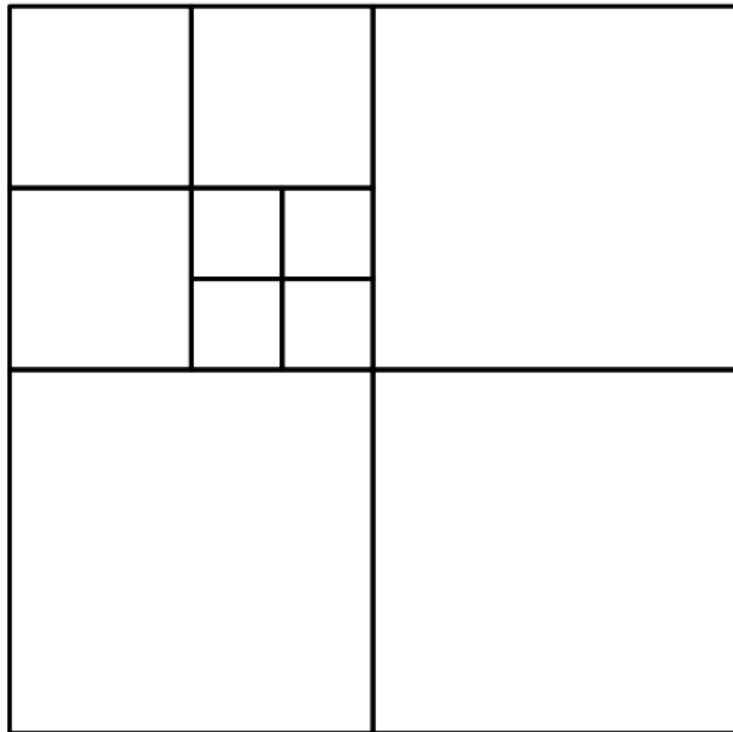
Superfície paramétrica – discretização do domínio



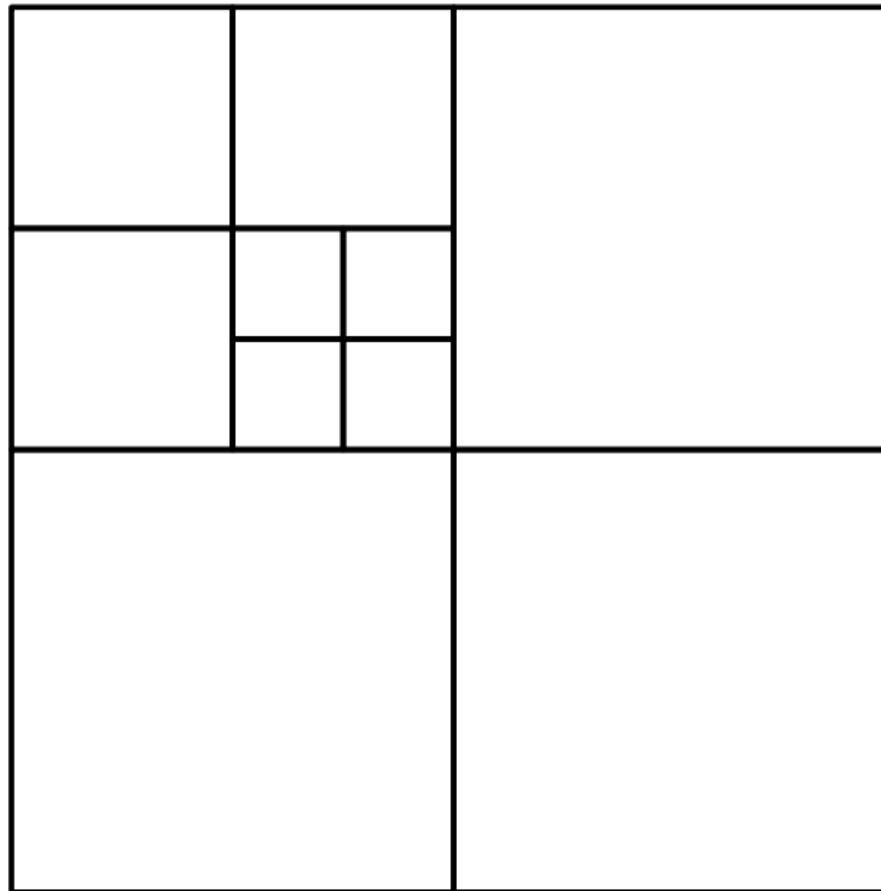
Superfície paramétrica – discretização do domínio: quadtree



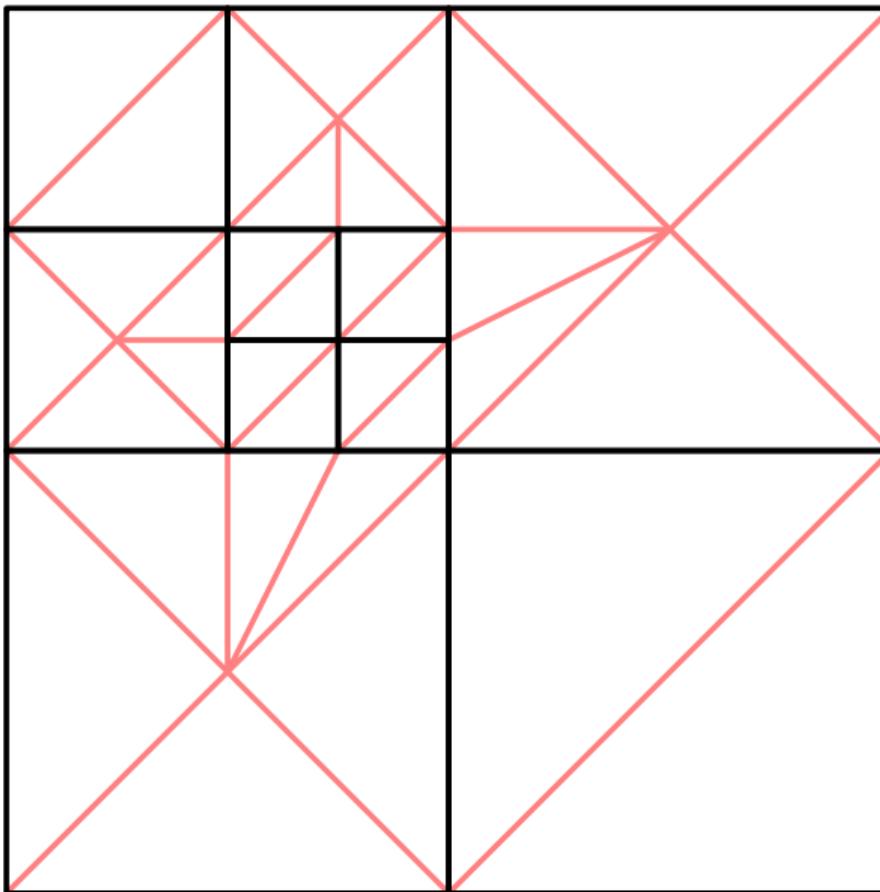
Superfície paramétrica – discretização do domínio: quadtree



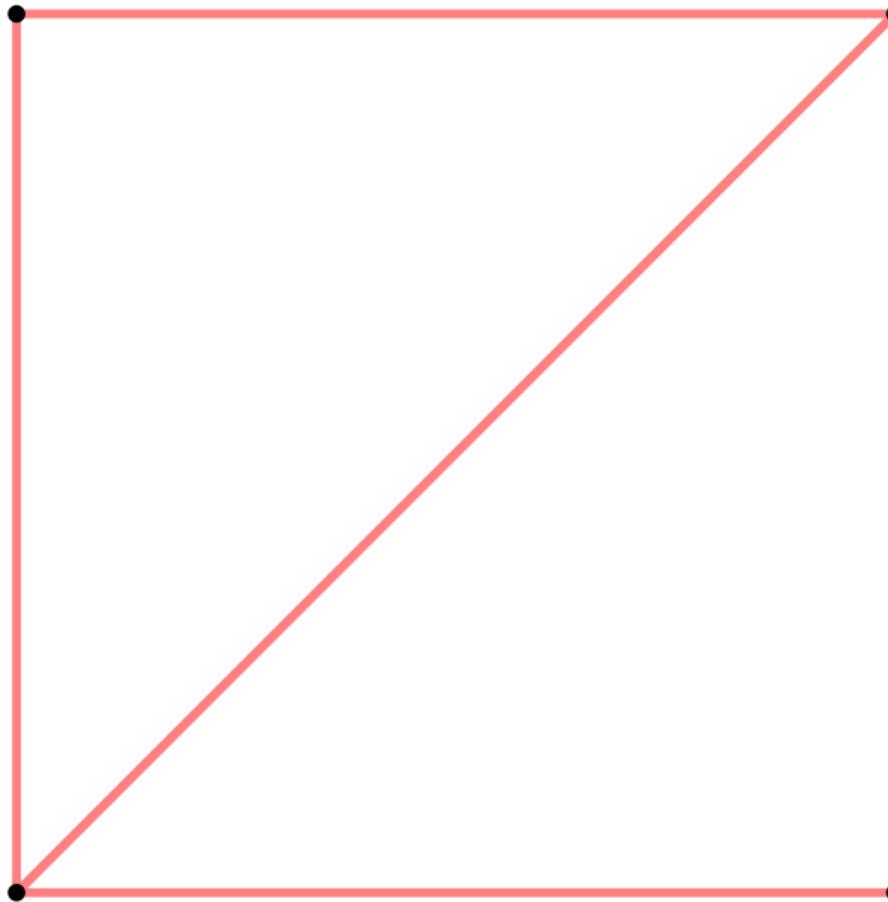
Superfície paramétrica – discretização do domínio: quadtree



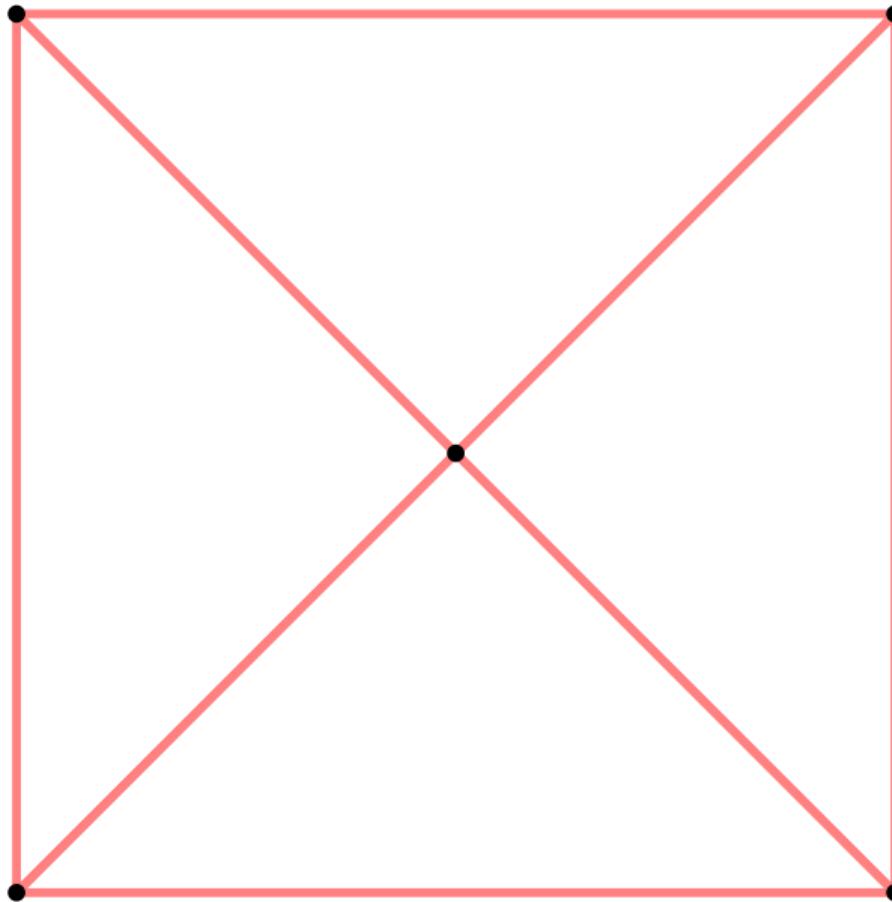
Superfície paramétrica – discretização do domínio: quadtree triangulada



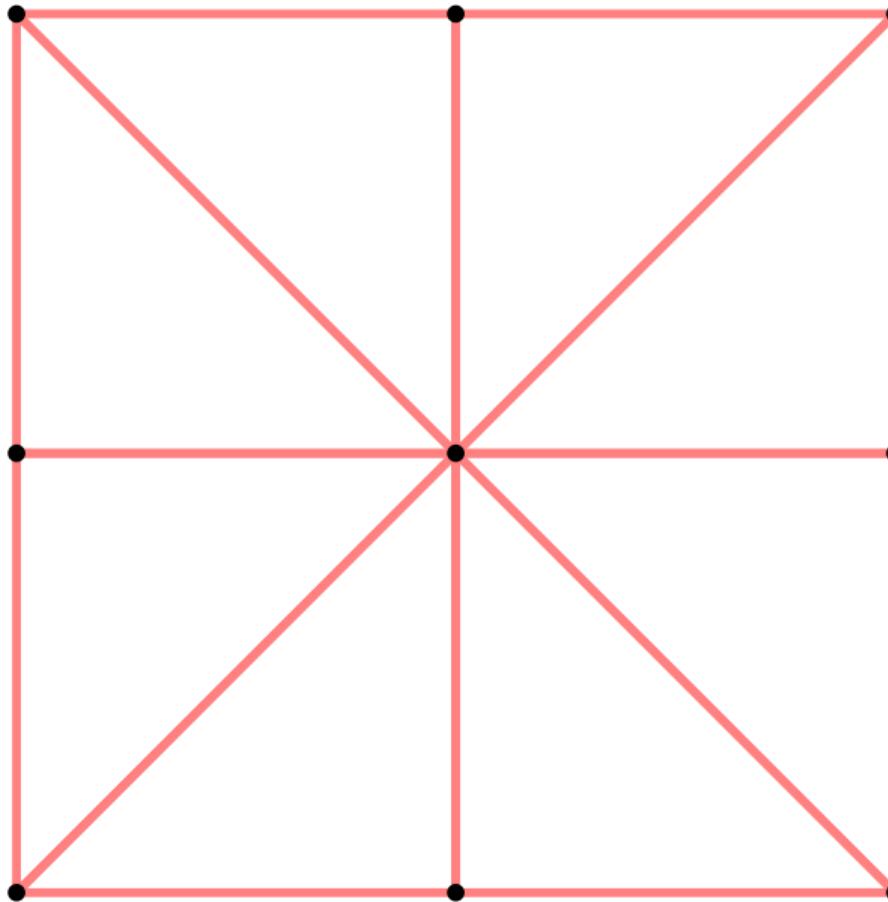
Malha 4-8 uniforme



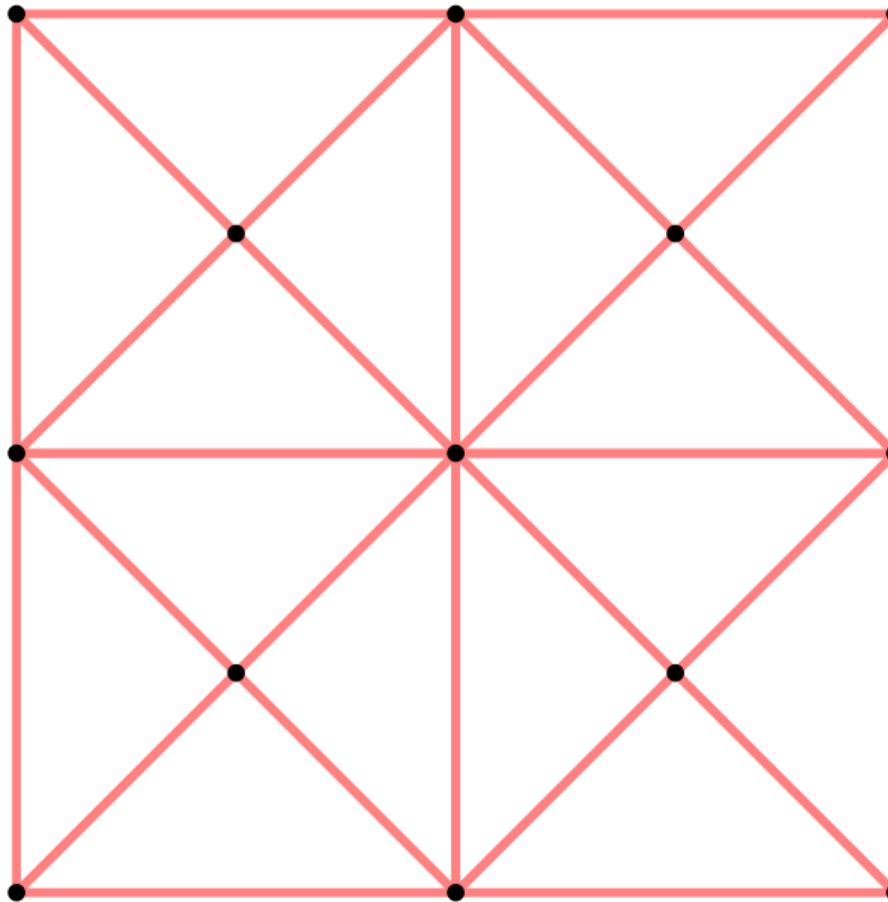
Malha 4-8 uniforme



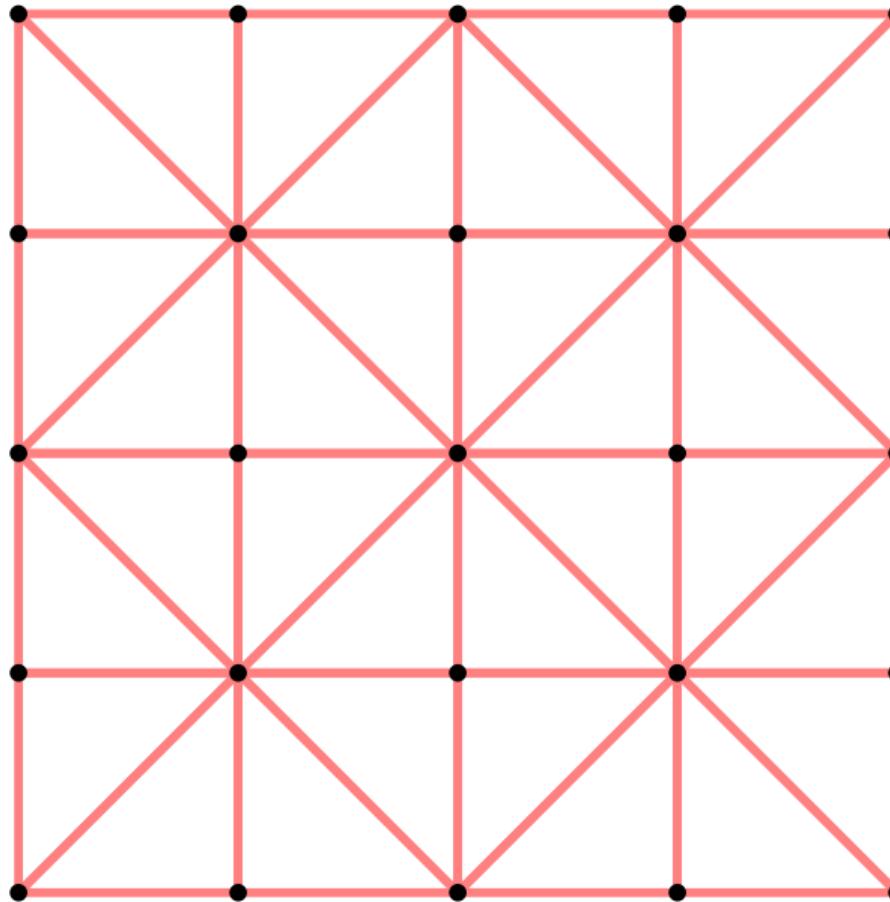
Malha 4-8 uniforme



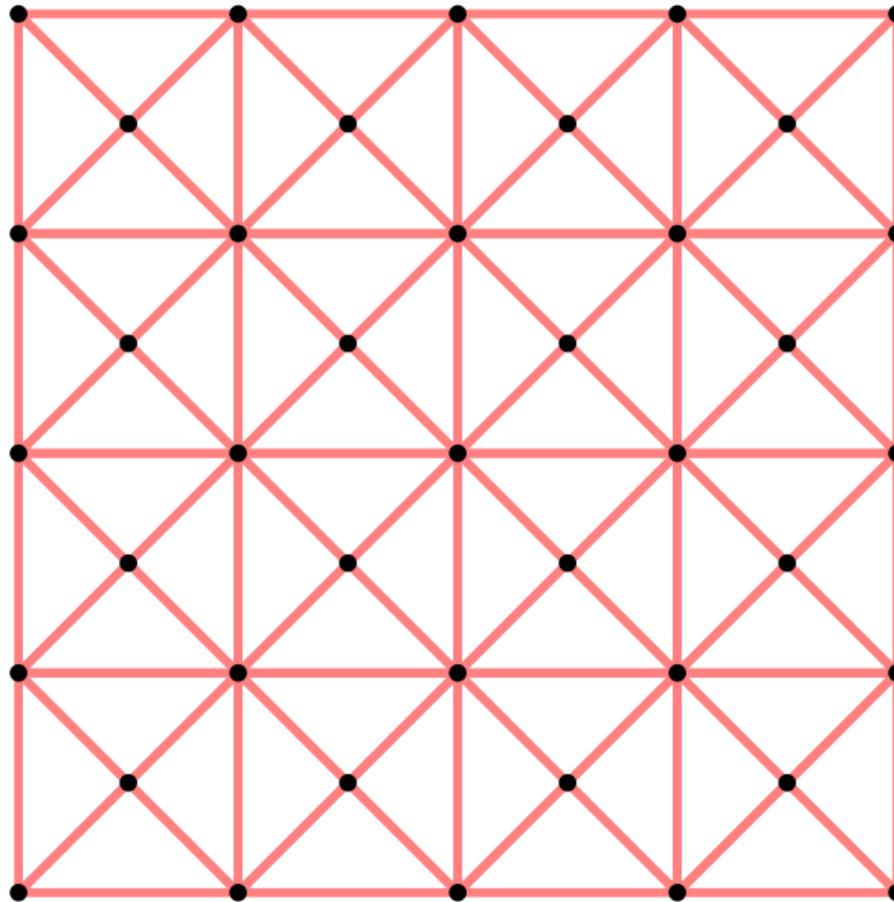
Malha 4-8 uniforme



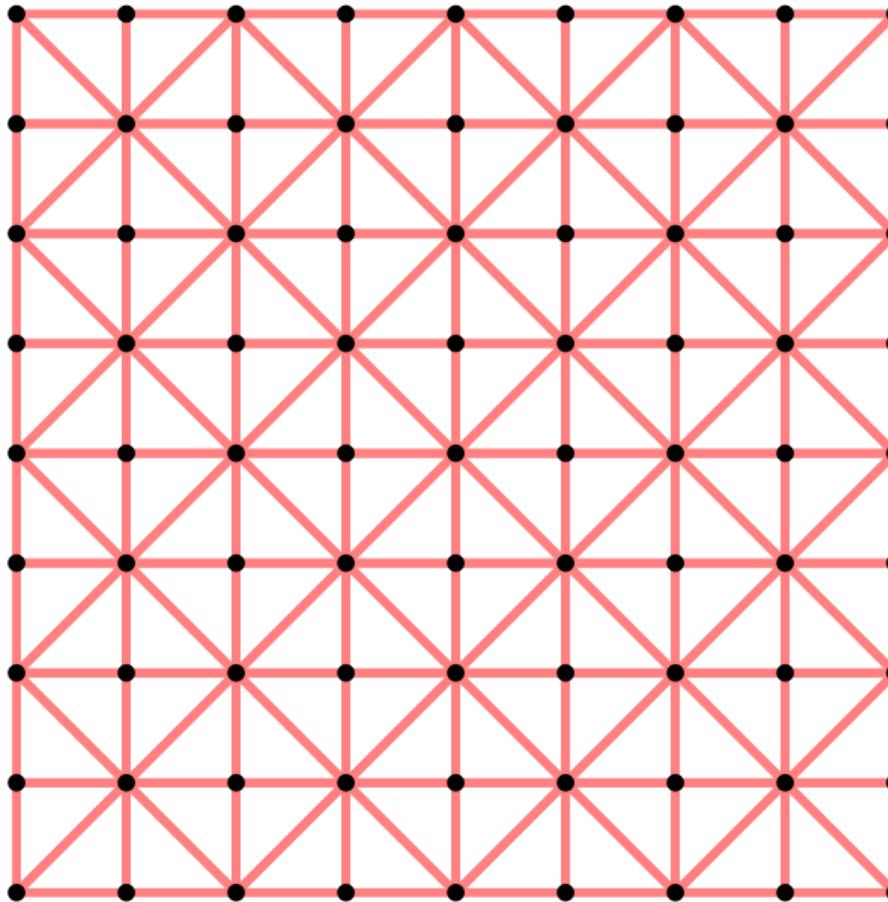
Malha 4-8 uniforme



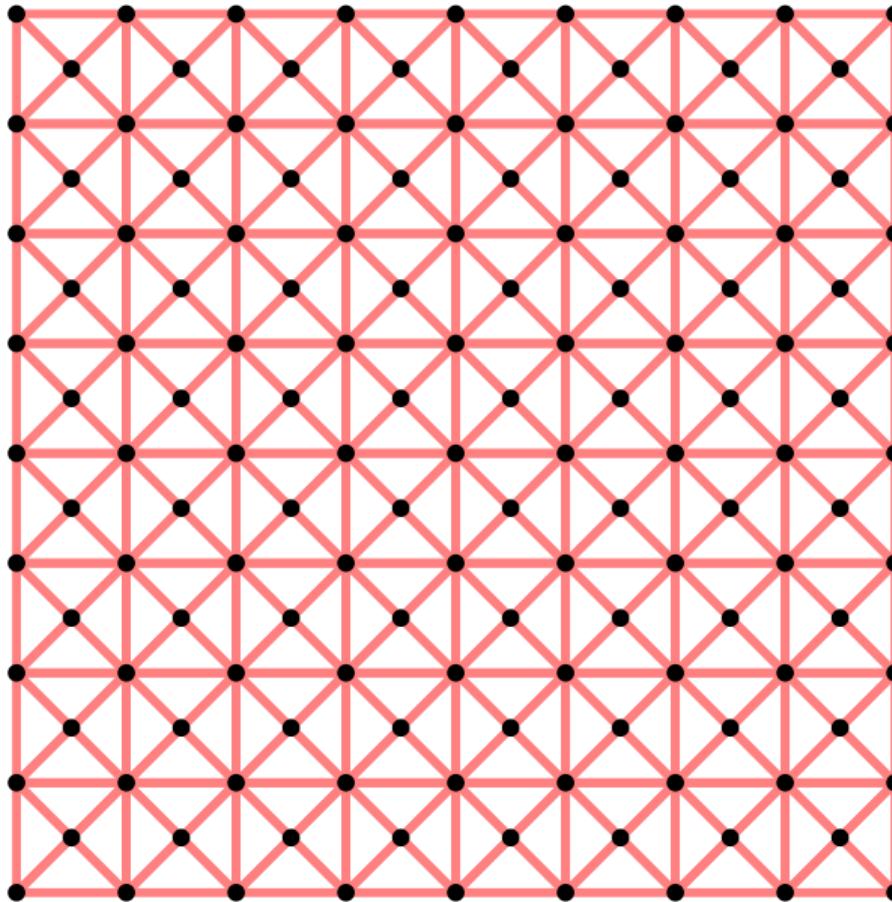
Malha 4-8 uniforme



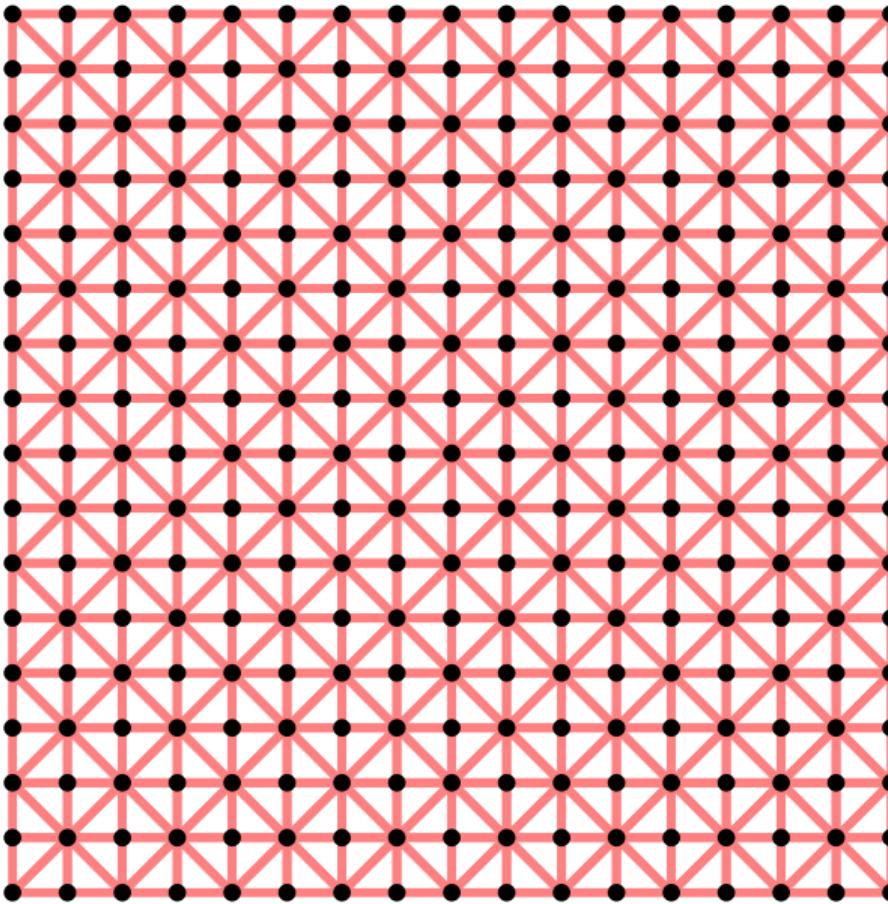
Malha 4-8 uniforme



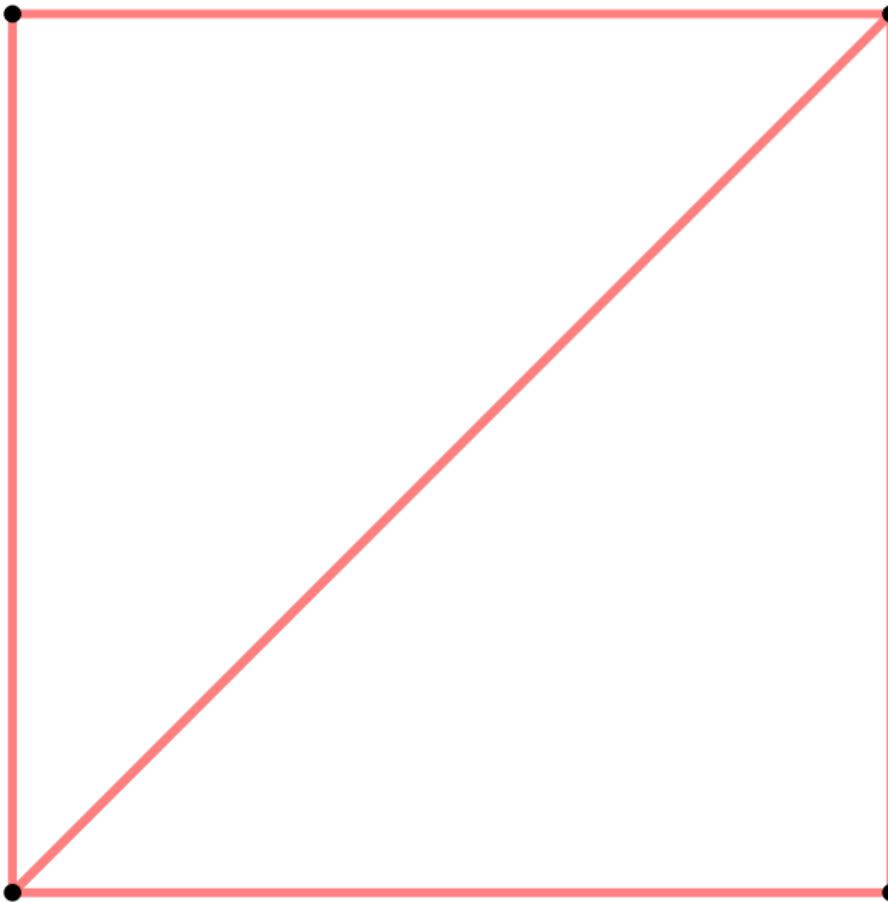
Malha 4-8 uniforme



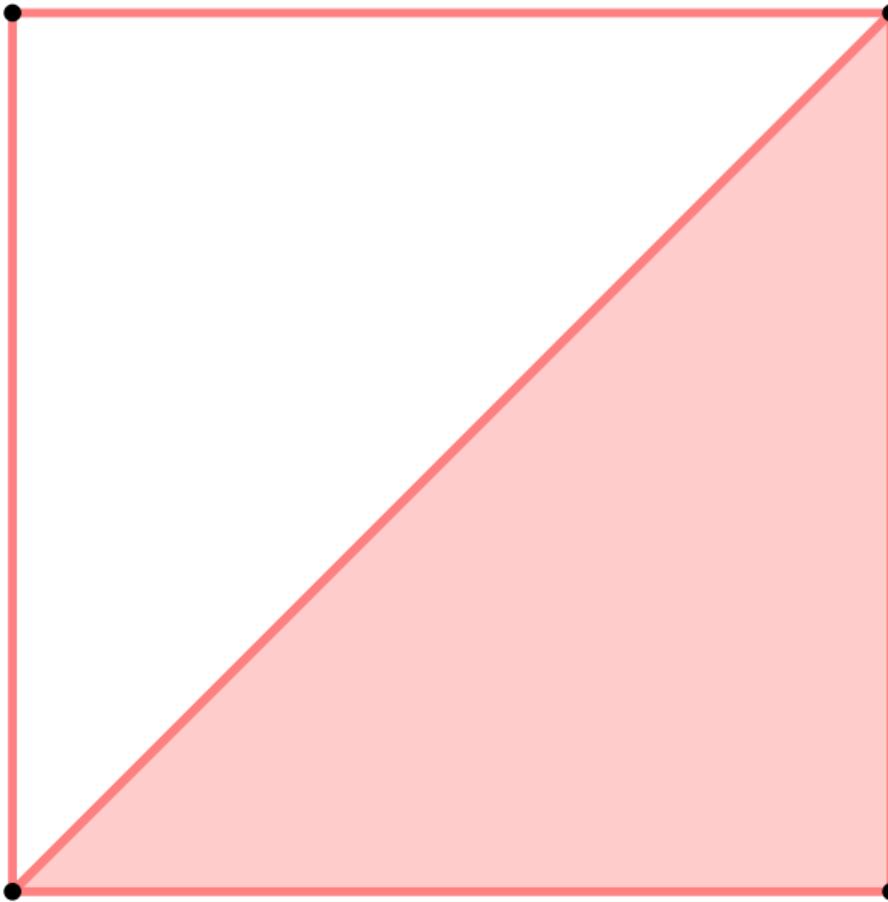
Malha 4-8 uniforme



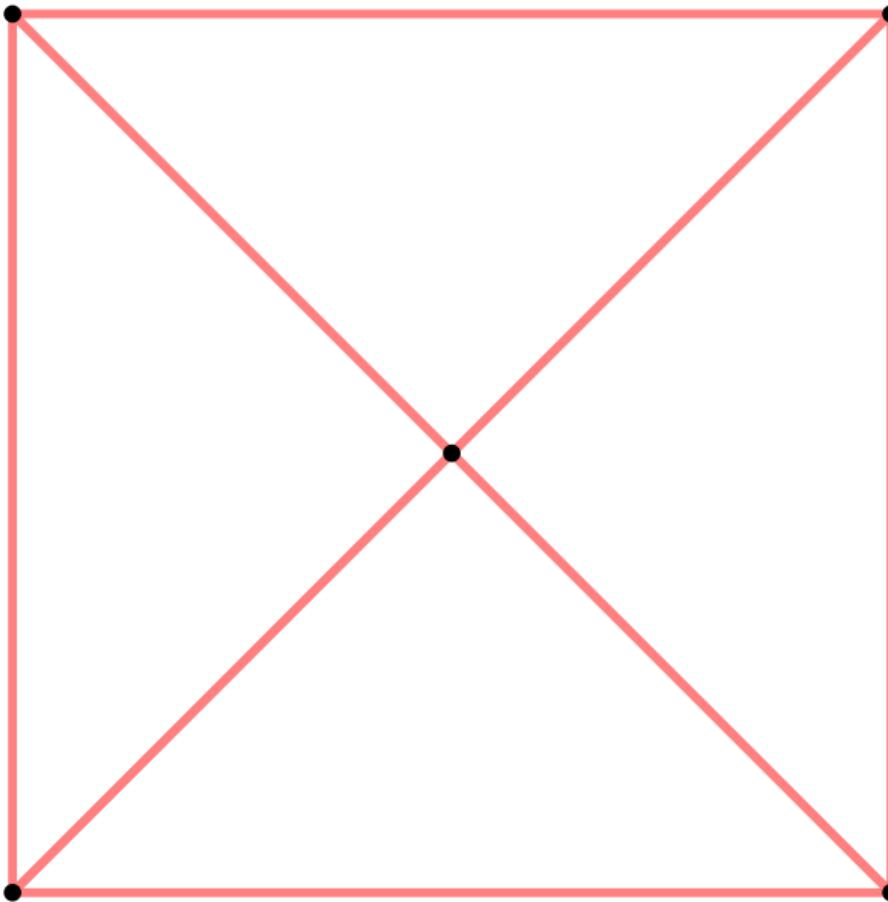
Malha 4-8 adaptativa



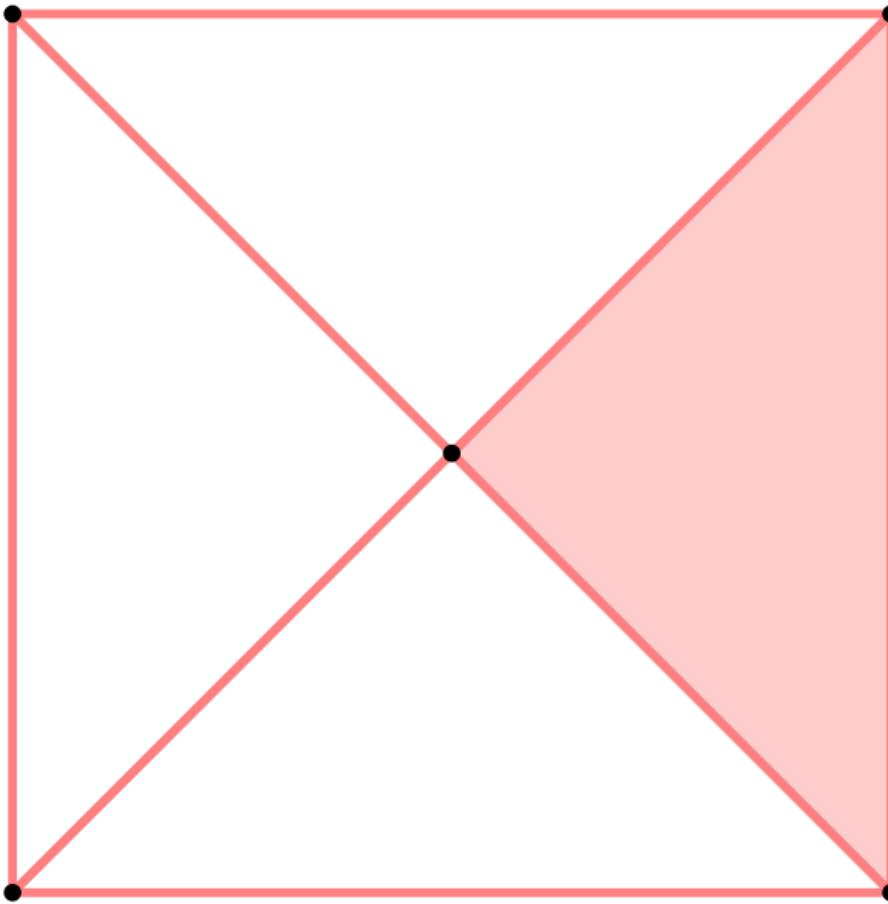
Malha 4-8 adaptativa



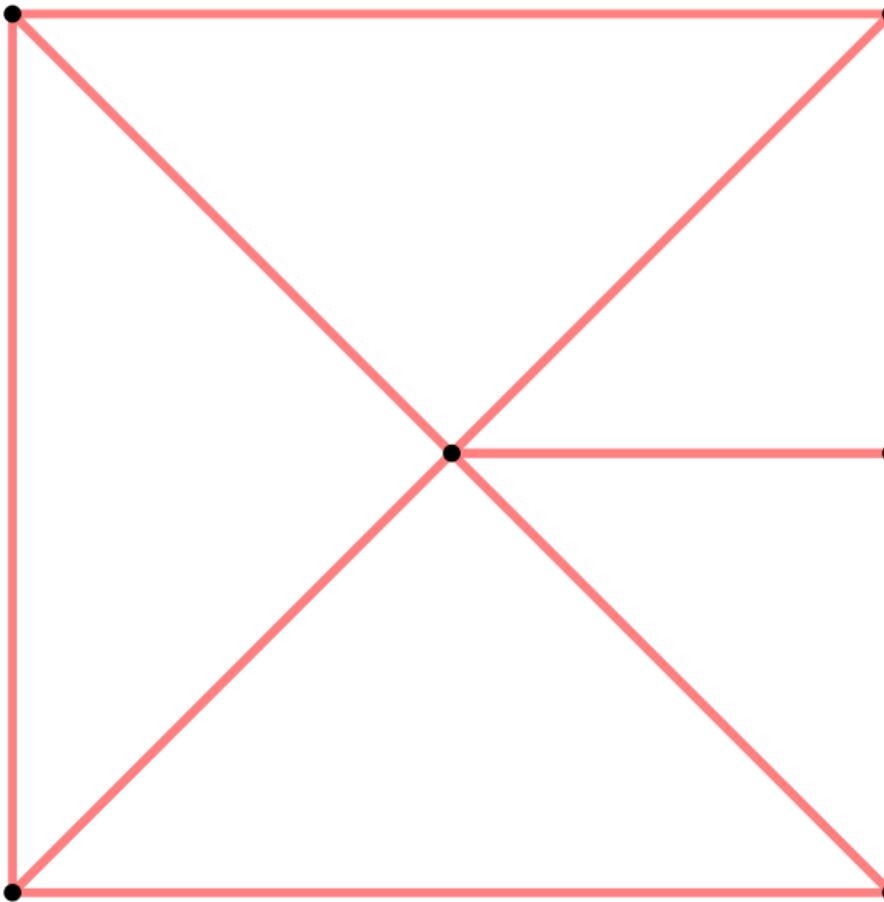
Malha 4-8 adaptativa



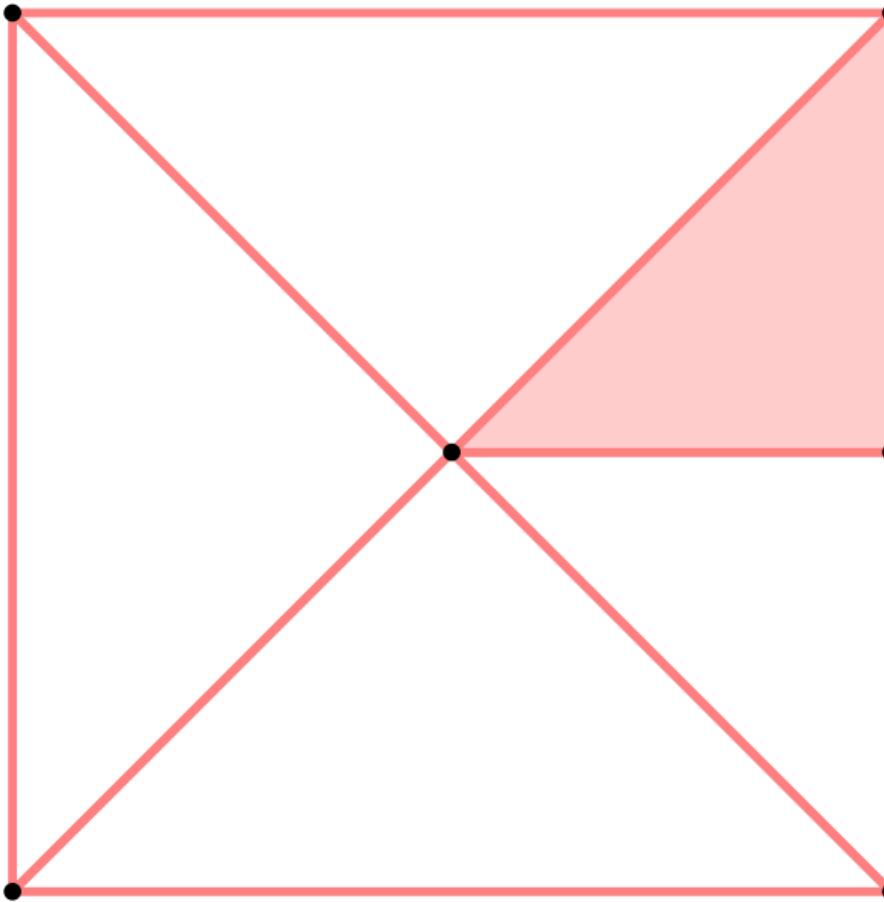
Malha 4-8 adaptativa



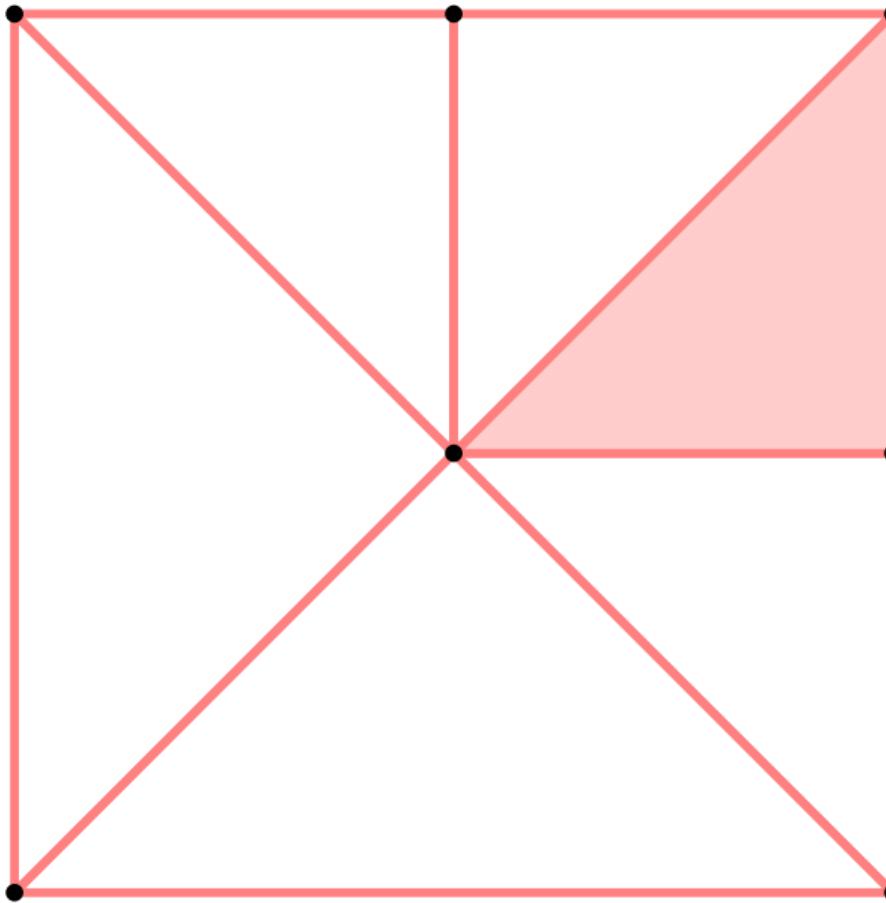
Malha 4-8 adaptativa



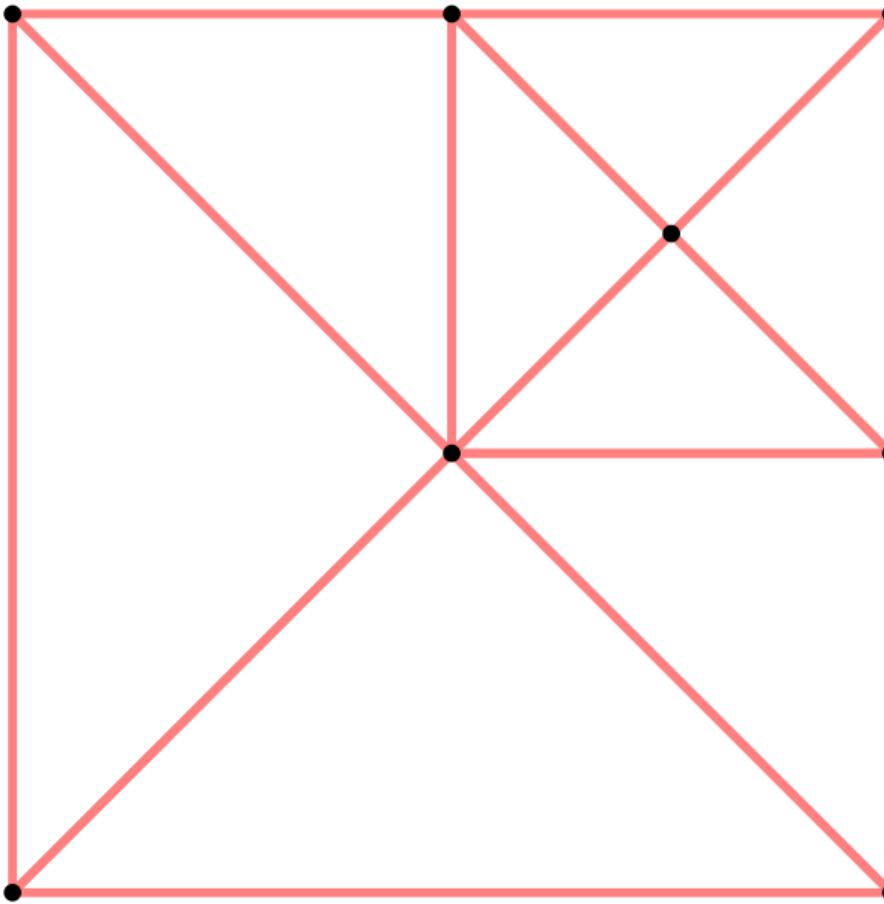
Malha 4-8 adaptativa



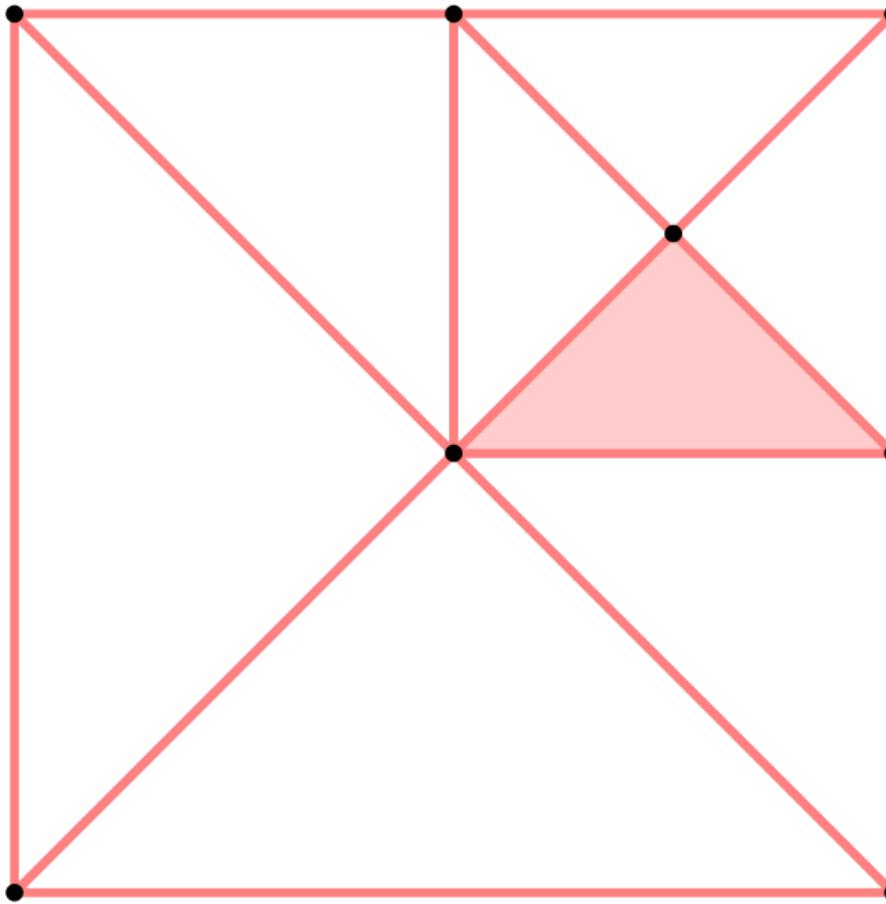
Malha 4-8 adaptativa



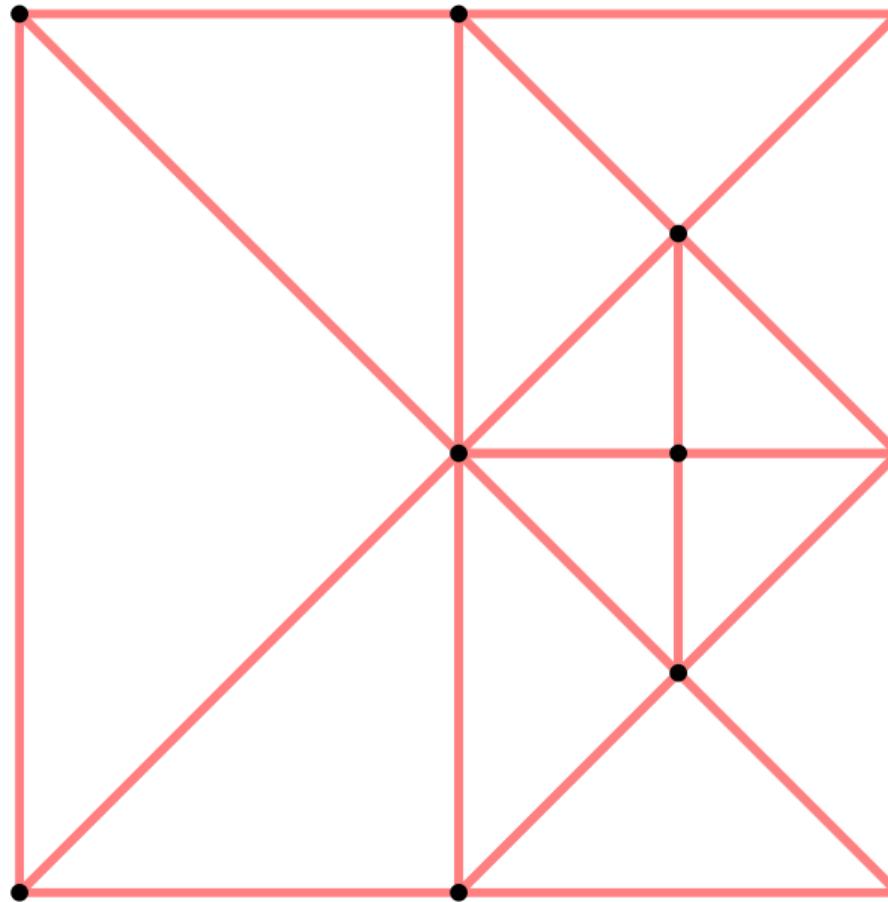
Malha 4-8 adaptativa



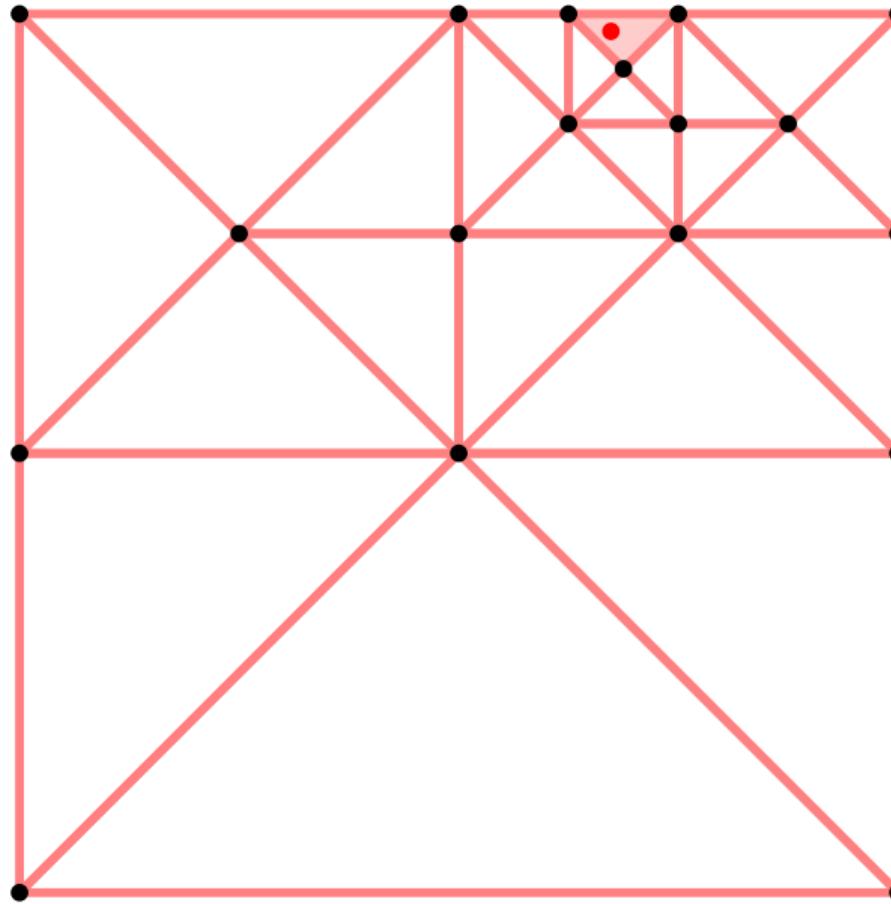
Malha 4-8 adaptativa



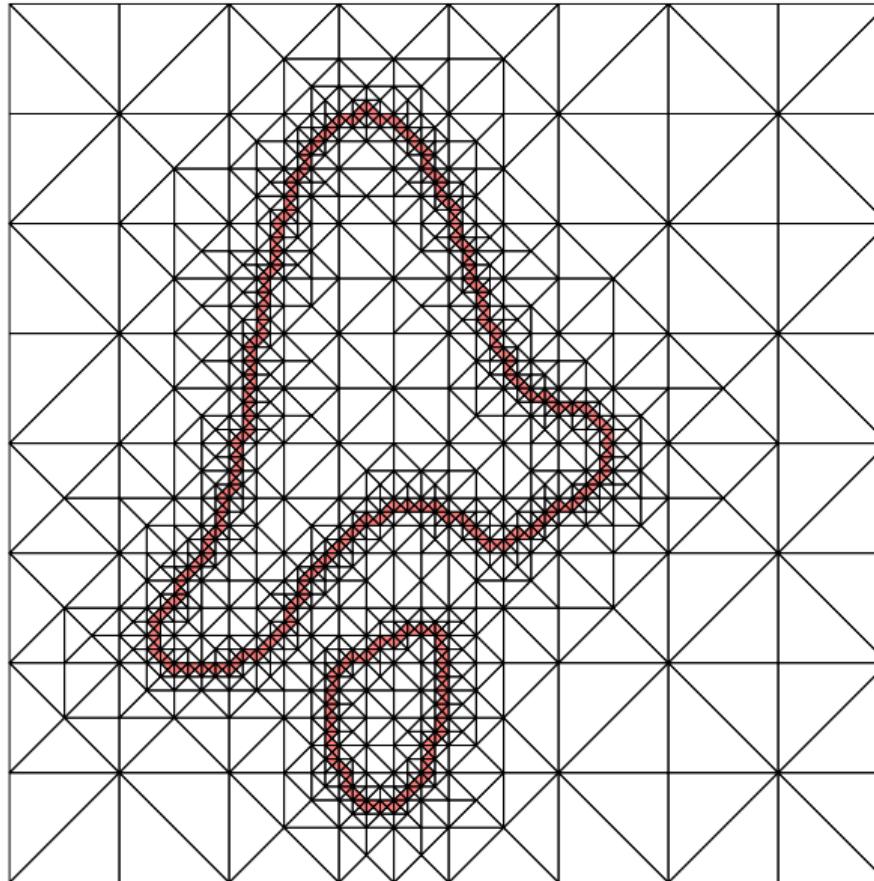
Malha 4-8 adaptativa



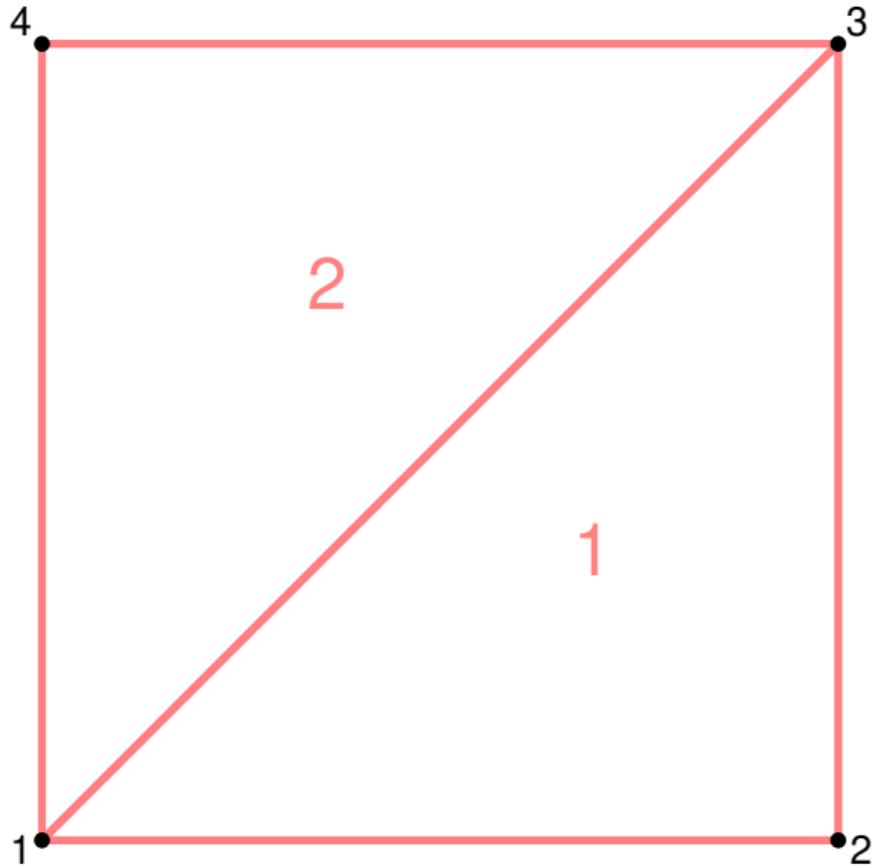
Malha 4-8 adaptativa



Malha 4-8 adaptativa – curva implícita

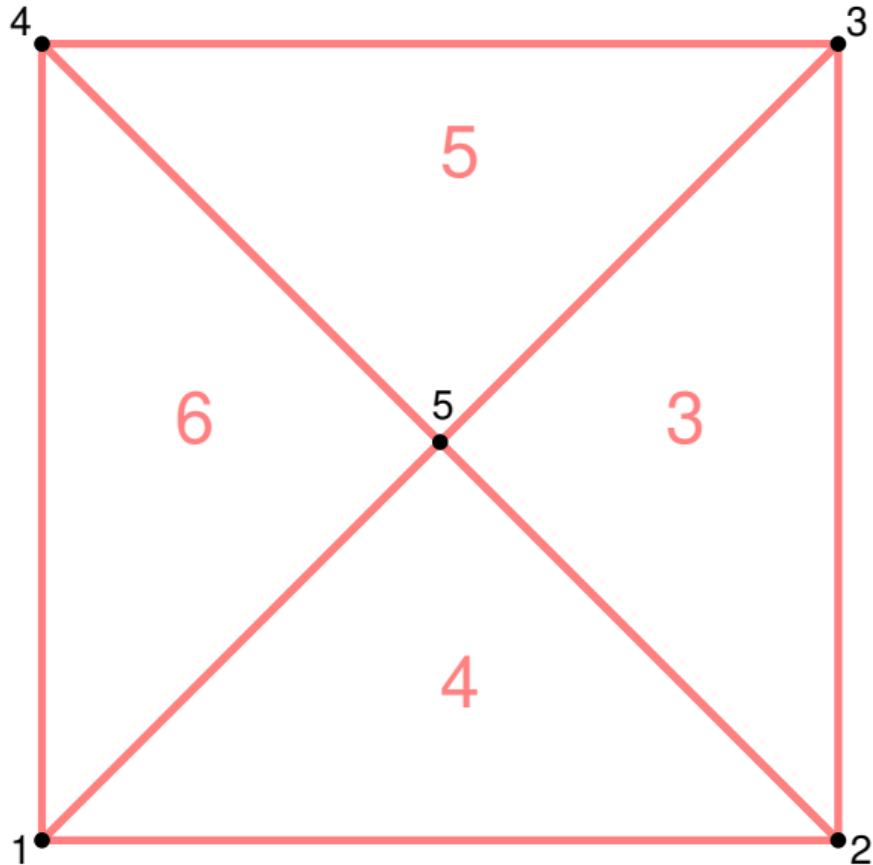


Malha 4-8 – representação



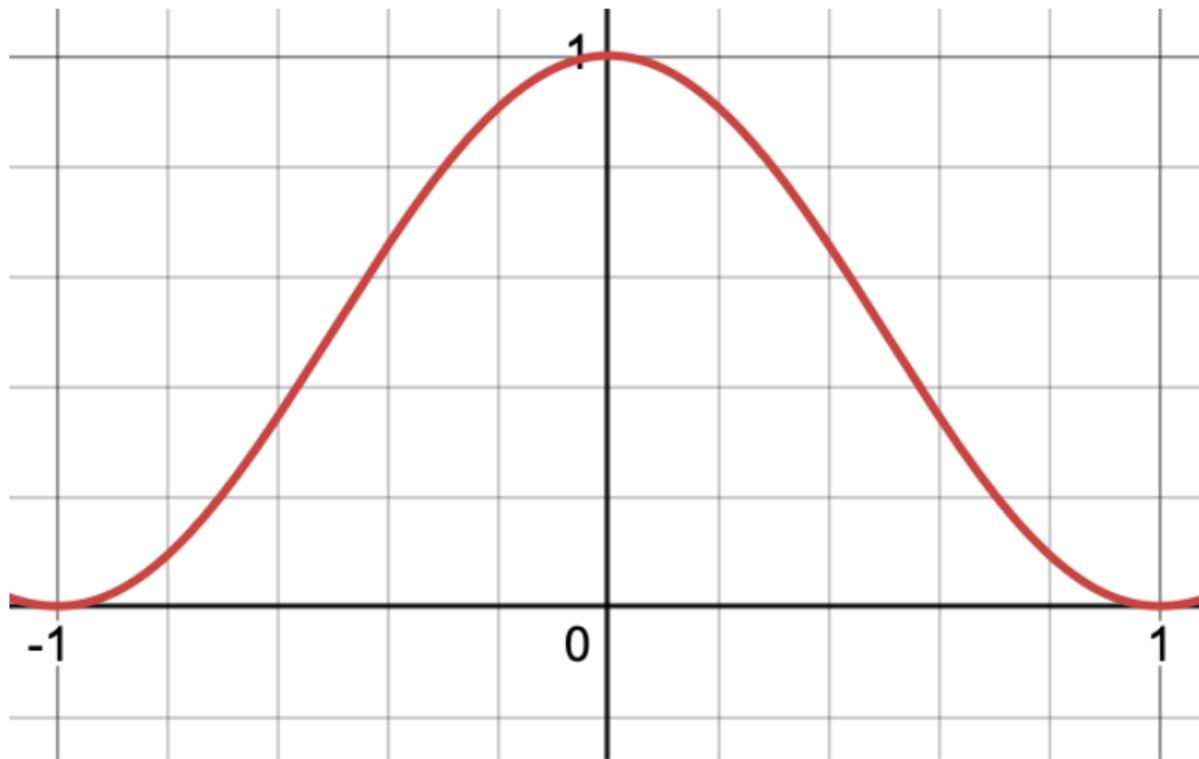
v	1	0	0	0
v	2	1	0	0
v	3	1	1	0
v	4	0	1	0
f	1	2	3	1
		2	–	–
f	2	4	1	3
		1	–	–

Malha 4-8 – representação



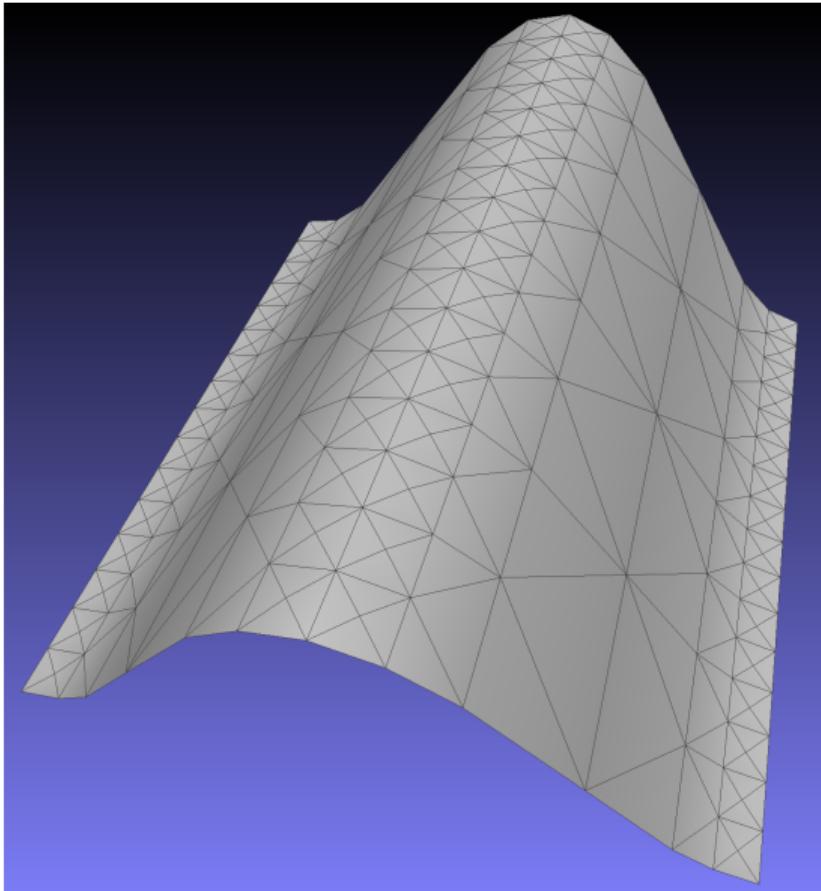
v	1	0	0	0	
v	2	1	0	0	
v	3	1	1	0	
v	4	0	1	0	
v	5	0.5	0.5	0	
f	1	2	3	1	x
f		2	—	—	
f	2	4	1	3	x
f		1	—	—	
f	3	5	2	3	
f		—	5	4	
f	4	5	1	2	
f		—	3	6	
f	5	5	3	4	
f		—	6	3	
f	6	5	4	1	
f		—	4	5	

Malha 4-8 adaptativa – superfície paramétrica

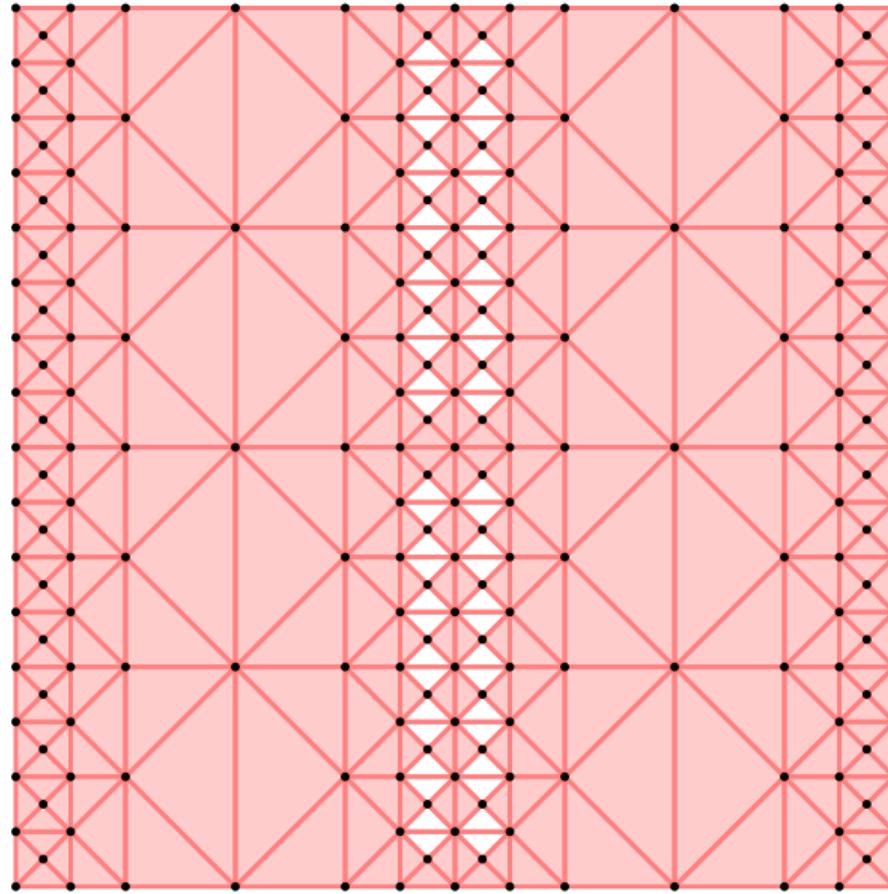


$$f(u, v) = (u, v, g(u)) \quad g(u) = -\frac{4}{9}x^6 + \frac{17}{9}x^4 - \frac{22}{9}x^2 + 1$$

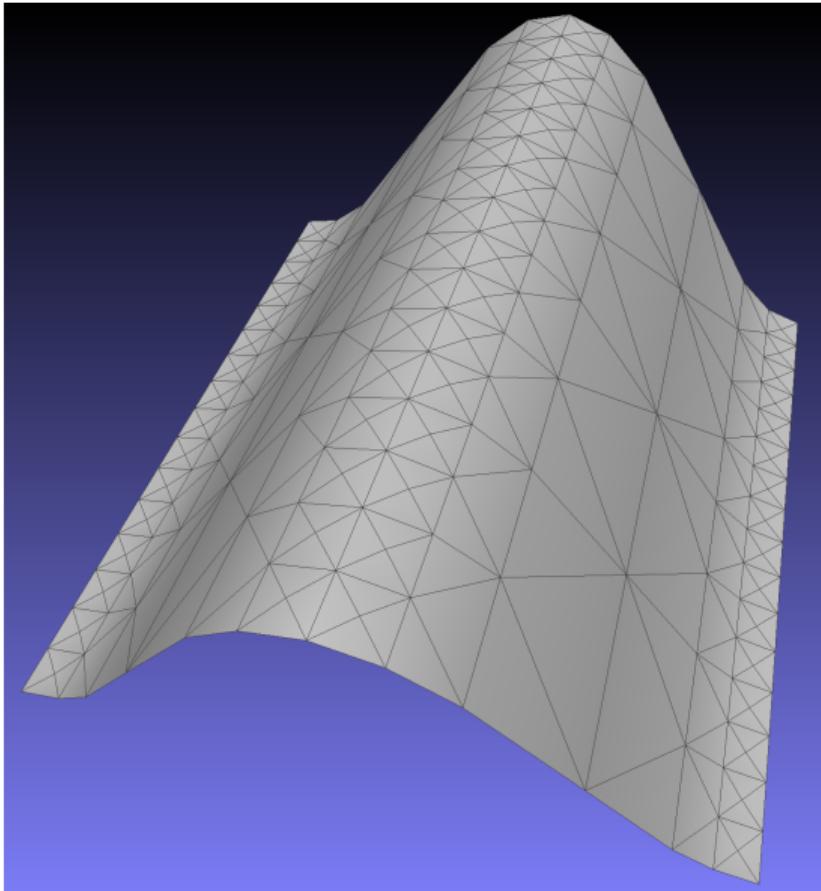
Malha 4-8 adaptativa – superfície paramétrica



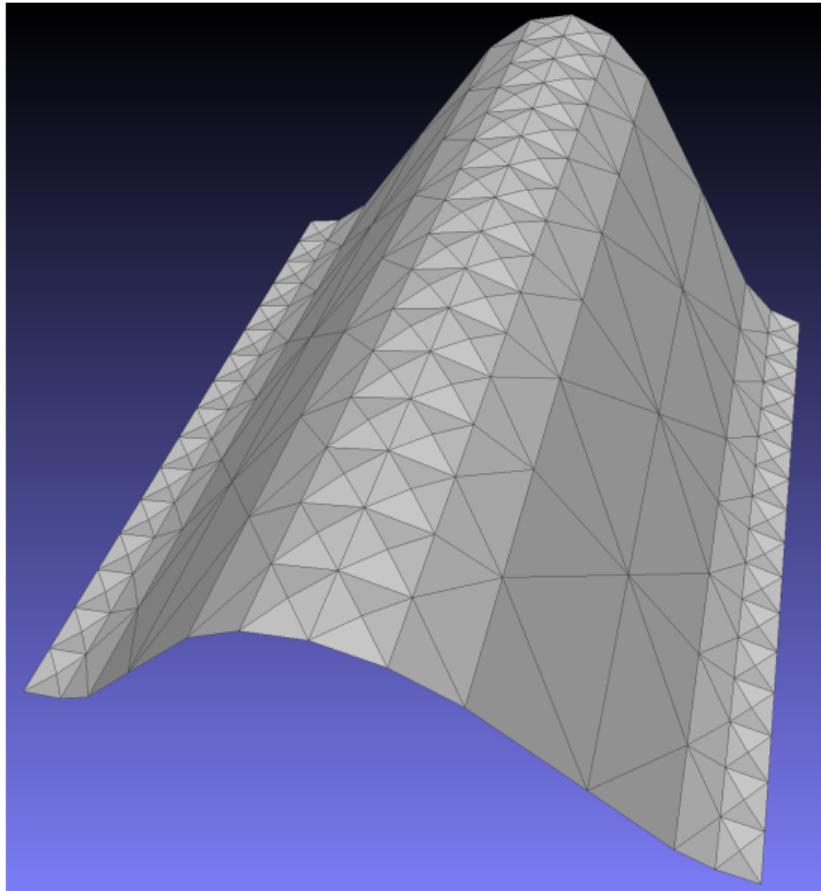
Malha 4-8 adaptativa – superfície paramétrica



Malha 4-8 adaptativa – superfície paramétrica

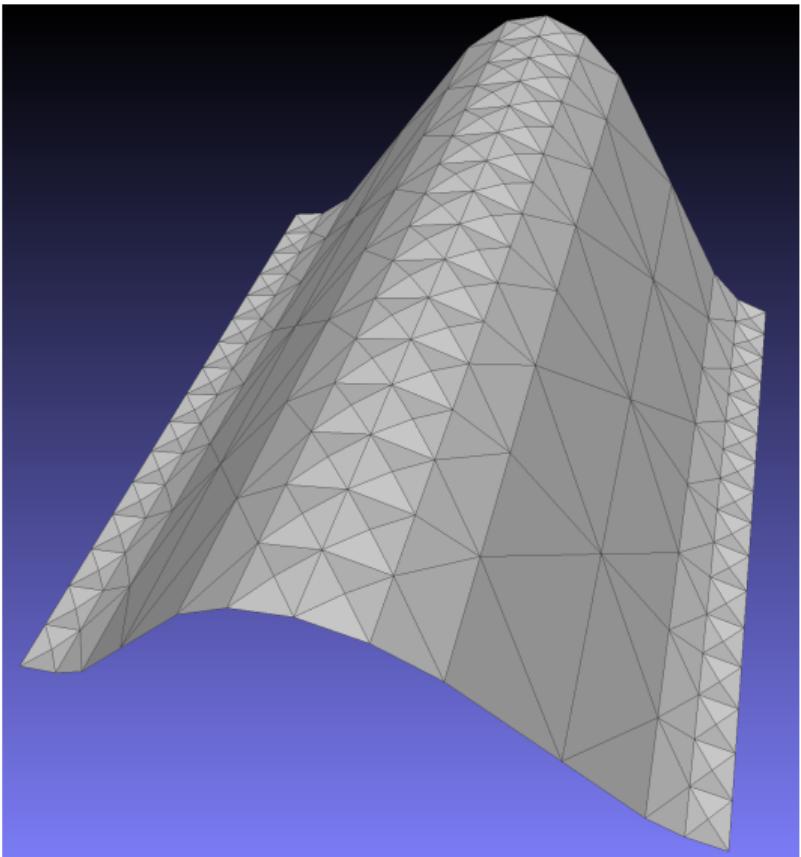


Malha 4-8 adaptativa – superfície paramétrica



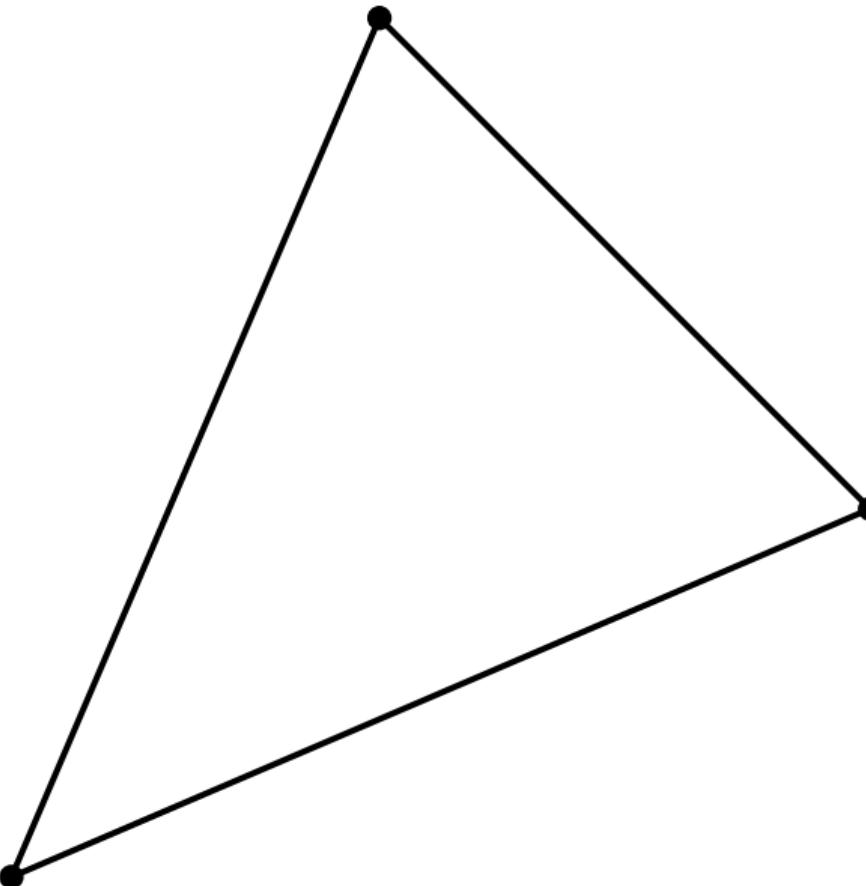
Superfície paramétrica – representação

OBJ

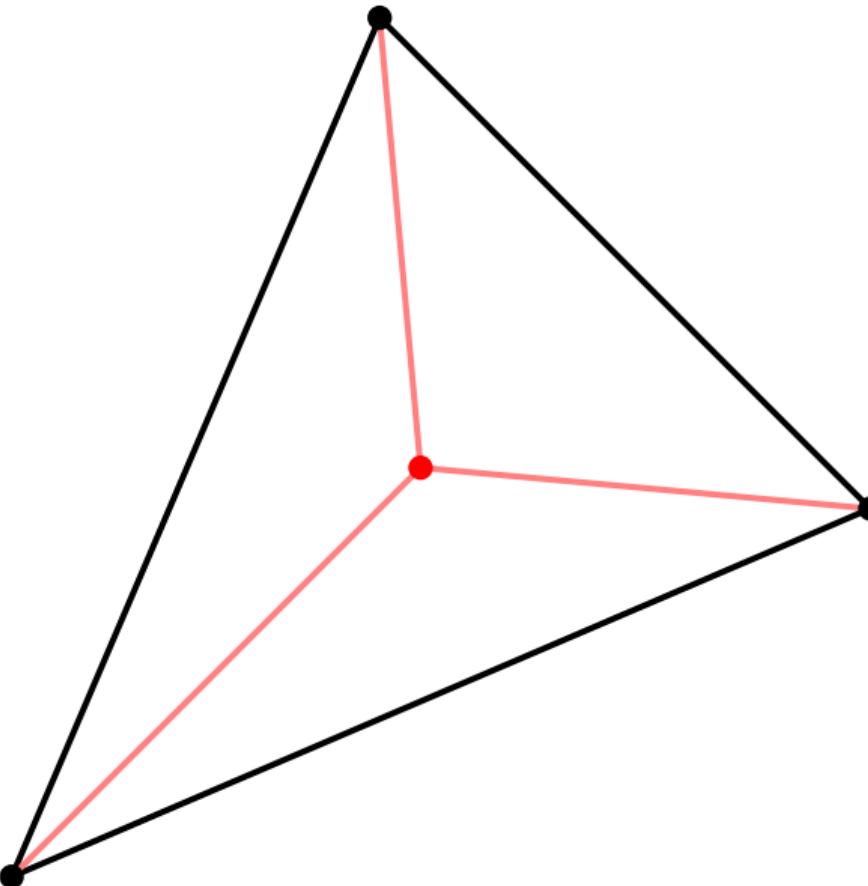


geometria	topologia
v -0.25 -0.75 0.8544921875	f 1 2 3
v -0.5 -0.5 0.5	f 4 5 2
v -0.5 -1.0 0.5	f 6 7 8
v -0.25 -0.25 0.8544921875	f 9 8 10
v -0.5 0.0 0.5	f 11 2 5
v 0.25 -0.75 0.8544921875	f 12 3 2
v 0.5 -1.0 0.5	f 13 8 7
v 0.5 -0.5 0.5	f 14 10 8
v 0.25 -0.25 0.8544921875	f 15 16 17
v 0.5 0.0 0.5	f 18 10 16
v -0.75 -0.25 0.1435546875	f 19 20 21
v -0.75 -0.75 0.1435546875	f 22 21 5
v 0.75 -0.75 0.1435546875	f 23 16 10
v 0.75 -0.25 0.1435546875	f 24 17 16
v 0.25 0.75 0.8544921875	f 25 21 20
...	...

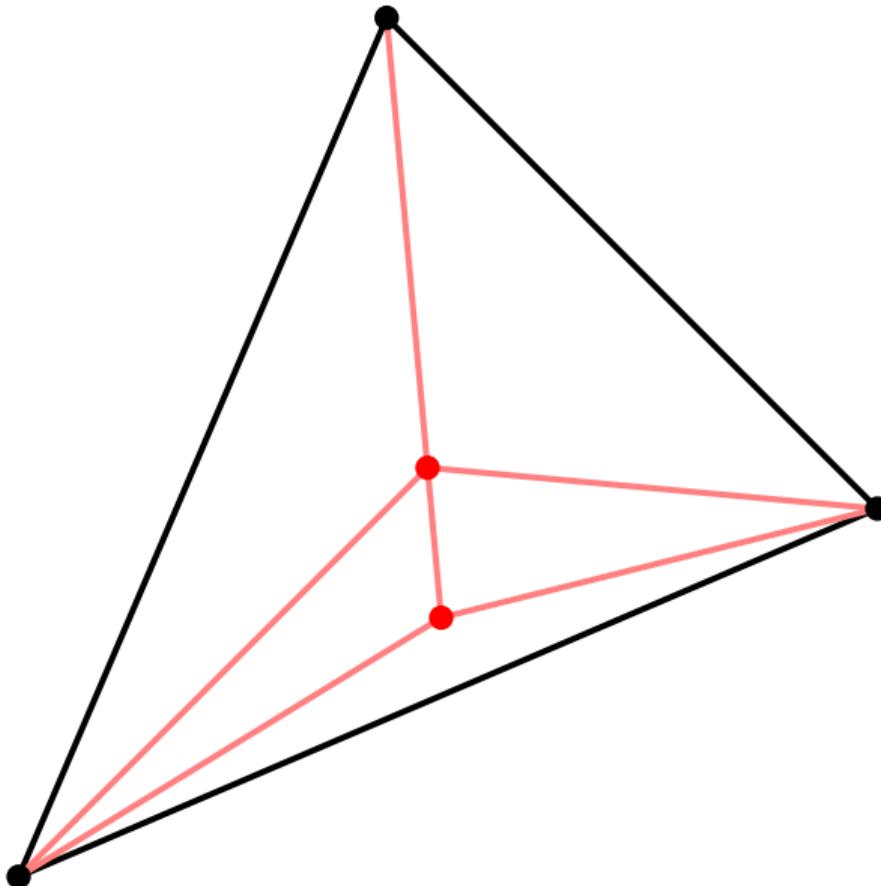
Subdivisão de triângulos



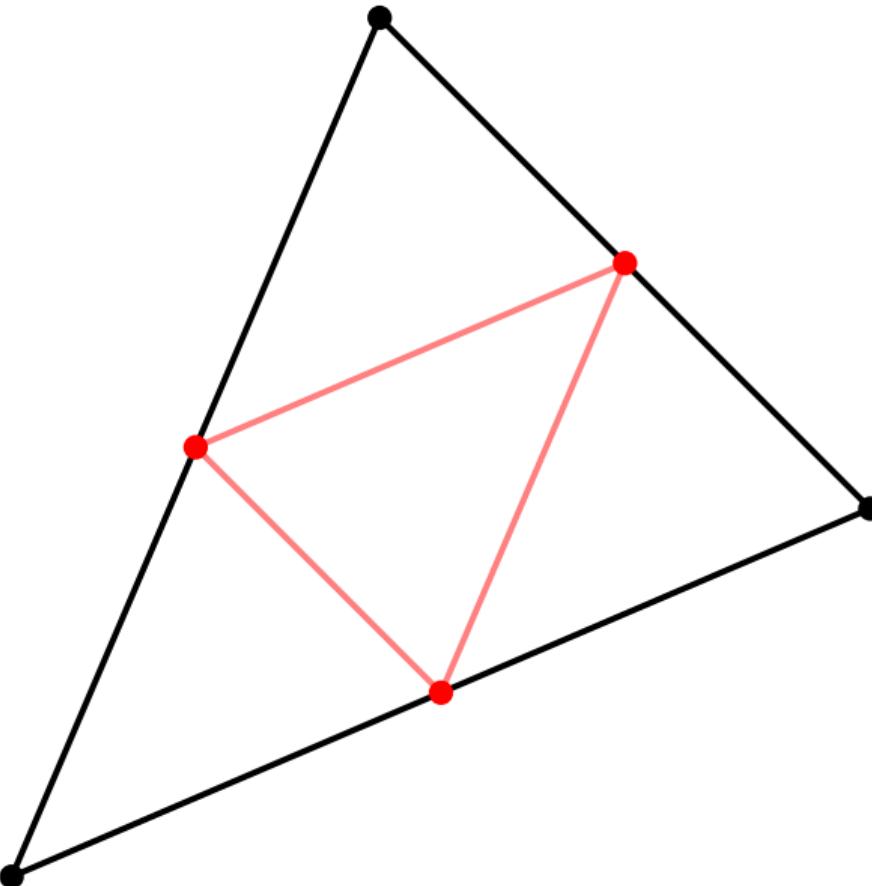
Subdivisão de triângulos



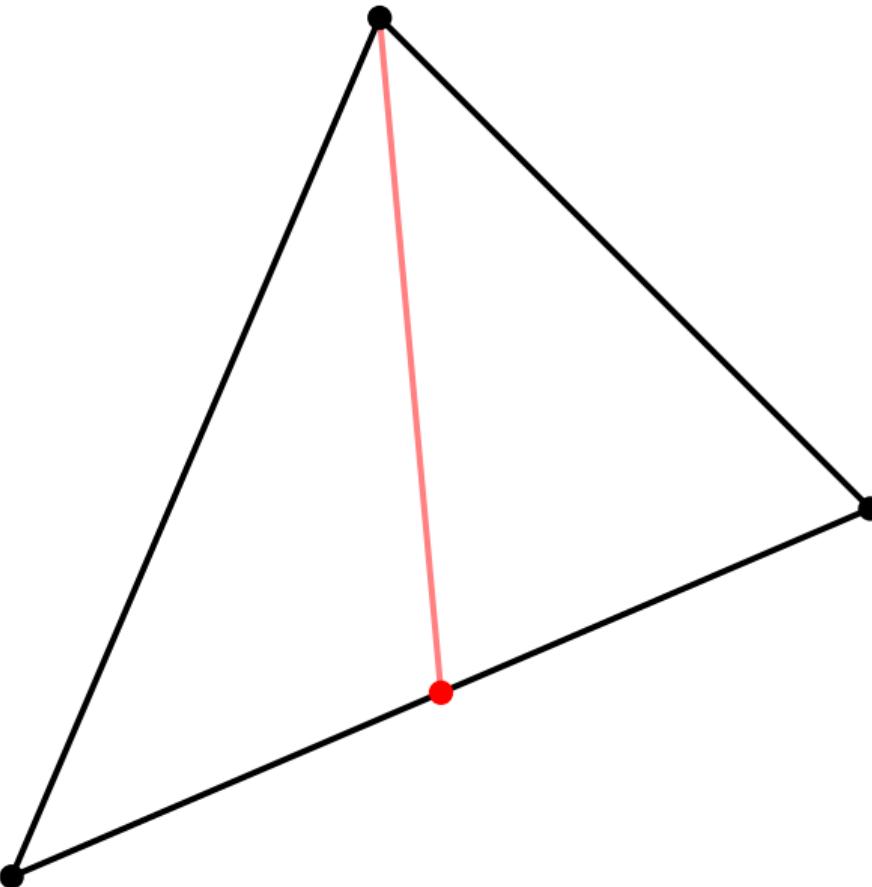
Subdivisão de triângulos



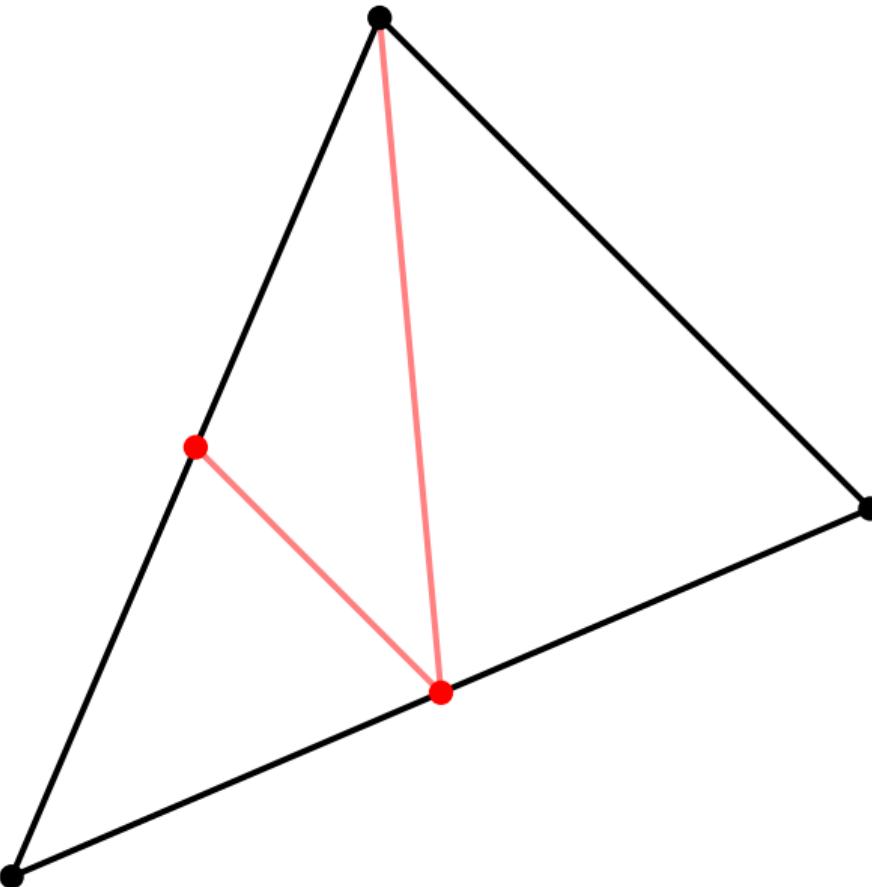
Subdivisão de triângulos



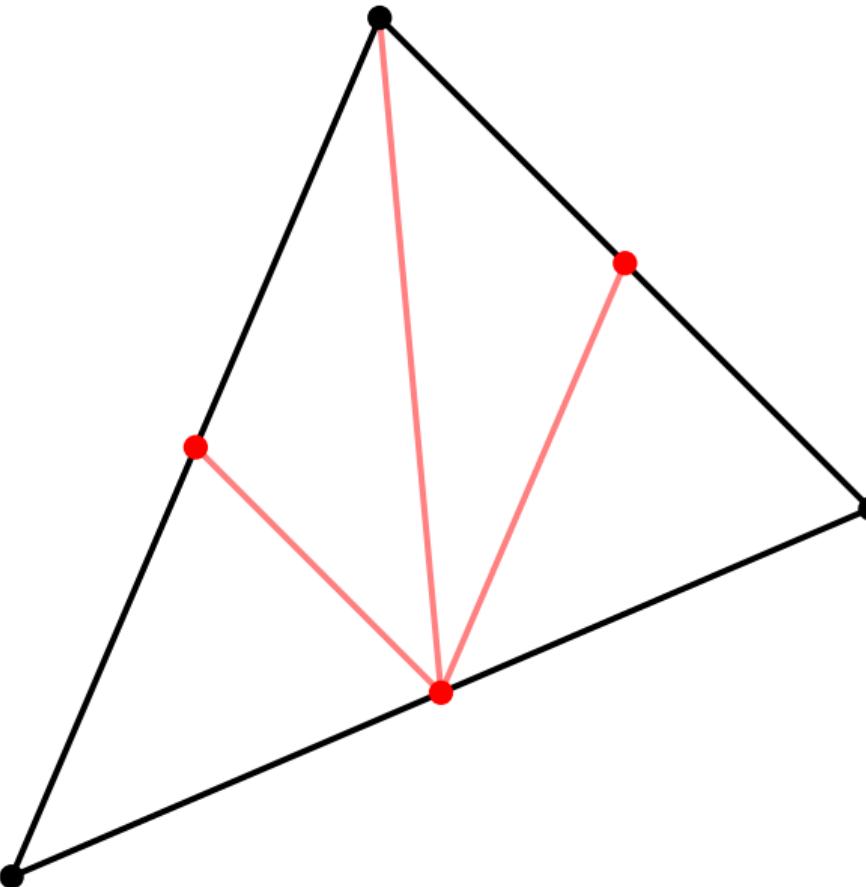
Subdivisão de triângulos



Subdivisão de triângulos



Subdivisão de triângulos



curvas e superfícies de forma livre

Requisitos para representação de formas

- pontos de controle
- intrínseca
- versatilidade
- controle de continuidade
- controle local
- redução de variação



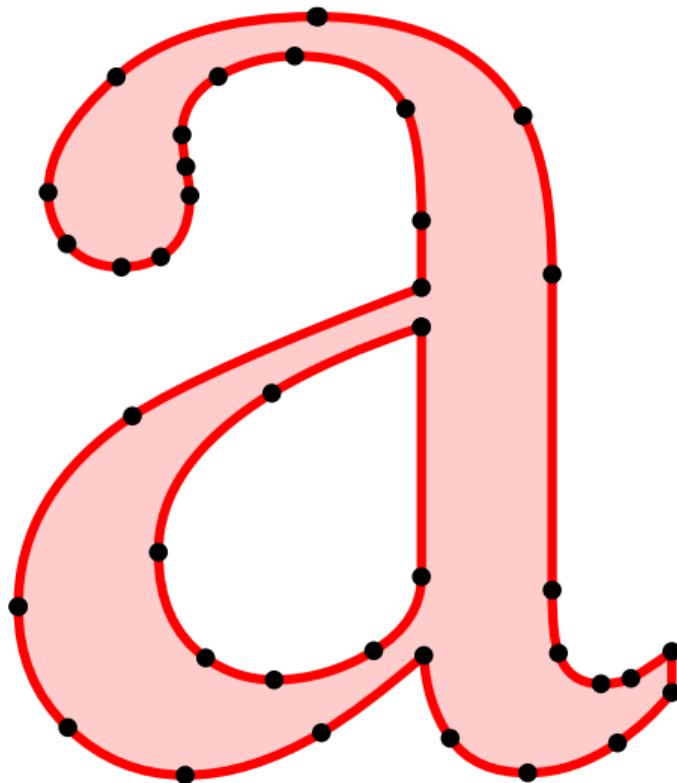
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- pontos de controle
- intrínseca
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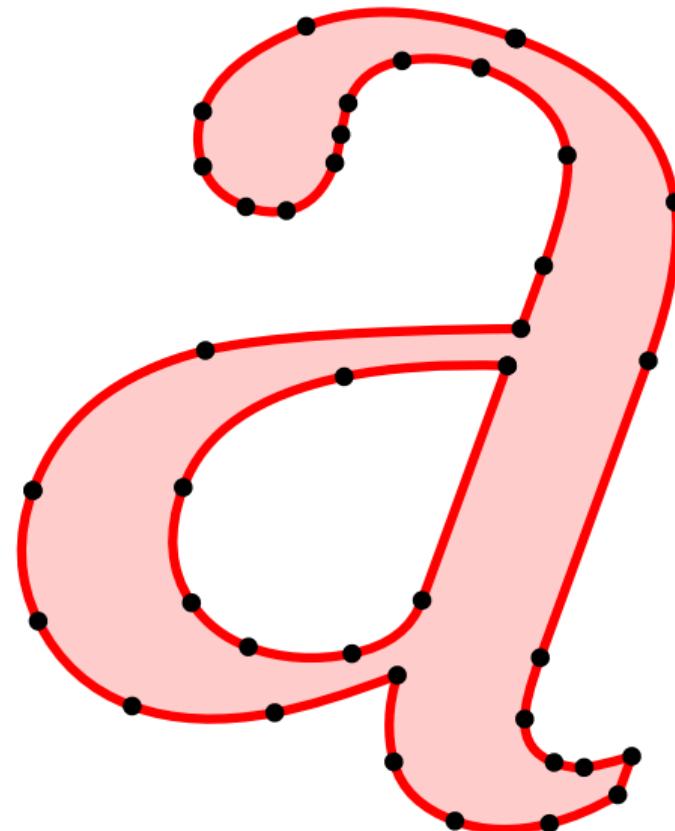
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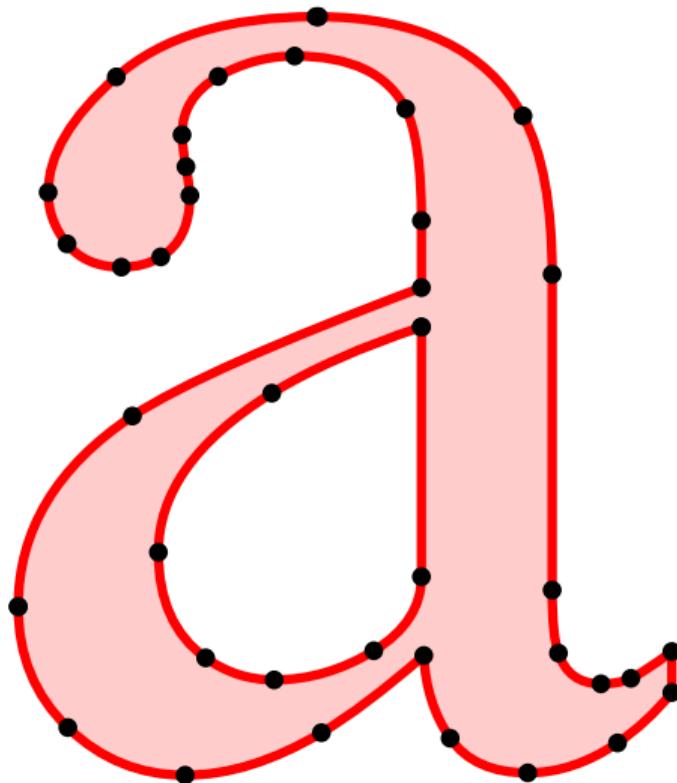
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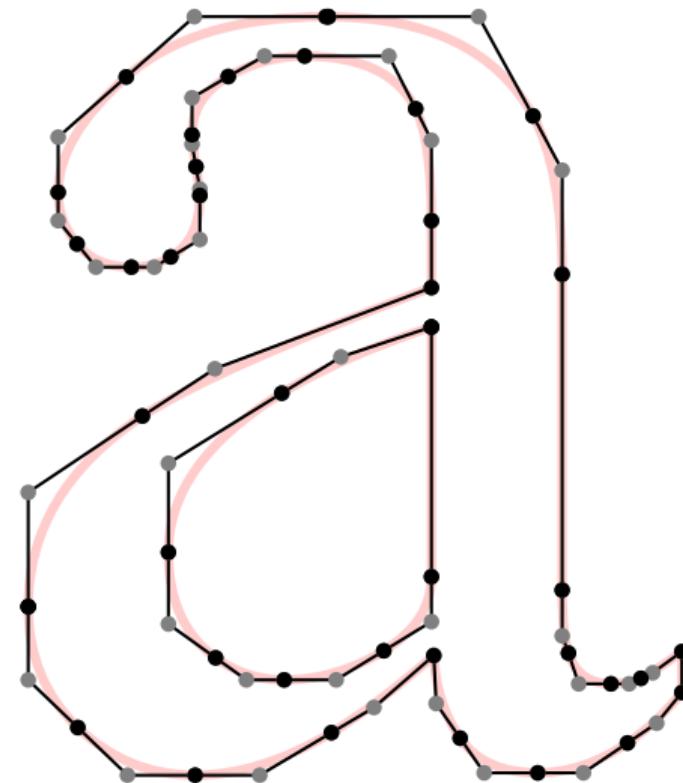
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Requisitos para representação de formas

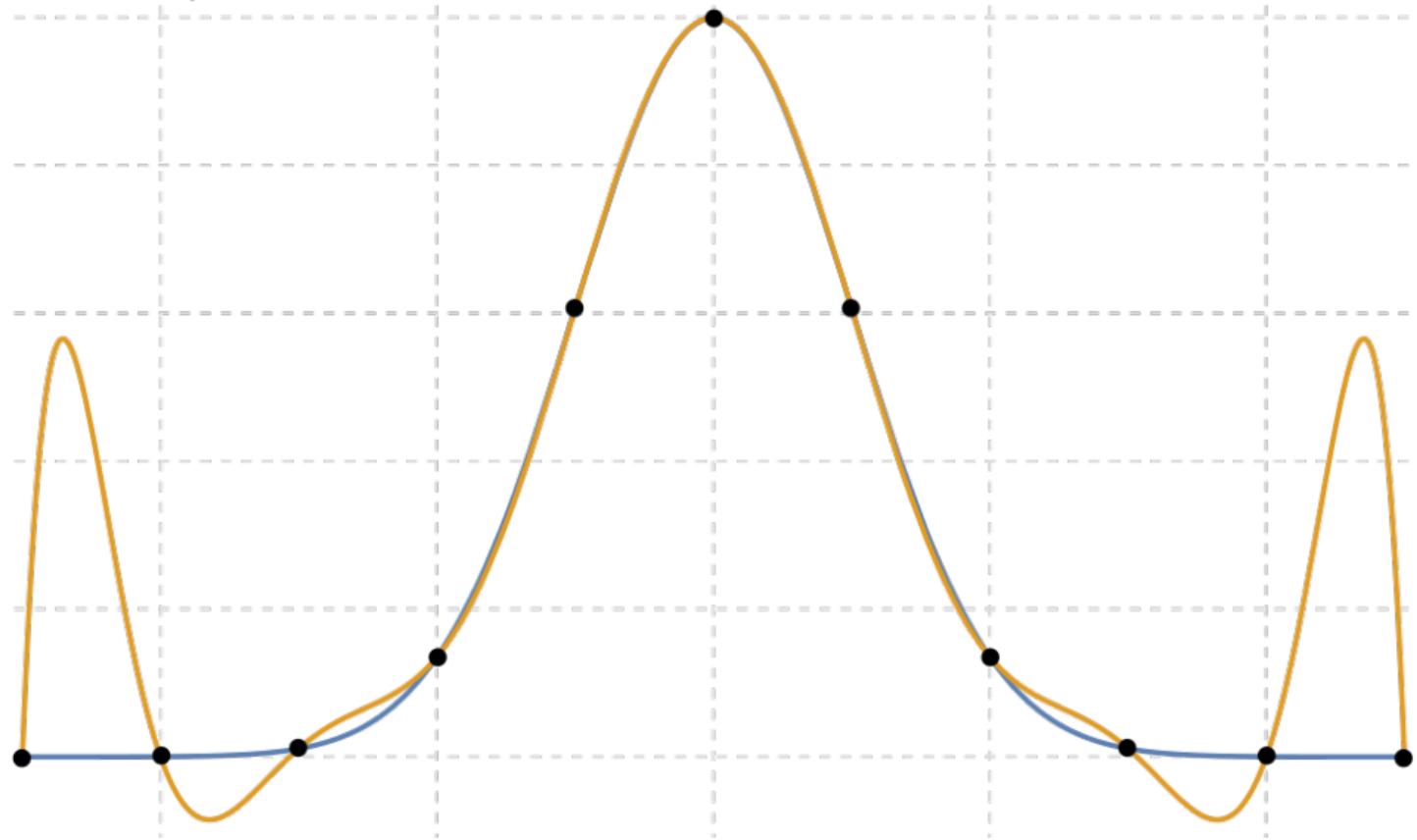
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- versatilidade
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- redução de variação



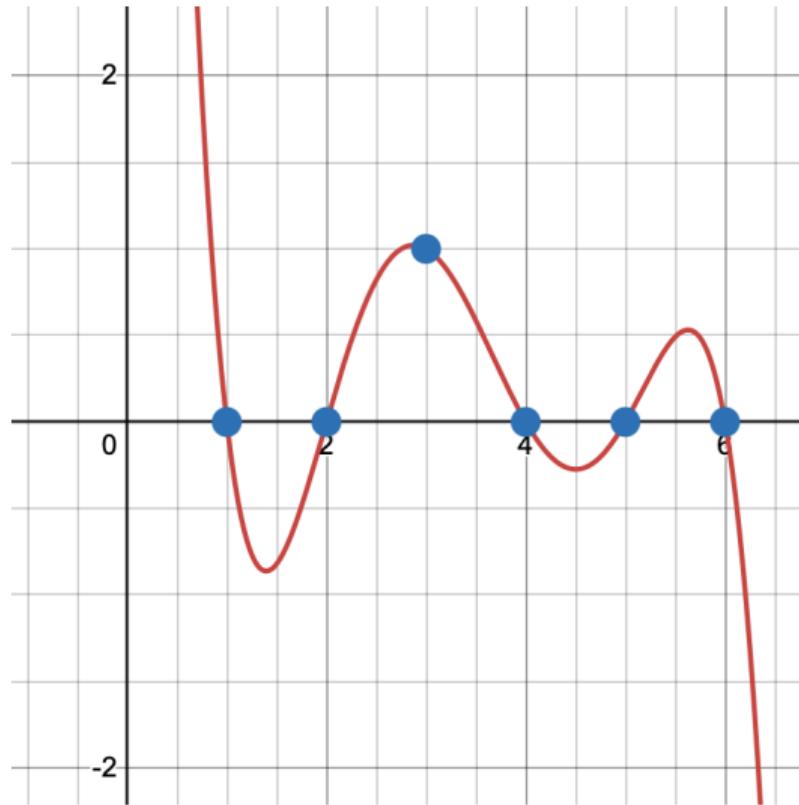
Requisitos para representação de formas

■ pontos de controle	M 146.75 138.5
	Q 118.0 129.0 99.25 117.5
	Q 63.25 95.25 63.25 67.0
	Q 63.25 44.25 78.25 33.5
	Q 88.0 26.5 100.0 26.5
	Q 116.5 26.5 131.625 35.75
	Q 146.75 45.0 146.75 59.25
■ intrínseca	L 146.75 138.5
■ versatilidade	M 18.75 49.75
■ controle de continuidade	Q 18.75 86.0 55.0 110.25
■ controle local	Q 78.0 125.25 146.75 151.0
■ redução de variação	L 146.75 172.25
	Q 146.75 197.75 141.75 207.75
	Q 133.25 224.5 106.5 224.5
	Q 93.75 224.5 82.25 218.0
	...

Aumento de variação



Aumento de variação



Objetos paramétricos

curvas paramétricas

$$\gamma: [a, b] \rightarrow \mathbf{R}^2$$

$$\gamma(t) = (x(t), y(t))$$

$$\gamma: [a, b] \rightarrow \mathbf{R}^3$$

$$\gamma(t) = (x(t), y(t), z(t))$$

superfícies paramétricas

$$\sigma: [a, b] \times [c, d] \rightarrow \mathbf{R}^3$$

$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

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$$\sigma(u, v) = (x(u, v), y(u, v), z(u, v))$$

funções polinomiais e racionais

avaliação computacional exata e eficiente

Funções polinomiais

avaliação ingênua

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$v \leftarrow 0$$

$$v \leftarrow v + a_0$$

$$v \leftarrow v + a_1 t$$

$$v \leftarrow v + a_2 t^2$$

$$v \leftarrow v + a_3 t^3$$

...

$$v \leftarrow v + a_n t^n$$

$$v = f(t)$$

$$t^k = \text{pow}(t, k)$$

Funções polinomiais

avaliação menos ingênuas

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$v \leftarrow 0 \qquad \qquad s \leftarrow 1$$

$$v \leftarrow v + a_0 s \qquad s \leftarrow s t$$

$$v \leftarrow v + a_1 s \qquad s \leftarrow s t$$

$$v \leftarrow v + a_2 s \qquad s \leftarrow s t$$

$$v \leftarrow v + a_3 s \qquad s \leftarrow s t$$

...

$$v \leftarrow v + a_n s \qquad s \leftarrow s t$$

$$v = f(t)$$

$$t^{k+1} = t^k \cdot t, \quad t^0 = 1$$

Funções polinomiais

avaliação eficiente

Horner

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$= a_0 + t(a_1 + t(a_2 + t(a_3 + \cdots + t(a_{n-1} + t a_n) \cdots)))$$

$$v \leftarrow a_n$$

$$v \leftarrow a_{n-1} + t v$$

...

$$v \leftarrow a_3 + t v$$

$$v \leftarrow a_2 + t v$$

$$v \leftarrow a_1 + t v$$

$$v \leftarrow a_0 + t v$$

$$v = f(t)$$

fused multiply-add (FMA)

Funções polinomiais

interpretação dos coeficientes

$$f(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

expansão de Taylor em torno de $t = 0$

$$a_k = \frac{f^{(k)}(0)}{k!}$$

informação pontual profunda determina forma global

não intuitivo

Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots + b_n t^n$$

$$\gamma(t) = P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots + P_n t^n \quad P_k = (a_k, b_k)$$

Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

$$y(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3 + \cdots + b_n t^n$$

$$\begin{aligned}\gamma(t) &= P_0 + P_1 t + P_2 t^2 + P_3 t^3 + \cdots + P_n t^n & P_k &= (a_k, b_k) \\ &= P_0 + tP_1 + t^2P_2 + t^3P_3 + \cdots + t^n P_n\end{aligned}$$

combinação linear de **pontos de controle** com pesos $1, t, t^2, t^3, \dots, t^n$

não intuitivo

Curvas polinomiais

interpretação dos coeficientes

$$x(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \cdots + a_n t^n$$

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combinação linear de **pontos de controle** com pesos $1, t, t^2, t^3, \dots, t^n$

não intuitivo

como escolher pontos de controle e pesos intuitivos?

coeficientes geométricos

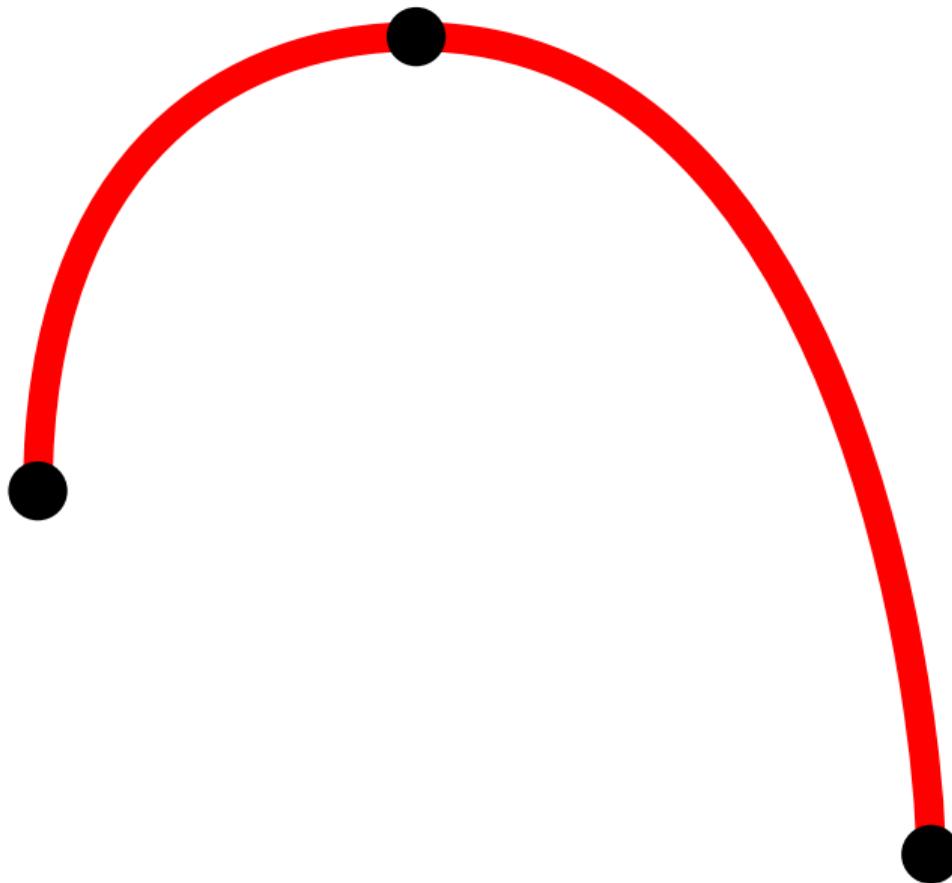
curvas de Hermite

Continuidade geométrica

- $\alpha, \beta: [0, 1] \rightarrow \mathbf{R}^d$
- continuidade C^0
 $\alpha(1) = \beta(0)$
- continuidade C^1
 $\alpha'(1) = \beta'(0)$ paramétrica
- continuidade G^1
 $\alpha'(1) \parallel \beta'(0)$ geométrica

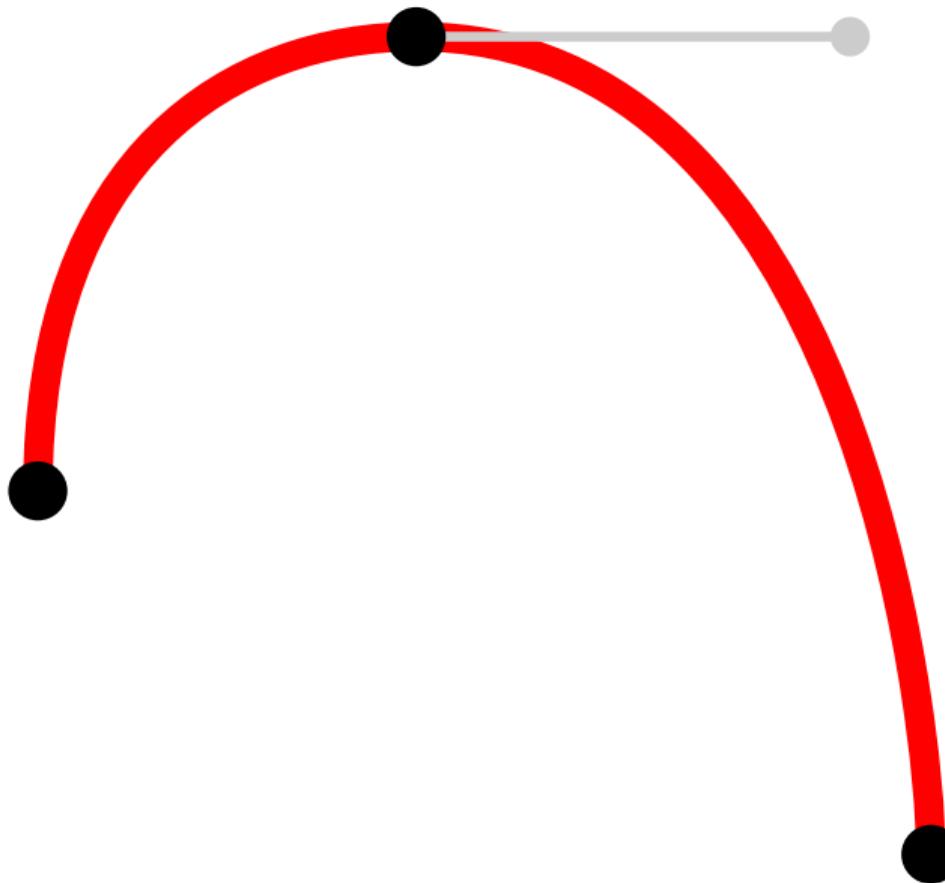
Continuidade C^0

$$\alpha(1) = \beta(0)$$



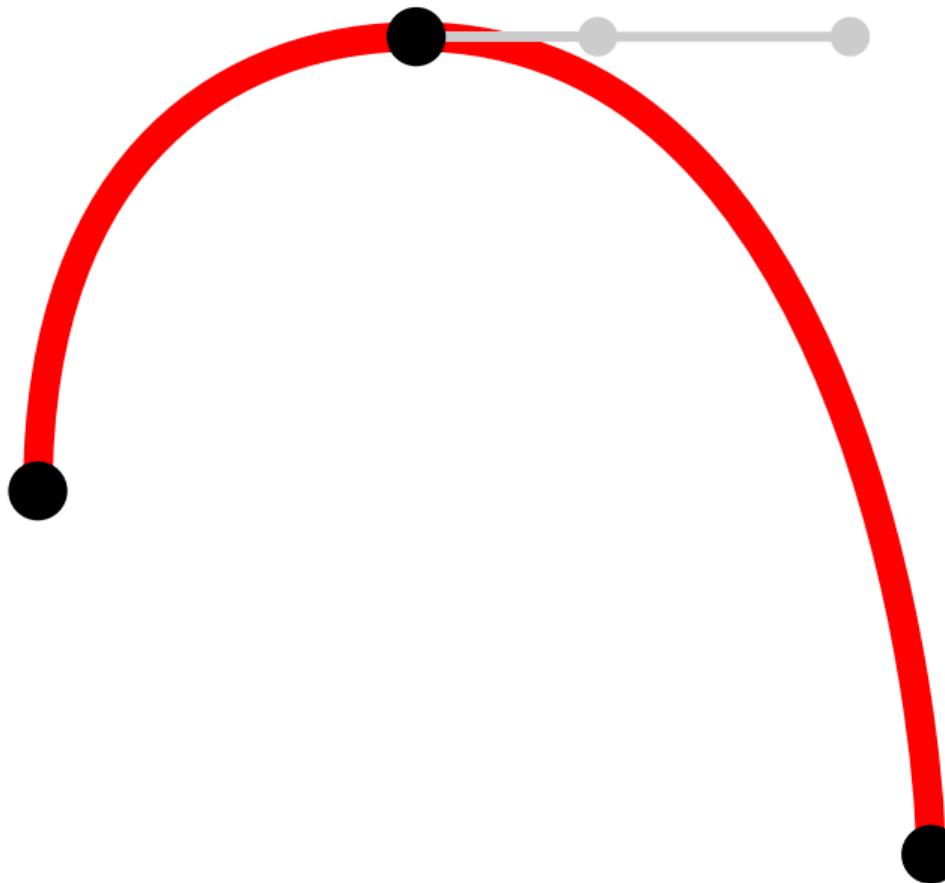
Continuidade paramétrica C^1

$$\alpha'(1) = \beta'(0)$$



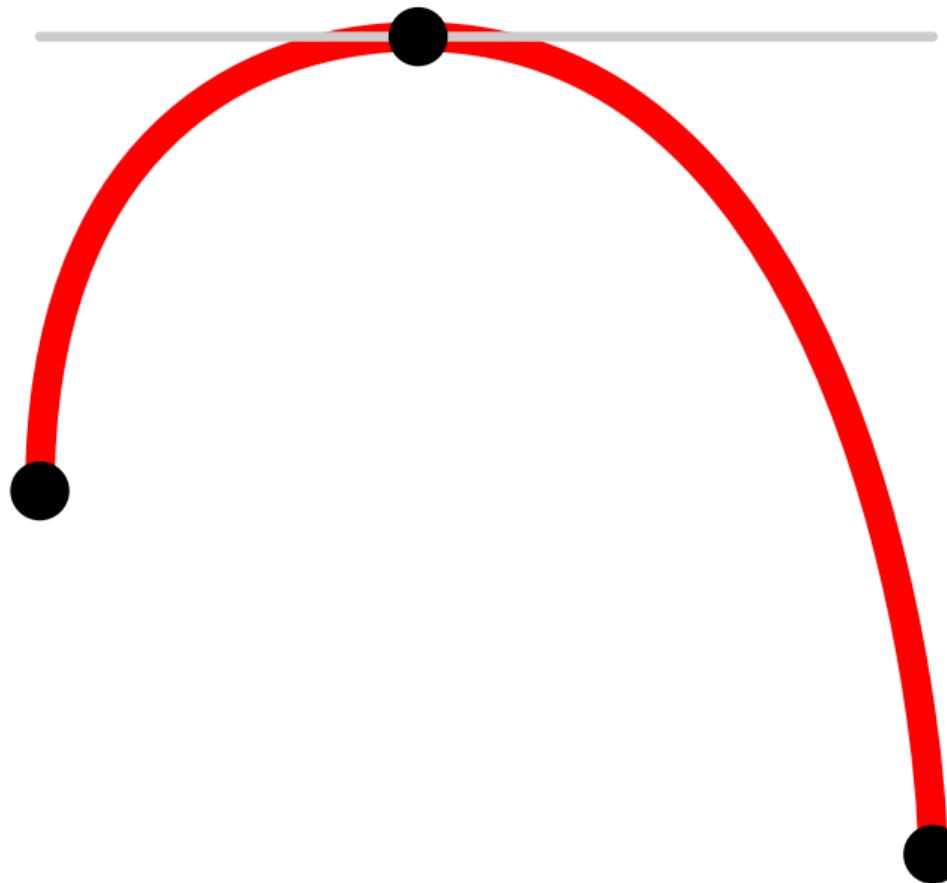
Continuidade geométrica G^1

$\alpha'(1) \parallel \beta'(0)$

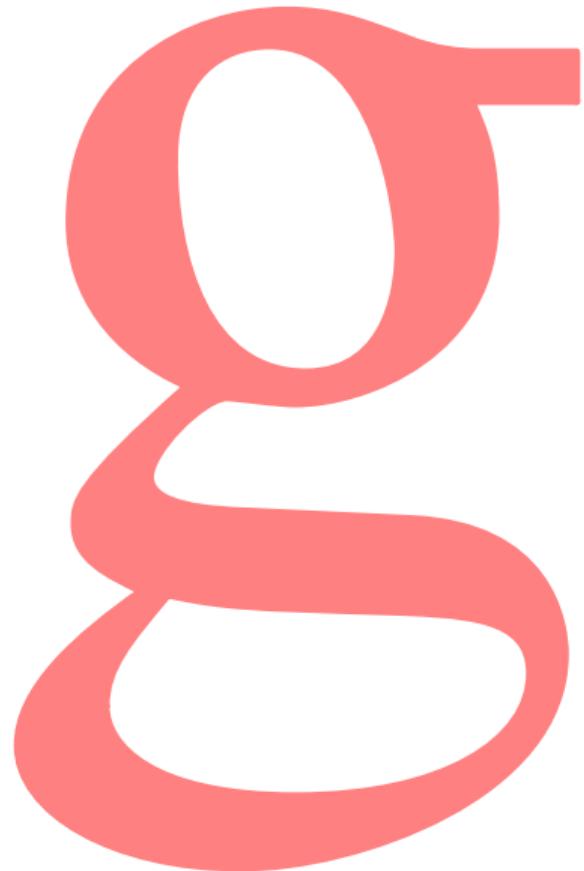


Continuidade geométrica G^1

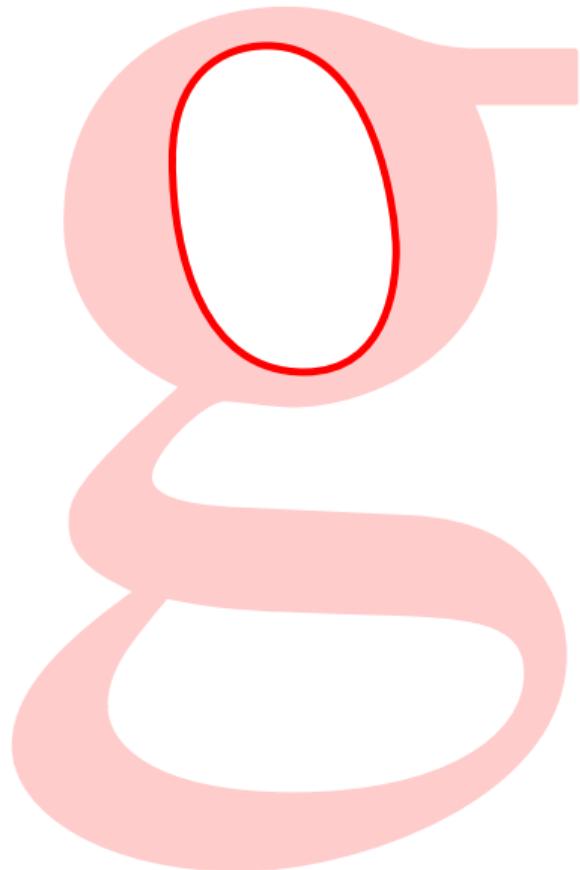
$\alpha'(1) \parallel \beta'(0)$



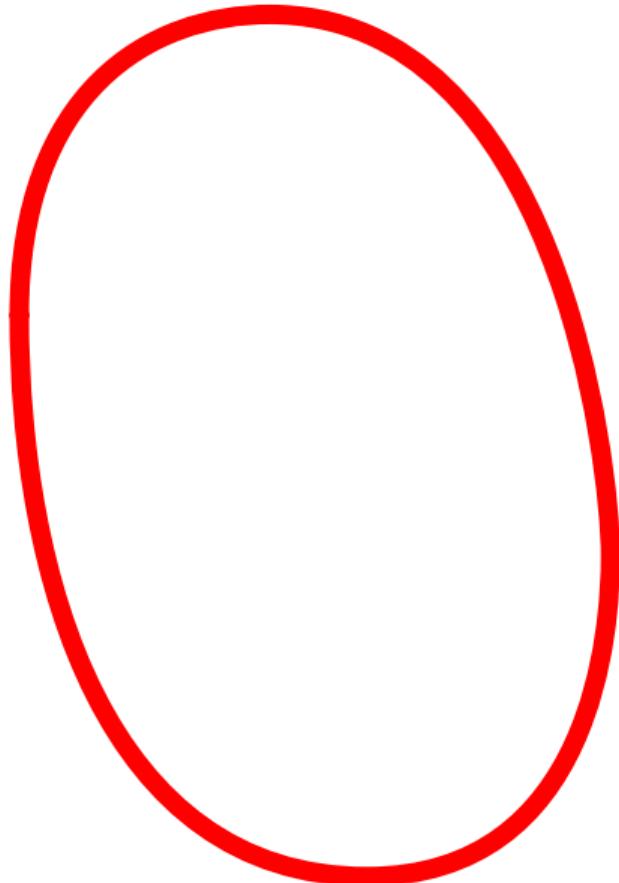
Continuidade geométrica



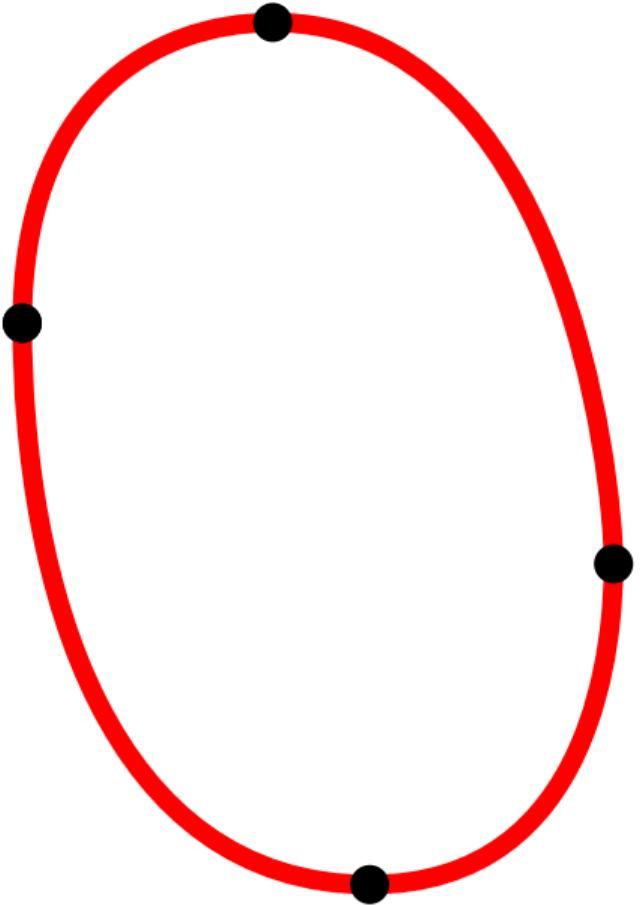
Continuidade geométrica



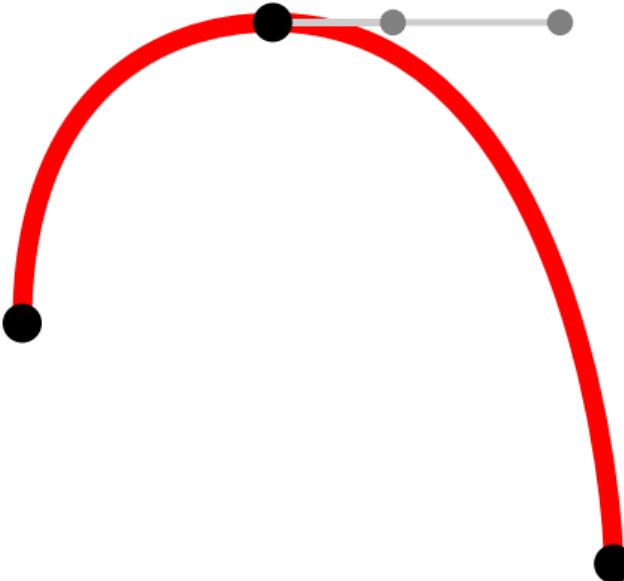
Continuidade geométrica



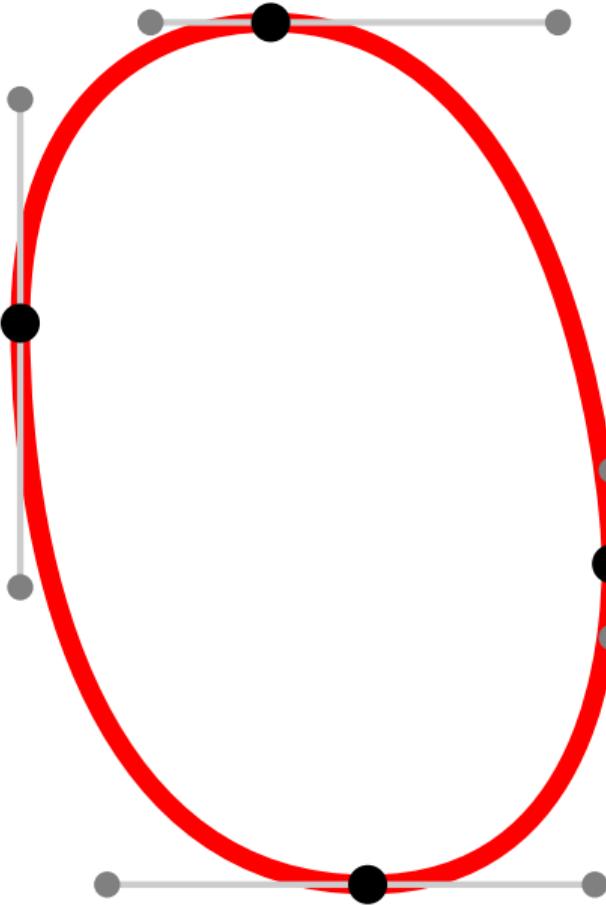
Continuidade geométrica



Continuidade geométrica

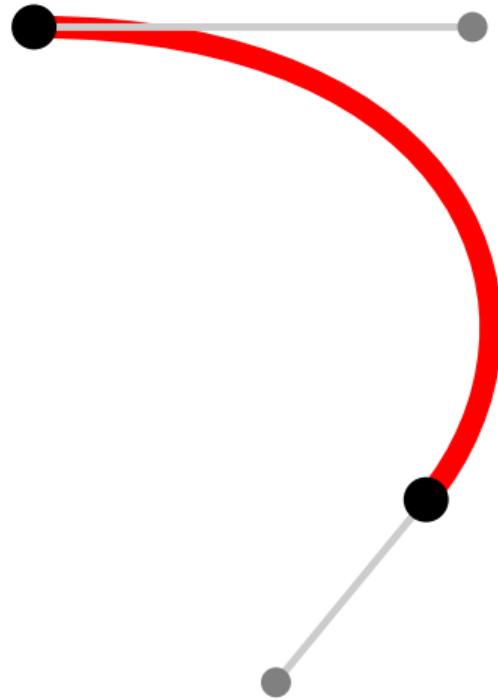


Continuidade geométrica



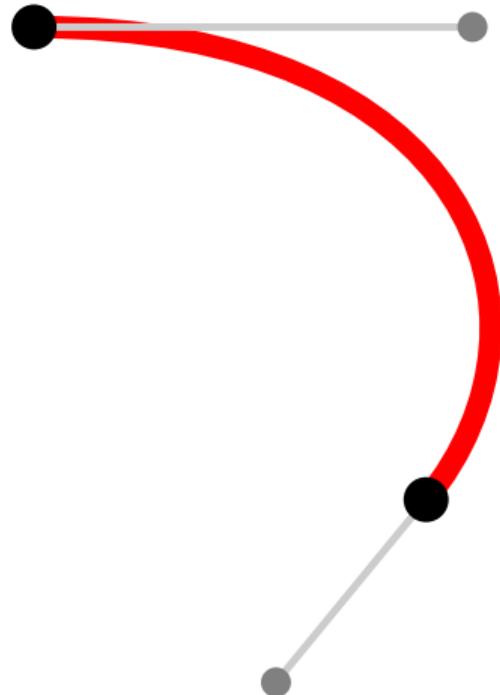
Cúbica de Hermite

- pontos extremos p_0, p_1
- vetores tangentes v_0, v_1



Cúbica de Hermite

- pontos extremos p_0, p_1
- vetores tangentes v_0, v_1
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
- $\gamma(1) = p_1$
- $\gamma'(0) = v_0$
- $\gamma'(1) = v_1$



Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad a_k \in \mathbf{R}^d$$

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$$\gamma \mapsto (\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)) \quad \text{transformação linear}$$

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$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix}$$

Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

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$$\gamma \mapsto (\gamma(0), \gamma(1), \gamma'(0), \gamma'(1)) \quad \text{transformação linear}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{pmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix} \implies \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 3 & -2 & -1 \\ 2 & -2 & 1 & 1 \end{pmatrix} \begin{bmatrix} p_0 \\ p_1 \\ v_0 \\ v_1 \end{bmatrix}$$

Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

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monomial

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monomial

$$= (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

Hermite

Cúbica de Hermite

$$\gamma: [0, 1] \rightarrow \mathbf{R}^d \quad \gamma(0) = p_0, \quad \gamma(1) = p_1, \quad \gamma'(0) = v_0, \quad \gamma'(1) = v_1$$

$$\gamma(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

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monomial

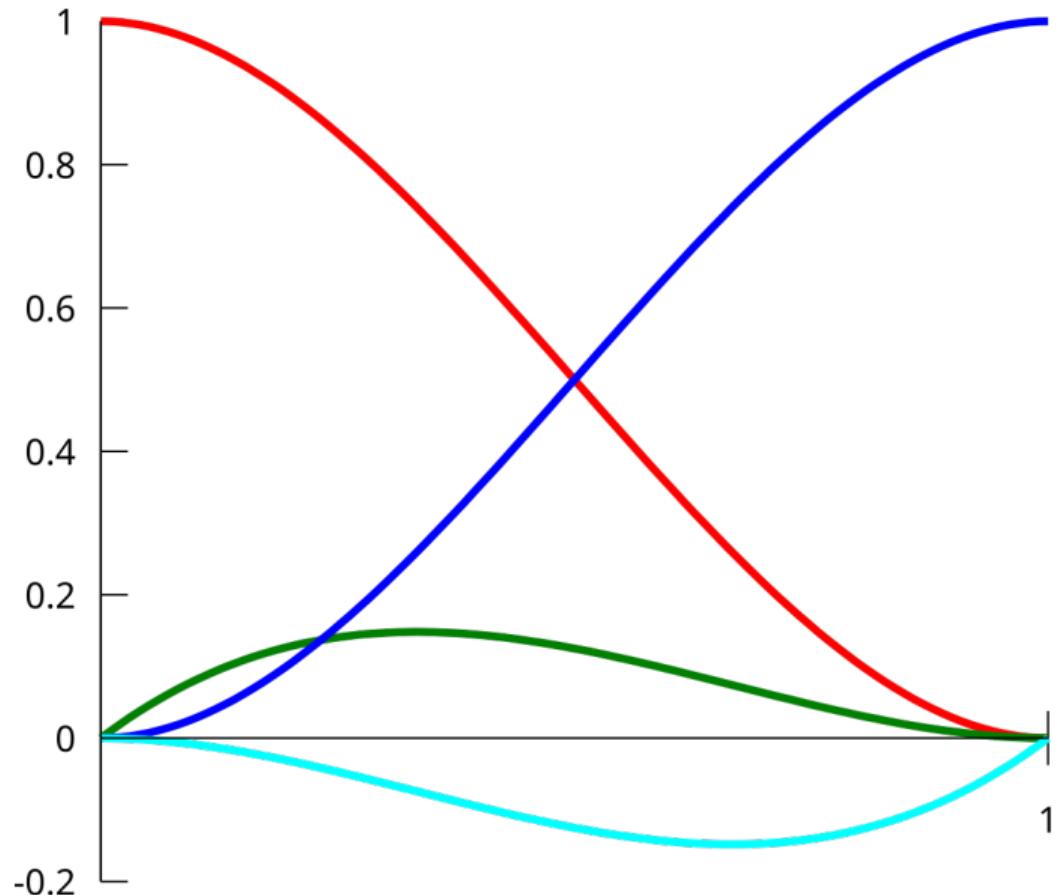
$$= (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

Hermite

$$= B^3[p_0, p_0 + \frac{1}{3}v_0, p_1 - \frac{1}{3}v_1, p_1]$$

Bézier

Base de Hermite



Base de Hermite

$$\gamma(t) = (2t^3 - 3t^2 + 1)p_0 + (-2t^3 + 3t^2)p_1 + (t^3 - 2t^2 + t)v_0 + (t^3 - t^2)v_1$$

$$= H_1(t)p_0 + H_2(t)p_1 + H_3(t)v_0 + H_4(t)v_1$$

pontos de controle mistos

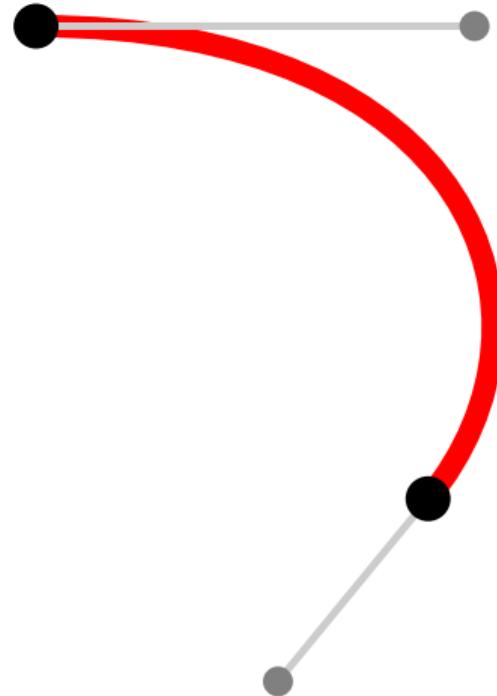
f	H_1	H_2	H_3	H_4
$f(0)$	1	0	0	0
$f(1)$	0	1	0	0
$f'(0)$	0	0	1	0
$f'(1)$	0	0	0	1

$$H_1(t) = H_2(1-t), \quad H_3(t) = -H_4(1-t)$$

base dual

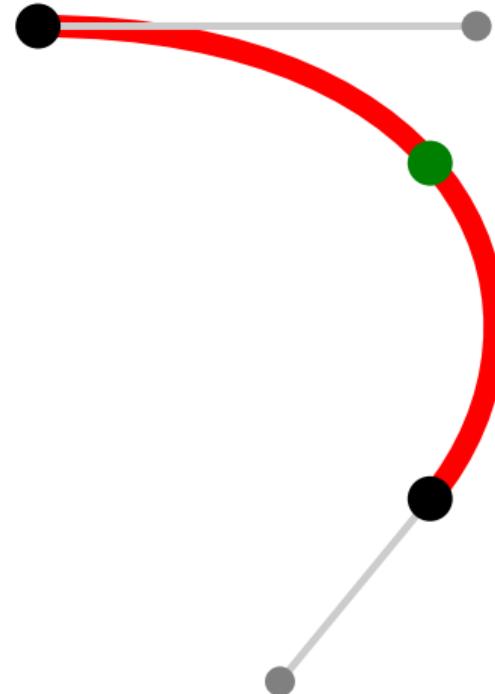
Cúbica de Hermite geométrica

- pontos extremos p_0, p_1
- direções tangentes v_0, v_1
- $\gamma: [0, 1] \rightarrow \mathbf{R}^d$
- $\gamma(0) = p_0$
- $\gamma(1) = p_1$
- $\gamma'(0) = \lambda_0 v_0 \quad \lambda_0, \lambda_1 \in \mathbf{R}$ a determinar
- $\gamma'(1) = \lambda_1 v_1$



Cúbica de Hermite geométrica

- pontos extremos p_0, p_1
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- $\gamma'(0) = \lambda_0 v_0$ $\lambda_0, \lambda_1 \in \mathbf{R}$ a determinar
- $\gamma'(1) = \lambda_1 v_1$
- $\gamma(t^*) = p^*$ $d = 2$: t^* dado / $d = 3$: equação cúbica em t^*



curvas de Bézier

Formulação geométrica

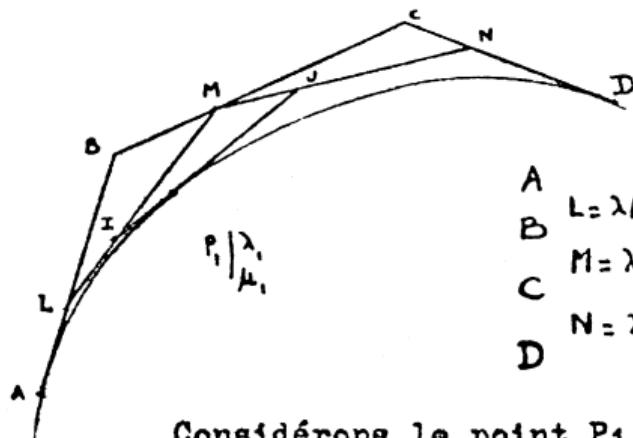
- de Casteljau (Citroën, 1959) interpolação linear repetida
 - Bézier (Renault, 1967) ponto + deslocamentos
 - Forrest (1972) polinômios de Bernstein
 - Boehm (1975) de Casteljau

Formulação geométrica

de Casteljau (Citroën, 1963)

1.5.- Sous-Pôles d'une courbe

1.5.1.- Définition des sous-pôles.



Considérons une cubique des pôles A, B, C, D. Nous avons vu précédemment que la construction par la méthode des barycentres donnent les différents points L, M, N, I, J, P.

$$A \quad L = \lambda A + \mu B$$

$$B \quad M = \lambda B + \mu C \quad I = \lambda^2 A + 2\lambda\mu B + \mu^2 C$$

$$C \quad J = \lambda^2 B + 2\lambda\mu C + \mu^2 D$$

$$D \quad N = \lambda C + \mu D \quad P = \lambda^3 A + 3\lambda^2 \mu B + 3\lambda\mu^2 C + \mu^3 D$$

Considérons le point P_1 de paramètres λ_1 et μ_1 (avec $\lambda_1 + \mu_1 = 1$) (λ_1 varie de 0 à 1 et μ_1 de 0 à 1, lorsque P va en D).

Cherchons les pôles de la cubique P_1 , D. Cette courbe dérive de la cubique initiale AD par changement des paramètres.

$$P = \lambda^3 A + 3\lambda^2 \mu B + 3\lambda\mu^2 C + \mu^3 D, \quad \lambda + \mu = 1, \quad \lambda = 1 - \mu, \quad \mu \in [0, 1]$$

Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

segmento se $t \in [0, 1]$

Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

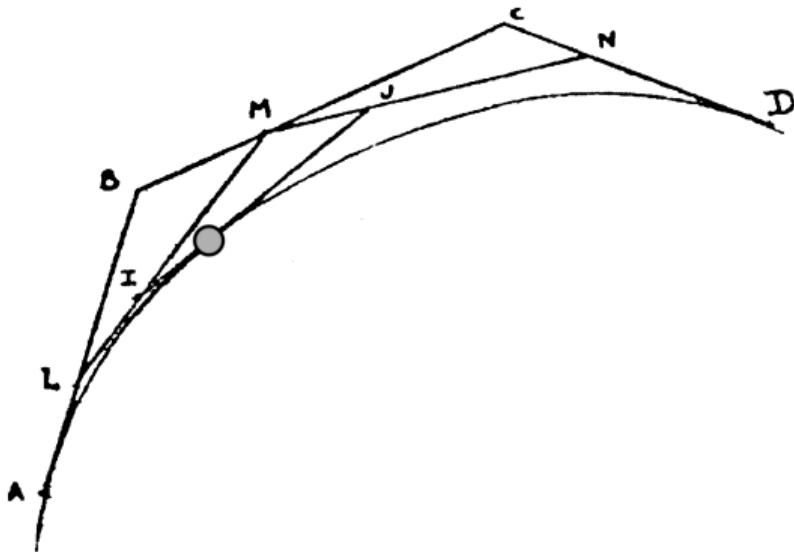
interpolação linear repetida

A

B

C

D



Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

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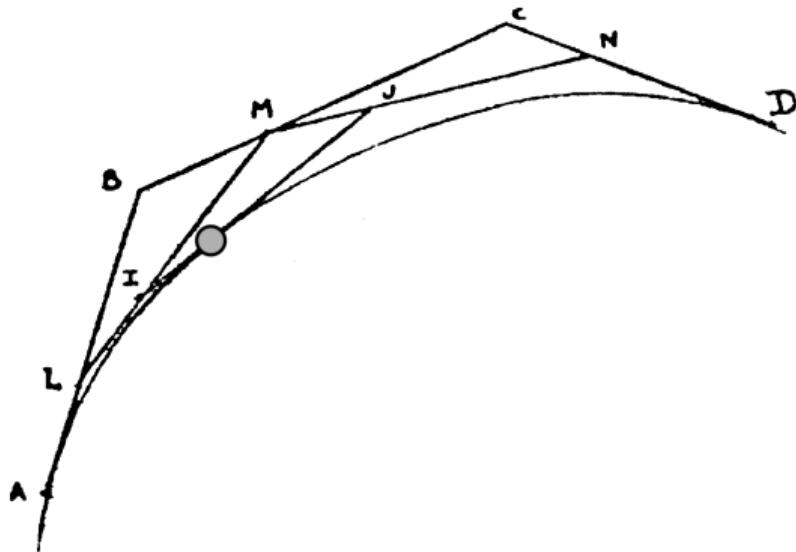
interpolação linear repetida

A $L = \text{lerp}(t, A, B)$

B $M = \text{lerp}(t, B, C)$

C $N = \text{lerp}(t, C, D)$

D



Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

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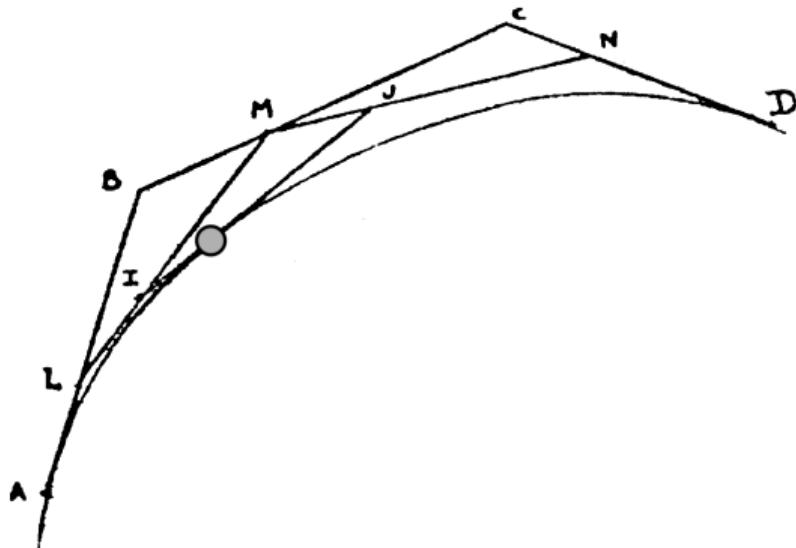
interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

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D



Formulação geométrica

de Casteljau (Citroën, 1963)

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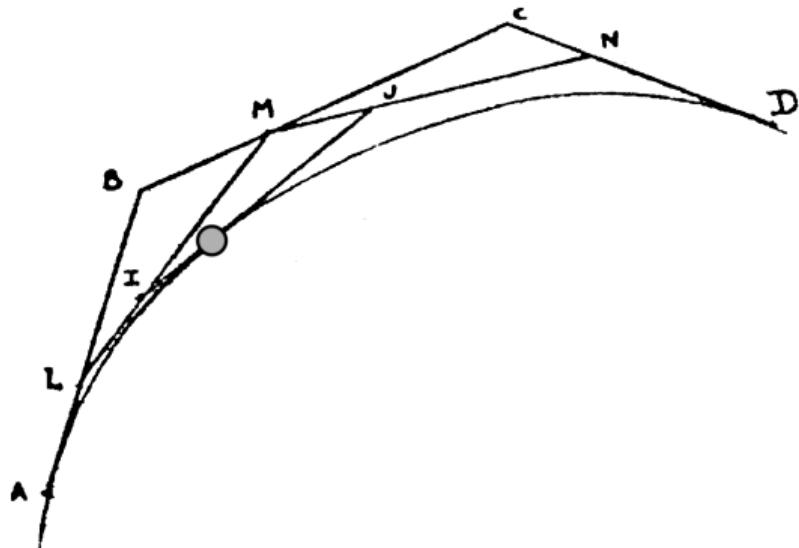
interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

$$B \quad M = \text{lerp}(t, B, C)$$

$$C \quad N = \text{lerp}(t, C, D)$$

D



$$I = \text{lerp}(t, L, M)$$

$$J = \text{lerp}(t, M, N)$$

$$P = \text{lerp}(t, I, J)$$

Formulação geométrica

de Casteljau (Citroën, 1963)

interpolação linear

$$\text{lerp}(t, v_0, v_1) = (1 - t)v_0 + tv_1$$

$$\text{lerp}(t, P_0, P_1) = (1 - t)P_0 + tP_1$$

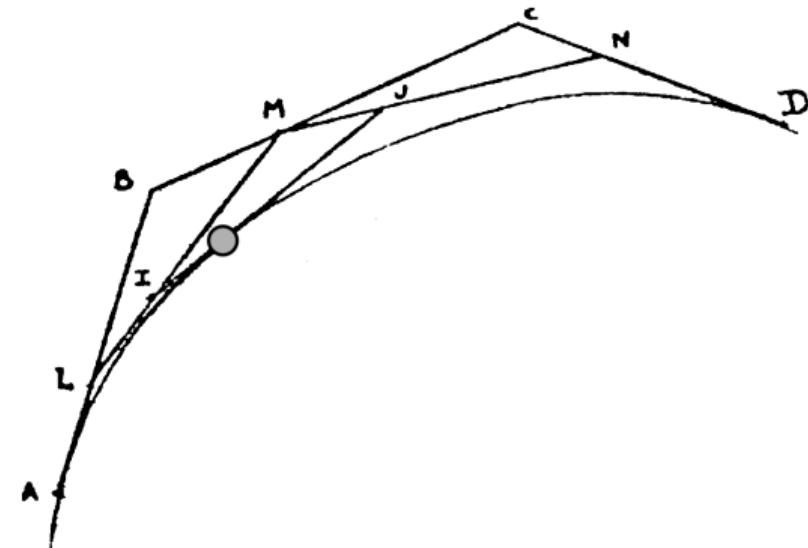
interpolação linear repetida

$$A \quad L = \text{lerp}(t, A, B)$$

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D



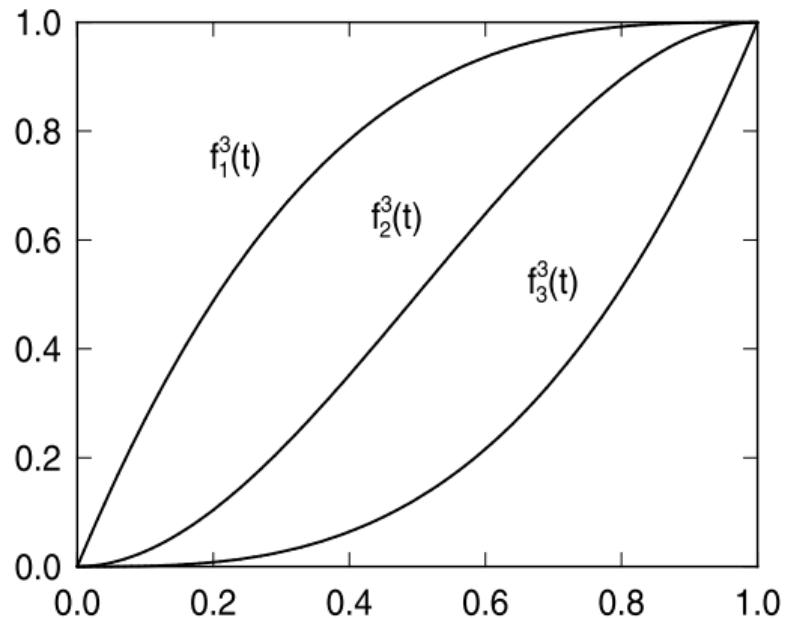
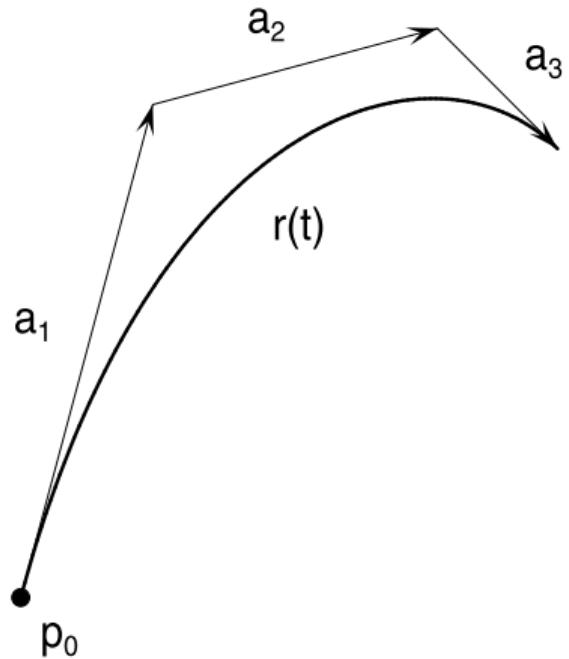
$$P = (1 - t)^3 A + 3(1 - t)^2 t B + 3(1 - t)t^2 C + t^3 D$$

Formulação geométrica

Bézier (Renault, 1967)

ponto + deslocamentos:

$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$



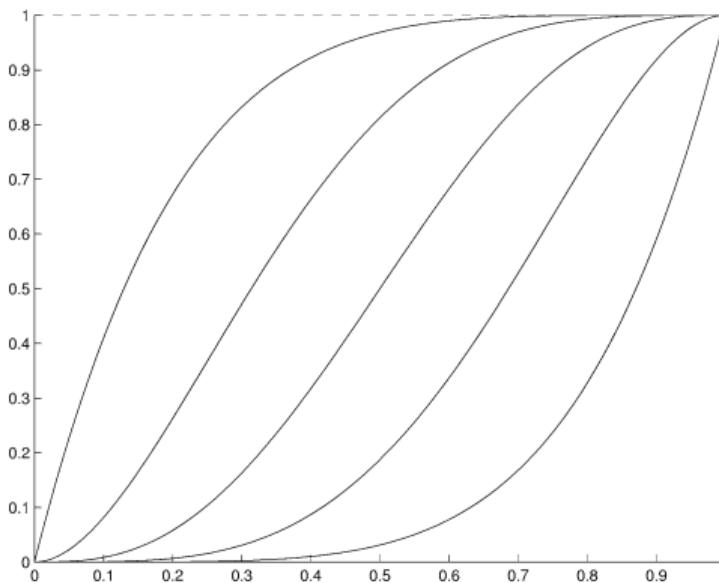
Formulação geométrica

Bézier (Renault, 1967)

ponto + deslocamentos: $B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$

- f_k^n polinômio de grau n
- $f_k^n(0) = 0$
- $f_k^n(1) = 1$
- $(f_k^n)^{(r)}(0) = 0, \quad r = 1, \dots, k - 1$
- $(f_k^n)^{(r)}(1) = 0, \quad r = 1, \dots, n - k$

interpolação de Hermite–Birkhoff



Laurent–Sablonnière

Onésime Duran

Formulação geométrica

Forrest (1972)

$$B(t) = p_0 + \sum_{k=1}^n f_k^n(t) a_k$$

$$p_k = p_0 + \sum_{j=1}^k a_j$$

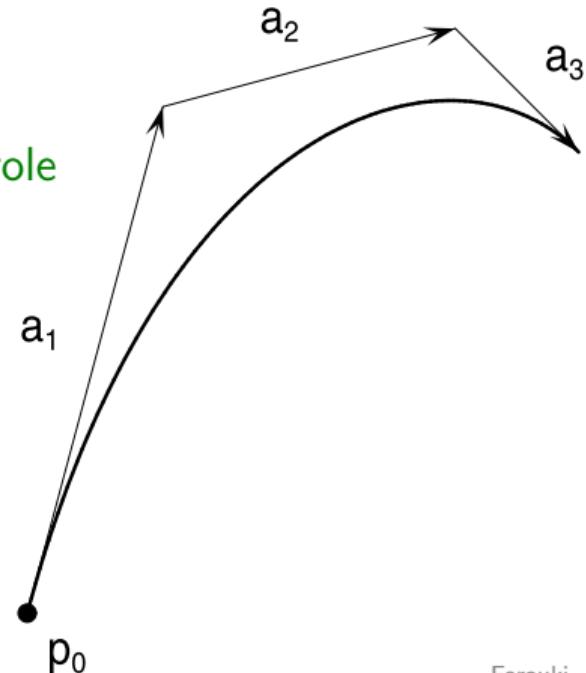
polígono de controle

$$a_k = p_k - p_{k-1}$$

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

curva de Bézier

$$B_k^n(t) = f_k^n(t) - f_{k+1}^n(t)$$



Farouki

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \text{polinômios de Bernstein}$$

Curvas de Bézier

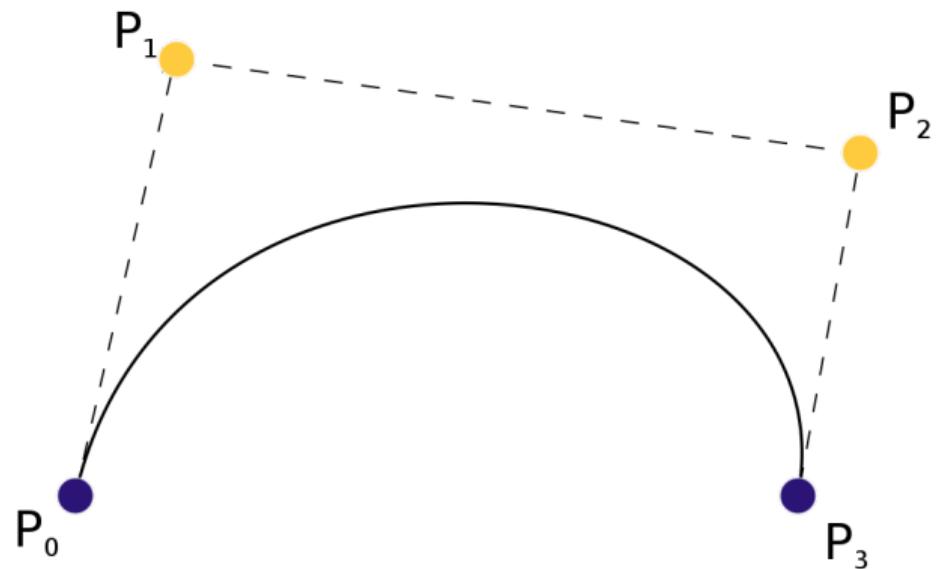
$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

$$B^0(t) = p_0$$

$$B^1(t) = (1-t)p_0 + tp_1$$

$$B^2(t) = (1-t)^2 p_0 + 2(1-t)t p_1 + t^2 p_2$$

$$B^3(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

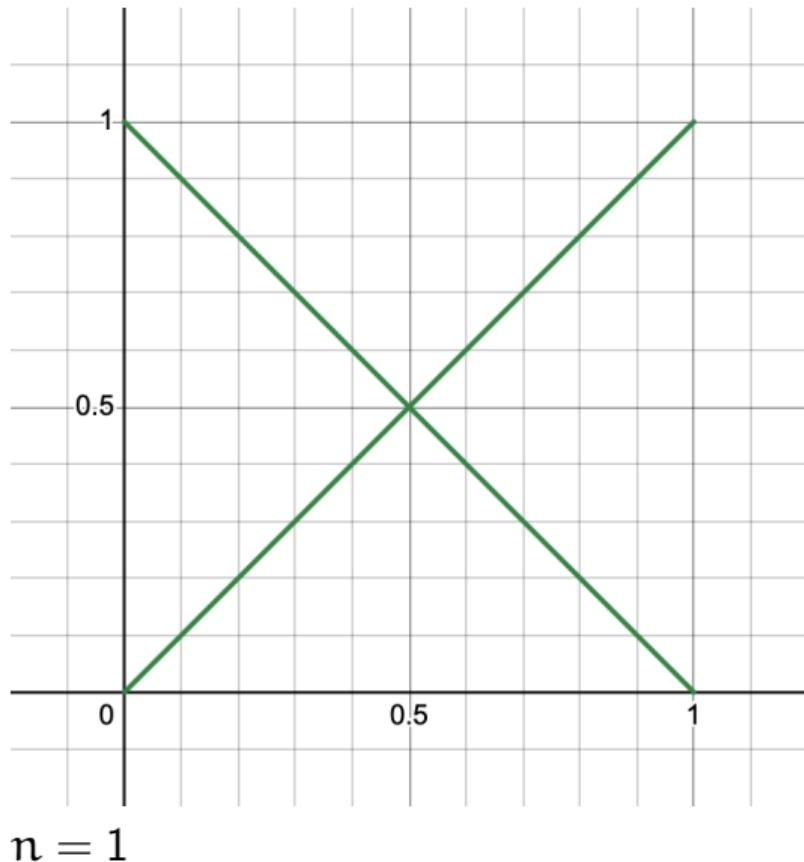


demo

Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

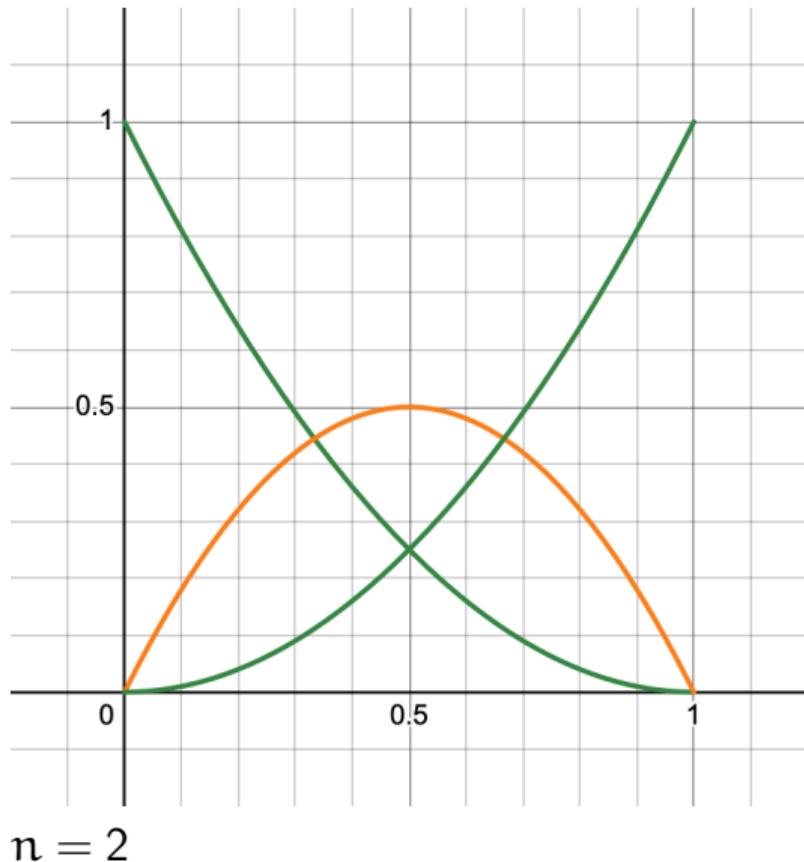
n				
0	1			
1	1 - t	t		
2	$(1 - t)^2$	$2(1 - t)t$	t^2	
3	$(1 - t)^3$	$3(1 - t)^2t$	$3(1 - t)t^2$	t^3
k	0	1	2	3



Polinômios de Bernstein

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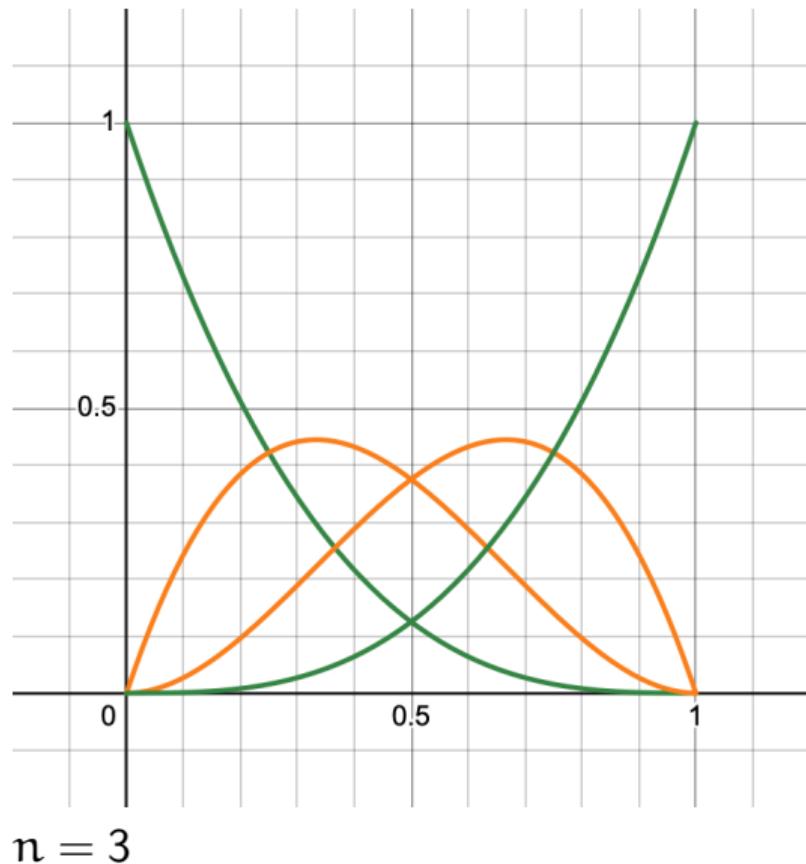
n				
0				
1	1 - t	t		
2	$(1 - t)^2$	$2(1 - t)t$	t^2	
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Polinômios de Bernstein

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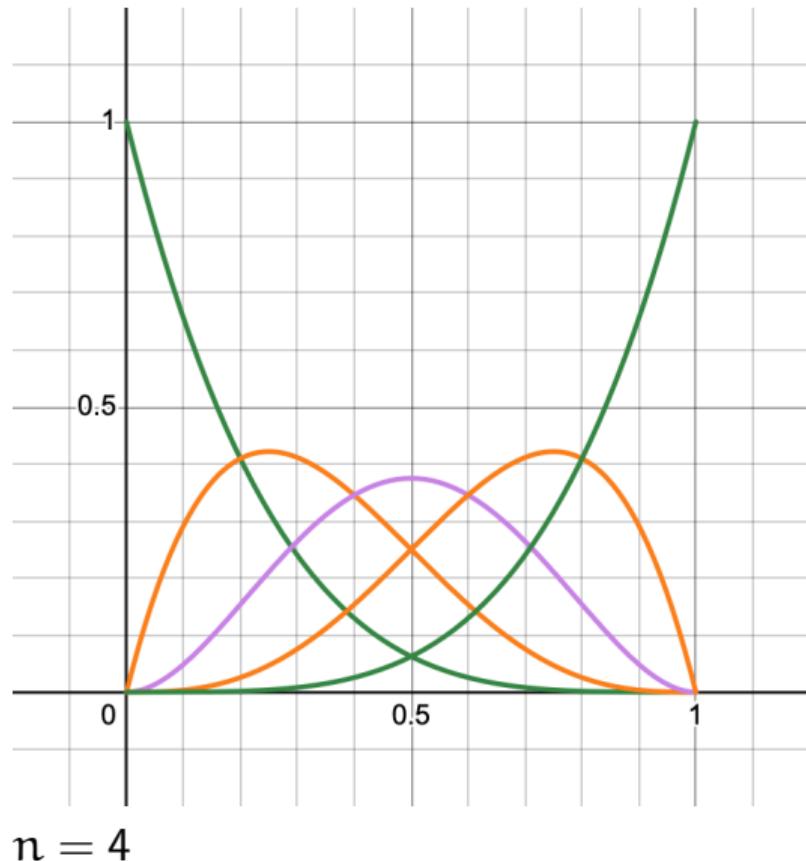
n				
0				
1	1 - t	t		
2	$(1 - t)^2$	$2(1 - t)t$	t^2	
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Polinômios de Bernstein

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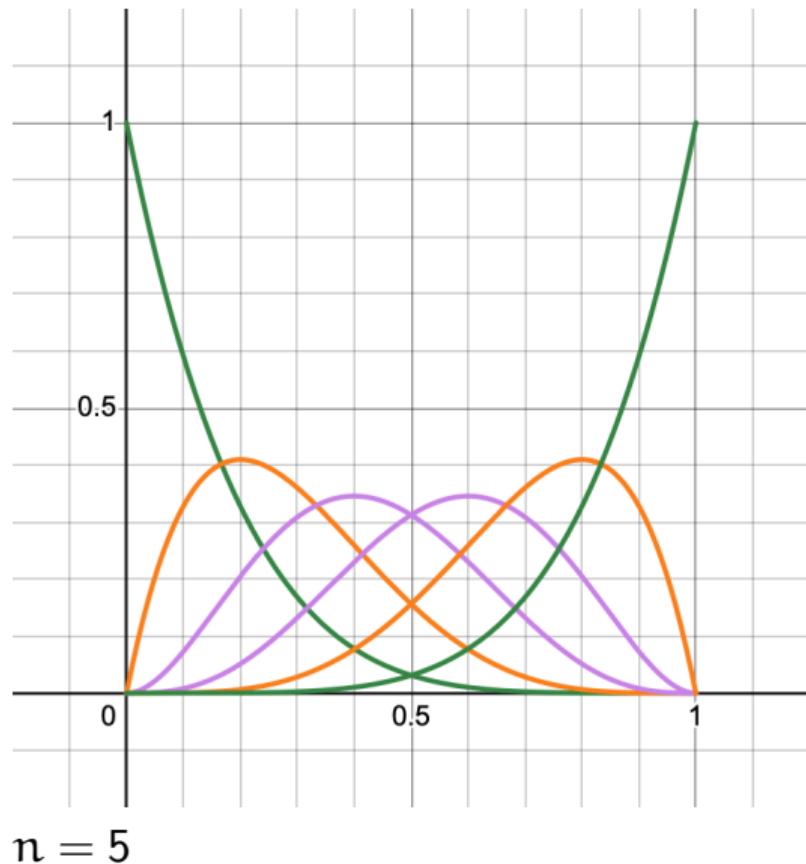
n				
0				
1	1 - t	t		
2	$(1-t)^2$	$2(1-t)t$	t^2	
3	$(1-t)^3$	$3(1-t)^2t$	$3(1-t)t^2$	t^3
k	0	1	2	3



Polinômios de Bernstein

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

n	
0	1
1	$1 - t$ t
2	$(1 - t)^2$ $2(1 - t)t$ t^2
3	$(1 - t)^3$ $3(1 - t)^2t$ $3(1 - t)t^2$ t^3
k	0 1 2 3



Teorema de aproximação

Weierstrass (1885)

Dada $f \in C[0, 1]$, para todo $\varepsilon > 0$ existe um polinômio p tal que

$$|f(t) - p(t)| < \varepsilon \quad \text{para todo } t \in [0, 1]$$

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Equivalentemente, existe uma sequência de polinômios p_n tal que

$$\lim_{n \rightarrow \infty} p_n = f \quad \text{uniforme}$$

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prova construtiva:

Bernstein (1912)

$$B_n(f)(t) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_k^n(t) \implies \lim_{n \rightarrow \infty} B_n(f) = f \quad \text{uniforme}$$

$$f \in C^r \implies \lim_{n \rightarrow \infty} B_n(f)^{(r)} = f \quad \text{uniforme}$$

Teorema de aproximação de Weierstrass

Bernstein (1912)

Démonstration du théorème de Weierstrass fondée sur le calcul des probabilités.

Je me propose d'indiquer une démonstration fort simple du théorème suivant de Weierstrass:

Si $F(x)$ est une fonction continue quelconque dans l'intervalle 01 , il est toujours possible, quel que petit que soit ε , de déterminer un polynôme $E_n(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$ de degré n assez élevé, tel qu'on ait

$$|F(x) - E_n(x)| < \varepsilon$$

en tout point de l'intervalle considéré.

A cet effet, je considère un événement A , dont la probabilité est égale à x . Supposons qu'on effectue n expériences et que l'on convienne de payer à un joueur la somme $F\left(\frac{m}{n}\right)$, si l'événement A se produit m fois. Dans ces conditions, l'espérance mathématique E_n du joueur aura pour valeur

$$E_n = \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}. \quad (1)$$

Or, il résulte de la continuité de la fonction $F(x)$ qu'il est possible de fixer un nombre δ , tel que l'inégalité

$$|x - x_0| \leq \delta$$

entraîne

$$|F(x) - F(x_0)| < \frac{\varepsilon}{2};$$

de sorte que, si $\bar{F}(x)$ désigne le maximum et $\underline{F}(x)$ le minimum de $F(x)$ dans l'intervalle $(x - \delta, x + \delta)$, on a

$$\bar{F}(x) - F(x) < \frac{\varepsilon}{2}, \quad F(x) - \underline{F}(x) < \frac{\varepsilon}{2}. \quad (2)$$

— 2 —

Soit de plus η la probabilité de l'inégalité $|x - \frac{m}{n}| > \delta$, et L le maximum de $|F(x)|$ dans l'intervalle 01 .

On aura alors

$$\underline{F}(x) \cdot (1 - \eta) - L \cdot \eta < E_n < \bar{F}(x) \cdot (1 - \eta) + L \cdot \eta. \quad (3)$$

Mais, en vertu du théorème de Bernoulli, on pourra prendre n assez grand pour avoir

$$\eta < \frac{\varepsilon}{4L}. \quad (4)$$

L'inégalité (3) se mettra donc successivement sous la forme

$$\underline{F}(x) + (\bar{F}(x) - F(x)) - \eta(L + \underline{F}(x)) < E_n < F(x) + (\bar{F}(x) - F(x)) + \eta(L - \bar{F}(x))$$

et ensuite

$$F(x) - \frac{\varepsilon}{2} - \frac{2L}{4L} \varepsilon < E_n < F(x) + \frac{\varepsilon}{2} + \frac{2L}{4L} \varepsilon;$$

donc

$$|F(x) - E_n| < \varepsilon \quad (5)$$

Or E_n est manifestement un polynôme de degré n .

Le théorème est donc démontré.

J'ajouterais seulement deux remarques.

Les polynomes approchés $E_n(x)$ sont surtout commodes, il me semble, lorsqu'on connaît exactement ou approximativement les valeurs de $F(x)$ pour $x = \frac{m}{n}$ ($m = 0, 1, \dots, n$).

La formule (1) et l'inégalité (5) montrent que, quelle que soit la fonction continue $F(x)$, on a

$$F(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^{m=n} F\left(\frac{m}{n}\right) \cdot C_n^m x^m (1-x)^{n-m}.$$

S. Bernstein.

Polinômios de Bernstein – base do espaço de polinômios

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} t^{k+j}$$

base canônica = base monomial: $1, t, t^2, t^3, \dots, t^n$

Polinômios de Bernstein – base do espaço de polinômios

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k = \binom{n}{k} \sum_{j=0}^{n-k} (-1)^j \binom{n-k}{j} t^{k+j}$$

$$\begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix} = \begin{bmatrix} (1-t)^3 \\ 3(1-t)^2t \\ 3(1-t)t^2 \\ t^3 \end{bmatrix} = \begin{pmatrix} 1 & -3 & 3 & -1 \\ 0 & 3 & -6 & 3 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

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$$t^k = \frac{1}{\binom{n}{k}} \sum_{j=k}^n \binom{j}{k} B_j^n(t)$$
$$\begin{bmatrix} 1 \\ t \\ t^2 \\ t^3 \end{bmatrix} = \frac{1}{3} \begin{pmatrix} 3 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{bmatrix} B_0^3(t) \\ B_1^3(t) \\ B_2^3(t) \\ B_3^3(t) \end{bmatrix}$$

Polinômios de Bernstein – propriedades

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

- $B_k^n(t) = B_{n-k}^n(1-t)$ simetria
- $B_k^n(0) = 0, \quad B_k^n(1) = 0, \quad B_0^n(0) = 1, \quad B_n^n(1) = 1$ pontos extremos
- $B_k^n(t)$ tem único máximo em $[0, 1]$, em $t = k/n$ unimodal
- $B_k^n(t) \geq 0$ para $t \in [0, 1]$ não negativo
- $\sum_{k=0}^n B_k^n(t) = ((1-t) + t)^n = 1$ partição da unidade
- $\sum_{k=0}^n \frac{k}{n} B_k^n(t) = t$ precisão linear

Curvas de Bézier – propriedades

$$B(t) = \sum_{k=0}^n B_k^n(t) p_k$$

- $B(t) = B(1 - t)$ simetria
- $B[p_0, \dots, p_n](t) = B[p_n, \dots, p_0](1 - t)$
- $B(0) = p_0, \quad B(1) = p_n$ pontos extremos
- a influência de p_k é maior perto de $t = k/n$ unimodal
- $B(t)$ é combinação convexa de p_0, \dots, p_n não negativo
 - curva contida no fecho convexo + partição da unidade
- $B(t)$ é gráfico de função quando $x_k = k/n$ precisão linear
 - segmento de reta se pontos de controle uniformemente espaçados

Curvas de Bézier – invariância afim

- Combinação afim

$$a_1 p_1 + \cdots + a_n p_n \text{ com } a_1 + \cdots + a_n = 1$$

- T transformação afim

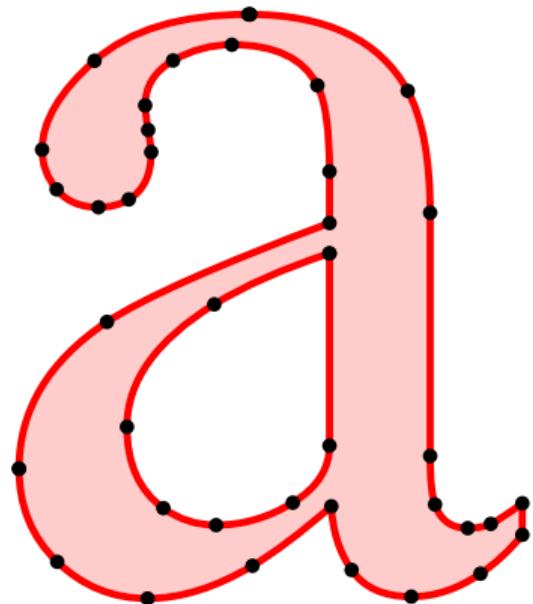
$$T(a_1 p_1 + \cdots + a_n p_n) = a_1 T(p_1) + \cdots + a_n T(p_n)$$

- invariância afim

$$T(B[p_0, \dots, p_n]) = B[T(p_0), \dots, T(p_n)]$$

$$T(p) = L(p) + v$$

$$\sum_{k=0}^n B_k^n(t)v = v$$



Curvas de Bézier – invariância afim

- Combinação afim

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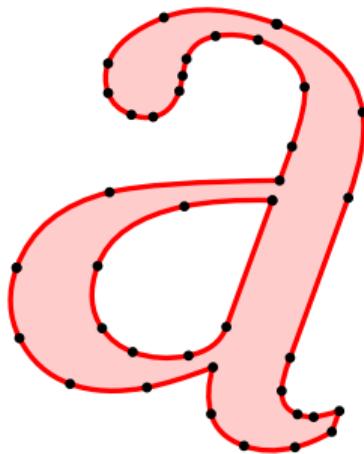
$$T(a_1 p_1 + \cdots + a_n p_n) = a_1 T(p_1) + \cdots + a_n T(p_n)$$

- invariância afim

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$$T(p) = L(p) + v$$

$$\sum_{k=0}^n B_k^n(t)v = v$$



Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k$$

Polinômios de Bernstein – recursão

$$B_k^n(t) = \binom{n}{k} (1-t)^{n-k} t^k \quad \binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{Pascal}$$

Polinômios de Bernstein – recursão

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Polinômios de Bernstein – recursão

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Polinômios de Bernstein – recursão

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Polinômios de Bernstein – recursão

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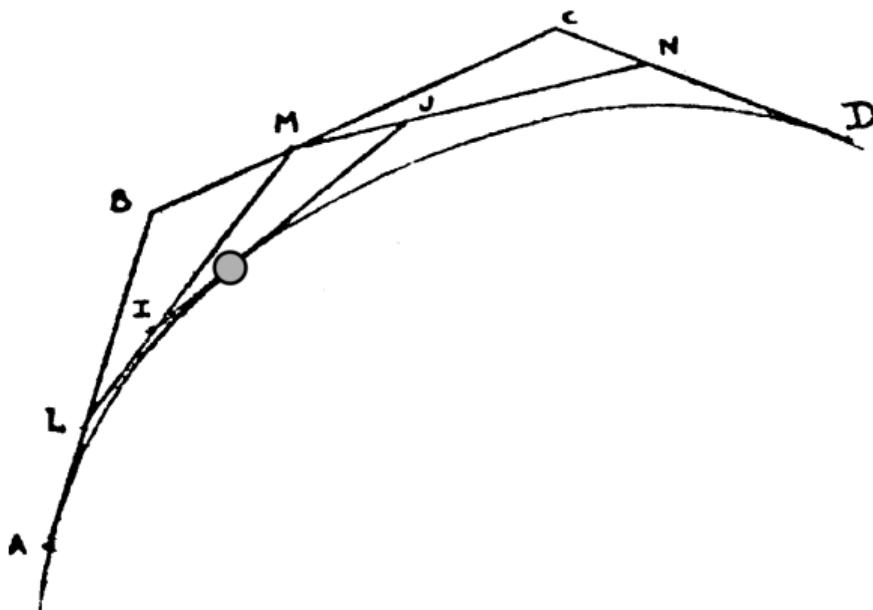
$$\begin{aligned} B_k^{n+1}(t) &= \binom{n+1}{k} (1-t)^{n+1-k} t^k \\ &= \binom{n}{k} (1-t)^{n+1-k} t^k + \binom{n}{k-1} (1-t)^{n+1-k} t^k \\ &= (1-t) \binom{n}{k} (1-t)^{n-k} t^k + t \binom{n}{k-1} (1-t)^{n-(k-1)} t^{k-1} \end{aligned}$$

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t) \quad \text{interpolação linear repetida}$$

Curvas de Bézier – algoritmo de de Casteljau

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t) \quad \text{recursão}$$

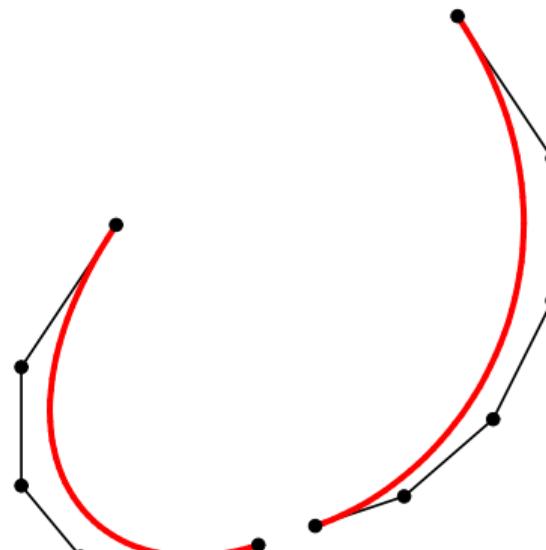
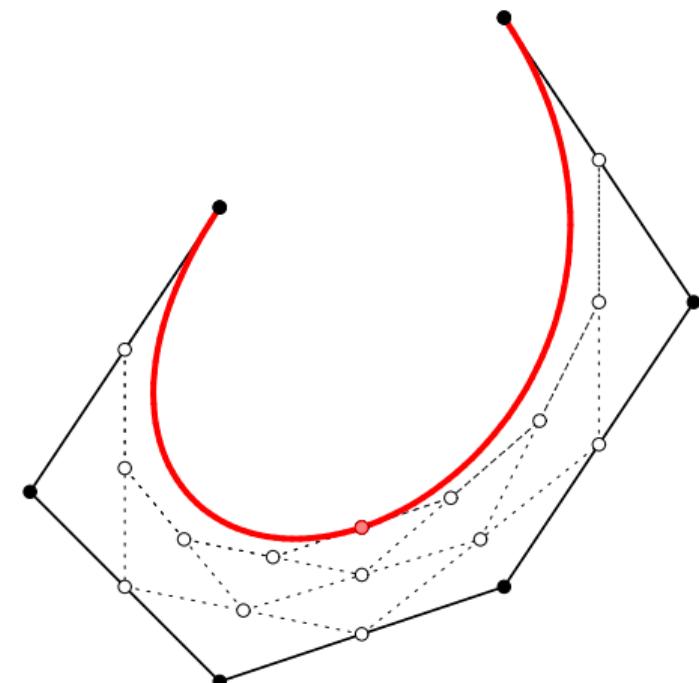
$$B^{n+1}[p_0, \dots, p_{n+1}](t) = (1-t) B^n[p_0, \dots, p_n](t) + t B^n[p_1, \dots, p_{n+1}](t)$$



Curvas de Bézier – subdivisão adaptativa

$$B_k^{n+1}(t) = (1-t) B_k^n(t) + t B_{k-1}^n(t)$$

recursão



Curvas de Bézier – presença

grau	PDF	SVG	JavaScript
1	●	●	●
2	●	●	●
3	●	●	●
4+	●	●	●

Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

■ Bézier quadrática

$$\gamma(t) = (1-t)^2 q_0 + 2(1-t)t q_1 + t^2 q_2$$

... multiplicando por $((1-t) + t) = 1$

■ Bézier cúbica

$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

■ Bézier de grau n

$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

Curvas de Bézier – elevação de grau

$$B^2[q_0, q_1, q_2] = B^3[p_0, p_1, p_2, p_3] ?$$

■ Bézier quadrática

$$\gamma(t) = (1-t)^2 q_0 + 2(1-t)t q_1 + t^2 q_2$$

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$$\gamma(t) = (1-t)^3 p_0 + 3(1-t)^2 t p_1 + 3(1-t)t^2 p_2 + t^3 p_3$$

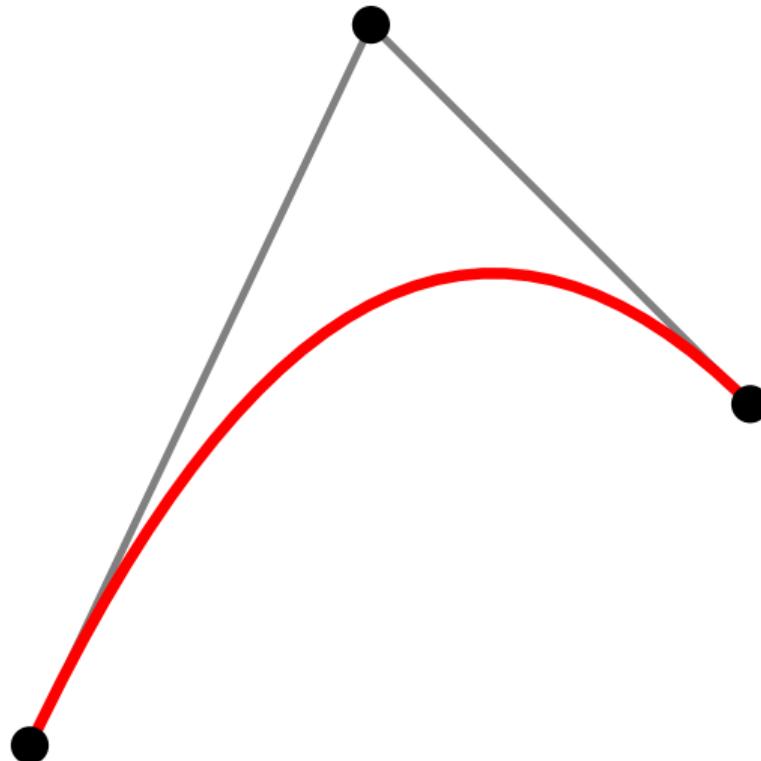
$$p_0 = q_0, \quad p_1 = \frac{1}{3}q_0 + \frac{2}{3}q_1, \quad p_2 = \frac{1}{3}q_2 + \frac{2}{3}q_1, \quad p_3 = q_2$$

■ Bézier de grau n

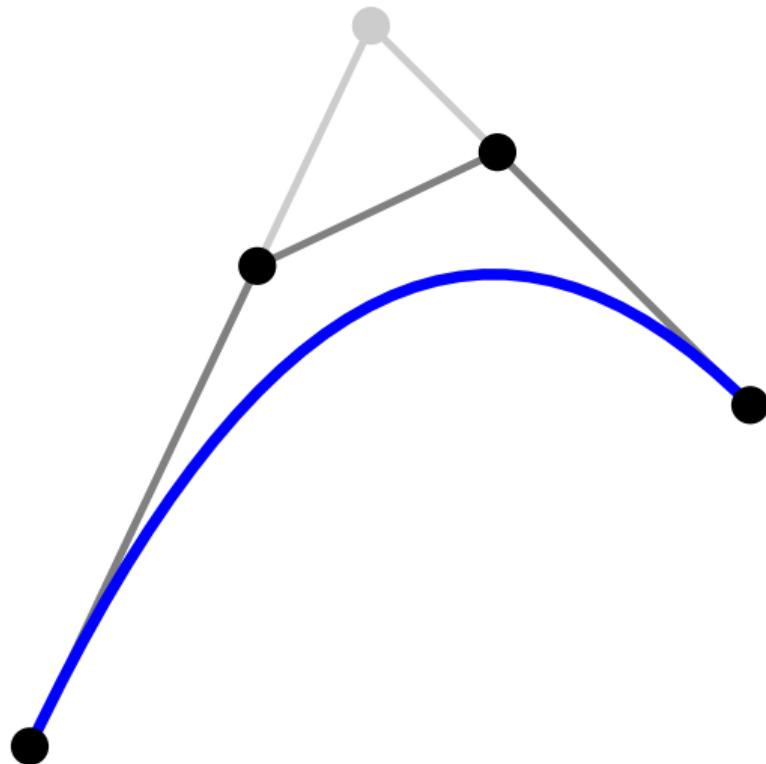
$$p_k = (1-t) q_k + t q_{k-1} \quad t = \frac{k}{n+1}$$

$$P^{n+r} \rightarrow B^n$$

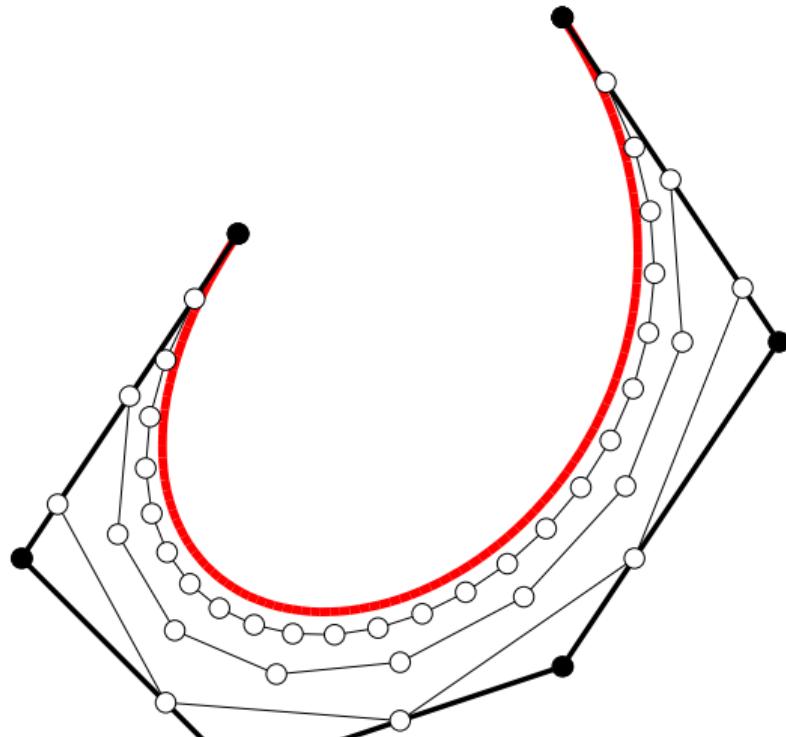
Curvas de Bézier – elevação de grau



Curvas de Bézier – elevação de grau



Curvas de Bézier – elevação de grau



$n = 5 \rightarrow 6, 10, 25$

Farouki

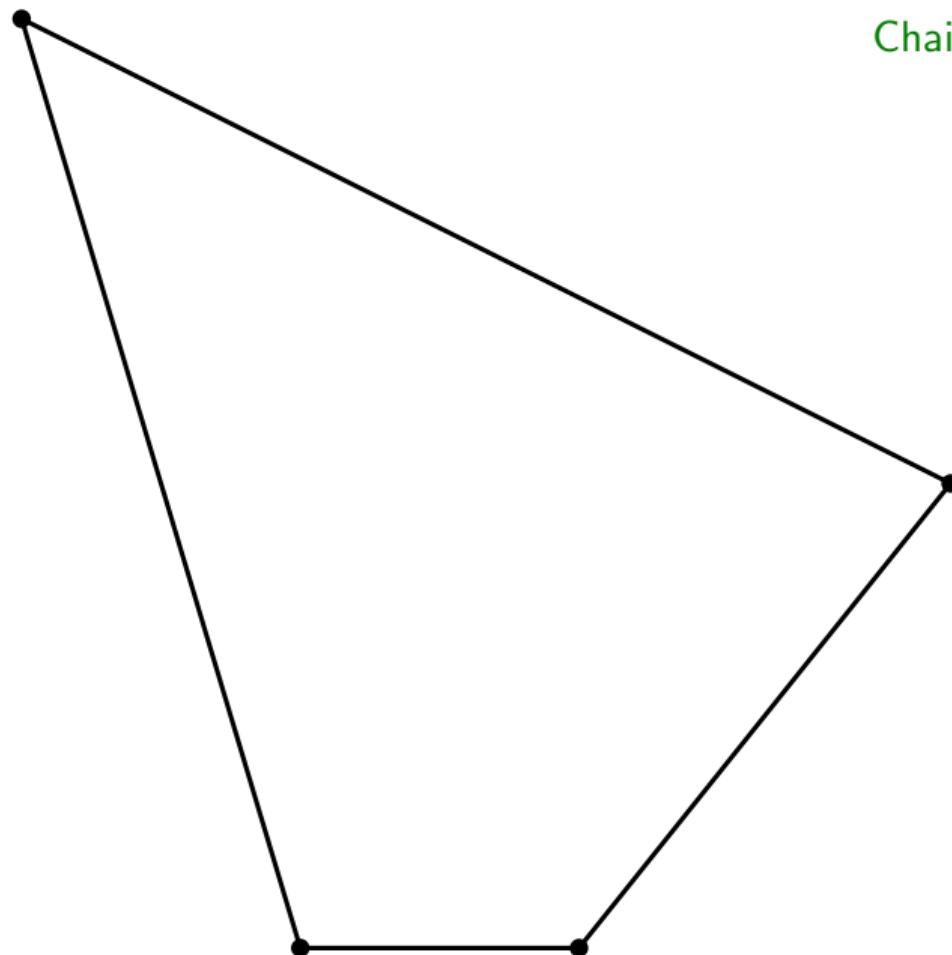
subdivisão

Curvas de subdivisão

- de Rham (1947, 1956) cantos a $1/3$
- de Casteljau (1963) Bézier quadrática
- Chaikin (1974) cantos a $1/4$
- Riesenfeld (1975) spline de Bézier quadráticas
- de Boor (1987) Comer cutting always works

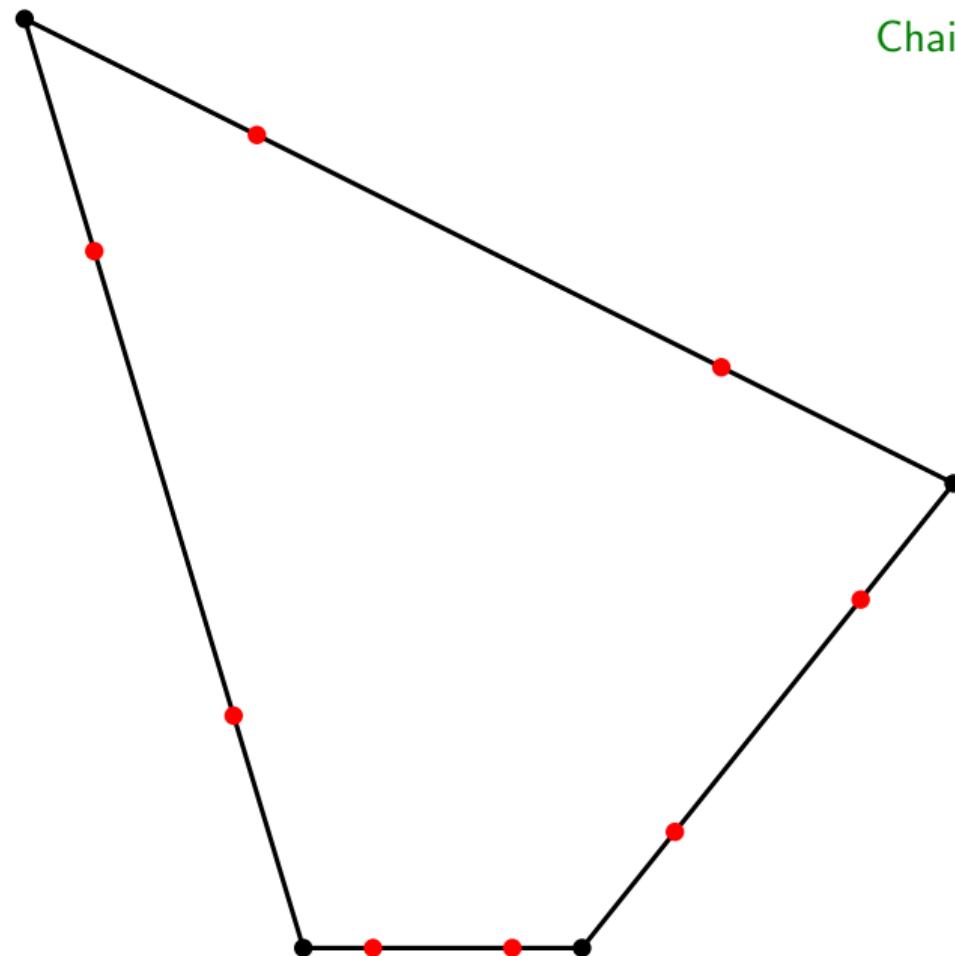
Curvas de subdivisão

Chaikin (1974)



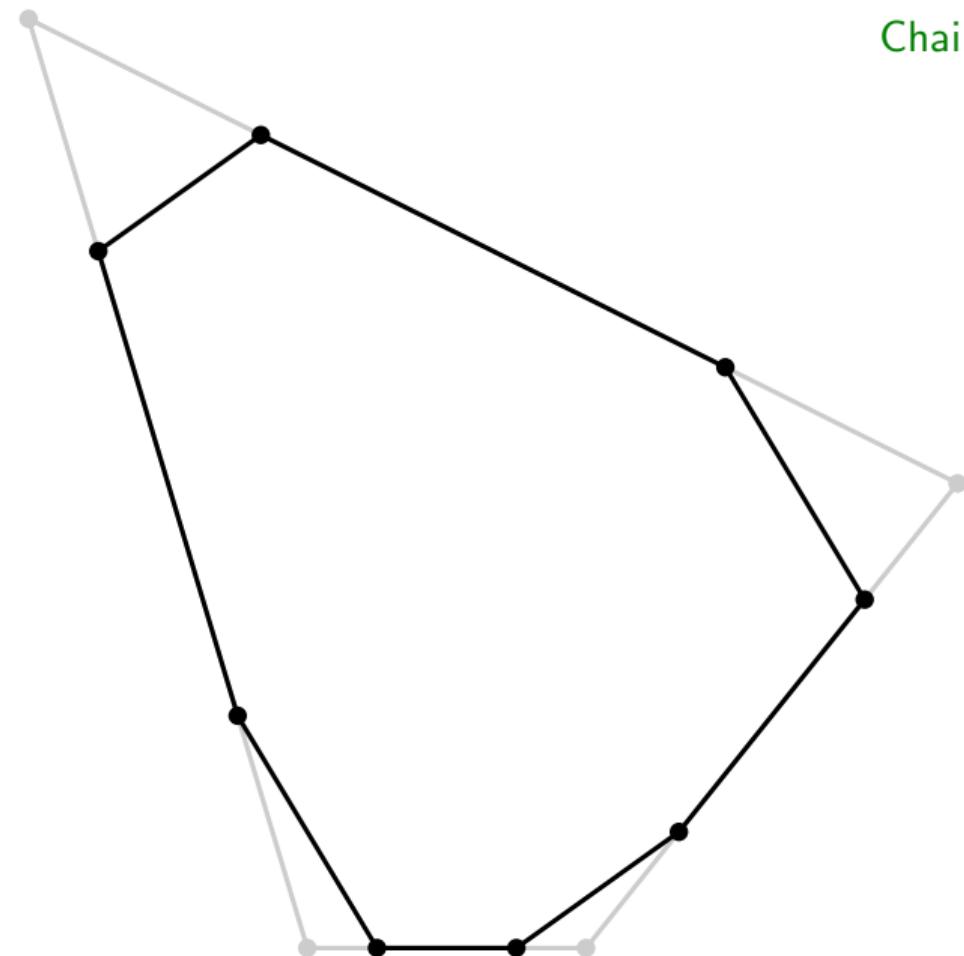
Curvas de subdivisão

Chaikin (1974)



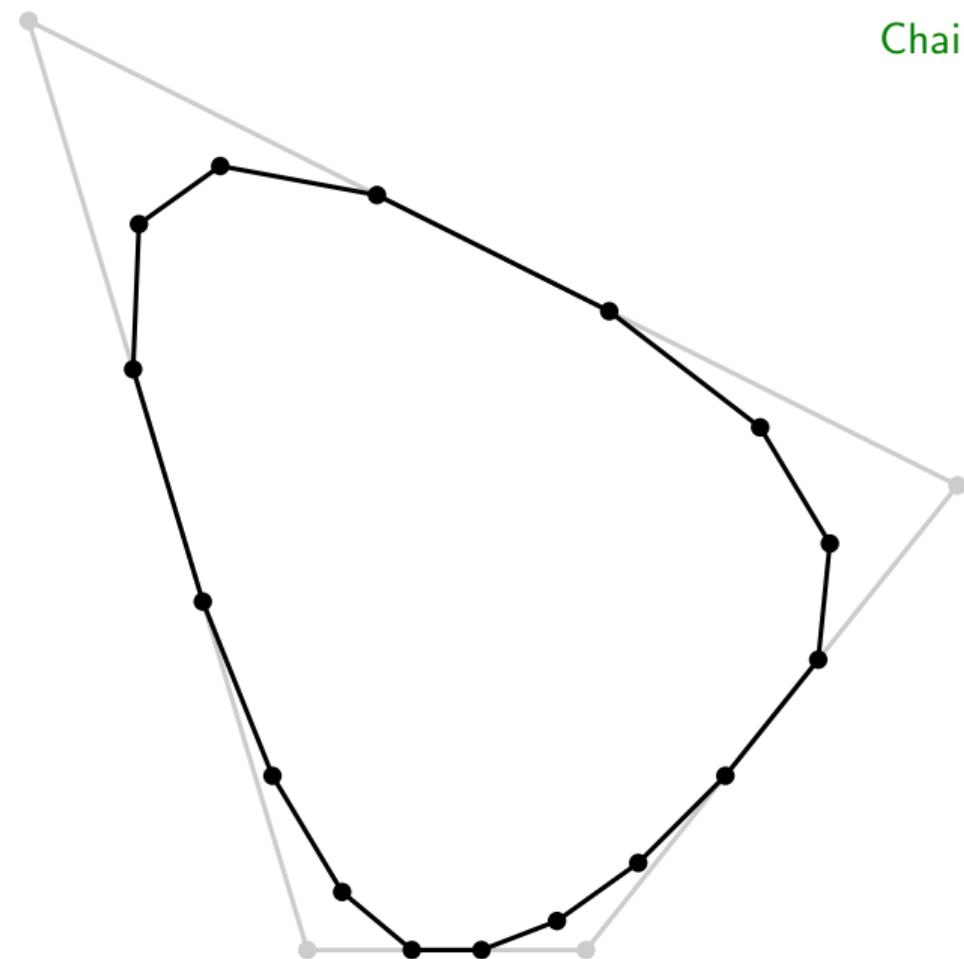
Curvas de subdivisão

Chaikin (1974)



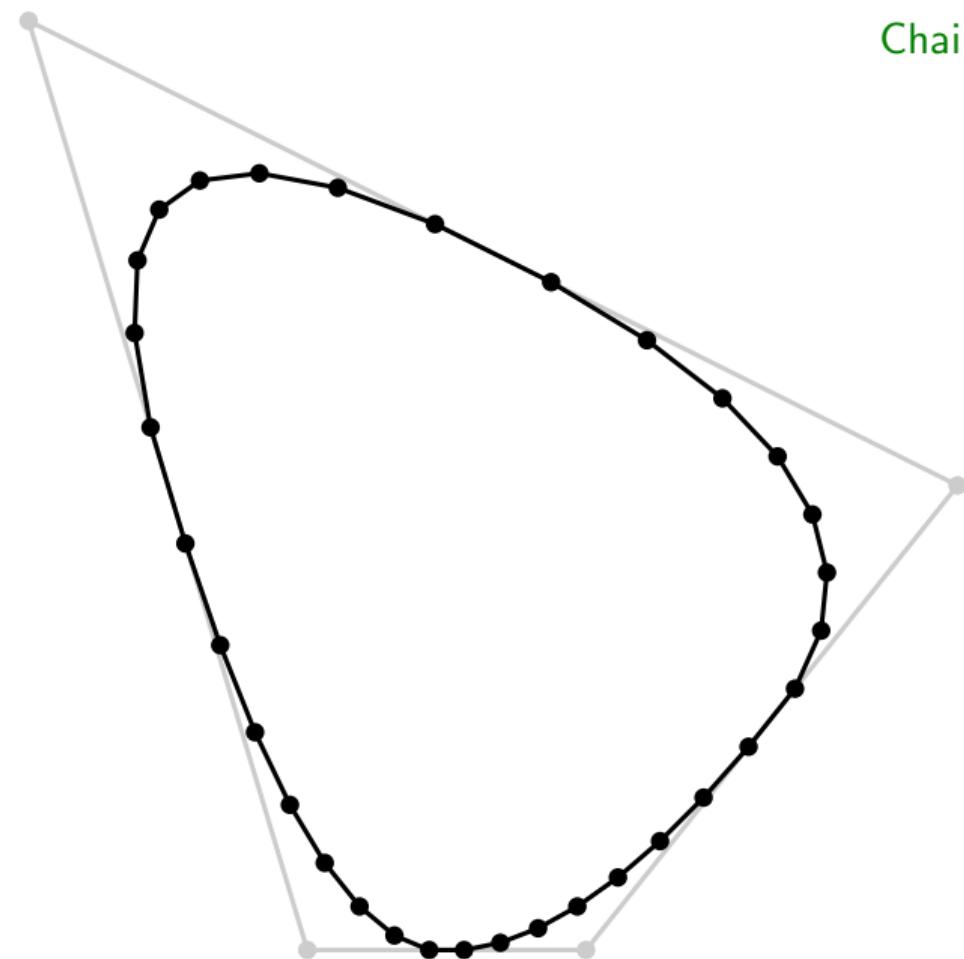
Curvas de subdivisão

Chaikin (1974)



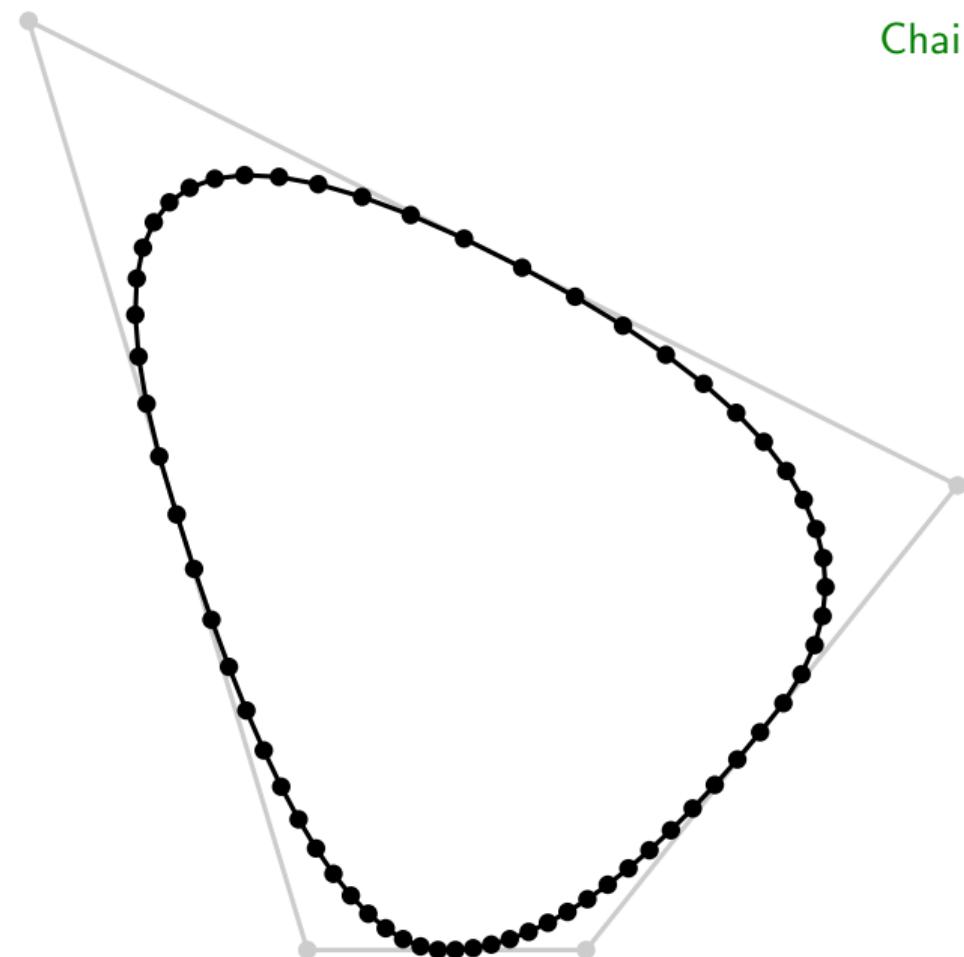
Curvas de subdivisão

Chaikin (1974)



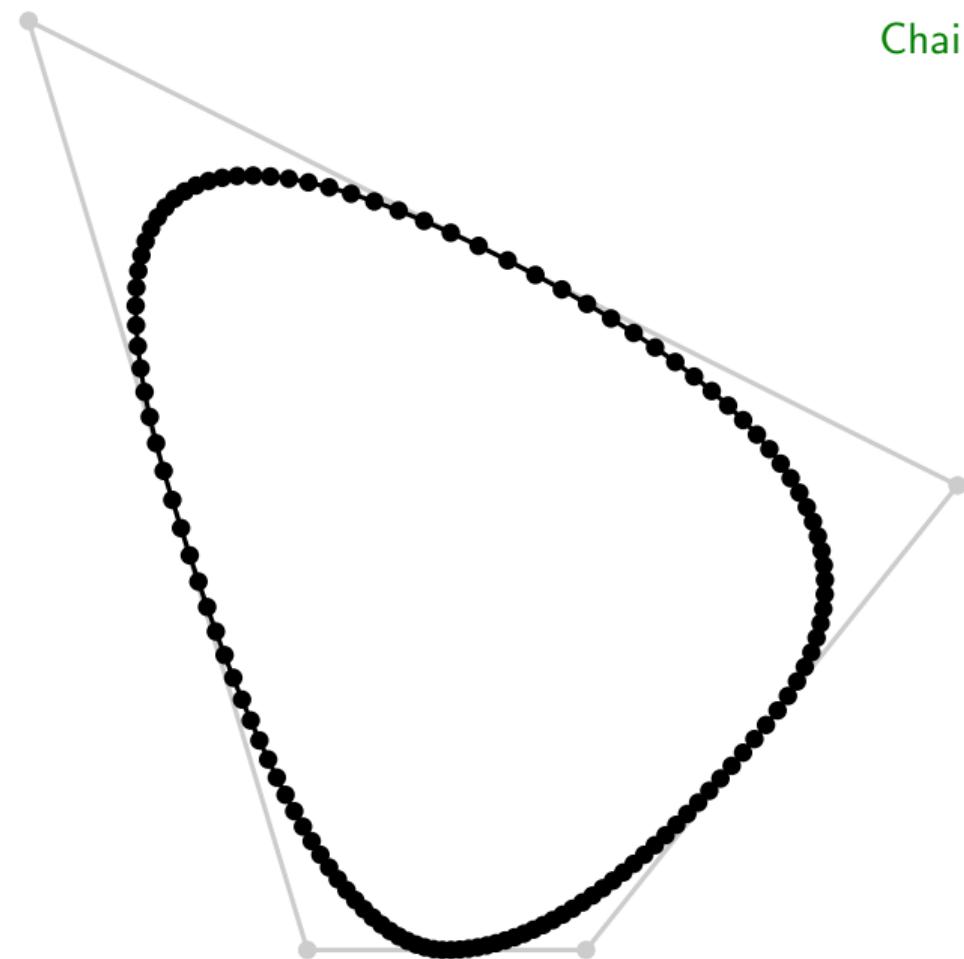
Curvas de subdivisão

Chaikin (1974)



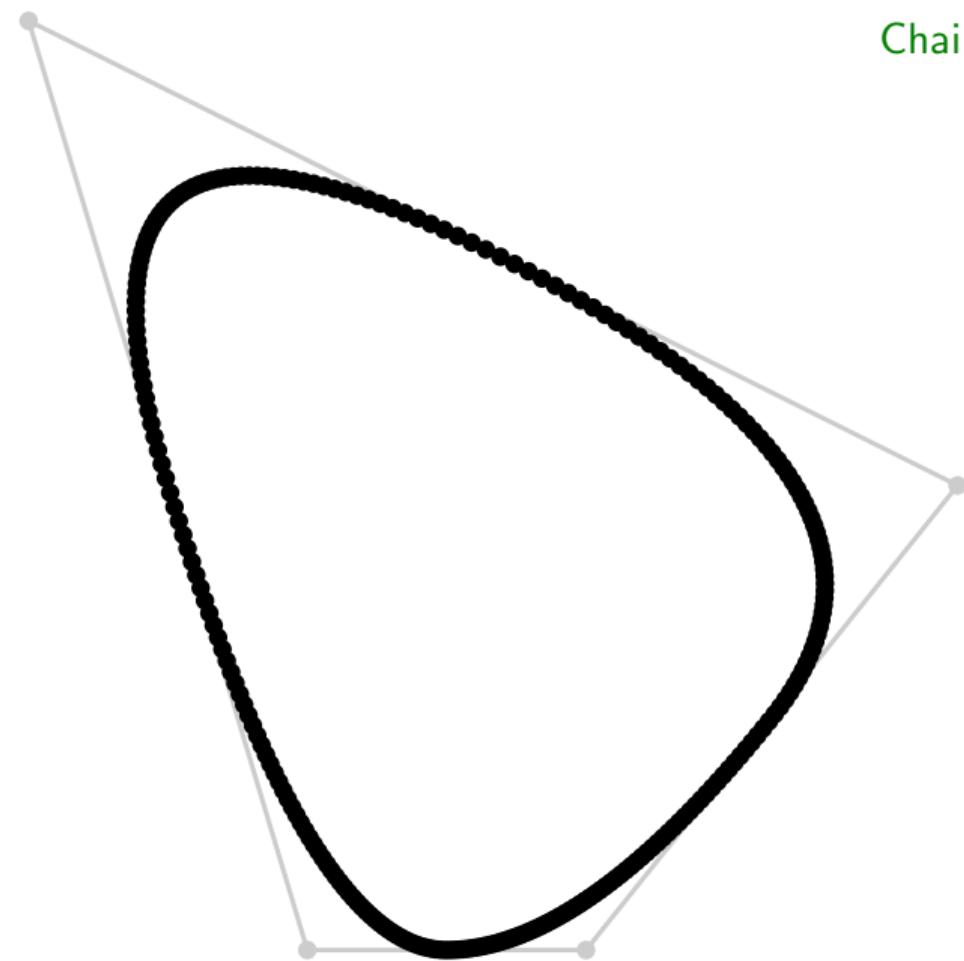
Curvas de subdivisão

Chaikin (1974)



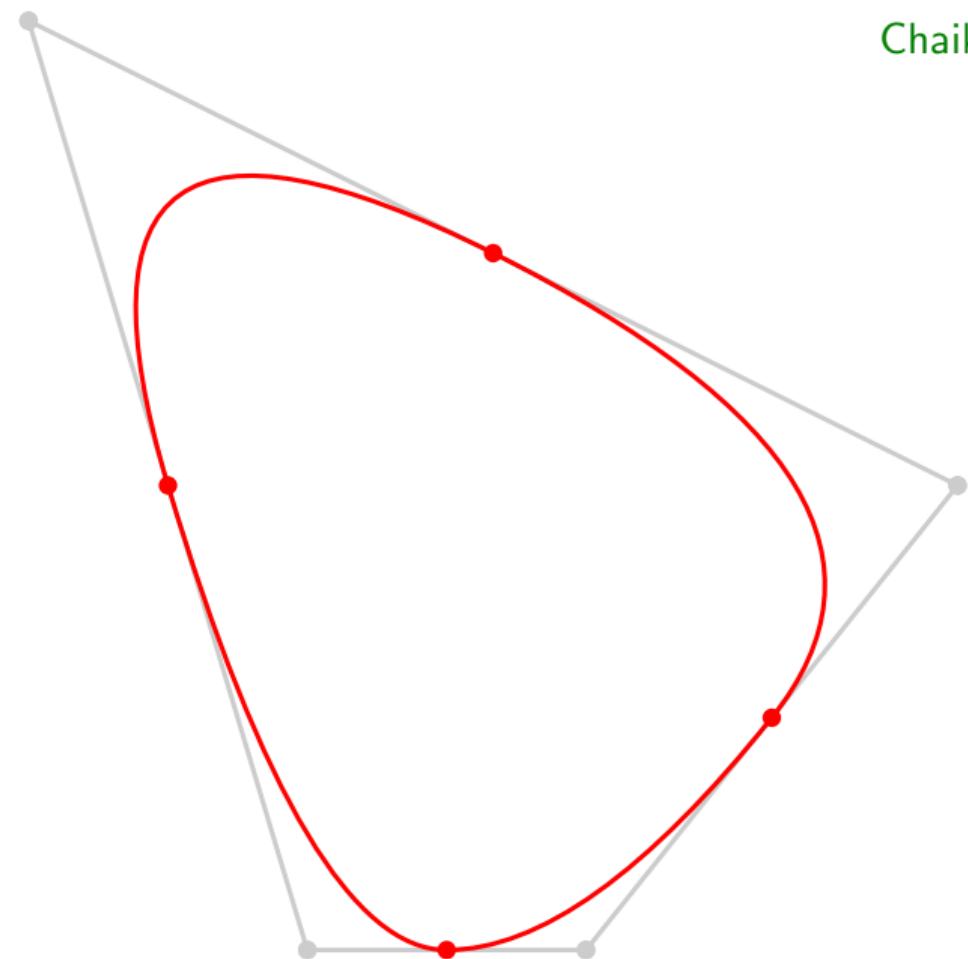
Curvas de subdivisão

Chaikin (1974)



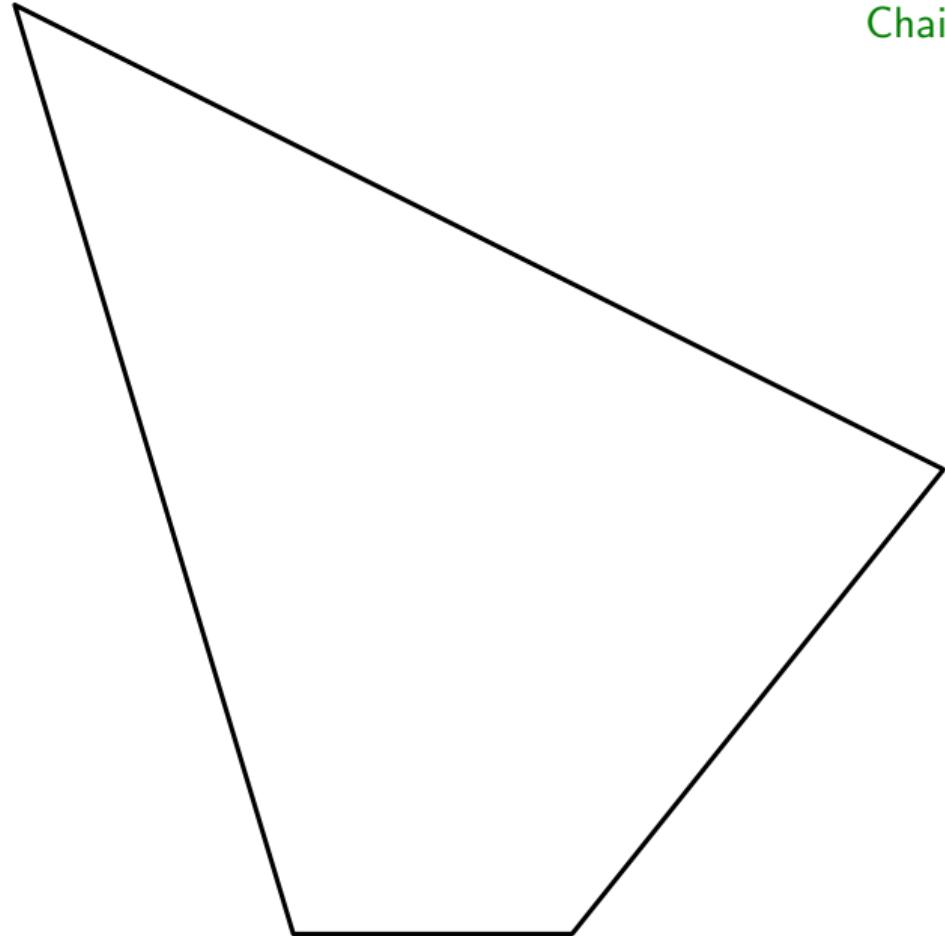
Curvas de subdivisão

Chaikin (1974)



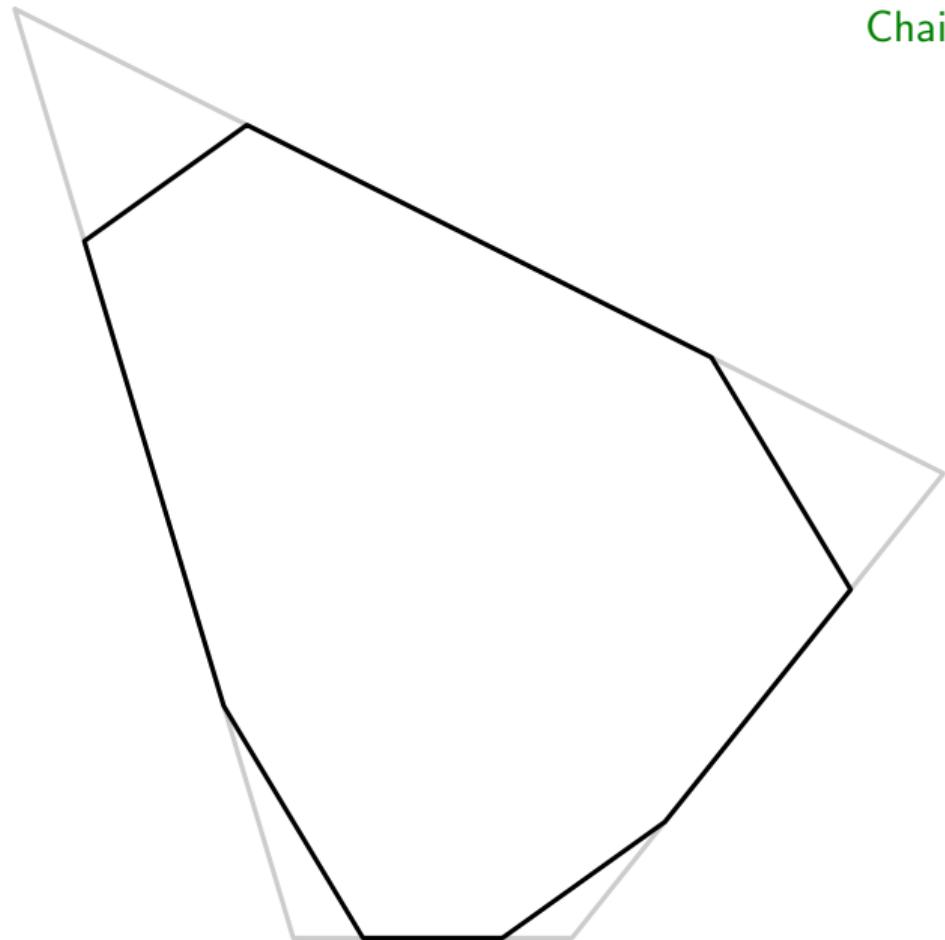
Curvas de subdivisão

Chaikin (1974)



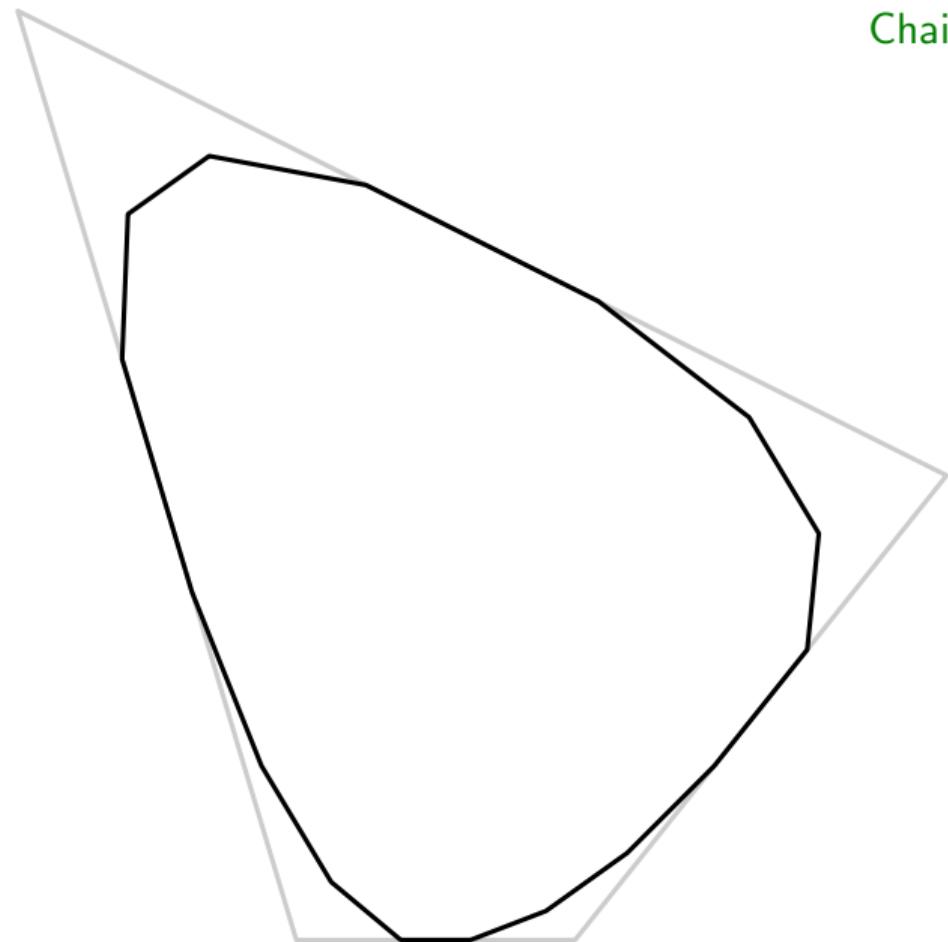
Curvas de subdivisão

Chaikin (1974)



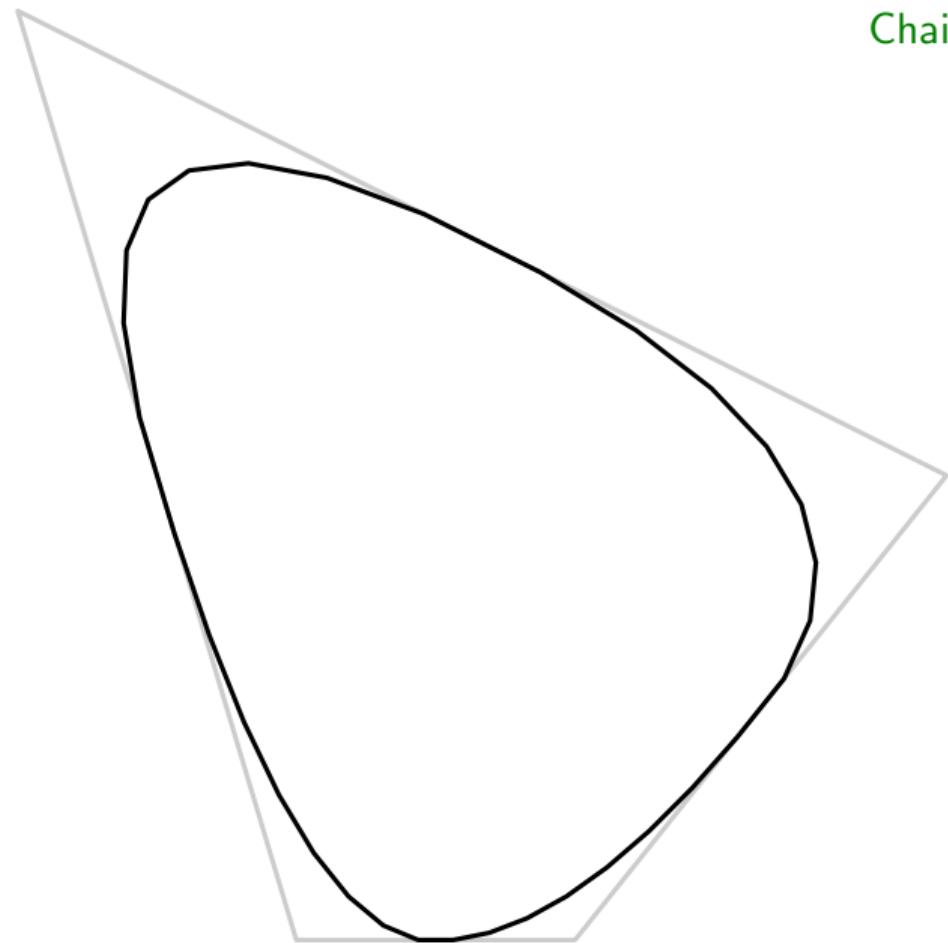
Curvas de subdivisão

Chaikin (1974)



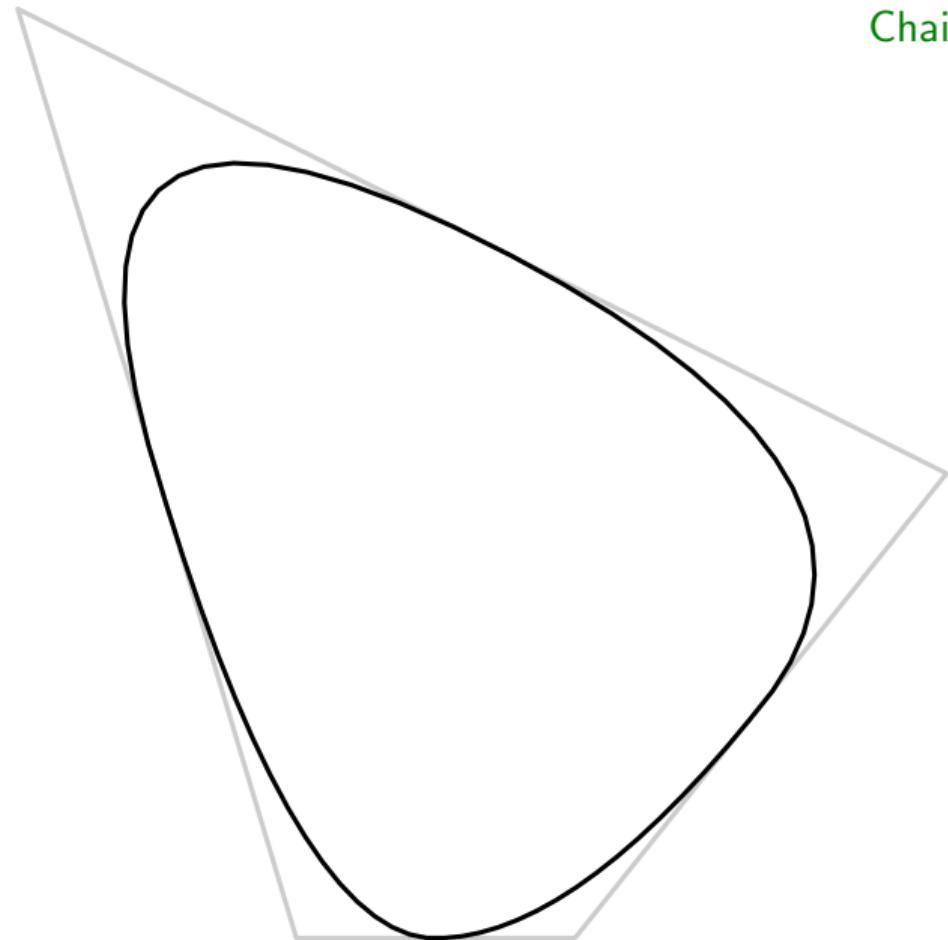
Curvas de subdivisão

Chaikin (1974)



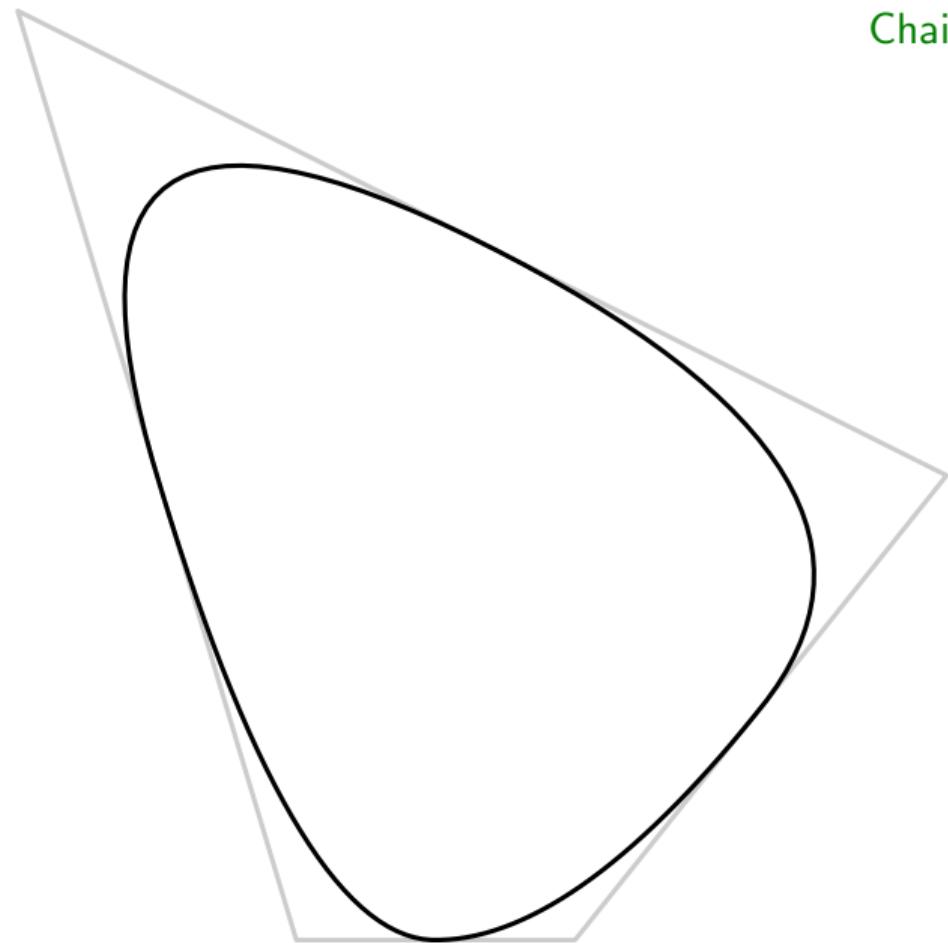
Curvas de subdivisão

Chaikin (1974)



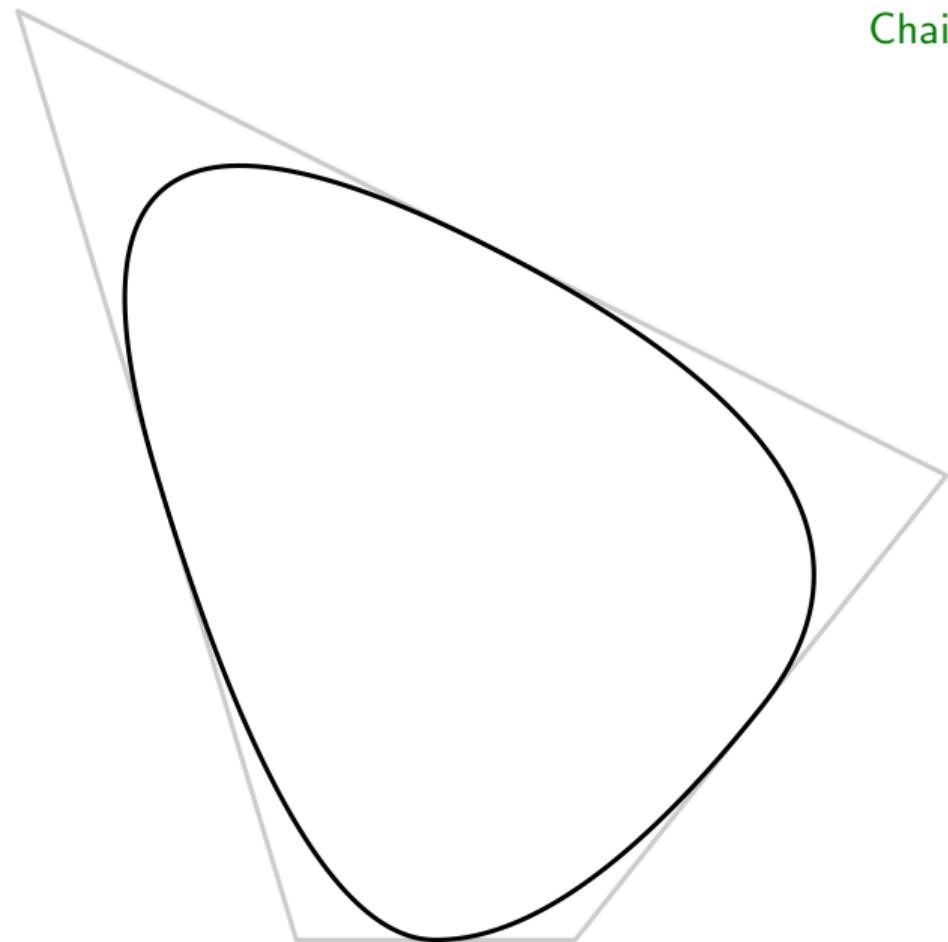
Curvas de subdivisão

Chaikin (1974)



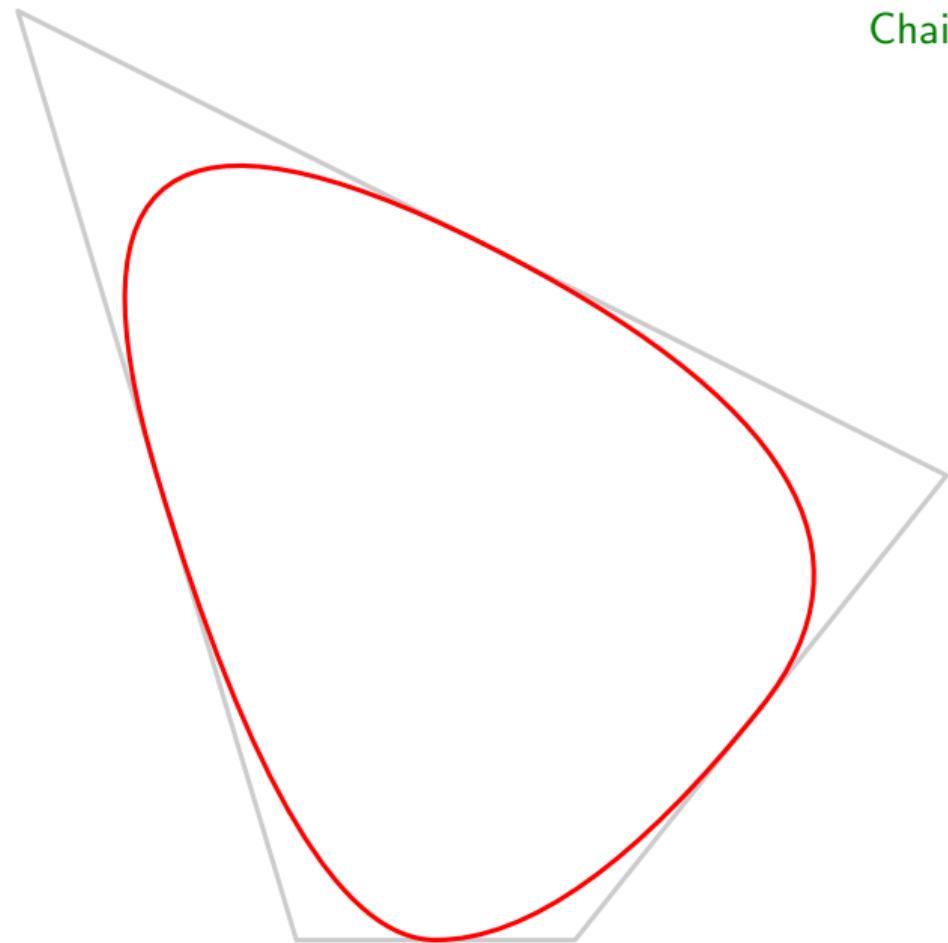
Curvas de subdivisão

Chaikin (1974)



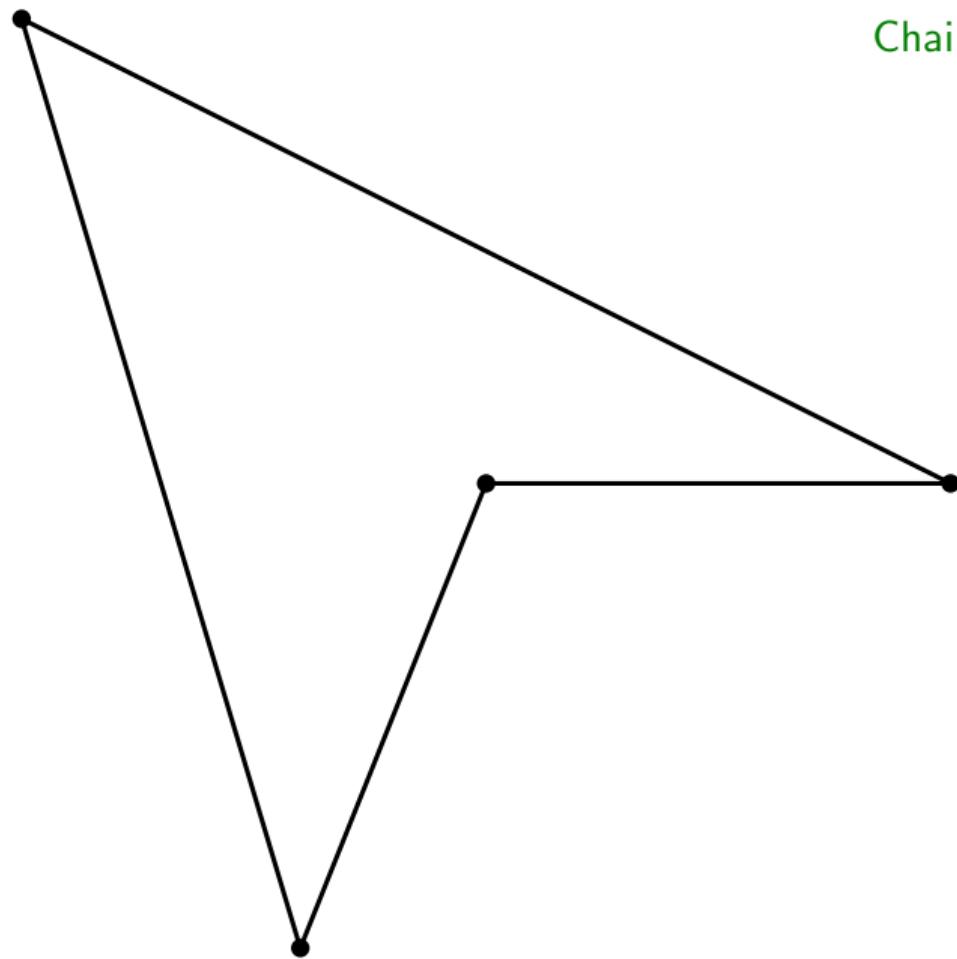
Curvas de subdivisão

Chaikin (1974)



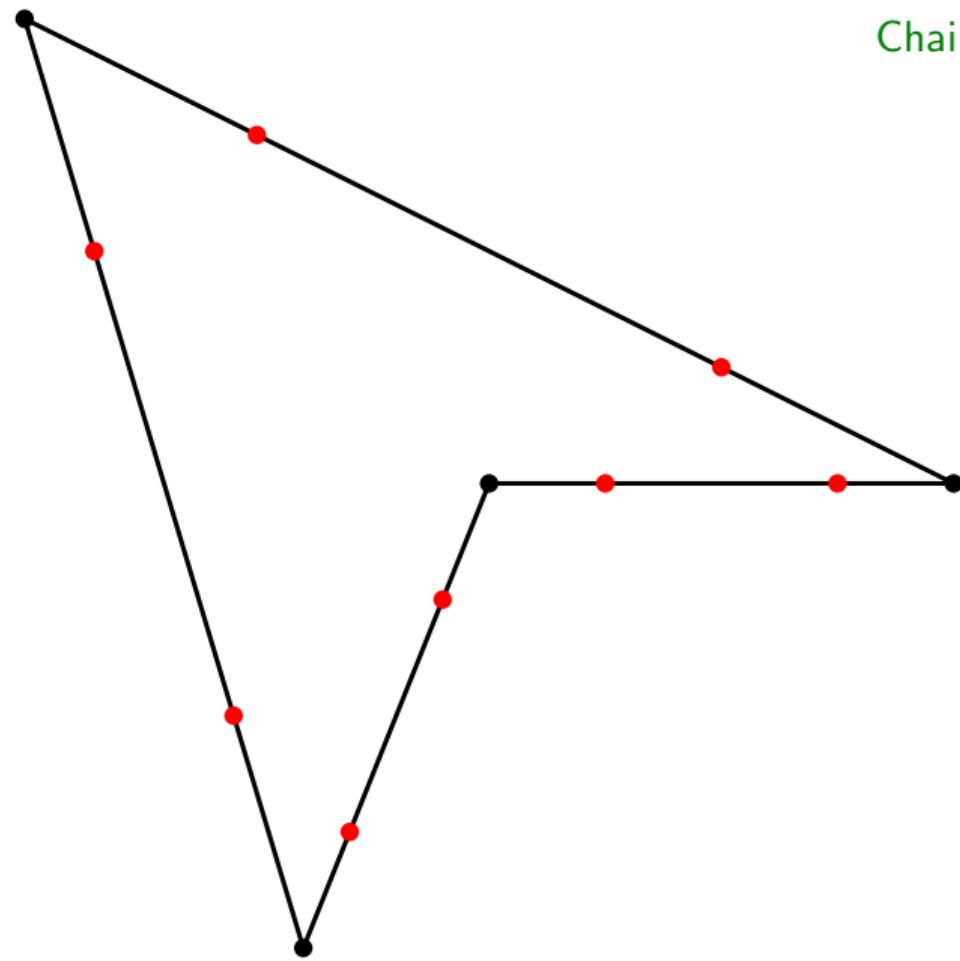
Curvas de subdivisão

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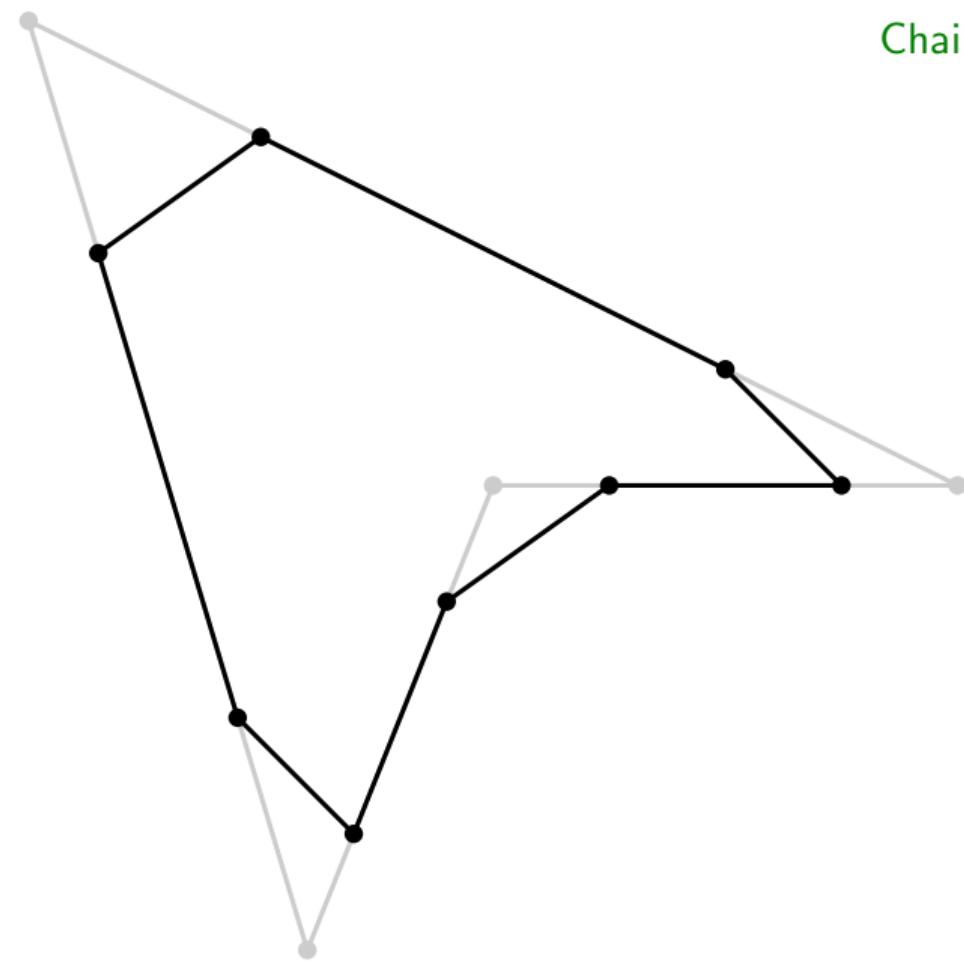
Curvas de subdivisão

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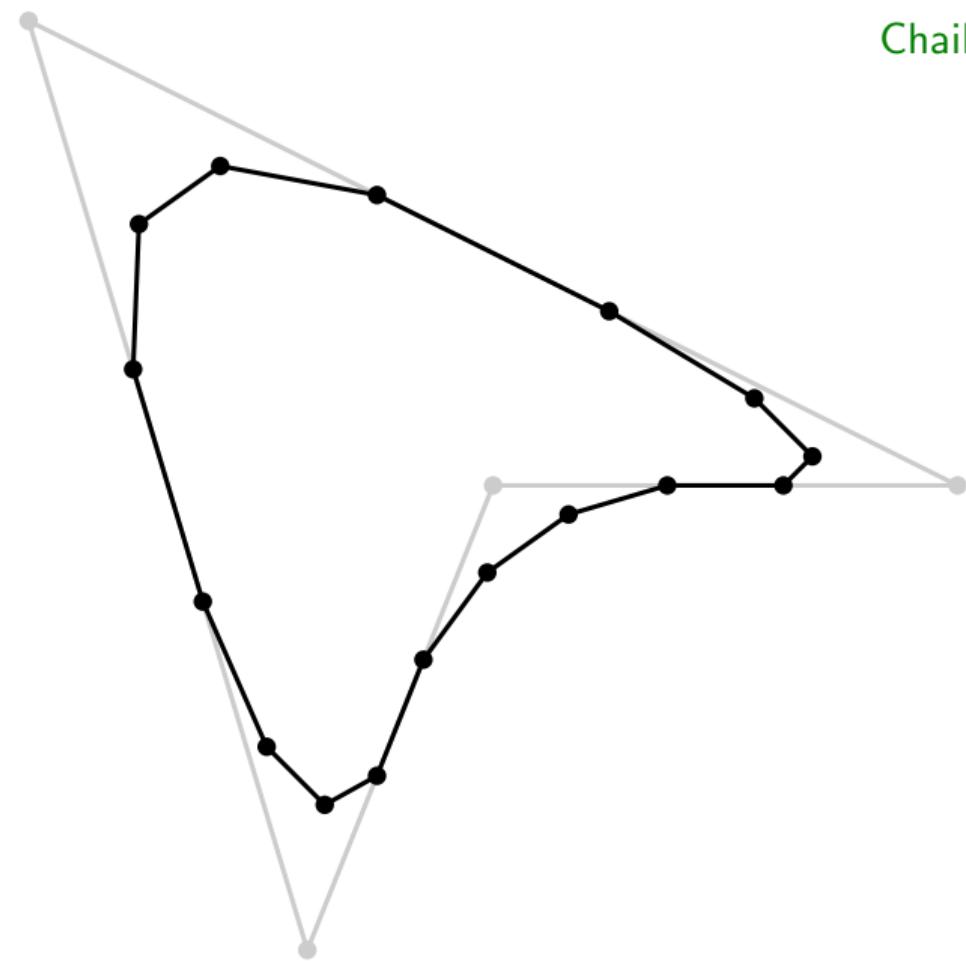
Curvas de subdivisão

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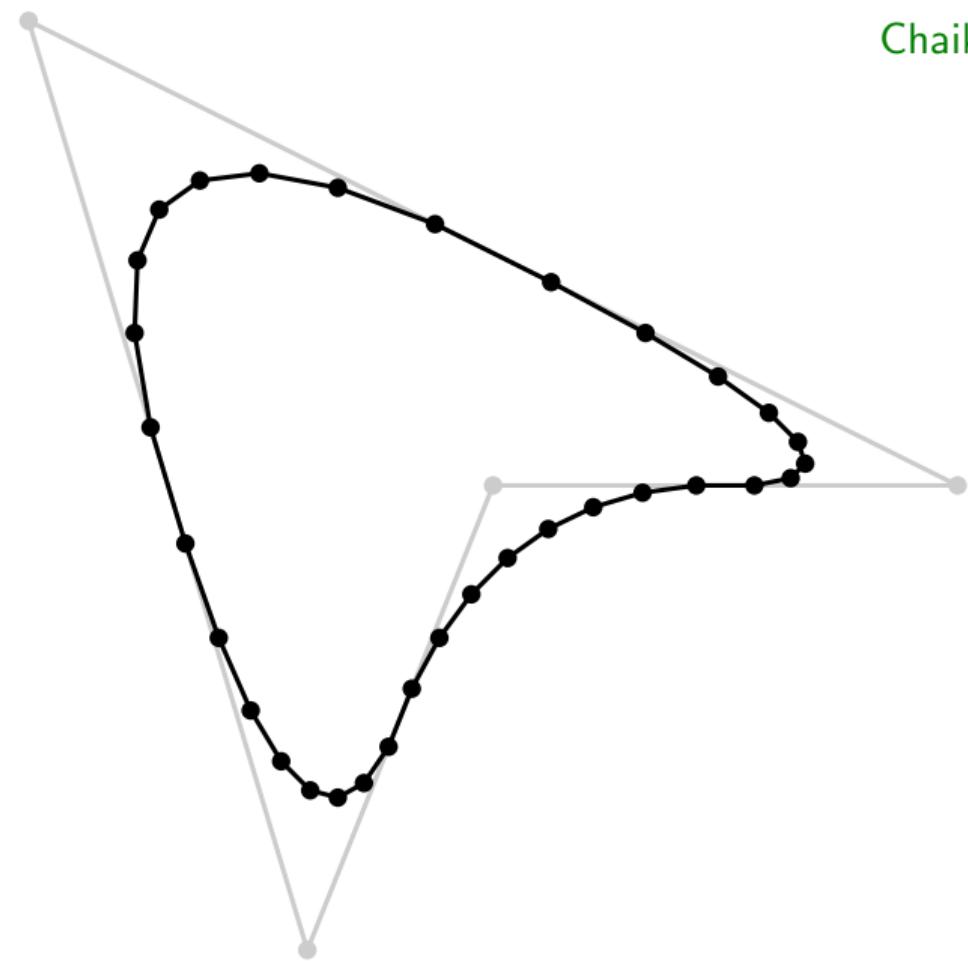
Curvas de subdivisão

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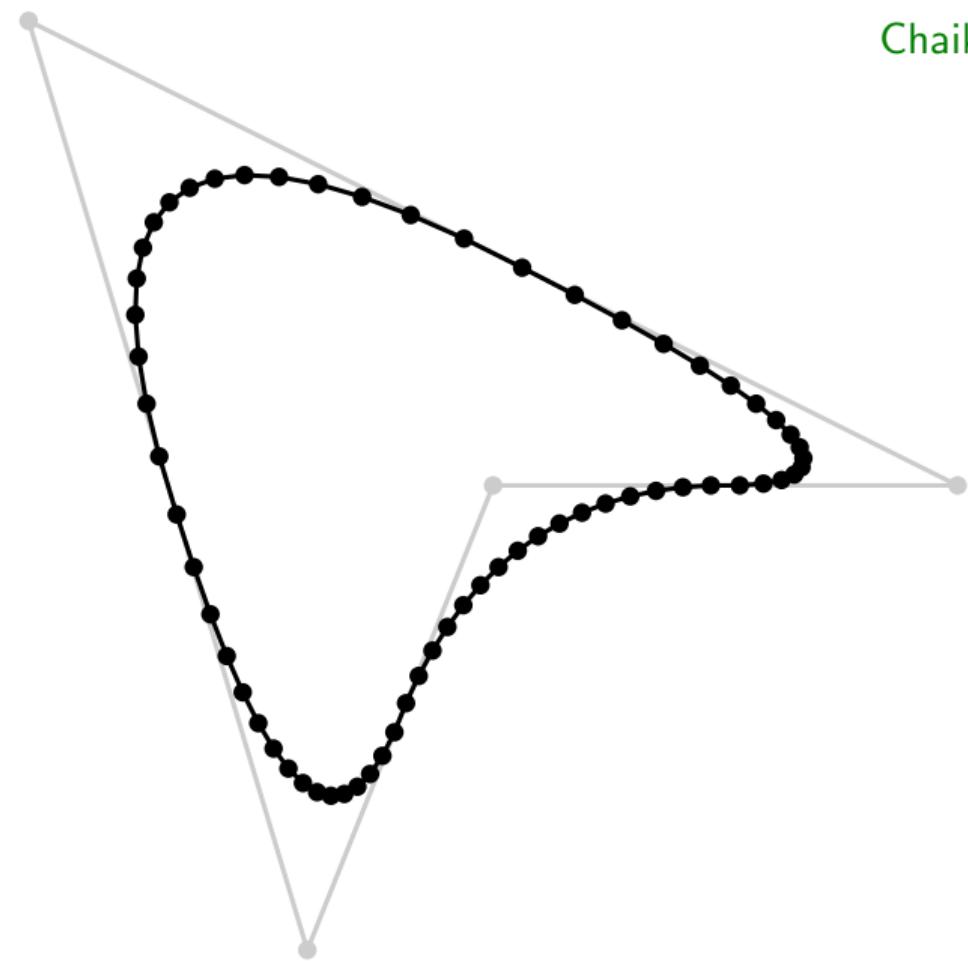
Curvas de subdivisão

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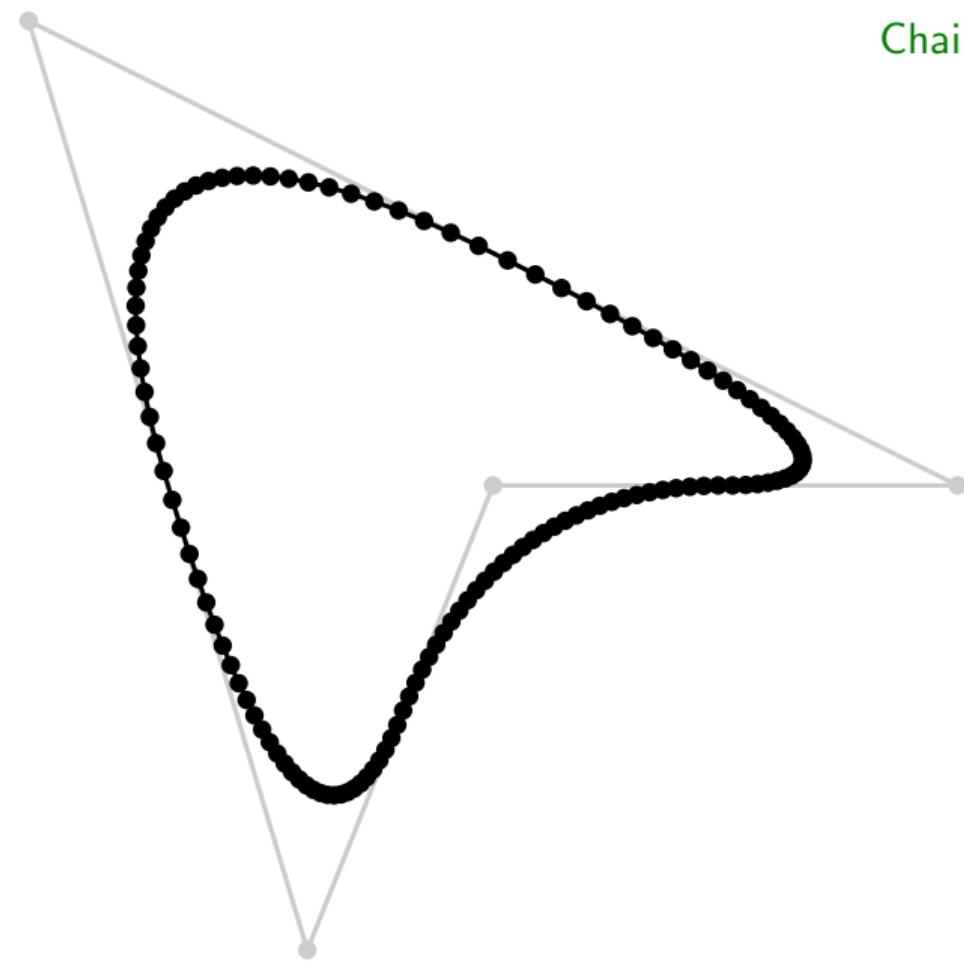
Curvas de subdivisão

Chaikin (1974)



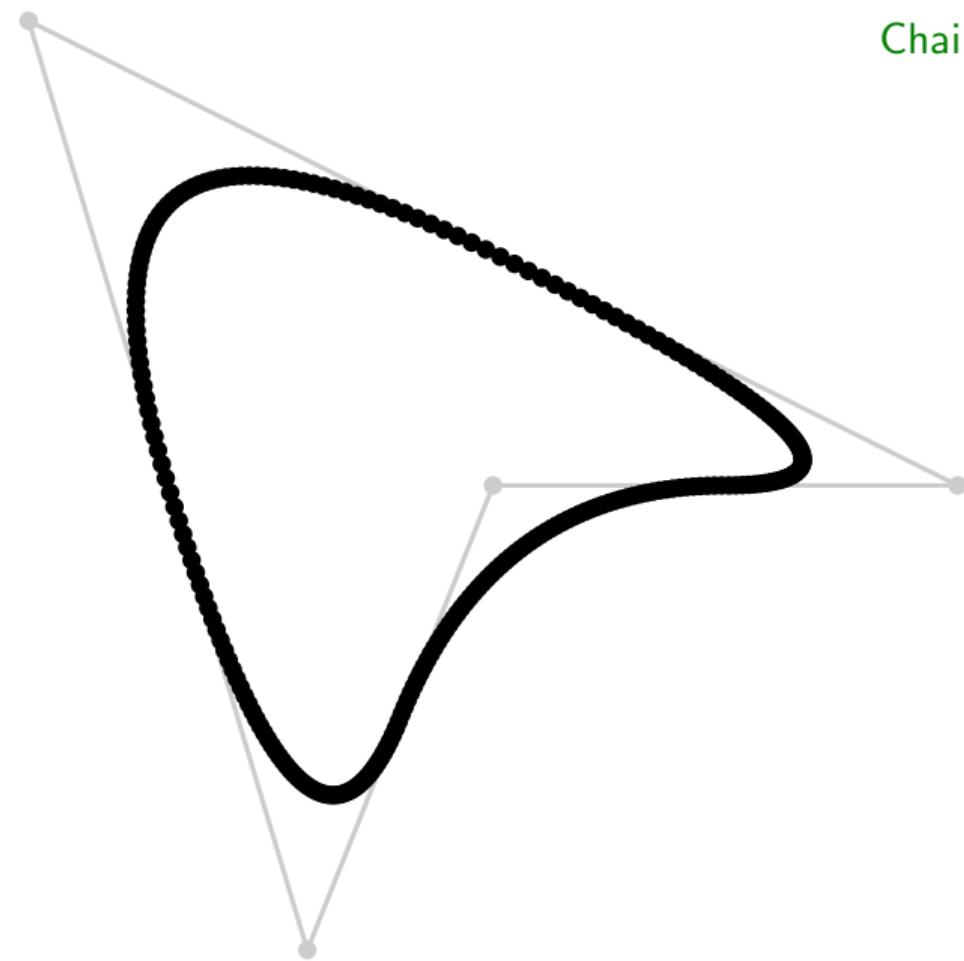
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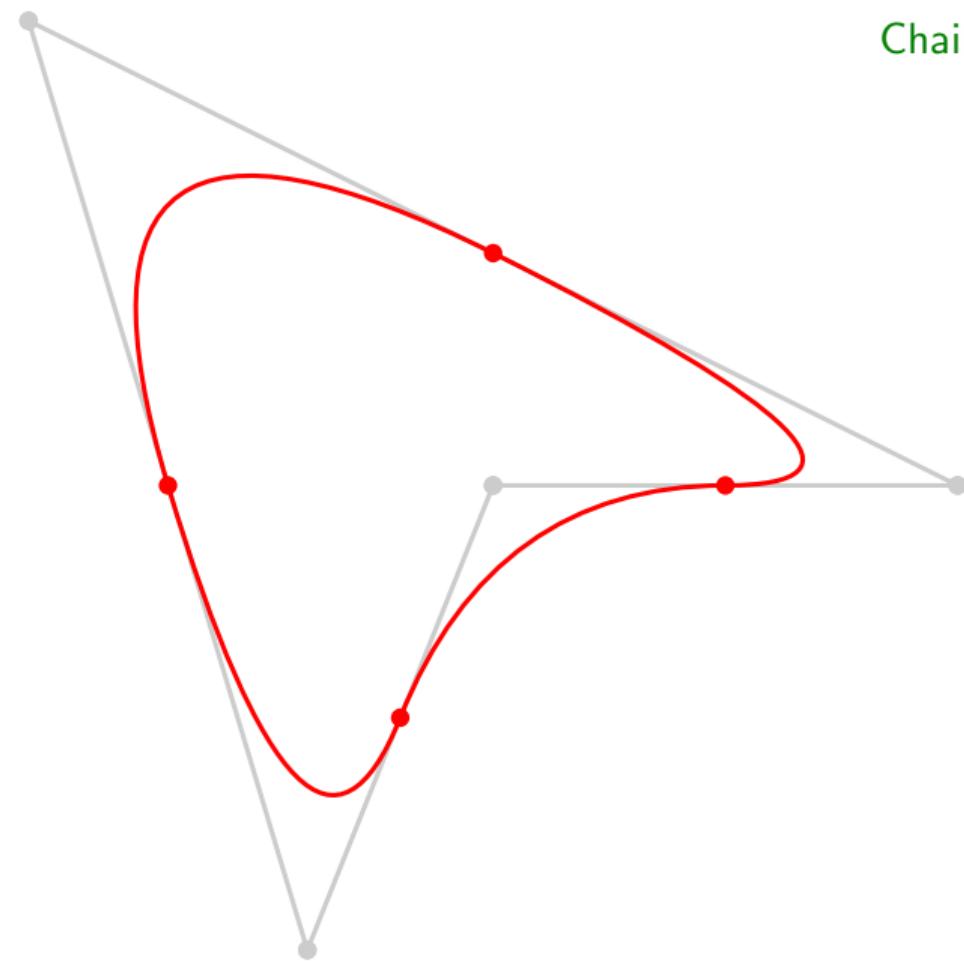
Curvas de subdivisão

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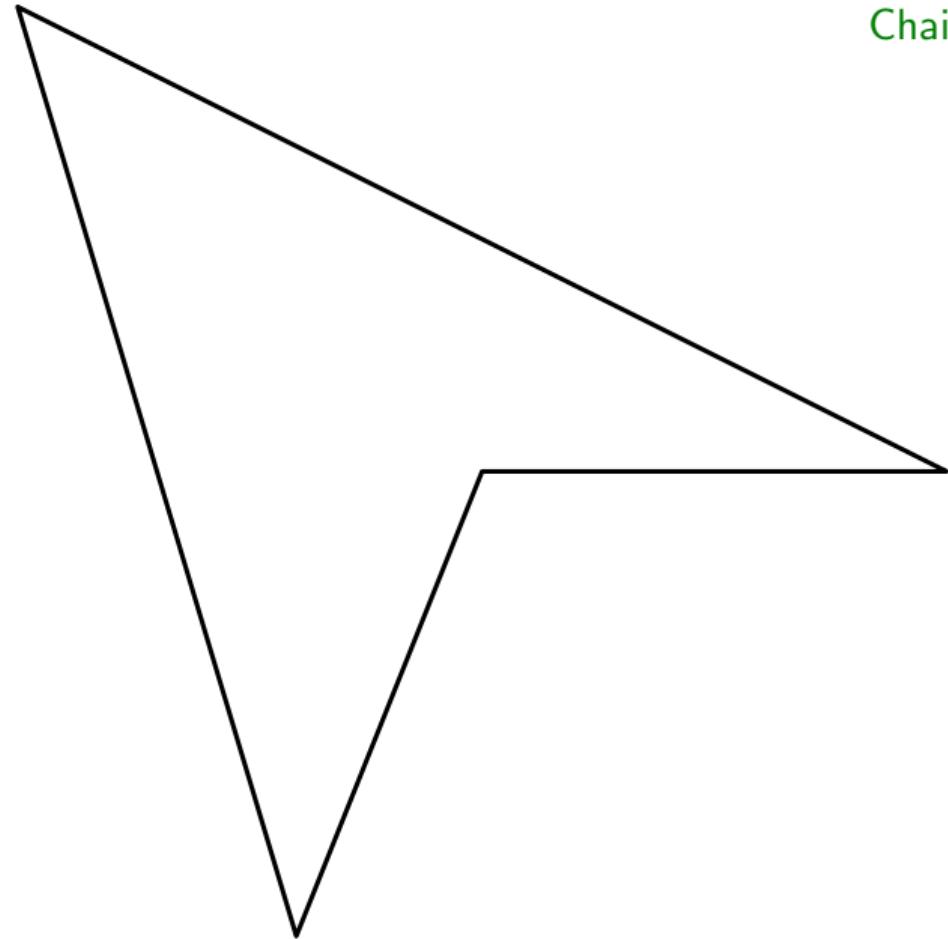
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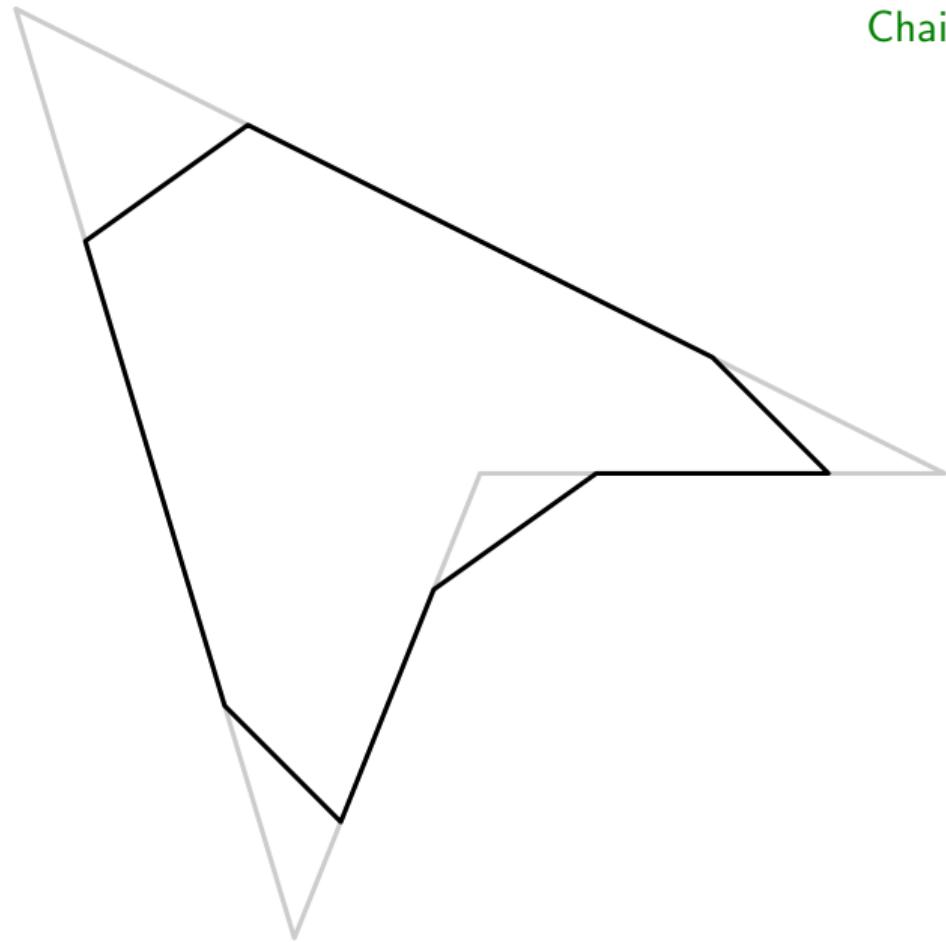
Curvas de subdivisão

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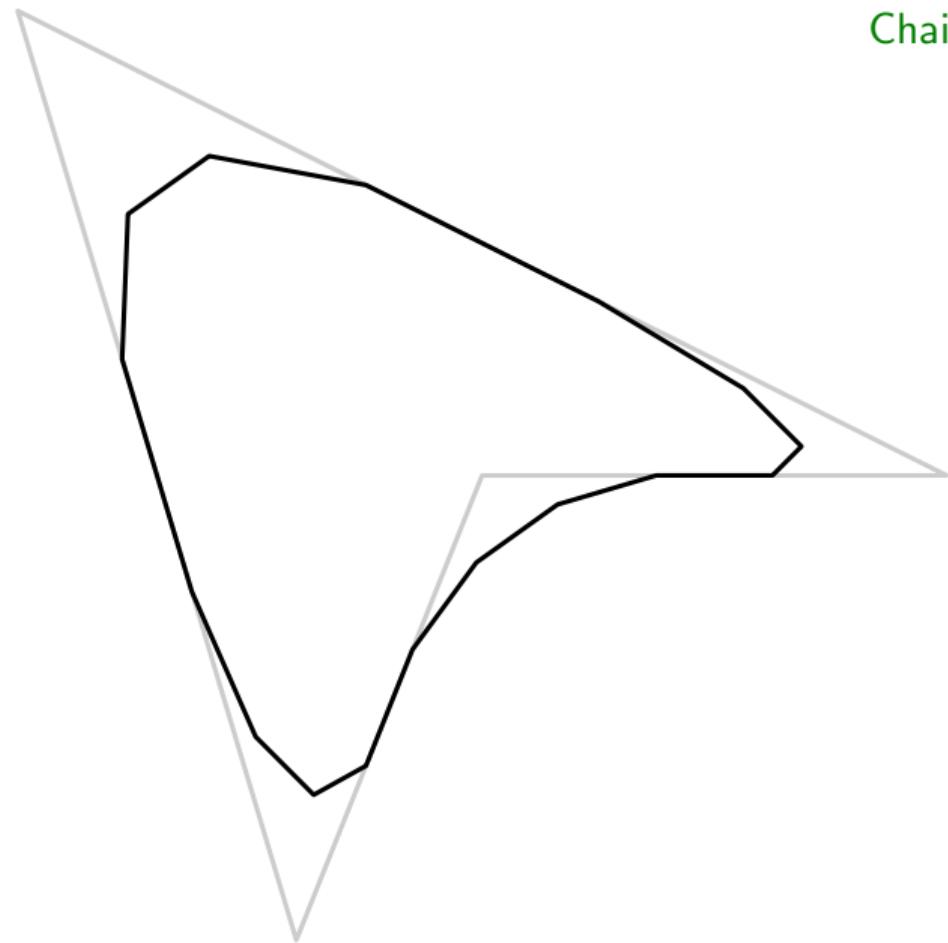
Curvas de subdivisão

Chaikin (1974)



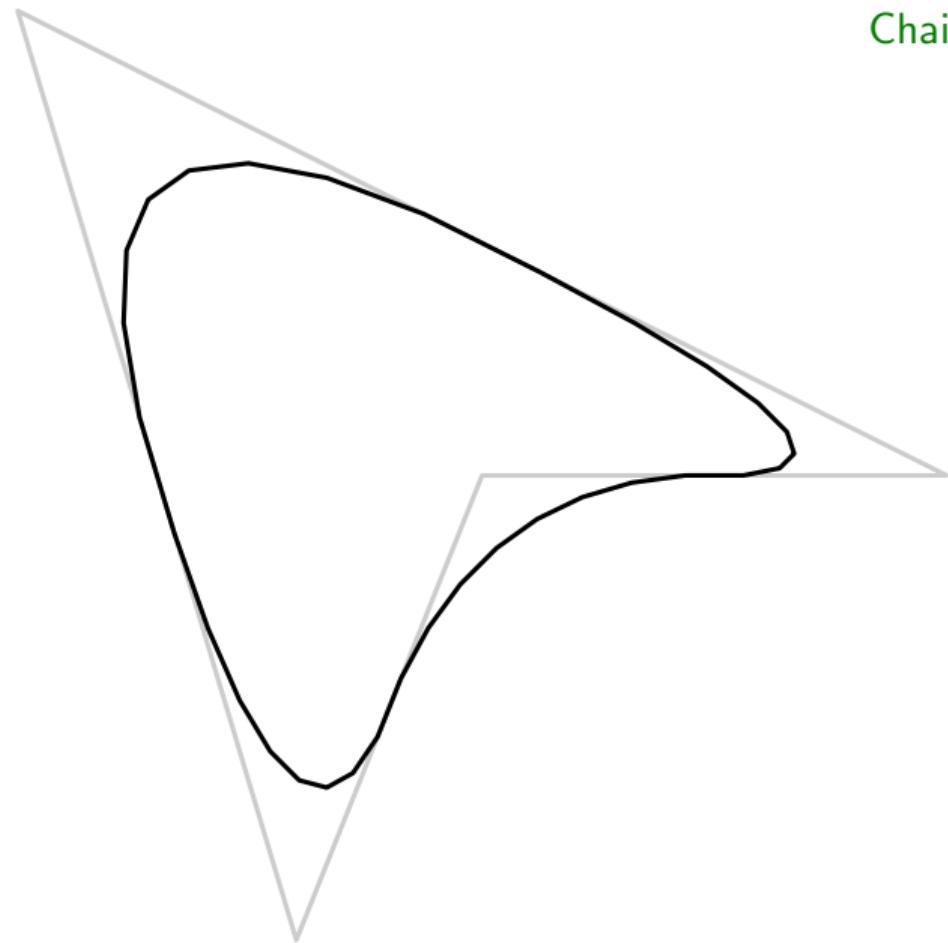
Curvas de subdivisão

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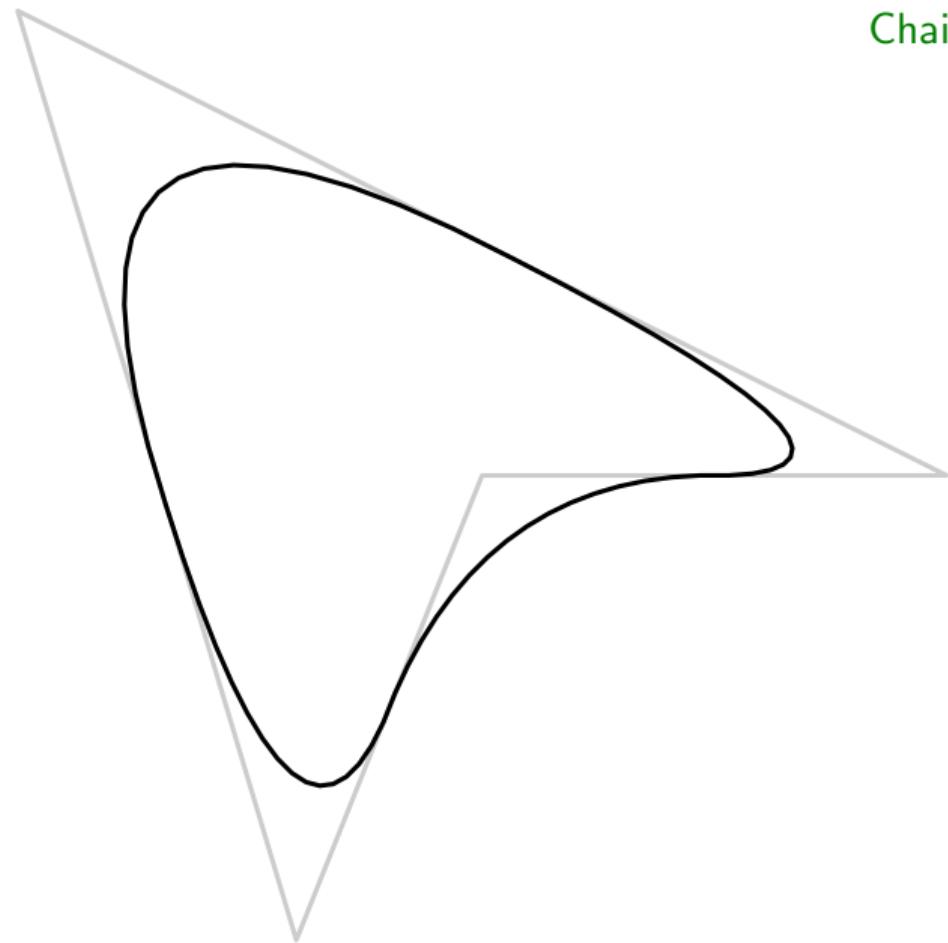
Curvas de subdivisão

Chaikin (1974)



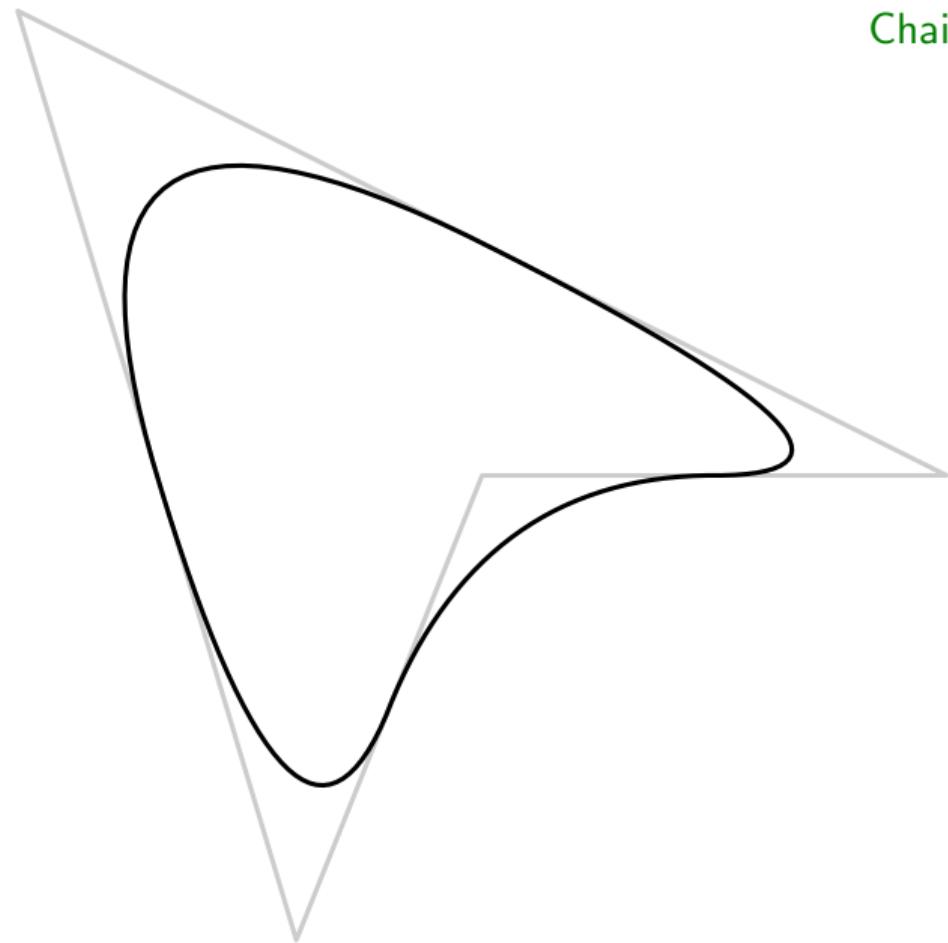
Curvas de subdivisão

Chaikin (1974)



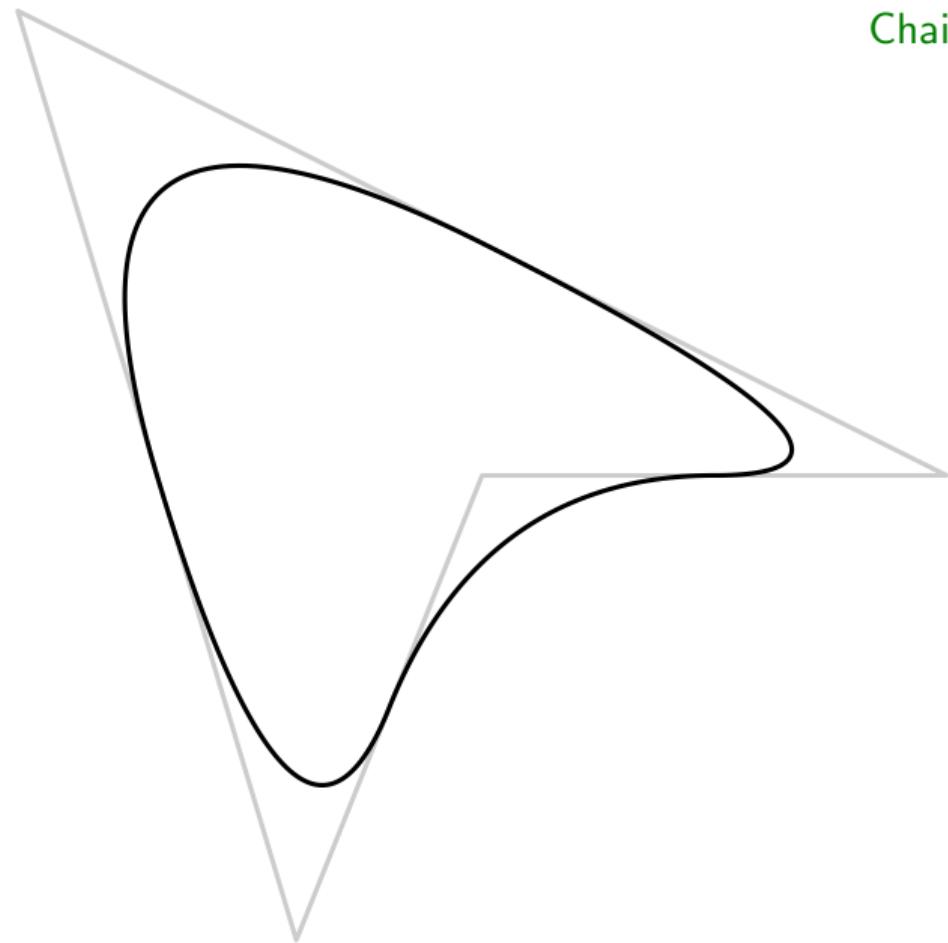
Curvas de subdivisão

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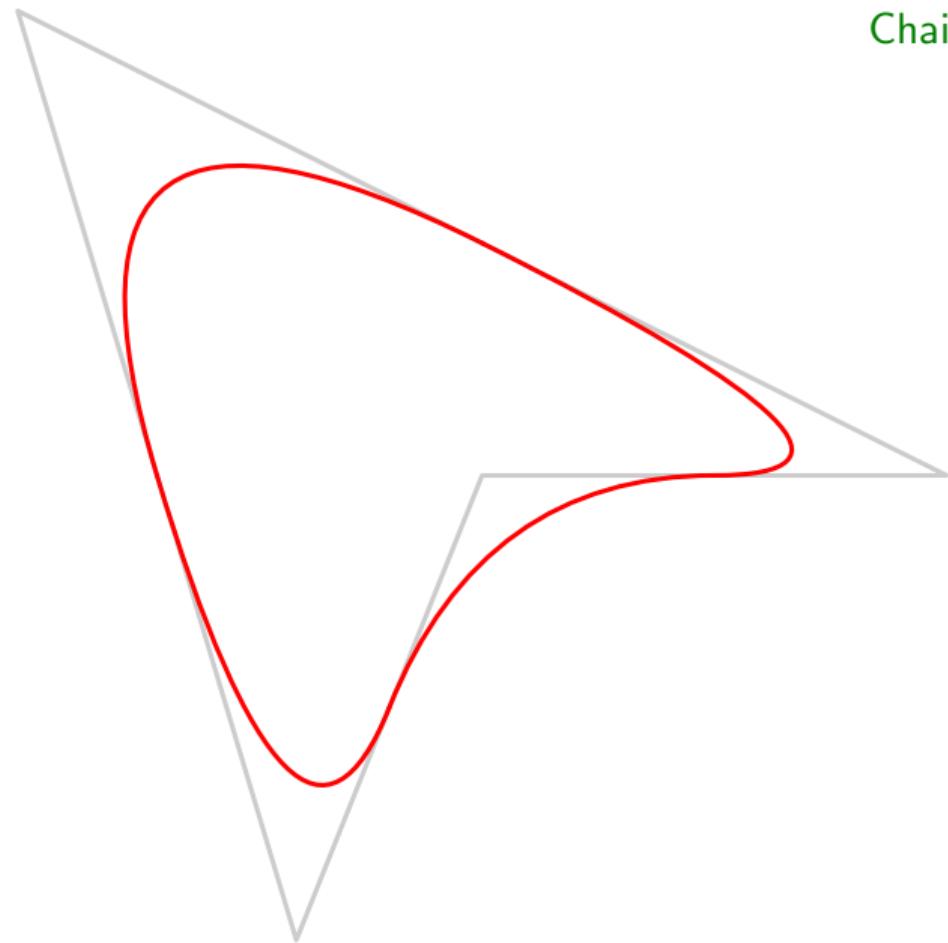
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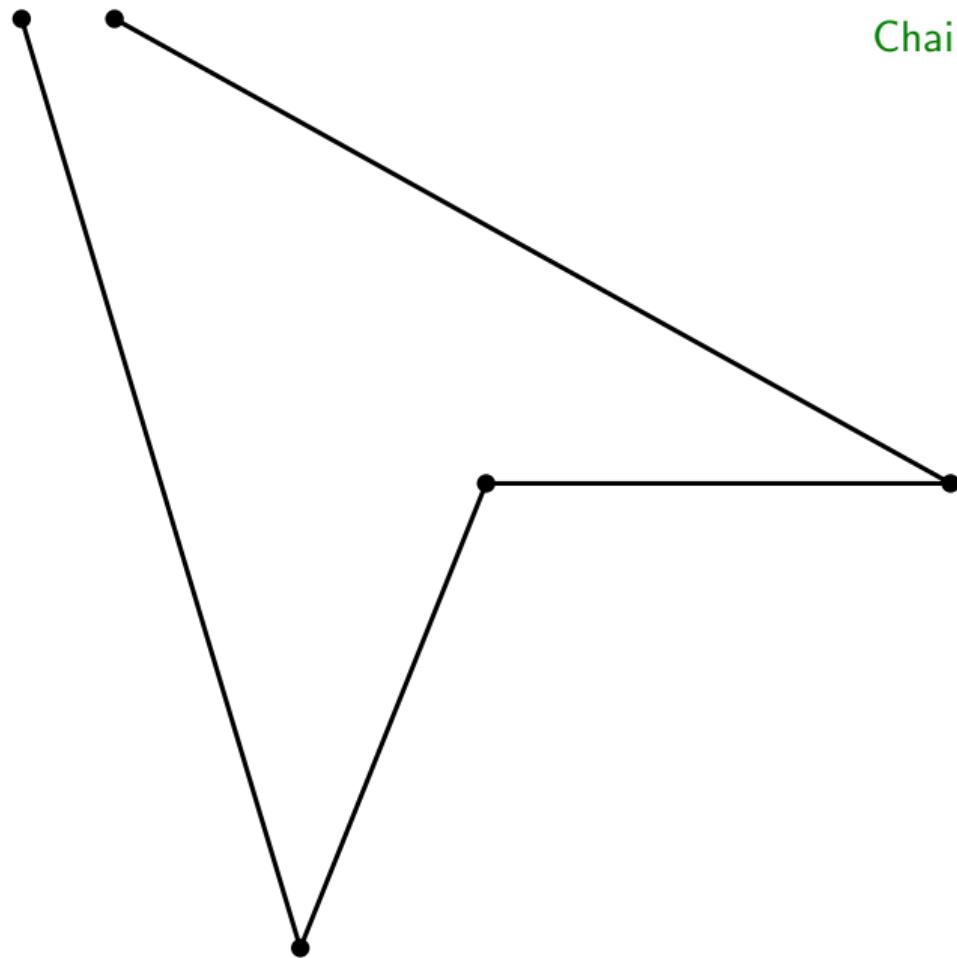
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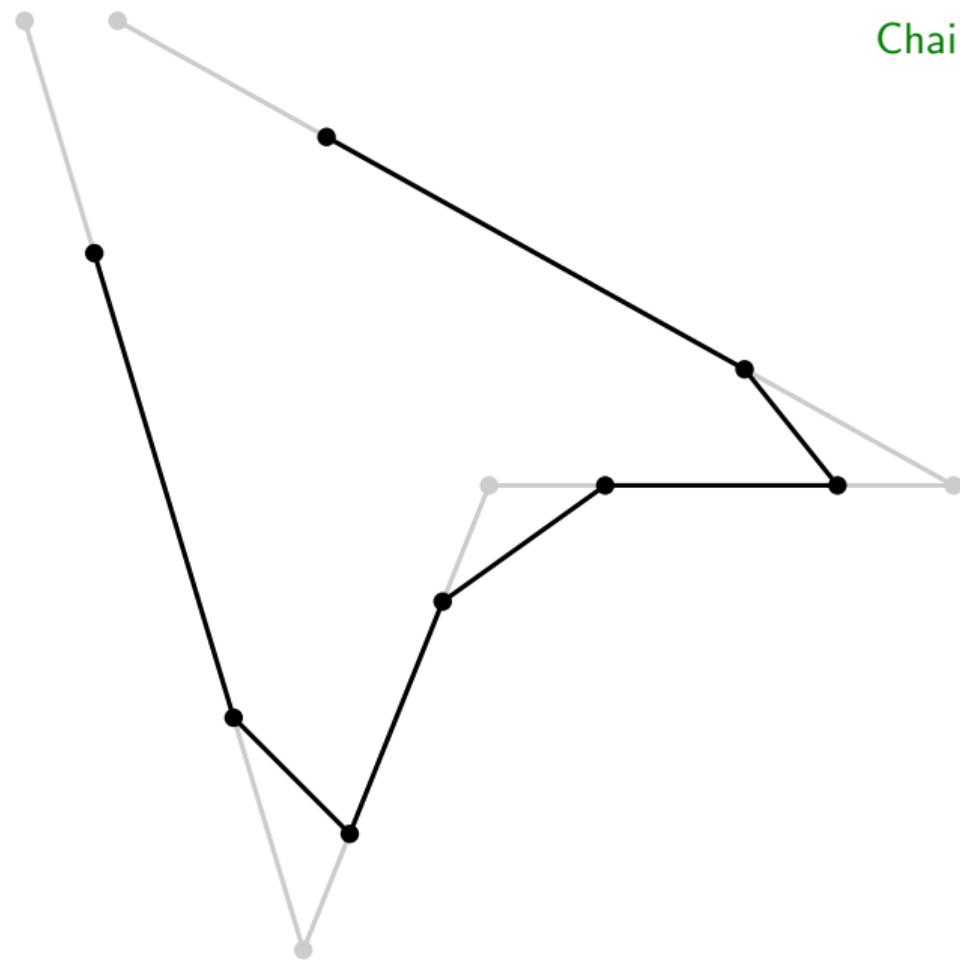
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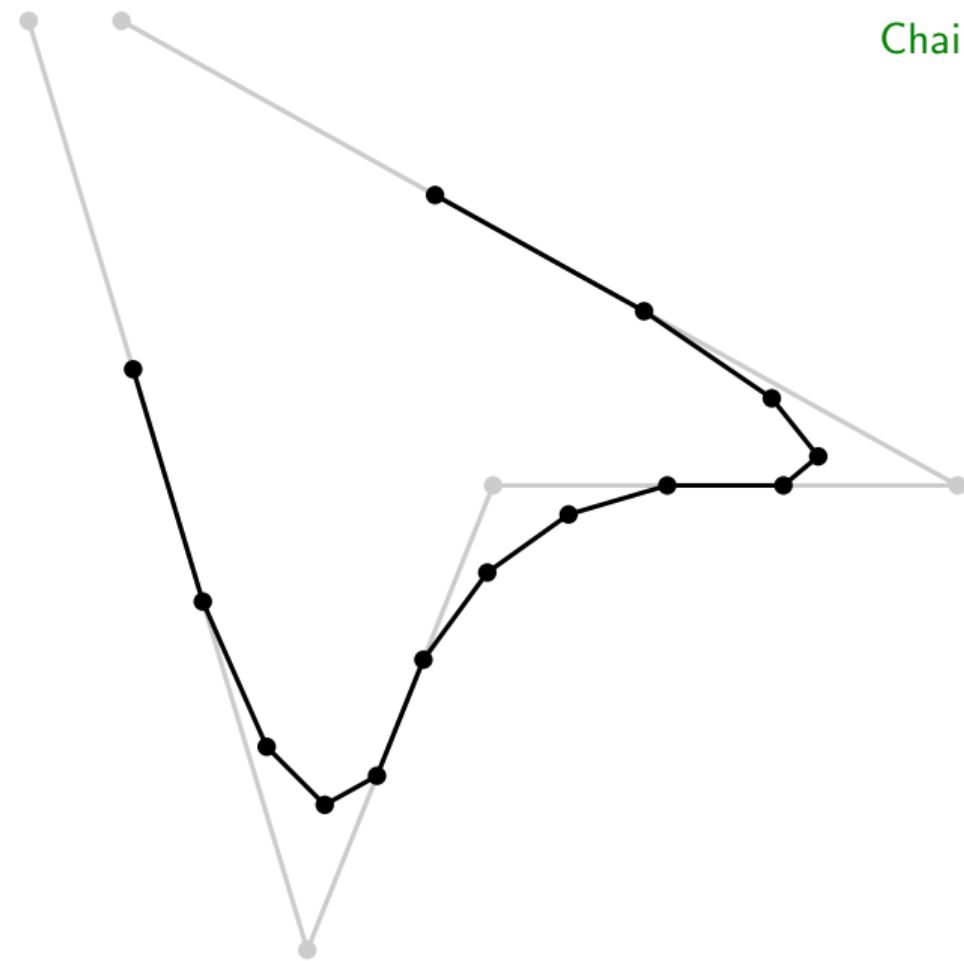
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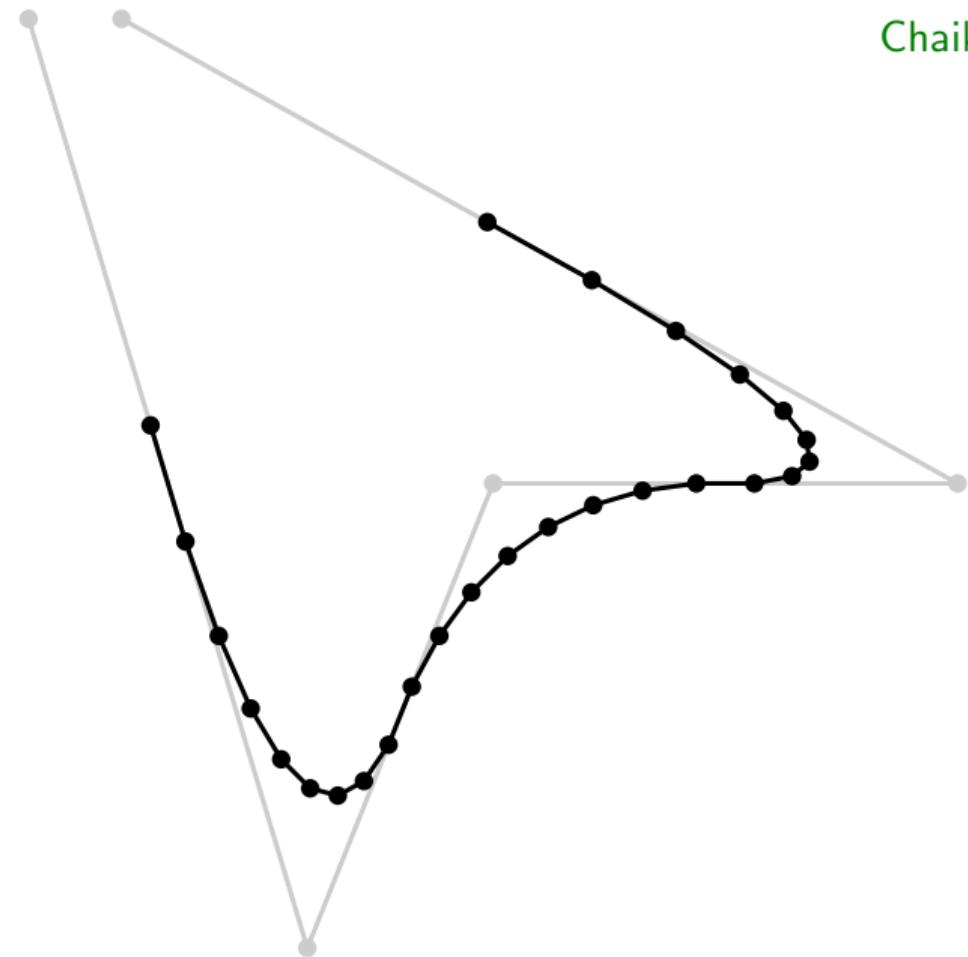
Curvas de subdivisão

Chaikin (1974)



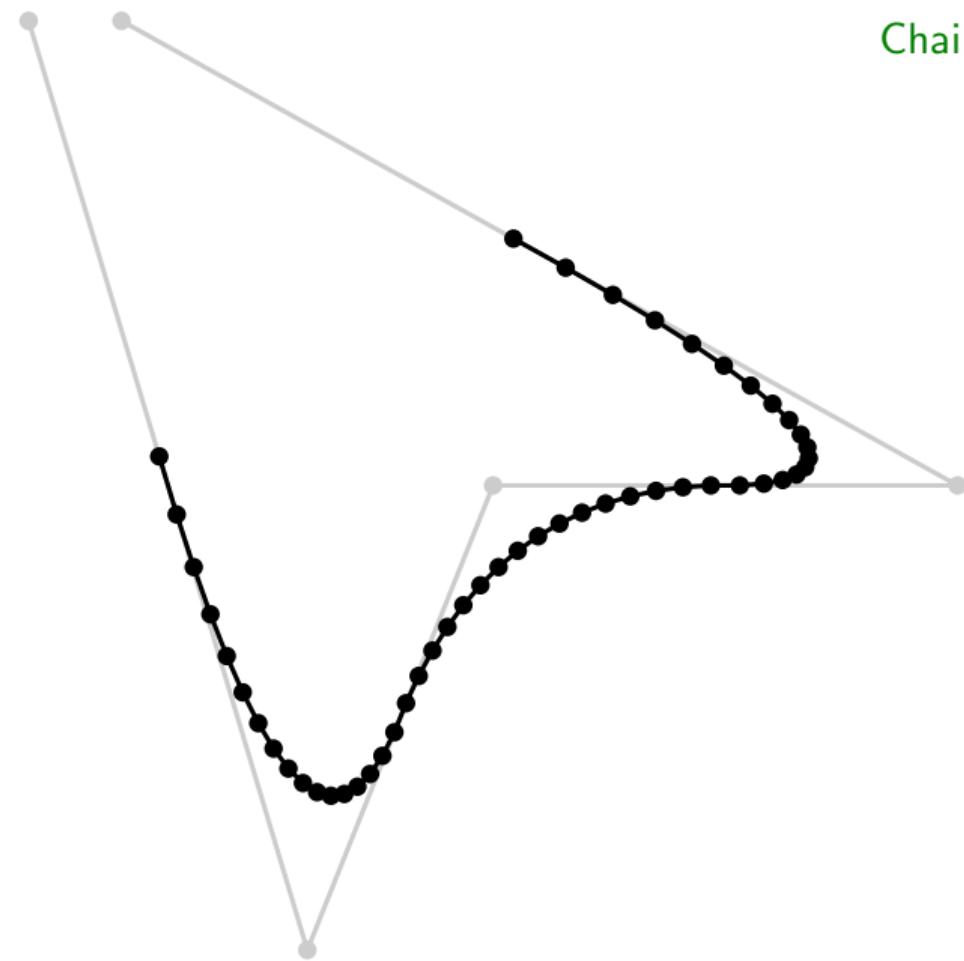
Curvas de subdivisão

Chaikin (1974)



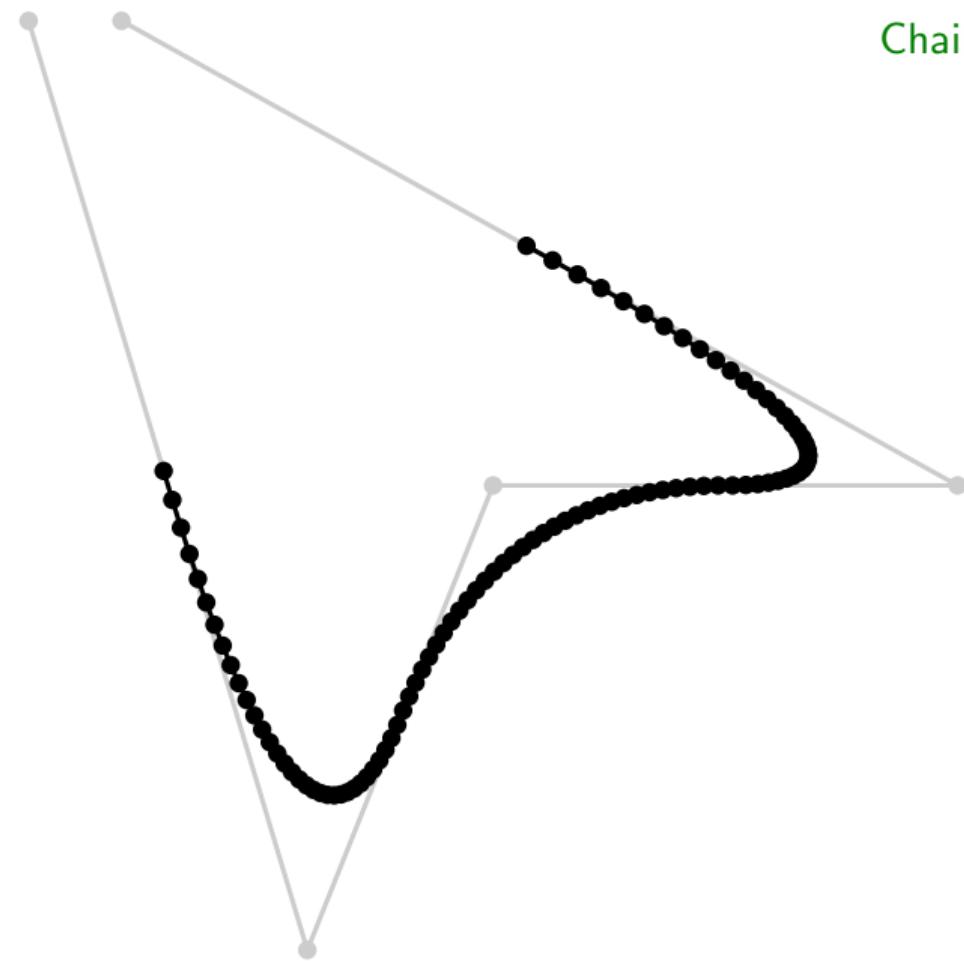
Curvas de subdivisão

Chaikin (1974)



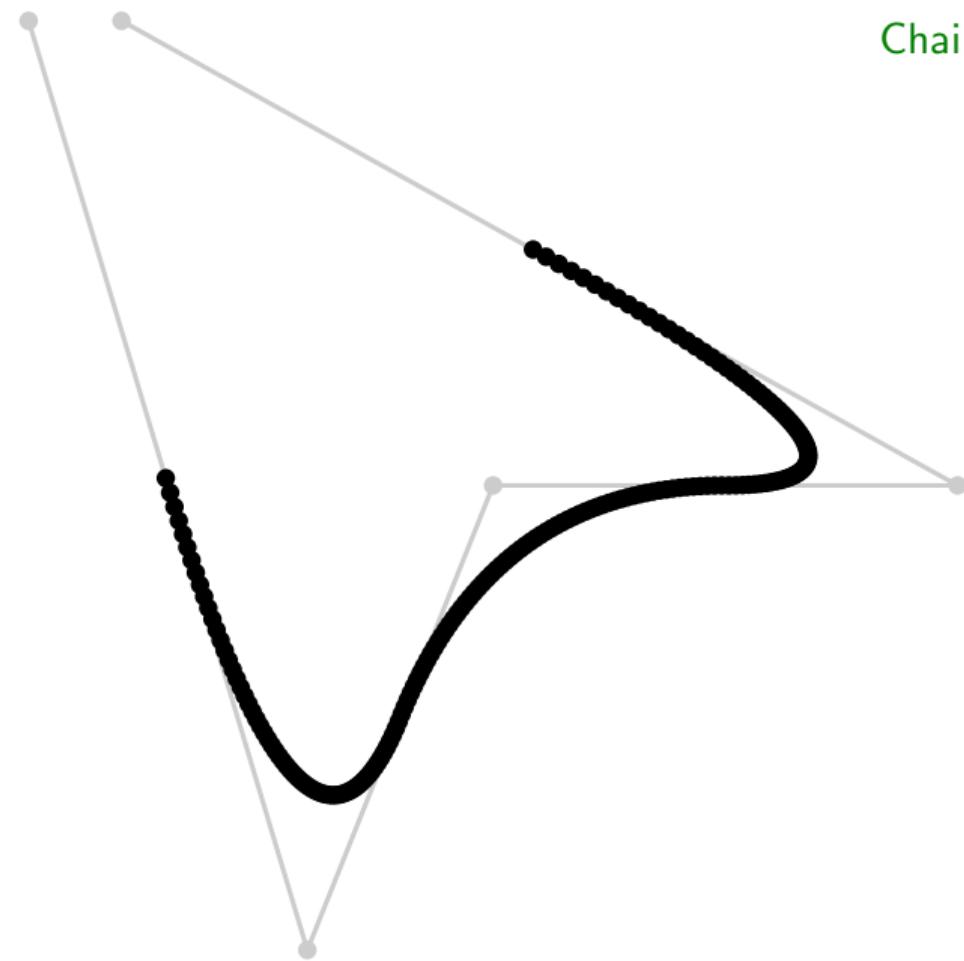
Curvas de subdivisão

Chaikin (1974)



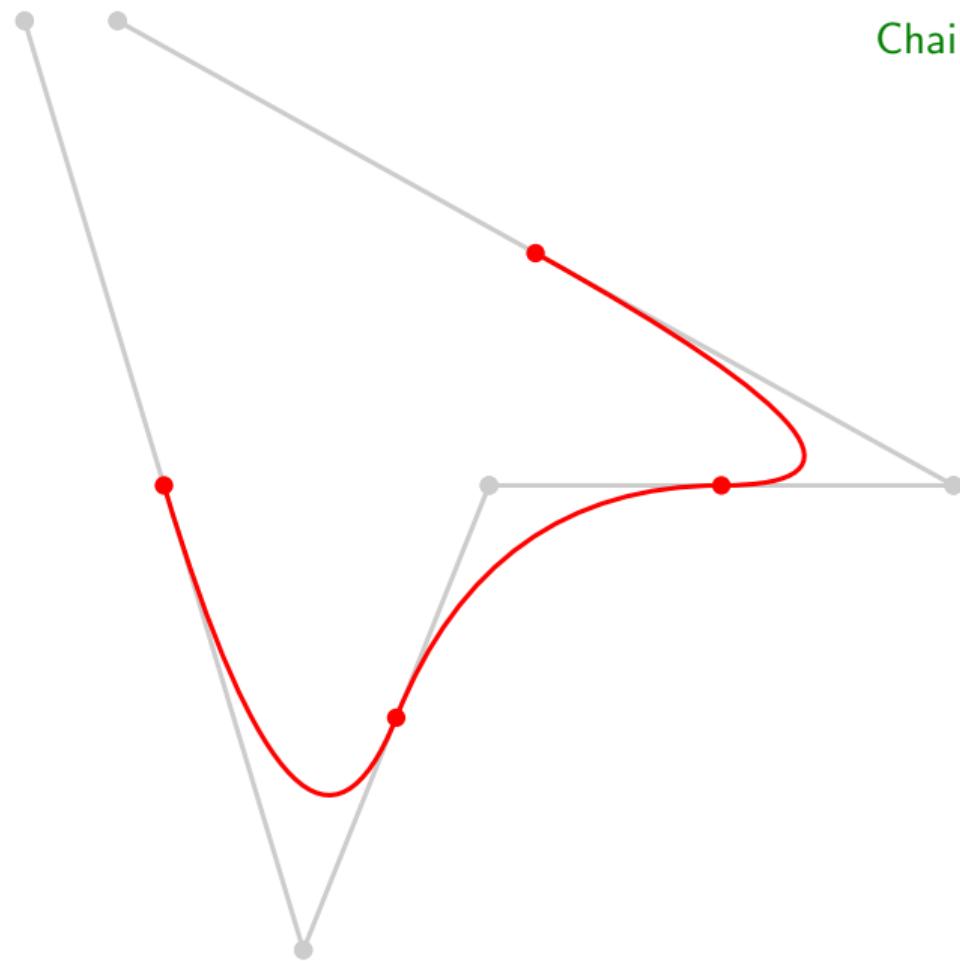
Curvas de subdivisão

Chaikin (1974)



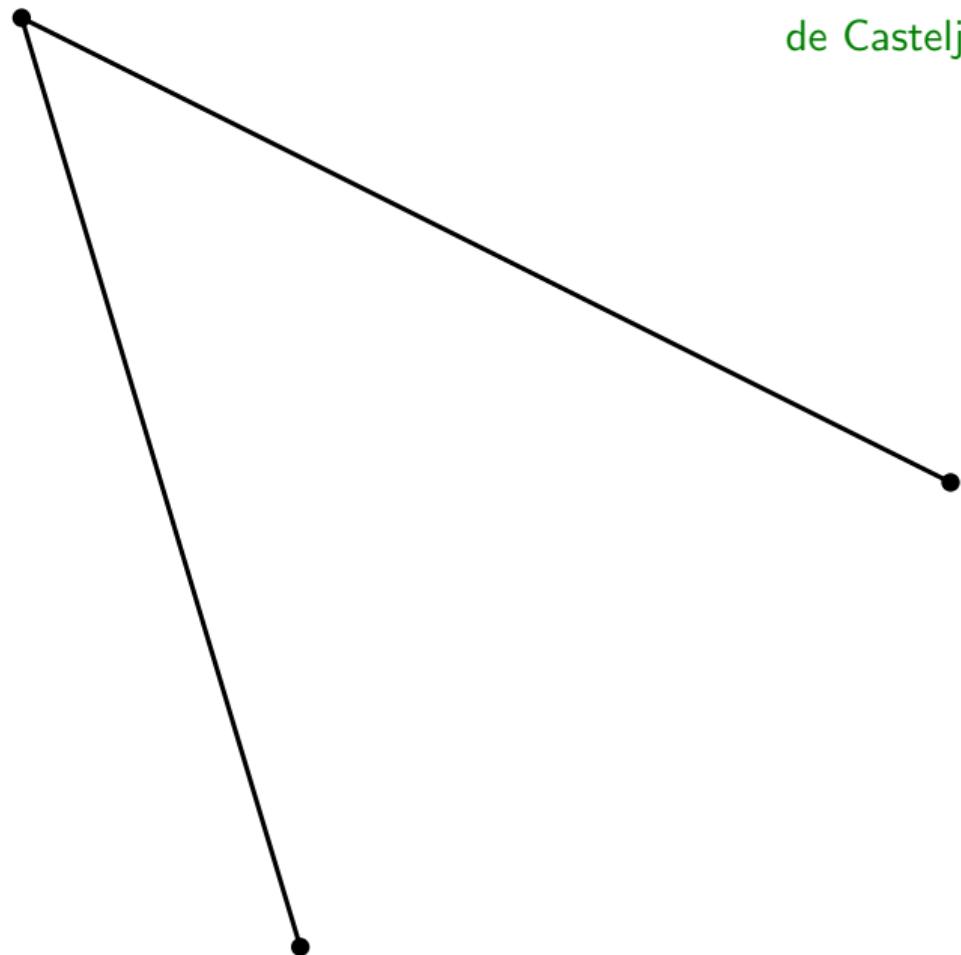
Curvas de subdivisão

Chaikin (1974)



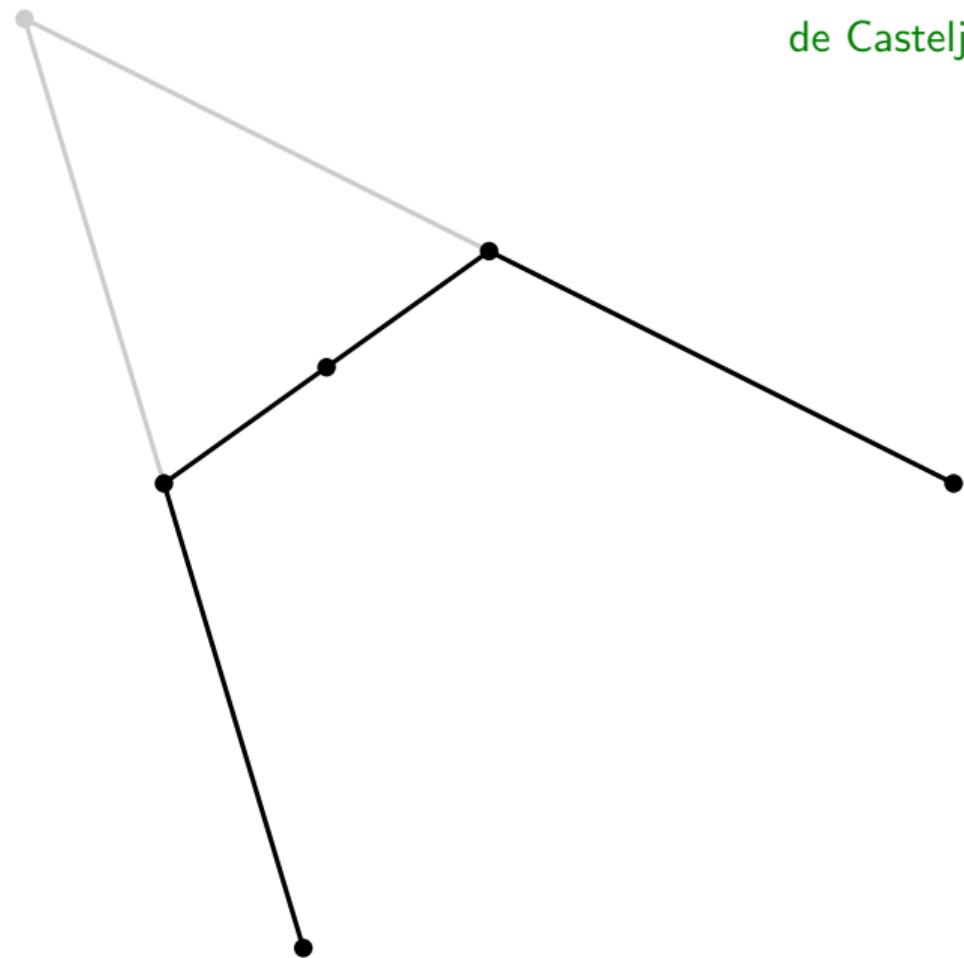
Curvas de subdivisão

de Casteljau (1963)



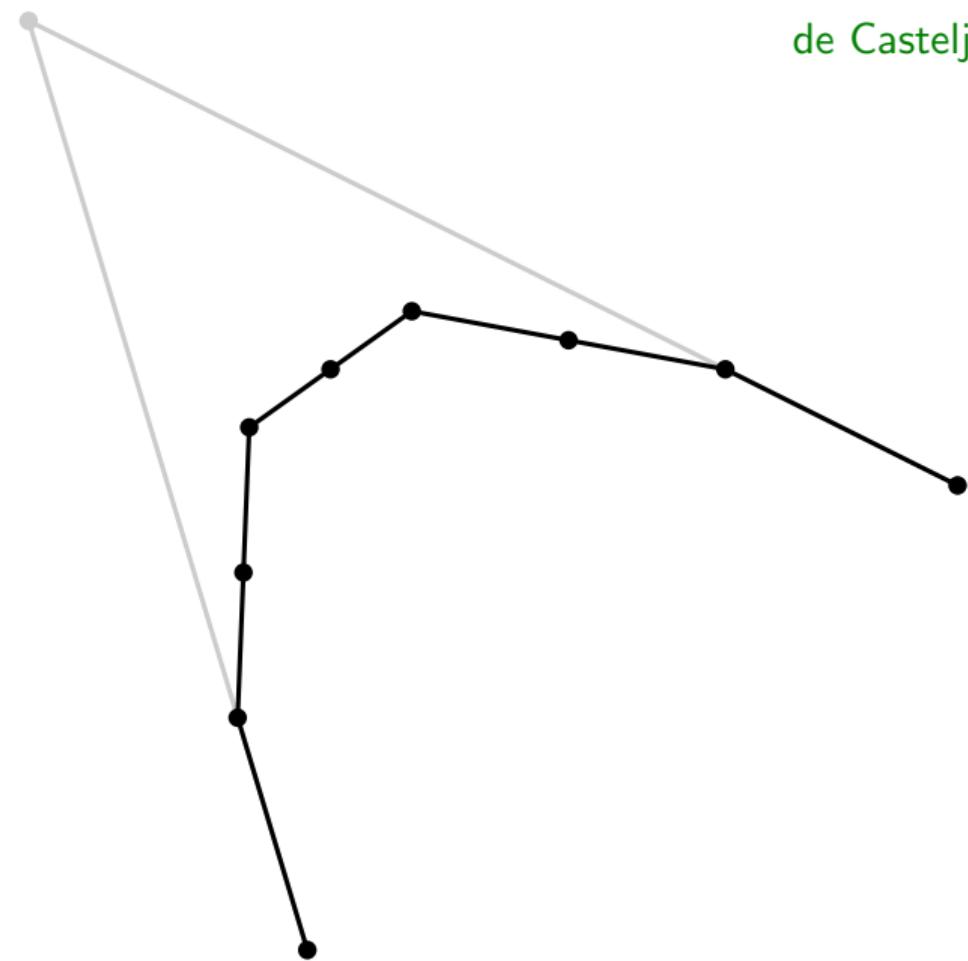
Curvas de subdivisão

de Casteljau (1963)



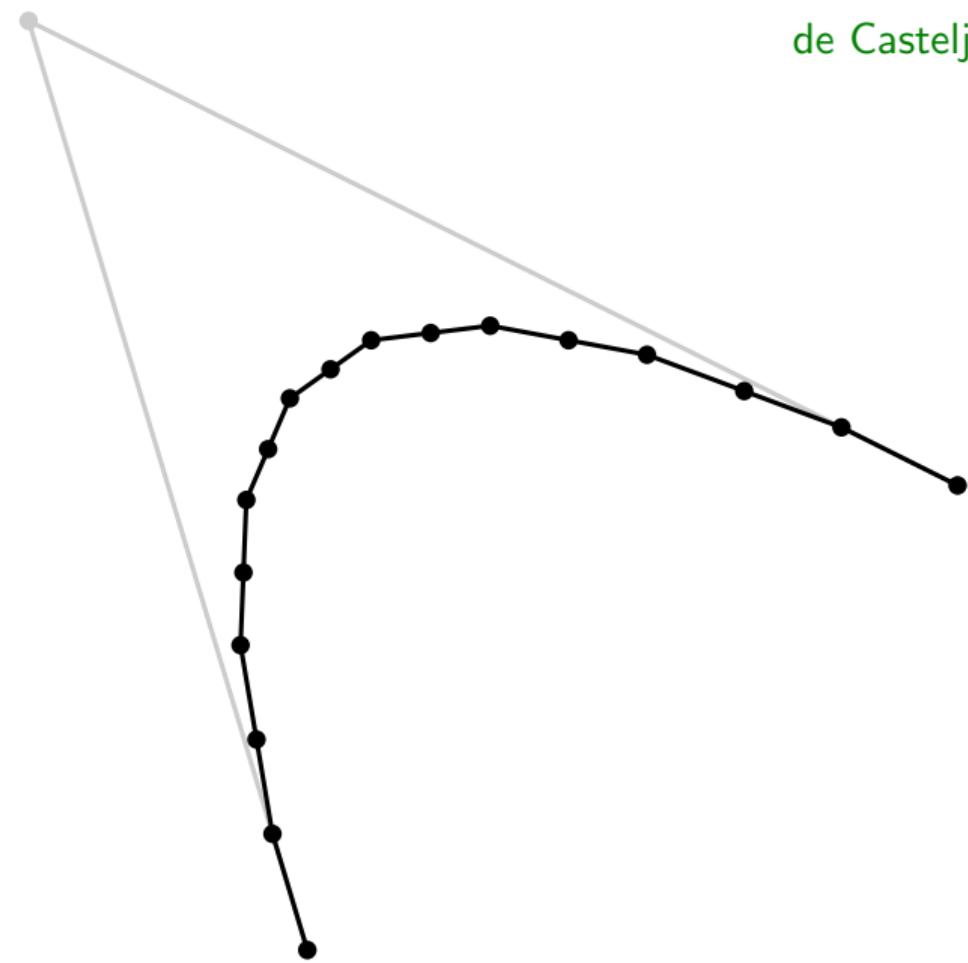
Curvas de subdivisão

de Casteljau (1963)



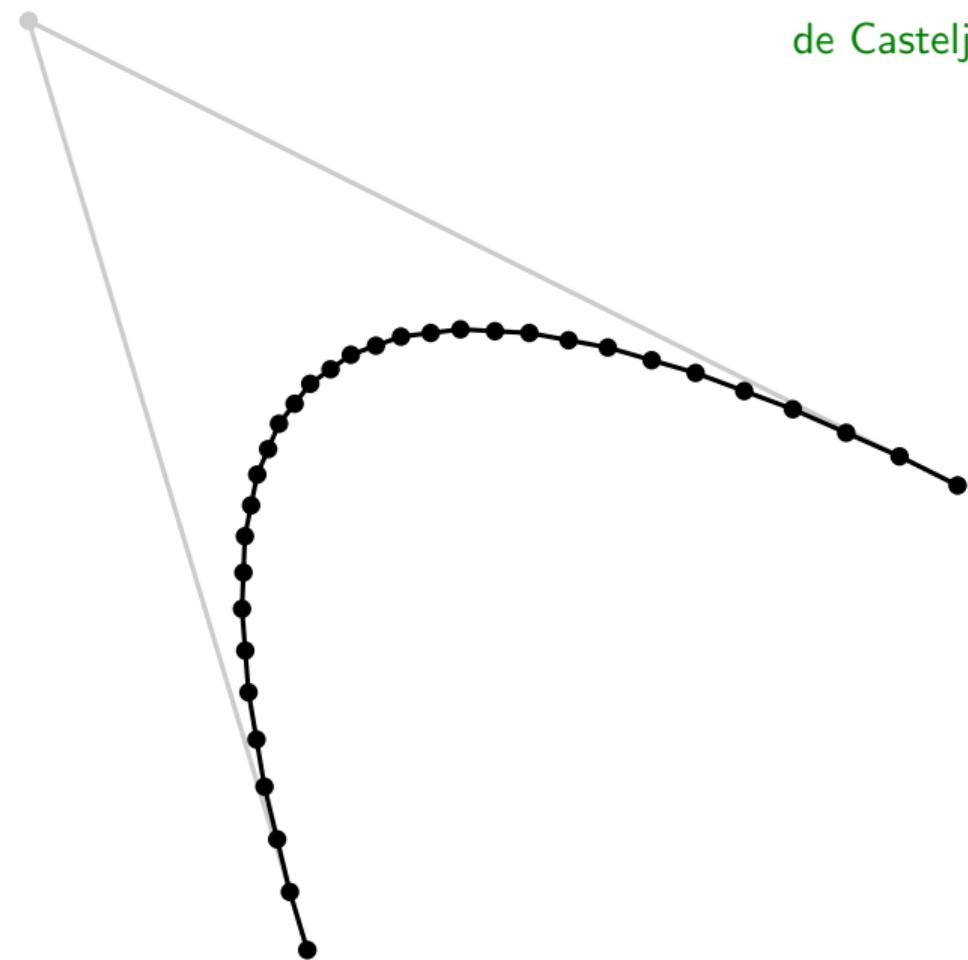
Curvas de subdivisão

de Casteljau (1963)



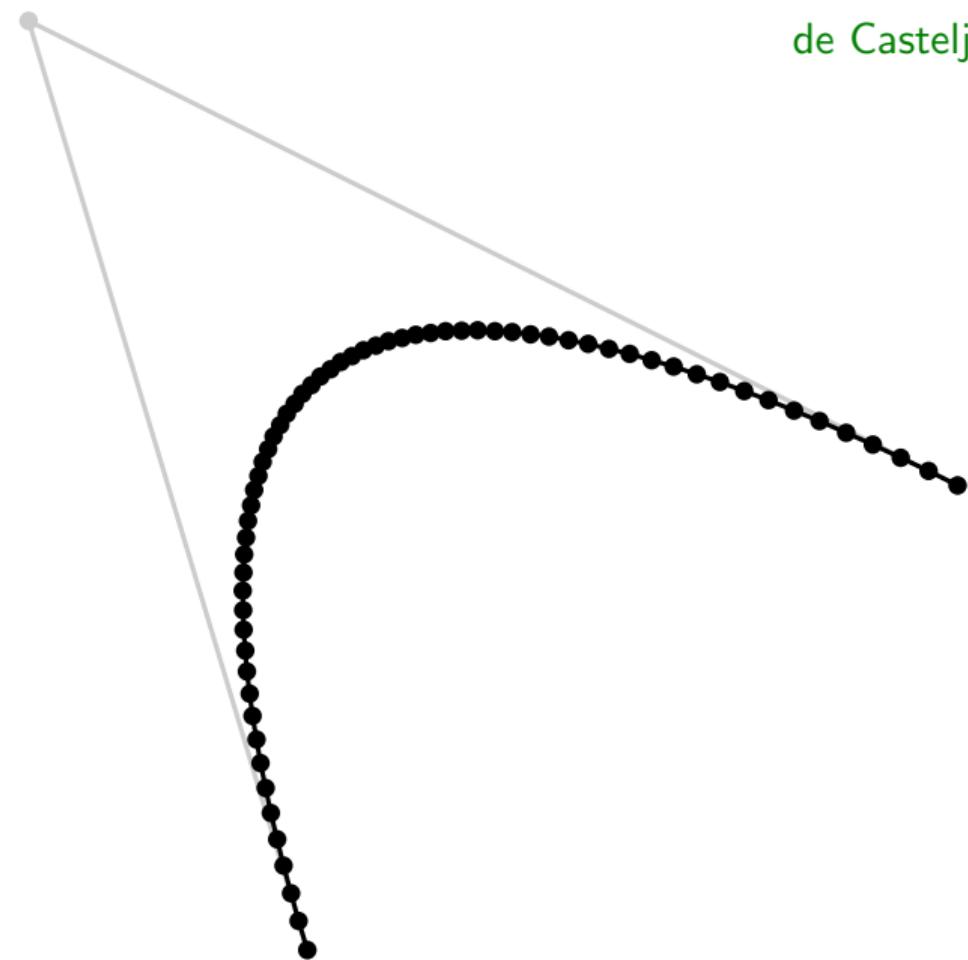
Curvas de subdivisão

de Casteljau (1963)



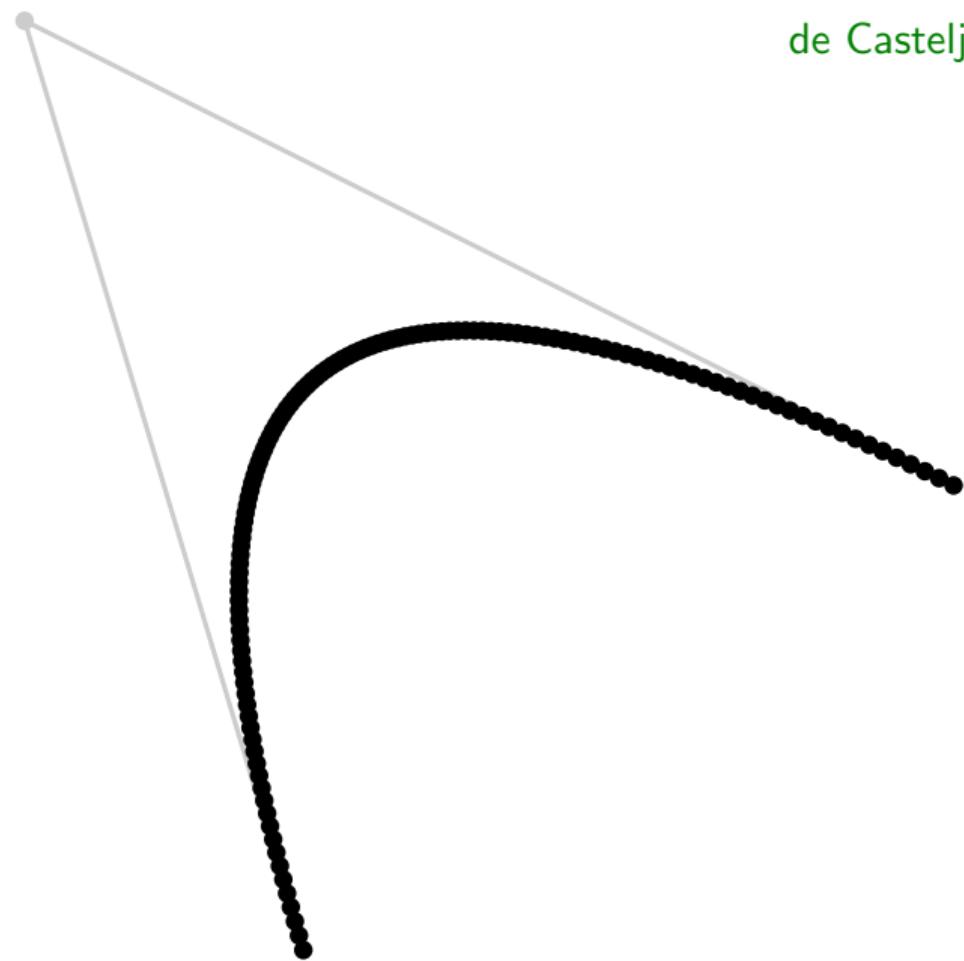
Curvas de subdivisão

de Casteljau (1963)



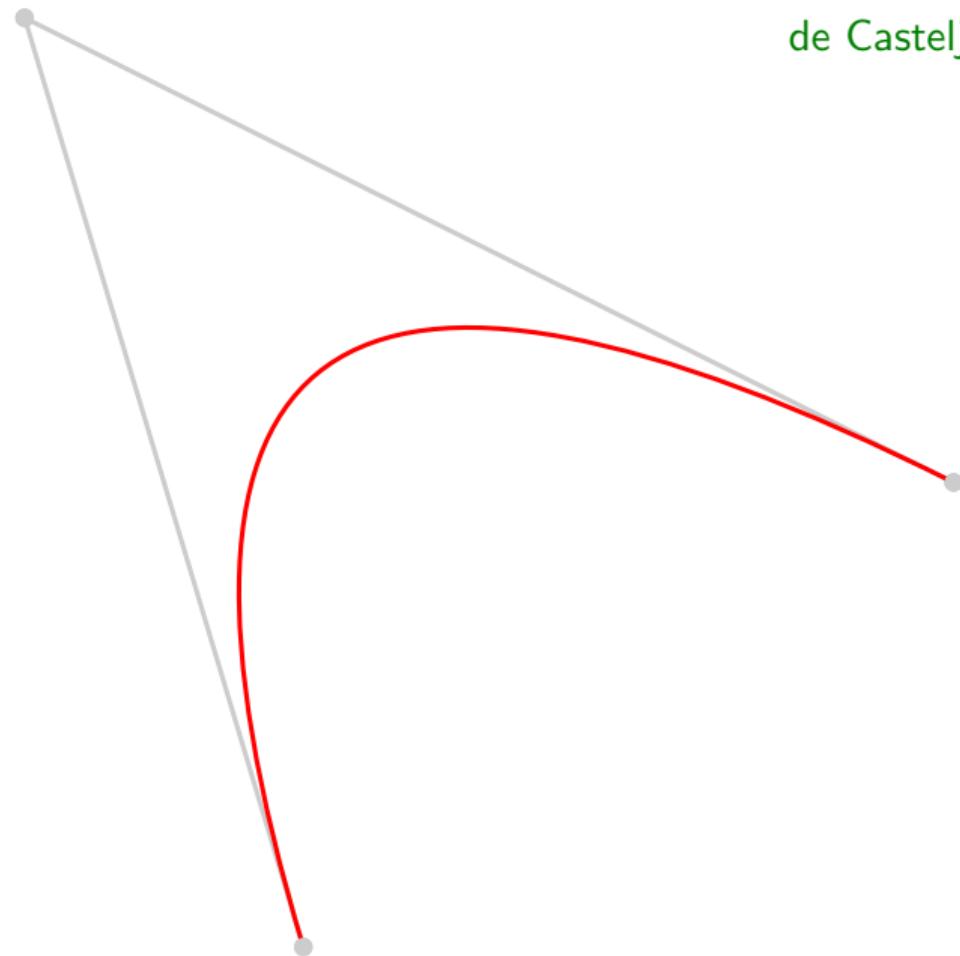
Curvas de subdivisão

de Casteljau (1963)



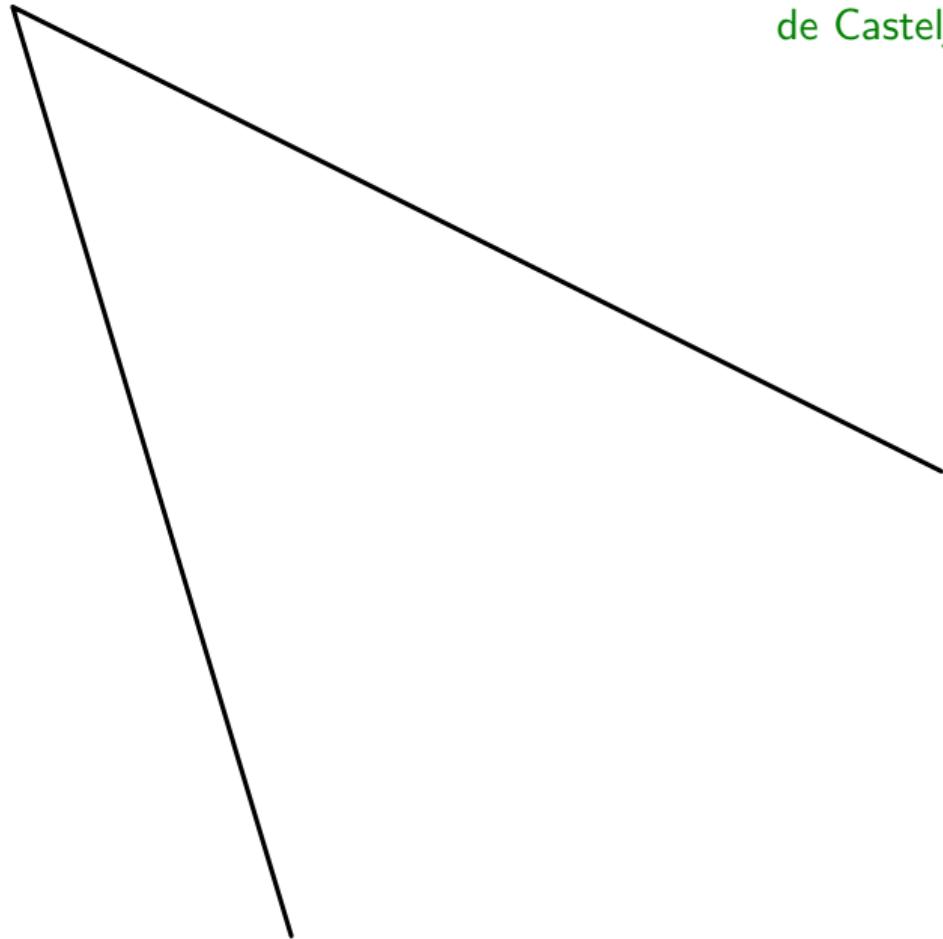
Curvas de subdivisão

de Casteljau (1963)



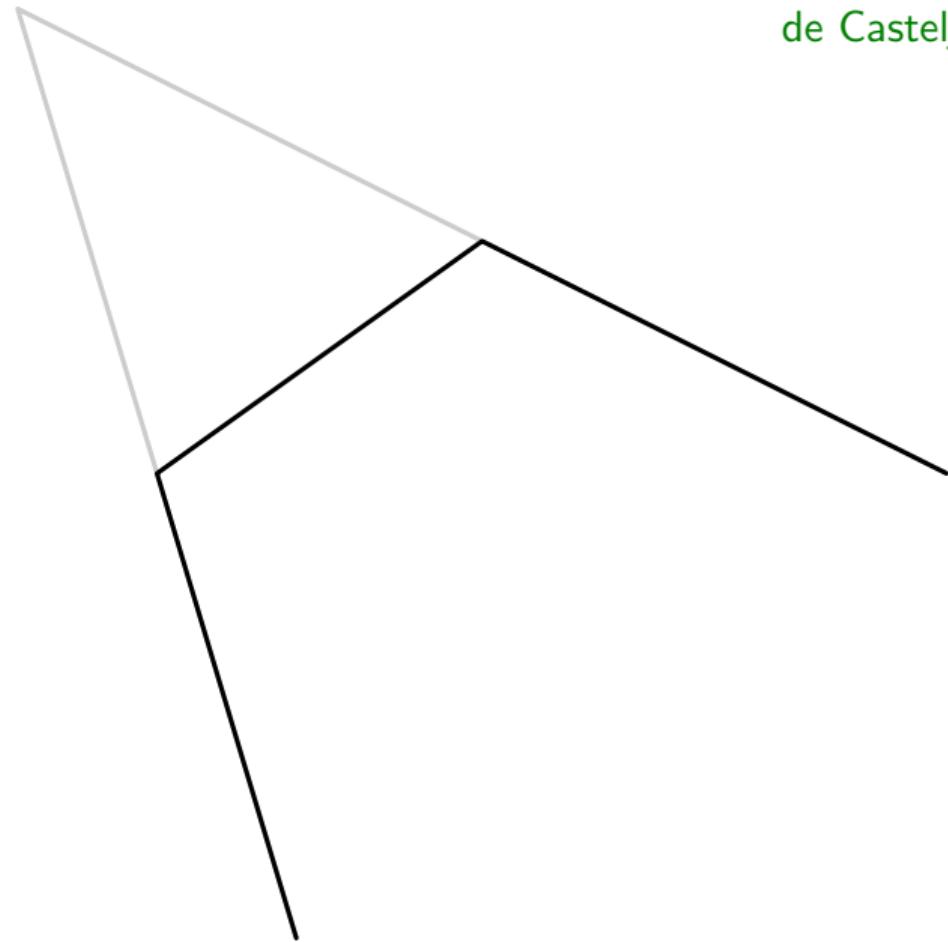
Curvas de subdivisão

de Casteljau (1963)



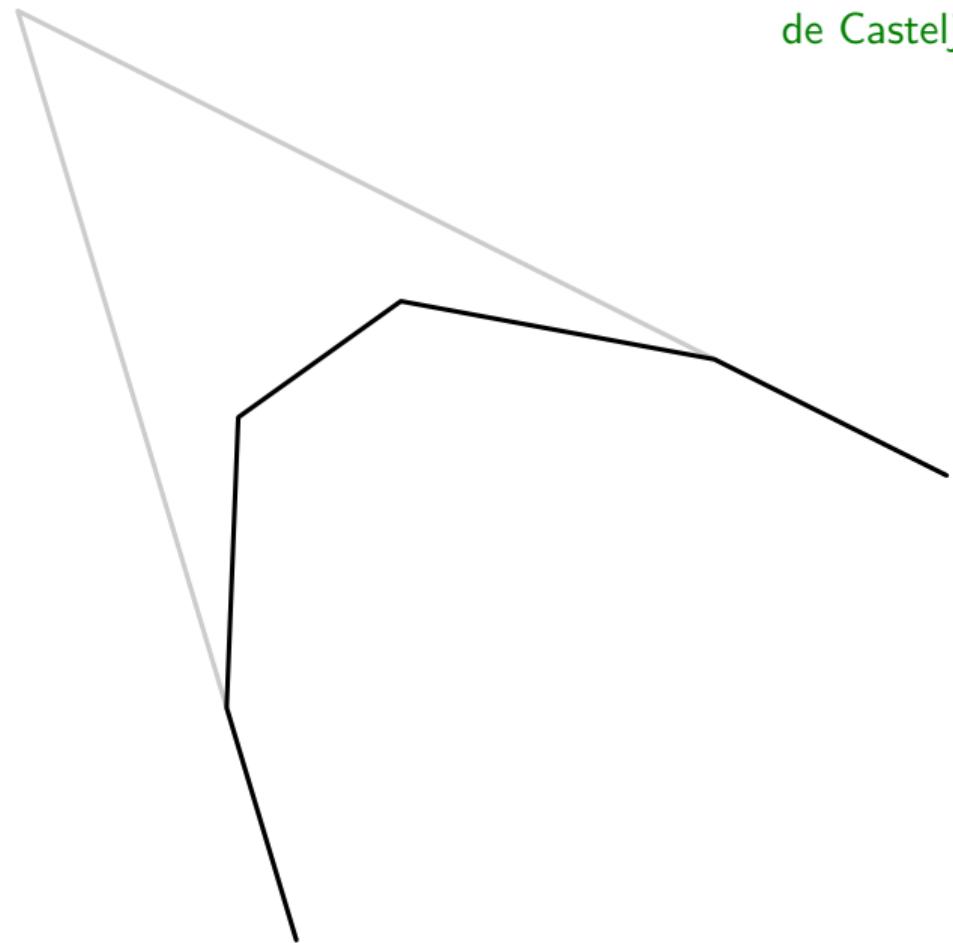
Curvas de subdivisão

de Casteljau (1963)



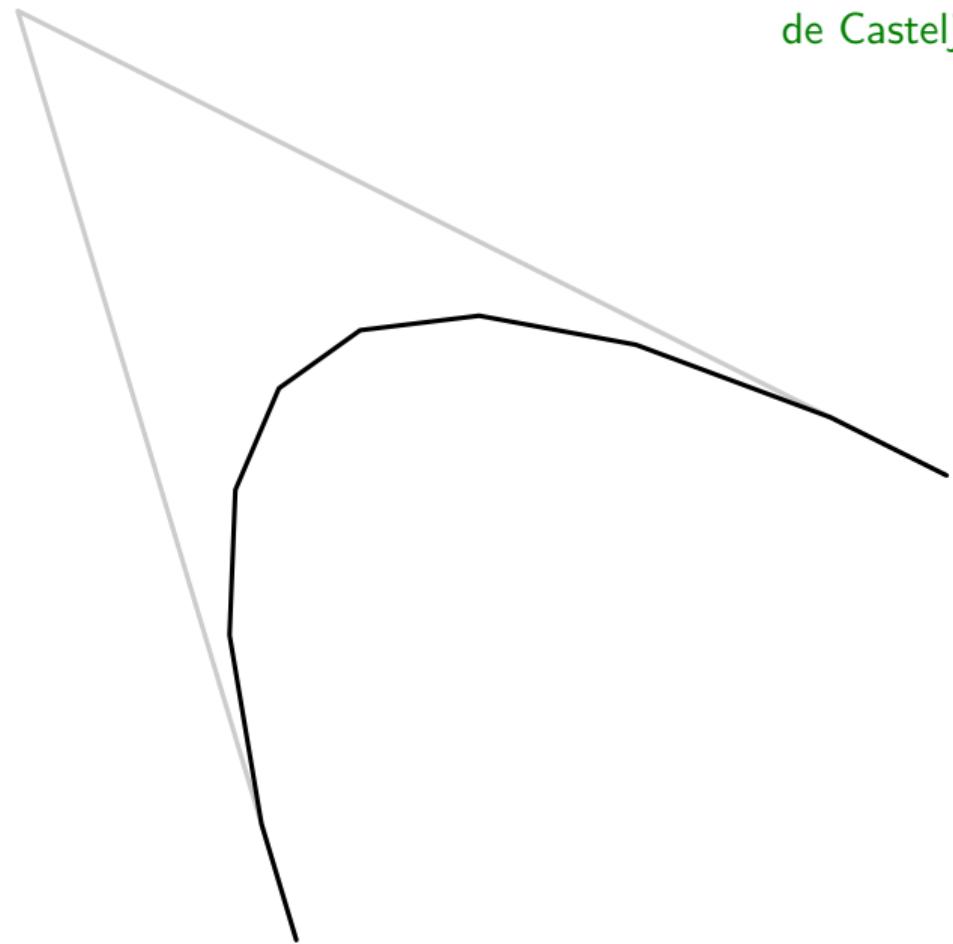
Curvas de subdivisão

de Casteljau (1963)



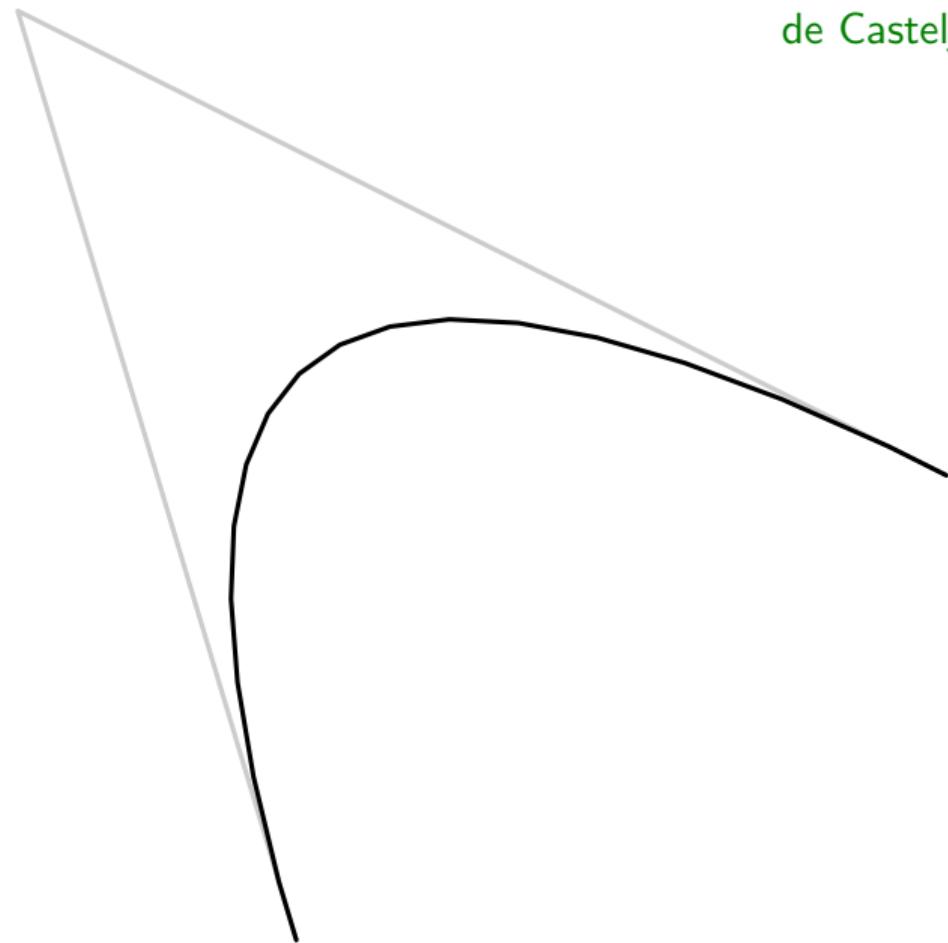
Curvas de subdivisão

de Casteljau (1963)



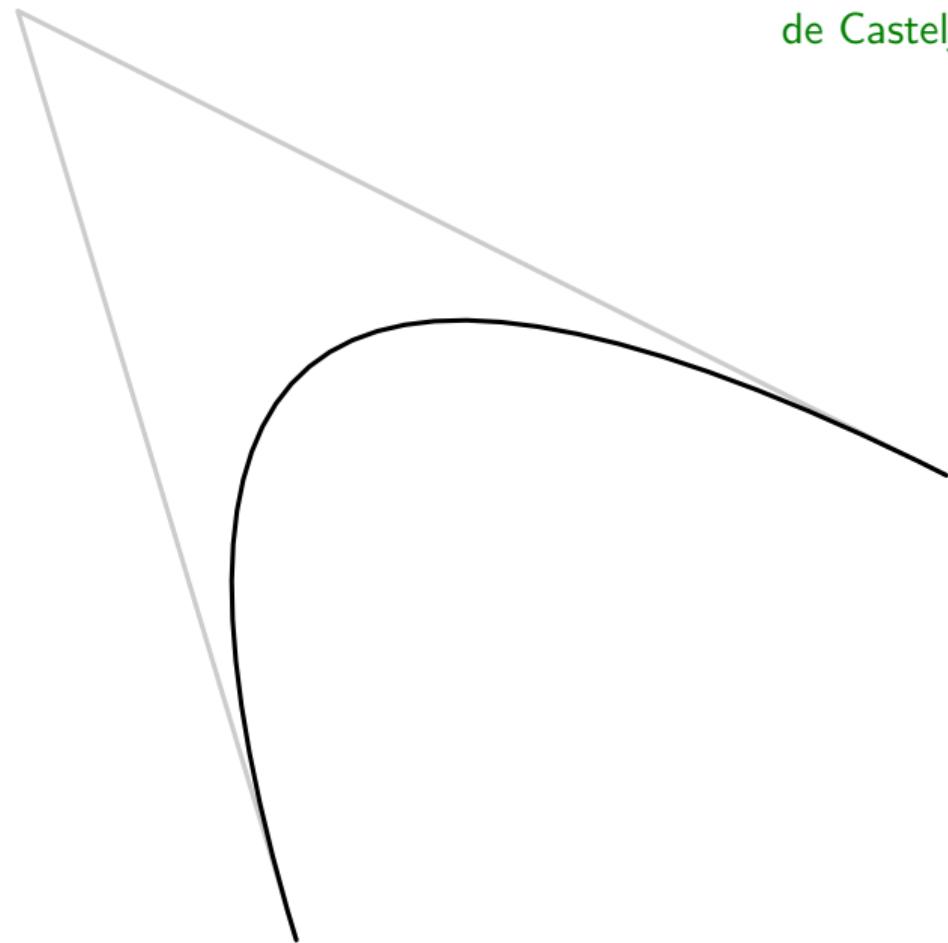
Curvas de subdivisão

de Casteljau (1963)



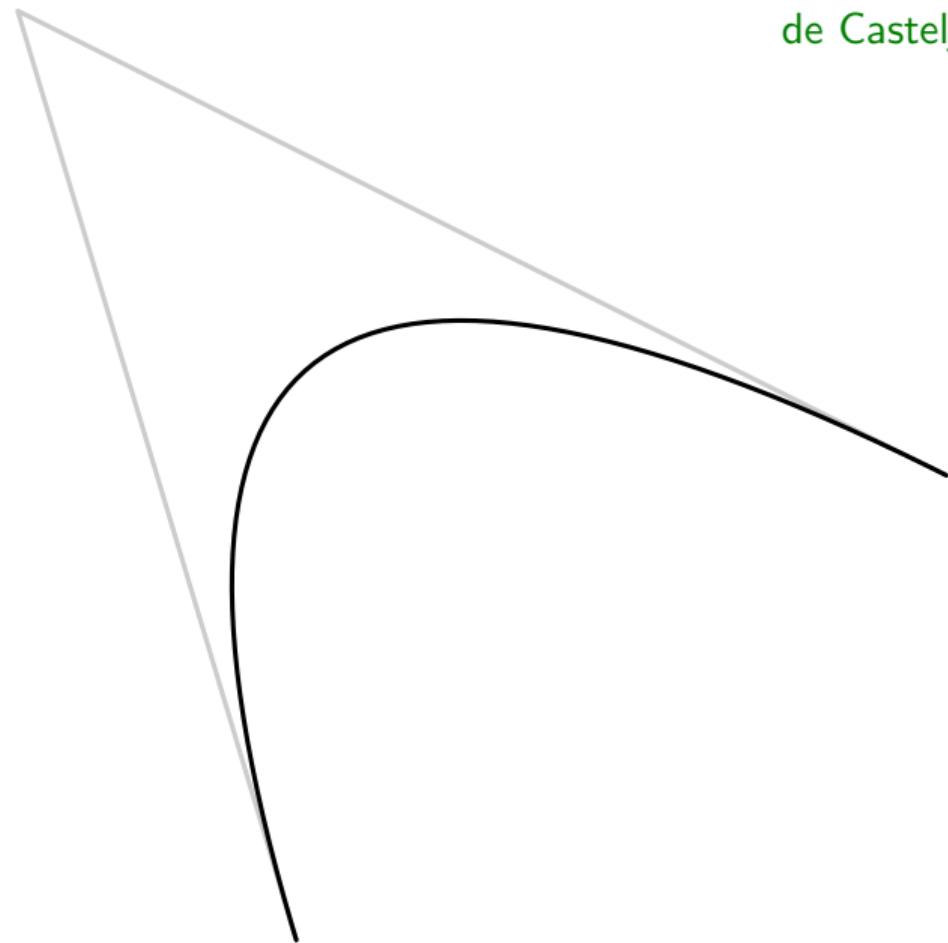
Curvas de subdivisão

de Casteljau (1963)



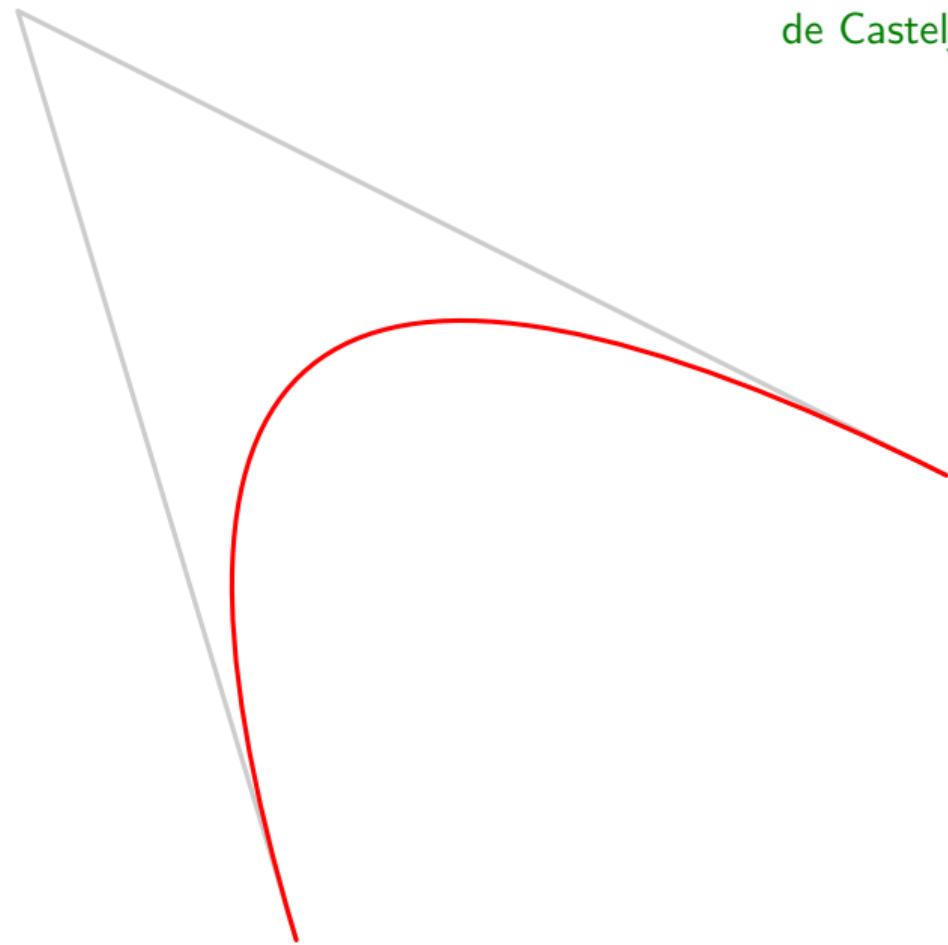
Curvas de subdivisão

de Casteljau (1963)



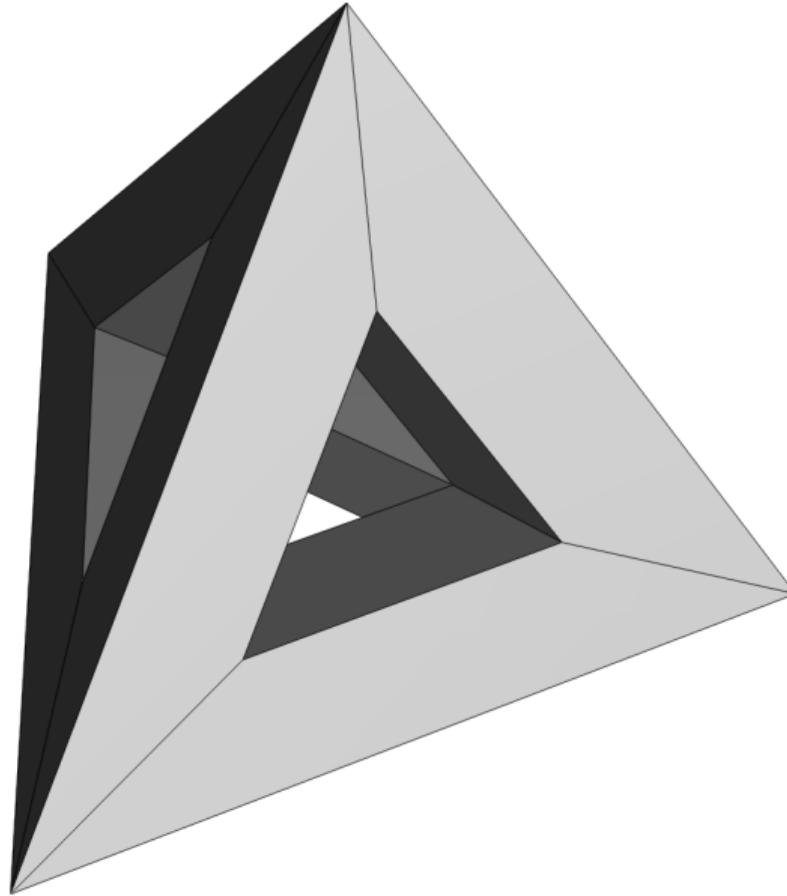
Curvas de subdivisão

de Casteljau (1963)



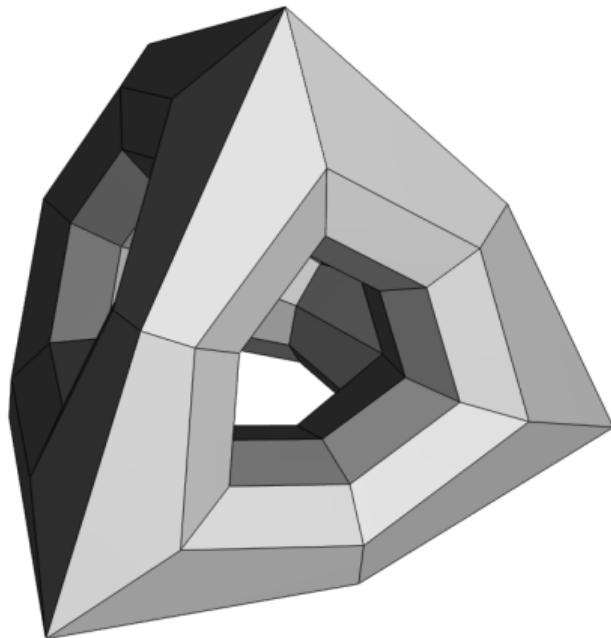
Superfícies de subdivisão

Catmull–Clark (1978)



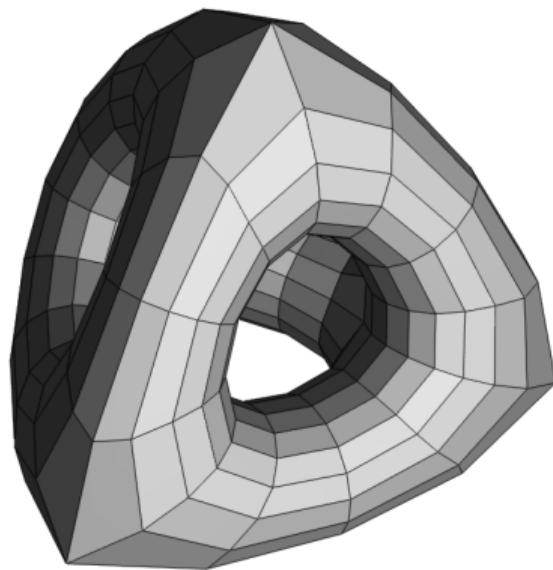
Superfícies de subdivisão

Catmull–Clark (1978)



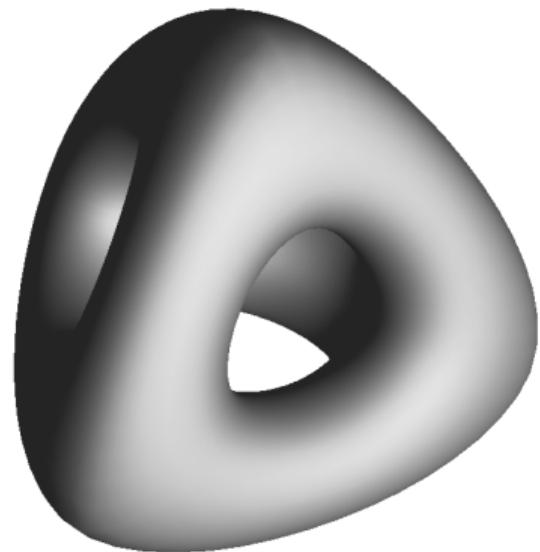
Superfícies de subdivisão

Catmull–Clark (1978)



Superfícies de subdivisão

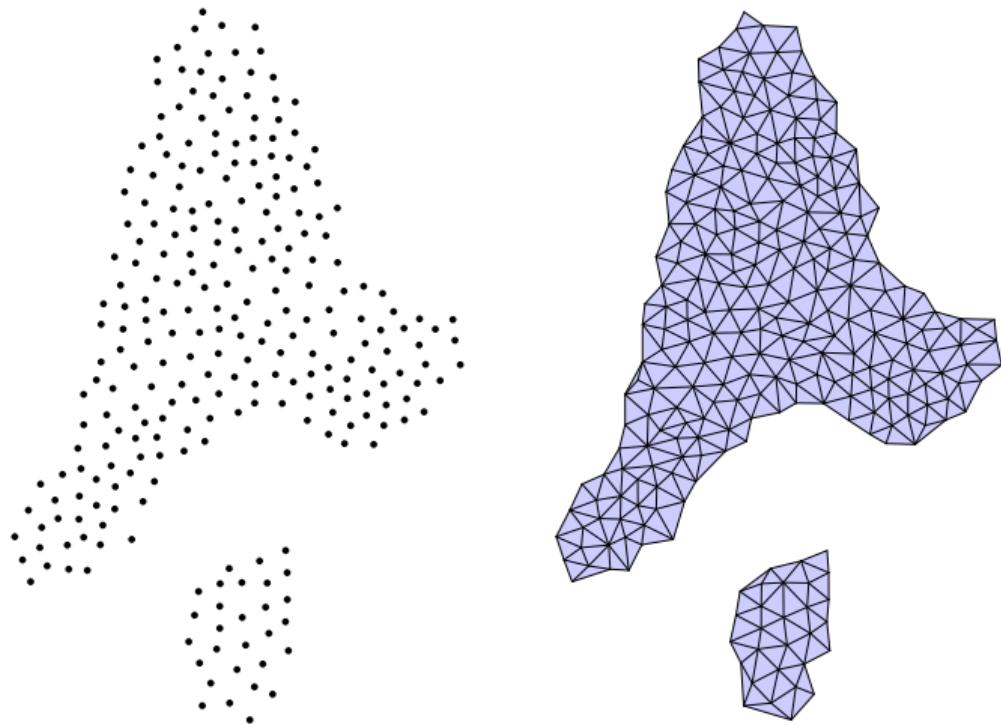
Catmull–Clark (1978)



Aquisição de formas



Aquisição de formas



Aquisição de formas

