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Production.

Q1) State the theorems which link line integral with surface integral & surface integral with volume integral.

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① Theorem which link line integral with surface integral

Stokes theorem :-

Let S be the piecewise smooth oriented surface in space & let boundary of S be smooth simple closed curve C. let, $F(x,y,z)$ be a continous vector function that has continous first partial derivative in a domain in space containing S. Then

$$\iint_S \text{curl}(F) \cdot n \, dA = \oint F \cdot r'(s) \, ds$$

where,

n = Unit normal vector of S

$r' = \frac{dr}{ds}$ = Unit tangent vector

s = arc length of C.

In components,

$$\iint_R \left[\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) N_1 + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) N_2 + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) N_3\right] du\,dv =$$

$$= \oint_C (F_1 dx + F_2 dy + F_3 dz)$$

Where,

$F = [F_1, F_2, F_3]$ & $N = [N_1, N_2, N_3]$

Also,

$n\,dA = N$ & $r' ds = [dx, dy, dz]$

Curve C is represented by $r(u,v)$.

② Theorem which link surface integral with volume

Divergence theorem of Gauss,

Let T be a closed bounded region in space whose boundary is a piecewise smooth orientable surface S. Let $F(x,y,z)$ be a vector function that is continous & has continous first partial derivative in some domain containing T. Then,

$$\iiint_T \operatorname{div} F \, dV = \iint_S F \cdot n \, dA$$

In components,

$$\iiint_T \left(\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} \right) dx\,dy\,dz$$

$$= \iint_S (f_1 \cos\alpha + f_2 \cos\beta + f_3 \cos\gamma)\, dA$$

$$= \iint_S (F_1\, dy\,dz + F_2\, dz\,dx + F_3\, dx\,dy)$$

Where,

$F = [f_1, f_2, f_3]$

$n = [\cos\alpha, \cos\beta, \cos\gamma]$

Q2) Evaluate the following.

a) Evaluate the line integral $\oint$ $\oint \bar{F}(\bar{r}) \cdot d\bar{r}$ Counter clock wise around the boundry C of the region R, where $\bar{F} = [3y^2, x - y^4]$, R is the square with vertices (1,1), (-1,1) (-1,-1), (1,-1)

→ Here,

$\bar{F} = [3y^2, x - y^4]$

i.e

$F_1 = 3y^2$

$F_2 = x - y^4$

Then,

by green's theorem.

$$\iint_R \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \oint_C \bar{F}(\bar{r})\, d\bar{r} \quad \text{—— (1)}$$

Now,

$\frac{\partial f_2}{\partial x} = 1$

$\frac{\partial f_1}{\partial y} = 6y$

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from ①.

$$\oint \bar{F}(\bar{r})\, d\bar{r} = \iint_R (1-6y)\, dx\, dy$$

Now,

x & y both varies from -1 to 1.

i.e.

$$\oint_C \bar{F}(\bar{r})\, d\bar{r} = \int_{-1}^{1}\int_{-1}^{1} (1-6y)\, dx\, dy.$$

$$= \int_{-1}^{1} (x-6xy)\Big]_{-1}^{1}\, dy$$

$$= \int_{-1}^{1} (2-12y)\, dy.$$

$$= 2y-6y^2\Big]_{-1}^{1}$$

$$= -4+8$$

$$\oint_C \bar{F}(\bar{r})\, d\bar{r} = 4.$$

b) Evaluate $\iint_S \bar{F}\cdot\hat{n}\, dA$, where $\bar{F} = [x^3, y^3, z^3]$

S is the surface of the sphere $x^2+y^2+z^2 = 4$

→ Here,

$$\bar{F} = [x^3, y^3, z^3]$$

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by gauss divergence theorem,

$$\iint_S \bar{F}\cdot\hat{n}\,dA = \iiint_T \operatorname{div} F\,dv.$$

Now,

$$\operatorname{div} f = 3x^2 + 3y^2 + 3z^2.$$

Converting into spherical co-ordinate (ρ, ϕ, θ)

$$x^2+y^2+z^2 = \rho^2$$

$$dv = \rho^2 \sin\phi$$

Also,

ρ varies from 0 to 2.

ϕ varies from 0 to π

θ varies from 0 to 2π

$$\therefore \iint_S \bar{F}\cdot\hat{n}\,dA = \iiint_T \operatorname{div} f\,dv.$$

$$\iiint_T \operatorname{div} f\,dv = \iiint_T 3(x^2+y^2+z^2)\,dv.$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} 3\rho^2\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^4 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= 3\int_0^{2\pi}\int_0^{\pi} \frac{\rho^5}{5} \sin\phi \Big|_0^2 \, d\phi \, d\theta$$

$$= \frac{3\times 32}{5}\int_0^{2\pi}\int_0^{\pi} \sin\phi \, d\phi \, d\theta$$

$$= \frac{3\times 32}{5}\int_0^{2\pi} (-\cos\phi)\Big|_0^{\pi} d\theta$$

$$= \frac{3\times 32}{5}\times 2\int_0^{2\pi} d\theta$$

$$= \frac{6\times 32}{5}\times 2\pi$$

$$= \frac{384\pi}{5}$$

i.e.

$$\iint_S \bar{F}\cdot\hat{n}\, dA = \frac{384\pi}{5}$$

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c) Evaluate $\int_C y\,dx + z\,dy + x\,dz$;

C is the curve of intersection of $x^2+y^2+z^2 = a^2$ & $x+z = a$.

→

Here,

C is the intersection of

$x^2+y^2+z^2 = a^2$ & $x+z = a$.

i.e

$z = a - x$

$x^2+y^2+(a-x)^2 = a^2$

$x^2+y^2+a^2-2ax+x^2 = a^2$

$2x^2+y^2-2ax = 0$

$\left(x-\frac{a}{2}\right)^2 + \frac{y^2}{2} = \frac{a^2}{4}$

$$\frac{\left(x-\frac{a}{2}\right)^2}{a^2/4} + \frac{y^2}{a^2/2} = 1$$

$x = \frac{a}{2}(1+\cos\theta)$

$y = \frac{a}{\sqrt{2}}\sin\theta$

$z = \frac{a}{2}(1-\cos\theta)$

$dx = -\frac{a}{2}\sin\theta$

$dy = \frac{a}{\sqrt{2}}\cos\theta$

$dz = \frac{a}{2}\sin\theta$

AS. θ varies from 0 to 2π

$$\int_c y\,dx + z\,dy + x\,dz$$

$$= \int_0^{2\pi} -\frac{a}{\sqrt{2}}\sin\theta\,\frac{a}{2}\sin\theta + \frac{a}{2}(1-\cos\theta) \times \frac{a}{\sqrt{2}}\cos\theta + \frac{a}{2}(1+\cos\theta)\frac{a}{2}\sin\theta$$

$$= \int_0^{2\pi}\left[-\frac{a^2}{2\sqrt{2}}\sin^2\theta + \frac{a^2}{2\sqrt{2}}\cos\theta + \frac{a^2}{4} \times (1+\cos\theta)\sin\theta\right]d\theta$$

$$= -\frac{a^2}{2\sqrt{2}} \times 4 \times \frac{\pi}{4} + 0 - \frac{a^2}{2\sqrt{2}} \times 4 \times \frac{\pi}{4}$$

$$= -\frac{a^2\pi}{\sqrt{2}}$$

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Q3)

a) Prove that a line integral $\int_C F(\bar{r})\cdot d\bar{r} = \int_C (f_1 dx + f_2 dy + f_3 dz)$ with

Continous F_1, F_2, F_3 in a domain D in space is independent of Path in D if the vector function $\bar{F} = [F_1, F_2, F_3]$ is the gradient of Some function f in D i.e $F = \text{grad}(f)$

→

To Prove:→
line integral is independent of Path if in domain if $\bar{F}$ is the gradient of Some function F

Proof:→
If F is gradient of Some function F.
i.e.

$$F_1 = \frac{\partial f}{\partial x}$$

$$F_2 = \frac{\partial f}{\partial y}$$

$$F_3 = \frac{\partial f}{\partial z}$$

We assume that,
$F = \text{grad } f$ holds for Some function f in D. & Show that this implies Path independent

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Let, c be any Path in D from any point A to any Point B in D. given by,

$$r(t) = [x(t), y(t), z(t)]$$

where,

$$a \le t \le b.$$

then,

$$\int_C f_1 dx + f_2 dy + f_3 dz =$$

$$\int_C \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz.$$

$$= \int_a^b \left(\frac{\partial f}{\partial x} \frac{dx}{dt} + \frac{\partial f}{\partial y} \frac{dy}{dt} + \frac{\partial f}{\partial z} \frac{dz}{dt} \right) dt$$

$$= \int_a^b \frac{df}{dt} dt$$

$$= F(x(t), y(t), z(t)) \Big|_{t=a}^{t=b}$$

$$= F(B) - f(A).$$

Which implies,
that. $\oint f(z) dz$ is path independent

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b) Parametrize the following curve
C: the intersection of
$x^2+y^2+z^2=a^2$ & $z=y^2$.

→ Here,
$$x^2+y^2+z^2=a^2$$
$$x^2+z+z^2=a^2.$$

$$x^2+z^2+z=a^2.$$
$$x^2+\left(z+\frac{1}{2}\right)^2=a^2+\frac{1}{4}$$

which is circle &
$r=\sqrt{a^2+\frac{1}{4}}$ & centre $=(0,0,-\frac{1}{2})$

let, curve represented by $r(\theta,\phi)$
then,
$$x=r\cos\theta=\sqrt{a^2+\frac{1}{4}}\cos\theta.$$

$$z=r\sin\theta=\sqrt{a^2+\frac{1}{4}}\sin\theta$$
&
$$y=\phi$$
i.e.
$$\cancel{r(\theta,\phi)=\left[\sqrt{a^2+\frac{1}{4}}\cos\theta,\sqrt{a^2+\frac{1}{4}}\sin\theta,\right.}$$

$$r(\theta,\phi)=\left[\sqrt{a^2+\frac{1}{4}}\cos\theta,\ \phi,\ \sqrt{a^2+\frac{1}{4}}\sin\theta\right.$$