Types of Numbers

Natural Numbers:

The group of numbers starting from 1 and including 1, 2, 3, 4, 5, and so on. Zero, negative numbers, and decimals are not included this group.

EXAMPLE

1. If n is an odd natural number, what is the highest number that always divides $n(n^2 - 1)$?

Answer: $n \cdot (n^2 - 1) = (n - 1) \cdot n \cdot (n + 1)$, which is a product of three consecutive numbers. Since n is odd, the numbers $(n - 1)$ and $(n + 1)$ are both even. One of these numbers will be a multiple of 2 and the other a multiple of 4 as they are two consecutive even numbers. Hence, their product is a multiple of 8. Since one out of every three consecutive numbers is a multiple of 3, one of the three numbers will be a multiple of three. Hence, the product of three numbers will be a multiple of 8 ´ 3 = 24.

Hence, the highest number that always divides $n \cdot (n^2 - 1)$ is 24.

2. For every natural number n, the highest number that $n \cdot (n^2 - 1) \cdot (5n + 2)$ is always divisible by is

(a) 6 (b) 24 (c) 36 (d) 48

Answer:

Case 1: If n is odd, $n \cdot (n^2 - 1)$ is divisible by 24 as proved in the earlier question.

Case 2: If n is even, both $(n - 1)$ and $(n + 1)$ are odd. Since product of three consecutive natural numbers is always a multiple of 3 and n is even, the product $n \cdot (n^2 - 1)$ is divisible by 6. Since n is even 5n is even. If n is a multiple of 2, 5n is a multiple of 2 and hence $5n + 2$ is a multiple of 4. If n is a multiple of 4, $5n + 2$ is a multiple of 2. Hence, the product $n \cdot (5n + 2)$ is a multiple of 8.

Hence, the product $n \cdot (n^2 - 1) \cdot (5n + 2)$ is a multiple of 24.

Hence, [b]

Rule: The product of n consecutive natural numbers is divisible by n!, where $n! = 1 \times 2 \times 3 \times 4 \times 5 \dots \times n$

EXAMPLE

3. Prove that $(2n)!$ is divisible by $(n!)^2$.

Answer: $(2n)! = 1\cdot2\cdot3\cdot4\cdot \ldots \cdot(n-1)\cdot n\cdot(n+1)\cdot \ldots\cdot 2n$

$= (n)!\cdot(n+1)\cdot(n+2)\cdot \ldots\cdot 2n$.

Since $(n+1)\cdot(n+2)\cdot \ldots\cdot 2n$ is a product of n consecutive numbers, it is divisible by n!. Hence, the product $(n)!\cdot(n+1)\cdot(n+2)\cdot \ldots\cdot 2n$ is divisible by $n!\cdot n! = (n!)^2$.

Whole Numbers:

All Natural Numbers plus the number 0 are called as Whole Numbers.

Integers:

All Whole Numbers and their negatives are included in this group.

Rational Numbers:

Any number that can be expressed as a ratio of two integers is called a rational number.

This group contains decimal that either do not exist (as in 6 which is 6/1), or terminate (as in 3.4 which is 34/10), or repeat with a pattern (as in 2.333... which is 7/3).

Irrational Numbers:

Any number that can not be expressed as the ratio of two integers is called an irrational number (imaginary or complex numbers are not included in irrational numbers).

These numbers have decimals that never terminate and never repeat with a pattern.

Examples include pi, e, and $\sqrt{2}$. $2 + \sqrt{3}$, $5 - \sqrt{2}$ etc. are also irrational quantities called **Surds.**

EXAMPLE

4. Express the value of $\dfrac{1}{\sqrt{5}+\sqrt{6}-\sqrt{11}}$ as a fraction whose denominator is rational.

Answer: $\dfrac{1}{\sqrt{5}+\sqrt{6}-\sqrt{11}} = \dfrac{\sqrt{5}+\sqrt{6}+\sqrt{11}}{(\sqrt{5}+\sqrt{6}-\sqrt{11})\times(\sqrt{5}+\sqrt{6}+\sqrt{11})}$

$$= \frac{\sqrt{5}+\sqrt{6}+\sqrt{11}}{[(\sqrt{5}+\sqrt{6})^2-(\sqrt{11})^2]} = \frac{\sqrt{5}+\sqrt{6}+\sqrt{11}}{2\sqrt{30}} = \frac{\sqrt{30}(\sqrt{5}+\sqrt{6}+\sqrt{11})}{60}$$

5. If p = $\frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}-\sqrt{7}}$ and q = $\frac{\sqrt{8}-\sqrt{7}}{\sqrt{8}+\sqrt{7}}$, then the value of $p^2 + pq + q^2$ is

Answer: p = $\frac{\sqrt{8}+\sqrt{7}}{\sqrt{8}-\sqrt{7}} = \frac{(\sqrt{8}+\sqrt{7})^2}{(\sqrt{8}-\sqrt{7})(\sqrt{8}+\sqrt{7})} = (\sqrt{8}+\sqrt{7})^2 = 15+2\sqrt{56}$

Similarly, q = $(\sqrt{8}-\sqrt{7})^2 = 15-2\sqrt{56}$

$p^2 + pq + q^2 = (15+2\sqrt{56})^2 + (15)^2 - (2\sqrt{56})^2 + (15-2\sqrt{56})^2$.

= 675 + 224 = 899

Real Numbers:

This group is made up of all the Rational and Irrational Numbers. The ordinary number line encountered when studying algebra holds real numbers.

Imaginary Numbers:

These numbers are formed by the imaginary number i ($i = \sqrt{-1}$). Any real number times i is an imaginary number.

Examples include i, 3i, −9.3i, and (pi)i. Now $i^2 = -1$, $i^3 = i^2 \times i = -i$, $i^4 = 1$.

EXAMPLE

6. What is the value of $\frac{i^4+i^6+i^8+i^{10}+i^{12}}{i^{14}+i^{16}+i^{18}+i^{20}+i^{22}}$?

Answer: $i^4 = 1$, $i^6 = i^4 \times i^2 = -1$, $i^8 = 1$, $i^{10} = -1$, and so on.

Hence, $\frac{i^4+i^6+i^8+i^{10}+i^{12}}{i^{14}+i^{16}+i^{18}+i^{20}+i^{22}} = \frac{1-1+1-1+1}{-1+1-1+1-1} = -1$

Complex Numbers:

A Complex Numbers is a combination of a real number and an imaginary number in the form a + bi. a is called the real part and b is called the imaginary part.

Examples include 3 + 6i, 8 + (−5)i, (often written as 8 - 5i).

Note: a number in the form $\frac{1}{a+ib}$ is written in the form of a complex number by multiplying both numerator and denominator by the conjugate of a + ib, i.e. a − ib.

Hence, $\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{ib}{a^2+b^2}$, which is in the form p + iq.

EXAMPLE

7. The value of $(\frac{1+i}{1-i})^7$ is

Answer: $\frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+i^2+2i}{1-i^2}$ = 2i/2 = i

Hence, $(\frac{1+i}{1-i})^7 = (i)^7 = -i$

Prime Numbers:

All the numbers that have only two divisors, 1 and the number itself, are called prime numbers. Hence, a prime number can only be written as the product of 1 and itself. The numbers 2, 3, 5, 7, 11…37, etc. are prime numbers.

Note: 1 is not a prime number.

EXAMPLE

8. If $x^2 - y^2 = 101$, find the value of $x^2 + y^2$, given that x and y are natural numbers.

Answer: $x^2 - y^2 = (x + y)(x - y) = 101$. But 101 is a prime number and cannot be written as product of two numbers unless one of the numbers is 1 and the other is 101 itself.

Hence, x + y = 101 and x − y = 1. -> x = 51, y = 50.

-> $x^2 + y^2 = 51^2 + 50^2 = 5101$.

9. What numbers have exactly three divisors?

Answer: The squares of prime numbers have exactly three divisors, i.e. 1, the prime number, and the square itself.

To find whether a number N is prime or not
Find the root R (approximate) of the number N, i.e. R = √N. Divide N by every prime number less than or equal to R. If N is divisible by at least one of those prime numbers it is not a prime number. If N is not divisible by any of those prime numbers, it is a prime number.

Odd and Even Numbers:

All the numbers divisible by 2 are called even numbers whereas all the numbers not divisible by 2 are called odd numbers. 2, 4, 6, 8… etc. are even numbers and 1, 3, 5, 7.. etc. are odd numbers.

Remember! Odd + Odd = Even Even + Even = Even Odd + Even = Odd $(\text{Odd})^{\text{Even}}$ = Odd	$(\text{Even})^{\text{Odd}}$ = Even Even × Odd = Even Even × Even = Even Odd × Odd = Odd $(\text{Odd})^{\text{Even}} \times (\text{Even})^{\text{Odd}}$ = Even $(\text{Odd})^{\text{Even}} + (\text{Even})^{\text{Odd}}$ = Odd

Remainders

Examples:

1.What is the remainder when the product 1998 × 1999 × 2000 is divided by 7?
Answer: the remainders when 1998, 1999, and 2000 are divided by 7 are 3, 4, and 5 respectively. Hence the final remainder is the remainder when the product 3 × 4 × 5 = 60 is divided by 7.
Answer = 4

2. What is the remainder when 2^{2004} is divided by 7?
2^{2004} is again a product (2 × 2 × 2... (2004 times)). Since 2 is a number less than 7 we try to convert the product into product of numbers higher than 7. Notice that 8 = 2 × 2 × 2. Therefore we convert the product in the following manner-
$2^{2004} = 8^{668}$ = 8 × 8 × 8... (668 times).
The remainder when 8 is divided by 7 is 1.
Hence the remainder when 8^{668} is divided by 7 is the remainder obtained when the product 1 × 1 × 1... is divided by 7
Answer = 1

3. What is the remainder when 2^{2006} is divided by 7?
This problem is like the previous one, except that 2006 is not an exact multiple of 3 so we cannot convert it completely into the form 8^{x}. We will write it in following manner-
$2^{2006} = 8^{668} \times 4$.
Now, 8^{668} gives the remainder 1 when divided by 7 as we have seen in the previous problem. And 4 gives a remainder of 4 only when divided by 7.
Hence the remainder when 2^{2006} is divided by 7 is the remainder when the product 1 × 4 is divided by 7.
Answer = 4

4.What is the remainder when 25^{25} is divided by 9?
Again $25^{25} = (18 + 7)^{25} = (18 + 7)(18 + 7)$...25 times $= 18K + 7^{25}$
Hence remainder when 25^{25} is divided by 9 is the remainder when 7^{25} is divided by 9.
Now $7^{25} = 7^3 \times 7^3 \times 7^3$ .. (8 times) $\times 7 = 343 \times 343 \times 343$... (8 times) $\times$ 7.
The remainder when 343 is divided by 9 is 1 and the remainder when 7 is divided by 9 is 7.
Hence the remainder when 7^{25} is divided by 9 is the remainder we obtain when the product $1 \times 1 \times 1$... (8 times) $\times$ 7 is divided by 9. The remainder is 7 in this case. Hence the remainder when 25^{25} is divided by 9 is 7.

Some Special Cases:
2.1A When both the dividend and the divisor have a factor in common.
Let N be a number and Q and R be the quotient and the remainder when N is divided by the divisor D.
Hence, $N = Q \times D + R$.
Let $N = k \times A$ and $D = k \times B$ where k is the HCF of N and D and $k > 1$.
Hence $kA = Q \times kB + R$.
Let Q_1 and R_1 be the quotient and the remainder when A is divided by B.
Hence $A = B \times Q_1 + R_1$.
Putting the value of A in the previous equation and comparing we get-
$k(B \times Q_1 + R_1) = Q \times kB + R$ --> $R = kR_1$.
Hence to find the remainder when both the dividend and the divisor have a factor in common,
Take out the common factor (i.e. divide the numbers by the common factor)
Divide the resulting dividend (A) by resulting divisor (B) and find the remainder (R_1).
The real remainder R is this remainder R1 multiplied by the common factor (k).
Examples

5.What the remainder when 2^{96} is divided by 96?
The common factor between 2^{96} and 96 is $32 = 2^5$.
Removing 32 from the dividend and the divisor we get the numbers 2^{91} and 3 respectively.
The remainder when 2^{91} is divided by 3 is 2.
Hence the real remainder will be 2 multiplied by common factor 32.
Answer = 64

2.1B THE CONCEPT OF NEGATIVE REMAINDER
$15 = 16 \times 0 + 15$ or
$15 = 16 \times 1 – 1$.
The remainder when 15 is divided by 16 is 15 the first case and -1 in the second case.
Hence, the remainder when 15 is divided by 16 is 15 or -1.
--> When a number $N < D$ gives a remainder R (= N) when divided by D, it gives a negative remainder of $R - D$.

For example, when a number gives a remainder of −2 with 23, it means that the number gives a remainder of 23 – 2 = 21 with 23.

EXAMPLE

6.Find the remainder when 7^{52} is divided by 2402.

Answer: $7^{52} = (7^4)^{13} = (2401)^{13} = (2402 - 1)^{13} = 2402K + (-1)^{13} = 2402K - 1$.

Hence, the remainder when 7^{52} is divided by 2402 is equal to −1 or 2402 – 1 = 2401.

Answer: 2401.

2 .1C When dividend is of the form $a^n + b^n$ or $a^n - b^n$:

Theorem1: $a^n + b^n$ is divisible by a + b when n is **ODD**.
Theorem 2: $a^n - b^n$ is divisible by a + b when n is **EVEN**.
Theorem 3: $a^n - b^n$ is ALWAYS divisible by a – b.

EXAMPLES

What is the remainder when $3^{444} + 4^{333}$ is divided by 5?

Answer:

The dividend is in the form $a^x + b^y$. We need to change it into the form $a^n + b^n$.

$3^{444} + 4^{333} = (3^4)^{111} + (4^3)^{111}$.

Now $(3^4)^{111} + (4^3)^{111}$ will be divisible by $3^4 + 4^3 = 81 + 64 = 145$.

Since the number is divisible by 145 it will certainly be divisible by 5.

Hence, the remainder is 0.

7.What is the remainder when $(5555)^{2222} + (2222)^{5555}$ is divided by 7?

Answer:

The remainders when 5555 and 2222 are divided by 7 are 4 and 3 respectively.

Hence, the problem reduces to finding the remainder when $(4)^{2222} + (3)^{5555}$ is divided by 7.

Now $(4)^{2222} + (3)^{5555} = (4^2)^{1111} + (3^5)^{1111} = (16)^{1111} + (243)^{1111}$.

Now $(16)^{1111} + (243)^{1111}$ is divisible by 16 + 243 or it is divisible by 259, which is a multiple of 7.

Hence the remainder when $(5555)^{2222} + (2222)^{5555}$ is divided by 7 is zero.

8.$20^{2004} + 16^{2004} - 3^{2004} - 1$ is divisible by:

(a) 317 (b) 323 (c) 253 (d) 91

Solution: $20^{2004} + 16^{2004} - 3^{2004} - 1 = (20^{2004} - 3^{2004}) + (16^{2004} - 1^{2004})$.

Now $20^{2004} - 3^{2004}$ is divisible by 17 (Theorem 3) and $16^{2004} - 1^{2004}$ is divisible by 17 (Theorem 2).

Hence the complete expression is divisible by 17.

$20^{2004} + 16^{2004} - 3^{2004} - 1 = (20^{2004} - 1^{2004}) + (16^{2004} - 3^{2004})$.

Now $20^{2004} - 1^{2004}$ is divisible by 19 (Theorem 3) and $16^{2004} - 3^{2004}$ is divisible by 19 (Theorem 2).

Hence the complete expression is also divisible by 19.

Hence the complete expression is divisible by $17 \times 19 = 323$.

2.1D When $f(x) = a + bx + cx^2 + dx^3 + \ldots$ is divided by $x - a$

The remainder when $f(x) = a + bx + cx^2 + dx^3 + ..$ is divided by $x - a$ is $f(a)$.
So, If $f(a) = 0$, $(x - a)$ is a factor of $f(x)$.

EXAMPLES

9.What is the remainder when $x^3 + 2x^2 + 5x + 3$ is divided by $x + 1$?
Answer: The remainder when the expression is divided by $(x - (-1))$ will be $f(-1)$.
Remainder = $(-1)^3 + 2(-1)^2 + 5(-1) + 3 = -1$
If $2x^3 - 3x^2 + 4x + c$ is divisible by $x - 1$, find the value of c.
Since the expression is divisible by $x - 1$, the remainder $f(1)$ should be equal to zero.
Or $2 - 3 + 4 + c = 0$, or $c = -3$.

2 .1E Fermat's Theorem

If p is a prime number and N is prime to p, then $N^p - N$ is divisible by p.

EXAMPLE

10.What is the remainder when $n^7 - n$ is divided by 42?
Answer: Since 7 is prime, $n^7 - n$ is divisible by 7.
$n^7 - n = n(n^6 - 1) = n(n + 1)(n - 1)(n^4 + n^2 + 1)$
Now $(n - 1)(n)(n + 1)$ is divisible by $3! = 6$
Hence $n^7 - n$ is divisible by $6 \times 7 = 42$.
Hence the remainder is 0.

2.1F Wilson 's Theorem

If p is a prime number, $(p - 1)! + 1$ is divisible by p.

EXAMPLE

11.Find the remainder when 16! Is divided by 17.
$16! = (16! + 1) - 1 = (16! + 1) + 16 - 17$
Every term except 16 is divisible by 17 in the above expression. Hence the remainder = the remainder obtained when 16 is divided by 17 = 16
Answer = 16

2.1G TO FIND THE NUMBER OF NUMBERS, THAT ARE LESS THAN OR EQUAL TO A CERTAIN NATURAL NUMBER N, AND THAT ARE DIVISIBLE BY A CERTAIN INTEGER

To find the number of numbers, less than or equal to n, and that are divisible by a certain integer p, we divide n by p. The quotient of the division gives us the number of numbers divisible by p and less than or equal to n.

EXAMPLE

12. How many numbers less than 400 are divisible by 12?
Answer: Dividing 400 by 12, we get the quotient as 33. Hence the number of numbers that are below 400 and divisible by 12 is 33.

13. How many numbers between 1 and 400, both included, are not divisible either by 3 or 5?
Answer: We first find the numbers that are divisible by 3 or 5. Dividing 400 by 3 and 5, we get the quotients as 133 and 80 respectively. Among these numbers divisible by 3 and 5, there are also numbers which are

divisible both by 3 and 5 i.e. divisible by 3 x 5 = 15. We have counted these numbers twice. Dividing 400 by 15, we get the quotient as 26.
Hence the number divisible by 3 or 5 = 133 + 80 – 26 = 187
Hence, the numbers not divisible by 3 or 5 are = 400 – 187 = 213.

14. How many numbers between 1 and 1200, both included, are not divisible by any of the numbers 2, 3 and 5?
Answer: as in the previous example, we first find the number of numbers divisible by 2, 3, or 5. from set theory we have
n(AUBUC) = n(A) + n(B) + n(C) – n(A ∩ B) – n(B ∩ C) – n(A ∩ C) + n(A ∩ B ∩ C)
n(2U3U5) = n(2) + n(3) + n(5) – n(6) – n(15) – n(10) + n(30)
à n(2U3U5) = 600 + 400 + 240 – 200 – 80 – 120 + 40 = 880
Hence number of numbers not divisible by any of the numbers 2, 3, and 5 = 1200 – 880 = 320.

Base System

There are two kinds of operations associated with conversion of bases:

1.A conversion from any base to base ten

The number (pqrstu) b is converted to base 10 by finding the value of the number. i.e. (pqrstu) b = u + tb + sb 2 + rb 3 + qb 4 + pb 5 .

Example

Q1. Convert (21344) 5 to base 10.

Answer: (21344) 5 = 4 + 4 × 5 + 3 × 25 + 1 × 125 + 2 × 625 = 1474

1.b conversion from base 10 to any base

A number written in base 10 can be converted to any base 'b' by first dividing the number by 'b', and then successively dividing the quotients by 'b'. The remainders, written in reverse order, give the equivalent number in base 'b'.

Example

Q2.Write the number 25 in base 4.

$$\begin{array}{r|ll} 4 & 25 & \\ \hline & 6 & 1 \\ \hline & 1 & 2 \\ \hline & 0 & 1 \\ \hline \end{array}$$

Writing the remainders in reverse order the number 25 in base 10 is the number 121 in base 4.

1.c Addition, subtraction and multiplication in bases:

Example

Q3. Add the numbers $(4235)_7$ and $(2354)_7$

Answers: The numbers are written as

$$\begin{array}{cccc} 4 & 2 & 3 & 5 \\ 2 & 3 & 5 & 4 \end{array}$$

The addition of 5 and 4 (at the units place) is 9, which being more than 7

would be written as 9 = 7 × 1 + 2. The Quotient is 1 and written is 2.

The Remainder is placed at the units place of the answer and the Quotient

gets carried over to the ten's place. We obtain

$$\begin{array}{cccc} & +1 & +1 & \\ 4 & 2 & 3 & 5 \\ 2 & 3 & 5 & 4 \\ \hline 6 & 6 & 2 & 2 \\ \hline \end{array}$$

At the tens place: 3 + 5 + 1 (carry) = 9

Similar procedure is to be followed when multiply numbers in the same base

Example

4. Multiply $(43)_8 \times (67)_8$

Answer:

7 × 3 = 21 = 8 × 2 + 5

7 × 4 + 2 = 30 = 8 × 3 + 6

6 × 3 = 18 = 8 × 2 + 2

6 × 4 + 2 = 26 = 8 × 3 + 2

$$
\begin{array}{r}
(4\ \ 3)_8 \\
(6\ \ 7)_8 \\
\hline
365 \\
322\ \ \\
\hline
3605
\end{array}
$$

For subtraction the procedure is same for any ordinary subtraction in base 10 except for the fact that whenever we need to carry to the right we carry the value equal to the base.

EXAMPLE

5. Subtract 45026 from 51231 in base 7.

Answer:

$$
\begin{array}{rr}
 & 5\ 1\ 2\ 3\ 1 \\
- & 4\ 5\ 0\ 2\ 6 \\
\hline
 & 3\ 2\ 0\ 2 \\
\hline
\end{array}
$$

In the units column since 1 is smaller than 6, we carry the value equal to the base from the number on the left. Since the base is 7 we carry 7. Now, 1 + 7 = 8

and 8 – 6 = 2. Hence we write 2 in the units column. We proceed the same way in the rest of the columns.

1D. IMPORTANT RULES ABOUT BASES

Rule1. A number in base N when written in base 10 is divisible by N – 1 when the sum of the digits of the number in base N is divisible by N – 1.

EXAMPLE

6. The number 35A246772 is in base 9. This number when written in base 10 is divisible by 8. Find the value of digit A.

Answer: The number written in base 10 will be divisible by 8 when the sum of the digits in base 9 is divisible by 8.

Sum of digits = 3 + 5 + A + 2 + 4 + 6 + 7 + 7 + 2 = 36 + A. The sum will be divisible by 8 when A = 4.

Rule2. When the digits of a k-digits number written in base N are rearranged in any order to form a new k-digits number, the difference of the two numbers, when written in base 10, is divisible by N – 1.

EXAMPLE

7. A four-digit number N1 is written in base 13. A new four-digit number N2 is formed by rearranging the digits of N1 in any order. Then the difference N1 – N2 when calculated in base 10 is divisible by

(a) 9 (b) 10 (c) 12 (d) 13

Answer: c

Divisibility Rules

A) Divisibility by 2, 4, 8, 16, 32..

A number is divisible by 2, 4, 8, 16, 32,.. 2 n when the number formed by the last one, two, three, four, five...n digits is divisible by 2, 4, 8, 16, 32,..2 n respectively.

Example: 1246384 is divisible by 8 because the number formed by the last three digits i.e. 384 is divisible by 8. The number 89764 is divisible by 4 because the number formed by the last two digits, 64 is divisible by 4.

B) Divisibility by 3 and 9

A number is divisible by 3 or 9 when the sum of the digits of the number is divisible by 3 or 9 respectively.

Example: 313644 is divisible by 3 because the sum of the digits- 3 + 1 + 3 + 6 + 4 + 4 = 21 is divisible by 3.

The number 212364 is divisible by 9 because the sum of the digit- 2 + 1 + 2 + 3 + 6 + 4 = 18 is divisible by 9.

c) Divisibility by 6, 12, 14, 15, 18..

Whenever we have to check the divisibility of a number N by a composite number C, the number N should be divisible by all the prime factors (the highest power of every prime factor) present in C .

divisibility by 6: the number should be divisible by both 2 and 3.

divisibility by 12: the number should be divisible by both 3 and 4.

divisibility by 14: the number should be divisible by both 2 and 7.

divisibility by 15: the number should be divisible by both 3 and 5.

divisibility by 18: the number should be divisible by both 2 and 9.

EXAMPLES

1. The six-digit number 73A998 is divisible by 6. How many values of A are possible?

Answer: Since the number is ending in an even digit, the number is divisible by 2. To find divisibility by 3, we need to consider sum of the digits of the number.

The sum of the digits = 7 + 3 + A + 9 + 9 + 8 = 36 + A.

For the number to be divisible by 3, the sum of the digits should be divisible by 3. Hence A can take values equal to 0, 3, 6, and 9.

Answer = 4

d) Divisibility by 7, 11, and 13

Let a number be ....kjlhgfedcba where a, b, c, d, are respectively units digits, tens digits, hundreds digits, thousands digits and so on. Starting from right to left, we make groups of three digit numbers successively and continue till the end. It is not necessary that the leftmost group has three digits.

Grouping of the above number in groups of three, from right to left, is done in the following manner à kj,ihg,fed,cba

We add the alternate groups (1 st , 3 rd , 5 th etc.. and 2 nd , 4 th , 6 th , etc..) to obtain two sets of numbers, N 1 and N 2 .

In the above example, N 1 = cba + ihg and N 2 = fed + kj

Let D be difference of two numbers, N 1 and N 2 i.e. D = N 1 – N 2 .

à If D is divisible by 7, then the original number is divisible by 7.

à If D is divisible by 11, then the original number is divisible by 11

à If D is divisible by 13 then the original number is divisible by 13.

Corollary:

Any six-digit, or twelve-digit, or eighteen-digit, or any such number with number of digits equal to multiple of 6, is divisible by **EACH** of 7, 11 and 13 if all of its digits are **same** .

For example 666666, 888888888888 etc. are all divisible by 7, 11, and 13.

Example

2. Find if the number 29088276 is divisible by 7.

Answer: We make the groups of three as said above- 29,088,276

N 1 = 29 + 276 = 305 and N 2 = 88

D = N 1 – N 2 = 305 – 88 = 217. We can see that D is divisible by 7. Hence, the original number is divisible by 7.

3. Find the digit A if the number 888…888A999…999 is divisible by 7, where both the digits 8 and 9 are 50 in number.

Answer: We know that 888888 and 999999 will be divisible by 7. Hence 8 written 48 times in a row and 9 written 48 times in a row will be divisible by 7. Hence we need to find the value of A for which the number 88A99 is divisible by 7. By trial we can find A is = 5.

Answer = 5.

<u>**Divisors** :</u>

All the numbers, including 1 and the number itself itself, which divide N complet ely are called divisors of the number N.

Example: The number 24 is divisible by 1, 2, 3, 4, 6, 8, 12, and 24. Hence all these numbers are divisors of 24.

Let N be a composite number such that N = (x) a (y) b (z) c .. where x, y, z.. are prime factors.

- The number of divisors of N = (a + 1)(b + 1)(c + 1)..

- The sum of the divisors $= \frac{x^{a+1} - 1}{x - 1} \times \frac{y^{b+1} - 1}{y - 1} \times \frac{z^{c+1} - 1}{z - 1} \ldots$

- The product of divisors $= (N)^{\frac{(a+1)(b+1)(c+1)}{2}}$

- The number of ways in which a composite number $N = (x)^a(y)^b(z)^c\ldots$ can be resolved into two factors $= \frac{(a+1)(b+1)(c+1)\ldots}{2}$

  If the composite number $N = (x)^a(y)^b(z)^c\ldots$ is a perfect square the number of ways it can be resolved into two factors $= \frac{(a+1)(b+1)(c+1)\ldots+1}{2}$

· The number of ways in which a composite number can be resolved into two factors which are prime to each other = 2 n – 1 , where n is the number of different prime factors of the number.

· The number of distinct prime factors of N = count of x, y, z..

· Assuming that x = 2, number of odd factors of N = (b + 1)(c + 1)..

· Assuming that x = 2, number of even factors of N = a(b + 1)(c + 1)..

EXAMPLES

1.Find the number of divisors of 60.

$60 = 2^2 \times 3 \times 5$

Therefore number of divisors of 60 = (2 + 1)(1 + 1)(1 + 1) = 12

30. Find the sum of divisors of 60.

The sum of the divisors $= \frac{2^3 - 1}{2 - 1} \times \frac{3^2 - 1}{3 - 1} \times \frac{5^2 - 1}{5 - 1} = 168$

2.Find the product of the divisors of 60.

Product of the divisors = (60) (2 + 1)(1 + 1)(1 + 1)/2 = (60) 6 .

3.How many pairs (a, b) are there such that a x b = 60?

The divisors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. We can see that 1 × 60 = 2 × 30 = 3 × 20 = 4 × 15 = 60 and so on. Hence the number of such pairs (a, b) which give $a \times b = 60$ are $1/2 \times$ (number of divisors of 60) = $1/2 \times 12 = 6$

4.How many ordered pairs of integers, (x, y) satisfy the equation $xy = 110$?

(a) 6 (b) 8 (c) 12 (d) 16

Answer: $110 = 2 \times 5 \times 11$. Hence, the number of divisors of 110 is = $2 \times 2 \times 2 = 8$. Hence, the number of positive pairs of x and y = 8. (Since (2, 55) is not same as (55, 2)). Also since we are asked for integers, the pair consisting of two negative integers will also suffice. Hence the total number of pairs = $2 \times 8 = 16$.

5.How many even positive divisors does 720 have?

Answer: $720 = 2^4 \times 3^2 \times 5$.

Therefore, the number of even divisors of 720 = $4 \times (2 + 1) \times (1 + 1) = 24$.

6.Find the number of odd divisors (divisors which are odd numbers) of 15000.

Answer: $15000 = 2^3 \times 3 \times 5^4$. Odd numbers are not divisible by 2, and therefore in an odd divisor, there will not be a power of 2. Therefore, we will only consider powers of 3 and 5. Hence, the total number of combinations- and hence the total number of divisors- in this case will be $2 \times 5 = 10$.

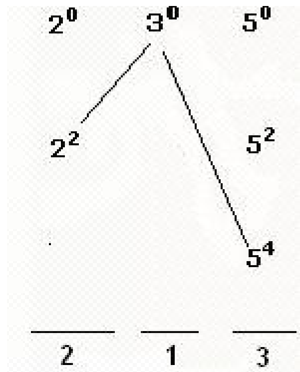
7.Find the number of even divisors of 15000.

Number of even divisors = Total number of divisors - number of odd divisors

= 40 - 10 = 30.

8.Find the number of divisors of 15000 that are perfect squares.

Answer: **In a perfect square, power of every prime factor is even**. Therefore, we will only consider those divisors of 15000 in which powers of prime factors are even. In other words, we will only consider even powers of prime factors. We write down only the even powers of prime factors



We can see that now the number of possible combinations = 2 × 1 × 3 = 6.

Therefore, 6 divisors of 15000 are perfect squares.

9.Find the number of divisors of 7! that are odd.

Answer: This question is same as the earlier question except that now we will first have to do prime factorization of 7!. For that, we will have to find powers of the prime factors in 7!. The prime factors present in 7! will be 2, 3, 5 and 7. Their respective powers are:

$$\text{Powers of } 2 \text{ in } 7! = [\frac{7}{2}] + [\frac{7}{4}] = 4$$
$$\text{Powers of } 3 \text{ in } 7! = [\frac{7}{3}] = 2$$
$$\text{Powers of } 5 \text{ in } 7! = 1$$
$$\text{Powers of } 7 \text{ in } 7! = 1$$

Therefore, $7! = 2^4 \times 3^2 \times 5 \times 7$. To find odd divisors of 7!, we ignore the powers of 2 and then calculate the number of combinations of powers of 3, 5, and 7. The number of combinations = 3 × 2 × 2 = 12. Therefore, there are 12 odd divisors of 7!.

HCF AND LCM

What is highest common factor (HCF) and least common multiple (LCM)? How do you calculate HCF and LCM of two or more numbers? Are you looking for problems on HCF and LCM? This chapter will answer all these questions.

HIGHEST COMMON FACTOR (HCF)

The largest number that divides two or more given numbers is called the highest common factor (HCF) of those numbers. There are two methods to find HCF of the given numbers:

Prime Factorization Method- When a number is written as the product of prime numbers, the factorization is called the prime factorization of that number. For example, 72 = 2 × 2 × 2 × 3 × 3 = 23 × 32
To find the HCF of given numbers by this method, we perform the prime factorization of all the numbers and then check for the common prime factors. For every prime factor common to all the numbers, we choose the least index of that prime factor among the given number. The HCF is product of all such prime factors with their respective least indices.
EXAMPLE
Find the HCF of 72, 288, and 1080
Answer:
72 = 23 × 32,
288 = 25 × 32,
1080 = 23 × 33 × 5
The prime factors common to all the numbers are 2 and 3. The lowest indices of 2 and 3 in the given numbers are 3 and 2 respectively.
Hence, HCF = 23 × 32 = 72.

Find the HCF of 36x3y2 and 24x4y.
Answer:
36x3y2 = 22∙32∙x3∙y2
24x4y = 23∙3∙x4∙y.
The least index of 2, 3, x and y in the numbers are 2, 1, 3 and 1 respectively.
Hence the HCF = 22∙3∙x2∙y = 12x2y.

Division method- To find HCF of two numbers by division method, we divide the higher number by the lower number. Then we divide the lower number by the first remainder, the first remainder by the second remainder... and so on, till the remainder is 0. The last divisor is the required HCF.
EXAMPLE
Find the HCF of 288 and 1080 by the division method.
Answer:

```
288|1080|3
     864
     216|288|1
         216
          72|216|3
             216
               0
```

Hence, the last divisor 72 is the HCF of 288 and 1080.

CONCEPT OF CO-PRIME NUMBERS: Two numbers are co-prime to each other if they have no common factor except 1. For example, 15 and 32, 16 and 5, 8 and 27 are the pairs of co-prime numbers. If the HCF of two numbers N1 and N2 be H, then, the numbers left after dividing N1 and N2 by H are co-prime to each other.
Therefore, if the HCF of two numbers be A, the numbers can be written as Ax and Ay, where x and y will be co-prime to each other.

SOLVED PROBLEMS ON HCF

1.Three company of soldiers containing 120, 192, and 144 soldiers are to be broken down into smaller groups such that each group contains soldiers from one company only and all the groups have equal number of soldiers. What is the least number of total groups formed?
Answer: The least number of groups will be formed when each group has number of soldiers equal to the HCF. The HCF of 120, 192 and 144 is 24. Therefore, the numbers of groups formed for the three companies will be 5, 8, and 6, respectively. Therefore, the least number of total groups formed = 5 + 8 + 6 = 19.

2.The numbers 2604, 1020 and 4812 when divided by a number N give the same remainder of 12. Find the highest such number N.
Answer: Since all the numbers give a remainder of 12 when divided by N, hence (2604 – 12), (1020 – 12) and (4812 – 12) are all divisible by N. Hence, N is the HCF of 2592, 1008 and 4800. Now 2592 = 25× 34, 1008 = 24 × 32 × 7 and 4800 = 26 × 3 × 52. Hence, the number N = HCF = 24 × 3 = 48.

3.The numbers 400, 536 and 645, when divided by a number N, give the remainders of 22, 23 and 24 respectively. Find the greatest such number N.
Answer: N will be the HCF of (400 – 22), (536 – 23) and (645 – 24). Hence, N will be the HCF of 378, 513 and 621. à N = 27.

4.The HCF of two numbers is 12 and their sum is 288. How many pairs of such numbers are possible?
Answer: If the HCF if 12, the numbers can be written as 12x and 12y, where x and y are co-prime to each other. Therefore, 12x + 12y = 288 --> x + y = 24.
The pair of numbers that are co-prime to each other and sum up to 24 are (1, 23), (5, 19), (7, 17) and (11, 13). Hence, only four pairs of such numbers are possible. The numbers are (12, 276), (60, 228), (84, 204) and (132, 156).
5.The HCF of two numbers is 12 and their product is 31104. How many such numbers are possible?
Answer: Let the numbers be 12x and 12y, where x and y are co-prime to each other. Therefore, 12x × 12y = 31104 --> xy = 216. Now we need to find co-prime pairs whose product is 216.

216 = 23 × 33. Therefore, the co-prime pairs will be (1, 216) and (8, 27). Therefore, only two such numbers are possible.

LEAST COMMON MULTIPLE (LCM)

The least common multiple (LCM) of two or more numbers is the lowest number which is divisible by all the given numbers.
To calculate the LCM of two or more numbers, we use the following two methods:

Prime Factorization Method: After performing the prime factorization of the numbers, i.e. breaking the numbers into product of prime numbers, we find the highest index, among the given numbers, of all the prime numbers. The LCM is the product of all these prime numbers with their respective highest indices.
EXAMPLE
1.Find the LCM of 72, 288 and 1080.
Answer: 72 = 23 × 32, 288 = 25 × 32, 1080 = 23 × 33 × 5
The prime numbers present are 2, 3 and 5. The highest indices (powers) of 2, 3 and 5 are 5, 3 and 1, respectively.
Hence the LCM = 25 × 33 × 5 = 4320.

2.Find the LCM of 36x3y2 and 24x4y.
Answer: 36x3y2 = 22•32•x3•y2 24x4y = 23•3•x4•y.
The highest indices of 2, 3, x and y are 3, 2, 4 and 2 respectively.
Hence, the LCM = 23•32•x4•y2 = 72x4y2.

Division Method: To find the LCM of 72, 196 and 240, we use the division method in the following way:

```
2 | 72, 240, 196
2 | 36, 120, 98
2 | 18, 60 , 49
3 | 9 , 30 , 49
  | 3  , 10 , 49
```

L.C.M. of the given numbers = product of divisors and the remaining numbers
= 2 × 2 × 2 × 3 × 3 × 10 × 49 = 72 × 10 × 49 = 35280.

Remember!
For **TWO** numbers, HCF x LCM = product of the two numbers

For example, the HCF of 288 and 1020 is 72 and the LCM of these two numbers is 4320. We can see that 72 x 4320 = 288 x 1080 = 311040.

Note- This formula is applicable only for two numbers.

PROPERTIES OF HCF AND LCM

· The HCF of two or more numbers is smaller than or equal to the smallest of those numbers.
· The LCM of two or more numbers is greater than or equal to the

largest of those numbers

- If numbers N1, N2, N3, N4 etc. give remainders R1, R2, R3, R4, respectively, when divided by the same number P, then P is the HCF of (N1 – R1), (N2 – R2), (N3 – R3), (N4 – R4) etc.
- If the HCF of numbers N1, N2, N3 … is H, then N1, N2, N3… can be written as multiples of H (Hx, Hy, Hz..). Since the HCF divides all the numbers, every number will be a multiple of the HCF.
- If the HCF of two numbers N1 and N2 is H, then, the numbers (N1 + N2) and (N1 – N2) are also divisible by H. Let N1 = Hx and N2 = Hy, since the numbers will be multiples of H. Then, N1 + N2 = Hx + Hy = H(x + y), and N1 – N2 = Hx – Hy = H(x – y). Hence both the sum and differences of the two numbers are divisible by the HCF.
- If numbers N1, N2, N3, N4 etc. give an equal remainder when divided by the same number P, then P is a factor of (N1 – N2), (N2 – N3), (N3 – N4)…
- If L is the LCM of N1, N2, N3, N4.. all the multiples of L are divisible by these numbers.
- If a number P always leaves a remainder R when divided by the numbers N1, N2, N3, N4 etc., then P = LCM (or a multiple of LCM) of N1, N2, N3, N4.. + R.

SOLVED PROBLEMS ON LCM

1.Find the highest four-digit number that is divisible by each of the numbers 24, 36, 45 and 60.
Answer: 24 = 23 × 3, 36 = 22 × 32, 45 = 32 × 5 and 60 = 23 × 32 × 5.
Hence, the LCM of 24, 36, 45 and 60 = 23 × 32 × 5 = 360.

2.The highest four-digit number is 9999. 9999 when divided by 360 gives the remainder 279. Hence, the number (9999 – 279 = 3720) will be divisible by 360.
Hence the highest four-digit number divisible by 24, 36, 45 and 60 = 3720.

3.Find the highest number less than 1800 that is divisible by each of the numbers 2, 3, 4, 5, 6 and 7.
Answer: The LCM of 2, 3, 4, 5, 6 and 7 is 420. Hence 420, and every multiple of 420, is divisible by each of these numbers. Hence, the number 420, 840, 1260, and 1680 are all divisible by each of these numbers. We can see that 1680 is the highest number less than 1800 which is multiple of 420.
Hence, the highest number divisible by each one of 2, 3, 4, 5, 6 and 7, and less than 1800 is 1680.

4.Find the lowest number which gives a remainder of 5 when divided by any of the numbers 6, 7, and 8.
Answer: The LCM of 6, 7 and 8 is 168. Hence, 168 is divisible by 6, 7 and 8. Therefore, 168 + 5 = 173 will give a remainder of 5 when divided by these numbers.
5.What is the smallest number which when divided by 9, 18, 24 leaves a remainder of 5, 14 and 20 respectively?

Answer: The common difference between the divisor and the remainder is 4 (9 - 5 = 4, 18 - 14 = 4, 24 - 20 = 4). Now the LCM of 9, 18, and 24 is 72. Now 72 - 4 = 72 - 9 + 5 = 72 - 18 + 14 = 72 - 24 + 20. Therefore, if we subtract 4 from 72, the resulting number will give remainders of 5, 14, and 20 with 9, 18, and 24.
Hence, the number = 72 - 4 = 68.

6.A number when divided by 3, 4, 5, and 6 always leaves a remainder of 2, but leaves no remainder when divided by 7. What is the lowest such number possible?
Answer: the LCM of 3, 4, 5 and 6 is 60. Therefore, the number is of the form 60k + 2, i.e. 62, 122, 182, 242 etc. We can see that 182 is divisible by 7. Therefore, the lowest such number possible = 182.

<u>Unit Digit</u>

To find the units digit of x^y we only consider the units digits of the number x.

To find the units digit of a × b, we only consider the units digits of the numbers a and b.

For example , to calculate units digit of 237^{234} we only consider the units digit of 237 . Hence, we find the units digit of 7^{234} .

To calculate units digit of 233 × 254, we only consider the units digit of 233 and 254 i.e. 3 and 4 , respectively. Hence, we find the units digit of 3 × 4.

To calculate units digit of x^y where x is a single digit number

To calculate units digit of numbers in the form x^y such 7^{253} , 8^{93} , 3^{74} etc.

Case 1: When y is NOT a multiple of 4

We find the remainder when y is divided by 4. Let y = 4q + r where r is the remainder when y is divided by 4, and 0 < r < 4.

The units digit of x^y is the units digit of x^r .

Case 2: When y is a multiple of 4

We observe the following conditions:

Even numbers 2, 4, 6, 8 when raised to powers which are multiple of 4 give the units digit as 6.

Odd numbers 3, 7, and 9 when raised to powers which are multiple of 4 give the units digit as 1.

EXAMPLES

1.Find the units digit of 7^{33}.

The remainder when 33 is divided by 4 is 1. Hence the units digit of 7^{33} is the unit digit of $7^{1} = 7$

2.Find the units digit of 43^{47}.

The units digit of 43^{47} can be found by finding the units digit of 3^{47}. 47 gives a remainder of 3 when divided by 4. Hence units digit = units digit of $3^3 = 7$

3.Find the units digit of $28^{28} – 24^{24}$.

We have to find the units digit of $8^{28} – 4^{24}$. Since 28 and 24 are both multiples of 4, the units digits of both 8^{28} and 4^{24} will be 6. Hence the units digit of the difference will be 0.

4.Find the units digit of $43^{43} – 22^{22}$:

Units digit of 43^{43} is 7 and units digit of 22^{22} is 4. Hence the units digit of the expression will be 7 – 4 = 3.