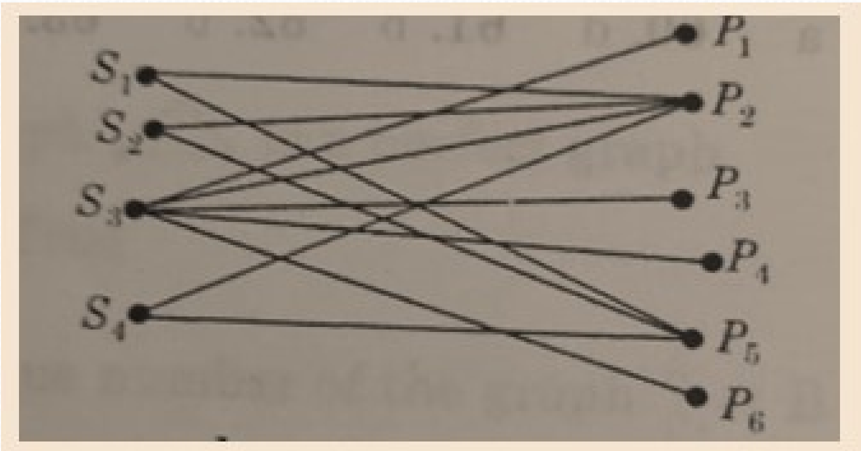
Lecture 3
Graph theory
Course: Discrete Mathematics

Matching

Suppose that four sales representatives S_1, S_2, S_3 and S_4 Are available to be sent to six places P_1, P_2, P_3, P_4, P_5 and P_6. Due to some limitations it is not possible that everyone can be sent to every place. The arrangement can be represented by a graph.



If we consider $V_1 = \{S_1, S_2\ S_3, S_4\}$ and $V_2 = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ then this graph is a bipartite graph.

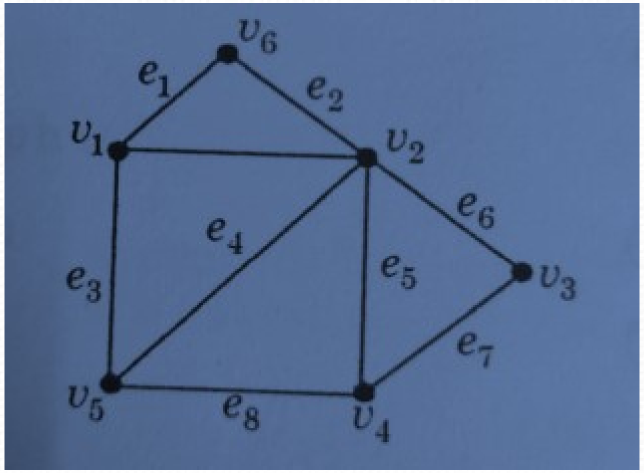
Now the question is “Is it possible to send all sale representatives to some place?” If no then what is the maximum number of places which can be covered.

Here we need the theory of matching.

Definition

A matching in a graph is a subset of edges in which no two edges are adjacent i.e. no two edges have common vertex.

Example



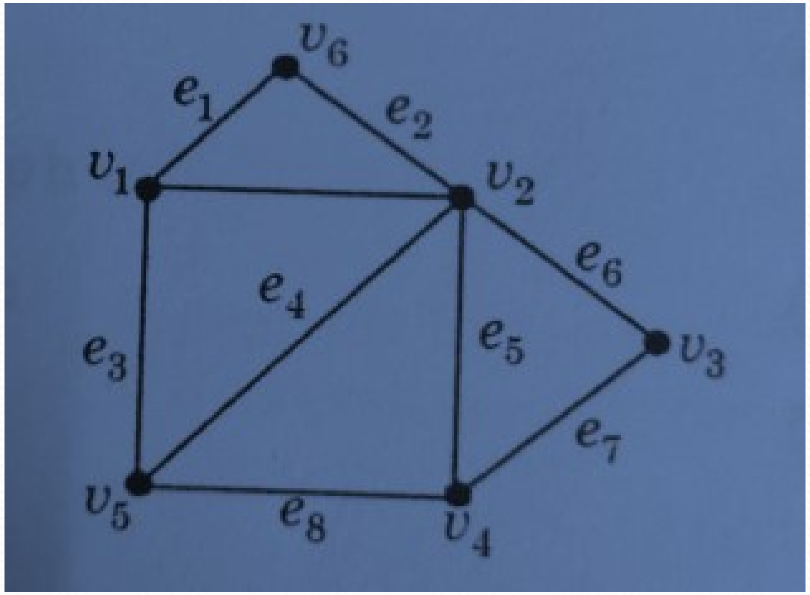
For this graph $M_1 = \{e_1, e_4, e_7\}$, $M_2 = \{e_6, e_8\}$ and $M_3 = \{e_2, e_3, e_7\}$ are matchings.

Definition

Let M be a matching in a graph G with end set E.

M is called a maximal matching if $M \cup \{e\}$ is not a matching for arbitrary $e \in E - M$.

Example



For this graph $M_1 = \{e_1, e_4, e_7\}$ and $M_3 = \{e_2, e_3, e_7\}$

are maximal matchings.

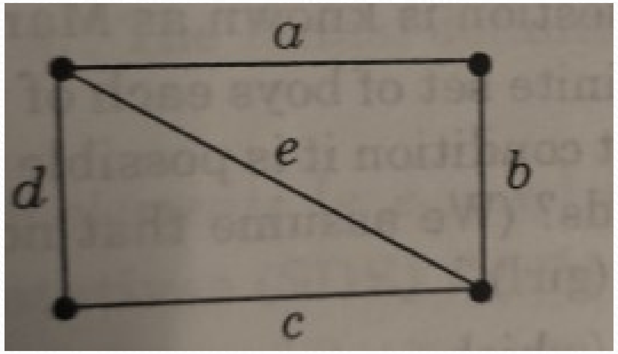
In K_3 every single edge forms a maximal matching.

A graph may have different maximal matchings with different cardinal number.

Definition

The maximal matchings with the largest number of edges are called Maximum Matching.

Example



$\{a, c\}$ and $\{e\}$ are both maximal matchings but $\{a, c\}$ is maximum matching.

Definition

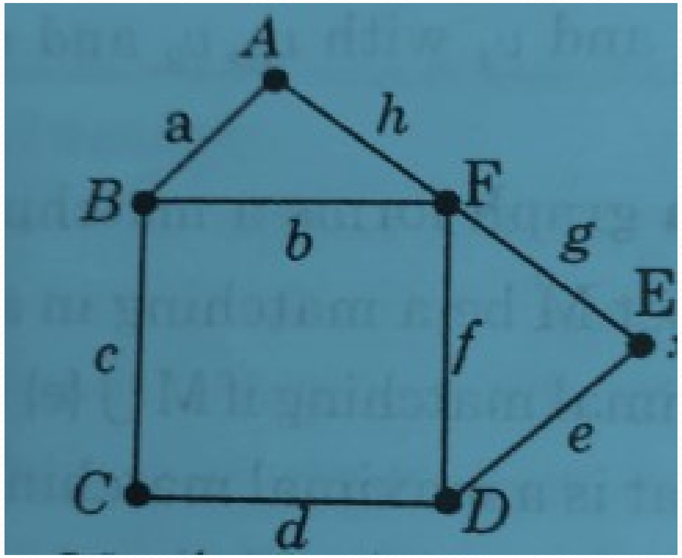
The cardinal number of a maximum matching of a graph is called matching number of the graph.

In the previous example matching number is 2.

Definition

A matching M in a graph G is called a perfect matching if every vertex of the graph G is the end vertex of some edge in M.

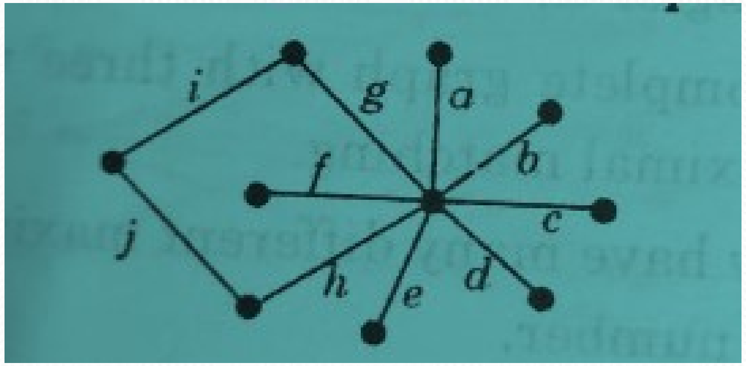
Example



For this graph $\{a, g, d\}$ is a perfect matching.

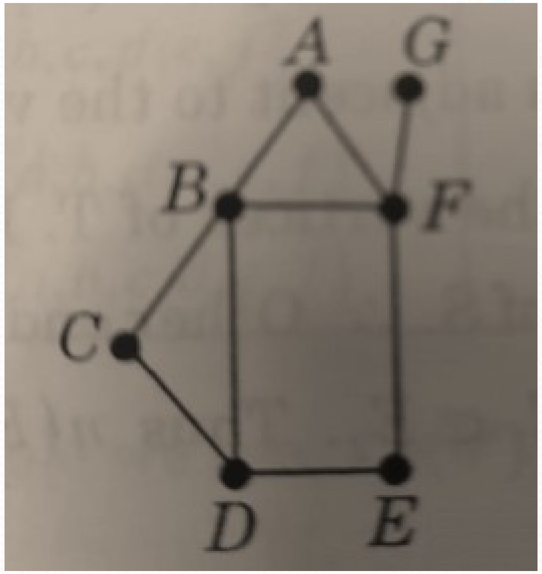
A perfect matching is a maximum matching. But the converse is not true.

Example



In this example $\{j, d\}$ is a maximum matching but it is not perfect.

Find a maximal matching in the following graph

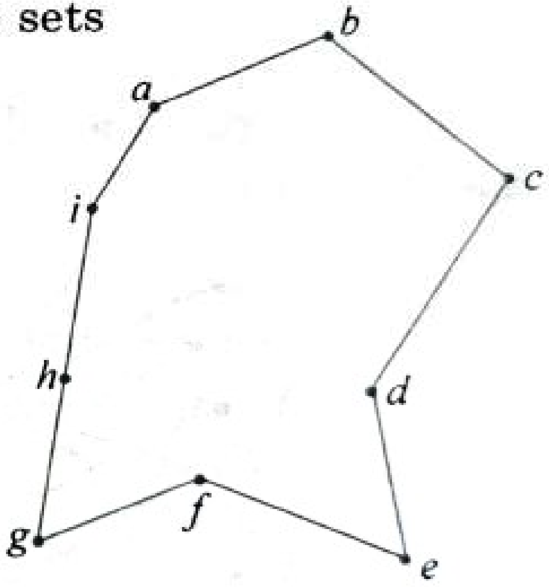


Solution: The set M={BC,DE,FG} is a maximal matching because if we add any one of the edges AB,AF,BF,BD,CD,EF to M it will no longer be a matching.

Ex.3. C_9, is a cycle(i.e.a circular chair)with the nine vertices a,b,c,d,e,f,g,h,i. How many distinct maximal matchings of size four in C_9 contain the edge ab? [WBUT 2012, 2014, 2016]

Solution. In the adjacent figure we have C_9.

The maximal matching are the edg sub sets



$M_1=\{ab,cd,ef,gh\}$, $M_2=\{bc,de,fg,hi\}$

$M_3=\{cd,ef,gh,ia\}$, $M_4=\{de,fg,hi,ab\}$

$M_5=\{ef,gh,ia,bc\}$, $M_6=\{fg,hi,ab,cd\}$

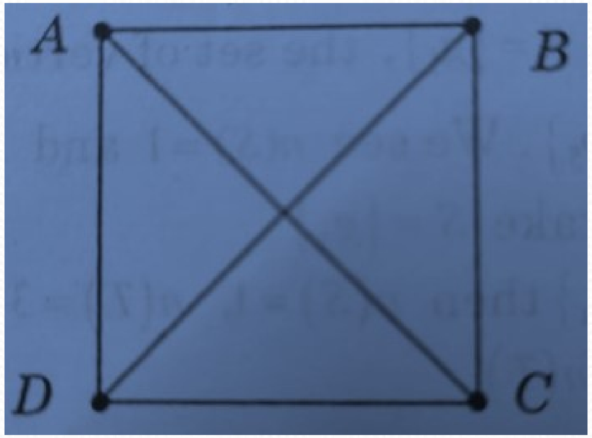
$M_7=\{gh,ia,bc,de\}$, $M_8=\{hi,ab,cd,ef\}$

$M_9=\{ia,bc,de,fg\}$, $M_{10}=\{ab,cd,ef,gh\}=M_1$.

$\therefore$ the required maximal matchins are M_1, M_4, M_6, M_8, and M_{10}.

How many different perfect matching are there in K_4.

Solution:



The perfect matchings are {AC,BD} and {AB,DC}.

4.2.3 The Marriage Problem

The following question is known as Marriage Problem :

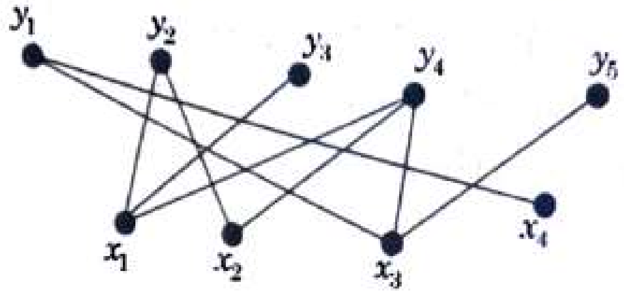
"Let we have a finite set of boys each of whom has several girl friends. Under what condition it is possible that each boy marries one of his girl friends? (We assume that no girl (boy) can marry more than one boy (girl)."

Hall's Theorem (which is going to be stated in this Article) answers this questions. Before that we go through the following example and pose the problem in graph theoretical term :

Let V_1 = set of four boys $=\{x_1, x_2, x_3, x_4\}$ and V_2 = set of five girls $=\{y_1, y_2, y_3, y_4, y_5\}$. In the following table their relationship are given

boy	girl friends
x_1	y_2, y_3, y_4
x_2	y_2, y_4
x_3	y_1, y_4, y_5
x_4	y_1

From this we get the following bipartite graph



A particular solution of this problem is the matching $M = \{(x_1y_3), (x_2y_2), (x_3y_4), (x_4y_1)\}$ that is x_1 to marry y_3; x_2 to marry y_2, x_3 to marry y_4 and x_4 to marry y_1.

Note that though the vertex y_5 is not an end vertex of any edge of M, every vertex of V_1 is end vertex of some edge of M.

If we consider the edge- subsets of the above graph,

$A_1 = \{(x_1y_2),(x_1y_3),(x_1y_4)\}$, $A_2 = \{(x_2y_2),(x_2y_4)\}$

$A_3 = \{(x_3y_1),(x_3y_4),(x_3y_5)\}$ and $A_4 = \{(x_4y_1)\}$

then the above matching M is called a **System of distinct Representatives (SDR)** of the family of finite sets $\{A_1, A_2, A_3, A_4\}$ because the element $(x_1 y_3) \in A_1$, $(x_2 y_2) \in A_2$, $(x_3y_4) \in A_3$ and $(x_4y_1) \in A_4$. The formal definition of SDR is given below:

A set of n *number of elements* $\{e_1, e_2, \ldots, e_n\}$ *is called a* **System of Distinct Representatives (SDR)** *for the family of finite set* $S = \{A_1, A_2, \ldots, A_n\}$ *if* $e_i \in A_i$ *for* $i = 1, 2, \ldots n$.

For example if $A_1 = \{1, 2, 3\}$, $A_2 = \{1, 4, 5\}$, $A_3 = \{3, 5\}$ *them* $\{1, 4, 5\}$ *is an SDR but* $\{1, 2, 5\}$ *is not.*

Marriage Condition (MC)

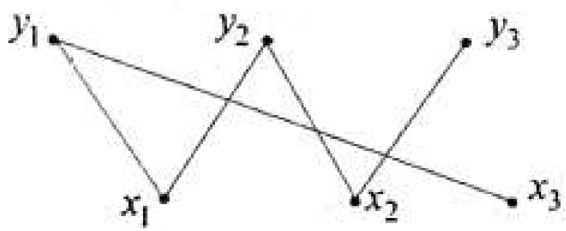
G be a bipartite graph with two partitioned vertex set V_1 and V_2. Marriage Condition states that for arbitrary set $S \subset V_1$

$n(S) \leq n(T)$ where T is the set of all vertices adjacent to vertices in S.

Speaking more generally: This condition states that for any family of finite sets $S = \{A_1, A_2, \ldots, A_n\}$ if $W \subset S$ be arbitrary we have $n(W) \leq n\left\{\bigcup_i A_i ; A_i \in W\right\}$.

Illustration.

(1) Consider the following bipartite graph G

G: 

From the above table we see $n(S) \le n(T)$ for arbitrary set $S \subset V_1$. So for this graph Marriage condition is satisfied.

$V_1 = \{x_1, x_2, x_3\}$, $V_2 = \{y_1, y_2, y_3\}$

We construct the following table (S is a subset of V_1 and T is set of all vertices adjacent to the vertices in S):

S	T	$n(S)$	$n(T)$
$\{x_1\}$	$\{y_1, y_2\}$	1	2
$\{x_2\}$	$\{y_2, y_3\}$	1	2
$\{x_3\}$	$\{y_1\}$	1	1
$\{x_1, x_2\}$	$\{y_1, y_2, y_3\}$	2	3
$\{x_1, x_3\}$	$\{y_1, y_2\}$	2	2
$\{x_2, x_3\}$	$\{y_2, y_3, y_1\}$	2	3
$\{x_1, x_2, x_3\}$	$\{y_1, y_2, y_3\}$	3	3

(2) Consider the family of finite set $S=\{A_1, A_2, A_3, A_4, A_5\}$ where the finite sets are $A_1=\{a,b,c,d,e\}$, $A_2=\{c,d,e\}$, $A_3=\{d,e\}$, $A_4=\{c,e\}$, and $A_5=\{e\}$.

Here consider the subset $W=\{A_2, A_3, A_4, A_5,\}$.

We see $n(W)=4$.

and $\bigcup_i \{A_i : A_i \in W\} = A_2 \cup A_3 \cup A_4 \cup A_5$

$$= \{c,d,e\}$$

$$\therefore n\left(\bigcup_i \{A_i : A_i \in W\}\right) = 3$$

$$\therefore n(W) \nleq n\left(\bigcup_i \{A_i : A_i \in W\}\right)$$

$\therefore$ S does not satisfy marriage condition.

Hall's Marriage Theorem

Let G be a bipartite graph with two partitioned vertex set V_1 and V_2. There exists a matching in G such that every vertex of V_1 is an end vertex of some edge in the matching if and only if G satisfies Marriage condition.

Speaking more generally: A family of finite set S has a SDR if and only if S satisfies the marriage condition.

Illustration. To illustrate the graphical version of Hall's theorem we refer some Ex. in the section of Illustrative Examples.

To illustrate the general-version of Hall's theorem, consider the family of finite set $S=\{A_1, A_2, A_3, A_4\}$ where $A_1=(a,b,d,e)$, $A_2=\{b,c,d,e\}$, $A_3=\{c,f\}$ and $A_4=\{b,c,f\}$. [WBUT 2012]

Construct the following table where W is arbitrary subset of S.

W	$\bigcup_i \{A_i : A_i \in W\}$	$n(W)$	$n\left(\bigcup_i \{A_i : A_i \in W\}\right)$
$\{A_1\}$	$\{a,b,d,e\}$	1	4
$\{A_2\}$	$\{b,c,d,e\}$	1	4
$\{A_3\}$	$\{c,f\}$	1	4
$\{A_4\}$	$\{b,c,f\}$	1	3
$\{A_1,A_2\}$	$\{a,b,d,e,c\}$	2	5
$\{A_1,A_3\}$	$\{a,b,d,e,c,f\}$	2	6
$\{A_1,A_4\}$	$\{a,b,d,e,c,f\}$	2	5
$\{A_2,A_3\}$	$\{b,c,d,e,f\}$	2	5
$\{A_2,A_4\}$	$\{b,c,d,e,f\}$	2	5
$\{A_3,A_4\}$	$\{c,f,b\}$	2	3
$\{A_1,A_2,A_3\}$	$\{a,b,d,e,c,f\}$	3	6
$\{A_1,A_2,A_4\}$	$\{a,b,d,e,c,f\}$	3	6
$\{A_2,A_3,A_4\}$	$\{b,c,d,e,f\}$	3	5
$\{A_1,A_3,A_4\}$	$\{a,b,d,e,c,f\}$	3	6
$\{A_1,A_2,A_3,A_4\}$	$\{a,b,c,d,e,f\}$	4	6

From the above table we see $n(W) \le n\left(\bigcup_i \{A_i : A_i \in W\}\right)$ for

arbitrary $W \subset S$.

$\therefore$ Marriage condition is satisfied by S. Theorefore by Hall's theorem there exist a SDR of S. We see one SDR is $\{a, b, e, f\}$ and another SDR is $\{d, c, f, b\}$.

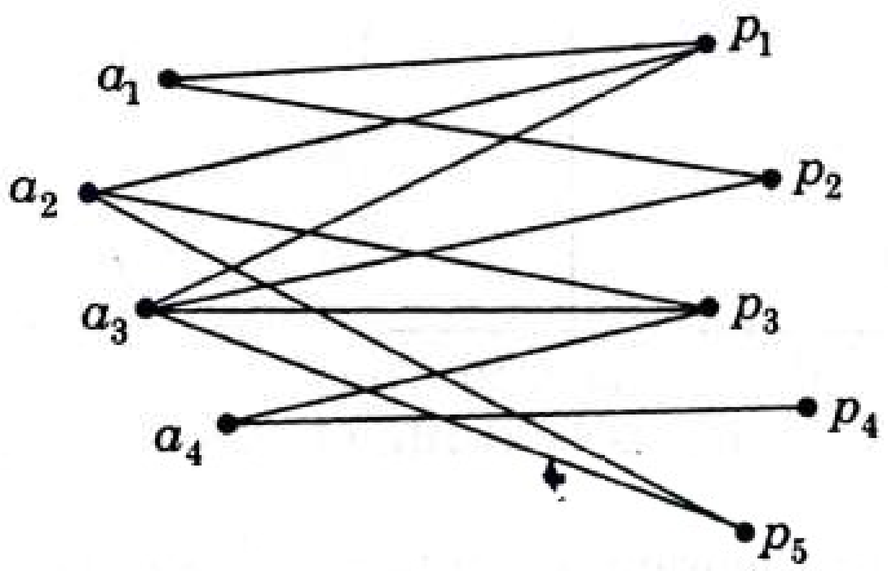
Ex. 5. Applicants a_1, a_2, a_3 and a_4 apply for five posts p_1, p_2, p_3, p_4 and p_5. The applications are done as follows :

$$a_1 \to \{p_1, p_2\},$$
$$a_2 \to \{p_1, p_3, p_5\},$$
$$a_3 \to \{p_1, p_2, p_3, p_5\} \quad \text{and}$$
$$a_4 \to \{p_3, p_4\}.$$

Using graph theory find (i) whether there is any perfect matching of the set of applicants into the set of posts.

If yes, find matching (ii) whether every applicant can be offered a single post. [WBUT 2013]

Solution. The given correspondance can be represented in the following way :



This is a bipartite graph with the two disjoint vertex set $V_1 = \{a_1, a_2, a_3, a_4\}$ and $V_2 = \{p_1, p_2, p_3, p_4, p_5\}$.

(i) Here we see the edge set $\{a_1p_1, a_2p_3, a_3p_5, a_4p_4\}$ is maximum matching which is not perfect.

This graph has no perfect matching

(ii) This can be treated as a Marriage Problem. We construct the following table :

Sub-set of vertices in V_1 (1)	Sub-set of adjacent vertices in V_2 (2)
$\{a_1\}$	$\{p_1, p_2\}$
$\{a_2\}$	$\{p_1, p_3, p_5\}$
$\{a_3\}$	$\{p_1, p_2, p_3, p_5\}$
$\{a_4\}$	$\{p_3, p_4\}$
$\{a_1a_2\}$	$\{p_1, p_2, p_3, p_5\}$
$\{a_1a_3\}$	$\{p_1, p_2, p_3, p_5\}$
$\{a_1a_4\}$	$\{p_1, p_2, p_3, p_4\}$
$\{a_2a_3\}$	$\{p_1, p_3, p_5, p_2\}$
$\{a_2a_4\}$	$\{p_1, p_3, p_5, p_4\}$
$\{a_3a_4\}$	$\{p_1, p_2, p_3, p_5, p_4\}$
$\{a_1a_2a_3\}$	$\{p_1, p_2, p_3, p_5\}$
$\{a_1a_2a_4\}$	$\{p_1, p_2, p_3, p_5, p_4\}$
$\{a_2a_3a_4\}$	$\{p_1, p_3, p_5, p_2, p_4\}$

In the above table we see, in every row cardinal number of every set in 2nd column is $\geq$ cardinal number of the set in 1st column.

Thus by Hall's theorem a solution of this Marriage problem exist. That is every applicant can be offered a single post.

Ex.5. In an area the

7. Suppose there are 5 councillors c_1, c_2, c_3, c_4 and c_5 who are members of three subcommittees s_1, s_2 and s_3.

The membership are as follow : $s_1 \rightarrow \{c_1, c_2\}$, $s_2 \rightarrow \{c_1, c_3, c_4\}$, $s_3 \rightarrow \{c_3, c_4, c_5\}$. It is required to form Borough committee in which one member (councilor) from each of the subcommittees is to represent. Is it possible to send one distinct councilor to the Borough committee from each of the subcommittees.

Solve the problem by using Hall's Theorem and if found, obtain a match.

8. Consider $S = \{A_1, A_2, A_3, A_4\}$ where $A_1 = \{a, b, d, e\}$, $A_2 = \{b, c, d, e, f\}$, $A_3 = \{e, f\}$, $A_4 = \{b, e, f\}$. Show whether S satisfies marriage condition. If yes find valid SDR of S

Ex.5. In an area the custom is that girls give-their choice to whom they will marry. One such choice is given below. Give your decision for matching and decide whether each girl will be married within thrir choice.

$g_1 \to (b_1, b_3, b_5)$, $g_2 \to (b_1, b_2, b_4)$, $g_3 \to (b_3, b_4, b_5)$

$g_4 \to (b_2, b_3, b_5)$, $g_5 \to (b_2, b_3, b_4)$, [WBUT 2014]

To be continued……

Thanks for watching

Have a nice day