Topic: Propositional Logic

Lec-4

Course: Discrete Mathematics

Argument

An argument in propositional logic is a sequence of propositions.

All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true.

Rules of Inference

- Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

- Modus tollens

$$\begin{array}{r} \neg q \\ p \rightarrow q \\ \hline \therefore\ \neg p \end{array}$$

- Hypothetical syllogism

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

- Disjunctive syllogism

$$\begin{array}{rl} & p \vee q \\ & \neg p \\ \hline \therefore & q \end{array}$$

- Addition

$$\begin{array}{rl} & p \\ \hline \therefore & p \vee q \end{array}$$

- Simplification

$$\begin{array}{rl} & p \wedge q \\ \hline \therefore & p \end{array}$$

- Conjunction

$$\begin{array}{ll} & p \\ & q \\ \hline \therefore & p \wedge q \end{array}$$

- Resolution

$$\begin{array}{ll} & p \vee q \\ & \neg p \vee r \\ \hline \therefore & q \vee r \end{array}$$

Predicates and Quantifiers

Predicates

Statements involving variables, such as

$$\text{“}x > 3,\text{”} \quad \text{“}x = y + 3,\text{”} \quad \text{“}x + y = z,\text{”}$$

Definition

$P(x)$ is called Predicate or Propositional function if $P(x)$ is a proposition for each value of x.

Quantifiers

In English, the words all, some, many, none, and few are used in quantifications.

We will focus on two types of quantification here: universal quantification and existential quantification.

The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Definition

The universal quantification of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.

We read $\forall x P(x)$ as "for all $x P(x)$" or "for every $x P(x)$."

An element for which $P(x)$ is false is called

counter example of $\forall x P(x)$.

Remark

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \cdots \land P(x_n),$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2), \ldots, P(x_n)$ are all true.

Definition

The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Here $\exists$ is called the existential quantifier.

A domain must always be specified when a statement $\exists x P(x)$ is used.

Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Remark

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \vee P(x_2) \vee \cdots \vee P(x_n),$$

Logical Equivalences Involving Quantifiers

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Rules of Inference for Quantified Statements

- Universal instantiation

$$\frac{\forall x P(x)}{\therefore\ P(c)}$$

- Universal generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$$

❖ Existential instantiation

$$\begin{array}{rl} & \exists x\, P(x) \\ \hline \therefore & P(c) \text{ for some element } c \end{array}$$

❖ Existential generalization

$$\begin{array}{rl} & P(c) \text{ for some element } c \\ \hline \therefore & \exists x\, P(x) \end{array}$$

Example

Show that the premises

“A student in this class has not read the book,”

and “Everyone in this class passed the first exam”

imply the conclusion “Someone who passed the first exam has not read the book.”

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and

$P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \land \neg B(x))$ and $\forall x(C(x) \rightarrow P(x))$.

The conclusion is
$\exists x(P(x) \land \neg B(x))$.

These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \land \neg B(x))$	Premise
2. $C(a) \land \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \to P(x))$	Premise
5. $C(a) \to P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \land \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \land \neg B(x))$	Existential generalization from (8)

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

(In Section 1.6 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion," "x is fierce," and "x drinks coffee," respectively. Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

Solution: We can express these statements as:

$$\forall x(P(x) \to Q(x)).$$
$$\exists x(P(x) \land \neg R(x)).$$
$$\exists x(Q(x) \land \neg R(x)).$$

Notice that the second statement cannot be written as $\exists x(P(x) \to \neg R(x))$. The reason is that $P(x) \to \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \to \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \to \neg R(x)).$$

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect.

Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

d) Does (c) follow from (a) and (b)?

Nested Quantifiers

Assume that the domain for the variables x and y consists of all real numbers. The statement

$$\forall x \forall y(x + y = y + x)$$

says that $x + y = y + x$ for all real numbers x and y. This is the commutative law for addition

$$\forall x \exists y(x + y = 0)$$

says that for every real number x there is a real number y such that $x + y = 0$. This states that every real number has an additive inverse. Similarly, the statement

$$\forall x \forall y \forall z(x + (y + z) = (x + y) + z)$$

is the associative law for addition of real numbers.

Translate into English the statement

$$\forall x \forall y((x > 0) \land (y < 0) \rightarrow (xy < 0)),$$

where the domain for both variables consists of all real numbers.

Solution: This statement says that for every real number x and for every real number y, if $x > 0$ and $y < 0$, then $xy < 0$. That is, this statement says that for real numbers x and y, if x is positive and y is negative, then xy is negative. This can be stated more succinctly as "The product of a positive real number and a negative real number is always a negative real number." ◀

The Order of Quantifiers

Let $P(x, y)$ be the statement "$x + y = y + x$." What are the truth values of the quantifications $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\forall x \forall y P(x, y)$$

denotes the proposition

"For all real numbers x, for all real numbers y, $x + y = y + x$."

Because $P(x, y)$ is true for all real numbers x and y (it is the commutative law for addition, which is an axiom for the real numbers—see Appendix 1), the proposition $\forall x \forall y P(x, y)$ is true. Note that the statement $\forall y \forall x P(x, y)$ says "For all real numbers y, for all real numbers x, $x + y = y + x$." This has the same meaning as the statement "For all real numbers x, for all real numbers y, $x + y = y + x$." That is, $\forall x \forall y P(x, y)$ and $\forall y \forall x P(x, y)$ have the same meaning, Let $Q(x, y)$ denote "$x + y = 0$." What are the truth values of the quantifications $\exists y \forall x Q(x, y)$ and $\forall x \exists y Q(x, y)$, where the domain for all variables consists of all real numbers?

Solution: The quantification

$$\exists y \forall x Q(x, y)$$

denotes the proposition

"There is a real number y such that for every real number x, $Q(x, y)$."

No matter what value of y is chosen, there is only one value of x for which $x + y = 0$. Because there is no real number y such that $x + y = 0$ for all real numbers x, the statement $\exists y \forall x Q(x, y)$ is false.

The quantification

$$\forall x \exists y Q(x, y)$$

denotes the proposition

"For every real number x there is a real number y such that $Q(x, y)$."

Given a real number x, there is a real number y such that $x + y = 0$; namely, $y = -x$. Hence, the statement $\forall x \exists y Q(x, y)$ is true.

Normal Forms

- Normal forms are standard forms, sometimes called canonical or accepted forms.

- A logical expression is said to be in *disjunctive normal form (DNF)* if it is written as a disjunction, in which all *terms* are conjunctions of *literals*.

- Similarly, a logical expression is said to be in *conjunctive normal form (CNF)* if it is written as a conjunction of disjunctions of literals.

- Disjunctive Normal Form (DNF)

$$(.. \wedge .. \wedge ..) \vee (.. \wedge .. \wedge ..) \vee \ldots \vee (.. \wedge ..)$$

Term

Literal, i.e. P or $\neg$P

Examples: $(P \wedge Q) \vee (P \wedge \neg Q)$

$P \vee (Q \wedge R)$

- Conjunctive Normal Form (CNF)

$$(.. \vee .. \vee ..) \wedge (.. \vee .. \vee ..) \wedge \ldots \wedge (.. \vee ..)$$

Examples: $(P \vee Q) \wedge (P \vee \neg Q)$

$P \wedge (Q \vee R)$

- $(x \vee \neg z \vee y) \wedge (\neg x \vee \neg y) \wedge (\neg y)$ is a CNF
- $(x \wedge z) \vee (\neg y \wedge z \wedge x) \vee (\neg x \wedge z)$ is a DNF
- $(x \wedge \neg(y \vee z) \vee u)$ is neither a CNF nor DNF, but is equivalent to DNF $((x \wedge \neg y \wedge \neg z) \vee u)$

CNF examples

Consider $(p \Rightarrow (\neg q \wedge r)) \wedge (p \Rightarrow \neg q)$

$$\equiv (\neg p \vee (\neg q \wedge r)) \wedge (\neg p \vee \neg q)$$

$$\equiv ((\neg p \vee \neg q) \wedge (\neg p \vee r)) \wedge (\neg p \vee \neg q)$$

$$\equiv (\neg p \vee \neg q) \wedge (\neg p \vee r) \wedge (\neg p \vee \neg q)$$

Exercise

Convert the following formulas into CNF

1. $\neg((p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r)))$
2. $(p \Rightarrow (\neg q \Rightarrow r)) \wedge (p \Rightarrow \neg q) \Rightarrow (p \Rightarrow r)$

From truth table to DNF

If a function, e.g. **F**, is given by a truth table, we know exactly for which assignments it is true.

F is true for three assignments:

- p, q, r are all true, $(p \wedge q \wedge r)$
- p, ¬q, r are all true, $(p \wedge \neg q \wedge r)$
- ¬p, ¬q, r are all true, $(\neg p \wedge \neg q \wedge r)$

DNF of F: $(p \wedge q \wedge r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge r)$

p	q	r	F
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	0
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

- **Complementation** can be used to obtain **conjunctive normal forms** from truth tables.

- If A is a formula containing only the connectives $\neg$, $\vee$ and $\wedge$, then its complement is formed by
  - replacing all $\vee$ by $\wedge$
  - replacing all $\wedge$ by $\vee$
  - replacing all atoms by their complements.

- Example: Find the complement of the formula
  - $(p \wedge q) \vee \neg r$

    $(\neg p \vee \neg q) \wedge r$

From truth table to CNF

- Solution: $\neg$G is **true** for the following assignments.

  $p = 1;\ q = 0;\ r = 1$
  $p = 1;\ q = 0;\ r = 0$
  $p = 0;\ q = 0;\ r = 1$

- The **DNF** of $\neg$G is therefore:

$$(p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$

- The formula has the complement:

$$(\neg p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (p \vee q \vee \neg r)$$

  - It is the desired CNF of G

p	q	r	G
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	1
0	0	1	0
0	0	0	1

How to find the DNF of $(p \vee q) \to \neg r$

p	q	r	$(p \vee q)$	$\neg r$	$(p \vee q) \to \neg r$
T	T	T	T	F	F
T	*T*	*F*	*T*	*T*	*T*
T	F	T	T	F	F
T	*F*	*F*	*T*	*T*	*T*
F	T	T	T	F	F
F	*T*	*F*	*T*	*T*	*T*
F	*F*	*T*	*F*	*F*	*T*
F	*F*	*F*	*F*	*T*	*T*

From the truth table we can set up the DNF

$$(p \vee q) \to \neg r \Leftrightarrow (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \vee (\neg p \wedge \neg q \wedge \neg r)$$

To be continued……

Thanks for watching

Have a nice day