Topic: Propositional Logic

Lec-3

Course: Discrete Mathematics

So far we have learned

Conjunction : Let p and q be two propositions. Then the conjunction of p and q denoted by $p \wedge q$,is the statement

"p and q".

Disjunction : Let p and q be two propositions. Then the disjunction of p and q denoted by $p \vee q$,is the statement

"p or q".

Conditional Statement

Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q."

Argument

An argument in propositional logic is a sequence of propositions.

All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

An argument is valid if the truth of all its premises implies that the conclusion is true.

Rules of Inference

- Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

- Modus tollens

$$\begin{array}{r} \neg q \\ p \rightarrow q \\ \hline \therefore\ \neg p \end{array}$$

- Hypothetical syllogism

$$\begin{array}{r} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore\ p \rightarrow r \end{array}$$

- Disjunctive syllogism

$$\begin{array}{l} \quad p \vee q \\ \quad \neg p \\ \hline \therefore\ q \end{array}$$

- Addition

$$\begin{array}{l} \quad p \\ \hline \therefore\ p \vee q \end{array}$$

- Simplification

$$\begin{array}{l} \quad p \wedge q \\ \hline \therefore\ p \end{array}$$

- Conjunction

$$\begin{array}{rl} & p \\ & q \\ \hline \therefore & p \land q \end{array}$$

- Resolution

$$\begin{array}{rl} & p \lor q \\ & \neg p \lor r \\ \hline \therefore & q \lor r \end{array}$$

Predicates and Quantifiers

Predicates

Statements involving variables, such as

$$\text{“}x > 3\text{,”} \quad \text{“}x = y + 3\text{,”} \quad \text{“}x + y = z\text{,”}$$

"computer x is under attack by an intruder,"

"computer x is functioning properly,"

These statements are neither true nor false when the values of the variables are not specified.

Definition

$P(x)$ is called Predicate or Propositional function if $P(x)$ is a proposition for each value of x.

Once a value has been assigned to the variable x, the statement $P(x)$ becomes a proposition and has a truth value.

EXAMPLE

Let $P(x)$ denote the statement "$x > 3$." What are the truth values of $P(4)$ and $P(2)$?

Solution:

We obtain the statement $P(4)$ by setting $x = 4$ in the statement "$x > 3$." Hence, $P(4)$, which is the statement "$4 > 3$," is true.

However, $P(2)$, which is the statement "$2 > 3$," is false.

Example

Let $A(x)$ denote the statement "Computer x is under attack by an intruder." Suppose that of the computers on campus, only CS2 and MATH1 are currently under attack by intruders. What are truth values of A(CS1), A(CS2), and A(MATH1)?

Solution: We obtain the statement A(CS1) by setting $x =$ CS1 in the statement "Computer x is under attack by an intruder." Because CS1 is not on the list of computers currently under attack, we conclude that A(CS1) is false.

Similarly, because CS2 and MATH1 are on the list of computers under attack, we know that A(CS2) and A(MATH1) are true.

We can have Propositional function with two or more variables.

Example

Let $Q(x, y)$ denote the statement "$x = y + 3$." What are the truth values of the propositions $Q(1, 2)$ and $Q(3, 0)$?

Quantifiers

In English, the words all, some, many, none, and few are used in quantifications.

We will focus on two types of quantification here: universal quantification and existential quantification.

The area of logic that deals with predicates and quantifiers is called the **predicate calculus**.

Definition

The universal quantification of $P(x)$ is the statement

"$P(x)$ for all values of x in the domain."

The notation $\forall x P(x)$ denotes the universal quantification of $P(x)$.

We read $\forall x P(x)$ as "for all $xP(x)$" or "for every $xP(x)$."

An element for which $P(x)$ is false is called

counter example of $\forall x P(x)$.

Example

Let $P(x)$ be the statement "$x + 1 > x$." What is the truth value of the quantification $\forall x P(x)$, where the domain consists of all real numbers?

Solution:

Because $P(x)$ is true for all real numbers x, the quantification

$$\forall x P(x) \text{ is true.}$$

Example

Let $Q(x)$ be the statement "$x < 2$."

What is the truth value of the quantification $\forall x Q(x)$, where the domain consists of all real numbers?

$Q(x)$ is not true for every real number x, because, for instance, $Q(3)$ is false. That is,

$x = 3$ is a counterexample for the statement $\forall x Q(x)$.
Thus

$\forall x Q(x)$ is false.

Remark

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the universal quantification $\forall x P(x)$ is the same as the conjunction

$$P(x_1) \land P(x_2) \land \cdots \land P(x_n),$$

because this conjunction is true if and only if $P(x_1)$, $P(x_2), \ldots, P(x_n)$ are all true.

Definition

The existential quantification of $P(x)$ is the proposition "There exists an element x in the domain such that $P(x)$."

We use the notation $\exists x P(x)$ for the existential quantification of $P(x)$.

Here $\exists$ is called the existential quantifier.

A domain must always be specified when a statement $\exists x P(x)$ is used.

Furthermore, the meaning of $\exists x P(x)$ changes when the domain changes. Without specifying the domain, the statement $\exists x P(x)$ has no meaning.

Let P(x) denote the statement "$x > 3$."

What is the truth value of the quantification $\exists x P(x)$, where the domain consists of all real numbers?

Because "$x > 3$" is sometimes true—for instance, when $x = 4$, the existential quantification of $P(x)$, which is $\exists x P(x)$, is true.

Let $Q(x)$ denote the statement "$x = x + 1$."

What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

Solution: Because $\mathrm{Q(x)}$ is false for every real number x, the existential quantification of $Q(x)$, which is $\exists x Q(x)$, is false.

Remark

When all the elements in the domain can be listed—say, $x_1, x_2, \ldots, x_n$—it follows that the existential quantification $\exists x P(x)$ is the same as the disjunction

$$P(x_1) \lor P(x_2) \lor \cdots \lor P(x_n),$$

Precedence of Quantifiers

The quantifiers $\forall$ and $\exists$ have higher precedence than all logical operators from propositional calculus.

For example, $\forall x P(x) \lor Q(x)$ is the disjunction of $\forall x P(x)$ and $Q(x)$.

In other words, it means $(\forall x P(x)) \lor Q(x)$ rather than $\forall x (P(x) \lor Q(x))$.

Binding Variables

When a quantifier is used on the variable x, we say that this occurrence of the variable is **bound**.

An occurrence of a variable that is not bound by a quantifier or set equal to a particular value is said to be **free**.

Example

In the statement $\exists x(x + y = 1)$, the variable x is bound by the existential quantification $\exists x$, but the variable y is free because it is not bound by a quantifier and no value is assigned to this variable.

This illustrates that in the statement $\exists x(x + y = 1), x$ is bound, but y is free.

In the statement $\exists x(P(x) \wedge Q(x)) \vee \forall x R(x)$, all variables are bound.

Logical Equivalences Involving Quantifiers

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Translating from English into Logical Expressions

Example

Express the statement

“Every student in this class has studied calculus”

using predicates and quantifiers.

Solution:

We introduce $C(x)$, which is the statement
"x has studied calculus."

Consequently, if the domain for x consists of the students in the class, we can translate our statement as $\forall x C(x)$.

Example

Consider these statements, of which the first three are premises and the fourth is a valid conclusion.

"All hummingbirds are richly coloured."
"No large birds live on honey."
"Birds that do not live on honey are dull in colour."
"Hummingbirds are small."

Let $P(x), Q(x), R(x)$, and $S(x)$ be the statements "x is a hummingbird," "x is large," "x lives on honey," and "x is richly coloured," respectively.

Assuming that the domain consists of all birds, express the statements in the argument using quantifiers and $P(x), Q(x), R(x)$, and $S(x)$.

Solution:

We can express the statements in the argument as

$$\forall x(P(x) \rightarrow S(x)).$$

$$\neg \exists x(Q(x) \land R(x)).$$

$$\forall x(\neg R(x) \rightarrow \neg S(x)).$$

$$\forall x(P(x) \rightarrow \neg Q(x)).$$

Rules of Inference for Quantified Statements

- Universal instantiation

$$\frac{\forall x P(x)}{\therefore\ P(c)}$$

- Universal generalization

$$\frac{P(c) \text{ for an arbitrary } c}{\therefore\ \forall x P(x)}$$

❖ Existential instantiation

$$\frac{\exists x\, P(x)}{\therefore\ P(c) \text{ for some element } c}$$

❖ Existential generalization

$$\frac{P(c) \text{ for some element } c}{\therefore\ \exists x\, P(x)}$$

Example

Show that the premises

"Everyone in this discrete mathematics class has taken a course in computer science" and

"Marla is a student in this class"

imply the conclusion "Marla has taken a course in computer science."

Solution:
Let $D(x)$ denote "x is in this discrete mathematics class,"

and
let $C(x)$ denote "x has taken a course in computer science."

Then the premises are $\forall x(D(x) \rightarrow C(x))$ and D(Marla).
The conclusion is C(Marla).

The following steps can be used to establish the conclusion from the premises.

Step	**Reason**
1. $\forall x(D(x) \rightarrow C(x))$	Premise
2. $D(\text{Marla}) \rightarrow C(\text{Marla})$	Universal instantiation from (1)
3. $D(\text{Marla})$	Premise
4. $C(\text{Marla})$	Modus ponens from (2) and (3)

Example

Show that the premises

"A student in this class has not read the book,"

and "Everyone in this class passed the first exam"

imply the conclusion "Someone who passed the first exam has not read the book."

Solution: Let $C(x)$ be "x is in this class," $B(x)$ be "x has read the book," and

$P(x)$ be "x passed the first exam."

The premises are $\exists x(C(x) \wedge \neg B(x))$ and
$\forall x(C(x) \rightarrow P(x))$.

The conclusion is
$\exists x(P(x) \wedge \neg B(x))$.

These steps can be used to establish the conclusion from the premises.

Step	Reason
1. $\exists x(C(x) \wedge \neg B(x))$	Premise
2. $C(a) \wedge \neg B(a)$	Existential instantiation from (1)
3. $C(a)$	Simplification from (2)
4. $\forall x(C(x) \rightarrow P(x))$	Premise
5. $C(a) \rightarrow P(a)$	Universal instantiation from (4)
6. $P(a)$	Modus ponens from (3) and (5)
7. $\neg B(a)$	Simplification from (2)
8. $P(a) \wedge \neg B(a)$	Conjunction from (6) and (7)
9. $\exists x(P(x) \wedge \neg B(x))$	Existential generalization from (8)

What are the negations of the statements $\forall x(x^2 > x)$ and $\exists x(x^2 = 2)$?

Solution: The negation of $\forall x(x^2 > x)$ is the statement $\neg\forall x(x^2 > x)$, which is equivalent to $\exists x\neg(x^2 > x)$. This can be rewritten as $\exists x(x^2 \leq x)$. The negation of $\exists x(x^2 = 2)$ is the statement $\neg\exists x(x^2 = 2)$, which is equivalent to $\forall x\neg(x^2 = 2)$. This can be rewritten as $\forall x(x^2 \neq 2)$. The truth values of these statements depend on the domain. ◀

Consider these statements. The first two are called *premises* and the third is called the *conclusion*. The entire set is called an *argument*.

"All lions are fierce."
"Some lions do not drink coffee."
"Some fierce creatures do not drink coffee."

(In Section 1.6 we will discuss the issue of determining whether the conclusion is a valid consequence of the premises. In this example, it is.) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a lion," "x is fierce," and "x drinks coffee," respectively. Assuming that the domain consists of all creatures, express the statements in the argument using quantifiers and $P(x)$, $Q(x)$, and $R(x)$.

Solution: We can express these statements as:

$$\forall x(P(x) \to Q(x)).$$
$$\exists x(P(x) \wedge \neg R(x)).$$
$$\exists x(Q(x) \wedge \neg R(x)).$$

Notice that the second statement cannot be written as $\exists x(P(x) \to \neg R(x))$. The reason is that $P(x) \to \neg R(x)$ is true whenever x is not a lion, so that $\exists x(P(x) \to \neg R(x))$ is true as long as there is at least one creature that is not a lion, even if every lion drinks coffee. Similarly, the third statement cannot be written as

$$\exists x(Q(x) \to \neg R(x)).$$ ◀

Translate each of these statements into logical expressions using predicates, quantifiers, and logical connectives.

a) No one is perfect.

b) Not everyone is perfect.

c) All your friends are perfect.

d) At least one of your friends is perfect.

Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.

b) All Swedish movies are serious.

c) No one can keep a secret.

d) There is someone in this class who does not have a good attitude.

Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a clear explanation," "x is satisfactory," and "x is an excuse," respectively. Suppose that the domain for x consists of all English text. Express each of these statements using quantifiers, logical connectives, and $P(x)$, $Q(x)$, and $R(x)$.

a) All clear explanations are satisfactory.

b) Some excuses are unsatisfactory.

c) Some excuses are not clear explanations.

d) Does (c) follow from (a) and (b)?

To be continued……

Thanks for watching

Have a nice day