Topic: Propositional Logic
Lec-2
Course: Discrete Mathematics

Topic: -

Module 1: Propositional Logic:

Logical Connectives, Conjunction, Disjunction, Negation and their truth table. Conditional Connectives, Implication, Converse, Contra positive, Inverse, Bi-conditional statements with truth table, Logical Equivalence, Tautology, Normal forms-CNF, DNF; Predicates and Logical Quantifications of propositions and related examples.

Module 2: Theory of Numbers:

Well Ordering Principle, Divisibility theory and properties of divisibility; Fundamental theorem of Arithmetic;
Euclidean Algorithm for finding G.C.D and some basic properties of G.C.D with simple examples; Congruences, Residue classes of integer rmodulo ()nn Z and its examples;

Module 3: Order, Relation and Lattices:

POSET, Hasse Diagram, Minimal, Maximal, Greatest and Least elements in a POSET, Lattices and its properties, Principle of Duality, Distributive and Complemented Lattices.

Module 4: Counting Techniques:

Permutations, Combinations, Binomial coefficients, Pigeon- hole Principle, Principles of inclusion and exclusions; Generating functions, Recurrence Relations and their solutions using generating function, Recurrence relation of Fibonacci numbers and its solution, Divide-and-Conquer algorithm and its recurrence relation and its simple application in computer.

Module 5: Graph Coloring:
Chromatic Numbers and its bounds, Independence and Clique Numbers, Perfect Graphs-Definition and examples, Chromatic polynomial and its determination, Applications of Graph Coloring.

Module 6: Matchings:
Definitions and Examples of Perfect Matching, Maximal and Maximum Matching, Hall's Marriage Theorem (Statement only) and related problems.

Text Books:

1. Kenneth H.Rosen, Discrete Mathematics and Its Applications, McGrawHill.
2. Russell Merris, Combinatorics, WILEY-INTERSCIENCE SERIESINDISCRETEMATHEMATICSANDOPTIMIZATION
3. N.Chandrasekaranand, M.Umaparvathi, Discrete Mathematics, PHI
4. Gary Haggard, John Schlipfand Sue Whitesides, Discrete Mathematics for Computer Science, CENGAGE Learning.
5. Gary Chartrandand Ping Zhang–Introduction to Graph Theory, TMH

Propositional Logic

Propositions

A proposition is a declarative sentence which can be either true or false but not both.

Examples

All the following declarative sentences are propositions.

- Toranto is the capital of Canada
- 1+1=2
- 2+2=3

The first two propsitions are true but the third one is false.

Examples

Consider the following sentences.

- What time is it ?
- Read this book
- $x + 1 = 3$
- $x + y = z$

The first two are not propositions as they are not declarative sentences. Third and fourth sentences are not propositions as they are neither true or false. But they can be made proposition by considering particular values of x, y, z.

- **Notations:**

We use letters p,q,r,s…. to denote propositional variables.

Definition

- **Negation :** Let p be a proposition. Then negation of p, denoted by $\sim p$ or $\bar{p}$,is the statement

  " It is not the case that p".

The proposition $\sim p$ is read as not p and the truth value of $\sim p$ is opposite to the truth value of p.

Definition

- **Conjunction :** Let p and q be two propositions. Then the conjunction of p and q denoted by $p \wedge q$,is the statement

  "p and q".

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth Table of Conjunction

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	F

Definition

- **Disjunction :** Let p and q be two propositions. Then the disjunction of p and q denoted by $p \vee q$,is the statement "p or q".

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Truth Table of Disjunction

p	q	$p \vee q$
T	T	T
F	T	T
T	F	T
F	F	F

Conditional Statement

Let *p* and *q* be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q."

The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

In the conditional statement $p \rightarrow q, p$ is called the hypothesis (or antecedent or premise) and q is called the conclusion (or consequence).

Truth Table of Conditional Statement

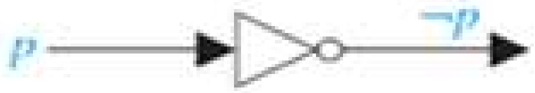
p	q	$p \to q$
T	T	T
F	T	T
T	F	F
F	F	T

Logic Circuits

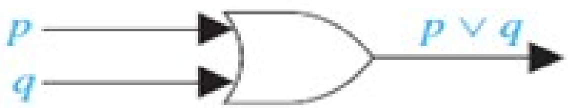
Propositional logic can be applied to the design of computer hardware. This was first observed in 1938 by Claude Shannon in his MIT master's thesis.

A **logic circuit** (or **digital circuit**) receives input signals $p_1, p_2 \ldots, p_n$, each a bit [either 0 (off) or 1 (on)], and produces output signals $s_1, s_2 \ldots, s_n$, each a bit.

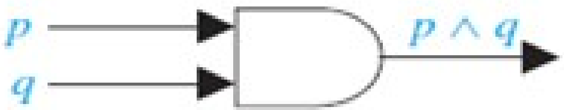
Basic logic gates



Inverter



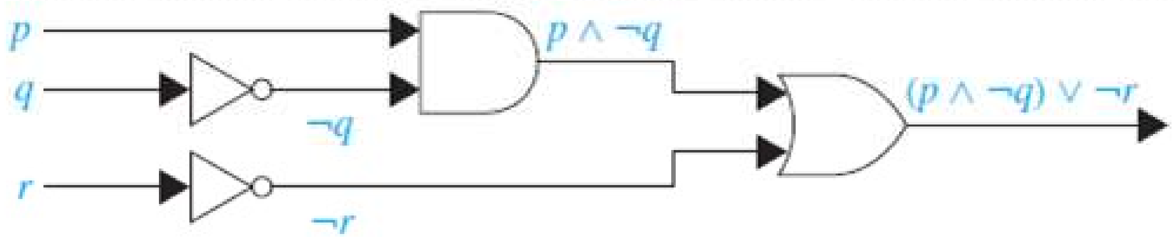
OR gate



AND gate

Example

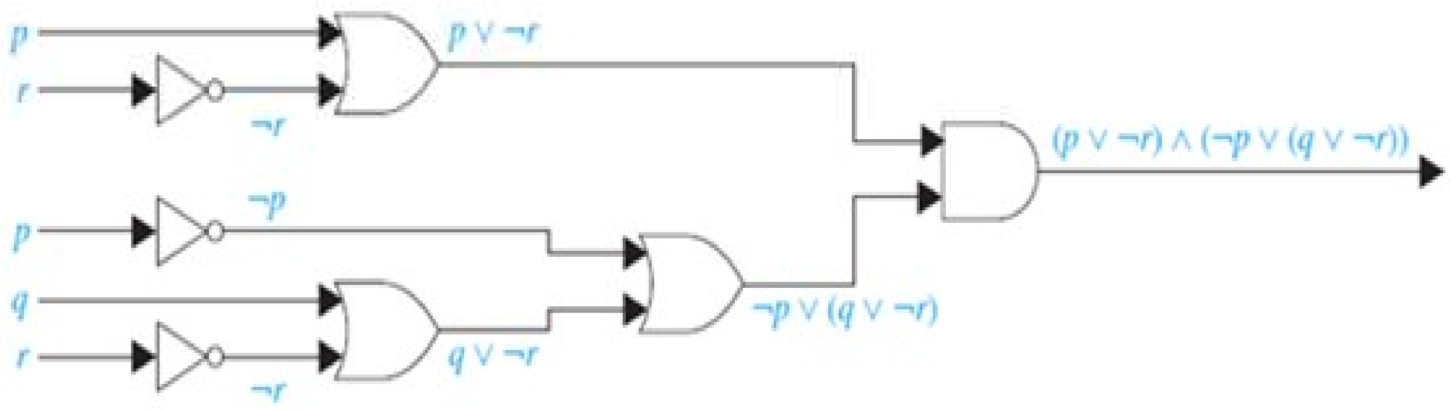
Determine the output of the combinatorial circuit



$$(p \wedge {\sim}q) \vee {\sim}r.$$

Example

Build a digital circuit that produces the output $(p \vee \sim r) \wedge (\sim p \vee (q \vee \sim r))$ when given input bits p, q, and r.



Tautology

A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.

Example

$$p \vee \sim p$$

p	$\sim p$	$p \vee \sim p$
T	F	T
F	T	T

Contradiction

A compound proposition that is always false is called a *contradiction*.

Example $p \wedge \sim p$

p	$\sim p$	$p \wedge \sim p$
T	F	F
F	T	F

Example

Show that each of these conditional statements is a tautology by using truth tables.

a) $(p \wedge q) \rightarrow p$ b) $p \rightarrow (p \vee q)$

c) $\neg p \rightarrow (p \rightarrow q)$ d) $(p \wedge q) \rightarrow (p \rightarrow q)$

Proof:

a)

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$
T	T	T	T
F	T	F	T
T	F	F	T
F	F	F	T

So $(p \wedge q) \rightarrow p$ is a tautology.

Proof:

b)

p	q	$p \vee q$	$p \to (p \vee q)$
T	T	T	T
F	T	T	T
T	F	T	T
F	F	F	T

So $p \to (p \vee q)$ is a tautology.

Logical Equivalence

Use truth tables to verify these equivalences.

a) $p \wedge \mathbf{T} \equiv p$

b) $p \vee \mathbf{F} \equiv p$

c) $p \wedge \mathbf{F} \equiv \mathbf{F}$

d) $p \vee \mathbf{T} \equiv \mathbf{T}$

e) $p \vee p \equiv p$

f) $p \wedge p \equiv p$

where $\mathbf{T}$ is a **tautology** and $\mathbf{F}$ is a **contradiction.**

a) $p \wedge \mathbf{T} \equiv p$

d) $p \vee \mathbf{T} \equiv \mathbf{T}$

p	T	$p \wedge T$	$p \vee T$
T	T	T	T
F	T	F	T

b) $p \vee \mathbf{F} \equiv p$

c) $p \wedge \mathbf{F} \equiv \mathbf{F}$

p	F	$p \wedge F$	$p \vee F$
T	F	F	T
F	F	F	F

Example

Show that $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

p	q	r	$p \to q$	$p \to r$	$q \wedge r$	$(p \to q) \wedge (p \to r)$	$p \to (q \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	F	F	F	F	F	F
F	F	T	T	T	F	T	T
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T
T	F	T	F	T	F	F	F
F	F	F	T	T	F	T	T

So
$(p \to q) \wedge (p \to r)$ and $p \to (q \wedge r)$ are logically equivalent.

Standard Logical Equivalences

$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws
$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws

$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences involving Conditional Statements

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Show that $\neg(p \to q)$ and $p \wedge \neg q$ are logically equivalent without using truth table.

Proof:

We know $p \to q \equiv \sim p \vee q$.

So $\neg(p \to q) \equiv \neg(\neg p \vee q)$

$\equiv \neg(\neg p) \wedge \neg q$

$\equiv p \wedge \neg q$

Argument

An argument in propositional logic is a sequence of propositions.

All but the final proposition in the argument are called premises and the final proposition is called the conclusion.

Valid argument

An argument is valid if the truth of all its premises implies that the conclusion is true.

Valid Arguments in Propositional Logic

Consider the following argument involving propositions (which, by definition, is a sequence of propositions):

"If you have a current password, then you can log onto the network."

"You have a current password."

Therefore,
"You can log onto the network."

We use p to represent "You have a current password"

and q to represent "You can log onto the network."

Then, the argument has the form

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore q \end{array}$$

When p and q are propositional variables, the statement $((p \rightarrow q) \land p) \rightarrow q$ is a tautology (can be checked).

In particular, when both $p \rightarrow q$ and p are true, we know that q must also be true.

We say this form of argument is **valid.**

How to check that an Argument is Valid or Not

From the definition of a valid argument form we see that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion q is valid, when

$(p_1 \wedge p_2 \wedge \cdots \wedge p_n) \rightarrow q$ is a tautology.

Rules of Inference

- Modus Ponens

$$\begin{array}{l} p \rightarrow q \\ p \\ \hline \therefore\ q \end{array}$$

Modus tollens

$$\begin{array}{rl} & \neg q \\ & p \to q \\ \hline \therefore & \neg p \end{array}$$

- Hypothetical syllogism

$$\begin{array}{r} p \to q \\ q \to r \\ \hline \therefore\ p \to r \end{array}$$

- Simplification

$$\begin{array}{r} p \wedge q \\ \hline \therefore\ p \end{array}$$

Example

Check the validity of the argument

If it rains today, then we will not have a barbecue today. If we do not have a barbecue today, then we will have a barbecue tomorrow. Therefore, if it rains today, then we will have a barbecue tomorrow.

Solution:

Let p be the proposition "It is raining today,"

let q be the proposition "We will not have a barbecue today," and

let r be the proposition "We will have a barbecue tomorrow."

Then this argument is of the form

$$\begin{array}{rl} & p \rightarrow q \\ & q \rightarrow r \\ \hline \therefore & p \rightarrow r \end{array}$$

which is a hypothetical syllogism. So the argument is valid.

Example

Show that the premises

“It is not sunny this afternoon and it is colder than yesterday,”

“We will go swimming only if it is sunny,”

“If we do not go swimming, then we will take a canoe trip,” and

“If we take a canoe trip, then we will be home by sunset” lead to the conclusion

“We will be home by sunset.”

Solution: Let p be the proposition "It is sunny this afternoon," q the proposition "It is colder than yesterday," r the proposition "We will go swimming," s the proposition "We will take a canoe trip," and t the proposition "We will be home by sunset."

Then the premises become

$\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$. The conclusion is simply t. We need to give a valid argument with premises $\neg p \wedge q$, $r \to p$, $\neg r \to s$, and $s \to t$ and conclusion t.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $\neg p$	Simplification using (1)
3. $r \rightarrow p$	Premise
4. $\neg r$	Modus tollens using (2) and (3)
5. $\neg r \rightarrow s$	Premise
6. s	Modus ponens using (4) and (5)
7. $s \rightarrow t$	Premise
8. t	Modus ponens using (6) and (7)

Show that the premises

"If you send me an e-mail message, then I will finish writing the program,"

"If you do not send me an e-mail message, then I will go to sleep early,"

and "If I go to sleep early, then I will wake up feeling refreshed"

lead to the conclusion
"If I do not finish writing the program, then I will wake up feeling refreshed."

Solution:

Let p be the proposition "You send me an e-mail message,"

q the proposition "I will finish writing the program,"

r the proposition "I will go to sleep early,"

s the proposition "I will wake up feeling refreshed."

Then the premises are $p \rightarrow q$, $\neg p \rightarrow r$, and $r \rightarrow s$.

The desired conclusion is $\neg q \rightarrow s$.

Step	Reason
1. $p \rightarrow q$	Premise
2. $\neg q \rightarrow \neg p$	Contrapositive of (1)
3. $\neg p \rightarrow r$	Premise
4. $\neg q \rightarrow r$	Hypothetical syllogism using (2) and (3)
5. $r \rightarrow s$	Premise
6. $\neg q \rightarrow s$	Hypothetical syllogism using (4) and (5)

To be continued……

Thanks for watching

Have a nice day