Lecture 1
Relation, POSET, Lattice
Course: Discrete Mathematics

Relation

Relationship between elements of sets occur in many contexts.

Every day we deal with relationships such as those between a business and its telephone number, an employee and his or her salary, and so on.

In mathematics we study relationships such as those between a positive integer and one that it divides, an integer and one that it is congruent to modulo 5, a real number x and the value $f(x)$ where f is a function, and so on.

Relationships such as that between a program and a variable it uses, and that between a computer language and a valid statement in this language often arise in computer science.

Definition

Let A and B be sets. A binary relation *from A to B* is a subset of $A \times B$.

That is, a binary relation from A to B is a set R of ordered pairs where the first element of each ordered pair comes from A and the second element comes from B.

We use the notation $a\ R\ b$ to denote that $(a, b) \in R$ and $a \not R\ b$ to denote that $(a, b) \notin R$.

Moreover, when (a, b) belongs to R, a is said to be **related to** b by R.

Example

Let $A = \{0, 1, 2\}$ and $B = \{a, b\}$.
Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from *A* to *B*.
This means, for instance, that $0 \; R \; a$, but that $1 \not{R} \, b$.

Example

Let A be the set of cities in the U.S.A., and let B be the set of the 50 states in the U.S.A.

Define the relation R by specifying that (a, b) belongs to R if a city with name a is in the state b.

For instance, (Boulder, Colorado), (Bangor, Maine), (Ann Arbor, Michigan), (Middletown, New Jersey), (Middletown, New York), (Cupertino, California), and (Red Bank, New Jersey) are in R.

Relations can be represented graphically

Example

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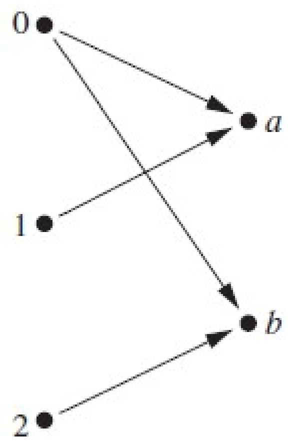
This means, for instance, that $0\ R\ a$, but that $1 \not{R}\, b$.



Another way to represent this relation is to use a table.

Example

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Then $\{(0, a), (0, b), (1, a), (2, b)\}$ is a relation from *A* to *B*.
This means, for instance, that $0 \; R \; a$, but that $1 \not R \, b$.

R	a	b
0	×	×
1	×	
2		×

Relations on a Set

Relations from a set A to itself are of special interest.

A relation on a set A is a subset of $A \times A$.

Example

Let A be the set $\{1, 2, 3, 4\}$.

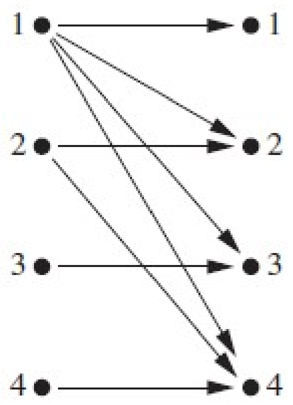
Which ordered pairs are in the relation

$$R = \{(a, b) \mid a \text{ divides } b\}?$$

Solution:

Because (a, b) is in R if and only if *a* and *b* are positive integers not exceeding 4 such that *a* divides *b*, we see that

$$R = \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,4),(3,3),(4,4)\}.$$



R	1	2	3	4
1	×	×	×	×
2		×		×
3			×	
4				×

Remark

How many relations are there on a set with n elements?

A relation on a set *A* is a subset of $A \times A$.
Because $A \times A$ has n^2 elements when *A* has n elements, and a set with m elements has 2^m subsets, there are 2^{n^2} subsets of *A* × *A*.

Thus, there are 2^{n^2} relations on a set with n elements.

For example, there are $2^{3^2} = 2^9 = 512$ relations on the set {*a*, *b*, *c*}.

Example

Consider these relations on the set of integers:

$R_1 = \{(a,b) \mid a \leq b\},$

$R_6 = \{(a,b) \mid a + b \leq 3\}.$

$R_2 = \{(a,b) \mid a > b\},$

$R_3 = \{(a,b) \mid a = b \text{ or } a \neq b\},$

$R_4 = \{(a,b) \mid a = b\},$

$R_5 = \{(a,b) \mid a = b + 1\},$

Which of these relations contain each of the pairs $(1,1), (1,2), (2,1), (1,-1), \text{and } (2,2)$?

Solution:

$R_1 = \{(a,b) \mid a \leq b\},$ Contains $(1,1),(1,2),(2,2)$ but not $(2,1),(1,-1)$.

$R_2 = \{(a,b) \mid a > b\},$ Contains $(2,1)$ but not , $(1,-1),(1,1),(1,2),(2,2)$

$R_3 = \{(a,b) \mid a = b \text{ or } a \neq b\},$ Contains all of them.

$R_4 = \{(a,b) \mid a = b\},$ Contains $(1,1),(2,2)$.

$R_5 = \{(a, b) \mid a = b + 1\},$

Contains (2,1) only

$R_6 = \{(a, b) \mid a + b \leq 3\}.$

Contains $(1, -1)$, $(1, 1)$, $(1, 2)$, $(2, 1)$ but not $(2, 2)$

Definition

A relation R on a set A is called reflexive if $(a, a) \in R$ for every element $a \in A$.

Remark:

Using quantifiers we see that the relation R on the set A is reflexive if $\forall a\big((a, a) \in R\big)$, where the universe of discourse is the set of all elements in A.

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1), (1,2), (2,1), (2,2), (3,4), (4,1), (4,4)\},$$

$$R_2 = \{(1,1), (1,2), (2,1)\},$$

$$R_3 = \{(1,1), (1,2), (1,4), (2,1), (2,2), (3,3), (4,1), (4,4)\},$$

Which of these relations are reflexive?

Solution:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$

Not Reflexive as $(3,3) \notin R_1$

$R_2 = \{(1,1),(1,2),(2,1)\},$ Not Reflexive as $(2,2) \notin R_2$

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},$

Reflexive

Example

Is the "divides" relation on the set of positive integers reflexive?

Solution:

Because $a \mid a$ whenever a is a positive integer, the "divides" relation is reflexive.

But if we replace the set of positive integers with the set of all integers the relation is not reflexive because by definition 0 does not divide 0.

Definition

A relation R on a set A is called symmetric if $(b, a) \in R$ whenever $(a, b) \in R$, for all $a, b \in A$.

A relation R on a set A such that for all $a, b \in A$, if $(a, b) \in R$ and $(b, a) \in R$, then $a = b$ is called antisymmetric.

Remark:

Using quantifiers, we see that the relation R on the set A is symmetric if

$$\forall a \forall b((a,b) \in R \rightarrow (b,a) \in R).$$

Similarly, the relation R on the set A is antisymmetric if

$$\forall a \forall b(((a,b) \in R \land (b,a) \in R) \rightarrow (a=b)).$$

Definition

A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$,

then $(a, c) \in R$, for all $a, b, c \in A$.

Remark:

Using quantifiers we see that the relation R on a set A is transitive if we have

$$\forall a \forall b \forall c(((a,b) \in R \land (b,c) \in R) \rightarrow (a,c) \in R).$$

Example

Consider the following relations on $\{1, 2, 3, 4\}$:

$$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$$

$$R_2 = \{(1,1),(1,2),(2,1)\},$$

$$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\},$$

Which of these relations are symmetric, antisymmetric or transitive?

Solution:

$R_1 = \{(1,1),(1,2),(2,1),(2,2),(3,4),(4,1),(4,4)\},$

Not symmetric as $(4,3) \notin R_1$

Not antisymmetric as $(1,2),(2,1) \in R_1$

Not transitive as $(3,4),(4,1) \in R_1$ but $(3,1) \notin R_1$

$R_2 = \{(1,1),(1,2),(2,1)\}$,

Symmetric

Not antisymmetric as $(1,2),(2,1) \in R_2$

Transitive

$R_3 = \{(1,1),(1,2),(1,4),(2,1),(2,2),(3,3),(4,1),(4,4)\}$,

Symmetric

Not antisymmetric as $(1,2),(2,1) \in R_3$

Transitive

Thanks for watching

Have a nice day