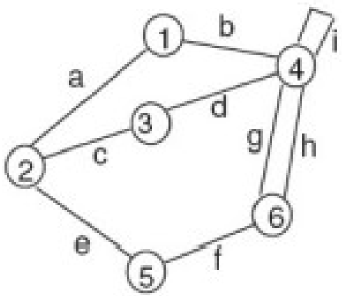
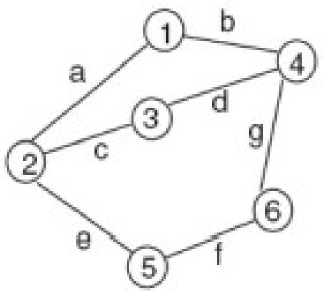
Lecture 1
Graph
Course: Discrete Mathematics

A graph $G = (V, E)$ is a pair of vertices (or nodes) V and a set of edges E, assumed finite i.e. $|V| = n$ and $|E| = m$.

An edge $e_k = (v_i, v_j)$ is incident with the vertices v_i and v_j.

A simple graph has no self-loops or multiple edges like below

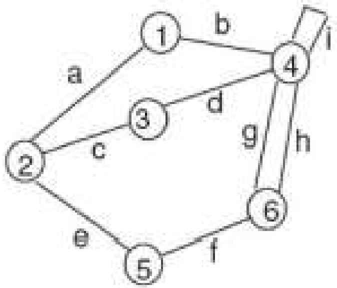


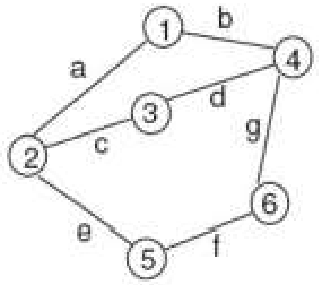


Here $V(G) = \{v_1, v_2, \ldots, v_5\}$ and $E(G) = \{e_1, e_2, \ldots, e_6\}$.

An edge $e_k = (v_i, v_j)$ is incident with the vertices v_i and v_j.

A simple graph has no self-loops or multiple edges like below





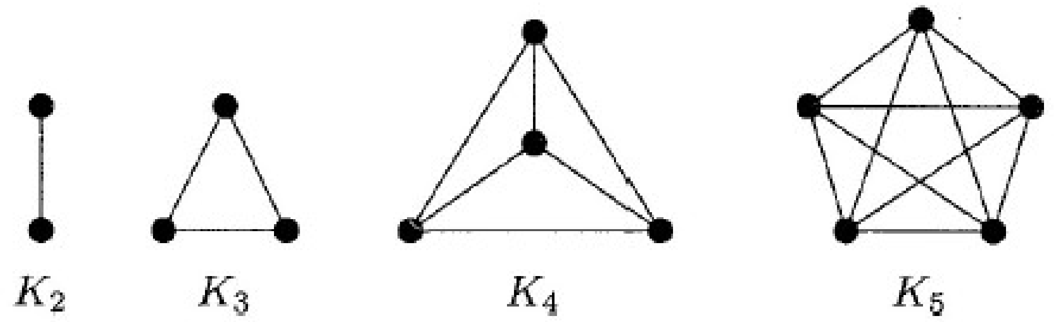
The degree $d(v)$ of a vertex V is its number of incident edges

A self-loop counts for 2 in the degree function.

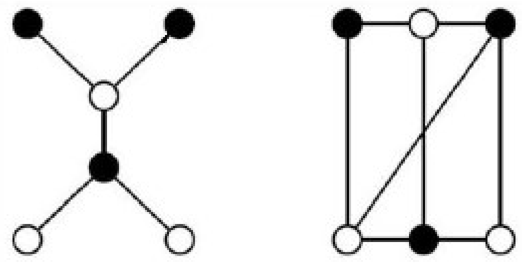
An isolated vertex has degree 0.

Proposition The sum of the degrees of a graph $G = (V, E)$ equals $2|E| = 2m$ (trivial)

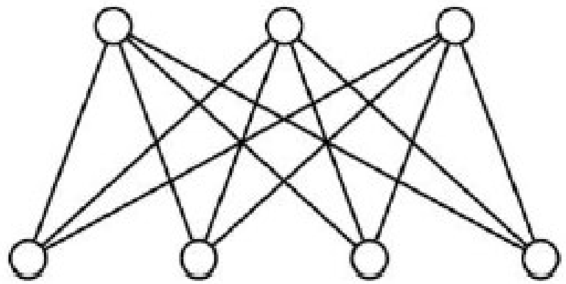
A complete graph K_n is a simple graph with all $B(n, 2) := \frac{n(n-1)}{2}$ possible edges, like the matrices below for $n = 2, 3, 4, 5$.



A bipartite graph is one where $V = V_1 \cup V_2$ such that there are no edges between V_1 and V_2 (the black and white nodes below)



A complete bipartite graph is one where all edges between V_1 and V_2 are present (i.e. $|E| = |V_1|.|V_2|$). It is noted as K_{n_1,n_2}.



Graph Colouring

Problems related to the colouring of maps of regions, such as maps of parts of the world, have generated many results in graph theory.

When a map is coloured, two regions with a common borders are customarily assigned different colours.

One way to ensure that two adjacent regions never have the same colour is to use a different colour for each region.

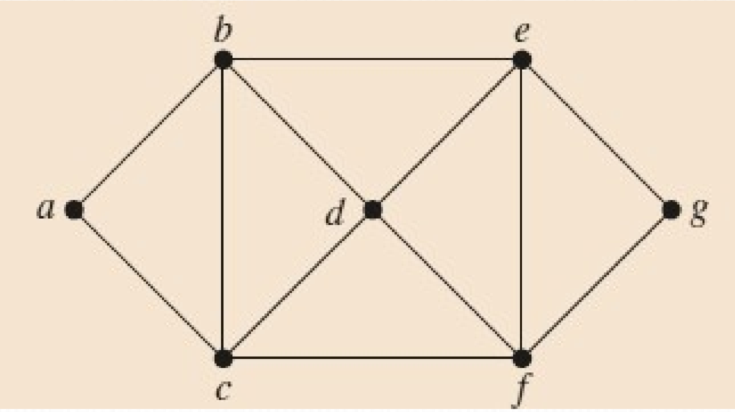
But this is inefficient, and on maps with many regions it would be hard to distinguish similar colours. Besides, a small number of colours should be used whenever possible.

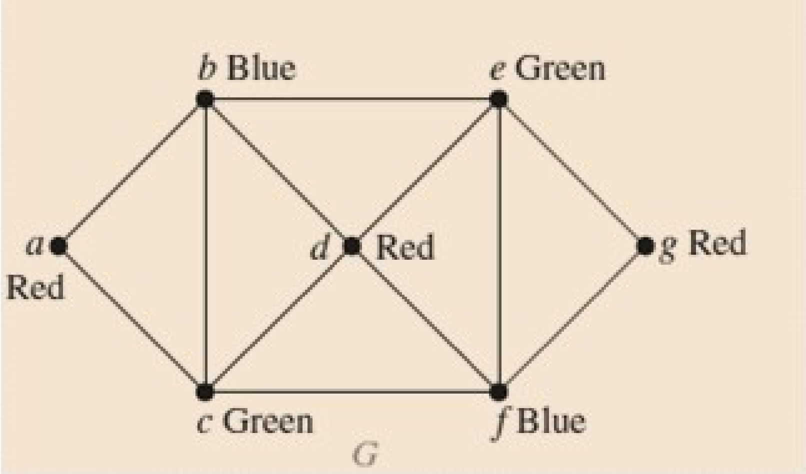
> *Graph coloring is arguably the most popular subject in graph theory.*
>
> *Noga Alon* (1993)

Definition

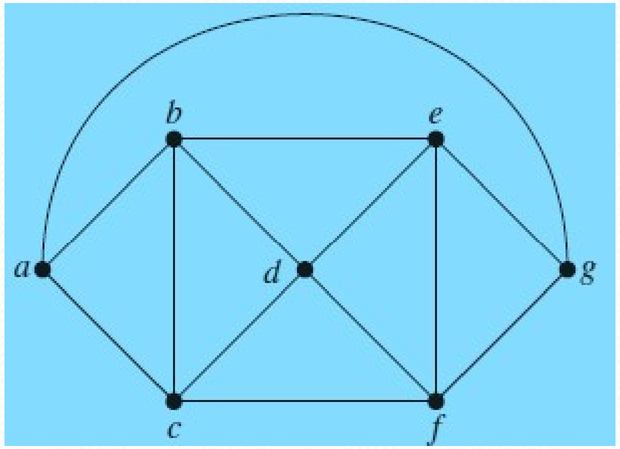
A colouring of a simple graph is the assignment of a colour to each vertex of the graph so that no two adjacent vertices are assigned the same colour.

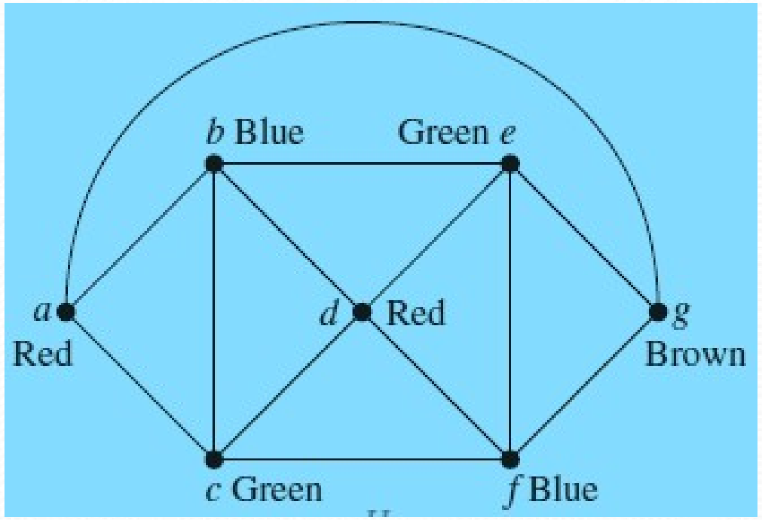
Example





Example





Definition

The chromatic number of a graph is the least number of colours needed for a colouring of this graph.

The chromatic number of a graph G is denoted by $\chi(G)$.

A graph G is k-colourable if there exists a colouring of G from a set of k colours.

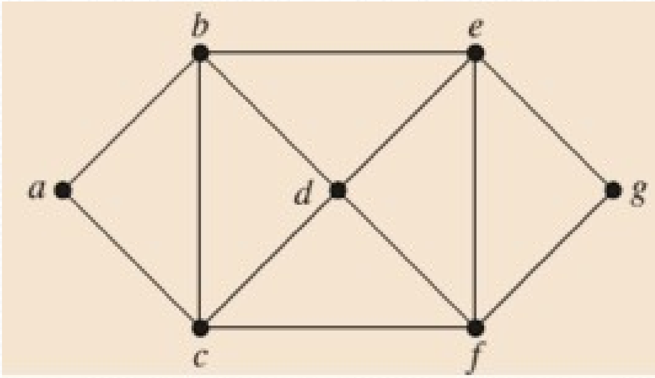
In other words, G is k-colourable if there exists a k-colouring of G.

The minimum positive integer k for which G is k-colourable is the chromatic number of G.

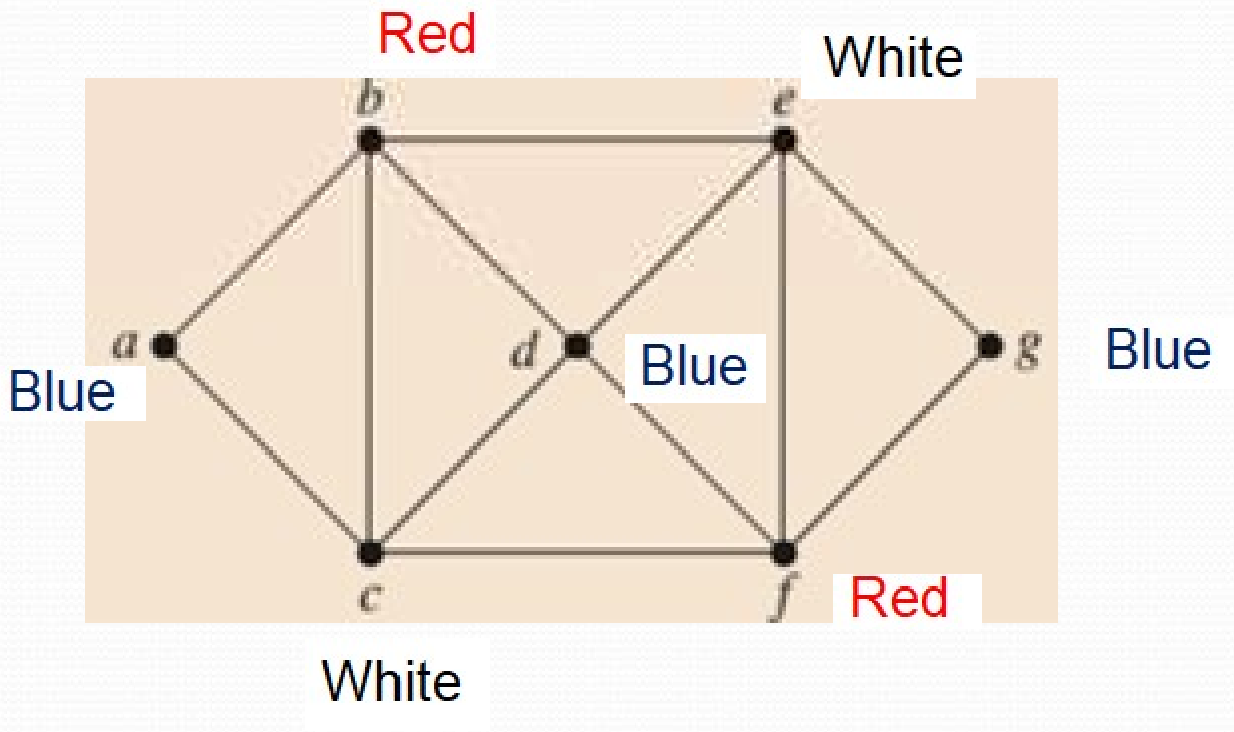
A graph G with chromatic number k is a k-chromatic graph.

Therefore, if $\chi(G) = k$, then there exists a k-colouring of G but not a (k−1)-colouring.

Find chromatic number of the following graph.

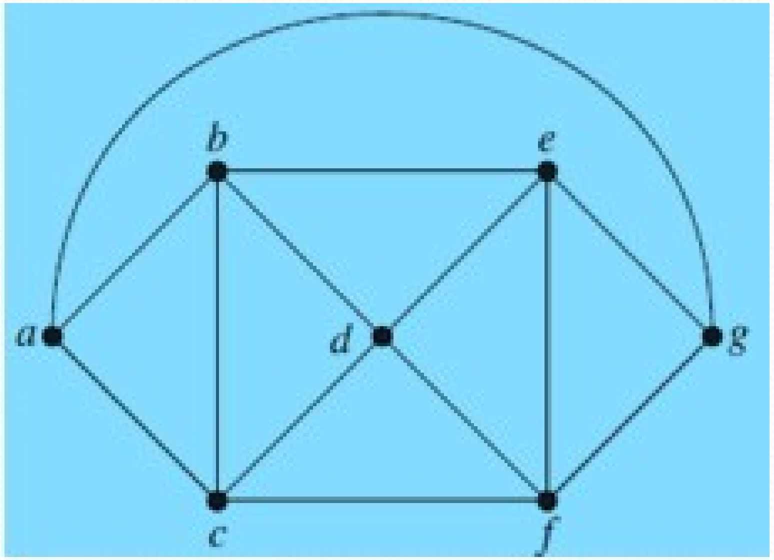


Solution:

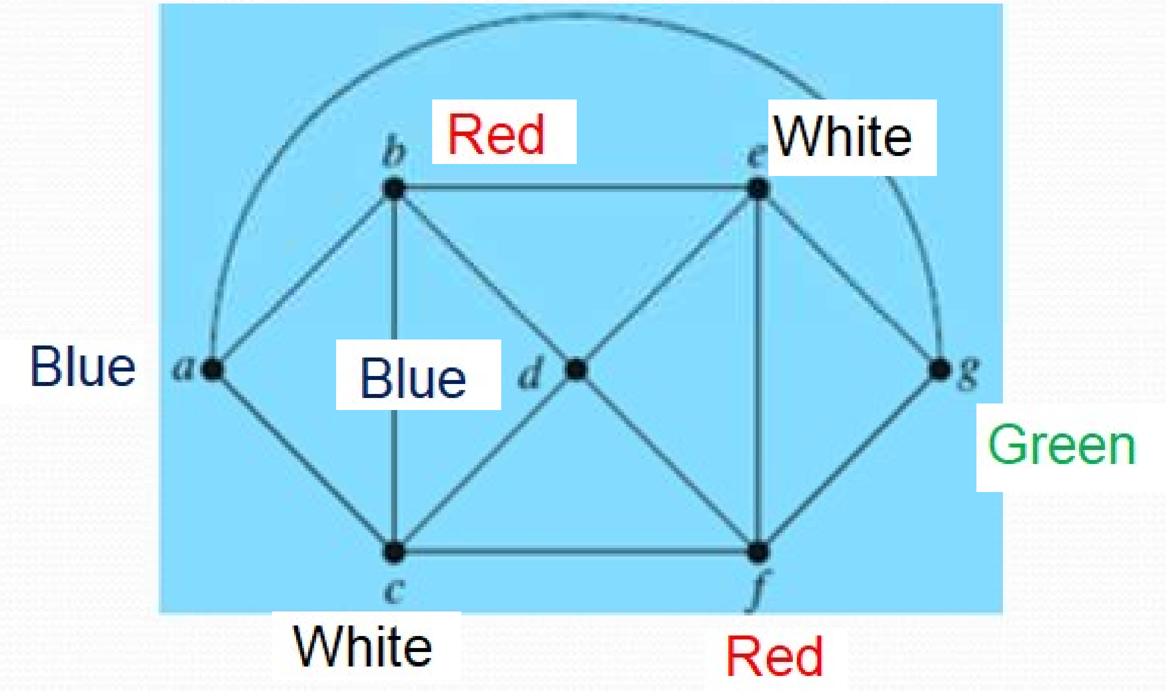


So chromatic number is 3.

Find chromatic number of the following graph.

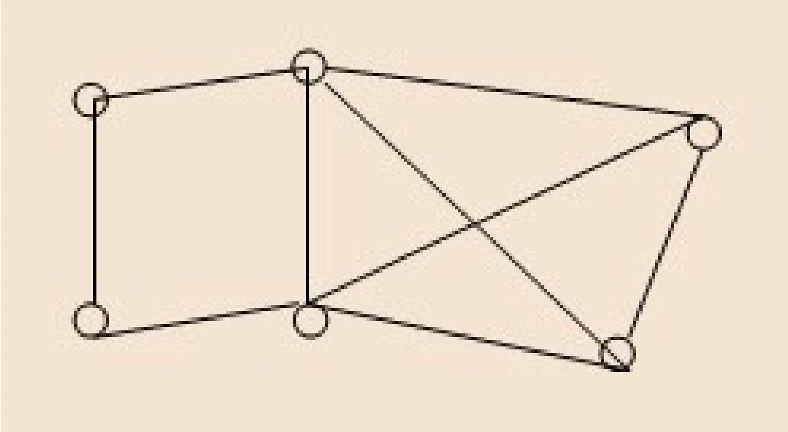


Solution:

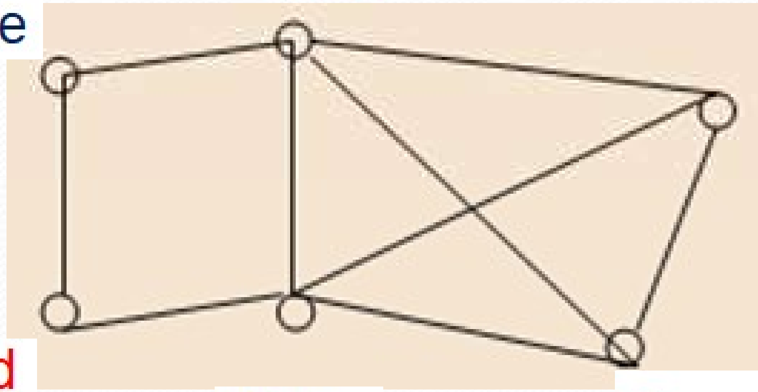


So chromatic number is 4.

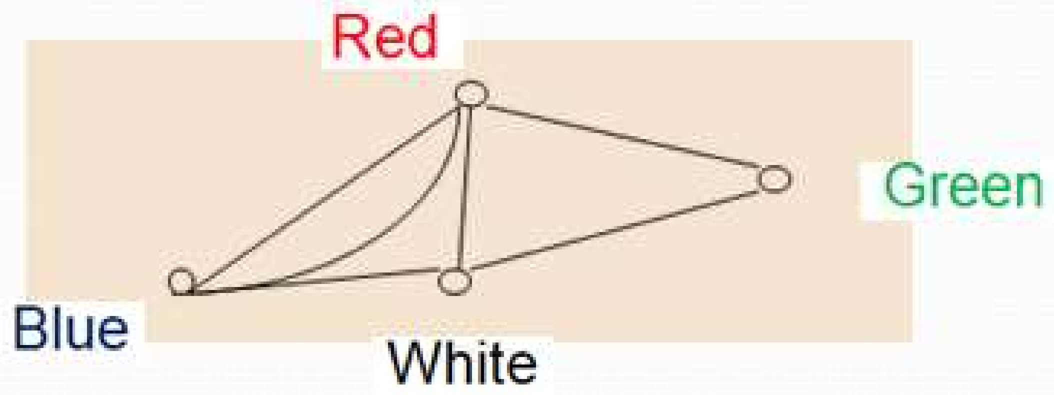
Find chromatic number of the following graph.



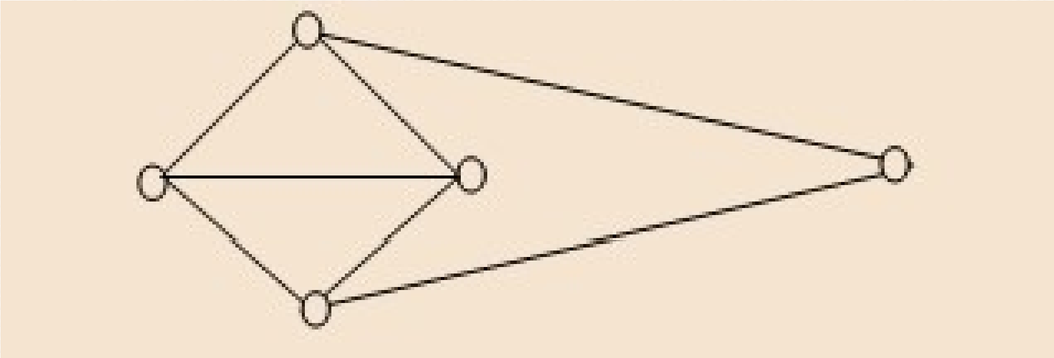
Solution:



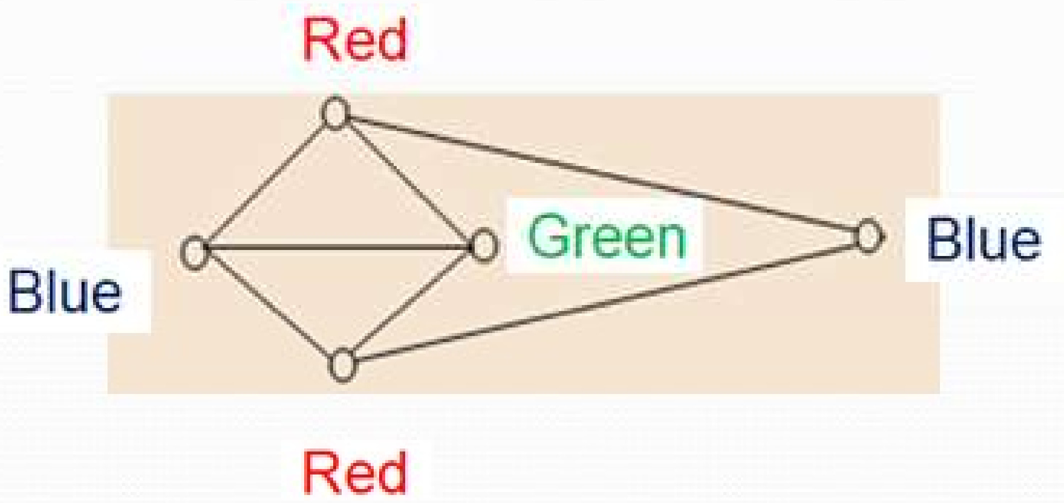
So chromatic number is 4.


Red
Green
Blue
White

Find chromatic number of the following graph.



Solution:



So chromatic number is 3.

What is the chromatic number of K_n ?

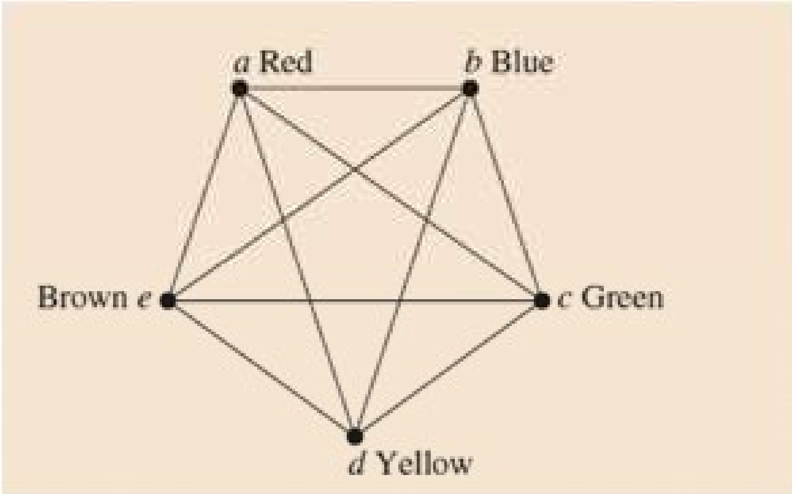
Solution:

In K_n , no two vertices can be assigned the same colour, because every two vertices of this graph are adjacent.

Hence, the chromatic number of K_n is n.

That is $\chi(K_n) = n$.

Example



So chromatic number is 5.

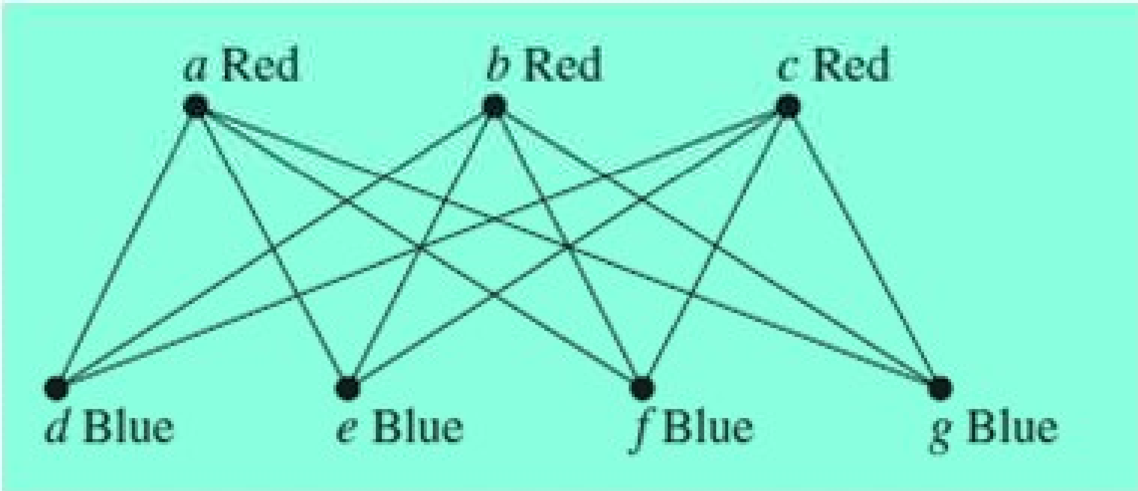
What is the chromatic number of the complete bipartite graph $K_{m,n}$, where *m* and *n* are positive integers?

Solution:

Only two colours are needed, because $K_{m,n}$ is a bipartite graph.

Hence, $\chi(K_{m,n}) = 2$.

Example



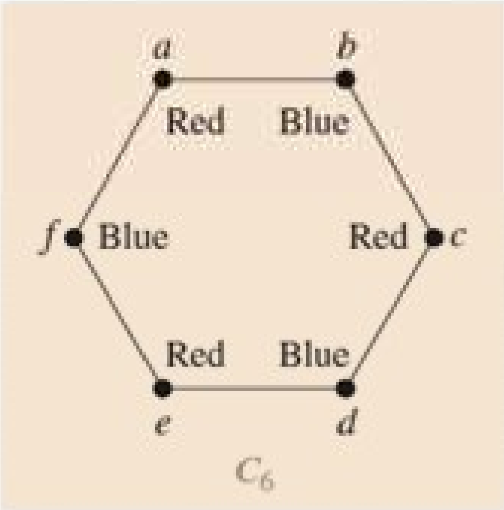
What is the chromatic number of the graph C_n, where $n \geq 3$?

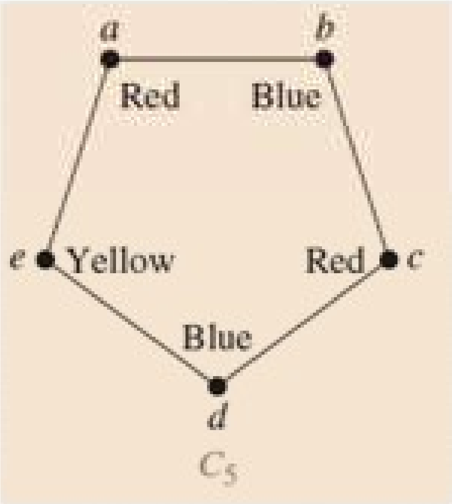
Solution:

$\chi(C_n) = 2$ if n is an even positive integer

$\chi(C_n) = 3$ if n is an odd positive integer

Example


a
b
Red
Blue
f
Blue
Red
c
Red
Blue
e
d
C_6


a
b
Red
Blue
e
Yellow
Red
c
Blue
d
C_5

Find the chromatic number of a circuit with 100 vertices.

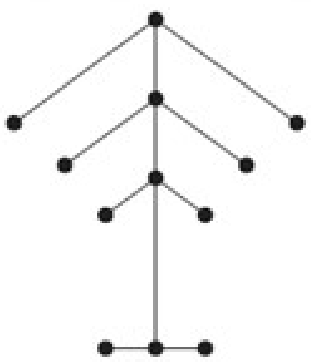
Ans. 2

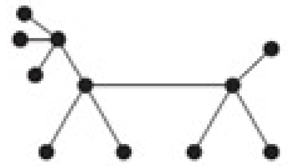
Find the chromatic number of a circuit with 111 vertices.

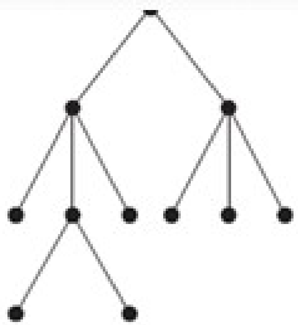
Ans. 3

Definition

A tree is a connected undirected graph with no simple circuits.

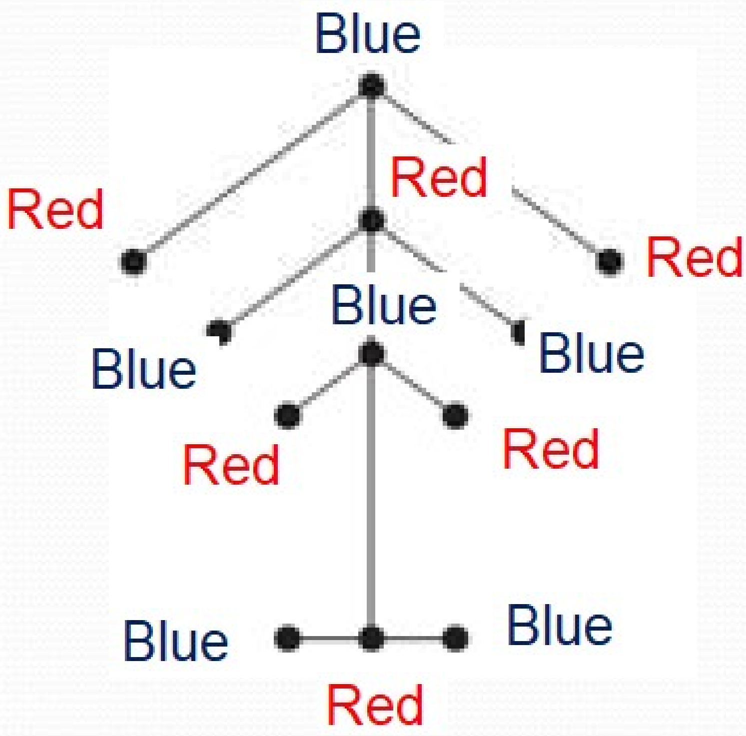






What is the chromatic number of a tree?

Solution:



Determining upper and/or lower bounds for the chromatic number of a graph.

Certainly, for every graph G of order n,

$$1 \leq \chi(G) \leq n.$$

And a useful, lower bound for the chromatic number of a graph involves the chromatic numbers of its subgraphs.

Theorem

If H is a subgraph of a graph G, then $\chi(H) \leq \chi(G)$.

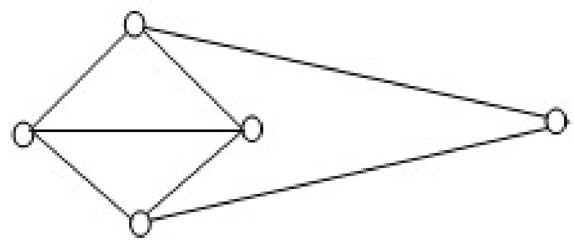
Definition

For a graph G, we define the clique number of $G, \omega(G)$, to be the largest value of k for which K_k is an induced subgraph of G.

Proposition

If G is a graph, $\chi(G) \geq \omega(G)$.

Example



Here $\omega(G) = 3$ as K_3 is a subgraph of it.

Also $\chi(G) = 3 = \omega(G)$.

Let’s talk about upper bounds

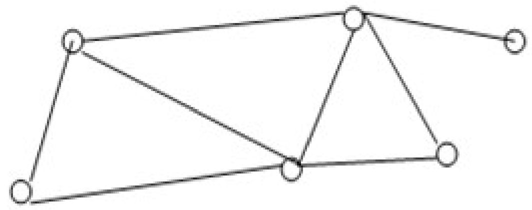
Definition

For a graph G, let $\Delta(G)$ denote the maximum degree of any of G's vertices, and $\delta(G)$ denote the minimum degree of any of G's vertices.

Proposition

For any graph G, $\chi(G) \leq \Delta(G)+1$.

Example

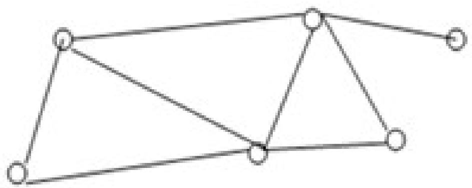


Here $\Delta(G) = 4$ and $\chi(G) = 3$. So $\chi(G) \leq \Delta(G) + 1$ is satisfied.

Brooks Theorem

Let G be a connected graph with $\Delta(G) \geq 3$. If G is not complete then $\chi(G) \leq \Delta(G)$.

Example



Here $\Delta(G) = 4$ and G is not complete.

and $\chi(G) = 3 \leq \Delta(G)$.

To be continued……

Thanks for watching

Have a nice day