Topic: Propositional Logic

Lec-1

Course: Discrete Mathematics

Propositional Logic

Propositions

A proposition is a declarative sentence which can be either true or false but not both.

Examples

All the following declarative sentences are propositions.

- Toranto is the capital of Canada
- 1+1=2
- 2+2=3

The first two propsitions are true but the third one is false.

Examples

Consider the following sentences.

- What time is it ?
- Read this book
- $x + 1 = 3$
- $x + y = z$

The first two are not propositions as they are not declarative sentences. Third and fourth sentences are not propositions as they are neither true or false. But they can be made proposition by considering particular values of x, y, z.

- **Notations:**

We use letters p,q,r,s…. to denote propositional variables.

Definition

- **Negation :** Let p be a proposition. Then negation of p, denoted by $\sim p$ or $\bar{p}$,is the statement

  " It is not the case that p".

The proposition $\sim p$ is read as not p and the truth value of $\sim p$ is opposite to the truth value of p.

Example

- Find the negation of the proposition

  " Ram's PC runs linux"

  and express in simple english.

Solution: The negation is

" Its not the case that Ram's PC runs linux"

or in simple english

" Ram's PC doesnot run linux"

- Find the negation of the proposition

“ Vandana’s cell phone has at least 32GB memory ”

and express in simple english.

Solution: The negation is

“ Its not the case that Vandana’s cell phone has at least 32GB memory”

or in simple english

“Vandana’s cell phone doesnot have at least 32GB memory” or more simpler

“Vandana’s cell phone has less than 32GB memory”

Truth Table of Negation

p	$\sim p$
T	F
F	T

Definition

- **Conjunction :** Let p and q be two propositions. Then the conjunction of p and q denoted by $p \wedge q$,is the statement

  "p and q".

The conjunction $p \wedge q$ is true when both p and q are true and is false otherwise.

Truth Table of Conjunction

p	q	$p \wedge q$
T	T	T
F	T	F
T	F	F
F	F	F

Example

- Statement p: Squares are rectangles
- Statement q: Rectangles have four sides

Then $p \wedge q$ is " squares are rectangles and rectangles have four sides"

Definition

- **Disjunction :** Let p and q be two propositions. Then the disjunction of p and q denoted by $p \vee q$,is the statement "p or q".

The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Truth Table of Disjunction

p	q	$p \vee q$
T	T	T
F	T	T
T	F	T
F	F	F

Example

- p: Some quadrilaterals are parallelograms
- q: Quadrilaterals have 11 sides

Then $p \vee q$ is

Some quadrilaterals are parallelograms, or quadrilaterals have 11 sides.

Definition

- **Exclusive Or :** Let p and q be two propositions. Then the exclusive or of p and q denoted by $p \oplus q$,is the proposition which is true when only one of p and q is true and is false otherwise.

Example

“Students who have taken calculus or computer science, but not both, can enroll in this class.”

Similarly, when a menu at a restaurant states, “Soup or salad comes with an entrée,” the restaurant almost always means that customers can have either soup or salad, but not both.

Truth Table of Exclusive Or

p	q	$p \oplus q$
T	T	F
F	T	T
T	F	T
F	F	F

Conditional Statement

Let p and q be propositions. The *conditional statement* $p \rightarrow q$ is the proposition "if p, then q."

The conditional statement $p \rightarrow q$ is false when p is true and q is false, and true otherwise.

In the conditional statement $p \rightarrow q, p$ is called the hypothesis (or antecedent or premise)
and q is called the conclusion (or consequence).

Truth Table of Conditional Statement

p	q	$p \to q$
T	T	T
F	T	T
T	F	F
F	F	T

Example

p: I am elected

q: I will lower taxes

$p \to q$: If I am elected, then I will lower taxes

Example

p: Its sunny

q: We will go to the beach

$p \rightarrow q$: If its sunny, then we will go to the beach

Converse, Contrapositive and Inverse

We can form some new conditional statements starting with a conditional statement $p \rightarrow q$.

The proposition $q \rightarrow p$ is called the **converse** of $p \rightarrow q$.

The **contrapositive** of $p \rightarrow q$ is the proposition $\sim q \rightarrow \sim p$.

The proposition $\sim p \rightarrow \sim q$ is called the **inverse** of $p \rightarrow q$.

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$q \rightarrow p$	$\sim q \rightarrow \sim p$	$\sim p \rightarrow \sim q$
T	T	F	F	T	T	T	T
F	T	T	F	T	F	T	F
T	F	F	T	F	T	F	T
F	F	T	T	T	T	T	T

Biconditional Statement

Let p and q be propositions. The biconditional statement $p \leftrightarrow q$ is the proposition

"p if and only if q."

Truth Table of Biconditional

p	q	$p \leftrightarrow q$
T	T	T
F	T	F
T	F	F
F	F	T

Example

Construct a truth table for each of these compound propositions

(i) $(p \vee q) \rightarrow (p \wedge q)$

(ii) $(p \vee \sim q) \rightarrow q$

p	q	$p \vee q$	$p \wedge q$	$p \vee q \rightarrow p \wedge q$
T	T	T	T	T
F	T	T	F	F
T	F	T	F	F
F	F	F	F	T

Logical Equivalence

The notation $p \equiv q$ denotes that *p* and *q* are logically equivalent.

How to check two propositions are equivalent

One way to determine whether two compound propositions are equivalent is to use a truth table.

In particular, the compound propositions *p* and *q* are equivalent if and only if the columns are same.

$$p \to q \equiv \sim p \vee q$$

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
F	T	T	T	T
T	F	F	F	F
F	F	T	T	T

p	q	$\sim p$	$\sim q$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
F	T	T	F	T	T
T	F	F	T	F	F
F	F	T	T	T	T

So

$$p \to q \equiv \sim q \to \sim p$$

.

Use a truth table to verify the distributive law

$$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r).$$

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Use a truth table to prove the Demorgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Use a truth table to prove the Absorption Laws

$$p \vee (p \wedge q) \equiv p$$

$$p \wedge (p \vee q) \equiv p$$

To be continued……

Thanks for watching

Have a nice day