Identification of Critical Node Clusters in Large-Scale Spatial Networks

Vishaal Krishnan Sonia Martínez

Mechanical and Aerospace Enginering
University of California, San Diego
v6krishn@ucsd.edu

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Outline

Large-scale spatial networks

Why large-scale networks? Increasingly pervasive due to advances in low-cost sensing, communication and computational capabilities

Essential characteristics

- Consist of a large number of nodes

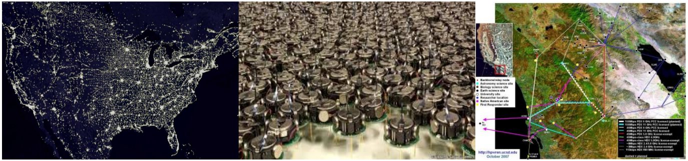


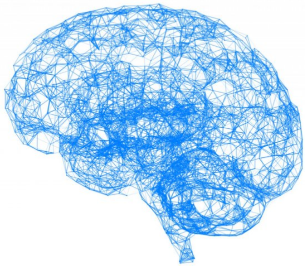
Figure: The US Powergrid; Swarm of Kilobots; Sensor network for forest fire monitoring

- Nodes embedded in a physical (metric) space - **underlying geometry cannot be ignored**
- Deployed to perform specific functions - **represented as dynamical processes on the network**

Network vulnerability and robustness

Structure affects Function

- Network topological structure (**determined by geometry**) $\rightarrow$ Function
- Network performance characterized by suitable metrics - e.g. Diameter, Average shortest path length, λ_{max} and λ_2 **of Laplacian**, node connectivity, etc



Robustness

- Performance deteriorates when nodes fail or are attacked
- Some nodes more important than others - **need to identify critical nodes to be protected**

What are critical node clusters?

Definition (Critical node clusters)

Set of nodes of given cardinality, whose removal results in the **maximum deterioration of a chosen metric**

Selected Literature

- M. Ventresca and D. Aleman, "*Efficiently identifying critical nodes in large complex networks*", 2015
- X. Chen, "*Critical nodes identification in complex systems*", 2015

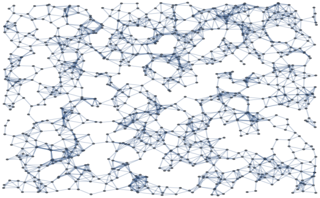
Drawbacks

- Identifying critical nodes formulated as a combinatorial optimization problem - **Complexity $\uparrow$ with network size N**
- But $N \to \infty$ is not necessarily bad, allows for simplifying approx. - algorithms should exploit this
- Some performance metrics harder to handle, but more important
- For spatial networks, geometry of embedding is important - may even help simplify the problem

Outline

In this work..

- Spatial network described by a **Random Geometric Graph**
  - Reasonable to describe large-scale network by spatial distribution of nodes
  - Nearest spatial neighbor interactions as a starting point



- Dynamical process on network - **Laplacian Consensus**

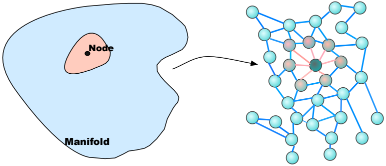
  Consensus (or agreement) is often crucial to achieving reliability in distributed systems

- Choice of performance metric - λ_2 **of graph Laplacian**
  - Governs **convergence rate** (more on it later)
  - Algebraic connectivity - measure of how well connected the graph is

In this work..

Continuum abstraction of large-scale network

- Large-scale network viewed as a discretization of a continuous space (manifold) - **N nodes sampled uniformly from $\Omega \subset \mathbb{R}^d$**
- The nodes are indexed by their position $x \in \Omega \subset \mathbb{R}^d$



- Object of interest for consensus - **Laplacian matrix (L_N) of graph**

In the limit $N \to \infty$[1]

- $L_N \to \mathcal{L}$ a bounded operator on $L^{(*)}(\Omega)$
- As the communication radius $r \to 0$, $\mathcal{L} \to -\Delta$ the **Laplace operator**

[1] Belkin, Niyogi

In this work..

Definition (What are critical node clusters?)

Set of nodes of given cardinality, whose removal results in the **maximum deterioration of chosen metric**

Metric chosen: λ_2 of Laplacian L_N

Following the continuum approximation

- Critical node clusters are now **subsets of Ω of given measure** - we consider balls of a given radius r
- The metric $\lambda_2(L_N)$ is now $\mu_2(-\Delta(\Omega))$ - the second eigenvalue of the Laplace operator on Ω

Outline

Laplacian consensus in functional form

Continuous-time Laplacian consensus on $\phi^d = (\phi_1^d, \ldots, \phi_N^d)$:

$$\frac{d}{dt}\phi^d = -L_N \phi^d$$

With the **continuum approximation ($N \to \infty$ and $r \to 0$)**:

$$\partial_t \phi = -\mathcal{L}\phi = \Delta\phi$$

where $\phi_i^d = \phi(t, x_i)$.

Boundary condition In the discrete case, **the sum is conserved**:

$$\frac{d}{dt} 1_N^\top \phi^d = -1_N^\top L_N \phi^d = 0$$

Equivalently, we need in the continuum case:

$$\frac{d}{dt}\int_\Omega \phi \; d\nu = \int_\Omega \partial_t \phi \; d\nu = \int_\Omega \Delta\phi \; d\nu = \int_{\partial\Omega} \nabla\phi \cdot \mathbf{n} \; dS = 0$$

This motivates a **Neumann boundary condition** $\nabla\phi \cdot \mathbf{n} = 0$ on $\partial\Omega$

Second eigenvalue μ_2 of $\mathcal{L} = -\Delta$

For the Neumann problem,

- $\mathcal{L} = -\Delta$ has an infinite sequence of eigenvalues $0 = \mu_1 \leq \mu_2 \leq \ldots \leq \mu_m \leq \ldots$
- The eigenfunctions $\{\psi_i\}_{i=1}^{\infty}$ form an orthonormal basis for $L^2(\Omega)$

The energy functional for consensus:

$$E = \frac{1}{2}\langle \phi, \mathcal{L}\phi \rangle_{L^2(\Omega)}$$

Convergence rate:

$$\frac{d}{dt}E \leq -2\mu_2 E$$

How to compute μ_2?

The Rayleigh quotient for operator $\mathcal{L} = -\Delta$

$$Q(\mathcal{L}) = \frac{\langle \psi, \mathcal{L}\psi \rangle_{L^2(\Omega)}}{\langle \psi, \psi \rangle_{L^2(\Omega)}} = \frac{\int_\Omega |\nabla \psi|^2 d\nu}{\int_\Omega |\psi|^2 d\nu}$$

The **min-max theorem** gives a variational characterization of eigenvalues

For the operator $\mathcal{L} = -\Delta$, we get:

$$\mu_2 = \inf_{\substack{\int_\Omega \psi d\nu = 0, \\ \int_\Omega |\psi|^2 d\nu = 1}} \int_\Omega |\nabla \psi|^2 d\nu$$

Problem formulation

Problem statement Identify a ball $B_r(x) \subset \Omega$ of radius r, $x \in \tilde{\Omega}$ such that $\mu_2(\Delta(\Omega \setminus B_r(x)))$ is an infimum

Equivalently, identify an x^* below

$$x^* \in \arg \inf_{x \in \tilde{\Omega}_r} \inf_{\substack{\int_{\Omega \setminus B_r(x)} \psi d\nu = 0, \\ \int_{\Omega \setminus B_r(x)} |\psi|^2 d\nu = 1}} \int_{\Omega \setminus B_r(x)} |\nabla \psi|^2 d\nu$$

The outer optimization problem is **non-convex**

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Approach

The idea is to set up a flow to guide a point $x \in \Omega$ towards x^*

- Design an algorithm to compute μ_2 for an arbitrary domain
- Use it to guide (set up a dynamics for) center of ball - x of $B_r(x)$
- Prove that the center of the ball converges to x^* (defined earlier)

Outline

Determination of μ_2 for a fixed domain

For μ_2, we have the optimization problem:

$$\inf_{\psi \in H^1(\Omega)} J(\psi) = \int_\Omega |\nabla \psi|^2,$$
$$\text{s.t} \quad \int_\Omega |\psi|^2 = 1, \quad \int_\Omega \psi = 0,$$
$$\nabla \psi \cdot \mathbf{n} = 0 \text{ on } \partial\Omega.$$

$\boxed{\mathcal{S}_\Omega \text{ for the feasible set}}$

Critical points of $J(\psi)$

$$\Delta \psi^* + \mu^* \psi^* = 0$$

Lemma

Of all the critical points ψ^* of the functional $J(\psi)$, the second eigenfunction ψ_2 of $\Delta(\Omega)$ is the only minimizer of $J(\psi)$ in $\mathcal{S}_\Omega$.

Determination of μ_2 for a fixed domain

We consider the dynamics:

$$\partial_t \psi = \Delta \psi + J(\psi)\psi \tag{1}$$

Lemma

The set $\mathcal{S}_\Omega$ is invariant with respect to (1).

Lemma

Solutions to (1) in $\mathcal{S}_\Omega$ converge in L^2-norm to the set of equilibria of (1).

Lemma

The second eigenfunction ψ_2 is the only locally asymptotically stable equilibrium in $\mathcal{S}_\Omega$ for (1).

μ_2 of residual domains

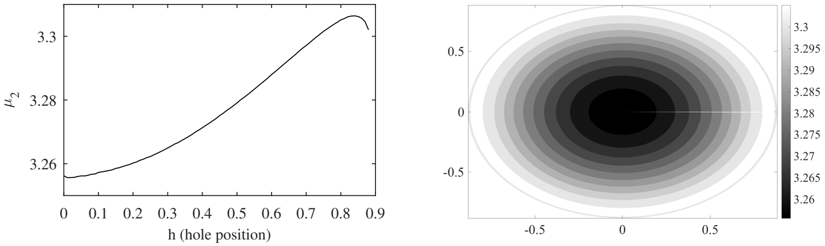

Figure: μ_2 as a function of h for a disk-shaped domain.

μ_2 of residual domains

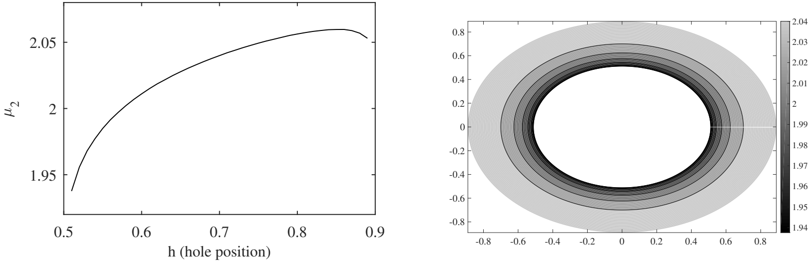

Figure: μ_2 as a function of h for a ring-shaped domain of inner radius 0.4 and unit outer radius.

Identification of critical node cluster

Lemma

The first-order condition for a critical point x^* of the functional μ_2 in the interior of the domain is given by:

$$\mu_2^* \left(\int_{\partial B_r(x^*)} |\psi_2^*|^2 \mathbf{n} \right) = \int_{\partial B_r(x^*)} |\nabla \psi_2^*|^2 \mathbf{n}, \tag{2}$$

where $(\mu_2^* = \mu_2(x^*), \psi_2^*)$ is the second eigenpair and $\mathbf{n}$ is the outward normal to $\partial B_r(x^*)$.

Identification of critical node cluster

We consider the following dynamics for the center of the ball:

$$
\begin{aligned}
&\frac{dx}{dt} = \mathbf{v} = \begin{cases} \mathbf{v}_{int}, & x \in \text{int } \tilde{\Omega}_r \\ \mathbf{v}_{int} - (\mathbf{v}_{int} \cdot \tilde{\mathbf{n}})\tilde{\mathbf{n}}, & x \in \partial\tilde{\Omega}_r \end{cases} \\
&\mathbf{v}_{int} = \int_{\partial B_r(x)} |\nabla\psi|^2 \mathbf{n} - J(\psi) \int_{\partial B_r(x)} |\psi|^2 \mathbf{n}, \\
&\partial_t \psi = \Delta\psi + J(\psi)\psi + a\psi + b, \\
&\nabla\psi \cdot \mathbf{n} = 0, \text{ on } \partial\Omega \cup \partial B_r(x)
\end{aligned} \tag{3}
$$

Theorem

The set Ψ is invariant with respect to the dynamics (3). The solutions to the dynamics (3) converge to a critical point of the objective functional μ_2. A critical point of μ_2 is locally asymptotically stable with respect to the dynamics (3) only if it is a strict local minimum.

Simulation

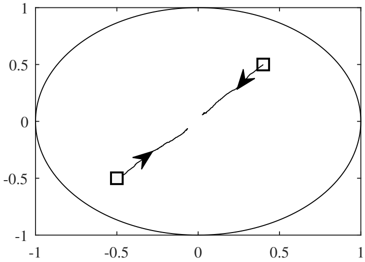

Figure: Path of the center of the hole, $x(t)$ from two different initial conditions $x(0) = (0.4, 0.5)$ and $x(0) = (-0.5, -0.5)$

Outline

Conclusions

- We studied the problem of identifying critical nodes for consensus in large-scale spatial networks
- Approximated the graph Laplacian matrix by the Laplace operator on the domain
- Formulation **does not conceal the geometry** of the problem - important for spatial networks
- Looked at the removal of balls of given radius - **to minimize μ_2 of residual domain**
- **Future work**: Generalization to arbitrary sets, and non-uniform distributions of nodes in the domain