1 Metric spaces

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We assume the reader has had ample experience with calculus on the real line, including in-depth coverage of convergence, of open and closed sets, and of continuous functions. In this chapter, we begin the process of lifting these concepts to arbitrary metric spaces.

1.1 Definitions and examples

Many of the concepts of analysis on the real line depend only on the notion of closeness (convergence, for example). When we are working on the real line, we usually take the distance between $x, y \in \mathbb{R}$ to be $|x - y|$. However, many of the results of calculus hold for any set X and any distance function $d(x, y)$ that satisfies four axioms.

Definition 1.1.1. A *metric space* (X, d) consists of a set X and a *metric* $d : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$,

(i) $d(x, y) \geq 0$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = d(y, x)$; and
(iv) $d(x, z) \leq d(x, y) + d(y, z)$.

Items (i) and (ii) together are sometimes expressed by saying that d is *positive definite*.
Item (iii) says that d is *symmetric*.
Item (iv) is the *triangle inequality*, and is usually the only metric space axiom that requires much effort to verify.

A set X can have a multitude of metrics placed on it, which is why we must include the metric when we specify the metric space we are considering. As a first example, consider $X = \mathbb{R}$. We know that the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = |x - y|$$

is a metric on $\mathbb{R}$, and it is called the *standard metric on* $\mathbb{R}$. However, the function $d(x, y) = 2|x - y|$ would work just as well as a metric on $\mathbb{R}$. A more exotic metric is given below.

Example 1.1.2 (Discrete metric)**.** Given any set X, the *discrete metric* on X is the function $d_{\text{disc}} : X \times X \to \mathbb{R}$ defined by

$$d_{\text{disc}}(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

A metric space equipped with the discrete metric is called a *discrete metric space*.

Exercise 1.1.2 asks the reader to verify that the discrete metric is in fact a metric.

Because we have the discrete metric, we can turn *every* set into a metric space in at least one way.

We could invent quite a few more metrics on $\mathbb{R}$. For example, the function $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$d(x, y) = \min\{1, |x - y|\},$$

or the function

$$d(x, y) = \frac{|x - y|}{1 + |x - y|}.$$

The reader is asked to verify that these are metrics in Exercises 1.1.4 and 1.1.5.

Thus, even on our familiar set $\mathbb{R}$, there are a number of choices for the metric to use, although if a metric on $\mathbb{R}$ is not specified, the standard metric is to be assumed.

Before continuing with more examples, we need to make two definitions.

Definition 1.1.3 (Subspace of a metric space)**.** Let (X, d) be a metric space and $Y \subseteq X$. The pair $(Y, d|_{Y \times Y})$ is itself a metric space (Exercise 1.1.6), and is called a *subspace* of (X, d).

With Definition 1.1.3, we can restrict any metric on $\mathbb{R}$ to obtain a metric on $[0, 1]$, $\mathbb{Q}$, $\mathbb{Z}$, etc. Frequently we denote the resulting metric simply by d, instead of by $d|_{Y \times Y}$, if no confusion could result by doing so.

Next we define neighborhoods. Except for the change in metric, these are defined exactly how they were on the real line.

Definition 1.1.4 (Neighborhood)**.** Let (X, d) be a metric space. Given a real number $\epsilon > 0$ and a point $x_0 \in X$, the *ϵ-neighborhood of x_0* is the set

$$N_\epsilon(x) = \{x \in X : d(x, x_0) < \epsilon\}.$$

In Definition 1.1.4, x_0 is the *center* of the neighborhood and ϵ is the *radius* of the neighborhood. Neighborhoods always contain their centers, because we insist on radii being positive.

Neighborhoods are often called *open balls*, because of what they look like in $\mathbb{R}^2$ with the *Euclidean distance* or *Euclidean metric* $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ defined by

$$d_2(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},$$

for points $x = (x_1, x_2)$ and $y = (y_1, y_2)$. Thus in this metric, the open neighborhood of the origin $0 = (0,0)$ with radius 1 is the open unit disc,

$$N_1((0,0)) = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

This is very special metric on $\mathbb{R}^2$, and we will have a lot more to say about it in the next section. Before that, we present two other useful metrics on $\mathbb{R}^2$ below. In each case the reader is asked to verify the metric axioms as an exercise.

It is easy to see that the Euclidean metric satisfies the first three axioms of a metric, but it not entirely straight-forward to prove that the Euclidean metric satisfies the triangle inequality, which is why we delay a proof of that fact until the end of the next section.

As we define further metrics, you should ask yourself what the neighborhoods of points look like.

Definition 1.1.5 (The ℓ^1 metric on $\mathbb{R}^2$)**.** The ℓ^1 *metric* is the function $d_1 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ defined, for points $x, y \in \mathbb{R}^2$, by

$$d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|.$$

Exercises 1.1.7 and 1.1.8 ask the reader to verify that the ℓ^1 and ℓ^∞ metrics are metrics on $\mathbb{R}^2$.

Definition 1.1.6 (The ℓ^∞ metric on $\mathbb{R}^2$)**.** Given points $x, y \in \mathbb{R}^2$, the ℓ^∞ *metric* is the function d_∞ defined by

$$d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

We consider metrics on one more set, the vector space $\mathcal{C}([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$.

We have defined the ℓ^1 metric (and the ℓ^∞ metric) on both $\mathbb{R}^2$ and $\mathcal{C}([0,1],\mathbb{R})$. These are different functions (their domains are different), but it should always be clear which metric we are discussing.

Definition 1.1.7 (The ℓ^1 metric on $\mathcal{C}([0,1],\mathbb{R})$)**.** The ℓ^1 *metric* on $\mathcal{C}([0,1],\mathbb{R})$ is the function $d_1 : \mathcal{C}([0,1],\mathbb{R}) \times \mathcal{C}([0,1],\mathbb{R}) \to \mathbb{R}$ defined, for functions $f, g \in \mathcal{C}([0,1],\mathbb{R})$, by

$$d_1(f,g) = \int_0^1 |f(t) - g(t)|\, dt.$$

Exercises 1.1.9 and 1.1.10 ask the reader to verify that the ℓ^1 and ℓ^∞ metrics are metrics on $\mathcal{C}([0,1],\mathbb{R})$.

Definition 1.1.8 (The ℓ^∞ metric on $\mathcal{C}([0,1],\mathbb{R})$)**.** Given functions $f, g \in \mathcal{C}([0,1],\mathbb{R})$, the ℓ^∞ *metric* is the function d_∞ defined by

$$d_\infty(f,g) = \max\{|f(t) - g(t)| : t \in [0,1]\}.$$

We conclude the section by proving a result.

Proposition 1.1.9. *Let (X,d) be a metric space and $x, y \in X$. If $d(x,y) < \epsilon$ for every $\epsilon > 0$, then $x = y$.*

Proof. Suppose that $x \neq y$. We know from part (i) of the definition of a metric space that $d(x,y) \geq 0$ always, and because $x \neq y$, part (ii) shows that $d(x,y) \neq 0$. Therefore $d(x,y) > 0$. Letting $\epsilon = d(x,y)$, we see that we cannot have $d(x,y) < \epsilon$, proving the proposition. $\square$

Exercises