1 1 Metric spaces

1.1 Definitions and examples

A metric space (X, d) consists of a set X and a metric $d : X \times X \to \mathbb{R}$ satisfying the following axioms for all $x, y, z \in X$:

- (i) $d(x, y) \geq 0$
- (ii) $d(x, y) = 0$ if and only if $x = y$
- (iii) $d(x, y) = d(y, x)$
- (iv) $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality)

1.2 Examples of metrics

- The standard metric on $\mathbb{R}$ is $d(x, y) = |x - y|$
- The discrete metric on any set X is defined by $ddisc(x, y) = 0$ if $x = y$ and 1 otherwise
- The Euclidean metric on $\mathbb{R}^2$ is $d(x, y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
- The ℓ^1 metric on $\mathbb{R}^2$ is $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$
- The ℓ^∞ metric on $\mathbb{R}^2$ is $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$

2 Neighborhoods and Subspaces

2.1 Neighborhoods

A neighborhood of a point $x \in X$ is an open set containing x. For metric spaces, neighborhoods are defined similarly to those in the real line.

2.2 Subspaces

A subspace $(Y, d|_{Y \times Y})$ of a metric space (X, d) is a metric space where the metric is restricted to $Y \times Y$. This allows defining metrics on subsets like $[0, 1]$, $\mathbb{Q}$, or $\mathbb{Z}$.

3 Exercises

- Verify that the discrete metric is a metric (Exercise 1.1.2)
- Verify the triangle inequality for the Euclidean metric (delayed until later)
- Verify the ℓ^1 and ℓ^∞ metrics on $\mathbb{R}^2$ and $C([0, 1], \mathbb{R})$ (Exercises 1.1.7, 1.1.8, 1.1.9, 1.1.10)