Real Analysis
Mid-Sem 2023
Time - 1.5 hours
Full marks 50

1.a) Prove that for each $n \geq 2$, $(n+1)! > 2^n$.
b) Prove that for all $n \in \mathbb{N}$, $(3+\sqrt{5})^n + (3-\sqrt{5})^n$ is an even integer. (4+6)

2.a) Prove that the set of natural numbers is not bounded from above.
b) Prove that there is an unique positive real number x, such that $x^2 = 2$ (5+5)

3.a) Prove that the union and intersection of finite number of open sets in $\mathbb{R}$ are open sets themselves.
b) Show that the set $\mathbb{N}$ has no limit points. (10+5)

4.a) Prove that $\lim_{n\to\infty} \frac{S_n}{t_n} = \frac{s}{t}$, given $\lim_{n\to\infty} S_n = s$ and $\lim_{n\to\infty} t_n = t$ with $t_n \neq 0$ $\forall n \in \mathbb{N}$ and $t \neq 0$.
b) Show whether the following sequence (x_n) with $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$ is convergent or not.
c) Given $x \geq 1$, show that $\lim_{n\to\infty} \left(2x^{1/n} - 1\right)^n = x^2$ (5+5+5)

Real Analysis
End-Sem 2023
Time - 3.00 hours
Full marks 100

1.a) Prove that a sequence can have atmost one limit
b) Consider $\{u_n\}$ and $\{v_n\}$ are two converging sequences which converges to u and v respectively. Then prove the following identities.

i) $\lim_{n\infty}(u_n + v_n) = u + v$
ii) if $c \in \mathbb{R}$, $\lim_{n\infty}(cu_n) = cu$
iii) $\lim_{n\infty}(u_n v_n) = uv$
iv) $\lim_{n\infty}(u_n/v_n) = u/v$ providing $\{v_n\}$ is a sequence of non zero elements and it does not converge to 0.

(5+15)

2. Test the convergences of the following two series:

(i)
$$S_1 = \sum_{n=1}^{\infty}\left(\frac{1}{2^n} + \frac{1}{3^n}\right)$$

(ii)
$$S_2 = 1 + \frac{1}{2^3} + \frac{1}{2^2} + \frac{1}{2^5} + \frac{1}{2^4}\cdots\cdots\cdots$$

(5+5)

3.a) State and prove the Sandwitch theorem of limits.
b) State and prove the Cauchy principle of limit. (10+15)

4. Use the definition of continuity at a point to prove that
i) $f(x) = 3x - 5$ is continuous at $x = 2$.
ii) $f(x) = x^2$ is continuous at $x = 3$.
iii) $f(x) = 1/x$ is continuous at $x = 1/2$. (5+5+5)

5.a) From the definition of differentiation prove that $(fg)'(x) = f(x)g'(x) + f'(x)g(x)$, where $f(x)$ and $g(x)$ are differentiable functions in the interval I.
b) Let $I \subset \mathbb{R}$ and $f : I \to \mathbb{R}$ is a real valued function differentiable at $c \in I$. Then prove that if $f'(x) > 0$ (*or* $f'(x) < 0$) at c, then the function is increasing (or decreasing) at c.
c) State and prove Taylor's theorem. (5+10+15)