INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

Monsoon Semester (Aug-Nov), 2023
Discrete Structures (DS, Section B)

Quiz I
30.08.2023

1. **(Sets)** [4]

   1. For all sets A and B, prove that

      $$(A \cap B) \cup (A \cap B') = A.$$

      Here A' denotes complement of set A.
   2. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find power set $P(A \times B)$. Here $A \times B$ denotes the Cartesian product of A and B.
   3. Let $\mathbb{R}$ be the set of real numbers. Is $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ a partition of $\mathbb{R}$? Here $\mathbb{R}^+$ denote set of positive reals, $\mathbb{R}^-$ denote set of negative reals. Explain your answer.
   4. Let $S_i = \{x \in \mathbb{R} \,|\, 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$ for all **positive** integers i. Then find the following:
      (a) $\cup_{i=1}^{\infty} S_i = ?$ 1, 10
      (b) $\cap_{i=1}^{\infty} S_i = ?$

2. **(Induction Proofs)** [4]

   Prove the following using induction.

   145
   \- 72
   783

   1. Suppose $e_0, e_1, \ldots,$ is a sequence defined as follows

      $$e_0 = 12,\ e_1 = 29,\ e_k = 5e_{k-1} - 6e_{k-2},\ \forall k \geq 2$$

      Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.
   2. Show that

      $$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \cdots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)},$$

      where $m, n = 0, 1, 2, \ldots$

3. **(Pigeon hole principle)** [3]

   The **pigeon-hole principle** states that:

   If we put $N + 1$ pigeons in N pigeon-holes, then there will be atleast one pigeon hole with at least two pigeons. Prove this statement using contrapositive proof.

   A **general pigeon-hole principle** is stated as follows:

   If we must put $Nk + 1$ or more pigeons into N pigeon holes, then some pigeon-hole must contain at least $k + 1$ pigeons. Prove this using contrapositive proof.

   Prove the following using pigeon-hole principle.

   1. Show that among 5 people at a dinner table, there are two that have an identical number of friends among those at the table.

Student's name:

End of Quiz

MID SEMESTER EXAMINATION
IIIT, Hyderabad, Monsoon 2023
Subject: Discrete Structures
(Code: MA05.101)

INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
HYDERABAD

20.09.2023
8:30 - 10:00

Duration: 90 minutes

Maximum Marks: 30

Instructions and Notations

There are 3 pages and 14 questions. All questions are compulsory. Calculator is not allowed. You are not allowed to ask questions in exams to anyone including invigilators. If you find that there are doubts or errors in questions, then please write it in your answer script with proper justification. If the justification is correct, you will be awarded full marks.

1. Let $A = \mathbb{Z}$ and let $\mathcal{R} = \{(x, y) \in A \times A \mid xy + x^2 = x^2 + 1\}$. Which of the following are true? [3]

   a). $0\mathcal{R}0$.

   b). $1\mathcal{R}1$.

   c). $1\mathcal{R}0$.

   d). $1\mathcal{R}(-2)$.

   e). $3\mathcal{R}2$.

   Note: Here $x\mathcal{R}y$ means that $(x, y) \in \mathcal{R}$.

   A relation is called **symmetric** if whenever $x\mathcal{R}y$, then $y\mathcal{R}x$. A relation is called **transitive** if whenever $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$. Is the given relation $\mathcal{R}$ symmetric and transitive? Justify.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Define a function $f : A \to B$ as follows: [2]

$$f(1) = b, \quad f(2) = a, \quad f(3) = c, \quad f(4) = a.$$

   Let $g : B \to A$ be the function defined by the rule

$$g(a) = 2, \quad g(b) = 3, \quad g(c) = 1, \quad g(d) = 3.$$

   a). Find $(g \circ f)(x)$ for each element x of A.

   b). What is the domain, co-domain, and range of $g \circ f$?

   c). Does inverse of f, g, and $g \circ f$ exists? Justify.

   d). Is it possible to define functions p and q with suitable domain and co-domain such that $p \circ g \circ f \circ q$ is an identity map from A to A? Justify.

3. Let $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$, and define the function $f : \mathbb{N}_+ \times \mathbb{N}_+ \to \mathbb{N}$ by [3]

$$f(m, n) = 2^m(2n + 1).$$

   Prove or disprove the following:

   a). f is injective.

   b). f is surjective.

Student's name:

4. For three sets $A, B, C \subseteq U$ we have the following: [2]

$$B \subsetneq A,\quad C^c \cup A = U,\quad |C^c \cap A| = 7,\quad |B \Delta C| = 5,$$
$$|A \setminus (B \setminus C)| = 6,\quad |A \setminus (C \setminus B)| = 9,\quad |(B \cup C)^c| = 5.$$

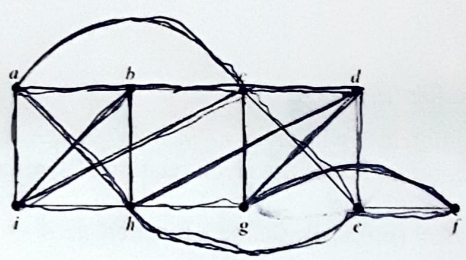
Find $|A|, |B|$, and $|C|$. If the given conditions above are inconsistent, then justify.

Here X^c denotes complement of the set X, and $\setminus$ denotes set difference, i.e., $X \setminus Y = \{x \in X \mid x \notin Y\}$, and Δ denotes symmetric set difference, i.e., $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Also, $|X|$ denotes number of elements in set X. You may use Venn diagram.

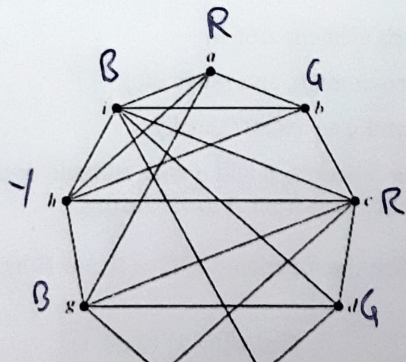
5. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 3. [2]

6. Using division algorithm, show that there exists integers x and y such that $19x + 43y = 1$. [2]

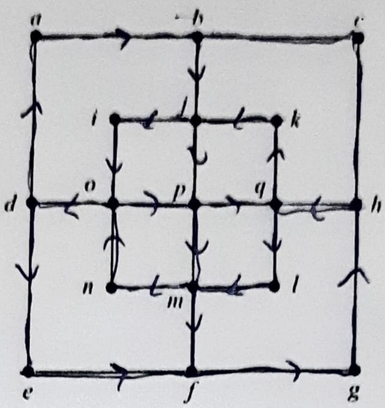
7. Determine whether the Euler tour (also called circuit) exists for the following graph. If Euler tour does not exist, then is there any Euler path? Justify your answer. [1]



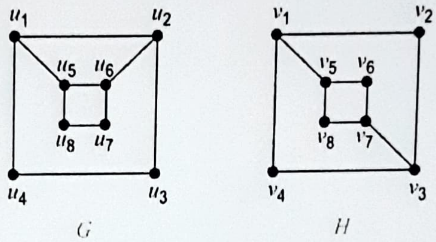
8. Recall that the chromatic number of a graph is the minimum number of colors needed to color the vertices of a graph, so that no two adjacent vertices have same color. Find the chromatic number of the following graph. [1]



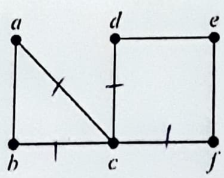
9. Determine whether the following graph has Hamiltonian circuit. State reasons why? [2]



10. Show whether the following graphs are isomorphic or not? [2]



11. Find all the cut vertices and cut edges of the following graph. A cut vertex is a vertex which when removed increases the number of components of graph. Similarly, cut edge is an edge which when removed increases the components of the graph. [Note when vertex gets removed, all edges incident on it is removed as well, whereas, when an edge is removed, the endpoints (vertices joining that edge) of the edges are not removed.] [2]



12. How many non-isomorphic connected simple graphs are possible with n vertices when n is 4? [2]

13. Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices. See cut vertex defined above. [3]

14. Show that if G is a connected graph, then it is possible to remove vertices (along with edges incident on it) to disconnect G if and only if G is not a complete graph. [3]

Quiz II

Discrete Structures
IIIT Hyderabad, Monsoon 2022

October 20, 2023

Consider the following system of congruent-recurrences:

$$
\begin{aligned}
a_n &\equiv a_{n-1} + 3n^2 && \text{with } a_0 \equiv 4 && \pmod 5 \\
a_n &\equiv 6a_{n-2} - a_{n-1} && \text{with } a_0 \equiv -1, a_1 \equiv 8 && \pmod{11} \\
a_n &\equiv 4a_{n-1} - 3a_{n-2} - 2 && \text{with } a_0 \equiv 2, a_1 \equiv 5 && \pmod 7
\end{aligned}
$$

Answer the following:

1. What is a_0 mod 385? ~~066~~ ~~338~~ 296 — 5 marks
2. What is a_2 mod 385? 19 — 5 marks
3. What is (a_{100} mod 5) ? — 15 marks
4. What is (a_{150} mod 35) ? — 20 marks
5. What is (a_{200} mod 385) ? — 25 marks
6. With the same initial/boundary conditions, how many values between 0 and 384 can (a_3 mod 385) take, if:
   - all the three congruences are satisfied? — 3 marks
   - *none of the three* congruences are satsfied? — 12 marks
   - exactly *one of the three* congruences are satsfied? — 15 marks

④ 56×8 = 448

11, 22, 33, 44, 55, 66, 77, 88, 99, 110, 121, 132, 143, 154, 165, 176.

③ 55×7 = 385

① 77×2 = 154

165 − 11

770 + 336 = 1106

660, 616, 1276, 210, 1066

175.

109, 7, 164

② 35×5 = 175

3, 349, 11

19, 209, 11

② 35×3

3, 274, 7

47, 329, 7

② 55×4 = 220

219

① 77×8 = 616

210

350

35, 70, 105, 140, 175, 210

1

1066, ① 385×3 = 1155

1066, 770, 2906

① 385×2 = 770

275

330, 551

End Semester Examination

Discrete Structures
IIIT Hyderabad, Monsoon 2023

November 25, 2023

There are *ten* questions, 10 marks each. Calculators are allowed. Maximum Marks: 100.

---

1. Fill in the following blanks: $10 \times 1 = 10$

   1. The coefficient of x^9y^3 in $(2x - 3y)^{12}$ is __________.
   2. The number of arrangements of the letters in `MISSISSIPPI` having no consecutive S's is __________.
   3. The number of positive integer solutions for $a + b + c + d = 10$ is __________.
   4. If two integers are selected at random without replacement from $\{1, 2, \ldots, 100\}$, the probability that the integers are consecutive is __________.
   5. If Pr(A) = 0.5, Pr(B) = 0.3, and $\Pr(A|B) + \Pr(B|A) = 0.8$, then $\Pr(A \cap B)$ is __________.
   6. State True or False: If eight people are in a room, at least two of them have birthdays that occur on the same day of the week: __________.
   7. State True or False: Let triangle ABC be equilateral with AB=1. If we select 10 points in the interiorof this triangle, there must be at least two whose distance apart is less than 1/3: __________.
   8. How many times must we roll a single die in order to get the same score at least thrice? __________.
   9. Solution to $a_{n+2} - 4a_{n+1} + 3a_n + 200 = 0$, with $a_0 = 2$, $a_1 = 104$ is __________.
   10. The chromatic number (minimum number of colors required to properly color a graph) of a connected bipartite graph is __________.

2. Give an example for each of the following: $2 + 3 + 5 = 10$

   1. A simple undirected graph G that has an Euler circuit (a circuit that has every edge once) and an example of a way to orient the edges (give directions to edges to obtain a directed graph) such that the resultant digraph does *not* have a Euler circuit and another way to orient the edges such that the digraph has a Euler circuit.
   2. A binary operation on graphs of n vertices such that the set of all graphs on n vertices forms a group (for that operation).
   3. Two binary operations on graphs of n vertices, say $+$ and $\star$, such that the set of all graphs on n vertices forms a ring (using $+, \star$).

3. Prove or disprove the following: $4 \times 2\frac{1}{2} = 10$

   1. If $\mathbb{F}$ is a finite field, the characteristic of $\mathbb{F}$ must be prime. However, the converse is not true.
   2. Any finite integral domain is a field.
   3. Any integral domain with finite characteristic must be of finite order.
   4. If U is an ideal of ring R and $1 \in U$, then $U = R$.

4. For any group G, let $A(G)$ denote the set of all automorphisms of G and let $F(G) = \{T_g \in A(G) \mid g \in G, T_g : G \to G \text{ where } \forall x \in G, T_g(x) = g^{-1}xg\}$. Prove the following: $1 + 2 + 3 + 4 = 10$

1. $A(G)$ is a group.
2. If $G = S_3$ (symmetric group of degree 3) then G is isomorphic to $F(G)$.
3. $F(G)$ is a normal subgroup of $A(G)$.
4. $F(G)$ is isomorphic to $G|Z$ where Z is the center of G.

5. Let G be a group in which , for some integer $n > 1$, $(ab)^n = a^n b^n$, for all $a, b \in G$. Prove the following: $4 \times 2\frac{1}{2} = 10$

1. $G^{(n)} = \{x^n \mid x \in G\}$ is a normal subgroup of G.
2. $G^{(n-1)} = \{x^{n-1} \mid x \in G\}$ is a normal subgroup of G.
3. $a^{n-1}b^n = b^n a^{n-1}$ for all $a, b \in G$.
4. $(aba^{-1}b^{-1})^{n(n-1)} = e$ for all $a, b \in G$.

6. Prove each of the following: (a) Lagrange's Theorem for finite groups (regarding order of a subgroup dividing the order of group), (b) If H and K are subgroups of group G then $(H \cap K)$ is a subgroup of G, and (c) any subgroup of a cyclic group is itself a cyclic group. $3 + 3 + 4 = 10$

7. Prove the following regarding simple planar graphs: $2 + 2 + 6 = 10$

1. Theory of planar graphs is popular only for undirected graphs and not *directed* graphs. Why?
2. Every planar graph is 6-colorable.
3. Let p_n be the probability that a simple graph on n vertices, chosen uniformly at random from all the $2^{\binom{n}{2}}$ possible simple undirected graphs, is planar. What are the values of p_4, p_5 and p_6?

8. Given n distinct objects, prove that: $3 + 5 + 2 = 10$

1. The number of *derangements* of n objects (arrangements where i^{th} object is not in i^{th} position, for all $1 \leq i \leq n$), is (approximately) $\frac{n!}{e}$.
2. The number of times you need to pick an object uniformly at random (one at a time with replacement), such that the probability that you pick the same object more than once is at least 0.5, is $O(\sqrt{n})$.
3. The number of ways in which the n objects can be permuted so that none of the following sequence of objects (assume objects are numbered as $1, \ldots, n$) occurs contiguously anywhere in the permutations: $1, 2, 3$ and $4, 5, 6, 7$, and $8, 9$ is ________________________. (Fill in the blanks and then prove it).

9. Given numbers a, b, c and d, suppose you have find $\gcd(a, b^{c^d})$ on your computer. How would you do it efficiently assuming that you know the prime factorization of a? (Note that machine may not be able to compute and store the value of b^{c^d}, even including all time/memory available in the Universe!)
Hint: Use Euclid's algorithm, Chinese Remainder Theorem and Fermat's Little Theorem together).

10. Write in detail with proofs and applications about any *two* among of the following: $2 \times 5 = 10$

1. Well-Ordering Principle
2. Pigeonhole Principle
3. Equivalence Relations and Partitions
4. Principle of Inclusion and Exclusion
5. Taxonomy of Recurrence Relations and Their Solutions
6. Platonic Solids and Planar Graphs

BEST OF LUCK