

1. (Sets)

[4]

1. For all sets A and B , prove that

$$(A \cap B) \cup (A \cap B') = A.$$

Here A' denotes complement of set A .

2. Suppose $A = \{1, 2\}$ and $B = \{2, 3\}$. Find power set $P(A \times B)$. Here $A \times B$ denotes the Cartesian product of A and B .
3. Let \mathbb{R} be the set of real numbers. Is $\{\mathbb{R}^+, \mathbb{R}^-, \{0\}\}$ a partition of \mathbb{R} ? Here \mathbb{R}^+ denote set of positive reals, \mathbb{R}^- denote set of negative reals. Explain your answer.
4. Let $S_i = \{x \in \mathbb{R} \mid 1 < x < 1 + \frac{1}{i}\} = (1, 1 + \frac{1}{i})$ for all positive integers i . Then find the following:

(a) $\bigcup_{i=1}^{\infty} S_i = ?$

$(1, 1]$

(b) $\bigcap_{i=1}^{\infty} S_i = ?$

2. (Induction Proofs)

[4]

Prove the following using induction.

1. Suppose e_0, e_1, \dots , is a sequence defined as follows

$$e_0 = 12, e_1 = 29, e_k = 5e_{k-1} - 6e_{k-2}, \forall k \geq 2$$

Prove that $e_n = 5 \cdot 3^n + 7 \cdot 2^n$ for all integers $n \geq 0$.

2. Show that

$$\frac{m!}{0!} + \frac{(m+1)!}{1!} + \dots + \frac{(m+n)!}{n!} = \frac{(m+n+1)!}{n!(m+1)},$$

where $m, n = 0, 1, 2, \dots$

3. (Pigeon hole principle)

[3]

The pigeon-hole principle states that:

If we put $N + 1$ pigeons in N pigeon-holes, then there will be atleast one pigeon hole with at least two pigeons. Prove this statement using contrapositive proof.

A general pigeon-hole principle is stated as follows:

If we must put $Nk + 1$ or more pigeons into N pigeon holes, then some pigeon-hole must contain at least $k + 1$ pigeons. Prove this using contrapositive proof.

Prove the following using pigeon-hole principle.

1. Show that among 5 people at a dinner table, there are two that have an identical number of friends among those at the table.

Instructions and Notations

There are 3 pages and 14 questions. All questions are compulsory. Calculator is not allowed. You are not allowed to ask questions in exams to anyone including invigilators. If you find that there are doubts or errors in questions, then please write it in your answer script with proper justification. If the justification is correct, you will be awarded full marks.

1. Let $A = \mathbb{Z}$ and let $\mathcal{R} = \{(x, y) \in A \times A \mid xy + x^2 = x^2 + 1\}$. Which of the following are true? [3]

- a). $0\mathcal{R}0$.
- b). $1\mathcal{R}1$.
- c). $1\mathcal{R}0$.
- d). $1\mathcal{R}(-2)$.
- e). $3\mathcal{R}2$.

Note: Here $x\mathcal{R}y$ means that $(x, y) \in \mathcal{R}$.

A relation is called **symmetric** if whenever $x\mathcal{R}y$, then $y\mathcal{R}x$. A relation is called **transitive** if whenever $x\mathcal{R}y$ and $y\mathcal{R}z$, then $x\mathcal{R}z$. Is the given relation \mathcal{R} symmetric and transitive? Justify.

2. Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c, d\}$. Define a function $f : A \rightarrow B$ as follows: [2]

$$f(1) = b, \quad f(2) = a, \quad f(3) = c, \quad f(4) = a.$$

Let $g : B \rightarrow A$ be the function defined by the rule

$$g(a) = 2, \quad g(b) = 3, \quad g(c) = 1, \quad g(d) = 3.$$

- a). Find $(g \circ f)(x)$ for each element x of A .
- b). What is the domain, co-domain, and range of $g \circ f$?
- c). Does inverse of f , g , and $g \circ f$ exists? Justify.
- d). Is it possible to define functions p and q with suitable domain and co-domain such that $p \circ g \circ f \circ q$ is an identity map from A to A ? Justify.

3. Let $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$, and define the function $f : \mathbb{N}_+ \times \mathbb{N}_+ \rightarrow \mathbb{N}$ by [3]

$$f(m, n) = 2^m(2n + 1).$$

Prove or disprove the following:

- a). f is injective.
- b). f is surjective.

4. For three sets $A, B, C \subseteq U$ we have the following:

[2]

$$B \subseteq A, \quad C^c \cup A = U, \quad |C^c \cap A| = 7, \quad |B \Delta C| = 5, \\ |A \setminus (B \setminus C)| = 6, \quad |A \setminus (C \setminus B)| = 9, \quad |(B \cup C)^c| = 5.$$

Find $|A|$, $|B|$, and $|C|$. If the given conditions above are inconsistent, then justify.

Here X^c denotes complement of the set X , and \setminus denotes set difference, i.e., $X \setminus Y = \{x \in X \mid x \notin Y\}$, and Δ denotes symmetric set difference, i.e., $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. Also, $|X|$ denotes number of elements in set X . You may use Venn diagram.

5. Show that a number is divisible by 9 if and only if the sum of its digits is divisible by 3.

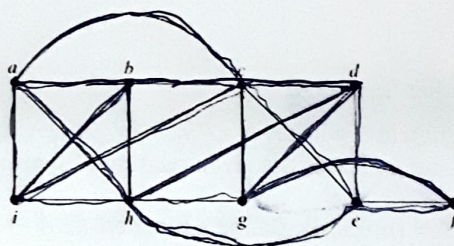
[2]

6. Using division algorithm, show that there exists integers x and y such that $19x + 43y = 1$.

[2]

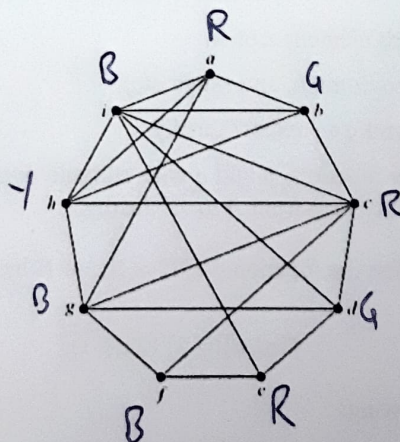
7. Determine whether the Euler tour (also called circuit) exists for the following graph. If Euler tour does not exist, then is there any Euler path? Justify your answer.

[1]



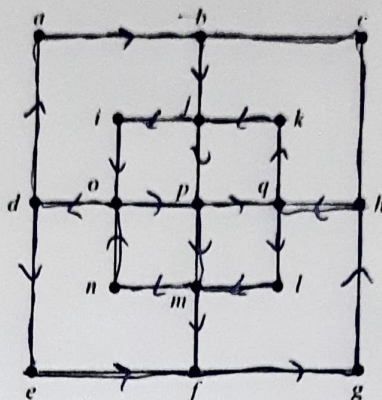
8. Recall that the chromatic number of a graph is the minimum number of colors needed to color the vertices of a graph, so that no two adjacent vertices have same color. Find the chromatic number of the following graph.

[1]



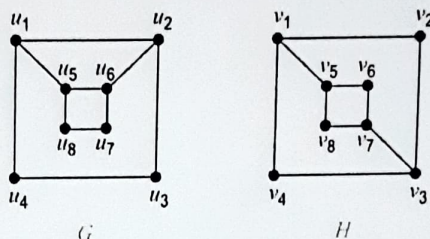
9. Determine whether the following graph has Hamiltonian circuit. State reasons why?

[2]



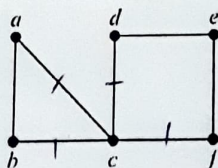
10. Show whether the following graphs are isomorphic or not?

[2]



11. Find all the cut vertices and cut edges of the following graph. A cut vertex is a vertex which when removed increases the number of components of graph. Similarly, cut edge is an edge which when removed increases the components of the graph. [Note when vertex gets removed, all edges incident on it is removed as well, whereas, when an edge is removed, the endpoints (vertices joining that edge) of the edges are not removed.]

[2]



12. How many non-isomorphic connected simple graphs are possible with n vertices when n is 4?

[2]

13. Show that a simple graph with at least two vertices has at least two vertices that are not cut vertices. See cut vertex defined above.

[3]

14. Show that if G is a connected graph, then it is possible to remove vertices (along with edges incident on it) to disconnect G if and only if G is not a complete graph.

[3]

Quiz II

Discrete Structures
IIIT Hyderabad, Monsoon 2022

October 20, 2023

Consider the following system of congruent-recurrences:

$$\begin{aligned} a_n &\equiv a_{n-1} + 3n^2 && \text{with } a_0 \equiv 4 && (\text{mod } 5) \\ a_n &\equiv 6a_{n-2} - a_{n-1} && \text{with } a_0 \equiv -1, a_1 \equiv 8 && (\text{mod } 11) \\ a_n &\equiv 4a_{n-1} - 3a_{n-2} - 2 && \text{with } a_0 \equiv 2, a_1 \equiv 5 && (\text{mod } 7) \end{aligned}$$

Answer the following:

1. What is $a_0 \text{ mod } 385$? 5 marks
2. What is $a_2 \text{ mod } 385$? 5 marks
3. What is $(a_{100} \text{ mod } 5)$? 15 marks
4. What is $(a_{150} \text{ mod } 35)$? 20 marks
5. What is $(a_{200} \text{ mod } 385)$? 25 marks
6. With the same initial/boundary conditions, how many values between 0 and 384 can $(a_3 \text{ mod } 385)$ take, if:
 - all the three congruences are satisfied? 3 marks
 - none of the three congruences are satisfied? 12 marks
 - exactly one of the three congruences are satisfied? 15 marks

Handwritten calculations and notes:

- ③ $55 \times 7 = 385$
- ① $77 \times 2 = 154$
- ④ $56 \times 8 = 448$
- ② $55 \times 4 = 220$
- ⑤ $77 \times 8 = 616$
- ⑥ $35 \times 3 = 105$
- ⑦ $35 \times 5 = 175$
- ⑧ $35 \times 8 = 280$
- ⑨ $35 \times 11 = 385$
- ⑩ $35 \times 14 = 490$
- ⑪ $35 \times 17 = 595$
- ⑫ $35 \times 20 = 700$
- ⑬ $35 \times 23 = 805$
- ⑭ $35 \times 26 = 910$
- ⑮ $35 \times 29 = 1015$
- ⑯ $35 \times 32 = 1120$
- ⑰ $35 \times 35 = 1225$
- ⑱ $35 \times 38 = 1330$
- ⑲ $35 \times 41 = 1435$
- ⑳ $35 \times 44 = 1540$
- ㉑ $35 \times 47 = 1645$
- ㉒ $35 \times 50 = 1750$
- ㉓ $35 \times 53 = 1855$
- ㉔ $35 \times 56 = 1960$
- ㉕ $35 \times 59 = 2065$
- ㉖ $35 \times 62 = 2170$
- ㉗ $35 \times 65 = 2275$
- ㉘ $35 \times 68 = 2380$
- ㉙ $35 \times 71 = 2485$
- ㉚ $35 \times 74 = 2590$
- ㉛ $35 \times 77 = 2695$
- ㉜ $35 \times 80 = 2800$
- ㉝ $35 \times 83 = 2905$
- ㉞ $35 \times 86 = 3010$
- ㉟ $35 \times 89 = 3115$
- ㊱ $35 \times 92 = 3220$
- ㊲ $35 \times 95 = 3325$
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- ㊴ $35 \times 101 = 3535$
- ㊵ $35 \times 104 = 3640$
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- ㊷ $35 \times 110 = 3850$
- ㊸ $35 \times 113 = 3955$
- ㊹ $35 \times 116 = 4060$
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- ㊻ $35 \times 122 = 4270$
- ㊼ $35 \times 125 = 4375$
- ㊽ $35 \times 128 = 4480$
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- ① $35 \times 137 = 4795$
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- ③ $35 \times 143 = 5005$
- ④ $35 \times 146 = 5110$
- ⑤ $35 \times 149 = 5215$
- ⑥ $35 \times 152 = 5320$
- ⑦ $35 \times 155 = 5425$
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- ⑥ $35 \times 302 = 10570$
- ⑦ $35 \times 305 = 10675$
- ⑧ $35 \times 308 = 10780$
- ⑨ $35 \times 311 = 10885$
- ⑩ $35 \times 314 = 10990$
- ⑪ $35 \times 317 = 11095$
- ⑫ $35 \times 320 = 11200$
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- ㊾ $35 \times 881 = 30835$
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- ⑥ $35 \times 902 = 31570$
- ⑦ $35 \times 905 = 31675$
- ⑧ $35 \times 908 = 31780$
- ⑨ $35 \times 911 = 31885$
- ⑩ $35 \times 914 = 31990$
- ⑪ $35 \times 917 = 32095$
- ⑫ $35 \times 920 = 32200$
- ⑬ $35 \times 923 = 32305$
- ⑭ $35 \times 926 = 32410$
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- ⑯ $35 \times 932 = 32620$
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- ⑱ $35 \times 938 = 32830$
- ⑲ $35 \times 941 = 32935$
- ⑳ $35 \times 944 = 33040$
- ㉑ $35 \times 947 = 33145$
- ㉒ $35 \times 950 = 33250$
- ㉓ $35 \times 953 = 33355$
- ㉔ $35 \times 956 = 33460$
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- ㉖ $35 \times 962 = 33670$
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- ㊱ $35 \times 992 = 34720$
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- ㊳ $35 \times 998 = 34930$
- ㊴ $35 \times 1001 = 35035$
- ㊵ $35 \times 1004 = 35140$
- ㊶ $35 \times 1007 = 35245$
- ㊷ $35 \times 1010 = 35350$
- ㊸ $35 \times 1013 = 35455$
- ㊹ $35 \times 1016 = 35560$
- ㊺ $35 \times 1019 = 35665$
- ㊻ $35 \times 1022 = 35770$
- ㊼ $35 \times 1025 = 35875$
- ㊽ $35 \times 1028 = 35980$
- ㊾ $35 \times 1031 = 36085$
- ㊿ $35 \times 1034 = 36190$
- ① $35 \times 1037 = 36295$
- ② $35 \times 1040 = 36400$
- ③ $35 \times 1043 = 36505$
- ④ $35 \times 1046 = 36610$
- ⑤ $35 \times 1049 = 36715$
- ⑥ $35 \times 1052 = 36820$
- ⑦ $35 \times 1055 = 36925$
- ⑧ $35 \times 1058 = 37030$
- ⑨ $35 \times 1061 = 37135$
- ⑩ $35 \times 1064 = 37240$
- ⑪ $35 \times 1067 = 37345$
- ⑫ $35 \times 1070 = 37450$
- ⑬ $35 \times 1073 = 37555$
- ⑭ $35 \times 1076 = 37660$
- ⑮ $35 \times 1079 = 37765$
- ⑯ $35 \times 1082 = 37870$
- ⑰ $35 \times 1085 = 37975$
- ⑱ $35 \times 1088 = 38080$
- ⑲ $35 \times 1091 = 38185$
- ⑳ $35 \times 1094 = 38290$
- ㉑ $35 \times 1097 = 38395$
- ㉒ $35 \times 1100 = 38500$
- ㉓ $35 \times 1103 = 38605$
- ㉔ $35 \times 1106 = 38710$
- ㉕ $35 \times 1109 = 38815$
- ㉖ $35 \times 1112 = 38920$
- ㉗ $35 \times 1115 = 39025$
- ㉘ $35 \times 1118 = 39130$
- ㉙ $35 \times 1121 = 39235$
- ㉚ $35 \times 1124 = 39340$
- ㉛ $35 \times 1127 = 39445$
- ㉜ $35 \times 1130 = 39550$
- ㉝ $35 \times 1133 = 39655$
- ㉞ $35 \times 1136 = 39760$
- ㉟ $35 \times 1139 = 39865$
- ㊱ $35 \times 1142 = 39970$
- ㊲ $35 \times 1145 = 40075$
- ㊳ $35 \times 1148 = 40180$
- ㊴ $35 \times 1151 = 40285$
- ㊵ $35 \times 1154 = 40390$
- ㊶ $35 \times 1157 = 40495$
- ㊷ $35 \times 1160 = 40600$
- ㊸ $35 \times 1163 = 40705$
- ㊹ $35 \times 1166 = 40810$
- ㊺ $35 \times 1169 = 40915$
- ㊻ $35 \times 1172 = 41020$
- ㊼ $35 \times 1175 = 41125$
- ㊽ $35 \times 1178 = 41230$
- ㊾ $35 \times 1181 = 41335$
- ㊿ $35 \times 1184 = 41440$
- ① $35 \times 1187 = 41545$
- ② $35 \times 1190 = 41650$
- ③ $35 \times 1193 = 41755$
- ④ $35 \times 1196 = 41860</$

End Semester Examination

Discrete Structures
IIIT Hyderabad, Monsoon 2023

November 25, 2023

There are ten questions, 10 marks each. Calculators are allowed.

Maximum Marks: 100.

1. Fill in the following blanks:

10 × 1 = 10

1. The coefficient of x^9y^3 in $(2x - 3y)^{12}$ is _____
2. The number of arrangements of the letters in MISSISSIPPI having no consecutive S's is _____
3. The number of positive integer solutions for $a + b + c + d = 10$ is _____
4. If two integers are selected at random without replacement from $\{1, 2, \dots, 100\}$, the probability that the integers are consecutive is _____
5. If $\Pr(A) = 0.5$, $\Pr(B) = 0.3$, and $\Pr(A|B) + \Pr(B|A) = 0.8$, then $\Pr(A \cap B)$ is _____
6. State True or False: If eight people are in a room, at least two of them have birthdays that occur on the same day of the week: _____
7. State True or False: Let triangle ABC be equilateral with $AB=1$. If we select 10 points in the interior of this triangle, there must be at least two whose distance apart is less than $1/3$: _____
8. How many times must we roll a single die in order to get the same score at least thrice? _____
9. Solution to $a_{n+2} - 4a_{n+1} + 3a_n + 200 = 0$, with $a_0 = 2$, $a_1 = 104$ is _____
10. The chromatic number (minimum number of colors required to properly color a graph) of a connected bipartite graph is _____

2. Give an example for each of the following:

2 + 3 + 5 = 10

1. A simple undirected graph G that has an Euler circuit (a circuit that has every edge once) and an example of a way to orient the edges (give directions to edges to obtain a directed graph) such that the resultant digraph does *not* have a Euler circuit and another way to orient the edges such that the digraph has a Euler circuit.
2. A binary operation on graphs of n vertices such that the set of all graphs on n vertices forms a group (for that operation).
3. Two binary operations on graphs of n vertices, say $+$ and $*$, such that the set of all graphs on n vertices forms a ring (using $+$, $*$).

3. Prove or disprove the following:

4 × 2½ = 10

1. If \mathbb{F} is a finite field, the characteristic of \mathbb{F} must be prime. However, the converse is not true.
2. Any finite integral domain is a field.
3. Any integral domain with finite characteristic must be of finite order.
4. If U is an ideal of ring R and $1 \in U$, then $U = R$.

4. For any group G , let $A(G)$ denote the set of all automorphisms of G and let $F(G) = \{T_g \in A(G) \mid g \in G, T_g : G \rightarrow G \text{ where } \forall x \in G, T_g(x) = g^{-1}xg\}$. Prove the following:

1 + 2 + 3 + 4 = 10

1. $A(G)$ is a group.
2. If $G = S_3$ (symmetric group of degree 3) then G is isomorphic to $F(G)$.
3. $F(G)$ is a normal subgroup of $A(G)$.
4. $F(G)$ is isomorphic to G/Z where Z is the center of G .
5. Let G be a group in which, for some integer $n > 1$, $(ab)^n = a^n b^n$, for all $a, b \in G$.
Prove the following: $4 \times 2\frac{1}{2} = 10$
 1. $G^{(n)} = \{x^n \mid x \in G\}$ is a normal subgroup of G .
 2. $G^{(n-1)} = \{x^{n-1} \mid x \in G\}$ is a normal subgroup of G .
 3. $a^{n-1} b^n = b^n a^{n-1}$ for all $a, b \in G$.
 4. $(aba^{-1}b^{-1})^{n(n-1)} = e$ for all $a, b \in G$.
6. Prove each of the following: (a) Lagrange's Theorem for finite groups (regarding order of a subgroup dividing the order of group), (b) If H and K are subgroups of group G then $(H \cap K)$ is a subgroup of G , and (c) any subgroup of a cyclic group is itself a cyclic group. $3 + 3 + 4 = 10$
7. Prove the following regarding simple planar graphs: $2 + 2 + 6 = 10$
 1. Theory of planar graphs is popular only for undirected graphs and not *directed* graphs. Why?
 2. Every planar graph is 6-colorable.
 3. Let p_n be the probability that a simple graph on n vertices, chosen uniformly at random from all the $2^{\binom{n}{2}}$ possible simple undirected graphs, is planar. What are the values of p_4 , p_5 and p_6 ?
8. Given n distinct objects, prove that: $3 + 5 + 2 = 10$
 1. The number of *derangements* of n objects (arrangements where i^{th} object is not in i^{th} position, for all $1 \leq i \leq n$), is (approximately) $\frac{n!}{e}$.
 2. The number of times you need to pick an object uniformly at random (one at a time with replacement), such that the probability that you pick the same object more than once is at least 0.5, is $O(\sqrt{n})$.
 3. The number of ways in which the n objects can be permuted so that none of the following sequence of objects (assume objects are numbered as $1, \dots, n$) occurs contiguously anywhere in the permutations: 1, 2, 3 and 4, 5, 6, 7, and 8, 9 is _____. (Fill in the blanks and then prove it).
9. Given numbers a, b, c and d , suppose you have find $\gcd(a, b^{c^d})$ on your computer. How would you do it efficiently assuming that you know the prime factorization of a ? (Note that machine may not be able to compute and store the value of b^{c^d} , even including all time/memory available in the Universe!)
Hint: Use Euclid's algorithm, Chinese Remainder Theorem and Fermat's Little Theorem together).
10. Write in detail with proofs and applications about any *two* among the following: $2 \times 5 = 10$
 1. Well-Ordering Principle
 2. Pigeonhole Principle
 3. Equivalence Relations and Partitions
 4. Principle of Inclusion and Exclusion
 5. Taxonomy of Recurrence Relations and Their Solutions
 6. Platonic Solids and Planar Graphs

BEST OF LUCK
