Linear Algebra(MA2.101), Spring 2024, IIIT Hyderabad

Quiz 1

Total Marks: 15

Answer any three questions out of five. Each question carries 5 marks.

1. Let

$$A = \begin{pmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{pmatrix}$$

   For which values of $Y = (y_1, y_2, y_3, y_4)^T$ do the system of equation $AX = Y$ has a solution and under what conditions do the systems $AX = Y$ don't have any solution? Use row-reduced echelon form of A to justify your answer. [5 marks]

2. Prove the following:

   (a) Suppose $a \in \mathbb{F}$ and $\vec{v} \in \mathbf{V}$, where $\mathbf{V}$ is a vector space defined over $\mathbb{F}$. If $a\,\vec{v} = \vec{0}$, then prove that either $a = 0$ or $\vec{v} = \vec{0}$. [2 marks]

   (b) For every $\vec{v} \in \mathbf{V}$, $-(-\vec{v}) = \vec{v}$. [1 marks]

   (c) Every element in a vector space has a unique additive inverse. [1 marks]

   (d) A vector space has a unique additive identity. [1 marks]

3. If $b \in \mathbb{F}$, then the set $\{(x_1, x_2, x_3, x_4) \in \mathbb{F}^4 : x_3 = 5x_4 + b\}$ is a subspace of $\mathbb{F}^4$ if and only if $b = 0$. [5 marks]

4. Using elementary row operations, prove that A, where

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

   is invertible if and only if $ad - bc \neq 0$. [5 marks]

MA2.101: Linear Algebra (Spring 2022)

Exam

Wednesday, 28 March 2024

Course outcomes: CO1, CO3, CO6.

1. ([4 marks]) Solve one of the following.

   - The system of equations

     $$\begin{aligned} x + y + z &= 6 \\ x + 4y + 6z &= 20 \\ x + 4y + \overset{8}{\cancel{\lambda}} z &= \cancel{\phi}.\ 7 \end{aligned}$$

     Find the values of λ and ϕ for which this system of equations has no solutions.

   - If $A\mathsf{x} = \mathsf{b}$ always has at least one solution, show that the only solution to $A^T\mathsf{y} = 0$ is $\mathsf{y} = 0$. Here A^T denotes the transposition of matrix A.

2. ([3 marks]) V is a finite-dimensional vector space and let $T : V \to V$ be a linear operator on V. Suppose that $\text{rank}(T^2) = \text{rank}(T)$. Prove that the range and nullspace of T have only the zero vector $\mathbf{0}$ in common.

3. ([4 marks]) Two vector spaces are called *isomprphic* if there exists an invertible linear transformation from one vector space onto the other one. Prove that two finite-dimensional vector spaces over $\mathbf{F}$ are isomorphic if and only if they have the same dimension.

4. ([4 marks]) Solve one of the following.

   (a) Prove both of the following statements.

   - The image or the range of a linear transformation $T : V \to W$ is a subspace of W.
   - A linear transformation $T : V \to W$ is one-to-one if and only if the nullspace of T only contains $0 \in V$.

(b) Consider the ordered bases $\mathcal{A} = \{(1,2),(-2,-3)\}$ and $\mathcal{B} = \{(2,1),(1,3)\}$ for a vector space $\mathbf{V}$. Then find the following

- Matrix P that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{A}$ to coordinates w.r.t. the ordered basis $\mathcal{B}$.
- Matrix Q that changes coordinates of any vector $\vec{\alpha} \in \mathbf{V}$ w.r.t. the ordered basis $\mathcal{B}$ to coordinates w.r.t. the ordered basis $\mathcal{A}$.

Best wishes for all your endeavours. Stay healthy and happy!