Machine replacement when transition probabilities are unknown

Problem statement

6.3. Some Examples

EXAMPLE 1. *A Machine Replacement Model.* Consider a machine which can be in any one of the states 0, 1, 2, ... Suppose that at the beginning of each day, the state of the machine is noted and a decision upon whether or not to replace the machine is made. If the decision to replace is made, then we assume that the machine is instantaneously replaced by a new machine whose state is 0.

The cost of replacing the machine will be denoted by R, and furthermore, we suppose that a maintenance cost $C(i)$ is incurred each day that the machine is in state i.

Also, we let P_{ij} represent the probability that a machine in state i at the beginning of one day will be in state j at the beginning of the next day.

It follows that the above is a two-action Markov decision model in which action 1 is the replacement action and action 2 the nonreplacement action. The one-stage costs and transition probabilities are given by:

$$C(i, 1) = R + C(0), \qquad C(i, 2) = C(i), \qquad i \geq 0$$
$$P_{ij}(1) = P_{0j}, \qquad P_{ij}(2) = P_{ij}, \qquad i \geq 0$$

Furthermore, we impose the following assumptions on the costs and transition probabilities:

(i) $\{C(i), i \geq 0\}$ is a bounded, increasing sequence
(ii) $\sum_{j=k}^{\infty} P_{ij}$ is an increasing function of i, for each $k \geq 0$.

Hence, (i) asserts that the maintenance cost is an increasing function of the state; and (ii) asserts that the probability of a transition into any block of states $\{k, k+1, \ldots\}$ is an increasing function of the present state.

In order to determine the structure of the optimal policy, we shall need the following lemma, whose proof is left to the reader.

Expected solution

> **Theorem 6.9**
>
> *Under Assumptions (i) and (ii), there exists an integer i^*, $i^* \leq \infty$, such that an α-optimal policy replaces for $i > i^*$ and does not replace for $i \leq i^*$.*

Environment information

We have chosen the following environment for the problem. Note, that P, R and C are unknown to the models.
We are also considering the initial state to be 1 instead of 0.

R

$R \geq 0$ is a hyperparameter

$C(i)$

We define $C(i)$ to be $2 - \frac{1}{i}$.

- This is an increasing sequence.
- This sequence is bounded by 2.

P

We define $P_{i,j} = \frac{e^{-i} . i^{j-1}}{(j-1)!}$.

- This is a pdf of a Poisson distribution with $\lambda = i$. We are using $j-1$ since Poisson distribution starts from 0 but we want our random variables to be from 1, so we just shift the distribution by 1.
- $\sum_{j=1}^{\infty} \frac{e^{-i} . i^{j-1}}{(j-1)!} \iff \sum_{l=0}^{\infty} \frac{e^{-i} . i^{l}}{(l)!} = 1$ (Poisson distribution)

- $\sum\limits_{j=k}^{\infty} \frac{e^{-i} \cdot i^{j-1}}{(j-1)!}$ is increasing in terms of i for all $k \geq 1$
  - Differentiate with respect to i, we get
  - $\sum\limits_{j=k}^{\infty} \frac{-e^{-i} \cdot i^{j-1} + e^{-i} \cdot (j-1) \cdot i^{j-2}}{(j-1)!}$
  - $\iff e^{-i} \cdot \big(- \sum\limits_{j=k}^{\infty} \frac{i^{j-1}}{(j-1)!} + \sum\limits_{j=k}^{\infty} \frac{(j-1) \cdot i^{j-2}}{(j-1)!} \big)$
  - $\iff e^{-i} \cdot \big(- \sum\limits_{j=k}^{\infty} \frac{i^{j-1}}{(j-1)!} + \sum\limits_{j=k}^{\infty} \frac{i^{j-2}}{(j-2)!} \big)$
  - $\iff e^{-i} \cdot \big(- \sum\limits_{j=k}^{\infty} \frac{i^{j-1}}{(j-1)!} + \sum\limits_{j=k-1}^{\infty} \frac{i^{j-1}}{(j-1)!} \big)$
  - $\iff e^{-i} \cdot \big(- \sum\limits_{j=k}^{\infty} \frac{i^{j-1}}{(j-1)!} + \frac{i^{k-2}}{(k-2)!} + \sum\limits_{j=k}^{\infty} \frac{i^{j-1}}{(j-1)!} \big)$
  - $\iff e^{-i} \cdot \frac{i^{k-2}}{(k-2)!}$
  - For $k \geq 2$, this value is positive, hence the function is increasing.
  - For $k = 1$, we know that $\sum\limits_{j=1}^{\infty} \frac{e^{-i} \cdot i^{j-1}}{(j-1)!} = 1 \, \forall \, i$.
  - Hence proved.