EE23010 NCERT Exemplar

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Let X_1, X_2, X_3....,X_n be a random sample of size $n\,(\geq 2)$ from a population having probability density function

$$f\,(x;\theta) = \begin{cases} \frac{2}{\theta x}\left(\log_e x\right) e^{-\frac{(\log_e x)^2}{\theta}} & ,0 < x < 1 \\ 0 & ,otherwise \end{cases}$$

where $\theta > 0$ is an unknown parameter. The maximum likelihood estimator of θ is,
Solution:

$$L\,(\theta) = f\,(x_1, x_2, .., x_n; \theta) \tag{1}$$

The likelihood function represents the relative probability of observing samples at different values of θ. So, the probability of observing the sample is the product of the probabilities of each observation.

$$L\,(\theta) = \prod_{i=1}^{n} f\,(x_i; \theta) \tag{2}$$

Maximizing $L\,(\theta)$ is equivalent to maximizing the the $\ln L\,(\theta)$ as ln is a monotonically increasing function.

$$l\,(\theta) = \ln L\,(\theta) \tag{3}$$

$$= \ln\left(\prod_{i=1}^{n} f\,(x_i; \theta)\right) \tag{4}$$

$$= \sum_{i=1}^{n} \ln f\,(x_i; \theta) \tag{5}$$

$$= -n\ln 2 - n\ln\theta + \sum_{i=1}^{n} \ln\left(-\ln x_i\right) - \sum_{i=1}^{n} \left(\ln x_i\right) - \sum_{i=1}^{n} \frac{\left(\ln x_i\right)^2}{\theta} \tag{6}$$

Maximizing $l\,(\theta)$ with respect to θ gives the MLE estimation, therefore

$$\frac{\partial l\,(\theta)}{\partial\theta} = 0 \tag{7}$$

$$\frac{-n}{\theta} + \frac{1}{(\theta)^2} \sum_{i=1}^{n} \left(\ln X_i\right)^2 = 0 \tag{8}$$

$$\theta = \frac{1}{n} \sum_{i=1}^{n} \left(\ln X_i\right)^2 \tag{9}$$