EE23010 NCERT Exemplar

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Let X_1, X_2, X_3....,X_n be a random sample of size $n\,(\geq 2)$ from a population having probability density function

$$f(x;\theta) = \begin{cases} \frac{2}{\theta x}(\log_e x)\, e^{-\frac{(\log_e x)^2}{\theta}} & ,0 < x < 1 \\ 0 & ,otherwise \end{cases}$$

where $\theta > 0$ is an unknown parameter. The maximum likelihood estimator of θ is,
Solution:

$$L(\theta; X_1, X_2, ..., X_n) = f(\theta; X_1, X_2, ..., X_n) \tag{1}$$

Maximizing $L(\theta)$ is equivalent to maximizing the the $\log L(\theta)$ as log is a monotonically increasing function.

$$l(\theta) = \log L(\theta) \tag{2}$$

$$= \log\left(\prod_{i=1}^{n} f(X_i|\theta)\right) \tag{3}$$

$$= \sum_{i=1}^{n} \log f(X_i|\theta) \tag{4}$$

$$= -n\ln 2 - n\ln\theta + \sum_{i=1}^{n} \ln(-\ln x_i) - \sum_{i=1}^{n} (\ln X_i) - \sum_{i=1}^{n} \frac{(\log_e X_i)^2}{\theta} \tag{5}$$

Maximizing $l(\theta)$ with respect to θ gives the MLE estimation, therefore

$$\frac{\partial l(\theta)}{\partial\theta} = 0 \tag{6}$$

$$\frac{-n}{\theta} + \frac{1}{(\theta)^2} \sum_{i=1}^{n} (\log_e X_i)^2 = 0 \tag{7}$$

$$\theta = \frac{1}{n} \sum_{i=1}^{n} (\log_e X_i)^2 \tag{8}$$