1. Laplace Transforms

Introduction: Laplace Transform is a particular type of definite integral as an operator which changes a function of one variable t into a function of another variable by s.

Definition: If f(t) is a function of t satisfying certain conditions, then definite integral

$$\phi(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{when it exists}$$

is called the "Laplace Transform of f(t)".
Thus

$$L[f(t)] = \phi(s) = \int_0^\infty e^{-st} f(t)\,dt$$

* Conditions for existance of Laplace transform:

(i) f(t) is continuous

(ii) $\lim_{t\to\infty} \left[e^{-at} f(t)\right]$, $s>a$ is finite

Then $L[f(t)]$ exists

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \cdots\cdots$$

dash – derivatives

suffix – integration

$v_1 = \int v\,dx,\quad v_2 = \int v_1\,dx, \ldots$

u — derivative will vanish

v — easily integrable

e.g. $\int \underset{u}{x^2}\,\underset{v}{\sin 2x}\,dx = x^2\left[\frac{-\cos 2x}{2}\right] - (2x)\left[\frac{-\sin 2x}{2^2}\right] + (2)\left[\frac{\cos 2x}{2^3}\right] - 0$

$$= -\frac{x^2}{2}\cos 2x + \frac{x\sin 2x}{2} + \frac{\cos 2x}{4}$$

Type I: Using Definition

Q. Find the L.T. of $f(t)$ where

① $f(t) = (t-1)^2$ for $t>1$

$\quad\;\; = 0$ for $0<t<1$

Ans: Using definition

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$0 \longrightarrow \infty \implies 0 - 1 - \infty$$

$$= \int_0^1 e^{-st} f(t)\,dt + \int_1^\infty e^{-st} f(t)\,dt$$

$$= \int_0^1 e^{-st}(0)\,dt + \int_1^\infty e^{-st}(t-1)^2\,dt$$

$$= 0 + \int_1^\infty \underset{u}{(t-1)^2}\, \underset{v}{e^{-st}}\,dt$$

$$= \left\{(t-1)^2\left[\frac{e^{-st}}{-s}\right] - [2(t-1)]\left[\frac{e^{-st}}{(-s)^2}\right] + [2(1)]\left[\frac{e^{-st}}{(-s)^3}\right] - 0\right\}_1^\infty$$

$$= \left[-\frac{(t-1)^2}{s}e^{-st} - 2(t-1)\frac{e^{-st}}{s^2} - \frac{2e^{-st}}{s^3}\right]_1^\infty$$

$$e^{-\infty} = \frac{1}{e^\infty} = \frac{1}{\infty} = 0$$

$$= [0-0-0] - \left[0-0-\frac{2e^{-s}}{s^3}\right]$$

$$= \frac{2e^{-s}}{s^3}$$

② $f(t) = \cos t$ $\quad 0<t<\pi$

$\quad\;\; = \sin t$ $\quad t>\pi$

Ans: Using definition

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$0 \longrightarrow \infty \implies 0 - \pi - \infty$$

$$L[f(t)] = \int_0^{\pi} e^{-st} f(t)\,dt + \int_{\pi}^{\infty} e^{-st} f(t)\,dt$$

$$= \int_0^{\pi} e^{-st} \cos t\,dt + \int_{\pi}^{\infty} e^{-st} \sin t\,dt$$

Using

$$\int e^{ax} \sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

$$\int e^{ax} \cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a \cos bx + b \cancel{\cos bx}\ \sin bx\right]$$

$a \longrightarrow -s \quad b \longrightarrow 1 \quad x \longrightarrow t$

$$= \left[\frac{e^{-st}}{(-s)^2+1^2}\left(-s \cancel{\sin t} \cos t + 1 \sin t\right)\right]_0^{\pi}$$

$$+ \left[\frac{e^{-st}}{(-s)^2+1^2}\left(-s \sin t - 1 \cos t\right)\right]_{\pi}^{\infty}$$

$$= \left[\frac{e^{-\pi s}}{s^2+1^2}\left(-s \cos \pi + \sin \pi\right) - \frac{1}{s^2+1}\left(-s+0\right)\right]$$

$$+ \left[0 - \frac{e^{-\pi s}}{s^2+1}\left(-0+1\right)\right]$$

$\sin \pi = 0$

$\cos \pi = -1$

$$= \frac{e^{-\pi s}}{s^2+1}(s+0) + \frac{s}{s^2+1} \cancel{+} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{s e^{-\pi s}}{s^2+1} + \frac{s}{s^2+1} \cancel{+} - \frac{e^{-\pi s}}{s^2+1}$$

$$= \frac{1}{s^2+1}\left[s e^{-\pi s} + s \cancel{+} - e^{-\pi s}\right]$$

$$= \frac{1}{s^2+1}\left[(s-1) e^{-\pi s} + s\right]$$

* Linearity property:

If k and l are constants then

$L[k f(t) + l\, g(t)] = k\, L[f(t)] + l\, L[g(t)]$

* Laplace Transforms of standard functions.

① $L[k]$: where k is constant

$$L[k] = \int_0^\infty e^{-st} k\, dt = k\int_0^\infty e^{-st}\, dt$$

$$= k\left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{-s}\left[e^{-st}\right]_0^\infty = \frac{k}{-s}\left[e^{-\infty} - e^{0}\right]$$

$$= \frac{k}{-s}[0-1] = \frac{k}{s}$$

$$\boxed{L[k] = \frac{k}{s}}$$

Note: $L(1) = \frac{1}{s}$

② $L(e^{at}) = \frac{1}{s-a}$ $(s>a)$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at} dt = \int_0^\infty e^{-st+at}\, dt$$

$$= \int_0^\infty e^{-(s-a)t}\, dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{(s-a)}\left[e^{-(s-a)t}\right]_0^\infty$$

$$= -\frac{1}{(s-a)}\left[e^{-\infty} - e^{0}\right]$$

$$= -\frac{1}{(s-a)}[0-1]$$

$$L[e^{at}] = \frac{1}{s-a}$$

Note: ① $L[e^{-at}] = \frac{1}{s+a}$

② $L[c^{at}] = \frac{1}{s - a\log c}$ $c>0,$ $s > a\log c$

3 $L(\sin at) = \frac{a}{s^2+a^2}$, $L(\cos at) = \frac{s}{s^2+a^2}$ $s>0$

Consider

$L[\cos at + i\sin at] = L[e^{iat}]$

$e^{i\theta} = \cos\theta + i\sin\theta$

$L[\cos at] + i\,L[\sin at] = L[e^{(ia)t}]$ .... by Linearity

$= \frac{1}{s-ia}$ .... $L[e^{at}] = \frac{1}{s-a}$

$= \frac{1}{s-ia} \times \frac{s+ia}{s+ia}$

$= \frac{s+ia}{(s)^2-(ia)^2} = \frac{s+ia}{s^2+a^2}$

$L[\cos at] + iL[\sin at] = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2}$

Comparing Real and imaginary parts

$L[\cos at] = \frac{s}{s^2+a^2}$, $L(\sin at) = \frac{a}{s^2+a^2}$

4 $L[\sinh at] = \frac{a}{s^2-a^2}$, $L[\cosh at] = \frac{s}{s^2-a^2}$

$L[\sinh at] = L\left[\frac{e^{at}-e^{-at}}{2}\right]$	$L(\cosh at) = L\left[\frac{e^{at}+e^{-at}}{2}\right]$
$=\frac{1}{2}L[e^{at}-e^{-at}]$	$=\frac{1}{2}L[e^{at}+e^{-at}]$
$=\frac{1}{2}\left[L(e^{at})-L(e^{-at})\right]$	$=\frac{1}{2}\left[L(e^{at})+L(e^{-at})\right]$
$=\frac{1}{2}\left[\frac{1}{s-a}-\frac{1}{s+a}\right]$	$=\frac{1}{2}\left[\frac{1}{s-a}+\frac{1}{s+a}\right]$
$=\frac{1}{2}\left[\frac{s+a-(s-a)}{(s-a)(s+a)}\right]$	$=\frac{1}{2}\left[\frac{s+a+s-a}{(s-a)(s+a)}\right]$
$=\frac{1}{2}\left[\frac{s+a-s+a}{s^2+a^2}\right]$	$=\frac{1}{2}\left[\frac{2s}{s^2+a^2}\right]$
$=\frac{1}{2}\left[\frac{2a}{s^2+a^2}\right]$	$=\frac{s}{s^2+a^2}$
$=\frac{a}{s^2+a^2}$	

(5) $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ $\quad (n+1)>0 \;,\; s>0$

Ans: $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

$L[t^n] = \int_0^\infty e^{-st} t^n\,dt$

putting $u = st$ $\quad t=0 \Rightarrow u=0$

$du = s\,dt$ $\quad t=\infty \Rightarrow u=\infty$

$\frac{du}{s} = dt$

$$L[t^n] = \int_0^\infty e^{-u} \left(\frac{u}{s}\right)^n \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^n} \frac{du}{s}$$

$$= \int_0^\infty e^{-u} \frac{u^n}{s^{n+1}}\,du$$

$$= \frac{1}{s^{n+1}} \int_0^\infty e^{-u} u^n\,du$$

$$= \frac{1}{s^{n+1}} \Gamma(n+1)$$

$\because \Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$

$$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$$

If n is a positive integer, $\boxed{\Gamma(n+1) = n!}$

$$\Rightarrow L(t^n) = \frac{n!}{s^{n+1}}$$

Note: $\Gamma(1/2) = \sqrt{\pi}$

$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\Gamma(n-2) = \ldots$

e.g.

$\Gamma(11/2) = 9/2\,\Gamma(9/2) = 9/2\; 7/2\,\Gamma(7/2) = 9/2\; 7/2\; 5/2\,\Gamma(5/2) = \ldots$

$= \frac{9}{2} \cdot \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma(1/2)$

$\Gamma(7/2) = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$

Type - II : Using std formulae:

Q. Find L.T. of the following

① $4t^2 + \sin 3t + 5e^{2t}$

Ans: $f(t) = 4t^2 + \sin 3t + 5e^{2t}$

$$L[f(t)] = L[4t^2 + \sin 3t + 5e^{2t}]$$

$$= 4\,L(t^2) + L(\sin 3t) + 5\,L(e^{2t})$$

$$= 4\,\frac{\lceil 3}{s^3} + \frac{3}{s^2+3^2} + 5\,\frac{1}{s-2}$$

$$= 4\,\frac{2!}{s^3} + \frac{3}{s^2+9} + \frac{5}{s-2}$$

$$= 4 \times \frac{2}{s^3} + \frac{3}{s^2+9} + \frac{5}{s-2}$$

$$= \frac{8}{s^3} + \frac{3}{s^2+9} + \frac{5}{s-2}$$

② $\cos 3t \sin 2t$

Ans: $f(t) = \cos 3t \sin 2t$

$$f(t) = \frac{1}{2}[2\cos 3t \sin 2t]$$

$$= \frac{1}{2}[\sin(3t+2t) - \sin(3t-2t)]$$

$$= \frac{1}{2}[\sin 5t - \sin t]$$

$$L[f(t)] = L[\cos 3t \sin 2t] = L\left[\frac{1}{2}(\sin 5t - \sin t)\right]$$

$$= \frac{1}{2}[L(\sin 5t) - \sin t)]$$

$$= \frac{1}{2}[L(\sin 5t) - L(\sin t)]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{5}{s^2+25} - \frac{1}{s^2+1}\right]$$

③ $\cos 2t \cos 5t$

Ans: $f(t) = \cos 2t \cos 5t$

$= \cos 5t \cos 2t$

$= \frac{1}{2}[2\cos 5t \cos 2t]$

$f(t) = \frac{1}{2}[\cos 7t + \cos 3t]$

$L[f(t)] = L\left[\frac{1}{2}(\cos 7t + \cos 3t)\right]$

$= \frac{1}{2}[L(\cos 7t + \cos 3t)]$

$= \frac{1}{2}[L(\cos 7t) + L(\cos 3t)]$

$= \frac{1}{2}\left[\frac{s}{s^2+7^2} + \frac{s}{s^2+3^2}\right]$

$= \frac{1}{2}\left[\frac{s}{s^2+49} + \frac{s}{s^2+9}\right]$

④ $\cos t \cos 2t \cos 3t$

Ans: $f(t) = \cos 3t \cos 2t \cos t$

$= \left[\frac{1}{2}(2\cos 3t \cos 2t)\right] \cos t$

$= \frac{1}{2}[\cos 5t + \cos t] \cos t$

$= \frac{1}{2}[\cos 5t \cos t + \cos^2 t]$

$= \frac{1}{2}\left\{\frac{1}{2}(2\cos 5t \cos t) + \frac{1+\cos 2t}{2}\right\}$

$= \frac{1}{2}\left[\frac{1}{2}(\cos 6t + \cos 4t) + \frac{1}{2}(1+\cos 2t)\right]$

$f(t) = \frac{1}{4}[\cos 6t + \cos 4t + 1 + \cos 2t]$

$L[f(t)] = L\left[\frac{1}{4}(\cos 6t + \cos 4t + 1 + \cos 2t)\right]$

$= \frac{1}{4}[L(\cos 6t) + L(\cos 4t) + L(1) + L(\cos 2t)]$

$= \frac{1}{4}\left[\frac{s}{s^2+36} + \frac{s}{s^2+16} + \frac{1}{s} + \frac{s}{s^2+4}\right]$

(5) $\left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3$

Ans: $f(t) = \left(\sqrt{t} + \frac{1}{\sqrt{t}}\right)^3 = \left(t^{1/2} + \frac{1}{t^{1/2}}\right)^3 = \left(t^{1/2} + t^{-1/2}\right)^3$

$(a+b)^3 = a^3 + 3ab^2 + 3a^2b + b^3$

$f(t) = (t^{1/2})^3 + 3\, t^{1/2} (t^{-1/2})^2 + 3(t^{1/2})^2 (t^{-1/2}) + (t^{-1/2})^3$

$= t^{3/2} + 3\, t^{1/2}\, t^{-1} + 3\, t\, t^{-1/2} + t^{-3/2}$

$f(t) = t^{\frac{3}{2}} + 3\, t^{-\frac{1}{2}} + 3\, t^{\frac{1}{2}} + t^{-\frac{3}{2}}$

$L[f(t)] = L[t^{3/2} + 3t^{1/2} + 3t^{-1/2} + t^{-3/2}]$

$= L[t^{3/2}] + 3L[t^{1/2}] + 3L[t^{-1/2}] + L(t^{-3/2})$

$L[t^n] = \frac{\overline{|n+1}}{s^{n+1}}$

$= \frac{\overline{|5/2}}{s^{5/2}} + 3\frac{\overline{|3/2}}{s^{3/2}} + 3\frac{\overline{|1/2}}{s^{1/2}} + \frac{\overline{|-1/2}}{s^{-1/2}}$

(6) $\cosh^5 t$

Ans: $f(t) = \cosh^5 t = (\cosh t)^5$

$= \left(\frac{e^t + e^{-t}}{2}\right)^5$

$= \frac{1}{2^5}[e^t + e^{-t}]^5$

Using Binomial thm

$= \frac{1}{32}\left[{}^5C_0 (e^t)^5 + {}^5C_1 (e^t)^4 (e^{-t}) + {}^5C_2 (e^t)^3 (e^{-t})^2 + {}^5C_3 (e^t)^2 (e^{-t})^3 + {}^5C_4 (e^t)(e^{-t})^4 + {}^5C_5 (e^{-t})^5\right]$

$= \frac{1}{32}\left[e^{5t} + 5e^{4t}e^{-t} + 10e^{3t}e^{-2t} + 10e^{2t}e^{-3t} + 5e^{t}e^{-4t} + e^{-5t}\right]$

$= \frac{1}{32}\left[e^{5t} + 5e^{3t} + 10e^{t} + 10e^{-t} + 5e^{-3t} + e^{-5t}\right]$

$= \frac{1}{16}\left[\frac{(e^{5t} + e^{-5t})}{2} + 5\frac{(e^{3t} + e^{-3t})}{2} + 10\frac{(e^{t} + e^{-t})}{2}\right]$

$f(t) = \frac{1}{16}[\cosh 5t + 5\cosh 3t + 10\cosh t]$

$$L[f(t)] = \frac{1}{16}\left[L(\cosh 5t) + 5L(\cosh 3t) + 10L(\cosh t)\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2-25} + 5\times\frac{3}{s^2-9} + 10\times\frac{1}{s^2-1}\right]$$

$$= \frac{1}{16}\left[\frac{5}{s^2-25} + \frac{15}{s^2-9} + \frac{10}{s^2-1}\right]$$

(7) $\frac{\cos\sqrt{t}}{\sqrt{t}}$

Ans: $f(t) = \frac{\cos\sqrt{t}}{\sqrt{t}} = \frac{1}{t^{1/2}}\cos\sqrt{t} = \frac{1}{t^{1/2}}\cos(t^{1/2})$

$\because \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\cdots$

$$\cos t^{1/2} = 1 - \frac{(t^{1/2})^2}{2!} + \frac{(t^{1/2})^4}{4!} - \frac{(t^{1/2})^6}{6!} + \cdots\cdots$$

$$\cos t^{1/2} = 1 - \frac{t}{2} + \frac{t^2}{24} - \frac{t^3}{720} + \cdots$$

$$\frac{1}{t^{1/2}}\cos t^{1/2} = t^{-1/2}\cos(t^{1/2})$$

$$= t^{-1/2}\left[1 - \frac{t}{2} + \frac{t^2}{24} - \frac{t^3}{720} + \cdots\right]$$

$$\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2} + \frac{t^{3/2}}{24} - \frac{t^{5/2}}{720} + \cdots\cdots$$

Appling L.T.

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L(t^{-1/2}) - \frac{1}{2}(t^{1/2}) + \frac{1}{24}L(t^{3/2}) - \frac{1}{720}L(t^{5/2}) + \cdots\cdots \quad \text{— (1)}$$

$L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$

$$L(t^{-1/2}) = \frac{\Gamma(1/2)}{s^{1/2}} = \frac{\sqrt{\pi}}{s^{1/2}}$$

$$L(t^{1/2}) = \frac{\Gamma(3/2)}{s^{3/2}} = \frac{\frac{1}{2}\Gamma\frac{1}{2}}{s^{3/2}} = \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} = \frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}}$$

$$L(t^{3/2}) = \frac{\Gamma(5/2)}{s^{5/2}} = \frac{3/2\cdot 1/2\,\Gamma(1/2)}{s^{5/2}} = \frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}}$$

$$L(t^{5/2}) = \frac{\Gamma 7/2}{s^{7/2}} = \frac{5/2 \; 3/2 \; 1/2 \; \Gamma 1/2}{s^{7/2}} = \frac{15}{8} \frac{\sqrt{\pi}}{s^{7/2}}$$

putting in ①

$$L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \frac{\sqrt{\pi}}{s^{1/2}} - \frac{1}{2}\frac{1}{2}\frac{\sqrt{\pi}}{s^{3/2}} + \frac{1}{24}\frac{3}{4}\frac{\sqrt{\pi}}{s^{5/2}} - \frac{1}{720}\frac{15}{8}\frac{\sqrt{\pi}}{s^{7/2}} \cdots\cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}} - \frac{1}{4}\frac{\sqrt{\pi}}{s^{1/2}s} + \frac{1}{32}\frac{\sqrt{\pi}}{s^{1/2}s^2} - \frac{1}{384}\frac{\sqrt{\pi}}{s^{1/2}s^3} \cdots\cdots$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \frac{1}{4}\frac{1}{s} + \frac{1}{32}\frac{1}{s^2} - \frac{1}{384}\frac{1}{s^3} \cdots\right]$$

$$= \frac{\sqrt{\pi}}{s^{1/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\frac{1}{(4s)^2} - \frac{1}{3!}\frac{1}{(4s)^3} \cdots\cdots\right]$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1/4s}{1!} + \frac{(1/4s)^2}{2!} - \frac{(1/4s)^3}{3!} + \cdots\right]$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} e^{-1/4s}$$

* change of Scale property:

If $L[f(t)] = \phi(s)$ then $L[f(at)] = \frac{1}{a}\phi\left(\frac{s}{a}\right)$

Example:

① If $L[f(t)] = \frac{3s}{s^2+9}$ then $L[f(2t)]$

Ans: $L[f(t)] = \frac{3s}{s^2+9} = \phi(s)$

$$\therefore L[f(2t)] = \frac{1}{2}\phi\left(\frac{s}{2}\right) = \frac{1}{2}\,\frac{3(s/2)}{(s/2)^2+9} = \frac{1}{2}\,\frac{3s/2}{\frac{s^2}{4}+9}$$

$$= \frac{1}{\cancel{4}}\,\frac{3s}{\frac{(s^2+36)}{\cancel{4}}} = \frac{3s}{s^2+36}$$

② If $f(t) = \sin 2t$ find $L[f(t)]$ & $L[f(3t)]$

Ans: $L[f(t)] = L(\sin 2t) = \frac{2}{s^2+4} = \phi(s)$

$$L[f(3t)] = \frac{1}{3}\phi\left(\frac{s}{3}\right) = \frac{1}{3}\,\frac{2}{(s/3)^2+4} = \frac{1}{3}\,\frac{2}{\frac{s^2}{9}+4}$$

$$= \frac{1}{3}\,\frac{2}{\frac{s^2+36}{9}} = \frac{6}{s^2+36}$$

③ If $f(t) = \operatorname{erf}\sqrt{t}$ and $L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}}$

find $L[\operatorname{erf} 2\sqrt{t}]$

Ans: $L[f(t)] = L[\operatorname{erf}\sqrt{t}] = \frac{1}{s\sqrt{s+1}} = \phi(s)$

$$L[\operatorname{erf} 2\sqrt{t}] = L[\operatorname{erf}\sqrt{4t}]\,.$$

$$= L[f(4t)]$$

$$= \frac{1}{4}\phi\left(\frac{s}{4}\right) = \frac{1}{\cancel{4}}\,\frac{1}{\frac{s}{\cancel{4}}\sqrt{\frac{s}{4}+1}}$$

$$= \frac{1}{s}\,\frac{1}{\sqrt{\frac{s+4}{4}}}$$

$$= \frac{2}{s\sqrt{s+4}}$$

* First shifting theorem:

$L[f(t)] = \phi(s)$ then $L[e^{at} f(t)] = \phi(s-a)$

$L[e^{-at} f(t)] = \phi(s+a)$

For example:

If $L(t^2) = \frac{\lfloor 3}{s^3} = \frac{2!}{s^3} = \frac{2}{s^3} = \phi(s)$

then $L[e^{-3t} t^2] = \frac{2}{(s+3)^3}$ ... by 1^{st} S.T.

$L[e^{5t} t^2] = \frac{2}{(s-5)^3}$ ... by 1^{st} S.T.

Type III : Using 1^{st} shifting th^m

Q. Find Laplace Transform of the following functions

① $e^{3t} \sin 2t$

Ans: $f(t) = \underline{e^{3t}}\ \underline{\sin 2t}$

$L(\sin 2t) = \frac{2}{s^2+4}$

$\therefore$ By 1^{st} S.T.

$L[e^{3t} \sin 2t] = \frac{2}{(s-3)^2+4} = \frac{2}{s^2-6s+9+4} = \frac{2}{s^2-6s+13}$

② $e^{-4t} \sin^3 t$

Ans: $f(t) = \underline{e^{-4t}}\ \underline{\sin^3 t}$

$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$

$\Rightarrow 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$

$\therefore \sin^3 t = \frac{3}{4}\sin t - \frac{1}{4}\sin 3t$

Applying L.T.

$L(\sin^3 t) = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$

$= \frac{3}{4} L(\sin t) - \frac{1}{4} L(\sin 3t)$

$$L(\sin^3 t) = \frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right]$$

By 1st S.T.

$$L(e^{-4t}\sin^3 t) = \frac{3}{4}\left[\frac{1}{(s+4)^2+1} - \frac{1}{(s+4)^2+9}\right]$$

$$= \frac{3}{4}\left[\frac{1}{s^2+8s+17} - \frac{1}{s^2+8s+25}\right]$$

(3) $\cos 3t \cosh 4t$

Ans: $f(t) = \cos 3t \cosh 4t$

$$= \cos 3t\left[\frac{e^{4t}+e^{-4t}}{2}\right]$$

$$= \frac{1}{2}\left[e^{4t}\cos 3t + e^{-4t}\cos 3t\right]$$

$$L[f(t)] = L\left[\frac{1}{2}\left(e^{4t}\cos 3t) + e^{-4t}\cos 3t\right)\right]$$

$$= \frac{1}{2}\left[L[e^{4t}\cos 3t) + L(e^{-4t}\cos 3t)\right]$$

$$\because L(\cos 3t) = \frac{s}{s^2+9}$$

$\therefore$ By using S.T.

$$L[f(t)] = \frac{1}{2}\left[\frac{s-4}{(s-4)^2+9} + \frac{s+4}{(s+4)^2+9}\right]$$

$$= \frac{1}{2}\left[\frac{s-4}{s^2-8s+25} + \frac{s+4}{s^2+8s+25}\right]$$

(4) $e^{-3t}\cosh 5t \sin 4t$

Ans: $f(t) = e^{-3t}\cosh 5t \sin 4t$

$$= e^{-3t}\left[\frac{e^{5t}+e^{-5t}}{2}\right]\sin 4t$$

$$= \frac{1}{2}\left[e^{2t} + e^{-8t}\right]\sin 4t$$

$$= \frac{1}{2}\left[e^{2t}\sin 4t + e^{-8t}\sin 4t\right]$$

$$L[f(t)] = L\left[\frac{1}{2}\left(e^{2t}\sin 4t + e^{-8t}\sin 4t\right)\right]$$

$$= \frac{1}{2}\left[L(e^{2t}\sin 4t) + L(e^{-8t}\sin 4t)\right]$$

$$\because L(\sin 4t) = \frac{4}{s^2+16}$$

By using 1^{st} shifting thm

$$L[f(t)] = \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$$

$$= \frac{4}{2}\left[\frac{1}{s^2-4s+20} + \frac{1}{s^2+16s+80}\right]$$

$$= 2\left(\frac{1}{s^2-4s+20} + \frac{1}{s^2+16s+80}\right)$$

⑤ $e^{-t}\cos 2t \sin 4t$

Ans: $f(t) = e^{-t}\sin 4t \cos 2t$

$$= e^{-t}\left[\frac{1}{2}(2\sin 4t \cos 2t)\right]$$

$$= \frac{e^{-t}}{2}\left[\sin 6t + \sin 2t\right]$$

$$= \frac{1}{2}\left(e^{-t}\sin 6t + e^{-t}\sin 2t\right)$$

$$L[f(t)] = L\left[\frac{1}{2}\left(e^{-t}\sin 6t + e^{-t}\sin 2t\right)\right]$$

$$= \frac{1}{2}\left[L(e^{-t}\sin 6t) + L(e^{-t}\sin 2t)\right]$$

$$L(\sin 6t) = \frac{6}{s^2+36} \qquad L(\sin 2t) = \frac{2}{s^2+4}$$

By using 1^{st} shifting thm

$$L[f(t)] = \frac{1}{2}\left[\frac{6}{(s+1)^2+36} + \frac{2}{(s+1)^2+4}\right]$$

$$= \frac{3}{s^2+2s+37} + \frac{1}{s^2+2s+5}$$

$$= \frac{3s^2+6s+15+s^2+2s+37}{(s^2+2s+37)(s^2+2s+5)}$$

$$= \frac{4s^2+8s+52}{(s^2+2s+37)(s^2+2s+5)}$$

⑥ $\sin 2t \cos t \cosh 2t$

Ans: $f(t) = \sin 2t \cos t \, [\cosh 2t]$

$$= \frac{1}{2}[2\sin 2t \cos t] \left[\frac{e^{2t} + e^{-2t}}{2} \right]$$

$$= \frac{1}{4}[\sin 3t + \sin t] (e^{t} + e^{-2t})$$

$$f(t) = \frac{1}{4}\left[e^{t}\sin 3t + e^{t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t\right]$$

Applying L.T.

$$L[f(t)] = L\left[\frac{1}{4}\left(e^{t}\sin 3t + e^{t}\sin t + e^{-2t}\sin 3t + e^{-2t}\sin t\right)\right]$$

$$= \frac{1}{4}\left[L(e^{t}\sin 3t) + L(e^{t}\sin t) + L(e^{-2t}\sin 3t) + L(e^{-2t}\sin t)\right]$$

$$L(\sin 3t) = \frac{3}{s^2+9} \qquad L(\sin t) = \frac{1}{s^2+1}$$

Using 1st shifting th^m

$$= \frac{1}{4}\left[\frac{3}{(s-1)^2+9} + \frac{1}{(s-1)^2+1} + \frac{3}{(s+2)^2+9} + \frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{4}\left[\frac{3}{s^2-2s+10} + \frac{1}{s^2-2s+2} + \frac{3}{s^2+4s+13} + \frac{1}{s^2+4s+5}\right]$$

⑦ $e^{2t}(1+t)^2$

Ans: $f(t) = e^{2t}(1+t)^2 = e^{2t}[1+2t+t^2]$

$$L(1+2t+t^2) = L(1) + 2L(t) + L(t^2)$$

$$= \frac{1}{s} + 2\frac{\Gamma 2}{s^2} + \frac{\Gamma 3}{s^3}$$

$$= \frac{1}{s} + 2\frac{1!}{s^2} + \frac{2!}{s^3}$$

$$= \frac{1}{s} + \frac{2}{s^2} + \frac{2}{s^3}$$

$$L[(1+t)^2] = L(1+2t+t^2) = \frac{s^2+2s+2}{s^3}$$

Using 1st S.T.

$$L[e^{2t}(1+t)^2] = \frac{(s-2)^2+2(s-2)+2}{(s-2)^3}$$

⑧ show that: $L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}}{2}t\right)\right] = \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}$

Ans: $f(t) = \sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t$

$$= \left[\frac{e^{t/2} - e^{-t/2}}{2}\right]\sin\frac{\sqrt{3}}{2}t$$

$$f(t) = \frac{1}{2}\left[e^{t/2}\sin\frac{\sqrt{3}}{2}t - e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right]$$

Applying L.T.

$$L[f(t)] = \frac{1}{2}\left[L\left(e^{t/2}\sin\frac{\sqrt{3}}{2}t\right) - L\left(e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right)\right] \quad \text{——} ①$$

$$\because L\left(\sin\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2} = \frac{\sqrt{3}/2}{s^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{\frac{4s^2+3}{4}}$$

Using 1^{st} S.T.

$$L\left(e^{t/2}\sin\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{(s-\frac{1}{2})^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2-s+\frac{1}{4}+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2-s+1}$$

$$L\left(e^{-t/2}\sin\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2+s+\frac{1}{4}+\frac{3}{4}} = \frac{\sqrt{3}/2}{s^2+s+1}$$

putting in ①

$$L[f(t)] = \frac{1}{2}\left[\frac{\sqrt{3}/2}{s^2-s+1} - \frac{\sqrt{3}/2}{s^2+s+1}\right]$$

$$= \frac{\sqrt{3}/2}{2}\left[\frac{(s^2+s+1)-(s^2-s+1)}{(s^2-s+1)(s^2+s+1)}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{s^2+s+1-s^2+s-1}{(s^2+1-s)(s^2+1+s)}\right]$$

$$= \frac{\sqrt{3}}{4}\left[\frac{2s}{(s^2+1)^2-(s)^2}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+2s^2+1-s^2}\right]$$

$$= \frac{\sqrt{3}}{2}\left[\frac{s}{s^4+s^2+1}\right]$$

* 2^{nd} shifting theorem:

If $L[f(t)] = \phi(s)$ and

$$g(t) = f(t-a) \quad \text{when } t > a$$
$$= 0 \quad \text{when } t < a$$

then

$$L[g(t)] = e^{-as}\phi(s)$$

Examples:

① $f(t) = \cos\left(t - \frac{\pi}{3}\right) \quad t > \pi/3$

$= 0 \quad t < \pi/3$

Ans: $f_1(t) = \cos t$

$$L[f_1(t)] = \frac{s}{s^2+1} = \phi(s)$$

By 2^{nd} S.T.

$$L[f(t)] = e^{-\pi/3}\phi(s) = e^{-\pi/3}\frac{s}{s^2+1}$$

② $f(t) = (t-1)^5 \quad t > 1$

$= 0 \quad t < 1$

Ans: $f_1(t) = t^5$

$$L[f_1(t)] = L(t^5) = \frac{\Gamma 6}{s^6} = \frac{5!}{s^6} = \frac{120}{s^6} = \phi(s)$$

By 2^{nd} S.T.

$$L[f(t)] = e^{-s}\phi(s) = e^{-s}\frac{120}{s^6}$$

③ $f(t) = e^{t-4} \quad t > 4$

$= 0 \quad t < 4$

Ans: $f_1(t) = e^t$

$$L[f_1(t)] = L(e^t) = \frac{1}{s-1} = \phi(s)$$

By 2^{nd} S.T.

$$L[f(t)] = e^{-4s}\phi(s) = e^{-4s}\frac{1}{s-1}$$