Martingales

Martingales are a key concept in probability and in mathematical finance. The term 'martingale' may refer to very different ideas e.g. a stochastic process that has no drift. Essentially, this is the idea of a fair (random) game. We encounter Martingales through three distinct, but closely connected ideas:

1. *Martingales* as a class of stochastic process;

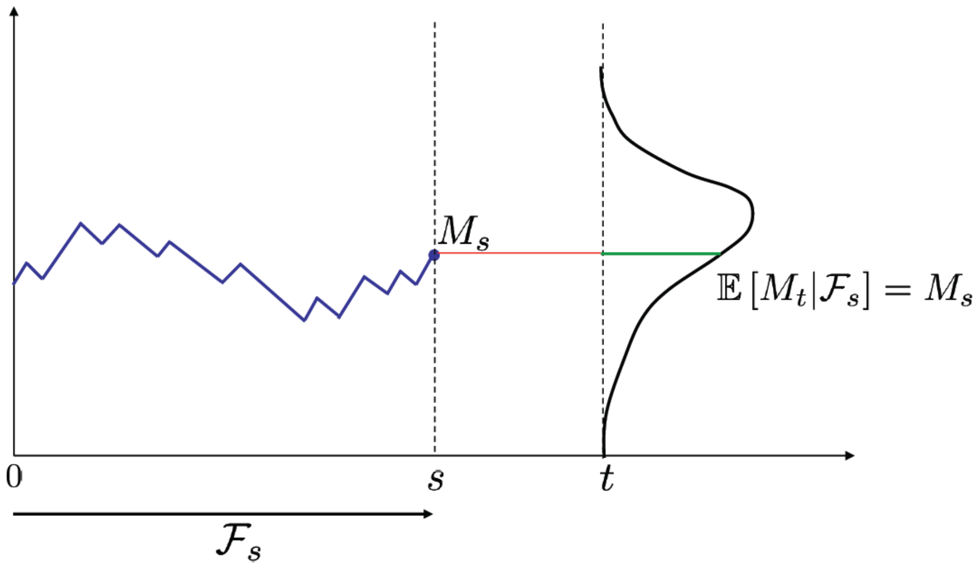
2. *Exponential martingales*, which are a specific and extremely useful example of a martingale;

3. *Equivalent martingale measures*, where we look for a probability measure $\mathbb{Q}$ such that a given stochastic process $S(t)$ is a martingale under $\mathbb{Q}$ regardless of its nature under $\mathbb{P}$. The correspondence between the measures $\mathbb{P}$ and $\mathbb{Q}$ is done through a change of measure.

Discrete Time Martingales

A discrete time stochastic process $\{M_t : t = 0, \ldots, T\}$ such that M_t is $\mathcal{F}_t$-measurable for $\mathbb{T} = \{0, \ldots, T\}$ is a **martingale** if $\mathbb{E}\,|M_t| < \infty$ and

$$\mathbb{E}\left[M_{t+1}|\mathcal{F}_t\right] = M_t \tag{1}$$



The first equation represents a standard integrability condition.

The second equation tells you that the expected value of M at time $t + 1$ conditional on all the information available up to time t is the value of M at time t. In short, a Martingale is a **driftless process**.

If we take expectation on both sides of eqn. 1, then

$$\mathbb{E}\left[M_{t+1}\right] = \mathbb{E}\left[M_t\right]$$

This is due to the **Tower Property** of conditional expectations.

Martingales are a very nice mathematical object. They "get rid of the drift" and enable us to focus on what probabilists consider is the most interesting part: the statistical properties of purely random processes.

In addition, Doob and Meyer have developed a powerful theory centred around martingales.

Continuous Time Martingales

Next, we generalize our definitions to continuous time: A continuous time stochastic process

$$\left\{ M_t : t \in \mathbb{R}^+ \right\}$$

such that M_t is $\mathcal{F}_t$-measurable for $t \in \mathbb{R}^+$ is a **martingale** if

$$\mathbb{E}\,|M_t| < \infty$$

and

$$\mathbb{E}\left[M_t | \mathcal{F}_s \right] = M_s, \quad 0 \leq s \leq t.$$

Lévy's Martingale Characterisation: Let $X_t,\ t > 0$ be a stochastic process and let $\mathcal{F}_t$ be the filtration generated by it. X_t is a Brownian motion iff the following conditions are satisfied:

1. $X_0 = 0$ a.s.;
2. the sample paths $t \mapsto X_t$ are continuous a.s.;
3. X_t is a martingale with respect to the filtration $\mathcal{F}_t$;
4. $|X_t|^2 - t$ is a martingale with respect to the filtration $\mathcal{F}_t$.

Quadratic Variation

The Lévy characterization can be contrasted with the classical definition of a Brownian motion as a stochastic process X_t satisfying:

1. $X_0 = 0$ a.s.;

2. the sample paths $t \mapsto X(t)$ are continuous a.s.;

3. **independent increments**: for $t_1 < t_2 < t_3 < t_4$ the increments $X_{t_4} - X_{t_3}$, $X_{t_2} - X_{t_1}$ are independent;

4. **normally distributed increments**: $X_t - X_s \sim N(0, |t-s|)$.

Lévy's characterization neither mentions independent increments nor normally distributed increments.

Instead, Lévy introduces two easily verifiable martingale conditions.

Itô Integrals and Martingales

Next, we explore the link between Itô integration and martingales.

Consider the stochastic process $Y(t) = X^2(t)$. By Itô, we have

$$X^2(T) = T + \int_0^T 2X(t)dX(t)$$

Taking the expectation, we get

$$\mathbb{E}[X^2(T)] = T + \mathbb{E}\left[\int_0^T 2X(t)dX(t)\right]$$

Now, the quadratic variation property of Brownian motions implies that

$$\mathbb{E}[X^2(T)] = T$$

and hence

$$\mathbb{E}\left[\int_0^T 2X(t)dX(t)\right] = 0$$

Therefore, the Itô integral

$$\int_0^T 2X(t)dX(t)$$

is a martingale.

In fact, this property is shared by all Itô integrals.

The Itô integral is a martingale

Let $g(t, X_t)$ be a function on $[0, T]$ and satisfying the technical condition. Then the Itô integral

$$\int_0^T g(t, X_t)dX_t$$

is a martingale.

So, Itô integrals are martingales.

But does the converse hold? Can we represent any martingale as an Itô integral?

The answer is yes!

Martingale Representation Theorem: If M_t is a martingale, then there exists a function $g(t, X_t)$ satisfying the technical condition such that

$$M_T = M_0 + \int_0^T g(t, X_t) dX_t$$

Example Using only Itô and the fact that Itô integrals are martingales, we will show that

$$\mathbb{E}\left[X^2(T)\right] = T.$$

Consider the function $F(t, X_t) = X_T^2$, then by Itô's lemma,

$$\begin{aligned} X_T^2 &= X_0^2 + \frac{1}{2}\int_0^T 2dt + \int_0^T 2X_t dX_t \\ &= \int_0^T dt + 2\int_0^T X_t dX_t \end{aligned}$$

since $X_0 = 0$

Taking the expectation,

$$\mathbb{E}\left[X_T^2\right] = \mathbb{E}\left[\int_0^T dt\right] + 2\mathbb{E}\left[\int_0^T X_t dX_t\right]$$

Now,

$$\int_0^T X_t dX_t$$

is an Itô integral and as a result $\mathbb{E}\left[\int_0^T X_t dX_t\right] = 0$

Moreover,

$$\mathbb{E}\left[\int_0^T dt\right] = \mathbb{E}\left[T\right] = T$$

We can conclude that

$$\mathbb{E}\left[X^2(T)\right] = T$$

As an aside, we can usually exchange the order of integration between the time integral and the expectation so that

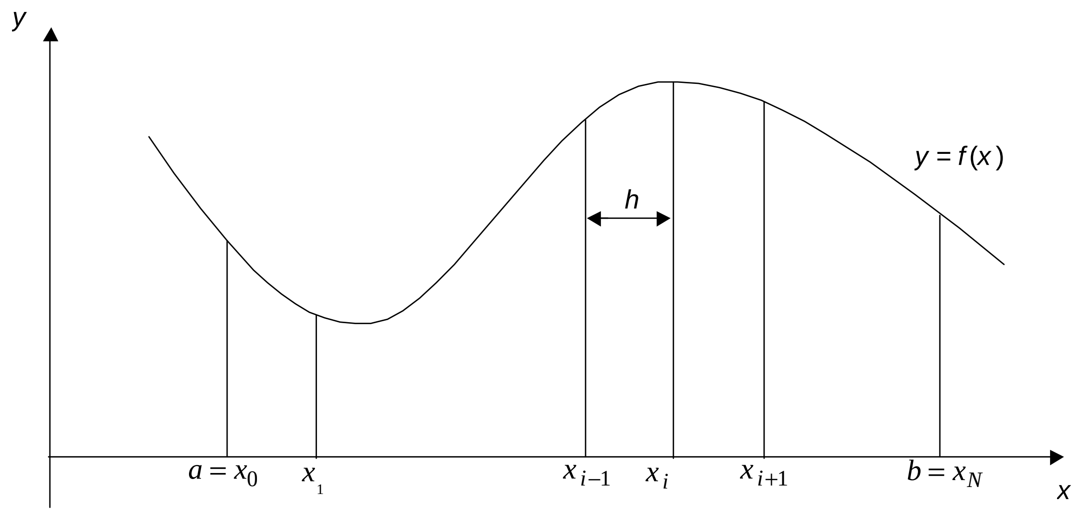
$$\mathbb{E}\left[\int_0^T f(X_t)dt\right] = \int_0^T \mathbb{E}\left[f(X_t)\right] dt$$

This is due to an analysis result known as **Fubini's Theorem**.

Itô Integral

Recall the usual Riemann definition of a definite integral

$$\int_a^b f(x)\,dx$$



which represents the area under the curve between $x = a$ and $x = b$, where the curve is the graph of $f(x)$ plotted against x.

Assuming f is a "well behaved" function on $[a,b]$, there are many different ways (which all lead to the same value for the definite integral).

Start by partitioning $[a,b]$ into N intervals with end points $x_0 = a < x_1 < x_2 < \ \ldots\ldots \ < x_{N-1} < x_N = b$, where the length of an interval $dx = x_i - x_{i+1}$ tends to zero as $N \to \infty$. So there are N intervals and $N+1$ points x_i.

Discretising x gives

$$x_i = a + idx$$

Now consider the definite integral

$$\int_0^T f(t)\,dt.$$

With Riemann integration there are a number of ways we can approximate this:

1. left hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i)\,(t_{i+1} - t_i)$$

2. right hand rectangle rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1})\,(t_{i+1} - t_i)$$

3. trapezium rule;

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} \frac{1}{2}\left(f(t_i) + f(t_{i+1})\right)(t_{i+1} - t_i)$$

4. midpoint rule

$$\int_0^T f(t)\,dt = \lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(\tfrac{1}{2}(t_i + t_{i+1})\right)(t_{i+1} - t_i)$$

In the limit $N \to \infty$, $f(t)$ we get the same value for each definition of the definite integral, provided the function is integrable.

Now consider the stochastic integral of the form

$$\int_0^T f(t, X)\,dX = \int_0^T f(t, X(t))\,dX(t)$$

where $X(t)$ is a Brownian motion. We can define this integral as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

where $X_i = X(t_i)$, or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i),$$

or as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f\left(t_{i+\frac{1}{2}}, X_{i+\frac{1}{2}}\right)(X_{i+1} - X_i),$$

where $t_{i+\frac{1}{2}} = \frac{1}{2}(t_i + t_{i+1})$ and $X_{i+\frac{1}{2}} = X\left(t_{i+\frac{1}{2}}\right)$ or in many other ways. So clearly drawing parallels with the above Riemann form.

Very Important: In the case of a stochastic variable $dX(t)$ the value of the stochastic integral **does** depend on which definition we choose.

In the case of a stochastic integral, the definition

$$I = \lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_i, X_i)(X_{i+1} - X_i),$$

is special. This definition results in the **Itô Integral**.

non-predictive

It is special because it is **non-anticipatory**; given that we are at time t_i we know $X_i = X(t_i)$ and therefore we know $f(t_i, X_i)$. The only uncertainty is in the $X_{i+1} - X_i$ term.

Compare this to a definition such as

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} f(t_{i+1}, X_{i+1})(X_{i+1} - X_i), \quad \text{(Stratonovich)}$$

predictive

which is **anticipatory**; given that at time t_i we know X_i but are uncertain about the future value of X_{i+1}. Thus we are uncertain about *both* the value of

$$f(t_{i+1}, X_{i+1})$$

and the value of $(X_{i+1} - X_i)$ — there exists uncertainty in both of these quantities. That is, evaluation of this integral requires us to anticipate the future value of X_{i+1} so that we may evaluate $f(t_{i+1}, X_{i+1})$.

The main thing to note about Itô integrals is that I is a random variable (unlike the deterministic case). Additionally, since I is essentially the limit of a sum of normal random variables, then by the CLT I is also normally distributed, and can be characterized by its mean and variance.

Example: Show that Itô's lemma implies that

$$3\int_0^T X^2 dX = X(T)^3 - X(0)^3 - 3\int_0^T X(t)\, dt.$$

Show that the result also can be found by writing the integral

$$3\int_0^T X^2 dX = \lim_{N\to\infty} 3\sum_{i=0}^{N-1} {X_i}^2 \left(X_{i+1} - X_i\right)$$

Hint: use $3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$.

The Itô integral here is defined as

$$\int_0^T 3X^2(t)\, dX(t) = \lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i^2 \left(X_{i+1} - X_i\right)$$

Now note the hint:

$$3b^2(a-b) = a^3 - b^3 - 3b(a-b)^2 - (a-b)^3$$

hence

$$\begin{aligned} &\equiv 3X_i^2(X_{i+1} - X_i) \\ &= X_{i+1}^3 - X_i^3 - 3X_i(X_{i+1} - X_i)^2 - (X_{i+1} - X_i)^3, \end{aligned}$$

so that

$$\sum_{i=0}^{N-1} 3X_i^2(X_{i+1} - X_i) =$$

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 - \sum_{i=0}^{N-1} 3X_i(X_{i+1} - X_i)^2$$
$$- \sum_{i=0}^{N-1} (X_{i+1} - X_i)^3$$

Now the first two expressions above give

$$\sum_{i=0}^{N-1} X_{i+1}^3 - \sum_{i=0}^{N-1} X_i^3 = X_N^3 - X_0^3$$
$$= X(T)^3 - X(0)^3 .$$

In the limit $N \to \infty$, i.e. $dt \to 0$, $(X_{i+1} - X_i)^2 \to dt$, so

$$\lim_{N\to\infty} \sum_{i=0}^{N-1} 3X_i (X_{i+1} - X_i)^2 = \int_0^T 3X(t)\, dt$$

Finally $(X_{i+1} - X_i)^3 = (X_{i+1} - X_i)^2 \cdot (X_{i+1} - X_i)$ which when $N \to \infty$ behaves like $dX^2 dX \sim O\left(dt^{3/2}\right) \longrightarrow 0.$

Hence putting together gives

$$X(T)^3 - X(0)^3 - \int_0^T 3X(t)\, dt$$

which is consistent with Itô's lemma.

The other important property that the Itô integral has is that it is a martingale. We know that

$$X_{i+1} - X_i$$

is a martingale; i.e. in the context

$$\mathbb{E}\left[X_{i+1} - X_i\right] = 0.$$

Since

$$\mathbb{E}\left[\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\left(X_{i+1} - X_i\right)\right] =$$
$$\sum_{i=0}^{N-1} f\left(t_i, X_i\right)\mathbb{E}\left[X_{i+1} - X_i\right] = 0$$

Thus

$$\mathbb{E}\left[\int_0^T f\left(t, X\left(t\right)\right) dX\left(t\right)\right] = 0.$$

This is, essentially a consequence of the Itô integral being non-anticipatory, as discussed earlier. No other stochastic integral has this property.

Exercise We know from Itô's lemma that

$$4\int_0^T X^3(t)\,dX(t) = X^4(T) - X^4(0) - 6\int_0^T X^2(t)\,dt$$

Show from the definition of the Itô integral that the result can also be found by initially writing the integral

$$4\int_0^T X^3 dX = \lim_{N\to\infty} 4\sum_{i=0}^{N-1} X_i^3 (X_{i+1} - X_i)$$

Hint: use $4b^3(a-b) = a^4 - b^4 - 4b(a-b)^3 - 6b^2(a-b)^2 - (a-b)^4$.

Proving that a Continuous Time Stochastic Process is a Martingale

Consider a stochastic process $Y(t)$ solving the following SDE:

$$dY(t) = f(Y_t, t)dt + g(Y_t, t)dX(t), \qquad Y(0) = Y_0$$

How can we tell whether $Y(t)$ is a martingale?

The answer has to do with the fact that Itô integrals are martingales.

$Y(t)$ is a martingale if and only if it satisfies the martingale condition

$$\mathbb{E}\left[Y_t|\mathcal{F}_s\right] = Y_s, \quad 0 \leq s \leq t$$

Let's start by integrating the SDE between s and t to get an exact form for $Y(t)$:

$$Y(t) = Y(s) + \int_s^t f(Y_u, u)du + \int_s^t g(Y_u, u)dX(u)$$

Taking the expectation conditional on the filtration at time s, we get

$$\begin{aligned}\mathbb{E}[Y_t|\mathcal{F}_s] &= \mathbb{E}\left[Y(s) + \int_s^t f(Y_u,u)du + \int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] \\ &= Y(s) + \mathbb{E}\left[\int_s^t f(Y_u,u)du|\mathcal{F}_s\right]\end{aligned}$$

where the last line follows from the fact that a Itô integral is a martingale, $\therefore$.

$$\mathbb{E}\left[\int_s^t g(Y_u,u)dX(u)|\mathcal{F}_s\right] = \int_s^s g(Y_u,u)dX(u) = 0.$$

So, $Y(t)$ is a martingale iff

martingale requires no drift

$$\mathbb{E}\left[\int_s^t f(u)du|\mathcal{F}_s\right] = 0$$

This condition is satisfied only if $f(Y_t,t) = 0$ for all t. Returning to our SDE, we conclude that $Y(t)$ is a martingale iff it is of the form

$$dY(t) = g(Y_t,t)dX(t), \qquad Y(0) = Y_0$$

Exponential Martingales

Let's start with a motivating example. no dependence on Y

Consider the stochastic process $Y(t)$ satisfying the SDE

$$[\text{log returns}] \quad dY(t) = f(t)dt + g(t)dX(t), \qquad Y(0) = Y_0 \tag{2}$$

where $f(t)$ and $g(t)$ are two time-dependent functions and $X(t)$ is a standard Brownian motion.

Define a new process $Z(t) = e^{Y(t)}$.

How should we choose $f(t)$ if we want the process $Z(t)$ to be a martingale?

Consider the process $Z(t) = e^{Y(t)}$. Applying Itô to the function we obtain:

$$\begin{aligned}
dZ(t) &= \frac{dZ}{dY}dY(t) + \frac{1}{2}\frac{d^2Z}{dY^2}dY^2(t) \\
&= \frac{dZ}{dY}(f(t)dt + g(t)dX(t)) + \frac{1}{2}\frac{d^2Z}{dY^2}g^2(t)dt \\
&= e^{Y(t)}\left(f(t) + \frac{1}{2}g^2(t)\right)dt + e^{Y(t)}g(t)dX(t) \\
&= Z(t)\left[\left(f(t) + \frac{1}{2}g^2(t)\right)dt + g(t)dX(t)\right]
\end{aligned}$$

$Z(t)$ is a martingale if and only if it is a driftless process.

Therefore for $Z(t)$ to be a martingale we must have

$$f(t) + \frac{1}{2}g^2(t) = 0$$

This is only possible if

$$f(t) = -\frac{1}{2}g^2(t)$$

Going back to the process $Y(t)$, we must have

$$dY(t) = -\frac{1}{2}g^2(t)dt + g(t)dX(t), \qquad Y(0) = Y_0$$

implying that

$$Y(T) = Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)$$

Hence, in terms of $Z(t)$:

$$dZ(t) = Z(t)g(t)dX(t).$$

Using the earlier relationship, we can write $Z(T) = e^{Y(T)}$.

Let's simplify this $Z(T) =$

$$\exp\left\{Y_0 - \frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

to give

$$Z(T) = Z_0 \exp\left\{-\frac{1}{2}\int_0^T g^2(t)dt + \int_0^T g(t)dX(t)\right\}$$

Because the stochastic process $Z(t)$ is the exponential of another process (namely $Y(t)$) and because it is a martingale, we call $Z(t)$ an **exponential martingale**.

We have actually just stumbled upon a much more general and very important result.

Key Condition (Novikov Condition)

A trading strategy $(\phi, \psi) = \{(\phi_t, \psi_t)\,; t \in [0, ..., T]\}$ is a previsible process in that $\phi_t \in \mathcal{F}_{t^-}$.

A stochastic process Υ_t satisfies the *Novikov condition* if

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \gamma_s^2 ds\right)\right] < \infty$$

where is a γ_t previsible process.

Key Fact

Given a process γ_t satisfying the Novikov condition, then the process M_t^γ defined as we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$M_t^\gamma = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

is a martingale.

In our earlier example $\gamma_t = -g(t)$; $M_t^\gamma = Z(t)$.

Key Fact (Girsanov's Theorem)

Given a process γ_t satisfying the Novikov condition, we can define the probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{F})$ equivalent to $\mathbb{P}$ through the Radon Nikodým derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^t \gamma_s dX_s - \frac{1}{2}\int_0^t \gamma_s^2 ds\right), \quad t \in [0, T]$$

In this case, the process $X_t^{\mathbb{Q}}$ defined as

$$X_t^{\mathbb{Q}} = X_t^{\mathbb{P}} + \int_0^t \gamma_s dX_s$$

as is a standard Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$.