Functions

input x $\longrightarrow$ $\boxed{\text{rule}}$ $\longrightarrow$ output y

Independent variable $\rightarrow x$

$y \leftarrow$ dependent variable

Defn

Every x maps to a <u>single</u> y

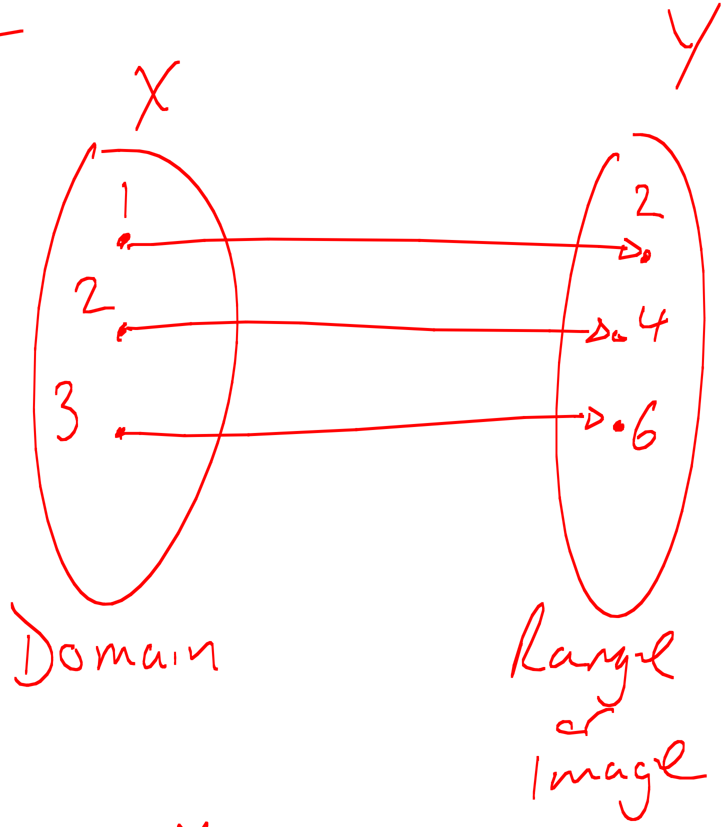
mappings

Consider two sets X & Y

X Y

one to one

is a function



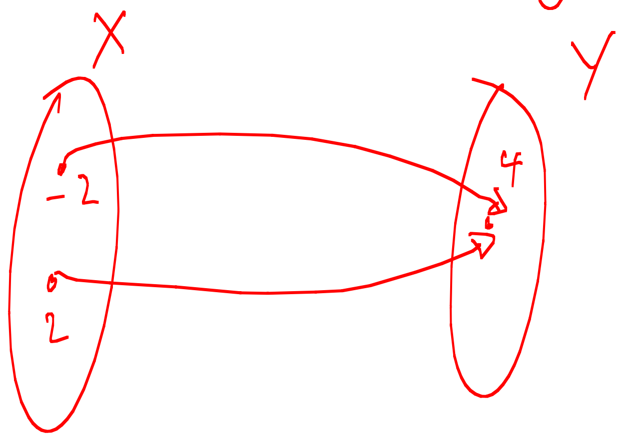
$y = 2x$

$f(x) = 2x$

$f : x \longmapsto 2x$

$f : X \longrightarrow Y$

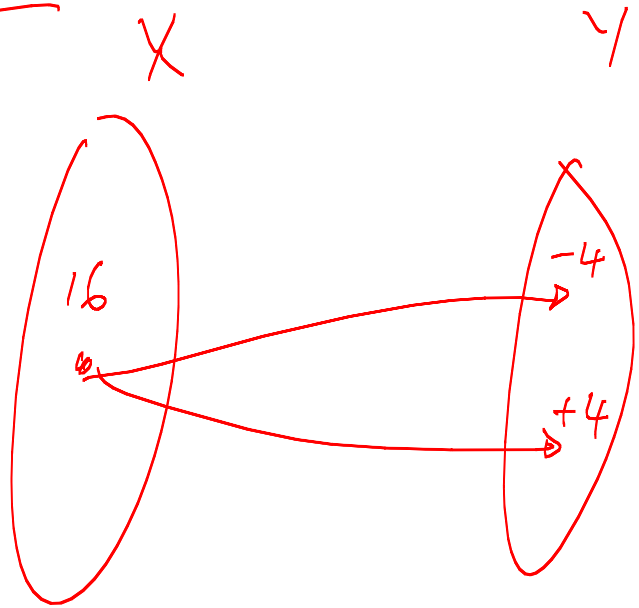
many to one



$f : x \longmapsto x^2$

is a function

one to many



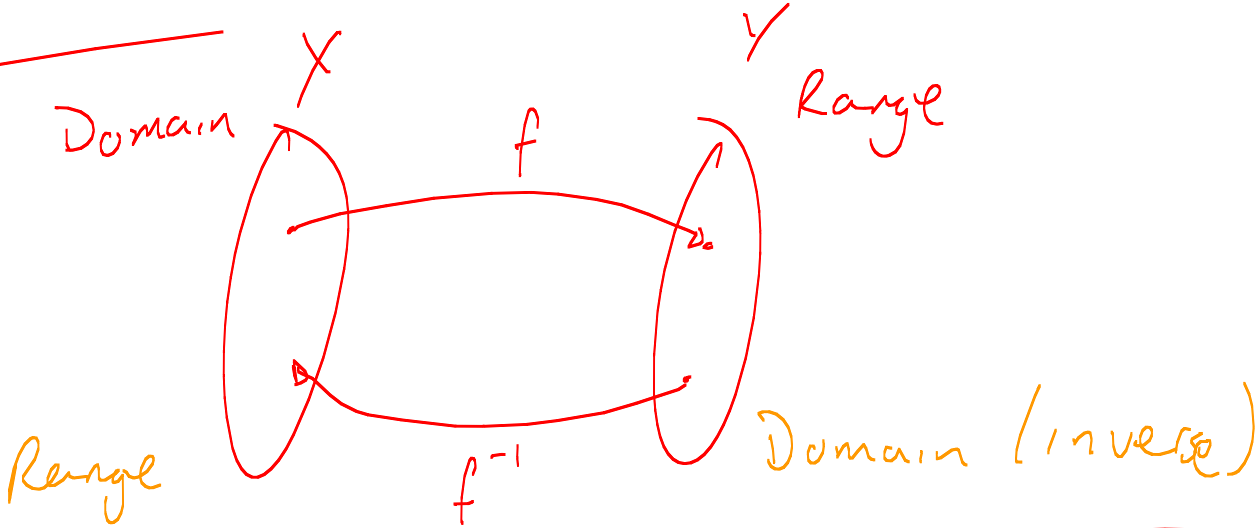
Not a function

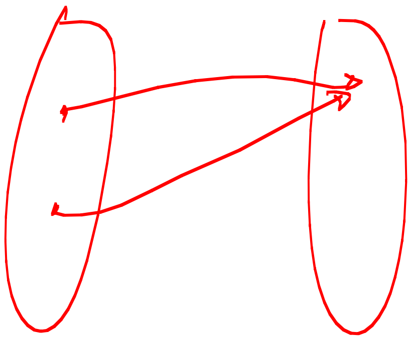
$$f : x \longmapsto \pm\sqrt{x}$$

$$f : x \longmapsto +\sqrt{x}$$

$$f : x \longmapsto -\sqrt{x}$$

Inverse function



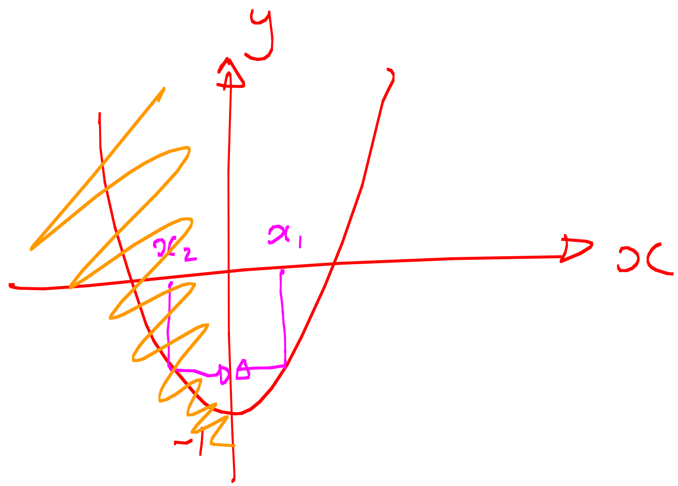


one to one

Example

Consider $y = 2x^2 - 1$; $x \in \mathbb{R}$

many to one.



restrict the domain $x \geq 0$ $y \geq -1$

find an inverse $y = 2x^2 - 1$ f

$$x = 2y^2 - 1$$

$$x \geq 0 \xrightarrow{f} y \geq -1$$

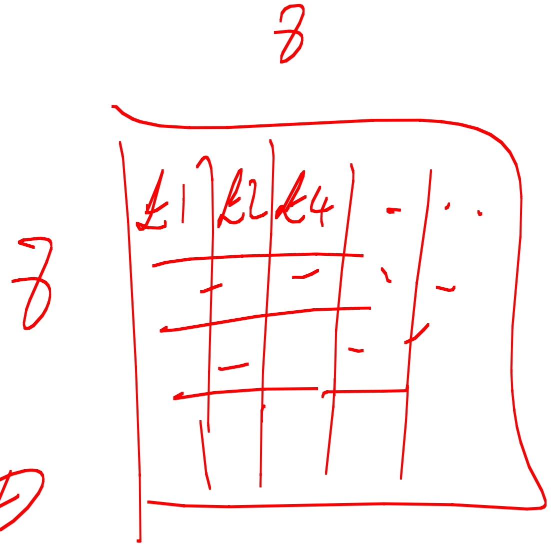
$$y \geq 0 \xleftarrow{f^{-1}} x \geq -1$$

make y the subject

$$2y^2 = x + 1$$

$$y^2 = \frac{x+1}{2}$$

$$y = \sqrt{\frac{x+1}{2}}$$



£100,000,000

2^0

2^1

2^2

2^3

$\vdots$

2^{63}

$\Big\}\ 2^{64}$

log laws

1. $\log xy = \log x + \log y$

2. $\log \frac{x}{y} = \log x - \log y$

3. $\log x^n = n \log x$

4. $\log 1 = 0$

chain rule

$$y = f[g(x)]$$

$u = g(x)$ $\qquad$ $y = f(u)$

$\dfrac{du}{dx}$ $\qquad$ $\dfrac{dy}{du}$

$$\boxed{\frac{dy}{dx} = \frac{dy}{\cancel{du}} \times \frac{\cancel{du}}{dx}}$$

chain rule

Example

$$y = e^{(4x^2)}$$

$u = 4x^2$ $\qquad$ $y = e^u$

$\frac{du}{dx} = 8x$ $\qquad$ $\frac{dy}{du} = e^u = e^{4x^2}$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = 8x e^{4x^2}$$

(2) $y = \sin(x^3)$

$u = x^3$ $\qquad$ $y = \sin u$

$\dfrac{du}{dx} = 3x^2$ $\qquad$ $\dfrac{dy}{du} = \cos u = \cos x^3$

$$\dfrac{dy}{dx} = 3x^2 \cos x^3$$

(3) $y = \ln(\sin x)$

$$\dfrac{dy}{dx} = \cos x \cdot \dfrac{1}{(\sin x)} = \dfrac{\cos x}{\sin x}$$

$$= \cot x$$

$y = \ln x$

$\dfrac{dy}{dx} = \dfrac{1}{x}$

$$y = e^{(4x^2)}$$

$$\frac{dy}{dx} = 8x\, e^{(4x^2)}$$

③ $y = \tan^2 x = (\tan x)^2$

$$\frac{dy}{dx} = 2\sec^2 x \,(\tan x)'$$

$$= 2\sec^2 x \tan x$$

$$y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

① $\frac{d}{dx}\left[\overset{u}{x}\,\overset{v}{e^{x}}\right]$ product

② $\frac{d}{dx}\left[\frac{e^{x}}{x}\right]$ Quotient

③ $\frac{d}{dx}\left[e^{-x^2}\right]$ chain

④ $\frac{d}{dx}\left[\overbrace{x^2 \underbrace{\ln(3x^2+2x-1)}_{\text{chain}}}^{\text{product}}\right]$

$$\frac{d}{dx}\left[\underset{u}{x^2}\, \underset{v}{\ln(3x^2+2x-1)}\right]$$

$$\frac{dv}{dx}$$

$$= 2x\ln(3x^2+2x-1)$$

$$+$$

$$x^2\cdot(6x+2)\frac{1}{(3x^2+2x-1)}$$

$$= 2x\ln(3x^2+2x-1) + \frac{x^2(\cancel{6}x+2)}{(3x^2+2x-1)}$$

$$4y^4 - 2\overset{u}{y^2}\overset{v}{x^2} - \overset{u}{y}\overset{v}{x^2} + x^2 + 3 = 0$$

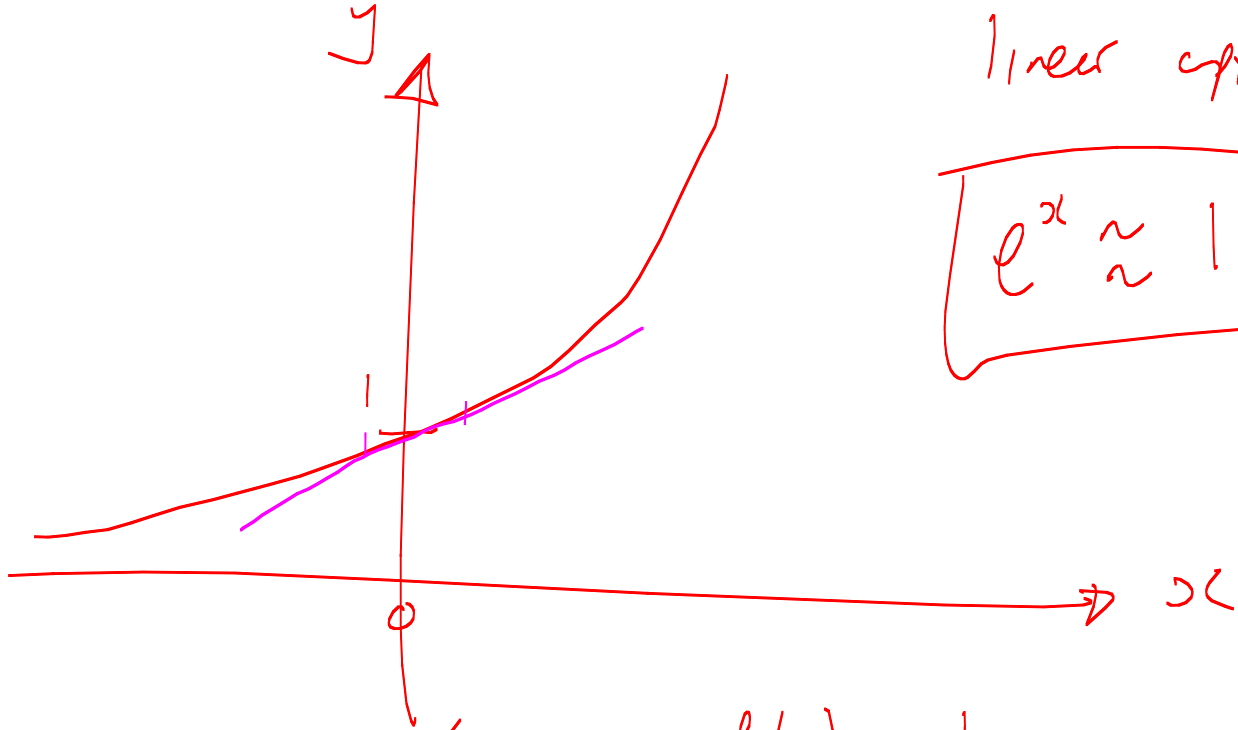
$$16y^3\frac{dy}{dx} - 2\left[x^2 2y\frac{dy}{dx} + y^2 2x\right]$$

$$- \left[x^2\frac{dy}{dx} + y2x\right] + 2x = 0$$

$$\frac{dy}{dx}\left(\qquad \right) =$$

Motivating example

$f(x) = e^x$



linear approx

$$\boxed{e^x \approx 1 + x}$$

$x = 0$

$f(x) = e^x$ $\qquad f(0) = 1$

$f'(x) = e^x$ $\qquad f'(0) = 1$

Linear approx

gradient $= 1$ $\qquad$ intercept $= 1$

grad $\qquad$ intercept

$y = mx + c$

$= x + 1$

① Quadratic approx

$$g(x) = ax^2 + bx + c$$

$$g'(x) = 2ax + b$$

$$g''(x) = 2a$$

make

$$g(0) = f(0) \qquad c = 1$$

$$g'(0) = f'(0) \qquad b = 1$$

$$g''(0) = f''(0) \qquad 2a = 1 \quad \therefore a = \frac{1}{2}$$

So $\boxed{e^x \approx 1 + x + \frac{1}{2}x^2}$

(i) $y = 2x$

$$\frac{dy}{dx} = 2$$

$$\int 2 \, dx = 2x + C$$

$C = 0$

(ii) $y = 2x + 5$

$$\frac{dy}{dx} = 2$$

$$\int 2 \, dx = 2x + C$$

$C = 5$

Improper Integrals

$$\Rightarrow \int_{-\infty}^{a} f(x).dx \quad \text{or} \quad \int_{-\infty}^{\infty} g(x).dx \quad \text{or} \quad \int_{b}^{\infty} h(x).dx$$

$$\lim_{a \to \infty} \int_{-a}^{+a} g(x).dx$$

Integration by Substitution

$$\rightarrow \quad I = \int \underbrace{g(f(x))} \, \underbrace{f'(x)\,.dx}$$

put $z = f(x)$

$$\frac{dz}{dx} = f'(x)$$

$$dz = f'(x)\,dx$$

$$I = \int g(z)\,dz$$

1) $\rightarrow I = \int \frac{x}{1+x^2} \cdot dx$

$z = 1 + x^2$

$\frac{dz}{dx} = 2x$

$\Downarrow$

$dx = \frac{dz}{2x}$

$I = \int \frac{\cancel{x}}{z} \cdot \frac{dz}{2\cancel{x}} = \int \frac{dz}{2z} = \frac{1}{2} \int \frac{dz}{z}$

$= \frac{1}{2} \ln z + C = \frac{1}{2} \ln(1+x^2) + C$

② $I = \int \frac{1}{x} \log x \cdot dx$

$z = \log x$

$\frac{dz}{dx} = \frac{1}{x}$ $\qquad dx = x\,dz$

$I = \int \frac{1}{\cancel{x}} z \cdot \cancel{x}\, dz = \int z \cdot dz$

$= \frac{z^2}{2} + C = \underline{\frac{1}{2}(\log x)^2 + C}$

(3)

$$\int_1^2 e^{x^2} \cdot 2x \cdot dx$$

change limits

$x = 1 \quad z = 1$

$x = 2 \quad z = 2^2 = 4$

$$z = x^2$$

$$\frac{dz}{dx} = 2x \qquad dx = \frac{dz}{2x}$$

$$\int_1^2 e^{x^2} 2x\, dx = \int_{z=1}^{z=4} e^z\, dz = \ldots$$

Important example

Can we find $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} \cdot dx$?

CDF for the standard Normal is

$$\longrightarrow \quad N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{s^2}{2}}\, ds$$

$$* \longrightarrow \quad \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2}}\, ds = 1 \qquad \text{let } x = \frac{s}{\sqrt{2}}$$

$$\frac{s^2}{2} = x^2 \qquad x = \frac{s}{\sqrt{2}} \qquad \frac{dx}{ds} = \frac{1}{\sqrt{2}}$$

$$ds = \sqrt{2}\, dx$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\overset{\frac{s^2}{2}}{x^2}} \underbrace{\sqrt{2}\, dx}_{ds} = 1 \qquad \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1 \quad \therefore \boxed{\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}}$$

Properties of even & odd functions

① $f(x)$ even $\quad \int_{-a}^{a} f(x) \cdot dx = 2\int_{0}^{a} f(x) \cdot dx$

② $g(x)$ odd $\quad \int_{-a}^{a} g(x) \cdot dx = 0$

$$I = \int x^2 e^{2x}\,dx \qquad \int u'v\,dx = uv - \int uv'\,dx + C$$

$$v \qquad u'$$

$$\rightarrow v = x^2 \qquad u' = e^{2x}$$

$$v' = 2x \qquad \rightarrow u = \frac{1}{2}e^{2x}$$

$$I = \frac{x^2 e^{2x}}{2} - \int \frac{1}{2}e^{2x} \cdot 2x \, dx$$

$$= \frac{x^2 e^{2x}}{2} - \int x e^{2x}\, dx$$

classic problem

$$\left(\int u'v\,dx = uv - \int uv'\,dx \right)$$

$$I = \int e^{x} \sin x \, . dx$$

v u'

$v = e^{x}$ $u' = \sin x$

$v' = e^{x}$ $u = -\cos x$

$$I = -e^{x}\cos x - \int -\cos x \, e^{x} \, . dx$$

$$= -e^{x}\cos x + \int e^{x}\cos x \, . dx$$

$$I = -e^x \cos x + \overbrace{\int e^x \cos x \, dx}$$

$$v = e^x \qquad u' = \cos x$$

$$v' = e^x \qquad u = \sin x$$

$$= -e^x \cos x + \left\{ e^x \sin x - \underbrace{\int e^x \sin x \, dx}_{I} \right\}$$

$$I = -e^x \cos x + e^x \sin x - I$$

$$2I = e^x \sin x - e^x \cos x$$

$$\therefore I = \frac{e^x \sin x - e^x \cos x}{2} + c$$

$$\int \frac{1}{cabin}\, d(cabin) = \text{Beach hut.}$$

$$= \log cabin + Sea$$

repeated factor

A
$$\frac{C}{(x+a)^2(x+b)^3} = \frac{A}{x+a} + \frac{B}{(x+a)^2} + \frac{C}{(x+b)} + \frac{D}{(x+b)^2} + \frac{E}{(x+b)^3}$$

linear linear

1 degree less

linear

B
$$\frac{2x+1}{(x^2+3x+2)(x-1)} = \frac{Ax+B}{x^2+3x+2} + \frac{C}{x-1}$$

Quadratic

chain rule

Single variable

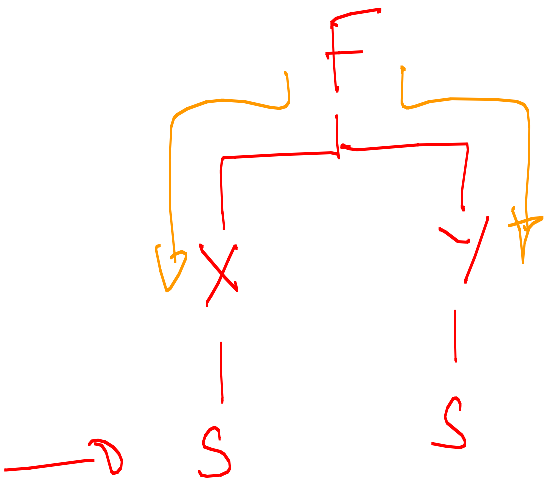
$f(g(x))$ $\qquad u = g(x)$

$$\boxed{\frac{df}{dx} = \frac{df}{du} \times \frac{du}{dx}}$$

multi variable:

$F(x, y)$ $\qquad x = x(s)$

$y = y(s)$ $\qquad$ chain rule I



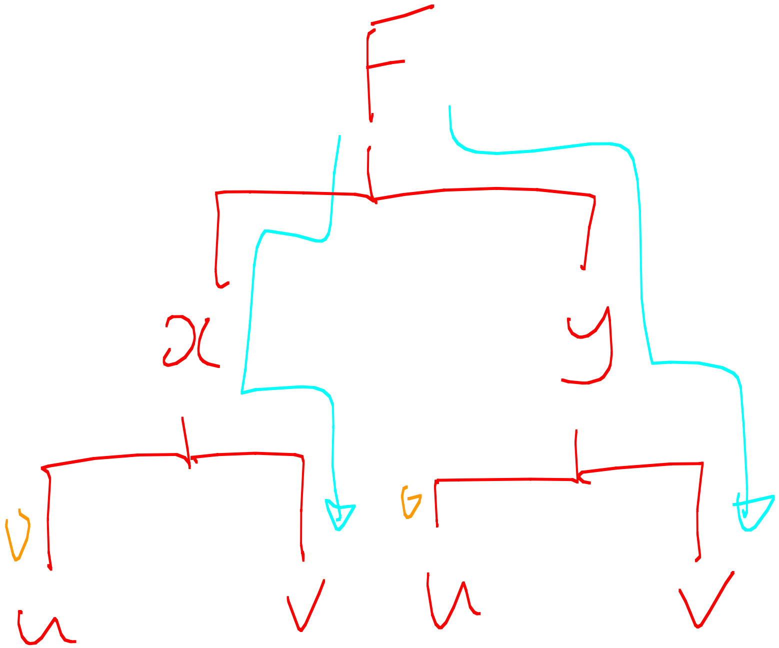
$$\boxed{\frac{dF}{ds} = \frac{\partial F}{\partial x} \cdot \frac{dx}{ds} + \frac{\partial F}{\partial y} \cdot \frac{dy}{ds}}$$

Chain rule II

$$F = f(x, y)$$

$$x = x(u, v)$$

$$y = y(u, v)$$



$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial F}{\partial v} = \frac{\partial F}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial F}{\partial y} \frac{\partial y}{\partial v}$$