Differential Equations

1^{st} order : • Var. sep. $\frac{dy}{dx} = f(x,y)$ with $f(x,y) = g(x)h(y)$

• Linear eq^{n} $\frac{dy}{dx} + P(x)y = Q(x)$ I.F $I(x) = e^{\int P(x)dx}$

2^{nd} order : • const. coeff. $ay'' + by' + cy = 0$ $a, b, c \in \mathbb{R}$

$y(x) = e^{\lambda x}$ $\lambda \in \mathbb{R}$

A.E $a\lambda^2 + b\lambda + c = 0$

• Cauchy-Euler eq^{n} $ax^2y'' + bxy' + cxy = 0$ $\exists$ sol^{n} $y(x) = x^{\lambda}$

A.E: $a\lambda^2 + (b-a)\lambda + c = 0$

Heat / Diffusion / Kolmogorov Eqn

(1) $$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u = u(x,t)$$

w.l.o.g put $\boxed{c^2 = 1}$ just to tidy up.

We will look for a separable solution of the form

$$u(x,t) = X(x)\,T(t) \quad (2)$$

$\bullet \equiv \frac{d}{dt}$ $\qquad$ $' \equiv \frac{d}{dx}$ $\qquad$ Subst. (2) into (1)

(Newtonian notation)

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\left(X(x)T(t)\right) = X\dot{T}\,; \quad \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(X(x)T(t)\right) = X'T$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(X'T\right) = X''T$$

$$X\dot{T} = X''T$$ now rearrange to get t dep. on LHS, x dep. on RHS

$$\frac{\dot{T}(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

LHS : indep. of x $\qquad$ RHS : indep. of t

t dep. $\qquad$ x dep.

Both sides equal a function indep. of both (x,t)

Only choice is a constant.

$\circ \equiv \frac{d}{dt}$

$$\frac{\mathring{T}}{T} = \frac{X''}{X} = \text{constant (say) } \lambda^2$$

i.e. some universal const.

$(-\lambda^2)$

$\rightarrow$ Two O.D.E.s

① $\frac{\mathring{T}}{T} = \lambda^2 \Longrightarrow \frac{dT}{dt} = \lambda^2 T$

② $\frac{d^2X}{dx^2} - \lambda^2 X = 0$

To solve ① $\frac{dT}{dt} = \lambda^2 T \rightarrow \int \frac{dT}{T} = \lambda^2 \int dt \quad \log T(t) = \lambda^2 t + C$

$T(t) = A\exp(\lambda^2 t)$

② $X'' - \lambda^2 X = 0$

Case (i) $\lambda^2 > 0$ $\quad X(x) = e^{mx}$ $\quad$ A.E: $m^2 - \lambda^2 = 0$ $\quad m = \pm\lambda$

G.S: $X(x) = \bar{A}e^{\lambda x} + \bar{B}e^{-\lambda x}$ $\quad \bar{A}, \bar{B} \in \mathbb{R}$

$\therefore \quad u(x,t) = X(x)T(t) = Ae^{\lambda^2 t}\left(\bar{A}e^{\lambda x} + \bar{B}e^{-\lambda x}\right)$

$= e^{\lambda^2 t}\left(\alpha e^{\lambda x} + \beta e^{-\lambda x}\right) \quad \alpha, \beta \in \mathbb{R}$

Case (ii) $\boxed{-\lambda^2 < 0}$ (i.e our const. is now negative)

$X''(x) + \lambda^2 X(x) = 0 \quad$ A.E: $m^2 + \lambda^2 = 0 \Rightarrow m = \pm i\lambda$

$\therefore \quad X(x) = \left(\tilde{A}\cos\lambda x + \tilde{B}\sin\lambda x\right) \quad \tilde{A}, \tilde{B} \in \mathbb{R}$

$\Rightarrow \quad u(x,t) = e^{\overset{\text{negative}}{\lambda^2 t}}\left(\delta\cos\lambda x + \varepsilon\sin\lambda x\right) \quad \varepsilon, \delta \in \mathbb{R}$

Case (iii) $\lambda^2 = 0 \quad X'' = 0$

$\int$: $X'(x) = a \qquad u(x,t) = (\theta x + \gamma) \quad \theta, \gamma \in \mathbb{R}$

$\int$: $X(x) = ax + b \qquad$ To get an particular sol^n we need 2 BCs, 1 I.C.

Applying this method to a different P.D.E.

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0 \qquad (3)$$

$$V = V(S,t)$$

Look for separable sol^{n}s of (3) of the form

$$V(S,t) = f(t)\,g(S)$$

$$\cdot \equiv \frac{d}{dt} \qquad {}' \equiv \frac{d}{dS} \qquad \frac{\partial V}{\partial t} = \dot{f}g\,; \quad \frac{\partial V}{\partial S} = fg'\,; \quad \frac{\partial^2 V}{\partial S^2} = fg''$$

$$\dot{f}g + \frac{1}{2}\sigma^2 S^2 fg'' + rSfg' - rfg = 0 \qquad \div \text{ thro by } fg$$

$$\frac{\dot{f}}{f} + \frac{1}{2}\sigma^2 S^2 \frac{g''}{g} + rS\frac{g'}{g} - r = 0$$

$$-\frac{\dot{f}}{f} = \frac{1}{2}\sigma^2 S^2 \frac{g''}{g} + rS\frac{g'}{g} - r = \text{constant } \lambda$$

$$-\frac{\dot{f}}{f} = \frac{1}{2}\sigma^2 s^2 \frac{g''}{g} + rs\frac{g'}{g} - r = \lambda$$

① $\frac{df}{dt} = -\lambda f$ easy to solve

② $\frac{1}{2}\sigma^2 S^2 \frac{d^2g}{dS^2} + rs\frac{dg}{ds} - (\lambda + r)g = 0$

Cauchy-Euler $\therefore \exists$ solⁿ of form $g(s) = s^m$

A.E: $\frac{1}{2}\sigma^2 m^2 + (r - \frac{1}{2}\sigma^2)m - (\lambda + r) = 0$

To convert LHS to a monic polynomial $\div$ thro by $\frac{1}{2}\sigma^2$

$$m^2 + \left(\frac{2}{\sigma^2}r - 1\right)m - \frac{2}{\sigma^2}(\lambda + r) = 0$$

Recall $ax^2 + bx + c = 0$ has solⁿs

$$m_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\left[m_\pm = \frac{1}{2}\left[\underbrace{\left(1 - \frac{2r}{\sigma^2}\right)}_{} \pm \sqrt{\left(\frac{2r}{\sigma^2} - 1\right)^2 + \frac{8}{\sigma^2}(\lambda + r)}\right]\right.$$

3 cases to consider.

For complex roots $x = \alpha \pm i\beta$

$$\underbrace{e^{\alpha x}\left[A\cos(\beta x) + B\sin(\beta x)\right]}$$

i/ $m_+ \neq m_- \in \mathbb{R}$

ii/ Single repeated root

iii/ Complex conjugate roots.