CQF Lecture One - Exercises

The Random Behaviour of Assets

The log-normal random walk.

$$dS = \mu \, S \, dt \, + \, \sigma \, S \, dX$$

In class, we have looked at the Stochastic Differential Equation as a popular model for stock price S.

On the dataset of prices from the Lecture Excel file – or any equity index of your own choice, we test the robustness of the assumption that

$$R_i \; = \; \left[\mu \delta t \, + \, \sigma \sqrt{\delta t} \, \phi_i \right]$$

if the drift is a negligible, very small and non-robust quantity, $\mu \delta t \approx 0$,

$$R_i \; = \; \sigma \sqrt{\delta t} \, \phi_i.$$

R_i represents the returns over timestep δt, the $\phi_i \sim N(0,1)$ is Standard Normal variable. The tasks below give sufficient detail to be preformed on your own – the solutions give more information but no separate computation provided.

1. Scaling of σ with time, to the size of δt: compute column(s) of returns (1D, 2D or 5D but no longer). For example, 5D will be $R_i = \frac{S_{t+5} - S_t}{S_t}$. For each, compute the std dev of the whole column.
   Adjust σ_{2D} by $1/\sqrt{2}$, and σ_{5D} by $1/\sqrt{5}$ – are these comparable to σ_{1D}?

2. Re-shuffle the dataset into two non-chronological halves (even / odd observations) and compute μ , σ separately for each half (1D returns only). Compare.

3. Construct Quantile-Quantile plots for 1D and 5D returns. The Q-Q plot assumes Normal distribution on horizontal axis – the better the fit between of the empirical returns to Normal distribution, the more observations will be on the diagonal line.

4. Construct a histogram over historical returns scaled to z-scores and compare to Normal distribution density.

   $$R_i = \frac{\delta S}{S} = \frac{S_{t+1} - S_t}{S_t} = \frac{S_{t+1}}{S_t} - 1$$

   This last point is implemented in Modelling Returns python lab (or alike content).

Exercises and Solutions edited by CQF Faculty, Richard.Diamond@fitchlearning.com

The Q-Q plot.

The plot is straightforward to build from the first principles in Excel, Python. There are ready functions, that can go as advanced as qqplot from statsmodels.api library in Python. However, going through the principles below you will gain further understanding of what the Q-Q plots signals.

Organise the data in columns that match Historic, Scaled and Standard returns. Historic stands for the actual or empirical S&P 500 return (our data and historic period, you can use any equity index). Scaled is the normalised return (also called z-score).

Standard refers the Normal Percentile that corresponds to the cumulative probability given by i/N_{obs}. It the value on axis X that cuts the requisite probability under the bell of Normal density *pdf*, or the axis Y result of Normal cumulative density *cdf*.

Historic	Scaled	i	i/N	Standard
-0.22900	-21.20462	1	0.00009	-3.74534
-0.09470	-8.78146	2	0.00018	-3.56758
-0.09354	-8.67429	3	0.00027	-3.45987
-0.09219	-8.54970	4	0.00036	-3.38162
-0.08642	-8.01584	5	0.00045	-3.31983
-0.07922	-7.35036	6	0.00054	-3.26858
...	...	...	...	...

Table 1: Inputs for a Q-Q plot.

1. Scale historic log-returns R_t to $Z_t = \frac{r_t - \mu}{\sigma}$. Why we use log and not simple returns here is explained in Solutions. Notice our original SDE is continuous-time model.
2. Sort the scaled returns in the ascending order and create an index column $i = 1, \ldots, N$.
3. The cumulative density – percentage of observations below this – will be simply i/N.
4. The standardised percentile is obtained with the inverse Normal CDF $\Phi^{-1}(i/N)$. Each observation adds' density (probability mass) of $1/N$.

Plot the scaled returns (Z-scores) from Step 1 against the Normal percentiles from Step 4. For the perfectly Normal log-returns the Q-Q plot would be a straight line.

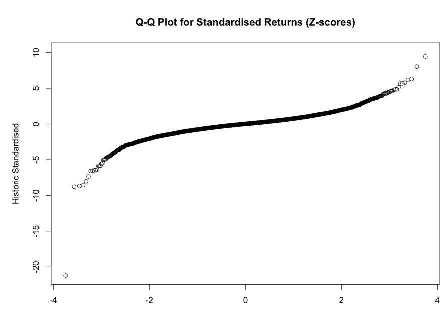

Figure 1: Q-Q Plot built step by step.