CQF Module 2
Fundamentals of Optimization and Application to Portfolio Selection

Exercises

1. Using the Lagrange multipliers optimization approach, parametrize the boundary of the opportunity set in terms of the expected portfolio returns, m, for the following set of assets:

Asset	μ	σ
A	0.08	0.12
B	0.10	0.12
C	0.10	0.15
D	0.14	0.20

in the following three cases:

a). The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 0.2 & 0.5 & 0.3 \\ 0.2 & 1 & 0.7 & 0.4 \\ 0.5 & 0.7 & 1 & 0.9 \\ 0.3 & 0.4 & 0.9 & 1 \end{pmatrix}$$

b). The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

c). The correlation coefficients between the four assets are given by

$$\rho = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2. Consider a three asset risky economy where the covariance matrix of expected returns is given by

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} = \begin{pmatrix} 9 & 3 & 0 \\ 3 & 16 & 5 \\ 0 & 5 & 25 \end{pmatrix}$$

Show that the covariance matrix is strictly positive definite in the sense that

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} > 0$$

if $\begin{pmatrix} x & y & z \end{pmatrix} \neq \begin{pmatrix} 0 & 0 & 0 \end{pmatrix}$. [Hint: Show that the resulting quadratic form is a sum of perfect squares.]

Show by calculation that the inverse of the covariance matrix is

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}^{-1} = \frac{1}{210} \begin{pmatrix} 25 & -5 & 1 \\ -5 & 15 & -3 \\ 1 & -3 & 9 \end{pmatrix}$$

3. Where should we be on the efficient frontier in question 1.a. if we wish to minimise the chance of a return less than 0.05?

   For simplicity, in this question, you may assume that the returns are normally distributed.

4. Consider a Markowitz world with a two asset risky economy. Assets A and B have an expected return over a one year time horizon, of 0.1 and 0.2 respectively. The standard deviation for asset A is 0.2 and for asset B is 0.3, over that year. The correlation between returns is 0.5. If W is the weighting attached to asset A, calculate the *boundary of the opportunity set*, by obtaining the expected return $r(W)$ and the standard deviation $\sigma_{\Pi}(W)$, for this portfolio Π. By varying W plot the boundary on a risk/reward diagram, and mark the *efficient frontier* (you may use Excel for the plot).

   Now introduce a risk free asset that has an annual return of 5%. Obtain the *Capital Market Line* and *Market Portfolio*.