Module 3: Black's Model - Solution

CQF

1 Introduction

In 1976, Black published an option model in which the underlying asset is not traded spot (i.e. bought and sold for value today) but is a futures contract (i.e. a derivative bought and sold today for value at a later date).

What is particularly interesting for us is that by comparing the valuation of options on spot instruments and on futures or forwards, we can see the important role played by the Time Value of Money in holding together instruments and market across time.

This in turn, is an added motivation for looking at better ways to model interest rates instead of keeping them constant or simply time-dependent.

2 Futures Forwards and Forward Price

Forward contracts are OTC derivatives securities in which the long party has the obligation to buy an agreed upon quantity of an underlying asset (securities, commodities or others) at an agreed upon time and at an agreed upon price called the forward price. Forward contracts are symmetrical contracts. Therefore, the obligations of the short party mirror those of the long party. The contract is settled at maturity and typically no cash flow is exchanged in the meantime. As they are OTC derivatives, forward contracts are subject to counterparty risk.

Futures contracts are an exchange-traded type of forwards. Since they are traded on an exchange, futures are heavily standardized. Counterparty risk is mitigated by the exchange's clearinghouse. In particular the clearinghouse requires that exchange participants posts margin and to settle the their contracts daily. The combination of daily settlement and margins generates a stream of daily cash-flow through the life of the contract.

2.1 Futures Forwards and Forward Price

Forward contracts can be priced easily through the no-arbitrage condition. The time t forward price for a contract maturing at time T on an underlying S is given by:

$$F(t;T) = S_t e^{r(T-t)}$$

where r is the (constant) risk-free rate.

As a result of the cash flows generated by margin and settlement, futures and forward prices only coincides when interest rates are modelled as constant or time dependent, but not when interest rates are stochastic. However, although the forward and futures price do not often coincide, the forward price still plays a role, at least as an approximation, since it enables us to bridge the time gap between time t and maturity T.

For more information on pricing forwards and futures, please refer to Paul Wilmott's book.

3 Pricing Options on Futures

We will keep the same setting and assumptions as for the Black-Scholes model, including the hypothesis that the asset's dynamics is given by a geometric Brownian motion.

3.1 Step 1: Spot, Futures and Forward Price Dynamics

The dynamics of the time t futures price, f_t, evolves according to a Geometric Brownian Motion:

$$df_t = \mu_f f_t dt + \sigma_f f_t dX, \quad f(0) = f_0$$

where μ_f and σ_f represent respectively the drift and diffusion of the futures.

Since interest rates are deterministic, the futures price is equal to the forward price and we can write:

$$f_t = F(t;T) = S_t e^{r(T-t)} \tag{1}$$

and in particular

$$f_0 = S_0 e^{rT}$$

Recall that

$$dS_t = \mu S_t dt + \sigma S_t dX, \quad S(0) = S_0$$

Applying Itô to the relationship (1), we can now express the dynamics of f_t as

$$df_t = (\mu - r) f_t dt + \sigma f_t dX, \quad f(0) = S_0 e^{rT}$$

and thus we can see now that

$$\begin{aligned} \mu_f &= \mu - r \\ \sigma_f &= \sigma \end{aligned}$$

namely, the volatility of the futures is equal to the volatility of the spot and the drift of the futures is the discounted drift of the spot.

As a result we can also see clearly that the dynamics (SDE) for f_t is of the same form as the dynamics (SDE) for $\frac{S_t}{B_t} = S_t^*$. In fact, we even have

$$f(t) = \frac{S(t)}{B(t)} e^{rT} = S^*(t) e^{rT}$$

where T is fixed by the contract. This relationship reveals that the futures price is already a (naturally) discounted process. The immediate conclusion from this observation is that we will not need to consider the discounted futures price process. We can just go ahead with the futures price process as it stands. As a result, we can directly go to the next step: the change of measure.

However, before doing this we need to redefine self-trading strategies. The reason is that taking a position on the futures market can be done at 0 cost. Hence, the value of the investor's wealth at any point in time is simply the amount he/she has invested in the risk-free asset:

$$V_t(\phi) = \phi_t^B B_t, \quad \forall t \in [0, T]$$

Therefore, when the underlying asset is a futures (modelled simply) the following definition applies:

Definition 1. *Self-Financing Trading Strategy A trading strategy* $\phi_t = (\phi_t^f, \phi_t^B)$ *defined over the time interval* $[0, T]$ *is* ***self-financing*** *if its wealth process* $V(\phi)$ *given by*

$$V_t(\phi) = \phi_t^B B_t, \quad \forall t \in [0, T]$$

satisfies the condition:

$$V_t(\phi) = V_0(\phi) + \int_0^t \phi_u^f df_u + \int_0^t \phi_u^B dB_u, \quad \forall t \in [0, T] \tag{2}$$

Notice that the self-financing condition remains unchanged.

3.2 Step 2: Change of Measure

In our case, we want to find a stochastic process θ such that f is a martingale under an equivalent martingale measure $\mathbb{Q}$ to be specified.

Applying Girsanov for an arbitrary process θ, we see that under $\mathbb{Q}$, the dynamics of f is given by

$$\frac{df(t)}{f(t)} = \mu_f dt + \sigma_f \left(-\theta dt + dX^{\mathbb{Q}}\right)$$

Hence, we also have

$$\frac{dS^*(t)}{S^*(t)} = (\mu - r - \sigma\theta)dt + \sigma dX^{\mathbb{Q}}$$

Recall that our objective is to find an equivalent martingale measure, i.e. a measure $\mathbb{Q}$ in which S^* is a martingale. One of the requirements for f to be a martingale is that its dynamics should be driftless:

$$\mu_f - \sigma_f\theta = 0$$

i.e.

$$\theta = \frac{\mu_f}{\sigma_f} = \frac{\mu_f}{\sigma}$$

Now, θ satisfies the Novikov condition. Invoking **Girsanov's theorem**, we can define the equivalent martingale measure $\mathbb{Q}$ called the **futures martingale measure** via the Radon Nikodým derivative as:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{\mu_f}{\sigma}X_t - \frac{1}{2}\frac{(\mu_f)^2}{\sigma^2}t\right), \quad t \in [0,T]$$

Moreover, the $\mathbb{Q}$-Brownian Motion, $X^{\mathbb{Q}}$, is defined as

$$X_t^{\mathbb{Q}} = X_t + \frac{\mu_f}{\sigma}t, \quad t \in [0,T]$$

and the asset process is effectively a $\mathbb{Q}$-martingale:

$$df_t = \sigma f_t dX_t^{\mathbb{Q}}, \quad f_0 > 0$$

3.3 Step 3: No-Arbitrage Valuation

A bit more work is required in this step. The reason is that the wealth process for the Black model and for Black-Scholes are different because positions in futures contracts can be entered at at now cost.

Let $\chi(t, f_t)$ be the time t arbitrage-free value of the derivative we are attempting to price.

We now define the time t discounted value of the the replicating portfolio as

$$V_t^*(\phi) = \frac{V_t(\phi)}{B_t}, \quad t \in [0, T]$$

Note that to prevent arbitrage, the value of the replicating portfolio must be equal to the value of the derivative:

$$\chi(t, f_t) = V_t, \quad t \in [0, T]$$

and hence

$$\frac{\chi(t, f_t)}{B_t} = V_t^*(\phi), \quad t \in [0, T] \tag{3}$$

In particular, for $t = T$, we have

$$\frac{\chi(T, f_T)}{B_T} = \frac{G(f_T)}{B_T} = V_T^*$$

taking the conditional expectation under $\mathbb{Q}$

$$\mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi)|\mathcal{F}_t\right] = \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1}G(f_T)|\mathcal{F}_t\right], \quad t \in [0, T] \tag{4}$$

Note that

$$dB_t^{-1} = -rB_t^{-1}dt$$

By the **Itô Product Rule**,

$$\begin{aligned}
dV_t^* &= d(V_t B_t^{-1}) \\
&= V_t dB_t^{-1} + B_t^{-1} dV_t \\
&= \phi_t^B B_t dB_t^{-1} + B_t^{-1}\left(\phi_t^f df_t + \phi_t^B dB_t\right) \\
&= -rB_t^{-1} V_t dt + B_t^{-1}\left(\phi_t^f df_t + \phi_t^B dB_t\right) \qquad (5)
\end{aligned}$$

Recall that $V_t = \Phi_t^B B_t$, so

$$rV_t dt = r\Phi_t^B B_t dt.$$

Moreover, recall that $rB_t dt = dB_t$, so the previous equation becomes

$$\Phi_t^B dB_t - rV_t dt = 0$$

Discounting and noticing that since $V_t = \Phi_t^B B_t$ then $\Phi_t^B = B_t^{-1} V_t$, we see that

$$B_t^{-1}\left(\Phi_t^B dB_t - rV_t dt\right) = B_t^{-1}\left(B_t^{-1} V_t dB_t - rV_t dt\right) = 0$$

Going back to the equation for dV_t^*, we conclude that

$$\begin{aligned} dV_t^* &= \phi_t^f B_t^{-1} df_t \\ &= \phi_t^f \frac{1}{B(0)} e^{-rt} df_t \end{aligned}$$

Integrating, we see that

$$\begin{aligned} V_t^* &= V_0^* + \int_0^t \phi_u^f \frac{1}{B(0)} e^{-ru} df_u \\ &= V_0^* + \int_0^t \phi_u^f \frac{1}{B(0)} e^{-ru} \sigma f_u dX_u^{\mathbb{Q}} \end{aligned} \tag{6}$$

Since

$$\int_0^t \phi_t^f \frac{1}{B(0)} e^{-ru} \sigma f_u dX_u^{\mathbb{Q}}$$

is an Itô integral, V^* is a martingale.

Since V^* is a martingale, it becomes obvious that

$$V_t^* = \mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi) | \mathcal{F}_t\right], \quad t \in [0, T] \tag{7}$$

Considering in addition relationships (3) and (4), we have

$$\begin{aligned} \frac{\chi(t, f_t)}{B_t} &= V_t^*(\phi) \\ &= \mathbf{E}^{\mathbb{Q}}\left[V_T^*(\phi) | \mathcal{F}_t\right] \\ &= \mathbf{E}^{\mathbb{Q}}\left[B_T^{-1} G(f_T) | \mathcal{F}_t\right], \quad t \in [0, T] \end{aligned} \tag{8}$$

Equation (8) is the cornerstone of no-arbitrage valuation.

Expressing it as

$$\chi(t, f_t) \quad = \quad B_t \mathbf{E}^{\mathbb{Q}} \left[B_T^{-1} G(f_T) | \mathcal{F}_t \right], \quad t \in [0, T] \tag{9}$$

we see that the value at time t of a derivative maturing at time T is the expected value under the $\mathbb{Q}$ measure of the discounted terminal value of the contract.

3.4 Step 4: Solve the Black Call Option on Futures Problem

3.4.1 Direct Derivation

This is were the real work starts. Since r is constant, we can write (8) as

$$\chi(t, f_t) \quad = \quad e^{-r(T-t)} \mathbf{E}^{\mathbb{Q}} \left[G(f_T) | \mathcal{F}_t \right], \quad t \in [0, T]$$

Going back to our futures price process, f_t, recall that under the measure $\mathbb{Q}$, we have

$$\frac{df(t)}{f(t)} = \sigma dX^{\mathbb{Q}}.$$

Thus, at time T,

$$f_T = f_t \exp \left\{ -\frac{1}{2}\sigma^2 (T-t) + \sigma \left(X_T^{\mathbb{Q}} - X_t^{\mathbb{Q}} \right) \right\} \tag{10}$$

Note: that f_T is an exponential martingale.

Setting $Y_T = \ln \frac{f_T}{f_t}, \quad \forall t \in [0, T]$, we see that

$$Y_T \sim \mathcal{N} \left(-\frac{1}{2}\sigma^2 (T-t), \sigma^2 (T-t) \right)$$

i.e. the log return of the asset over the period $[0, T]$, Y_T, is normally distributed with mean

$$-\frac{1}{2}\sigma^2 (T-t)$$

and variance

$$\sigma^2 (T-t)$$

Our pricing formula can now be rewritten as

$$V_t(\phi) = e^{-r(T-t)} \int_{-\infty}^{\infty} G(f_0 e^y) p(y) dy$$

where p is the PDF of Y.

In order to alleviate our computations, we are going to normalize our expression. First, define

$$\tilde{\sigma} = -\frac{1}{2}\sigma^2$$
$$\tau = T - t$$

With this notation, we have

$$Y_T \sim \mathcal{N}\left(\tilde{\sigma}\tau, \sigma^2\tau\right)$$

Now, define the standardized normal random variable Z as

$$Z = \frac{Y - \tilde{\sigma}\tau}{\sigma\sqrt{\tau}}$$

(Recall that $Z \sim \mathcal{N}(0, 1)$)

The payoff function for a call is given by

$$G(f_T) = \max\left[f_T - E, 0\right]$$

Writing $X(\tau)$ as $\sqrt{\tau}Z$ and substituting into the pricing equation, we get

$$V_t(\phi) = e^{-r\tau}\int_{-\infty}^{\infty} \max\left[f_t e^{\tilde{\sigma}\tau + \sigma\sqrt{\tau}z} - E, 0\right]\varphi(z)dz$$

where φ is the standard normal PDF:

$$\varphi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$$

We can get rid of the max by noticing that the integral vanishes when

$$f_t e^{\tilde{\sigma}\tau + \sigma\sqrt{\tau}z} < E$$

i.e. when

$$z < z_0 := \frac{\ln\left(\frac{E}{f_t}\right) - \tilde{\sigma}\tau}{\sigma\sqrt{\tau}}$$

The pricing formula becomes

$$\begin{aligned}
\chi(t, f_t) &= e^{-r\tau}\int_{z_0}^{\infty}\left(f_t e^{\tilde{\sigma}\tau + \sigma\sqrt{\tau}z} - E\right)\varphi(z)dz \\
&= e^{-r\tau}\int_{z_0}^{\infty} f_t e^{\tilde{\sigma}\tau + \sigma\sqrt{\tau}z}\varphi(z)dz - e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz
\end{aligned}$$

Evaluating the second term on the right-hand side yields:

$$\begin{aligned} -e^{-r\tau}\int_{z_0}^{\infty} E\varphi(z)dz &= -Ee^{-r\tau}\int_{z_0}^{\infty}\varphi(z)dz \\ &= -Ee^{-r\tau}P[Z \geq z_0] \end{aligned}$$

By symmetry of the normal distribution, this can also be written as

$$-Ee^{-r\tau}P[Z \leq -z_0] = -Ee^{-r\tau}N(-z_0)$$

where N is the standard normal CDF.

To evaluate the first term, we need to complete the square in the exponent:

$$\begin{aligned} e^{-r\tau}\int_{z_0}^{\infty} f_t e^{\tilde{\sigma}\tau+\sigma\sqrt{\tau}z}\varphi(z)dz &= \frac{e^{(\tilde{\sigma}-r)\tau}f_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{\sigma\sqrt{\tau}z-\frac{1}{2}z^2}dz \\ &= \frac{e^{(\tilde{\sigma}-r)\tau}f_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2+\frac{1}{2}\sigma^2\tau}dz \\ &= e^{-r\tau}\frac{f_t}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}(z-\sigma\sqrt{\tau})^2}dz \\ &= e^{-r\tau}f_t P[U \geq z_0] \end{aligned}$$

where $U = Z - \sigma\sqrt{\tau}$ so that $U \sim \mathcal{N}(\sigma\sqrt{\tau}, 1)$. Standardizing, we see that the first term is actually equal to

$$e^{-r\tau}f_t N(-z_0 + \sigma\sqrt{\tau})$$

In order to write this in the more familiar form, all we need to do is to define d_1 and d_2 as

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{f_t}{E}\right) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= \frac{\ln\left(\frac{f_t}{E}\right) - \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}} \end{aligned}$$

$$\chi(t, f_t) = e^{-r(T-t)}\left[f_t N(d_1) - EN(d_2)\right]$$

3.4.2 Alternative Derivation Through Change of Measure

In the case of a call, the valuation equation (8) is

$$\begin{aligned}\chi(t, f_t) &= B_t \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} \left[f_T - K \right]^+ | \mathcal{F}_t \right] \\ &= B_t \mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} \left[f_t \exp \left\{ -\frac{1}{2}\sigma^2 (T-t) + \sigma \left(X_T^{\mathbb{Q}} - X_t^{\mathbb{Q}} \right) \right\} - K \right]^+ | \mathcal{F}_t \right],\end{aligned}$$

where, as in the course notes, we denote the strike price by K to avoid any confusion with $\mathbb{E}$, the expectation.

As in the lecture, we focus on the case $t = 0$ for convenience. The conditional expectation becomes an unconditional expectation:

$$\begin{aligned}&\mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} \left[f_T - K \right]^+ \right] \\ = &\underbrace{\mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} f_T \mathbf{1}_{\{f_T > K\}} \right]}_{(1)} - \underbrace{\mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} K \mathbf{1}_{\{f_T > K\}} \right]}_{(2)},\end{aligned}$$

and equation (10) simplifies to:

$$f_T = f_0 \exp \left\{ -\frac{1}{2}\sigma^2 T + \sigma X_T^{\mathbb{Q}} \right\}. \tag{11}$$

We start with the easier Term (2) in (11):

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} K \mathbf{1}_{\{f_T > K\}} \right] &= e^{-rT} K P^{\mathbb{Q}} \left[f_T > K \right] \\ &= e^{-rT} K P^{\mathbb{Q}} \left[f_0 e^{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T} > K \right] \\ &= e^{-rT} K P^{\mathbb{Q}} \left[\ln \left(\frac{f_0}{K} \right) - \frac{1}{2}\sigma^2 T > -\sigma X_T^{\mathbb{Q}} \right] \\ &= e^{-rT} K P^{\mathbb{Q}} \left[\frac{\ln \left(\frac{f_0}{K} \right) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}} > \xi \right] \\ &= e^{-rT} K N(d_2),\end{aligned}$$

where we have emphasized the fact that the probability P is taken with respect to the measure $\mathbb{Q}$ and have defined $\xi = -X_T^{\mathbb{Q}} / \sqrt{T}$, which is standard Normal random variable $\xi \sim \mathcal{N}(0, 1)$, and where

$$d_2 = \frac{\ln \left(\frac{f_0}{K} \right) - \frac{1}{2}\sigma^2 T}{\sigma \sqrt{T}}.$$

Now, let's look at the "tricky" Term (1) in (11):

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}} \left[B_T^{-1} f_T \mathbf{1}_{\{f_T > K\}} \right] &= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[f_T \mathbf{1}_{\{f_T > K\}} \right] \\ &= e^{-rT} \mathbb{E}^{\mathbb{Q}} \left[f_0 \exp \left\{ \sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T \right\} \mathbf{1}_{\{f_T > K\}} \right]\end{aligned}$$

Notice that the term $\exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}$ is the same Doléans exponential we identified in class when we derived the Black-Scholes formula.

So we can apply the second version of Girsanov's Theorem (for stochastic exponentials) to define a new probability measure $\bar{\mathbb{Q}}$ via the Radon Nikodym derivative

$$\frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} = \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}$$

Under the $\bar{\mathbb{Q}}$ measure,

$$X_t^{\bar{\mathbb{Q}}} = X_t^{\mathbb{Q}} - \sigma t, \qquad t \in [0, T]$$

is a standard Brownian motion. Moreover,

$$f_T = f_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\}. \tag{12}$$

Hence, equation (12) for Term (1) becomes:

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[B_T^{-1} f_T \mathbf{1}_{\{f_T > K\}}\right] &= e^{-rT}\mathbb{E}^{\mathbb{Q}}\left[f_0 \exp\left\{\sigma X_T^{\mathbb{Q}} - \frac{1}{2}\sigma^2 T\right\}\mathbf{1}_{\{f_T > K\}}\right] \\
&= e^{-rT} f_0 \int \frac{d\bar{\mathbb{Q}}}{d\mathbb{Q}} \mathbf{1}_{\{S_T > K\}} d\mathbb{Q} \\
&= e^{-rT} f_0 P^{\bar{\mathbb{Q}}}\left[f_T > K\right] \\
&= e^{-rT} f_0 P^{\bar{\mathbb{Q}}}\left[f_T > K\right] \\
&= e^{-rT} f_0 P^{\bar{\mathbb{Q}}}\left[f_0 \exp\left\{\sigma X_T^{\bar{\mathbb{Q}}} + \frac{1}{2}\sigma^2 T\right\} > K\right] \\
&= e^{-rT} f_0 P^{\bar{\mathbb{Q}}}\left[\ln\left(\frac{f_0}{K}\right) + \frac{1}{2}\sigma^2 T > -\sigma X_T^{\bar{\mathbb{Q}}}\right] \\
&= e^{-rT} f_0 P^{\bar{\mathbb{Q}}}\left[\frac{\ln\left(\frac{f_0}{K}\right) + \frac{1}{2}\sigma^2 T}{\sigma\sqrt{T}} > \xi\right] \\
&= e^{-rT} f_0 N(d_1),
\end{aligned}$$

where again $\xi \sim \mathcal{N}(0,1)$ and

$$d_1 = \frac{\ln\left(\frac{f_0}{K}\right) + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}.$$

3.5 The Boundary Value Problem for Black's Model

Applying the **Feynman-Kač formula** to

$$\chi(t, f_t) \;=\; e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}\left[G(f_T)|\mathcal{F}_t\right], \quad t \in [0, T]$$

with

$$df_t = \sigma f_t dX_t^{\mathbb{Q}}, \quad f_0 > 0$$

we conclude that the value $V(t, f_t)$ of the option on a futures solves the boundary value problem

$$\begin{aligned} \frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 f^2 \frac{\partial^2 V}{\partial f^2} - rV &= 0 \\ V(T, f_T) = G(f_T) & \end{aligned}$$