M1L6 Exercises

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We will use a few versions of Itô's lemma, each one generalizes the previous:

Lemma 0.1. *Define a twice differentiable function $F(W_t)$ where W_t is a Brownian motion. Then the following version of Itô's lemma holds:*

$$dF = \frac{1}{2}\frac{d^2F}{dW^2}\,dt + \frac{\partial F}{\partial W}\,dW_t$$

Lemma 0.2. *Define a twice differentiable function $F(t, W_t)$ where W_t is a Brownian motion. Then the following version of Itô's lemma holds:*

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{d^2F}{dW^2}\right)dt + \frac{\partial F}{\partial W}\,dW_t$$

Lemma 0.3. *Define a twice differentiable function $F(t, S_t)$ on an Itô process:*

$$dS_t = A(t, S_t)\,dt + B(t, W_t)\,dW_t$$

where W_t is a Brownian motion, and A, B are functions. Then the following version of Itô's lemma holds:

$$dF = \left(\frac{\partial F}{\partial t} + A\frac{\partial F}{\partial S} + \frac{B^2}{2}\frac{d^2F}{dS^2}\right)dt + B\frac{\partial F}{\partial S}\,dW_t$$

We also have the Fokker-Planck equation and the related steady-state distributions:

Theorem 0.4. *Consider a stochastic process y_t such that it evolves as:*

$$dy_t = A(t, y_t)\,dt + B(t, y_t)\,dW_t$$

where W_t is a Brownian motion. Then its transition probability density function $p(y, t; y', t')$ satisfies the Fokker-Planck (Forward Kolmogorov Equation):

$$\frac{\partial p}{\partial t'} = \frac{1}{2}\frac{\partial^2}{\partial y'^2}\left(B(t', y')^2 p\right) - \frac{\partial}{\partial y'}\left(A(t', y')p\right) \tag{FP}$$

Additionally, the steady-state distribution p_∞ will satisfy the following ODE:

$$\frac{d}{dy'}\left(A(t', y')p_\infty\right) = \frac{1}{2}\frac{d^2}{dy'^2}\left(B(t', y')^2 p_\infty\right) \tag{SS}$$

1

Let a share price S satisfy

$$dS_t = A(t, S_t)\, dt + B(t, S_t)\, dW_t$$

If we had $g = g(S)$, we want to mold A, B such that the drift coefficient of Itô's lemma 0.3 is zero. That is, we require that:

$$A\frac{\partial F}{\partial S} + \frac{B^2}{2}\frac{d^2F}{dS^2} = 0$$

This is an ODE in S. In order to solve this ODE, we would require A, B to be in terms of S only, as any dependency on t would result in g having t-terms show up, contradicting $g = g(S)$.

3

Define F such that $\frac{\partial F}{\partial W} = W(1 - e^{-W^2})$. We would like to write the stochastic integral of this partial derivative in the form below:

$$\int_0^t W_\tau \left(1 - e^{-W_\tau^2}\right) dW_\tau = \overline{F}(W_t) + \int_0^t G(W_\tau)\, dW_\tau$$

To do so, we first find the rest of the derivatives and F such that we can use Itô's lemma 0.1:

$$F(W) = \frac{1}{2}W^2 + \frac{1}{2}e^{-W^2}$$
$$\frac{\partial^2 F}{\partial W^2} = 1 - e^{-W^2} + 2W^2e^{-W^2}$$

Applying to Itô's lemma 0.1 (in integral form):

$$\frac{1}{2}W_t^2 + \frac{1}{2}e^{-W_t^2} - \frac{1}{2} = \int_0^t \frac{1}{2}\left(1 - e^{-W_\tau^2} + 2W_\tau^2 e^{-W_\tau^2}\right) d\tau + \int_0^t W_\tau\left(1 - e^{-W_\tau^2}\right) dW_\tau$$
$$\implies \int_0^t W_\tau\left(1 - e^{-W_\tau^2}\right) dW_\tau = \frac{1}{2}W_t^2 + \frac{1}{2}e^{-W_t^2} - \frac{1}{2} + \int_0^t -\frac{1}{2}\left(1 - e^{-W_\tau^2} + 2W_\tau^2 e^{-W_\tau^2}\right) d\tau$$

We see that we have the required form, if we set:

$$\overline{F}(W_t) = \frac{1}{2}W_t^2 + \frac{1}{2}e^{-W_t^2} - \frac{1}{2}$$
$$G(W_t) = -\frac{1}{2}\left(1 - e^{-W_\tau^2} + 2W_\tau^2 e^{-W_\tau^2}\right)$$

4

Consider the process

$$d(\log y) = (\alpha - \beta \log y)\, dt + \delta\, dW_t$$

We define u such that $y(u) = e^u$ ($u(y) = \log y$). We see that u satisfies:

$$du = (\alpha - \beta u)\, dt + \delta\, dW_t$$

Noting that $\frac{\partial^n y}{\partial u} = u$ for all $n \in \mathbb{N}$, we can apply Itô's lemma 0.3 to $y = e^u$:

$$\begin{aligned} dy = d(e^u) &= \left[0 + (\alpha - \beta u)e^u + \frac{1}{2}\delta^2 e^u\right] dt + \delta e^u\, dW_t \\ \implies \frac{dy}{y} &= \left(\alpha - \beta u + \frac{1}{2}\delta^2\right) dt + \delta\, dW_t \\ \implies \frac{dy}{y} &= \left(\alpha - \beta \log y + \frac{1}{2}\delta^2\right) dt + \delta\, dW_t \end{aligned}$$

5

Set $G(t, W_t) = e^{t + ae^{W_t}}$ for a constant a. We note the following identity for later:

$$e^{W_t} = \frac{\log G - t}{a}$$

We calculate the partial derivatives of G:

$$\frac{\partial G}{\partial t} = e^{t + ae^W} = G$$

$$\begin{aligned} \frac{\partial G}{\partial W} &= ae^W \cdot e^{t + ae^W} \\ &= a \cdot \frac{\log G - t}{a} \cdot G \\ &= G(\log G - t) \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 G}{\partial W^2} &= \frac{\partial G}{\partial W}(\log G - t) + G \cdot \frac{1}{G}\frac{\partial G}{\partial W} \\ &= \frac{\partial G}{\partial W}(\log G - t + 1) \\ &= G(\log G - t)[(\log G - t) + 1] \\ &= G[(\log G - t)^2 + \log G - t] \end{aligned}$$

We apply Itô's lemma 0.2:

$$\begin{aligned} dG_t &= \left[G_t + \frac{1}{2}G_t[(\log G_t - t)^2 + \log G_t - t]\right] dt + G(\log G_t - t)\,dW_t \\ \implies \frac{dG_t}{G_t} &= \left[1 + \frac{1}{2}(\log G_t - t) + \frac{1}{2}(\log G_t - t)^2\right] dt + (\log G_t - t)\,dW_t \end{aligned}$$

6

A spot rate r_t evolves according to:

$$dr_t = u(r_t)\,dt + \nu r_t^{\beta}\,dW_t$$

To find the steady-state distribution p_∞, we apply (SS) with $A = u$, $B = \nu r^{\beta}$:

$$\begin{aligned} \frac{d}{dr}(u \cdot p_\infty) &= \frac{1}{2}\frac{d^2}{dr^2}\left(\nu^2 r_t^{2\beta} p\infty\right) \\ \implies u \cdot p_\infty &= \frac{1}{2}\frac{d}{dr}\left(\nu^2 r_t^{2\beta} p_\infty\right) \\ &= \nu^2 \beta r_t^{2\beta-1} p_\infty + \frac{1}{2}\nu^2 r_t^{2\beta}\frac{d}{dr}(p_\infty) \\ \implies u(r_t) &= \nu^2 \beta r_t^{2\beta-1} + \frac{1}{2}\nu^2 r_t^{2\beta} \cdot \frac{1}{p_\infty}\frac{d}{dr}(p_\infty) \\ &= \nu^2 \beta r_t^{2\beta-1} + \frac{1}{2}\nu^2 r_t^{2\beta} \cdot \frac{d}{dr}(\log p_\infty) \end{aligned}$$

Note that there are no constants of integration involved as $p_\infty, \frac{dp_\infty}{dr}$ are assumed to be sufficiently quickly approaching 0 as $r \to \infty$.