M1L2 Exercises

February 11, 2024

Binomial Method solution

Quick reminder for the binomial method: say we have a stock price S_t, with known initial value S_0 and known values at time T: S_T^u, S_T^d where $S_T^d < S_0 < S_T^u$. Let there be some option V_t with payoffs (value) at time T based on the stock prices at time T, V_T^u, V_T^d. Also assume interest rates of r, compounded continuously. We want to find the option value (at time 0) V.

We construct a portfolio long 1 option and short Δ stocks. It will have value $V - \Delta S_0$ at $t = 0$, and either $V_T^u - \Delta S_T^u$ or $V_T^d - \Delta S_T^d$ at time T. We would like to choose Δ such that both future values are equal, which requires:

$$\Delta = \frac{V_T^u - V_T^d}{S_T^u - S_T^d} \tag{1}$$

We then require that the present value of the portfolio at time T: $e^{-rT}(V_T^d - \Delta S_T^d)$ (equivalently, using $\cdot^u$) be equal to the portfolio at time 0, $V - \Delta S_0$, which gives us:

$$V = \Delta S_0 + e^{-rT}(V_T^d - \Delta S_T^d) \tag{2}$$

In the case of options with $V_T^d = 0$, this further simplifies to

$$V = \Delta \left(S_0 - e^{-rT} S_T^d\right) \tag{3}$$

1

Consider a 3 month European call option with $K = 79$, on a stock following the binomial tree can be described as below.

$$\begin{array}{cc} & 84 \\ 80 & \\ & 76 \end{array}$$

and no interest rates. We get the following option value tree:

$$\begin{array}{ccc} & & 5 \\ V & & \\ & & 0 \end{array}$$

We first calculate Δ:

$$\Delta = \frac{5-0}{84-76} = \frac{5}{8}$$

From which we can get the option price:

$$V = \frac{5}{8}(80-76) = 2.50$$

2

We have the share price structure across a year:

$$\begin{array}{ccc} & & 98 \\ 92 & & \\ & & 86 \end{array}$$

Say we have a 1 year European call with $K = 90$, then we have option pricing tree:

$$\begin{array}{ccc} & & 8 \\ V & & \\ & & 0 \end{array}$$

and there is an interest rate of 2% p.a (cts compounding). Then we calculate Δ:

$$\Delta = \frac{8-0}{98-86} = \frac{2}{3}$$

From which, after discounting we can get the option price:

$$V = \frac{2}{3}(92 - e^{-0.02*1} \cdot 86) \approx 5.14$$

3

We have the share price structure across 3 months ($T = 0.25$):

$$\begin{array}{ccc} & & 17 \\ 15 & & \\ & & 13 \end{array}$$

Say we have a 3 month power option with payoff $\max(S^2 - 159, 0)$, then we have option pricing tree:

$$\begin{array}{ccc} & & 130 \\ V & & \\ & & 10 \end{array}$$

with no interest rates. Then we calculate Δ:

$$\Delta = \frac{130 - 10}{17 - 13} = 30$$

From which, after discounting we can get the option price:

$$V = 30 \cdot 15 + (10 - 30 \cdot 13) = 70$$

4

We have the share price structure across 3 months ($T = 0.25$):

$$\begin{array}{ccc} & & 92 \\ 75 & & \\ & & 59 \end{array}$$

with no interest rates. We see that the risk neutral probability of the share going up, p satisfies:

$$p \cdot 92 + (1 - p) \cdot 59 = 75$$

i.e

$$p = \frac{75 - 59}{92 - 59} = 0.485$$

And the probability of a fall is $1 - p = 0.515$.

5

We have the share price structure across 3 months ($T = 0.25$):

$$\begin{array}{ccc} & & 84 \\ 80 & & \\ & & 76 \end{array}$$

Say we have a 3 month digital call option with $K = 79$, then we have option pricing tree:

$$\begin{array}{ccc} & & 1 \\ V & & \\ & & 0 \end{array}$$

with no interest rates. Then we calculate Δ:

$$\Delta = \frac{1-0}{84-76} = \frac{1}{8}$$

From which we can get the option price:

$$V = \frac{1}{8}(80-76) = 0.5$$

7

We have the share price structure across 6 months ($T = 0.5$):

$$\begin{array}{ccc} & & 69 \\ & 66 & \\ 63 & & 63 \\ & 60 & \\ & & 57 \end{array}$$

Say we have a 6 month digital put option with $K = 61$, then we have option pricing tree:

$$\begin{array}{ccc} & & 0 \\ & V^u & \\ V & & 0 \\ & V^d & \\ & & 4 \end{array}$$

with an interest rate of 4% with cts compounding.

We get the two later deltas:

$$\Delta^u = \frac{0-0}{69-63} = 0$$
$$\Delta^d = \frac{0-4}{63-57} = -\frac{2}{3}$$

From which, we can recover V^u, V^d:

$$V^u = 0$$
$$V^d = -\frac{2}{3}(60 - e^{-0.04 \cdot 0.25} \cdot 63) \approx 1.582$$

Getting us:

$$\begin{array}{ccc} & & 0 \\ & 0 & \\ V & & 0 \\ & 1.582 & \\ & & 4 \end{array}$$

From which we can calculate Δ:

$$\Delta = \frac{0 - 1.582}{66 - 60} = -0.264$$

Then we can finally get V:

$$V = -0.264(63 - e^{-0.04 \cdot 0.25} \cdot 66) \approx 0.615$$

About £0.62.

8

We have an asset S with value α today following a T-time step binomial tree:

$$\begin{array}{ccc}
 & & \alpha + 20 \\
 & \alpha + 10 & \\
\alpha & & \alpha \\
 & \alpha - 10 & \\
 & & \alpha - 20 \\
Time & T_1 & T
\end{array}$$

with $r = 0$. Consider a European call option with payoff $V(S, T) = \max(S - \alpha - 5, 0)$, we get option pricing tree:

$$\begin{array}{ccc}
 & & 15 \\
 & V_1 & \\
V & & 0 \\
 & V_{-1} & \\
 & & 0 \\
Time & T_1 & T
\end{array}$$

We immediately see that $V_{-1} = 0$, as $\Delta_{-1} = 0$. Therefore we just need to calculate V_1:

$$\Delta_1 = \frac{15}{20} = 0.75$$

And therefore:

$$V_1 = 0.75(\alpha + 10 - \alpha) = 7.5 = 15/2$$

Resulting in the tree:

$$\begin{array}{ccc}
 & & 15 \\
 & 15/2 & \\
V & & 0 \\
 & 0 & \\
 & & 0 \\
Time & T_1 & T
\end{array}$$

And so we calculate Δ, then V:

$$\Delta_1 = \frac{7.5}{20} = 0.375$$

$$V = 0.375(\alpha - (\alpha - 10)) = 3.75 = 15/4$$

So we get the requested tree:

$$\begin{array}{ccc}
 & & 15 \\
 & 15/2 & \\
15/4 & & 0 \\
 & 0 & \\
 & & 0 \\
Time & T_1 & T
\end{array}$$