Transition Density Functions

- Trinomial random walk
- Forward Kolmogorov equation
- Backward Kolmogorov equation
- Solution of Forward equation — Solve a PDE
    - Similarity reduction method
    - Use of Dirac-delta function

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Randomness in finance

Modern finance theory, especially derivatives theory, is based on the random movement of financial quantities.

We are now going to explore the simple idea of the random walk and see its relationship to differential equations.

This is achieved via the concept of a **transition probability density function**.

time dep. / unsteady

R.V.s dep. on time

1 • The trinomial random walk define

Solve the PDE in 3 for

3 • The transition probability density function

derive

2 • An equation for the transition probability density function

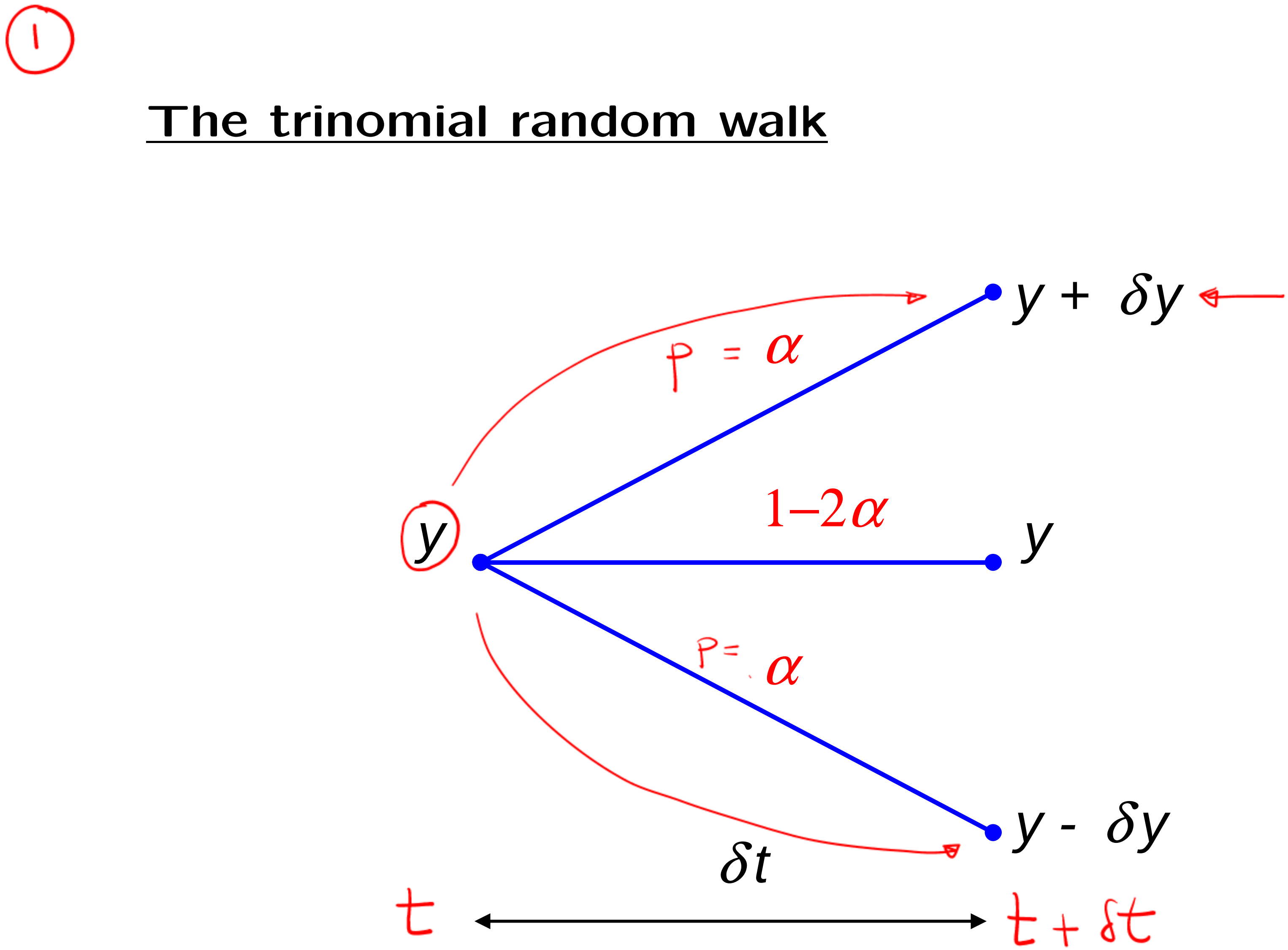
<u>State</u>

1 position/value y (y,t)

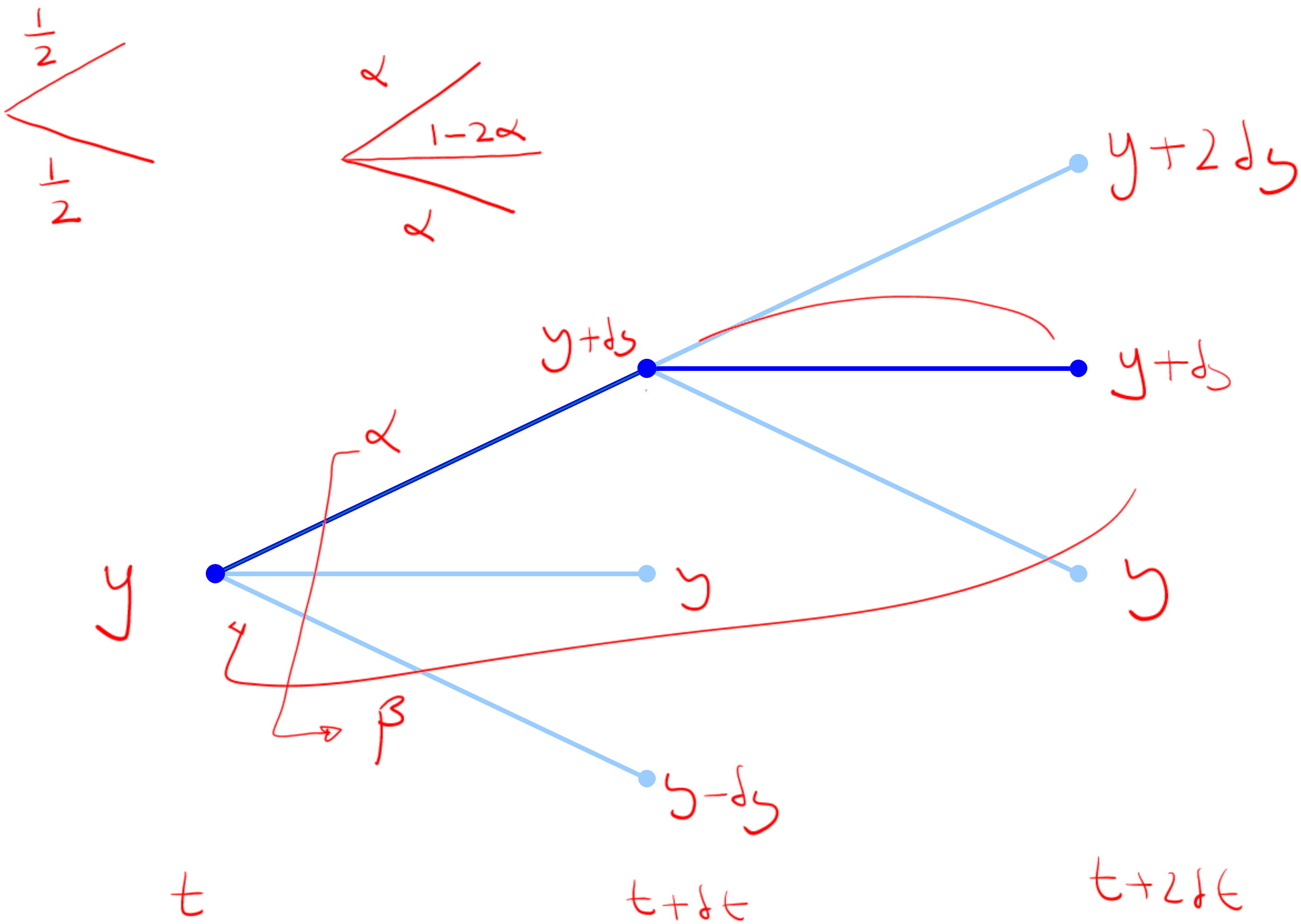
2 time t future (y',t')

past (y,t)

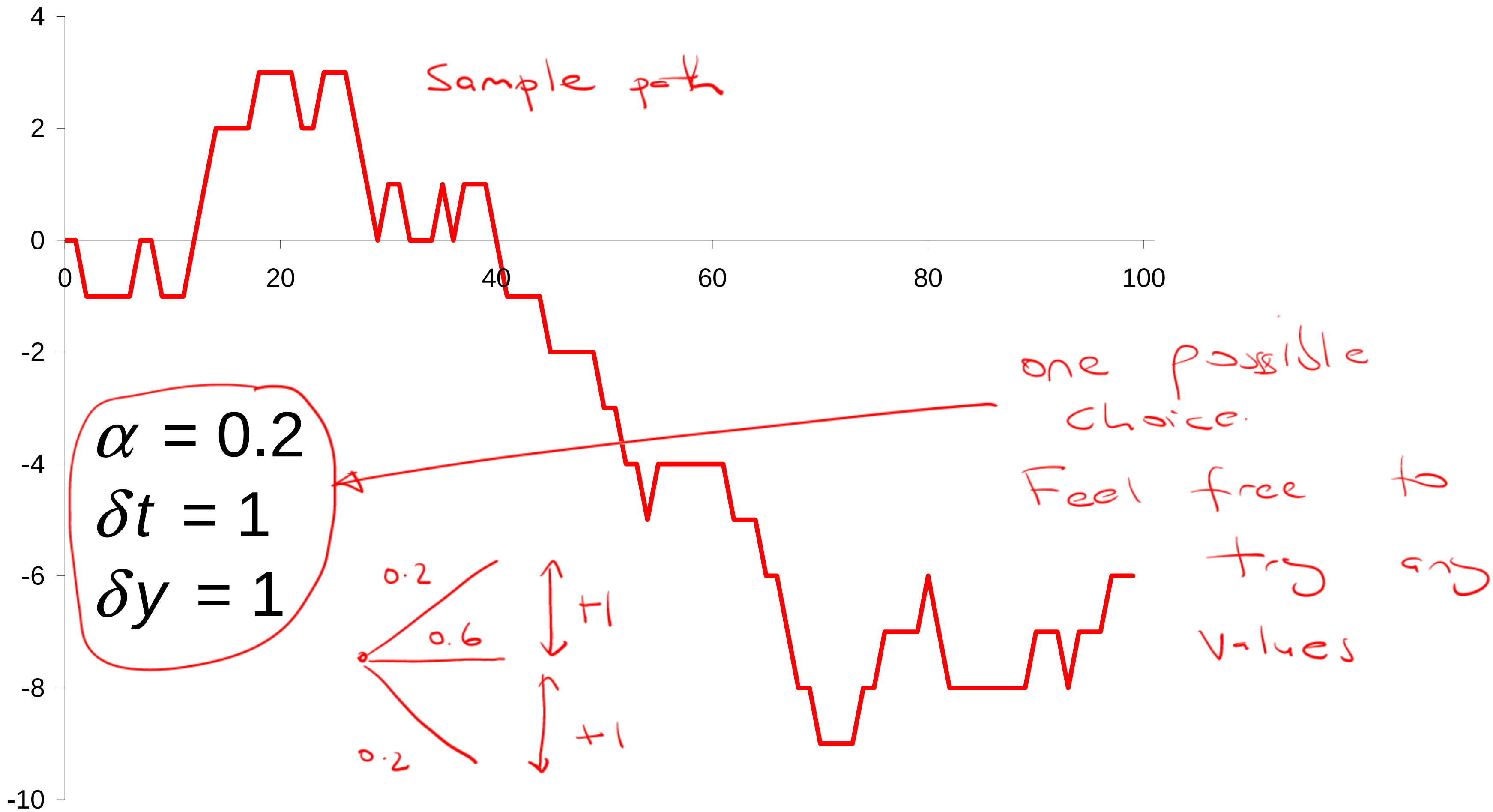
The trinomial random walk



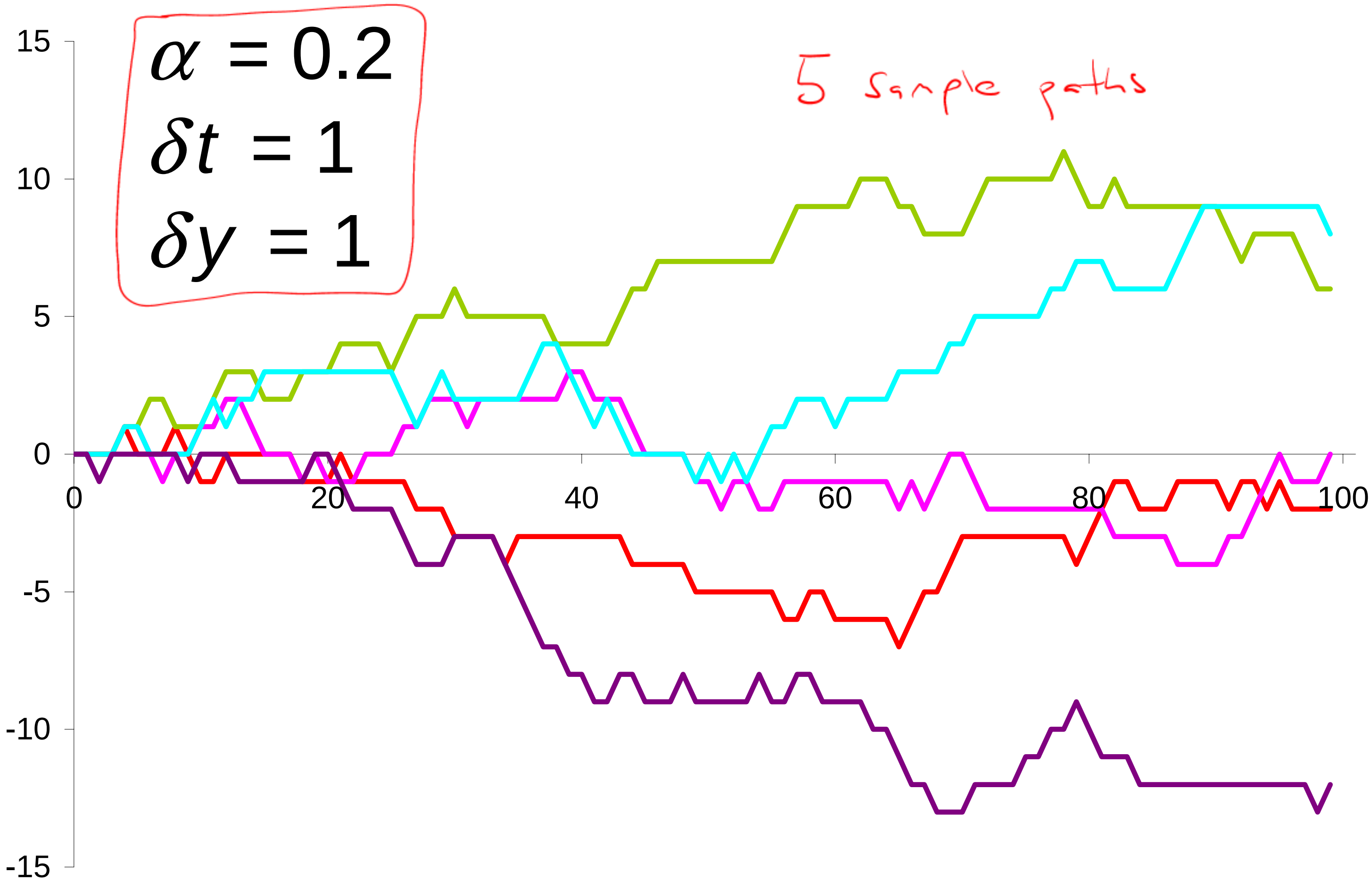
y is the value of our random variable. δt is a time step. α is a probability. δy is the size of the move in y.



Suppose the top branch is chosen after one time step. After the second there are three places that y could be.



After lots of time steps we might end up with a picture like this. This is a random walk.



Often we are interested in the probabilistic properties of the random walk rather than the outcome of a single realization.

The transition probability density function

To analyze the probabilistic properties of the random walk, we introduce the **transition probability density function** $p(y, t; y', t')$ defined by

$p(\underbrace{y, t}_{\text{past}}\ ;\ \underbrace{y', t'}_{\text{future}})$

$$\text{Prob}(a < \underbrace{y' < b \text{ at time } t'}_{\text{variables}} | y \text{ at time } t) = \int_a^b p(y, t; y', t')\, dy'.$$

parameters

In words this is "the probability that the random variable y' lies between a and b at time t' in the future, given that it started out with value y at time t."

y

t

$a < y' < b$

Warning: The trinomial random walk presented here is ‘discrete’ in the sense that time and random variable only take discrete values.

We will be moving over to a continuous-time and continuous-variable model shortly.

Think of y and t as being current values with y' and t' being future values. The transition probability density function can be used to answer the question,

> "What is the probability of the variable y' being in a specified range at time t' in the future given that it started out with value y at time t?"

The transition probability density function $p(y, t; y', t')$ satisfies two equations, one involving derivatives with respect to the future state and time (y' and t') and called the **forward equation**, and the other involving derivatives with respect to the current state and time (y and t) and called the **backward equation**.

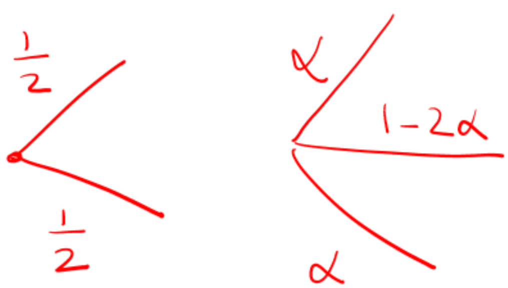
Fwd K.E. (y', t') vars (y, t) fixed

Bwd K.E (y, t) vars (y', t') fixed

From the trinomial model to the transition probability density function

The variable y can either rise, fall or take the same value after a time step δt. These movements have certain probabilities associated with them.

We are going to assume that the probability of a rise and a fall are both the same, $\alpha < \frac{1}{2}$. (But, of course, this can be generalized. Why would we want to generalize this?)

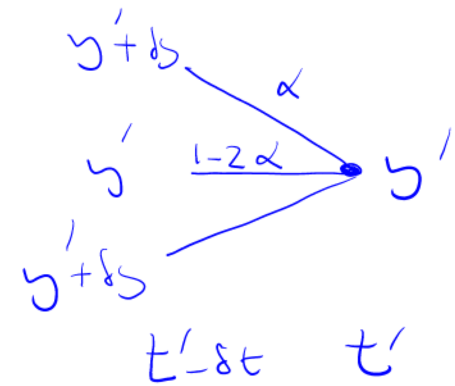


Derivation of:

The forward equation

The variable y takes the value y' at time t', but how did it get there?

$p(t'-\delta t, y'+\delta y) \times p(\text{down})$ α

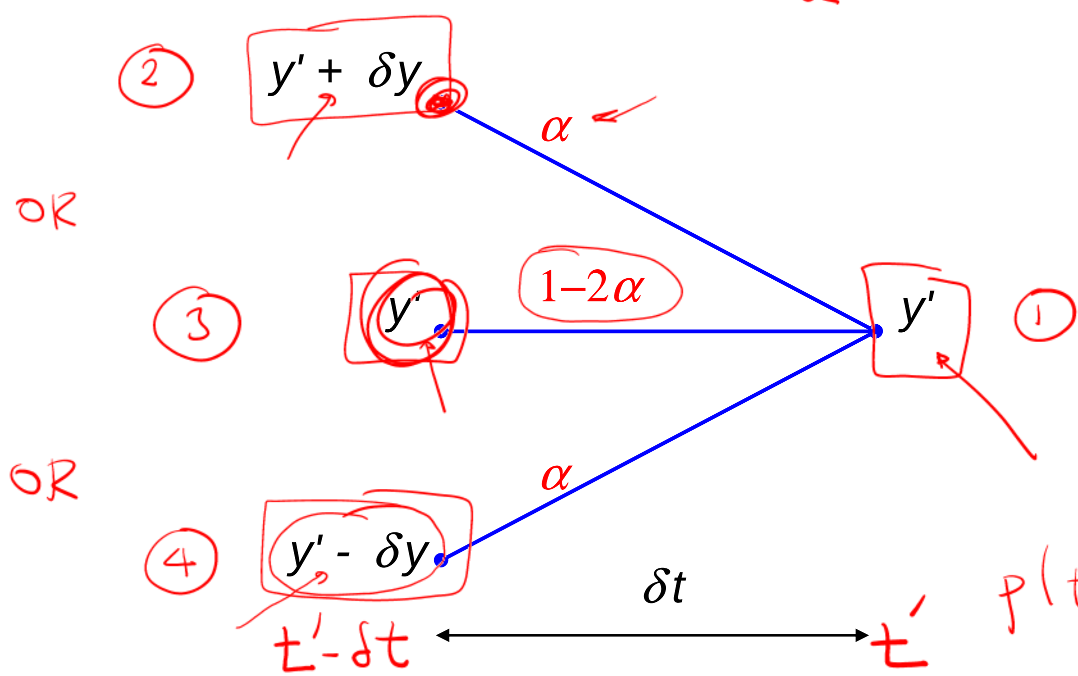


fixed

y.

t

I.C



$p(t'-\delta t, y'-\delta y) \times p(\text{up})$ α

In our trinomial walk we can only get to the point y' from the three values $y' + \delta y$, y' and $y' - \delta y$.

The probability of being at y' at time t' is related to the probabilities of being at the previous three values and *moving in the right direction*:

$$p(y,t;y',t') = \overset{p(\text{down})}{\alpha}\, p(y,t;\underline{y'+\delta y}, t'-\delta t) \qquad ① \quad ②$$

$$+\underbrace{(1-2\alpha)}_{p(\text{not moving})}p(y,t;y',t'-\delta t) + \underbrace{\alpha}_{p(\text{up})}\, \underbrace{p(y,t;y'-\delta y,t'-\delta t)}_{④}. \qquad ③ \tag{1}$$

$f(x,t)$ 2D T.S.E $\quad x \to x+\delta x,\ t \to t+\delta t$

$$f(x\pm\delta x, t\pm\delta t) = f(x,t) \pm \frac{\partial f}{\partial t}\delta t \pm \frac{\partial f}{\partial x}\delta x + \frac{1}{2!}\frac{\partial^2 f}{\partial x^2}\delta x^2 + \ldots$$

The above is in the maths primer typed notes

We can easily expand each of the terms in Taylor series about the point y', t'. For example,

$$p(y,t;y'+\delta y,t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} + \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial {y'}^2} + \cdots,$$

$$p(y,t;y'-\delta y,t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} - \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial {y'}^2} + \cdots$$

and

$$p(y,t;y',t'-\delta t) \approx p(y,t;y',t') - \delta t\frac{\partial p}{\partial t'} + \cdots.$$

Subst. the above in (1)

Equation (1) becomes (and we are using shorthand notation $p(y,t;y',t') = p$)

$y'+\delta y$ going down

$$p = \alpha(p - \delta t\frac{\partial p}{\partial t'} + \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots)$$

$$+(1-2\alpha)(p - \delta t\frac{\partial p}{\partial t'} + \cdots)$$

staying in same position

$$+\alpha(p - \delta t\frac{\partial p}{\partial t'} - \delta y\frac{\partial p}{\partial y'} + \tfrac{1}{2}\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots).$$

$y'-\delta y$ going up

Lots of these terms cancel out!

We are left with

$$\delta t\frac{\partial p}{\partial t'} = \alpha\,\delta y^2\frac{\partial^2 p}{\partial y'^2} + \cdots .$$

$(\delta t,\ \delta y^2)$

$$\frac{\partial p}{\partial t'} = \alpha\,\frac{\delta y^2}{\delta t}\,\frac{\partial^2 p}{\partial y'^2}$$

Now we explicitly mention that we are really interested in the continuous limit, as increments in time and y get smaller and smaller.

Consider $\delta y^2/\delta t$

The above equation only makes sense if

① numerator $\to 0$ quicker than $\delta t \to 0$: Random walk collapses to zero

② $\delta t \to 0$ quicker than num $\to 0$: Random walk grows indef.

$$\frac{\alpha\,\delta y^2}{\delta t}$$

③ $\frac{\delta y^2}{\delta t}$ = constant $\delta y^2 \sim O(\delta t)$ $\longrightarrow$ $\delta y \sim O(\delta t^{1/2})$ finite random walk.

tends to some finite limit as the time step and the y increment δy go to zero.

δy behaves like $\sqrt{\delta t}$

Let's define

-0.00638

-6.38×10^{-3}

$$\frac{\alpha\, \delta y^2}{\delta t} = c^2,$$

never negative

> 0

for some finite, non-zero c.

The final equation is now

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}.$$

variables (y', t')

initial

$t' > t$

1st order time

2nd order random variable

This is the **Fokker–Planck** or **forward Kolmogorov equation**. It is a forward parabolic partial differential equation, requiring initial conditions at time t and to be solved for $t' > t$.

This equation is to be used if there is some special state now and you want to know what could happen later. For example, you know the current value of y and want to know the distribution of values at some later date.

Observations:

- This is a partial differential equation for p as a function of two independent variables y' and t'.

- It is an example of a diffusion equation.

- y and t are rather like parameters in this problem, think of them as starting quantities for the random walk.

- This is a diffusion equation. You need that special relationship between α, δt and δy to get this equation.

$\delta y \sim O(\delta t^{1/2})$

- This is also an example of Brownian motion.

- When we get on to financial applications the quantity c will be related to volatility.

$\boxed{c^2} \longrightarrow \frac{1}{2}\sigma^2$ "forget I said this!"

The backward equation

Now we come to find the backward equation. This will be useful if we want to calculate probabilities of reaching a specified final state from various initial states. It will be a backward parabolic partial differential equation requiring conditions imposed in the future, and solved backwards in time.

fixed (s,t)

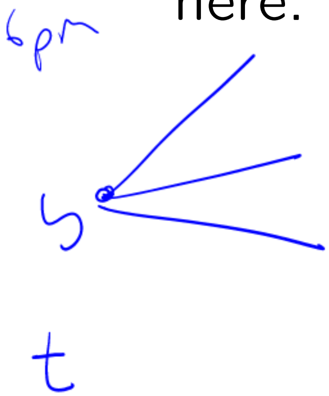
Whereas the forward equation had independent variable t' and y' the backward equation has variables t and y.

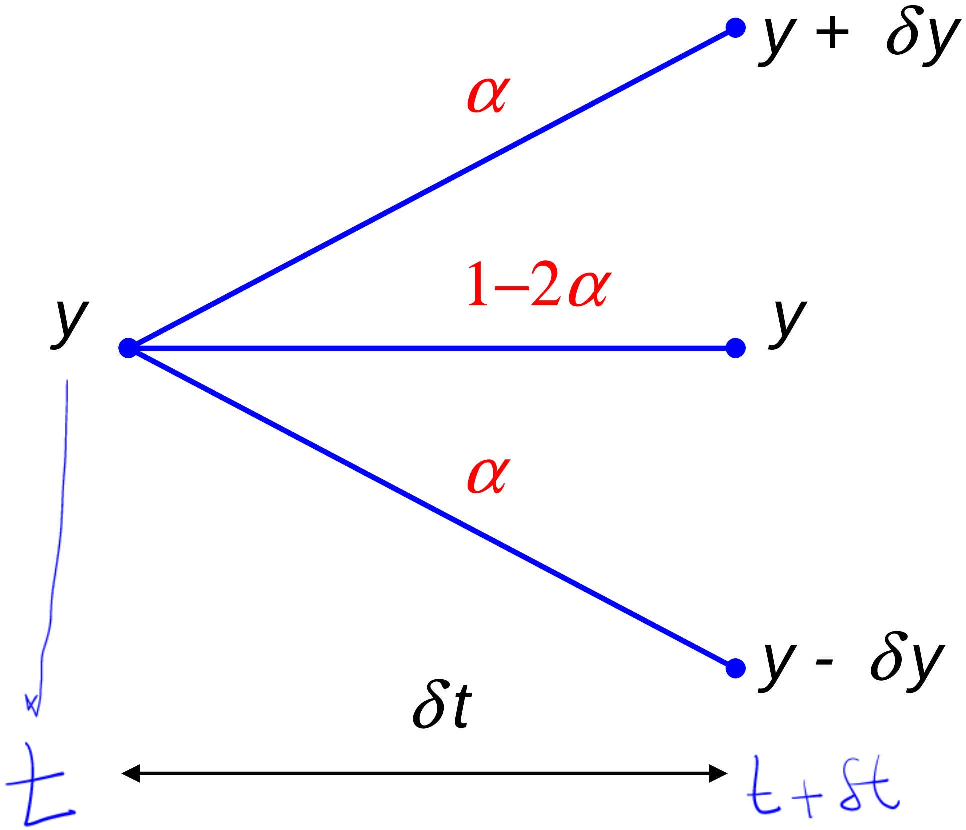
(s,t) $\quad$ (s,t) $\quad$ fixed (s',t') $\quad$ final state

y'

(s,t) $\quad$ (s^*,t) $\quad$ t'

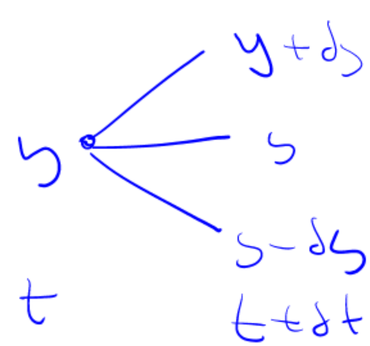
The derivation uses the trinomial random walk directly as drawn here.





s'

t'



The justification for the relationship between the probabilities of being at the four ‘nodes’ is more subtle than in the derivation of the forward equation.

So, let’s look at a concrete example.

At 7pm you are in the office. (This is the point (y, t).)

$y + \delta s$

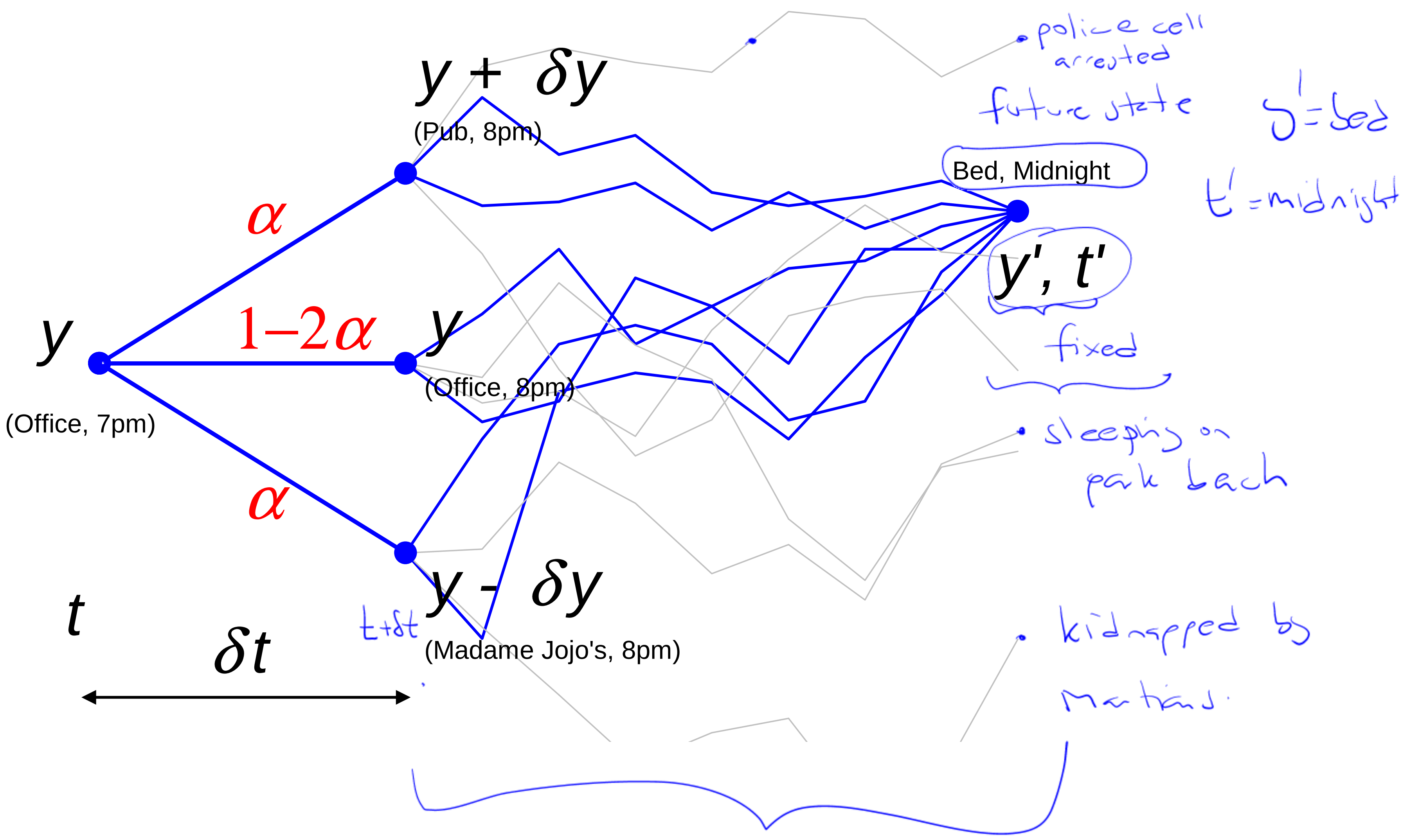
At 8pm you will be at one of three places: The Pub; Still at the office; Madame Jojo's. (These are the points $(y + \delta y, t + \delta t)$, $(y, t + \delta t)$ and $(y - \delta y, t + \delta t)$.)

Pub

office

MJJ

We are going to look at the probability that at midnight you are tucked up in bed. (This is the point (y', t').)


y + δy
(Pub, 8pm)
α
y
1–2α
y
(Office, 7pm)
(Office, 8pm)
α
y - δy
(Madame Jojo's, 8pm)
t
δt
t+δt
police cell
arrested
future state
y' = bed
Bed, Midnight
t' = midnight
y', t'
fixed
sleeping on
park bench
kidnapped by
Martians.

Remember that $p(y, t; y', t')$ represents the probability of being at the future point (y', t'), bed at midnight, given that you started at (y, t), the office at 7pm.

You can only get to the bed at midnight via either the pub, the office or Madame Jojo's at 8pm.

What happens after 8pm doesn't matter (you may not even remember!), we are only concerned with the probability that you are in bed at midnight, not how you got there.

In the previous figure there are lots of different paths, only the ones ending up in bed are of interest to us.

In words:

The probability of going from the office at 7pm to bed at midnight is the probability of going to the pub from the office and then to bed at midnight plus the probability of staying in the office and then getting to bed at midnight plus the probability of going to Madame Jojo's from the office and then to bed at midnight.

In symbols we can write this as

$$p(y,t;y',t') = \alpha\, p(y+\delta y, t+\delta t; y', t')$$
$$+(1-2\alpha)p(y,t+\delta t;y',t') + \alpha\, p(y-\delta y, t+\delta t; y', t').$$

The Taylor series expansion leads to the **backward Kolmogorov equation**.

Expand the above using a T.S.E

Use earlier scenario cases

The end result is

$$\frac{\partial p}{\partial t} + c^2 \frac{\partial^2 p}{\partial y^2} = 0.$$

Very basic scenario

Exactly the same as the forward equation, but with a sign change.

(The sign change makes all the difference between a forward and a backward diffusion equation.)

asset prices

Warning: More general random walks lead to slightly more complicated forward and backward equations, and their relationship is no longer as simple as a change of sign.

Generalization

We will look at more general random walks in later lectures. But why would we want to consider them?

$\boxed{\tau = T - t}$

- Financial quantities are more interesting than the trinomial model here.

- At the very least, equity prices cannot go negative, unlike the variable y here. (r can however)

- We might need different models for different financial quantities, equities, interest rates,. . .

Part III

Similarity solutions

Solve Forward Eqⁿ

That was our first partial differential equation.

As a general rule, we are not going to spend much time finding explicit solutions to equations—our emphasis will be on number crunching—but it is well worth looking at the forward diffusion equation in some detail. In particular, it is helpful to solve the equation in a simple case because

- it illustrates a very useful technique, similarity solutions
- it highlights the important role that the normal distribution plays

The equation to be solved is

To solve combine existing variables to create a new variable

$p(y', t')$

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial {y'}^2}.$$

p is a transi- density function

This equation has an infinite number of solutions. It has different solutions for different **initial conditions** and different **boundary conditions**

The initial condition tells you how the solution starts off. We must specify p as a function of y' at some point in time, t'.

Boundary conditions tell you how the function behaves on specified y' boundaries. Diffusion equations typically need two boundary conditions.

Warning: We are now going to do lots of (very fast) manipulations of functions, including solving ordinary differential equations. Do not panic!

We are now going to find a very simple solution. It is very simple and very special because unlike most solutions of the diffusion it does not depend on *two* independent variable y' and t' but on a combination of them.

Let us seek a solution of the form

p factored into a time dep. quantity t'^a and a new fn. $f\left(\frac{y'}{t'^b}\right)$

ansatz

$p = p(y', t')$

$$p = t'^a f\left(\frac{y'}{t'^b}\right).$$

where $\xi = \frac{y'}{t'^b}$ is a new variable, i.e R.V

Here a and b are constants.

Note that f is a function of only the one variable.

ξ - R.V $\quad f(\xi)$ p.d.f

$ay'' + by' + cy = 0 \qquad y = e^{\lambda x}$

$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$ $\xi = y' t'^{-b}$ $\frac{\partial \xi}{\partial y'} = \underline{t'^{-b}}$; $\frac{\partial \xi}{\partial t'} = -b y' t'^{-b-1}$

From

a, b to be determined

$$p = t'^a f\left(\frac{y'}{t'^b}\right) = \underline{t'^a f(\xi)}$$

we have $\frac{\partial p}{\partial y'} = t'^a \frac{\partial}{\partial y'} f(\xi) = t'^a \frac{df}{d\xi} \frac{\partial \xi}{\partial y'} = t'^{a-b} \frac{df}{d\xi}$

$$\frac{\partial p}{\partial y'} = t'^{a-b} \frac{df}{d\xi}$$

where

$$\xi = \frac{y'}{t'^b}.$$

Note: Derivatives of f are ordinary, not partial, since f only has one argument.

$p = t'^a f(\xi)$ $\quad \xi = y' t'^{-b}$

Then $\quad \dfrac{\partial \xi}{\partial t'} = -b y' t'^{-b-1}$

$$\frac{\partial^2 p}{\partial y'^2} = t'^{a-2b} \frac{d^2 f}{d\xi^2}.$$

goes in RHS of PDE

Also

$$\frac{\partial p}{\partial t'} = a t'^{a-1} f(\xi) - b y' t'^{a-b-1} \frac{df}{d\xi}.$$

put in LHS of

$$\frac{\partial p}{\partial t'} = \underbrace{a t'^{a-1} f(\xi)}_{} + \underbrace{t'^{a} \frac{df}{d\xi} \frac{\partial \xi}{\partial t'}}_{\text{product}}$$

$$\frac{\partial p}{\partial t'} = c^2 \frac{\partial^2 p}{\partial y'^2}$$

$P = t^{-1/2} f\left(\frac{y'}{\sqrt{t}}\right)$

Let's substitute these into our partial differential equation and see what happens.

$$at'^{a-1} f(\xi) - by' t'^{a-b-1} \frac{df}{d\xi} = c^2 t'^{a-2b} \frac{d^2 f}{d\xi^2}.$$

Set $a - 1 = a - 2b \Rightarrow \boxed{b = \frac{1}{2}}$

$\frac{y'}{t'^b} = \xi$

$y' t'^{a-b-1}$

$y' t'^{-b} \quad \xi$

$\xi t'^{a-1}$

Now all t' terms vanish

$$a f(\xi) - \frac{1}{2} \xi \frac{df}{d\xi} = c^2 \frac{d^2 f}{d\xi^2}$$

Let's cancel some t's and write $y' = {t'}^b \xi$:

$$af(\xi) - b\xi \frac{df}{d\xi} = c^2 {t'}^{-2b+1} \frac{d^2 f}{d\xi^2}.$$

So far we have only been fooling around 'changing variables.' The next step is important.

The left-hand side of this equation is only a function of ξ, whereas the right-hand side depends on both ξ and t'. This is only possible if the right-hand side is also independent of t'.

And this is only possible if $b = \frac{1}{2}$.

If we can solve

$$af(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2f}{d\xi^2}$$

then we have found a solution of our original equation in the form

$$\underbrace{p = t'^{a} f\left(\frac{y'}{t'^{b}}\right)}_{*} \qquad \boxed{b = \frac{1}{2}}$$

$$p = t'^{a} f\left(\frac{y'}{\sqrt{t'}}\right).$$

And this isn't just a single solution, it is a whole family of solutions because we can choose the constant a.

However, for our present problem, only one value of a is relevant.

Remember that p represents a probability. That means that its integral must be one:

See my scribbles on previous slide

$$\boxed{\int_{-\infty}^{\infty} p(y',t')\,dy' = 1} = \int_{-\infty}^{\infty} t'^a f\left(\frac{y'}{\sqrt{t'}}\right) dy'.$$

$\circledast$

must hold $\forall$ time

$$t'^a \int_{\mathbb{R}} f\left(\frac{y'}{\sqrt{t'}}\right) dy'$$

Change variables by writing $y' = t'^{1/2} u$ to get

$$u = \frac{y'}{\sqrt{t'}} \qquad \sqrt{t'}\,du = dy'$$

$t'^{1/2}$

$$t'^{a+\frac{1}{2}} = t'^{0}$$

$$\boxed{t'^{a+1/2}} \underbrace{\int_{-\infty}^{\infty} f(u)\ du = 1.}$$

Conclusion? This is only possible if $\boxed{a = -\frac{1}{2}.}$

Our ordinary differential equation is now

$$-\frac{1}{2}f(\xi) - \frac{1}{2}\xi\frac{df}{d\xi} = c^2\frac{d^2f}{d\xi^2}.$$

$$-\frac{1}{2}\left[f(\xi) + \xi\frac{df}{d\xi} \right] = c^2 \frac{d^2f}{d\xi^2}$$

This can be written as

$$-\frac{1}{2}\frac{d\,(\xi\ f(\xi))}{d\xi} = c^2\frac{d^2f}{d\xi^2}.$$

exact derivative $\frac{d}{d\xi}\left(\xi f(\xi) \right)$

(That was lucky!)

$$-\frac{1}{2}\frac{d}{d\xi}\left(\xi f(\xi) \right) = c^2 \frac{d^2f}{d\xi^2}$$

Integrate both sides

$$\xi \to \infty \quad \begin{cases} f(\xi) \to 0 \\ \frac{df}{d\xi} \to 0 \end{cases}$$

$$-\frac{1}{2}\left(\xi f(\xi) \right) = c^2\frac{df}{d\xi} + \text{constant}$$

This can be integrated once to give

$\therefore$ Const $= 0$

$$-\frac{1}{2}\xi\, f(\xi) = c^2\frac{df}{d\xi}.$$

$e^{A} \times e^{B} = e^{A+B}$

$e^{-\xi^2/4c^2} \times e^{C} \Rightarrow A$

$e^{-\xi^2/4c^2 + \text{Const}}$

(There's an arbitrary constant of integration that could go in here but for the answer we want this is zero.)

$\int \frac{1}{x}\, dx$ $\qquad$ $\int x \to \frac{x^2}{2}$

This can be rewritten as

$\int \frac{df}{f} = -\frac{1}{2c^2}\int \xi\, d\xi$

$$c^2\frac{d\,(\ln f)}{d\xi} = -\frac{1}{2}\xi$$

Now take exp. of both sides

$\log f(\xi) = -\frac{\xi^2}{4c^2} + \text{Const.}$ $\qquad$ $f(\xi) = A\, e^{-\xi^2/4c^2}$ $\quad$ A is a normalising const.

$\therefore \int_{\mathbb{R}} f(\xi) = 1$

. . . and integrated again to give

$$\ln f(\xi) = -\frac{\xi^2}{4c^2} + \text{ an arbitrary constant of integration}$$

$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ standard result

or

$$f(\xi) = A \exp\left(-\frac{\xi^2}{4c^2}\right).$$

The constant A is chosen so that the integral of f is one.

$A\int_{\mathbb{R}} e^{-\xi^2/4c^2}\,d\xi = 1$ set $x = \frac{\xi}{2c} \Rightarrow 2c\,dx = d\xi$

$2cA\underbrace{\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx}}_{\sqrt{\pi}} = 1$

$2c\sqrt{\pi}\,A = 1$ $\therefore A = \frac{1}{2c\sqrt{\pi}}$

$f(\xi) = \frac{1}{2c\sqrt{\pi}} \exp\left[-\frac{\xi^2}{4c^2}\right]$

Recall $\xi = y'/\sqrt{t'}$

Cutting to the chase, and going back to the original t' and y', we have

Replacing ξ by $\frac{y'}{\sqrt{t'}}$ and also $p = \frac{1}{\sqrt{t'}} f(\xi)$

$$p = \frac{1}{\underbrace{2c\sqrt{\pi\, t'}}}\exp\left(-\frac{y'^2}{4c^2t'}\right).$$

Do you recognize this expression?

$$e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} \qquad e^{-\frac{1}{2}\frac{(y'-0)^2}{2c^2t'}}$$

$y' \sim N(0, 2c^2t')$

It is very like

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\phi^2}{2}\right).$$

It is the probability density function for the normal distribution!

So

- y' is normally distributed
- with mean of zero
- and standard deviation of $c\sqrt{2t'}$ Variance = $\cancel{}\ 2c^2t'$

Minor generalization... suppose that y' has value y at time t then we have

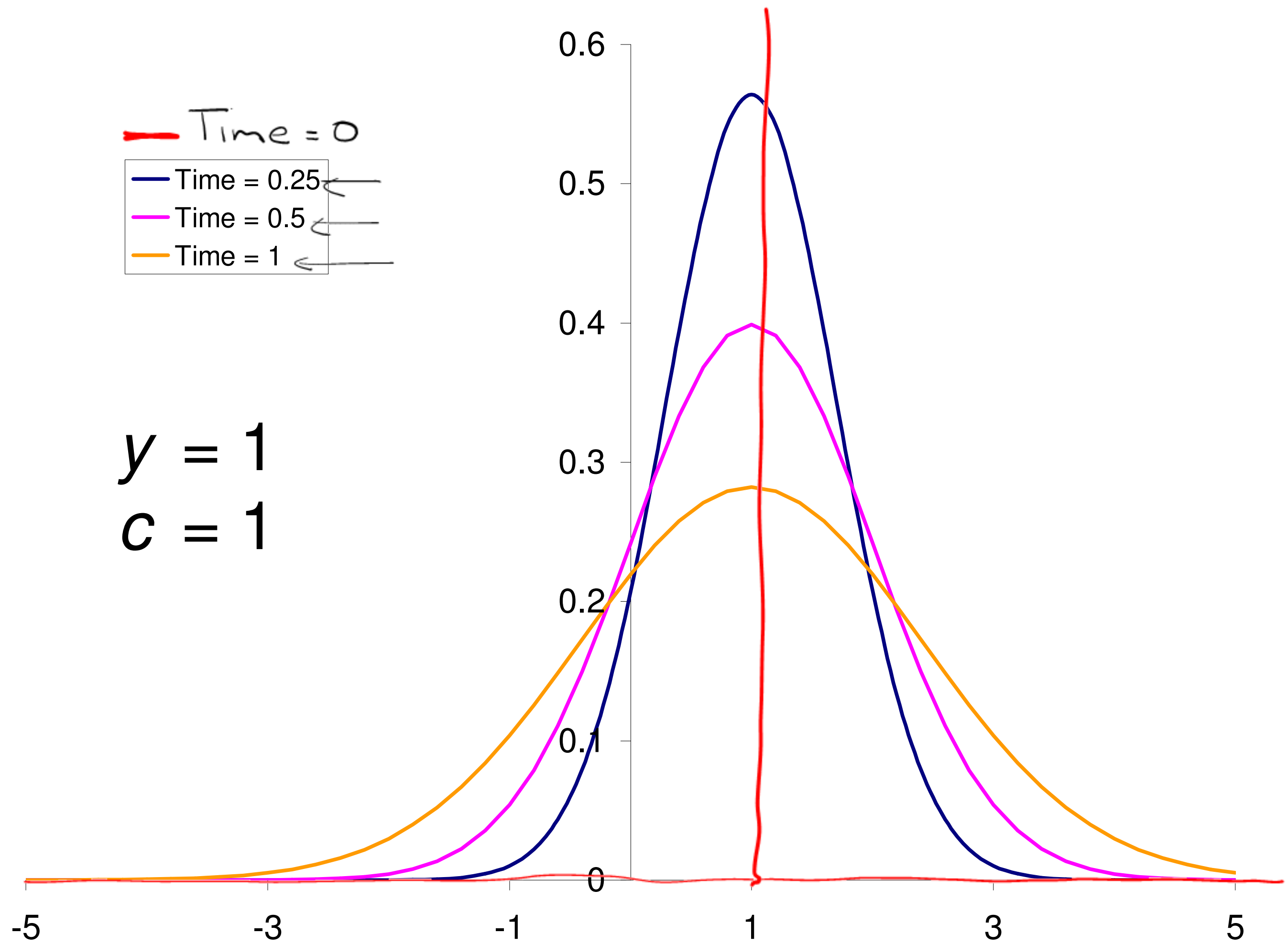
Now incorporate the I.C. i.e Random walker is y at t.

$$p(y,t;y',t') = \frac{1}{2c\sqrt{\pi(t'-t)}} \exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right).$$

And this is our transition probability density function for our random walk!

When $y' = y$ exp term switches off

leaving $\displaystyle p = \lim_{t' \to t} \frac{1}{2c\sqrt{\pi(t'-t)}}$ $\Big\}$ $t' \to t$?



$y' = y = 1$

$\tau = t' - t \qquad \tau \to 0$

Summary

Please take away the following important ideas

- Taylor series as a way of finding out values of a function knowing only local information (such as gradients, and higher derivatives). It is an approximation only.

- Random walks have associated differential equations for their probability density functions, and are naturally related to the normal distribution.

- Generally partial differential equations are hard to solve explicitly, but sometimes they can be simplified to ordinary differential equations.