1

Consider:

$$p(y,t;y',t') = \frac{1}{2c\sqrt{\pi(t'-t)}}\exp\left(-\frac{(y'-y)^2}{4c^2(t'-t)}\right)$$

We can drop y, t (Substitute $y'-y, t'-t$ etc.) so we only need to work on:

$$p(y',t') = \frac{1}{2c\sqrt{\pi t'}}\exp\left(-\frac{y'^2}{4c^2t'}\right)$$

Calculating the necessary derivatives for the FKE, we get:

$$\begin{aligned}
\frac{\partial p}{\partial t'} &= \frac{1}{2c\sqrt{\pi}}\cdot\left(-\frac{1}{2\sqrt{t'^3}}\right)\exp\left(-\frac{y'^2}{4c^2t'}\right) + \frac{1}{2c\sqrt{\pi t'}}\exp\left(-\frac{y'^2}{4c^2t'}\right)\cdot\left(\frac{y'^2}{4c^2t'^2}\right) \\
&= \frac{1}{2c\sqrt{\pi t'}}\exp\left(-\frac{y'^2}{4c^2t'}\right)\cdot\left(-\frac{1}{2t'}+\frac{y'^2}{4c^2t'^2}\right) \\
&= p\cdot\left(\frac{-2c^2t'+y'^2}{4c^2t'^2}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial p}{\partial y'} &= p\cdot\left(-\frac{y'}{2c^2t'}\right) \\
\frac{\partial^2 p}{\partial y'^2} &= p\cdot\left(-\frac{y'^2}{4c^4t'^2}\right) - \frac{p}{2c^2t'} \\
&= p\left(\frac{y'^2-2c^2t'}{4c^4t'^2}\right) \\
&= \frac{1}{c^2}\left[p\left(\frac{y'^2-2c^2t'}{4c^2t'^2}\right)\right] \\
&= \frac{1}{c^2}\frac{\partial p}{\partial t'}
\end{aligned}$$

Which is the FKE.

2

We consider a symmetric random walk. Consider sufficiently small step sizes δy and δt. Then we have two previous points to consider: $(y-\delta y, t-\delta t), (y+\delta y, t-\delta t)$. Let $p(x,s;y,t) = p(y,t)$ be the transition density function. We expand $p(y-\delta y, t-\delta t), p(y+\delta y, t-\delta t)$ w.r.t (y,t):

$$\begin{aligned}
p(y+\delta y, t-\delta t) &= p(y,t) + \delta y\frac{\partial p}{\partial y} - \delta t\frac{\partial p}{\partial t} + \frac{1}{2}\delta^2 y\frac{\partial^2 p}{\partial y^2} + o(\delta t, \delta y^2) \\
p(y-\delta y, t-\delta t) &= p(y,t) - \delta y\frac{\partial p}{\partial y} - \delta t\frac{\partial p}{\partial t} + \frac{1}{2}\delta^2 y\frac{\partial^2 p}{\partial y^2} + o(\delta t, \delta y^2)
\end{aligned}$$

We know that we want the expected position after the step to be y, so we take expectations:

$$\begin{aligned}
p(y,t) &= \frac{1}{2}\left(p(y+\delta y, t-\delta t) + p(y-\delta y, t-\delta t)\right) \\
p &= \frac{1}{2}\left(2p - 2\delta t\frac{\partial p}{\partial t} + \delta y^2\frac{\partial^2 p}{\partial y^2}\right) \\
\frac{\partial p}{\partial t} &= \frac{1}{2}\frac{\delta y^2}{\delta t}\frac{\partial^2 p}{\partial y^2}
\end{aligned}$$

For this PDE to be well-defined, we would require $\frac{\delta y^2}{\delta t} \sim O(1)$. We can do this as our definition of the symmetric random walk was independent of any values $\delta t, \delta y$ could take, apart from having to be sufficiently small. Therefore, it would have to hold true as $\delta t, \delta y \to 0$. We are also free to set $\delta t = \delta y^2$, as the limiting behaviour is the only concern, giving us:

$$\frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial y^2}$$

3

Let's solve the above equation. Assume we have:

$$p(y,t) = \frac{1}{\sqrt{t}}f(\eta)$$

Where $\eta = \frac{y}{\sqrt{t}}$. We can then find the necessary partial derivatives:

$$\begin{aligned}
\frac{\partial p}{\partial t} &= -\left(\frac{1}{2\sqrt{t^3}}f(\eta) + \frac{y}{2t^2}f'(\eta)\right) \\
\frac{\partial p}{\partial y} &= \frac{1}{t}f'(\eta) \\
\frac{\partial^2 p}{\partial y^2} &= \frac{1}{\sqrt{t^3}}f''(\eta)
\end{aligned}$$

Plugging them into the equation:

$$\begin{aligned}
-\left(\frac{1}{2\sqrt{t^3}}f(\eta) + \frac{y}{2t^2}f'(\eta)\right) &= \frac{1}{2\sqrt{t^3}}f''(\eta) \\
f(\eta) + \frac{y}{\sqrt{t}}f'(\eta) &= -f''(\eta) \\
f(\eta) + \eta f'(\eta) &= -f''(\eta)
\end{aligned}$$

This is an ODE, which we can start solving with use of the product rule and integrating:

$$\begin{aligned} f(\eta) + \eta f'(\eta) &= -f''(\eta) \\ \frac{d}{d\eta}(\eta f(\eta)) &= -f''(\eta) \\ \implies \eta f(\eta) &= -f'(\eta) + K \end{aligned}$$

For some constant K. Here, we consider the limiting behaviour of f, f' as $\eta \to \infty$. We know for a fact that as a pdf, $p, p' \to 0$ as $y \to \infty$. As η is just a rescaling of y, $f, f' \to 0$. We will also need: $f(\eta) \sim O(\eta^{-1})$. However, we cannot have $\lim\limits_{\eta\to\infty} \frac{f(\eta)}{\eta} = L$ for $L \neq 0$, as that would imply that $\int_1^\infty f(\eta)\, d\eta = \int_1^\infty L\eta^{-1}\, d\eta = \infty$, contradicting it's pdf status. Hence we require $f(\eta) \sim o(\eta^{-1})$ which implies that $\eta f(\eta) \to 0$. Hence, $K = 0$ and we can proceed to solve the remaining equation:

$$\begin{aligned} \eta f(\eta) &= -f'(\eta) \\ \int \frac{df}{f} &= \int -\eta\, d\eta \\ \ln f &= -\frac{\eta^2}{2} + C \\ f(\eta) &= Ae^{-\frac{\eta^2}{2}} \end{aligned}$$

For some constant A. Recall that $p = \frac{1}{\sqrt{t}} f(y/\sqrt{t})$ is a probability distribution, therefore, we use the substitution $u = y/\sqrt{2t}$ and $\int_{\mathbb{R}} p = 1$ to get our constant:

$$\begin{aligned} 1 &= \int_{\mathbb{R}} \frac{A}{\sqrt{t}} e^{-\frac{y^2}{2t}}\, dy \\ 1 &= A\sqrt{2} \int_{\mathbb{R}} e^{u^2}\, du \\ 1 &= A\sqrt{2\pi} \\ A &= \frac{1}{\sqrt{2\pi}} \end{aligned}$$

By the standard Gaussian integral. So, we finally have the solution:

$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$$