Stochastic Calculus and Itô's Lemma

In this lecture. . .

* Construct a Brownian Motion (B.M)

* Extending TSE to Itô's lemma

* Stoch. Diff Eqⁿ's

By the end of this lecture you will be able to

- understand where Brownian motion and diffusion processes come from

- manipulate functions of $\underbrace{\text{random variables}}_{\text{a B.M.}}$,

Introduction

Stochastic calculus is very important in the mathematical modeling of financial processes. This is because of the underlying (assumed) random nature of financial markets.

$x \in \mathbb{R}$

Stochastic calculus: A motivating example

$\mathbb{R}$

Toss a coin. Every time you throw a head you receive \$1, every time you throw a tail you pay out \$1. In the experiment below the sequence was THHTHT, and you finished even.

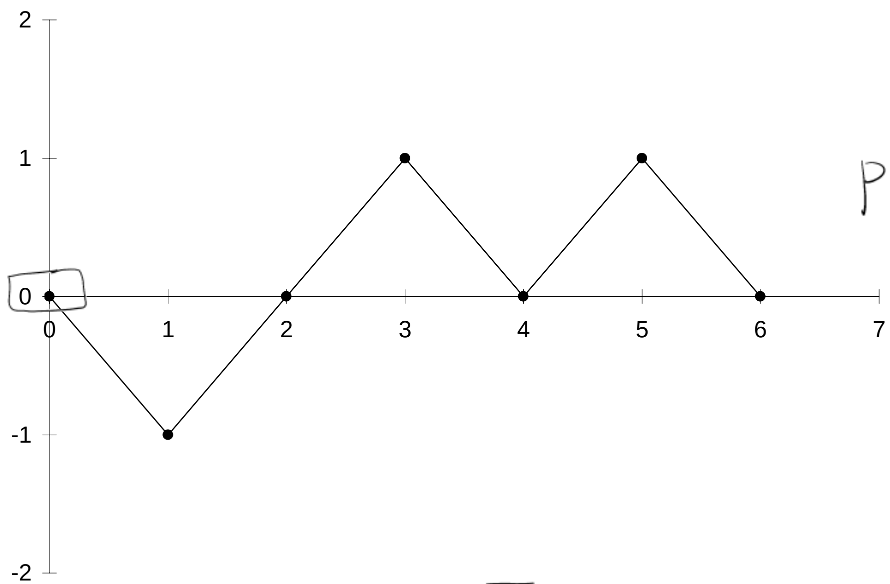
$X : \omega \in \Omega \to \mathbb{R}$

outcome

set of all possible outcomes.

starting condition zero money

$R_i : \begin{cases} +1 & \text{Head} \\ -1 & \text{Tail} \end{cases}$



$p(+1) = \frac{1}{2} = p(-1)$

$\mathbb{E}[X] = \sum x_i f_i = \mu$ $\mathbb{V}[X] = \sum x_i^2 f_i - \mu^2$

Certificate in Quantitative Finance

- R_i denotes the random amount, either \$1 or -\$1, you make on the ith toss:

Indep. of coin tosses

$\mathbb{V}[R_i]$

$$E[R_i] = 0, \quad E[R_i^2] = 1 \quad \text{and} \quad \boxed{E[R_iR_j] = 0.}$$

$= \frac{1}{2}(+1) + (-1)\frac{1}{2}$ $\qquad (+1)^2\frac{1}{2} + (-1)^2\frac{1}{2}$ $\qquad = \mathbb{E}(R_i)\mathbb{E}(R_j)$

In this example it doesn't matter whether or not these expectations are conditional on the past. In other words, if you threw five heads in a row it does not affect the outcome of the sixth toss.

- Introduce S_i to mean the total amount of money you have won up to and including the ith toss so that

$\boxed{R_0 = 0}$

new R.V.

$$S_i = \sum_{j=1}^{i} R_j; \quad \text{subject to } S_0 = 0$$

Later on it will be useful if we have $S_0 = 0$, i.e., you start with no money.

If we now calculate expectations of S_i it *does* matter what information we have. If we calculate expectations of future events before the experiment has even begun then

$$E\left[\sum_{j=1}^{i} R_j\right] = \sum_{j=1}^{i} E[R_j] \qquad V[S_i]$$

$$E[S_i] = 0 \quad \text{and} \quad E[S_i^2] = E[R_1^2 + 2R_1R_2 + \cdots] = i.$$

On the other hand, suppose there have been five tosses already, can we use this information and what can we say about expectations for the sixth toss?

$$S_i^2 = \left(\sum_{j=1}^{i} R_j\right)^2 = R_1^2 + \ldots + R_i^2 + \underbrace{2R_1R_2 + 2R_1R_3}_{\text{expected value of these is zero}}$$

$$E[R_1^2] + E[R_2^2] + \ldots$$

- This is the **conditional expectation**.

$$E\left[\left(\sum_{j=1}^{i} R_j\right)^2\right] = E\left[\sum_{j=1}^{i} R_j^2\right]$$

The expectation of S_6 conditional upon the previous five tosses gives

$$E_m[S_{m+1} | \ldots] \longrightarrow E_5[S_6|\overbrace{R_1, \ldots, R_5}^{S_5}] = S_5.$$

Quadratic variation

The **quadratic variation** of the random walk is defined by

a new way to measure changes

$$Q = \sum_{j=1}^{i} \underbrace{\left(S_j - S_{j-1}\right)}_{\pm 1}^2. \qquad \underbrace{\sum_{j=1}^{i} \left(S_j - S_{j-1}\right)}$$

Because you either win or lose an amount \$1 after each toss, $|S_j - S_{j-1}| = 1$. Thus the quadratic variation is always i:

Variation

$$V^2 = Q = \sum_{j=1}^{i} \underbrace{\left(S_j - S_{j-1}\right)}^2 = i. \qquad i \times 1 \qquad \boxed{S_{12}}$$

We are going to use the coin-tossing experiment for one more demonstration. And that will lead us to a continuous-time random walk.

0 t

Brownian motion

Now introduced time dep.

$S_0 = 0$. Change the rules of the coin-tossing experiment slightly.

Change bet size $R_i : \{$

First of all restrict the time allowed for the six tosses to a period t, so each toss will take a time $t/6$. Second, the size of the bet will not be \$1 but $\sqrt{t/6}$.

play game in time t

$\therefore$ each toss $t/6$ bet size $\sqrt{t/6}$

This new experiment clearly still possesses both the Markov and martingale properties, and its quadratic variation measured over the whole experiment is

$$R_i : \begin{cases} \sqrt{\frac{t}{6}} & H \\ -\sqrt{\frac{t}{6}} & T \end{cases}$$

$$\pm\sqrt{\frac{t}{6}} \qquad Q = \sum_{j=1}^{6} \underbrace{\left(S_j - S_{j-1}\right)}_{\sqrt{t/6}}^2 = 6 \times \left(\sqrt{\frac{t}{6}}\right)^2 = t.$$

finite quantity

Change the rules again, to speed up the game.

Game still played in time t. n tosses $\Rightarrow$ each toss $\frac{t}{n}$

- We will have n tosses in the allowed time t, with an amount $\sqrt{t/n}$ riding on each throw. Bet size $= \pm\sqrt{\frac{t}{n}}$

next week

Again, the Markov and martingale properties are retained and the quadratic variation is still

Total time t

each toss t/n

Winnings $\sqrt{t/n}$

$$Q = \sum_{j=1}^{n} \underbrace{\left(S_j - S_{j-1}\right)}_{\pm\sqrt{t/n}}^2 = n \times \left(\sqrt{\frac{t}{n}}\right)^2 = t$$

$= \cancel{n} \times \frac{t}{\cancel{n}}$

finite variation

indep. of n.

Now make n larger and larger. This speeds up the game, decreasing the time between tosses, with a smaller amount for each bet. But the new scalings have been chosen very carefully, the time step is decreasing like n^{-1} but the bet size only decreases by $n^{-1/2}$.

$\frac{t}{n}$ $\pm\sqrt{\frac{t}{n}}$

time $t \times \frac{1}{n}$

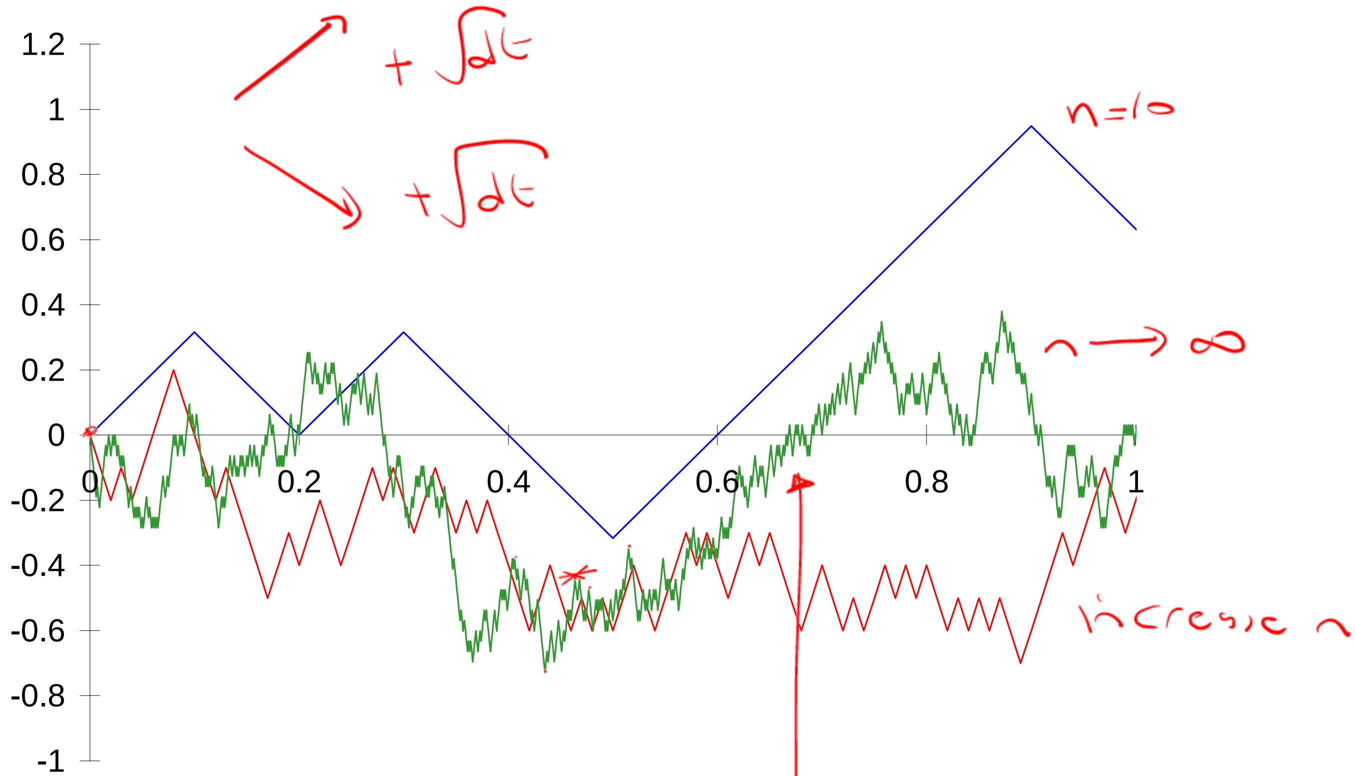
bet $\sqrt{\frac{t}{n}}$ $\sqrt{t} \cdot \sqrt{\frac{1}{n}}$

$\sqrt{\frac{t}{n}}$

$\sqrt{dt}$

$\uparrow \sqrt{\frac{t}{n}}$ $\sqrt{dt}$

$\downarrow \sqrt{\frac{t}{n}}$ $\sqrt{dt}$



B.M. cts everywhere

A series of coin tossing experiments. differentiable nowhere.

discrete S_i
cts $S(t)$ S_t

As we go to the limit $n = \infty$, the resulting random walk stays finite. It has an expectation, conditional on a starting value of zero, of

We fixed it

$$\text{mean} = E[S(t)] = \underline{0} \xrightarrow{\lim_{n\to\infty}} E\left[\sum_{i=1}^{n} R_i\right] \quad \underbrace{\lim_{n\to\infty} \sum_{i=1}^{n} E(R_i)}$$

$$\sqrt{\frac{t}{n}}\left(\frac{1}{2}\right) + \left(-\sqrt{\frac{t}{n}}\right)\left(\frac{1}{2}\right) = 0$$

and a variance

$$\boxed{\text{Variance:} \quad E[S(t)^2] = t.}$$

We use $S(t)$ to denote the amount you have won or the value of the random variable after a time t.

$$E\left[\sum_{i=1}^{n} R_i^2\right] \qquad \sqrt{\frac{t}{n}}$$

Conclusion

- The limiting process for this random walk as the time steps go to zero is called **Brownian motion**, and we will denote it by $X(t)$. X_t, W_t, B_t

$$\sum_{i=1}^{n} \left(\sqrt{\frac{t}{n}}\right)^2 \times \frac{1}{2} + \left(-\sqrt{\frac{t}{n}}\right)^2 \frac{1}{2}$$

$n \to \infty$ $\frac{t}{n}$ becomes smaller

$$\sum_{i=1}^{n} \frac{t}{n} \qquad n \times \frac{t}{n} = t$$

Having built up the idea and properties of Brownian motion from a series of experiments, we can discard the experiments, to leave the Brownian motion that is defined by its properties. These properties will be very important for our financial models.

Properties of a Standard Brownian motion

* $X_0 = 0$ almost surely

* B.M $t \longmapsto X_t$ is continuous everywhere

* X_t is a Normally distributed R.V $X_t \sim N(0, t)$

* A B.M has indep. increments. $dX_t = X_{t+dt} - X_t$

If $t > s$ $X_t - X_s \sim N(0, |t-s|)$

$dX_t \sim N(0, dt)$

$E[dX_t] = 0$

$V[dX_t] = dt$

Very Important Notation

$$\frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}\right]$$

We have seen X as the 'end result' of a random walk, up to some time t.

We will often work with the amount by which X changes from moment to moment.

$$\longrightarrow dX_t = X_{t+dt} - X_t = X(t+dt) - X(t) \sim N(0, \underbrace{|t_i - t_{i-1}|}_{dt})$$

- Think of dX as being an increment in X, i.e. a Normal random variable with mean zero and standard deviation $dt^{1/2}$.

$X_t - X_s$ trans' pdf: $$\frac{1}{\sqrt{2\pi|t-s|}} \exp\left[-\frac{1}{2}\frac{(x_t-x_s)^2}{|t-s|}\right]$$

$\sigma^2 =$ Variance

$N(0, dt)$

$\sigma = \sqrt{t-s}$ $\quad \sigma^2 = |t-s|$

Functions of stochastic variables and Itô's lemma

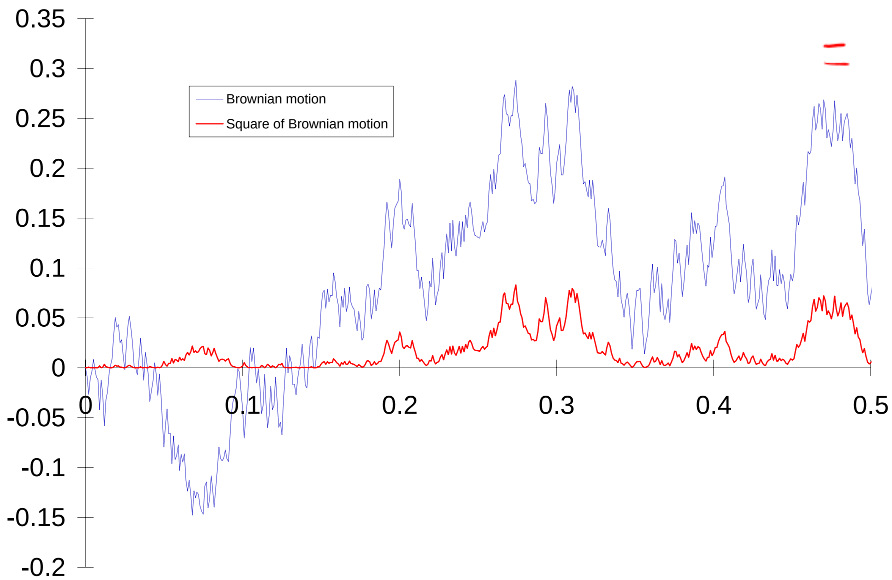
Now we'll see the idea of a function of a stochastic variable. Below is shown a realization of a Brownian motion $X(t)$ and the function $F(X) = X^2$.

$$\frac{dX}{dt} = \lim_{dt \to 0} \frac{X(t+dt) - X(t)}{dt}$$

$$= \lim_{dt \to 0} \frac{\sqrt{dt}}{dt}$$

$$= \lim_{dt \to 0} \frac{1}{\sqrt{dt}} \quad \text{does not exist}$$

$\sqrt{dt} \to 0$?



Whenever we have functions of a variable it is natural to want to know how to differentiate and manipulate these functions.

What are the rules of calculus when variables are stochastic?

$f = x^2 \qquad \frac{df}{dx} = 2x \quad \therefore \quad df = 2x\,dx$

$F(X) = X^2 \qquad dF = 2X\,dX \quad ?$

$X_t \sim O(\sqrt{dt}) \qquad \sqrt{\frac{t}{n}} \qquad dX \sim O(\sqrt{dt})$

$dX^2 \equiv (dX)^2 \quad \sqrt{dt} \qquad dX^2 \sim O(dt)$

$X_t \qquad dX_t \qquad \lim_{dt \to 0} \underbrace{dX^2}_{} \longrightarrow \boxed{dt}$

t	X_t	dX
0		
1		
2		
3		
4		
5		
6		

$\int dX^2$

The first point to note is that in the stochastic world we really have two 'variables.'

These are time t and the Brownian motion X.

We are used to writing ordinary and partial differential equations in the form

$$\frac{dF}{d\cdot}$$

or

$$\frac{\partial F}{\partial\cdot}$$

where the quantities on the bottom are the independent variables.

So might expect something similar in the stochastic world.

We immediately hit a problem, however.

Because dX is of size $\sqrt{dt}$ it is much bigger than dt.

This means that we have to be careful whenever we think about gradients/slopes/derivatives/sensitivities, since these are limits as dt goes to zero.

For this reason, in the stochastic world we instead work with stochastic differential equations.

These take the form

$$dF = \cdots dt + \cdots dX.$$

So, what are the rules of calculus?

Since X is stochastic, so is F, and we can ask 'what is the stochastic differential equation for F?'

If $F(X) = X^2$ what is the equation for dF?

If $F = X^2$ is it true that $dF = 2X\ dX$?

No.

- The ordinary rules of calculus do not generally hold in a stochastic environment.

Then what are the rules of calculus?

We are going to throw caution to the wind, pretend that there are no problems or subtleties, use Taylor series. . . and see what happens!

Taylor Series . . . and Itô

$\exists$ fn. f = B.M $F(x)$

$X \longrightarrow X + dX$

If we were to do a naive Taylor series expansion of F, completely disregarding the nature of X, and treating dX as a small increment in X, we would get

1D T.S.E

$$F(X + dX) = F(X) + \frac{dF}{dX}dX + \tfrac{1}{2}\frac{d^2F}{dX^2}dX^2,$$

ignoring higher-order terms.

$dF = F(X+dX) - F(X)$ is called a differential

We could argue that $F(X + dX) - F(X)$ was just the 'change in' F and so

dt is very small

$$\boxed{dF = \frac{dF}{dX}dX + \tfrac{1}{2}\frac{d^2F}{dX^2}dX^2.}$$

In everyday Calculus $dx^2 << 1$ so we ignore it

This is *almost* correct.

$\boxed{dt}$

Because of the way that we have defined Brownian motion, and have seen how the quadratic variation behaves, it turns out that the dX^2 term isn't really random at all.

The dX^2 term becomes (as all time steps become smaller and smaller) the same as its average value, dt.

$$f = f(x)$$

$$\frac{df}{dx} = \lim_{dx \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$f(x+\delta x) = f(x) + \frac{df}{dx}\delta x + \cancel{\frac{1}{2}\frac{d^2f}{dx^2}\delta x^2}$$

$\delta x << 1 \Rightarrow$

δx^2 even smaller

$$\lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x} = \frac{df}{dx}$$

Taylor series and the 'proper' Itô are very similar. The only difference being that the correct Itô's lemma has a dt instead of a dX^2.

$F = X^2$ $\quad dF = 2X\,dX$?

Use Itô

drift

S.D.E

$$dF = \frac{dF}{dX}dX + \frac{1}{2}\frac{d^2F}{dX^2}dt$$

diffusion

$$= 2X\,dX + \frac{1}{2} \times 2\,dt = 2X\,dX + dt$$

$$dF = 2X\,dX + dt$$

- You can, with little risk of error, use Taylor series with the 'rule of thumb'

To do stochastic calculus without shedding tears

mean square limit

$$dX^2 = dt.$$

$\mathbb{E}[dX^2] = dt$

and in practice you will get the right result.

Let's get some intuition now, and then shortly we will do Itô's lemma properly!

$$dF = \frac{dF}{dX} dX + \frac{1}{2} \frac{d^2F}{dX^2} dt$$

Itô I

T.S.E fns of a deterministic Var.

Itô's lemma fns of a stochastic variable.

We can now answer the question, "If $F = X^2$ what is dF?" In this example

$\{Y_t : t \in \mathbb{R}^+\}$ family of R.Vs indexed with time

X_t or X

$$\frac{dF}{dX} = 2X \quad \text{and} \quad \frac{d^2F}{dX^2} = 2.$$

$\frac{d}{dX}(X) = 1$

Therefore Itô's lemma tells us that

$\frac{d}{dt}$ of X does not exist

$$dF = dt + 2X\, dX.$$

drift $= 1$ diffusion $= 2X$

This is an example of a **stochastic differential equation**, and more specifically it is the S.D.E for $F = X^2$

$F = X^n$ or $\sin X$ or e^X or $X^2 e^X + \log X$

 use Itô I

Stochastic differential equations

Stochastic differential equations are used to model random quantities, a stock price for example.

They have two parts to them, a **deterministic** and a **random**.

Suppose $F(t, X) = t^2 + X_t^n$ or $t e^{X_t + t^2}$ or $t^2 \log t X_t$

What is the S.D.E for the above functions?

2D T.S.E for $f(t, X_t)$ $\quad t \to t + dt, \quad X_t \to X_t + dX_t$

$$f(t+dt, X+dX_t) = f(t, X_t) + \frac{\partial f}{\partial t} dt + \frac{\partial F}{\partial X_t} dX_t + \frac{1}{2} \frac{d^2 F}{d X_t^2} dX_t^2 \to dt$$

$$dF = \underbrace{\left(\frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial X_t^2} \right)}_{\text{drift}} dt + \underbrace{\frac{\partial F}{\partial X_t}}_{\text{diffusion}} dX_t$$

Itô II

$$\frac{\partial F}{\partial t} = 2t \qquad \frac{\partial F}{\partial X} = n X^{n-1}$$

$$\frac{\partial^2 F}{\partial X^2} = n(n-1) X^{n-2}$$

Suppose we want to model a stock price as a random quantity. Let's use S to denote that stock price.

A stochastic differential equation for S would look something like this:

drift

diffusion

$$dS = \overbrace{\text{Deterministic}}^{\mu S\,dt} + \overbrace{\text{Random}}^{\sigma S\,dX}.$$

In words: "The change in the stock price has a predictable component and a random component."

More precisely

$$dS = \text{ Something } dt + \text{ Something else } dX.$$

The randomness is captured by the dX term.

But what are these 'somethings'?

In the S.D.E above S_t is our stochastic process.

dS is a differential for S_t or SDE for S_t

In the standard models they would be functions of S and time, t.

drift $f(S,t)$ scales with time step

time step

sqrt of time step.

$$dS = f(S,t)\, dt + g(S,t)\, dX.$$

diffusion $g(S,t)$ scales with square root of time-step.

The function $f(S,t)$ captures how the predictable bit of the stock model varies with S and t and the $g(S,t)$ function captures the randomness.

$dS = f\, dt$ or $\boxed{\frac{dS}{dt} = f(S,t)}$ O.D.E

S.D.E is an ODE plus a $\underbrace{\text{random component}}_{\text{a B.M.}}$

$$dS = f(S,t)\,dt + g(S,t)\,dX.$$

We sometimes call the $f(S,t)$ function the **growth rate** or the **drift**.

The $g(S,t)$ is related to the **volatility** of S.

f - drift / growth rate

g - diffusion / volatility

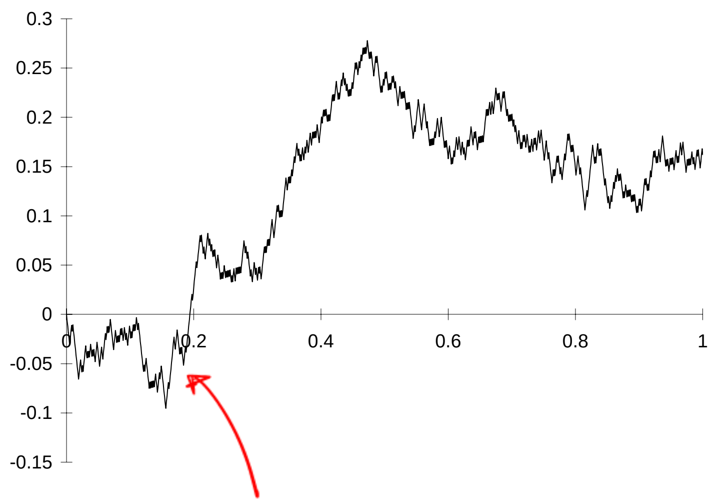
Some pertinent examples

The first example simple Brownian motion but with a drift:

drift $= \mu$

diffusion $= \sigma$

$$dS = \mu \, dt + \underbrace{\sigma \, dX}_{\text{random cpt.}}.$$



In this realization S has gone negative.

Our second example is similar to the above but the drift and randomness scale with S:

$ds = \mu s\, dt$

$\sigma = 0$ switches off randomness

$\int \frac{1}{s}\, ds = \mu \int dt$

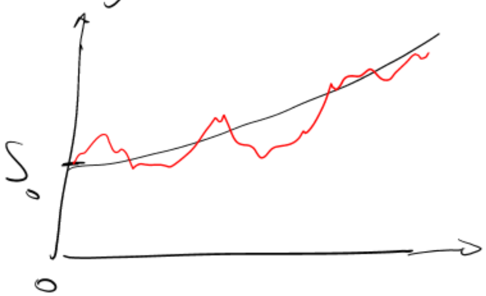
$$dS = \mu S\, dt + \sigma S\, dX.$$

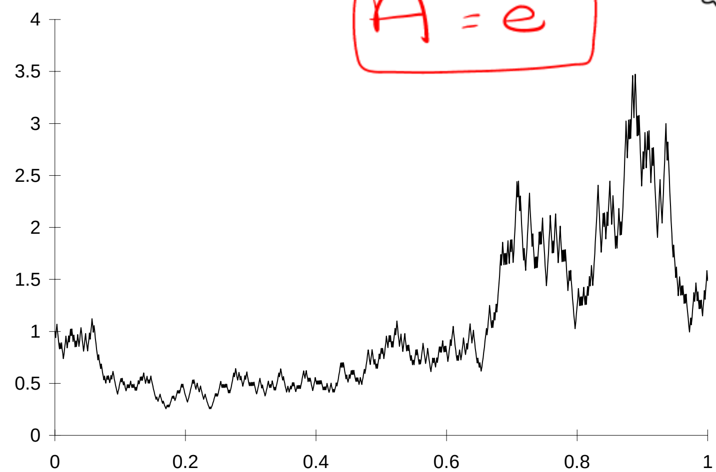
$\log s = \mu t + c \Rightarrow S_t = A e^{\mu t}$ $\quad \mu \to r$ $\quad S_t = A e^{rt}$

$\boxed{A = e^{c}}$

at time $t = 0$

$S_0 \Rightarrow S_t = S_0 e^{rt}$





If S starts out positive it can never go negative; the closer that S gets to zero the smaller the increments dS.

The third example is

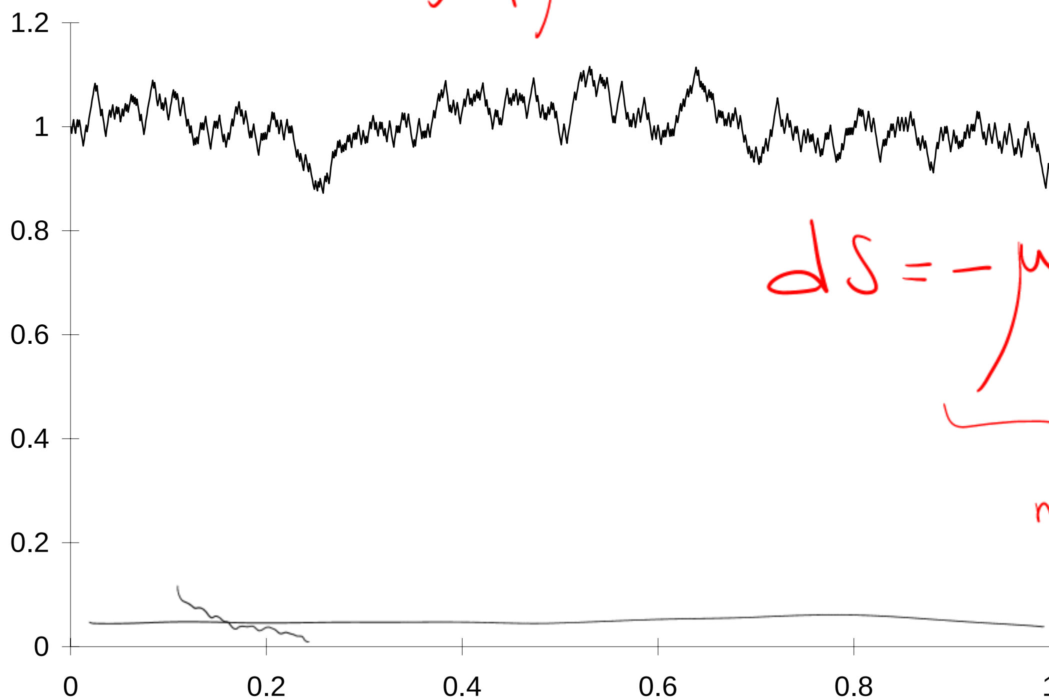
$$dS = (\nu - \mu S)dt + \sigma\, dX.$$

an I.R model

$$dS = \mu\left(\frac{\nu}{\mu} - S\right)dt + \sigma\, dX$$

$\frac{\nu}{\mu} = \bar{S}$ mean rate

μ - speed of reversion



$$dS = \underbrace{-\mu(S - \bar{S})}_{\text{mean reverting}}\, dt + \sigma\, dX$$

If we put $u = S - \bar{S}$ $\quad du = -\mu u\, dt + \sigma\, dX$ O-U process

The random walk

$$dS = (\nu - \mu S)dt + \sigma\, dX$$

is an example of a **mean-reverting** random walk.

If S is large, the negative coefficient in front of dt means that S will move down on average, if S is small it rises on average. There is still no incentive for S to stay positive in this random walk.

With r instead of S this random walk is the Vasicek model for the short-term interest rate.

Returning to Itô II

$$dF = \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\right)dt + \frac{\partial F}{\partial X_t}\, dX_t$$

Rearrange to get last term on the RHS on its own

$$\frac{\partial F}{\partial X_t}\, dX_t = dF - \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial X_t^2}\right)dt$$

Now integrate both sides on $(0, t)$

$$\int_0^t \frac{\partial F}{\partial X_s}\, dX_s = \int_0^t dF - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2}\right) ds$$

Now because Itô II is for $F(t, X) \Rightarrow \int_0^t dF = F(t, X_t) - F(0, X_0)$

$$\underbrace{\int_0^t \frac{\partial F}{\partial X_s}\, dX_s}_{\text{Itô integral}} = F(t, X_t) - F(0, X_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2}\right) ds$$

$$\int_0^t (s^2 + X_s^3)\, dX_s$$. To do this we use the derived formula on the previous slide (bottom)

comparing with $$\int_0^t \frac{\partial F}{\partial X_s} dX_s = F(t, X_t) - F(0, X_0) - \int_0^t \left(\frac{\partial F}{\partial s} + \frac{1}{2}\frac{\partial^2 F}{\partial X_s^2} \right) ds \quad \dagger$$

We see $\frac{\partial F}{\partial X_t} = t^2 + X_t^3 \longrightarrow F = t^2 X_t + \frac{X_t^4}{4} \longrightarrow \frac{\partial F}{\partial t} = 2t X_t$

$$\frac{\partial^2 F}{\partial X_t^2} = 3X_t^2$$

Now substitute everything above in $\dagger$

$$\int_0^t (s^2 + X_s^3)\, dX_s = t^2 X_t + \frac{X_t^4}{4} - \frac{X_0^4}{4} - \int_0^t \left(2s\underline{\underline{X_s}} + \frac{3}{2}\underline{\underline{X_s^2}} \right) \underline{\underline{ds}}$$

This is the stopping. Any further attempt to simplify the integral will start losing marks.

Itô I $\quad dF = \boxed{\frac{dF}{dX_t} dX_t} + \frac{1}{2} \frac{d^2F}{dX_t^2} dt$

$$\frac{dF}{dX_t} dX_t = dF - \frac{1}{2} \frac{d^2F}{dX_t^2} dt$$

Now integrate over $(0, T)$

$$\int_0^T \frac{dF}{dX_t} dX_t = \underbrace{\int_0^T dF}_{F(X_T) - F(X_0)} - \frac{1}{2} \int_0^T \frac{d^2F}{dX_t^2} dt$$

$$\int_0^T \frac{dF}{dX_t} dX_t = F(X_T) - F(X_0) - \frac{1}{2} \int_0^T \frac{d^2F}{dX_t^2} dt \qquad \dagger\dagger$$

<u>Example</u> : $\int_0^T X_t \, dX_t$ $\qquad \frac{dF}{dX_t} = X_t \Rightarrow F = \frac{1}{2} X_t^2$

$$\Downarrow$$

$$\frac{d^2F}{dX_t^2} = 1$$

Subst in $\dagger\dagger$

$$\int_0^T X_t\,dX_t = \frac{1}{2}\left(X_T^2 - X_0^2\right) - \frac{1}{2}\underbrace{\int_0^T 1\,dt}_{[t]_0^T}$$

$$= \frac{1}{2}\left(X_T^2 - \cancel{X_0^2}\right) - \underbrace{\frac{T}{2}}_{}$$

$= 0$

extra term comes from $dX^2 \to dt$ and not zero.

$dX \sim \phi\sqrt{dt}$ $\qquad \mathbb{E}[dX] = 0 \qquad \mathbb{V}[dX] = dt$

$\phi \sim \mathcal{N}(0,1)$ $\qquad \mathbb{E}[\phi] = 0 \qquad \mathbb{V}[\phi] = 1 = \mathbb{E}[\phi^2]$

$$\mathbb{E}[\phi\sqrt{dt}] = \sqrt{dt}\,\mathbb{E}[\phi] = \sqrt{dt} \times 0 = 0$$

$$\mathbb{V}[\phi\sqrt{dt}] = dt\,\mathbb{V}[\phi] = dt \times 1 = dt$$

Given a stoch. process G_t with S.D.E

$$(*) \quad dG_t = A(t, G_t)\,dt + B(t, G_t)\,dX_t$$

Now integrate over 0 and t

$$\int_0^t dG_s = \int_0^t A(s, G_s)\,ds + \underbrace{\int_0^t B(s, G_s)\,dX_s}$$

$$G_t = G_0 + \int_0^t A(s, G_s)\,ds + \int_0^t B(s, G_s)\,dX_s \quad (**)$$

$(*)$ SDE in differential form

$(**)$ is the S.D.E expressed in integral form

$$(*) \equiv (**)$$

The final example is similar to the third but we are going to adjust the random term slightly:

$$dS = (\nu - \mu S)dt + \sigma S^{1/2} dX.$$

Now if S ever gets close to zero the randomness decreases, perhaps this will stop S from going negative?

This particular stochastic differential equation for S will be important later on, it is the Cox, Ingersoll & Ross model for the short-term interest rate.

$$dS = -\mu(S - \bar{S})dt + \sigma S^{1/2} dX$$

$S \longrightarrow 0$ $\quad S^{1/2}$ in the diffusion $\longrightarrow 0$

switching off the randomness

$$dS = -\mu(0 - \bar{S})dt \rightarrow +ve$$

Pursuing this idea further, imagine what might be meant by

$$\int dW = \int g(t)\,dt + \int f(t)\,dX.$$

- Equations like this are called **stochastic differential equations**. Their precise meaning comes, however, from the technically more accurate equivalent stochastic integral.

This equation above is shorthand for

$$W(t) = \int_0^t g(\tau)\,d\tau + \int_0^t f(\tau)\,dX(\tau).$$

X_t or W_t Brownian Motion Wiener process

The mean square limit

This is useful in the precise definition of stochastic integration.

Examine the quantity

$$\left(E\left[\left(\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2-t\right)^2\right] \right) \to 0$$

where

$$t_j=\frac{jt}{n}.$$

$$E\left[\left(f(x)-l\right)^2\right]=0$$

$$\Rightarrow \quad f(x)=l \text{ in the mean square limit}$$

This can be expanded as

$$E\left[\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^4\right.$$

$$+2\sum_{i=1}^{n}\sum_{j<i}(X(t_i)-X(t_{i-1}))^2(X(t_j)-X(t_{j-1}))^2$$

$$\left.-2t\sum_{j=1}^{n}(X(t_j)-X(t_{j-1}))^2+t^2\right].$$

Since $X(t_j)-X(t_{j-1})$ is Normally distributed with mean zero and variance t/n we have

$$E\left[(X(t_j)-X(t_{j-1}))^2\right]=\frac{t}{n}$$

and

$$E\left[(X(t_j)-X(t_{j-1}))^4\right]=\frac{3t^2}{n^2}.$$

Thus the required expectation becomes

$$n\frac{3t^2}{n^2} + n(n-1)\frac{t^2}{n^2} - 2tn\frac{t}{n} + t^2 = O\left(\frac{1}{n}\right).$$

As $n \to \infty$ this tends to zero. We therefore say that

$$\sum_{j=1}^{n} (X(t_j) - X(t_{j-1}))^2 = t$$

in the 'mean square limit.' This is often written, for obvious reasons, as

$$\int_0^t (dX)^2 = t.$$

Whenever we talk about 'equality' in the following 'proof' we mean equality in the mean square sense.

Summary

Please take away the following important ideas

- Functions of random variables can't be differentiated in quite the same way as functions of deterministic variables.

- Instead of using Taylor series you must use Itô's lemma. However, they are very similar and a simple rule of thumb can usually be used to get from Taylor to Itô.