M1L4 Exercises

February 11, 2024

1

Let ϕ be a standard normal random variable, $\phi \sim \mathcal{N}(0,1)$. Also let $\psi = \sqrt{dt}\phi$. By linearity, we have that:

$$\mathbb{E}[\psi] = \mathbb{E}[\sqrt{dt}\phi] = \sqrt{dt}\mathbb{E}[\phi] = 0$$

And by scaling properties of variance:

$$\mathbb{V}[\psi] = \mathbb{V}[\sqrt{dt}\phi] = (\sqrt{dt})^2\mathbb{V}[\phi] = dt \cdot 1 = dt$$

2

We express the following in the standard form:

$$dG = A(G,t)\,dt + B(G,t)\,dW_t$$

a.

$$df = (1 - 2\mu t f)\,dt - (1 + 2\sqrt{f})\,dW_t$$

b.

$$dy = (Ay + By^2)\,dt + Cy^2\,dW_t$$

c.

$$dS = -\frac{\nu + \mu S}{3}\,dt - \frac{\sigma}{3}\,dW_t$$

3

We recall the time-independent Itô's lemma:

$$dF = \frac{1}{2}\frac{\partial^2 F}{\partial W^2}\,dt + \frac{\partial F}{\partial W}\,dW_t$$

We apply this equation to each of the following

a. $f(W_t) = W_t^n$:

$$df = \frac{n(n-1)}{2} W_t^{n-2}\, dt + nW_t^{n-1}\, dW_t$$

b. $y(W_t) = e^{W_t}$:

$$dy = \frac{1}{2} e^{W_t}\, dt + e^{W_t}\, dW_t$$
$$dy = \frac{y}{2}\, dt + y\, dW_t$$

c. $g(W_t) = \log W_t$:

$$dg = -\frac{1}{2W_t^2}\, dt + \frac{1}{W_t}\, dW_t$$

d. $h(W_t) = \sin W_t + \cos W_t$:

$$dh = -\frac{\sin W_t + \cos W_t}{2}\, dt + (\cos W_t - \sin W_t)\, dW_t$$

e. $f(W_t) = a^{W_t}$ for $a > 0$.

$$df = \frac{a^{W_t}(\log a)^2}{2}\, dt + a^{W_t}(\log a)\, dW_t$$
$$\frac{df}{f} = \frac{(\log a)^2}{2}\, dt + (\log a)\, dW_t$$

4

We can rewrite the time-dependent Itô formula to get an expression for stochastic integrals in terms of typical integrals:

$$\int_o^t \frac{\partial F}{\partial W_\tau}\, dW_\tau = F(t, W_t) - F(0, W_0) - \int_0^t \left(\frac{\partial F}{\partial \tau} + \frac{1}{2} \frac{\partial^2 F}{\partial W_\tau^2} \right) d\tau$$

Note that when solving for each of the derivatives, we do not need to add any constant function terms, i.e $F = b(t, W_t) + c(t)$ as we are implicitly calculating definite integrals. We then re-express the following stochastic integrals:

a. $\int_0^t W_\tau^3 \, dW_\tau$. We know that $\frac{\partial F}{\partial W_t} = W_t^3$, therefore:

$$
\begin{aligned}
F &= \frac{1}{4} W_t^4 \\
\frac{\partial F}{\partial t} &= 0 \\
\frac{\partial^2 F}{\partial W_t^2} &= 3W_t^2
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t W_\tau^3 \, dW_\tau &= \frac{W_t^4}{4} - 0 - \int_0^t \left(0 + \frac{3}{2} W_t^2 \right) \, d\tau \\
&= \frac{1}{4} W_t^4 - \frac{3}{2} \int_0^t W_\tau^2 d\tau
\end{aligned}
$$

b. $\int_0^t \tau \, dW_\tau$. We know that $\frac{\partial F}{\partial W_t} = t$, therefore:

$$
\begin{aligned}
F &= tW_t \\
\frac{\partial F}{\partial t} &= W_t \\
\frac{\partial^2 F}{\partial W_t^2} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\int_0^t \tau \, dW_\tau &= tW_t - 0 - \int_0^t \left(0 + W_t \right) \, d\tau \\
&= tW_t - \int_0^t W_t \, d\tau
\end{aligned}
$$

c. $\int_0^t (\tau + W_\tau) \, dW_\tau$. We know that $\frac{\partial F}{\partial W_t} = t + W_t$, therefore:

$$
\begin{aligned}
F &= \frac{1}{2} W_t^2 + tW_t \\
\frac{\partial F}{\partial t} &= W_t \\
\frac{\partial^2 F}{\partial W_t^2} &= 1
\end{aligned}
$$

$$
\int_0^t (\tau + W_\tau) \, dW_\tau = \frac{1}{2} W_t^2 + tW_t - \int_0^t \left(W_t + \frac{1}{2} \right) \, d\tau
$$

5

Define the PDE:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial x} + bu$$

For function $u(x,t)$, and constants a, b. We consider the substitution:

$$u(x,y) = e^{\alpha x + \beta t} v(x,t)$$

We calculate the necessary derivatives:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= e^{\alpha x + \beta t}\left(\beta v + \frac{\partial v}{\partial t}\right) \\
\frac{\partial u}{\partial x} &= e^{\alpha x + \beta t}\left(\alpha v + \frac{\partial v}{\partial x}\right) \\
\frac{\partial^2 u}{\partial x^2} &= e^{\alpha x + \beta t}\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right)
\end{aligned}$$

Substituting in the original equation:

$$\begin{aligned}
e^{\alpha x + \beta t}\left(\beta v + \frac{\partial v}{\partial t}\right) &= e^{\alpha x + \beta t}\left(\alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2}\right) + ae^{\alpha x + \beta t}\left(\alpha v + \frac{\partial v}{\partial x}\right) + be^{\alpha x + \beta t} v \\
\beta v + \frac{\partial v}{\partial t} &= \alpha^2 v + 2\alpha\frac{\partial v}{\partial x} + \frac{\partial^2 v}{\partial x^2} + a\alpha v + a\frac{\partial v}{\partial x} + bv \\
\frac{\partial v}{\partial t} &= \frac{\partial^2 v}{\partial x^2} + (2\alpha + a)\frac{\partial v}{\partial x} + (\alpha^2 + a\alpha + b - \beta)v
\end{aligned}$$

i.e we want:

$$\begin{aligned}
2\alpha + a &= 0 \\
\alpha^2 + a\alpha + b - \beta &= 0
\end{aligned}$$

Evidently, we see $\alpha = -\frac{a}{2}$, from which we get $\beta = b - \frac{a^2}{4}$. Plugging these in, we get the heat equation:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$$