M1L3 Exercises

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1

Consider:

$$p(y, t; y', t') = \frac{1}{2c\sqrt{\pi(t' - t)}} \exp\left(-\frac{(y' - y)^2}{4c^2(t' - t)}\right)$$

We can drop y, t (Substitute $y' - y, t' - t$ etc.) so we only need to work on:

$$p(y', t') = \frac{1}{2c\sqrt{\pi t'}} \exp\left(-\frac{y'^2}{4c^2 t'}\right)$$

Calculating the necessary derivatives for the FKE, we get:

$$\frac{\partial p}{\partial t'} = \frac{1}{2c\sqrt{\pi}} \cdot \left(-\frac{1}{2\sqrt{t'^3}}\right) \exp\left(-\frac{y'^2}{4c^2 t'}\right) + \frac{1}{2c\sqrt{\pi t'}} \exp\left(-\frac{y'^2}{4c^2 t'}\right) \cdot \left(\frac{y'^2}{4c^2 t'^2}\right)$$

$$= \frac{1}{2c\sqrt{\pi t'}} \exp\left(-\frac{y'^2}{4c^2 t'}\right) \cdot \left(-\frac{1}{2t'} + \frac{y'^2}{4c^2 t'^2}\right)$$

$$= p \cdot \left(\frac{-2c^2 t' + y'^2}{4c^2 t'^2}\right)$$

$$\frac{\partial p}{\partial y'} = p \cdot \left(-\frac{y'}{2c^2 t'}\right)$$

$$\frac{\partial^2 p}{\partial y'^2} = p \cdot \left(-\frac{y'^2}{4c^4 t'^2}\right) - \frac{p}{2c^2 t'}$$

$$= p \left(\frac{y'^2 - 2c^2 t'}{4c^4 t'^2}\right)$$

$$= \frac{1}{c^2} \left[p\left(\frac{y'^2 - 2c^2 t'}{4c^2 t'^2}\right)\right]$$

$$= \frac{1}{c^2} \frac{\partial p}{\partial t'}$$

Which is the FKE.

2

We consider a symmetric random walk. Consider sufficiently small step sizes δy and δt. Then we have two previous points to consider: $(y - \delta y, t - \delta t), (y + \delta y, t - \delta t)$. Let $p(x, s; y, t) = p(y, t)$ be the transition density function. We expand $p(y - \delta y, t - \delta t), p(y + \delta y, t - \delta t)$ w.r.t (y, t):

$$p(y + \delta y, t - \delta t) = p(y, t) + \delta y \frac{\partial p}{\partial y} - \delta t \frac{\partial p}{\partial t} + \frac{1}{2} \delta^2 y \frac{\partial^2 p}{\partial y^2} + o(\delta t, \delta y^2)$$

$$p(y - \delta y, t - \delta t) = p(y, t) - \delta y \frac{\partial p}{\partial y} - \delta t \frac{\partial p}{\partial t} + \frac{1}{2} \delta^2 y \frac{\partial^2 p}{\partial y^2} + o(\delta t, \delta y^2)$$

We know that we want the expected position after the step to be y, so we take expectations:

$$p(y, t) = \frac{1}{2} \left(p(y + \delta y, t - \delta t) + p(y - \delta y, t - \delta t) \right)$$

$$p = \frac{1}{2} \left(2p - 2\delta t \frac{\partial p}{\partial t} + \delta y^2 \frac{\partial^2 p}{\partial y^2} \right)$$

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\delta y^2}{\delta t} \frac{\partial^2 p}{\partial y^2}$$

For this PDE to be well-defined, we would require $\frac{\delta y^2}{\delta t} \sim O(1)$. We can do this as our definition of the symmetric random walk was independent of any values $\delta t, \delta y$ could take, apart from having to be sufficiently small. Therefore, it would have to hold true as $\delta t, \delta y \to 0$. We are also free to set $\delta t = \delta y^2$, as the limiting behaviour is the only concern, giving us:

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial y^2}$$

3

Let's solve the above equation. Assume we have:

$$p(y, t) = \frac{1}{\sqrt{t}} f(\eta)$$

Where $\eta = \frac{y}{\sqrt{t}}$. We can then find the necessary partial derivatives:

$$\frac{\partial p}{\partial t} = - \left(\frac{1}{2\sqrt{t^3}} f(\eta) + \frac{y}{2t^2} f'(\eta) \right)$$

$$\frac{\partial p}{\partial y} = \frac{1}{t} f'(\eta)$$

$$\frac{\partial^2 p}{\partial y^2} = \frac{1}{\sqrt{t^3}} f''(\eta)$$

Plugging them into the equation:

$$-\left(\frac{1}{2\sqrt{t^3}}f(\eta) + \frac{y}{2t^2}f'(\eta)\right) = \frac{1}{2\sqrt{t^3}}f''(\eta)$$

$$f(\eta) + \frac{y}{\sqrt{t}}f'(\eta) = -f''(\eta)$$

$$f(\eta) + \eta f'(\eta) = -f''(\eta)$$

This is an ODE, which we can start solving with use of the product rule and integrating:

$$f(\eta) + \eta f'(\eta) = -f''(\eta)$$

$$\frac{d}{d\eta}(\eta f(\eta)) = -f''(\eta)$$

$$\implies \eta f(\eta) = -f'(\eta) + K$$

For some constant K. Here, we consider the limiting behaviour of f, f' as $\eta \to \infty$. We know for a fact that as a pdf, $p, p' \to 0$ as $y \to \infty$. As η is just a rescaling of y, $f, f' \to 0$. We will also need: $f(\eta) \sim O(\eta^{-1})$. However, we cannot have $\lim\limits_{\eta \to \infty} \frac{f(\eta)}{\eta} = L$ for $L \neq 0$, as that would imply that $\int_1^\infty f(\eta)\, d\eta = \int_1^\infty L\eta^{-1}\, d\eta = \infty$, contradicting it's pdf status. Hence we require $f(\eta) \sim o(\eta^{-1})$ which implies that $\eta f(\eta) \to 0$. Hence, $K = 0$ and we can proceed to solve the remaining equation:

$$\eta f(\eta) = -f'(\eta)$$

$$\int \frac{df}{f} = \int -\eta\, d\eta$$

$$\ln f = -\frac{\eta^2}{2} + C$$

$$f(\eta) = Ae^{-\frac{\eta^2}{2}}$$

For some constant A. Recall that $p = \frac{1}{\sqrt{t}}f(y/\sqrt{t})$ is a probability distribution, therefore, we use the substitution $u = y/\sqrt{2t}$ and $\int_\mathbb{R} p = 1$ to get our constant:

$$1 = \int_\mathbb{R} \frac{A}{\sqrt{t}}e^{-\frac{y^2}{2t}}\, dy$$

$$1 = A\sqrt{2}\int_\mathbb{R} e^{u^2}\, du$$

$$1 = A\sqrt{2\pi}$$

$$A = \frac{1}{\sqrt{2\pi}}$$

3

By the standard Gaussian integral. So, we finally have the solution:

$$p(y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)$$