The Binomial Model - Discrete Time Finance

The model has made option pricing accessible to MBA students and finance practitioners preparing for the CFA. It is a very useful tool for conveying the ideas of delta hedging and no arbitrage, in addition to the subtle concept of risk neutrality and option pricing.

Here the model is considered in a slightly more mathematical way. The basic assumptions in option pricing theory consist of two forms, key:

- Short selling allowed
- No arbitrage opportunities

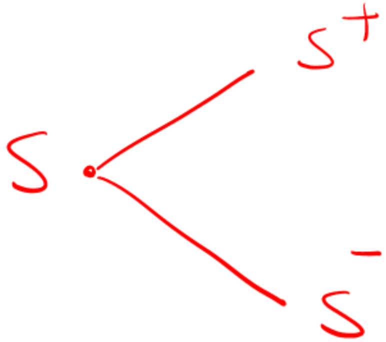
and relaxable

Black Scholes assumptions

- Frictionless markets - no transaction costs, limits to trading or taxes
- Perfect liquidity
- Known volatility and interest rates
- No dividends on the underlying

The key assumptions underlying the binomial model are:

- an asset value changes only at discrete time intervals $\therefore$ it is a discrete time model
- fractional trading is allowed 0.75 share (100) (100)
- an asset's worth can change to one of only two possible new values at each time step.



worth money

$$\Pi = V - \Delta S$$

$$\left[V = \Pi + \Delta S \right]$$

cash

amount of a risky asset

Consider a portfolio Π, long an option and short Δ assets. $V^{\pm}$ denotes the option value corresponding to asset price $S^{\pm}$:

stock price

$$\begin{tikzcd}[row sep=small] & S^{+} \\ S \arrow[ur] \arrow[dr] & \\ & S^{-} \end{tikzcd}$$

option price

$$\begin{tikzcd}[row sep=small] & V^{+} \\ V \arrow[ur] \arrow[dr] & \\ & V^{-} \end{tikzcd}$$

Portfolio:

$$\begin{tikzcd}[row sep=small] & \Pi^{+} \\ \Pi \arrow[ur, no head] \arrow[dr, no head] & \\ & \Pi^{-} \end{tikzcd}$$

at t

$$\Pi = V - \Delta S$$

At time T there are two possible outcomes:

to eliminate risk

$$\begin{aligned} \Pi^{-} &= V^{-} - \Delta S^{-} \\ \Pi^{+} &= V^{+} - \Delta S^{+} \end{aligned}$$

Choosing Δ so $\Pi^{-} = \Pi^{+}$ gives

$$\Delta = \frac{V^{+} - V^{-}}{S^{+} - S^{-}}$$

This choice of Δ makes Π risk-free, so no-arbitrage suggests that the return on Π equal the risk-free rate, i.e.

$\Pi = \Pi^{+}$

$\Pi = \Pi^{-}$

Compounding

$$
\begin{aligned}
e^{rT}\Pi &= \Pi^{-} = \Pi^{+} \\
PV \quad &= V^{-} - \Delta S^{-} = V^{-} - \left(\frac{V^{+} - V^{-}}{S^{+} - S^{-}}\right) S^{-} \\
&= \frac{V^{-}\left(S^{+} - S^{-}\right) - S^{-}\left(V^{+} - V^{-}\right)}{S^{+} - S^{-}} \\
e^{rT}\Pi \quad &= \frac{V^{-}S^{+} - S^{-}V^{+}}{S^{+} - S^{-}}.
\end{aligned}
$$

Hence we can write

$$
\begin{aligned}
e^{rT}\left(V - \Delta S\right) &= \frac{V^{-}S^{+} - S^{-}V^{+}}{S^{+} - S^{-}} \\
e^{rT}V &= \frac{V^{-}S^{+} - S^{-}V^{+}}{S^{+} - S^{-}} + \left(\frac{V^{+} - V^{-}}{S^{+} - S^{-}}\right) S e^{rT} \\
&= \frac{\left(e^{rT}S - S^{-}\right)}{S^{+} - S^{-}} V^{+} + \frac{\left(S^{+} - e^{rT}S\right)}{S^{+} - S^{-}} V^{-} \\
e^{rT}V &= qV^{+} + \left(1 - q\right) V^{-}
\end{aligned}
$$

replace Δ

$\{p, 1-p\}$ real

$\{q, 1-q\}$ risk-neutral

and finally we can write

$$\boxed{V = e^{-rT}\left(qV^{+} + (1-q)\,V^{-}\right)}$$

where we define

p q

real vs. risk-neutral

$$q = \frac{\left(e^{rT}S - S^{-}\right)}{S^{+} - S^{-}}$$

with $0 < q < 1$.

If compounding is discrete

$e^{-rT} \approx 1 - rT$

$$q = \frac{\left((1 + rT)\,S - S^{-}\right)}{S^{+} - S^{-}}$$

$(1+rT)^{-1}$

and the option price becomes

e^{-rT}

$$V = (1 - rT)\left(qV^{+} + (1-q)\,V^{-}\right).$$

Musiela - Martingale methods fn. modelling

$= P.V\ \mathbb{E}^{q}[\text{Payoff}]$

Shreve discrete time finance

Björk

↳ arb. theory in cts time

q is a *risk-neutral probability* which come about from insistence on no arbitrage.

ϕ Δ

ψ

time

The one period model - Replication

Another way of looking at the Binomial model is in terms of replication: we can replicate the option using only cash (or bonds) and the asset. That is, mathematically, simply a rearrangement of the earlier equations. It is, nevertheless a very important interpretation. We will consider **self-financing** portfolios - no cash inflows and no cash outflows.

Later you will consider in terms of a SDE

In one time step:

1. The asset moves from $S_0 = s$ to $S_1 = s_u$ or $S_1 = s_d$.

   time 0 time 1

2. An option X pays off x_u if the asset price is s_u and x_d if the price is s_d.

$s = S_0$ $\quad$ $s = S_1 \begin{cases} s_u & x_u \\ s_d & x_d \end{cases}$

$t = 0$ $\quad$ $t = 1$

$$\frac{dB}{dt} = rB$$

$$\begin{array}{cccc}
\text{Stock} & & & s_u \\
 & & \nearrow & \\
 & s & & \\
 & & \searrow & \\
 & & & s_d \\
 & t=0 & & t=1
\end{array}
\qquad\qquad
\begin{array}{cccc}
\text{Option:} & & & x_u \\
 & & \nearrow & \\
 & V & & \\
 & & \searrow & \\
 & & & x_d \\
 & t=0 & & t=1
\end{array}$$

3. There is a bond market in which a pound invested today is continuously compounded at a constant (risk-free) rate r and becomes e^r, one time-step later. The dynamics of the risk-free asset B_t satisfies

SDE with zero randomness

I.V.P. $$\boxed{dB_t = rB_t dt;\ \ B_0 = 1.}$$

Integrating over $[t, T]$, where $t = 0$, $T = 1$ gives $B_1 = e^r$.

$$\int_t^T \frac{dB_s}{B_s} = r\int_t^T ds \qquad \log \frac{B_T}{B_t} = r(T-t) \qquad B_T = e^{r(T-t)} B_t \qquad B_1 = e^r B_0$$

$\Pi = V - \Delta S \longrightarrow 1$ d.o.f.

Now consider a portfolio of ψ bonds and ϕ assets which at time $t = 0$, will have an initial value of

risky asset

at $t = 0$

$$\Pi(\phi, \psi) = \phi s + \psi \times 1$$

bonds

$\Rightarrow$ 2 parameters to calculate

Now with this money we can buy or sell bonds or stocks in order to obtain a new portfolio at time-step 1. At this new time-step there exist two possible outcomes

1 bond $\rightarrow e^r$

X $\quad x_u \quad x_d$

$s \nearrow s_u$

$s \searrow s_d$

$$\begin{aligned} \Pi^+ &= \phi s_u + \psi e^r \\ \Pi^- &= \phi s_d + \psi e^r \end{aligned}$$

In order to replicate the option insist that

also $\boxed{\Pi = V}$

$$\Pi^+ = x_u; \ \Pi^- = x_d$$

Can we construct a hedging strategy which will guarantee to pay-off the option, whatever happens to the asset price?

$\Pi^+ = \Pi^-$

$V = \phi s + \psi$

$\Pi = \phi s + \psi$

$\downarrow$

V

The Hedging Strategy

$\Pi^+ \to x_u$

$\Pi^- \to x_d$

We arrange the portfolio so that its value is exactly that of the required option pay-out at the terminal time regardless of whether the stock moves up or down. This is because having two unknowns ϕ, ψ, the amount of stock and bond, and we wish to match the two possible terminal values, x_u, x_d, the option payoffs. Thus we need to have

$$\left\{ \begin{aligned} x_u &= \phi s_u + \psi e^r, \quad (a) \\ x_d &= \phi s_d + \psi e^r. \quad (b) \end{aligned} \right.$$

Subtracting the two expressions and rearranging gives

(a) − (b) $\longrightarrow$

$$\phi = \frac{x_u - x_d}{s_u - s_d}. \equiv \Delta = \frac{V^+ - V^-}{S^+ - S^-}$$

Then substituting for ϕ in either equation yields

i.e (a), (b)

$$\psi = e^{-r} \frac{x_d s_u - x_u s_d}{s_u - s_d}$$

$+$ buy

$-$ sell

This is a *hedging strategy*.

At time step 1, the value of the portfolio is

$$X = \begin{cases} x_u & \text{if } S_1 = s_u \\ x_d & \text{if } S_1 = s_d \end{cases}$$

This is the option payoff. Thus, given $V = \phi s + \psi$ we can construct the above portfolio which has the same payoff as the option. Hence the price for the option must be V. Any other price would allow arbitrage as you could play this hedging strategy, either buying or selling the option, and make a guaranteed profit.

$\Pi = \phi s + \psi$

Thus the fair, arbitrage-free price for the option is given by

$$\begin{aligned} V &= (\phi s + \psi) \\ &= \frac{x_u - x_d}{s_u - s_d} s + e^{-r} \frac{x_d s_u - x_u s_d}{s_u - s_d} \\ &= e^{-r} \left(\underbrace{\frac{e^r s - s_d}{s_u - s_d}}_{q} x_u + \underbrace{\frac{s_u - e^r s}{s_u - s_d}}_{1-q} x_d \right). \end{aligned}$$

P.V

$\mathbb{Q} = \{q, 1-q\}$

Define

$$q = \frac{e^r s - s_d}{s_u - s_d},$$

then we conclude that

$$V = e^{-r}\left(q x_u + (1-q)\, x_d\right)$$

$$= e^{-r} \mathbb{E}^{\mathbb{Q}}[X]$$

where

$$0 < q < 1.$$

We can think of q as a probability induced by insistence on no-arbitrage, i.e. the so-called *risk-neutral probability*. It has nothing to do with the real probabilities of s_u and s_d occurring; these are p and $1-p$, in turn.

The option price can be viewed as the discounted expected value of the option pay-off X with respect to the probabilities q and $(1-q)$,

$$\begin{aligned} V &= e^{-r}\left(q x_u + (1-q)\, x_d\right) \\ &= \mathbb{E}^{\mathbb{Q}}\left[e^{-r} X\right]. \end{aligned}$$

$\boxed{2}$

Δ

hedged portfolio

$\boxed{1} = \boxed{2} \rightarrow$ replication

The set of probabilities $\mathbb{Q} = \{q, (1 - q)\}$ is called a **risk-neutral measure.**
The fact that the risk neutral/fair value (or $q-$value) of a call is less than the expected value of the call (under the real probability p), is not a puzzle.

Pricing a call using the real probability, p, you will probably make a profit, but you might also might make a loss. Pricing an option using the risk-neutral probability, q, you will certainly make neither a profit nor a loss.
So, under the risk-neutral measure, all assets are expected to grow at the same risk-free rate r.

done previously using Δ hedged.

Example: A stock is currently trading at 100. In one year it will have risen to 103 or fallen to 98. If interest rates are zero, use a replicating strategy to price a one year call option with strike $K = 100$.

Asset:

$$
\begin{array}{ccl}
 & & \omega_1 \\
 & & 103 \quad S(1) > 100 \\
 & \nearrow & \qquad \text{Payoff} = 3 \\
S(0) = 100 & & \\
 & \searrow & \\
 & & 98 \quad S(1) < 100 \\
 & & \omega_2 \quad \text{Payoff} = 0
\end{array}
$$

Option:

$$
\begin{array}{ccl}
 & & V_1 = 3 = X_u \\
 & \nearrow & \\
V_0 & & \\
 & \searrow & \\
 & & V_2 = 0 = X_d
\end{array}
$$

Portfolio

$$
\begin{array}{ccl}
 & & \Pi^+ = X_u \\
 & \nearrow & \\
\Pi = V & & \\
 & \searrow & \\
 & & \Pi^- = X_d
\end{array}
$$

ϕ stocks and ψ units of bonds. $V = \phi s + \psi$

$$x_u = \overset{s_u}{103}\phi + \psi = 3, \quad x_d = \underset{s_d}{98}\phi + \psi = 0 \;\rightarrow\; \underset{\text{buy}}{\phi = 3/5}; \quad \underset{\text{sell}}{\psi = -294/5}$$

$V = \phi s + \psi$

$$\therefore \quad V_0 = S(0) \times \phi + \psi \times 1 = \frac{300}{5} - \frac{294}{5} = 1.2$$

$$q(\text{up}) = \frac{e^{rt}s - s_d}{s_u - s_d}, \quad q(\text{down}) = \frac{s_u - e^{rt}s}{s_u - s_d} = 1 - q = \frac{s_u - s}{s_u - s_d}$$

where $r = 0$. $q = \frac{s - s_d}{s_u - s_d}$

up ω_1 100 103

$$q(\omega_1) = \frac{100 - 98}{103 - 98} = \frac{2}{5} = q \leftarrow x_u$$

down ω_2 100 98

$$q(\omega_2) = \frac{103 - 100}{103 - 98} = \frac{3}{5} = 1 - q \leftarrow x_d$$

$\mathbb{Q} = \{ \frac{2}{5}, \frac{3}{5} \}$

So the risk neutral probabilities are

$$\left(\frac{2}{5}, \frac{3}{5}\right)$$

So the expected value (under the risk-neutral probabilities/measure) is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}X\right]$$

where $r = 0, t = 0, T = 1$,

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[X] &= \sum_{\omega} q(\omega_i) X(\omega_i) \\ &= \frac{2}{5} \times 3 + \frac{3}{5} \times 0 = 1.2 \end{aligned}$$

Ex: Repeat with hedged portfolio

$C - P = S - E e^{-r(T-t)}$

Using Sample Paths

all or nothing. one unit if ITM at expiry else zero.

A *binary option* (also called a *digital option*) pays one dollar at time $t = T$ if the asset price is above a fixed strike level K and is worthless otherwise.

call

put at $t=T$ $S_T < K$

Consider the following model with $K = 5$, $r = 0$:

description of stock price

	t=0	t=1	t=2	
ω	$S(0)$	$S(1)$	$S(2)$	
ω_1	4	7	11	uu
ω_2	4	7	6	ud
ω_3	4	3	6	du
ω_4	4	3	1	uu

ω	$S(0)$	$S(1)$	$S(2)$
ω_1	s	u	u
ω_2	s	u	d
ω_3	s	d	u
ω_4	s	d	d

Asset:

$$
\begin{array}{ccccccc}
 & & & & & \omega_1 = uu & \\
 & & & & 11 & S(2) > K & \text{Payoff } = 1 \\
 & & & \nearrow & & & \\
 & & 7 & & & & \\
 & \nearrow & & \searrow & & \omega_2 = ud & \\
S(0) = 4 & & & & 6 & S(2) > K & \text{Payoff } = 1 \\
 & \searrow & & \nearrow & & \omega_3 = du & \\
 & & 3 & & & & \\
 & & & \searrow & & & \\
 & & & & 1 & S(2) < K & \text{Payoff } = 0 \\
 & & & & & \omega_4 = dd &
\end{array}
$$

Option:

$$
\begin{tikzcd}[row sep=small]
 & & 1 \\
 & V_1 \arrow[ur] \arrow[dr] & \\
V_0 \arrow[ur] \arrow[dr] & & 1 \\
 & V_2 \arrow[ur] \arrow[dr] & \\
 & & 0
\end{tikzcd}
$$

sub_model I

sub-model I

sub-model II

1, 1, 1, 0

Replicate backwards over each time period 'sub-model':

Model I

Stock
$$
\begin{tikzcd}[row sep=small]
 & 11 \\
7 \arrow[ur] \arrow[dr] & \\
 & 6
\end{tikzcd}
$$

Option:
$$
\begin{tikzcd}[row sep=small]
 & 1 \\
V_1 \arrow[ur] \arrow[dr] & \\
 & 1
\end{tikzcd}
$$

Replicate with ϕ units of stock and ψ units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$ $= 1 \therefore r = 0$

no action; buy

$$\left.\begin{array}{r} 11\phi + \psi = 1 \\ 6\phi + \psi = 1 \end{array}\right\} \to \quad \phi = 0; \ \psi = 1$$

$$\therefore \quad V_1 = S(1) \times \phi + \psi \times 1 = 7 \times 0 + 1 \times 1 = 1$$

$$\begin{tikzcd}[row sep=small] \text{Stock} & & 6 \\ & 3 \arrow[ur] \arrow[dr] & \\ & & 1 \end{tikzcd} \qquad \begin{tikzcd}[row sep=small] \text{Option:} & & 1 \\ & V_2 \arrow[ur] \arrow[dr] & \\ & & 0 \end{tikzcd}$$

Replicate with ϕ units of stock and units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$

buy; sell

$$\left.\begin{array}{r} 6\phi + \psi = 1 \\ \phi + \psi = 0 \end{array}\right\} \to \quad \phi = \tfrac{1}{5}; \ \psi = -\tfrac{1}{5}$$

$$\therefore \quad V_2 = S(1) \times \phi + \psi \times 1 = 3 \times \tfrac{1}{5} - \tfrac{1}{5} \times 1 = \tfrac{2}{5}.$$

$$\text{Stock} \quad \begin{tikzcd}[row sep=small] & 7 \\ 4 \arrow[ur] \arrow[dr] & \\ & 3 \end{tikzcd} \qquad\qquad \text{Option:} \quad \begin{tikzcd}[row sep=small] & 1 \\ V_0 \arrow[ur] \arrow[dr] & \\ & 0.4 \end{tikzcd}$$

Replicate with ϕ units of stock and units of bonds/riskless asset: $V = \phi S + \psi e^{rt}$

buy sell

$$\left.\begin{array}{r} 7\phi + \psi = 1 \\ 3\phi + \psi = 0.4 \end{array}\right\} \rightarrow \quad \phi = 0.15; \ \psi = -0.05$$

$$\therefore \quad V_0 = S(0) \times \phi + \psi \times 1 = 4 \times 0.15 - 0.05 \times 1 = 0.55.$$

So the binary call option struck at 5 is valued at 0.55

Now calculate the risk-neutral probabilities and use these to validate the option price calculated above.

$$q\,(\text{up}) = \frac{e^{rt}s - s_d}{s_u - s_d}, \quad q\,(\text{down}) = \frac{s_u - e^{rt}s}{s_u - s_d}$$

where $r = 0$.

$$= \frac{S - S_d}{S_u - S_d} \qquad 1 - q = \frac{S_u - S}{S_u - S_d}$$

ω_1 4 7 11

$$q\,(\omega_1) = \frac{4-3}{7-3} \times \frac{7-6}{11-6} = \frac{1}{20}$$

ω_2 4 7 6

$$q\,(\omega_2) = \frac{4-3}{7-3} \times \frac{11-7}{11-6} = \frac{1}{5}$$

ω_3 4 3 6

$$q\,(\omega_3) = \frac{7-4}{7-3} \times \frac{3-1}{6-1} = \frac{3}{10}$$

$$Q = \left\{ \frac{1}{20}, \frac{1}{5}, \frac{3}{10}, \frac{9}{20} \right\}$$

$\omega_4 \quad 4 \quad 3 \quad 1$

$$q(\omega_4) = \frac{7-4}{7-3} \times \frac{6-3}{6-1} = \frac{9}{20}$$

So the expected value (under the risk-neutral probabilities/measure) is

$$\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)}X\right]$$

where $r = 0, t = 0, T = 1$,

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}[X] &= \sum_{\omega} q(\omega_i)\, X(\omega_i) \\ &= \frac{1}{20} \times 1 + \frac{1}{5} \times 1 + \frac{3}{10} \times 1 + \frac{9}{20} \times 0 \\ &= 0.55\end{aligned}$$

hence verified.

Ex: re-do using Paul's Δ hedged portfolio