3 Differential Equations

3.1 Introduction

2 Types of Differential Equation (D.E)

(i) Ordinary Differential Equation (O.D.E)

Equation involving (ordinary) derivatives

$$x,\ y,\ \frac{dy}{dx},\ \frac{d^2y}{dx^2},\ \ldots\ldots..,\ \frac{d^ny}{dx^n} \quad \text{(some fixed } n\text{)}$$

y is some unknown function of x together with its derivatives, i.e.

$$F\left(x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{\ (n)}\right)=0 \tag{1}$$

Note $y^4 \neq y^{(4)}$

Also if $y = y\,(t)$, where t is time, then we often write

$$\dot{y}=\frac{dy}{dt},\ \ \ddot{y}=\frac{d^{\ 2}y}{dt^2},\ \ \ldots\ldots,\ \ddddot{y}=\frac{d^{\ 4}y}{dt^4}$$

(ii) Partial Differential Equation (PDE)

Involve partial derivatives, i.e. unknown function dependent on two or more variables,

e.g.

$$\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x \partial y}+\frac{\partial u}{\partial z}-u=0$$

So here we solving for the unknown function $u\left(x,y,z,t\right)$.

More complicated to solve - better for modelling real-life situations, e.g. finance, engineering & science.

In quant finance there is no concept of spatial variables, unlike other branches of mathematics.

Order of the highest derivative is the **order of the DE**

An ode is of **degree** r if $\dfrac{d^{\,n}y}{dx^n}$ (where n is the order of the derivative) appears with power r

$\left(r\epsilon\mathbb{Z}^+\right)$ $-$ the definition of n and r is distinct. Assume that any ode has the property that each

$\dfrac{d^\ell y}{dx^\ell}$ appears in the form $\left(\dfrac{d^\ell y}{dx^\ell}\right)^r \longrightarrow \left(\dfrac{d^n y}{dx^n}\right)^r$ order n and degree r.

Examples:

	DE	order	degree
(1)	$y' = 3y$	1	1
(2)	$(y')^3 + 4\sin y = x^3$	1	3
(3)	$\left(y^{(4)}\right)^2 + x^2\left(y^{(2)}\right)^5 + (y')^6 + y = 0$	4	2
(4)	$y'' = \sqrt{y' + y + x}$	2	2
(5)	$y'' + x(y')^3 - xy = 0$	2	1

Note - example (4) can be written as $(y'')^2 = y' + y + x$

We will consider ODE's of degree one, and of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)\, y = g(x)$$

$$\equiv \sum_{i=0}^{n} a_i(x)\, y^{(i)}(x) = g(x) \quad \text{(more pedantic)}$$

Note: $y^{(0)}(x)$ - zeroth derivative, i.e. $y(x)$.

This is a Linear ODE of order n, i.e. $r = 1\ \forall$ (for all) terms. Linear also because $a_i(x)$ not a function of $y^{(i)}(x)$ - else equation is Non-linear.

Examples:

	DE	Nature of DE
(1)	$2xy'' + x^2y' - (x+1)\,y = x^2$	Linear
(2)	$yy'' + xy' + y = 2$	$a_2 = y \Rightarrow$ Non-Linear
(3)	$y'' + \sqrt{y'} + y = x^2$	Non-Linear $\because (y')^{\frac{1}{2}}$
(4)	$\dfrac{d^4y}{dx^4} + y^4 = 0$	Non-Linear - y^4

Our aim is to solve our ODE either <u>explicitly</u> or by finding the most general $y\,(x)$ satisfying it or <u>implicitly</u> by finding the function y implicitly in terms of x, via the most general function g s.t $g\,(x, y) = 0$.

Suppose that y is given in terms of x and n arbitrary constants of integration c_1, c_2, ..., c_n.

So $\widetilde{g}\,(x,\ c_1,\ c_2, \ldots\ldots., c_n) = 0$. Differentiating $\widetilde{g}$, n times to get $(n+1)$ equations involving

$$c_1,\ c_2, \ldots\ldots., c_n,\ x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{\,(n)}.$$

Eliminating $c_1,\ c_2, \ldots\ldots., c_n$ we get an ODE

$$\widetilde{f}\ \left(x,\ y,\ y',\ y'',\ \ldots\ldots,\ y^{\,(n)}\right) = 0$$

Examples:

(1) $y = x^3 + ce^{-3x}$ (so 1 constant c)

$\Rightarrow \dfrac{dy}{dx} = 3x^2 - 3ce^{-3x}$, so eliminate c by taking $3y + y' = 3x^3 + 3x^2$, i.e.

$$-3x^2\left(x+1\right) + 3y + y' = 0$$

(2) $y = c_1 e^{-x} + c_2 e^{2x}$ (2 constant's so differentiate twice)

$y' = -c_1 e^{-x} + 2c_2 e^{2x} \Rightarrow y'' = c_1 e^{-x} + 4c_2 e^{2x}$

Now

$$\left.\begin{array}{ll} y + y' = 3c_2 e^{2x} & \text{(a)} \\ y' + y'' = 6c_2 e^{2x} & \text{(b)} \end{array}\right\}$$

and 2(a)=(b) $\therefore 2\left(\, y + y'\right) = y + y'' \rightarrow$

$$y'' - 2y' - y = 0.$$

Conversely it can be shown (under suitable conditions) that the general solution of an n$^{\text{th}}$ order ode will involve n arbitrary constants. If we specify values (i.e. boundary values) of

$$y,\ y',\ \ldots\ldots\ldots\ldots,\ y^{(n)}$$

for values of x, then the constants involved may be determined.

A solution $y = y(x)$ of (1) is a function that produces zero upon substitution into the lhs of (1).

Example:

$y'' - 3y' + 2y = 0$ is a 2$^{\text{nd}}$ order equation and $y = e^x$ is a solution.

$y = y' = y'' = e^x$ - substituting in equation gives $e^x - 3e^x + 2e^x = 0$. So we can verify that a function is the solution of a DE simply by substitution.

Exercise:

(1) Is $y(x) = c_1 \sin 2x + c_2 \cos 2x$ (c_1, c_2 arbitrary constants) a solution of $y'' + 4y = 0$

(2) Determine whether $y = x^2 - 1$ is a solution of $\left(\frac{dy}{dx}\right)^4 + y^2 = -1$

3.1.1 Initial & Boundary Value Problems

A DE together with conditions, an unknown function $y(x)$ and its derivatives, all given at the same value of independent variable x is called an **Initial Value Problem** (IVP).

e.g. $y'' + 2y' = e^x$; $y(\pi) = 1$, $y'(\pi) = 2$ is an IVP because both conditions are given at the same value $x = \pi$.

A **Boundary Value Problem** (BVP) is a DE together with conditions given at different values of x, i.e. $y'' + 2y' = e^x$; $y(0) = 1$, $y(1) = 1$.

Here conditions are defined at different values $x = 0$ and $x = 1$.

A solution to an IVP or BVP is a function $y(x)$ that both solves the DE and satisfies all given initial or boundary conditions.

Exercise: Determine whether any of the following functions

(a) $y_1 = \sin 2x$ (b) $y_2 = x$ (c) $y_3 = \frac{1}{2}\sin 2x$ is a solution of the IVP

$$y'' + 4y = 0; \quad y(0) = 0, \quad y'(0) = 1$$

3.2 First Order Ordinary Differential Equations

Standard form for a first order DE (in the unknown function $y(x)$) is

$$y' = f(x, y) \tag{2}$$

so given a 1^{st} order ode

$$F\left(x, y, y'\right) = 0$$

can often be rearranged in the form (2), e.g.

$$xy' + 2xy - y = 0 \Rightarrow \quad y' = \frac{y - 2x}{x}$$

3.2.1 One Variable Missing

This is the simplest case

y missing:

$$y' = f\,(x) \qquad \text{solution is } y = \int f(x)dx$$

x missing:

$$y' = f\,(y) \qquad \text{solution is } x = \int \frac{1}{f(y)}dy$$

Example:

$$y' = \cos^2 y\ , \quad y = \frac{\pi}{4} \text{ when } x = 2$$

$$\Rightarrow x = \int \frac{1}{\cos^2 y}dy = \int \sec^2 y\ dy \Rightarrow x = \tan y + c\ ,$$

c is a constant of integration.

This is the general solution. To obtain a particular solution use

$$y\,(2) = \frac{\pi}{4} \longrightarrow 2 = \tan\frac{\pi}{4} + c \Rightarrow c = 1$$

so rearranging gives

$$y = \arctan\,(x - 1)$$

3.2.2 Variable Separable

$$y' = g(x)\,h(y) \tag{3}$$

So $f(x, y) = g(x)\,h(y)$ where g and h are functions of x only and y only in turn. So

$$\frac{dy}{dx} = g(x)\,h(y) \rightarrow \int \frac{dy}{h(y)} = \int g(x)\,dx + c$$

$c -$ arbitrary constant.

Two examples follow on the next page:

$$\frac{dy}{dx} = \frac{x^2+2}{y}$$

$$\int y\ dy = \int \left(x^2+2\ \right) dx \rightarrow \frac{y^2}{2} = \frac{x^3}{3} + 2x + c$$

$$\frac{dy}{dx} = y \ln x \text{ subject to } y = 1 \text{ at } x = e\ (y(e) = 1)$$

$$\int \frac{dy}{y} = \int \ln x\ dx \quad \text{Recall: } \int \ln x\ dx = x(\ln x - 1)$$

$$\ln y = x(\ln x - 1) + c \rightarrow y = A \exp(x \ln x - x)$$

A $-$ arb. constant

now putting $x = e$, $y = 1$ gives $A = 1$. So solution becomes

$$y = \exp(\ln x^x)\exp(-x) \rightarrow y = \frac{x^x}{e^x} \Rightarrow y = \left(\frac{x}{e}\right)^x$$

3.2.3 Linear Equations

These are equations of the form

$$y' + P(x)\, y = Q(x) \tag{4}$$

which are similar to (3), but the presence of $Q(x)$ renders this no longer separable. We look for a function $R(x)$, called an **Integrating Factor** (I.F) so that

$$R(x)\ y' + R(x)P(x)\, y = \frac{d}{dx}\left(R(x)y\right)$$

So upon multiplying the lhs of (4), it becomes a derivative of $R(x)y$, i.e.

$$R\ y' + RPy = Ry' + R'y$$

from (4).

This gives $RPy = R'y \Rightarrow R(x)P(x) = \dfrac{dR}{dx}$, which is a DE for R which is separable, hence

$$\int \frac{dR}{R} = \int P dx + c \rightarrow \ln R = \int P dx + c$$

So $R(x) = K \exp\left(\int P\ dx\right)$, hence there exists a function $R(x)$ with the required property. Multiply (4) through by $R(x)$

$$\underbrace{R(x)\left[y' + P(x)y\right]}_{=\frac{d}{dx}(R(x)y)} = R(x)Q(x)$$

$$\frac{d}{dx}(Ry) = R(x)Q(x) \rightarrow R(x)y = \int R(x)Q(x)dx + B$$

B – arb. constant.

We also know the form of $R(x) \rightarrow$

$$yK \exp\left(\int P\ dx\right) = \int K \exp\left(\int P\ dx\right) Q(x)dx + B.$$

Divide through by K to give

$$y \,\mathsf{exp}\left(\int P\ dx\right) = \int \mathsf{exp}\left(\int P\ dx\right) Q(x)dx + \text{ constant.}$$

So we can take $K = 1$ in the expression for $R(x)$.

To solve $y' + P(x)\,y = Q(x)$ calculate $\boxed{R(x) = \mathsf{exp}\left(\int P\ dx\right)}$, which is the I.F.

Examples:

1. Solve $y' - \frac{1}{x}y = x^2$

In this case c.f (4) gives $P(x) \equiv -\frac{1}{x}$ & $Q(x) \equiv x^2$, therefore

I.F $R(x) = \exp\left(\int -\frac{1}{x}\, dx\right) = \exp\left(-\ln x\right) = \frac{1}{x}$. Multiply DE by $\frac{1}{x} \longrightarrow$

$$\begin{aligned}\frac{1}{x}\left(y' - \frac{1}{x}y\right) &= x \Rightarrow \frac{d}{dx}\left(\frac{y}{x}\right) = x \rightarrow \int d\left(x^{-1}y\right) \\ &= \int x dx + c\end{aligned}$$

$$\Rightarrow \frac{y}{x} = \frac{x^2}{2} + c \therefore \text{GS is } y = \frac{x^3}{2} + cx$$

2. Obtain the general solution of $(1+ye^x)\dfrac{dx}{dy}=e^x$

$$\frac{dy}{dx}=(1+ye^x)\,e^{-x}=e^{-x}+y\Rightarrow$$

$$\frac{dy}{dx}-y=e^{-x}$$

Which is a linear equation, with $P=-1$; $Q=e^{-x}$

$$\text{I.F } R(y)=\exp\left(\int -dx\right)=e^{-x}$$

so multiplying DE by I.F

$$\begin{aligned} e^{-x}\left(y'-y\right) &= e^{-2x}\rightarrow\frac{d}{dx}\left(ye^{-x}\right)=e^{-2x}\Rightarrow \\ \int d\left(ye^{-x}\right) &= \int e^{-2x}dx \end{aligned}$$

$$ye^{-x}=-\frac{1}{2}e^{-2x}+c\therefore y=ce^x-\frac{1}{2}e^{-x}\text{ is the GS.}$$

3.3 Second Order ODE's

Typical second order ODE (degree 1) is

$$y'' = f\left(x, y, y'\right)$$

solution involves two arbitrary constants.

3.3.1 Simplest Cases

A y', y missing, so $\boxed{y'' = f\ (x)}$

Integrate wrt x (twice): $y = \int \left(\int f\ (x)\ dx\right) dx$

Example: $y'' = 4x$

$$\text{GS } y = \int \left(\int 4x \; dx \right) dx = \int \left[2x^2 + C \right] dx = \frac{2x^3}{3} + Cx + D$$

B y missing, so $\boxed{y'' = f \; (y', \; x)}$

Put $P = y' \rightarrow y'' = \dfrac{dP}{dx} = f \; (P, x)$, i.e. $P' = f \; (P, x)$ - first order ode

Solve once $\rightarrow P(x)$

Solve again $\rightarrow y(x)$

Example: Solve $x\dfrac{d^{\,2}y}{dx^2} + 2\dfrac{dy}{dx} = x^3$

Note: $\boxed{\text{A}}$ is a special case of $\boxed{\text{B}}$

C y' and x missing, so $\boxed{y'' = f\ (y)}$

Put $p = y'$, then

$$\begin{aligned} \frac{d^2y}{dx^2} &= \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy} \\ &= f\ (y) \end{aligned}$$

So solve 1st order ode

$$p\frac{dp}{dy} = f\,(y)$$

which is separable, so

$$\int p\ dp = \int\ f\ (\ y)\ dy \rightarrow$$

$$\frac{1}{2}p^2 = \int f(y)\,dy + \text{const.}$$

Example: Solve $y^3 y'' = 4$

$\Rightarrow y'' = \frac{4}{y^3}$. Put $p = y' \rightarrow \frac{d^2y}{dx^2} = p\frac{dp}{dy} = \frac{4}{y^3}$

$\therefore \int p\ dp = \int \frac{4}{y^3}\ dy \Rightarrow p^2 = -\frac{4}{y^2} + D\ \ \therefore\ \ p = \frac{\pm\sqrt{Dy^2-4}}{y}$, so from our definition of p,

$$\frac{dy}{dx} = \frac{\pm\sqrt{Dy^2-4}}{y} \Rightarrow \int dx = \int \frac{\pm\ y}{\sqrt{Dy^2-4}}dy$$

Integrate rhs by substitution (i.e. $u = Dy^2 - 4$) to give

$$x = \frac{\pm\sqrt{Dy^2 - 4}}{D} + E \rightarrow \left[D\left(x - E\right)^2\right] = Dy^2 - 4$$

$$\therefore \text{ GS is } Dy^2 - D^2\left(x - E\right)^2 = 4$$

D x missing: $\quad y'' = f\ \ (y',\ y)$

Put $P = y'$, so $\dfrac{d^2y}{dx^2} = P\dfrac{dP}{dy} = f\ \ (P, y)$ - 1st order ODE

3.3.2 Linear ODE's of Order at least 2

General n^{th} order linear ode is of form:

$$a_n(x)\,y^{(n)} + a_{n-1}(x)\,y^{(n-1)} + ... + a_1(x)\,y' + a_0(x)\,y = g(x)$$

Use symbolic notation:

$$D \equiv \frac{d}{dx}\ ;\ \ D^r \equiv \frac{d^{\ r}}{dx^{\ r}} \quad \text{so } D^r y \equiv \frac{d^{\ r}y}{dx^{\ r}}$$

$\therefore\ \ a_r D^r \equiv a_r(x)\dfrac{d^{\ r}}{dx^{\ r}}$ so

$$a_r D^r y = a_r(x)\frac{d^{\ r}y}{dx^{\ r}}$$

Now introduce

$$L = a_n D^n + a_{n-1}D^{n-1} + a_{n-2}D^{n-2} + ... + a_1 D + a_0$$

so we can write a linear ode in the form

$$L\ y = g$$

$L-$ Linear Differential Operator of order n and its definition will be used throughout.

If $g(x) = 0 \ \forall x$, then $L\ y = 0$ is said to be **HOMOGENEOUS**.

$L\ y = 0$ is said to be the homogeneous part of $L\ y = g$.

L is a linear operator because as is trivially verified:

(1) $L(y_1 + y_2) = L\ (y_1) + L(y_2)$

(2) $L(cy) = cL(y) \qquad c \in \mathbb{R}$

GS of $Ly = g$ is given by

$$\boxed{y = y_c + y_p}$$

where y_c- Complimentary Function & y_p- Particular Integral (or Particular Solution)

$$\left.\begin{array}{l} y_c \text{ is solution of } \; Ly = 0 \\ y_p \text{ is solution of } \; Ly = g \end{array}\right\} \therefore \text{ GS } y = y_c + y_p$$

Look at homogeneous case $Ly = 0$. Put Ⓢ $=$ all solutions of $Ly = 0$. Then Ⓢ forms a vector space of dimension n. Functions $y_1(x)$, ..., $y_n(x)$ are LINEARLY DEPENDENT if $\exists\ \lambda_1$, ..., $\lambda_n \in \mathbb{R}$ (not all zero) s.t

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \ldots\ldots\ldots\ldots + \lambda_n y_n(x) = 0$$

Otherwise y_i's ($i = 1$, ..., n) are said to be LINEARLY INDEPENDENT (Lin. Indep.) $\Rightarrow$ whenever

$$\lambda_1 y_1(x) + \lambda_2 y_2(x) + \ldots\ldots\ldots\ldots + \lambda_n y_n(x) = 0 \quad \forall x$$

then $\lambda_1 = \lambda_2 =$ .... $= \lambda_n = 0$.

FACT:

(1) $L-$ n^{th} order linear operator, then $\exists$ n Lin. Indep. solutions $y_1, \ldots\ldots, y_n$ of $Ly = 0$ s.t GS of $Ly = 0$ is given by

$$y = \lambda_1 y_1 + \lambda_2 y_2 + \ldots\ldots\ldots\ldots + \lambda_n y_n \qquad \underset{1 \leq i \leq n}{\lambda_i \in \mathbb{R}} .$$

(2) Any n Lin. Indep. solutions of $Ly = 0$ have this property.

To solve $Ly = 0$ we need only find by "hook or by crook" n Lin. Indep. solutions.

3.3.3 Linear ODE's with Constant Coefficients

Consider Homogeneous case: $Ly = 0$.

All basic features appear for the case $n = 2$, so we analyse this.

$$L\ y = a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \qquad a,\ b,\ c \in \mathbb{R}$$

Try a solution of the form $\qquad y = \exp(\lambda x)$

$$L\left(e^{\lambda x}\right) = \left(aD^2 + bD + c\right)e^{\lambda x}$$

hence $a\lambda^2 + b\lambda + c = 0$ and so λ is a root of the quadratic equation

$a\lambda^2 + b\lambda + c = 0$ **AUXILLIARY EQUATION (A.E)**

There are three cases to consider:

(1) $b^2 - 4ac > 0$

So $\lambda_1 \neq \lambda_2 \in \mathbb{R}$, so GS is

$$\boxed{y = c_1 \exp(\lambda_1 x) + c_2 \exp(\lambda_2 x)}$$

c_1, c_2 − arb. const.

(2) $b^2 - 4ac = 0$

So $\lambda = \lambda_1 = \lambda_2 = -\dfrac{b}{2a}$

Clearly $e^{\lambda x}$ is a solution of $L\ y = 0$ - but theory tells us there exist two solutions for a 2nd order ode. So now try $y = x\exp(\lambda x)$

$$\begin{aligned} L\left(xe^{\lambda x}\right) &= \left(aD^2 + bD + c\right)\left(xe^{\lambda x}\right) \\ &= \underbrace{\left(a\lambda^2 + b\lambda + c\right)}_{=0}\left(xe^{\lambda x}\right) + \underbrace{(2a\lambda + b)}_{=0}\left(e^{\lambda x}\right) \\ &= 0 \end{aligned}$$

This gives a 2nd solution $\therefore$ GS is $y = c_1\exp(\lambda x) + c_2 x\exp(\lambda x)$, hence

$$\boxed{y = (c_1 + c_2 x)\exp(\lambda x)}$$

(3) $b^2 - 4ac < 0$

So $\lambda_1 \neq \lambda_2 \in \mathbb{C}$ - Complex conjugate pair $\lambda = p \pm iq$ where

$$p = -\frac{b}{2a}, \qquad q = \frac{1}{2a}\sqrt{\left|b^2 - 4ac\right|} \quad (\neq 0)$$

Hence

$$\begin{aligned} y &= c_1 \exp(p+iq)x + c_2 \exp(p-iq)x \\ &= c_1 e^{px} e^{iq} + c_2 e^{px} e^{-iq} = e^{px}\left(c_1 e^{iqx} + c_2 e^{-iqx}\right) \end{aligned}$$

Eulers identity gives $\exp(\pm i\theta) = \cos\theta \pm i \sin\theta$

Simplifying (using Euler) then gives the GS

$$y(x) = e^{px}(A\cos qx + B\sin qx)$$

Examples:

(1) $y'' - 3y' - 4y = 0$

Put $y = e^{\lambda x}$ to obtain A.E

A.E: $\lambda^2 - 3\lambda - 4 = 0 \rightarrow (\lambda - 4)(\lambda + 1) = 0 \qquad \Rightarrow \lambda = 4$ & -1 - 2 distinct $\mathbb{R}$ roots

$$\text{GS} \quad y(x) = Ae^{4x} + Be^{-x}$$

(2) $y'' - 8y' + 16y = 0$

A.E $\qquad \lambda^2 - 8\lambda + 16 = 0 \to (\lambda - 4)^2 = 0 \Rightarrow \lambda = 4\ ,\ 4$ (2 fold root)

'go up one', i.e. instead of $y = e^{\lambda x}$, take $y = xe^{\lambda x}$

$$\text{GS} \quad y(x) = (C + Dx)\, e^{4x}$$

(3) $y'' - 3y' + 4y = 0$

A.E: $\lambda^2 - 3\lambda + 4 = 0 \to \lambda = \dfrac{3 \pm \sqrt{9-16}}{2} = \dfrac{3 \pm i\sqrt{7}}{2} \equiv p \pm iq$

$\left(p = \dfrac{3}{2},\ \ q = \dfrac{\sqrt{7}}{2}\right)$

$$y = e^{\frac{3}{2}x}\left(a \cos \frac{\sqrt{7}}{2}x + b \sin \frac{\sqrt{7}}{2}x\right)$$

3.4 General n^{th} Order Equation

Consider

$$Ly = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \text{.........} + a_1 y' + a_0 y = 0$$

then

$$L \equiv a_n D^n + a_{n-1} D^{n-1} + a_{n-2} D^{n-2} + \text{.......} + a_1 D + a_0$$

so $Ly = 0$ and the A.E becomes

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + \text{............} + a_1 \lambda + a_0 = 0$$

Case 1 (Basic)

n distinct roots λ_1, ..., λ_n then $e^{\lambda_1 x}$, $e^{\lambda_2 x}$, ..., $e^{\lambda_n x}$ are n Lin. Indep. solutions giving a GS

$$y = \beta_1 e^{\lambda_1 x} + \beta_2 e^{\lambda_2 x} + \ldots\ldots + \beta_n e^{\lambda_n x}$$

$\beta_i -$ arb.

Case 2

If λ is a real $r-$ fold root of the A.E then $e^{\lambda x}$, $xe^{\lambda x}$, $x^2 e^{\lambda x}$,.........., $x^{r-1} e^{\lambda x}$ are r Lin. Indep. solutions of $Ly = 0$, i.e.

$$y = e^{\lambda x}\left(\alpha_1 + \alpha_2 x + \alpha_3 x^2 \ldots\ldots + \alpha_r\ x^{r-1}\right)$$

$\alpha_i -$ arb.

Case 3

If $\lambda = p + iq$ is a r - fold root of the A.E then so is $p - iq$

$$\left.\begin{array}{c} e^{px}\cos qx,\ \ xe^{px}\cos qx,\ \ldots\ldots\ldots\ldots, x^{r-1}e^{px}\cos qx \\ e^{px}\sin qx,\ xe^{px}\sin qx,\ \ldots\ldots\ldots\ldots, x^{r-1}e^{px}\sin qx \end{array}\right\}$$

$\rightarrow$ $2r$ Lin. Indep. solutions of $L\ y = 0$

$$\begin{aligned} \text{GS } y &= e^{px}\left(c_1 + c_2x + c_3x^2 + \ldots\ldots\ldots\ldots\right)\cos qx + \\ & \quad e^{px}\left(C_1 + C_2x + C_3x^2 + \ldots\ldots\ldots\ldots\right)\sin qx \end{aligned}$$

Examples: Find the GS of each ODE

(1) $y^{(4)} - 5y'' + 6y = 0$

A.E: $\lambda^4 - 5\lambda^2 + 6 = 0 \to \left(\lambda^2 - 2\right)\left(\lambda^2 - 3\right) = 0$

So $\lambda = \pm\sqrt{2}$, $\lambda = \pm\sqrt{3}$ - four distinct roots

$\therefore$ GS $y = Ae^{\sqrt{2}x} + Be^{-\sqrt{2}x} + Ce^{\sqrt{3}x} + De^{-\sqrt{3}x}$ (Case 1)

(2) $\dfrac{d^{\,6}y}{dx^6} - 5\dfrac{d^{\,4}y}{dx^4} = 0$

A.E: $\lambda^6 - 5\lambda^4 = 0$ roots: 0, 0, 0, 0, $\pm\sqrt{5}$

GS $y = Ae^{\sqrt{5}x} + Be^{-\sqrt{5}x} + \left(C + Dx + Ex^2 + Fx^3\right)$ ($\because \exp(0) = 1$)

(3) $\dfrac{d^{4}y}{dx^4} + 2\dfrac{d^{2}y}{dx^2} + y = 0$

A.E: $\lambda^4 + 2\lambda^2 + 1 = \left(\lambda^2 + 1\right)^2 = 0$ $\qquad \lambda = \pm i$ is a 2 fold root.

Example of Case (3)

$$y = A\cos x + Bx\cos x + C\sin x + Dx\sin x$$

3.5 Non-Homogeneous Case - Method of Undetermined Coefficients

$$\text{GS} \quad y = \text{C.F} + \text{P.I}$$

C.F comes from the roots of the A.E

There are three methods for finding P.I

(a) "Guesswork" - which we are interested in

(b) Annihilator

(c) D-operator Method

(a) Guesswork Method

If the rhs of the ode $g(x)$ is of a certain type, we can guess the form of P.I. We then try it out and determine the numerical coefficients.

The method will work when $g(x)$ has the following forms

i. Polynomial in x $\quad g(x) = p_0 + p_1 x + p_2 x^2 + \ldots\ldots\ldots + p_m x^m$.

ii. An exponential $\quad g(x) = Ce^{kx} \quad$ (Provided k is not a root of A.E).

iii. Trigonometric terms, $g(x)$ has the form $\sin ax$, $\cos ax$ $\quad$ (Provided ia is not a root of A.E).

iv. $g(x)$ is a combination of i. , ii. , iii. provided $g(x)$ does not contain part of the C.F (in which case use other methods).

Examples:

(1) $y'' + 3y' + 2y = 3e^{5x}$

The homogeneous part is the same as in (1), so $y_C = Ae^{-x} + Be^{-2x}$. For the non-homog. part we note that $g(x)$ has the form e^{kx}, so try $y_p = Ce^{5x}$, and $k = 5$ is not a solution of the A.E.

Substituting y_p into the DE gives

$$C\left(5^2 + 15 + 2\right)e^{5x} = 3e^{5x} \;\rightarrow C = \frac{1}{14}$$

$$\therefore y = Ae^{-x} + Be^{-2x} + \frac{1}{14}e^{5x}$$

(2) $y'' + 3y' + 2y = x^2$

GS $y = \text{C.F} + \text{P.I} = y_c + y_p$

C.F: A.E gives

$$\lambda^2 + 3\lambda + 2 = 0 \Rightarrow \lambda = -1,\ -2 \therefore y_c = ae^{-x} + be^{-2x}$$

P.I Now $g(x) = x^2$,

so try $y_p = p_0 + p_1 x + p_2 x^2 \quad \rightarrow y_p' = p_1 + 2p_2 x \quad \rightarrow y_p'' = 2p_2$

Now substitute these in to the DE, ie

$2p_2 + 3\left(p_1 + 2p_2 x\right) + 2\left(p_0 + p_1 x + p_2 x^2\right) = x^2$ and equate coefficients of x^n

$O\left(x^2\right)$: $\quad 2p_2 = 1 \Rightarrow p_2 = \frac{1}{2}$

$$O\left(x\right): \qquad 6p_2 + 2p_1 = 0 \Rightarrow p_1 = -\tfrac{3}{2}$$

$$O\left(x^0\right): \qquad 2p_2 + 3p_1 + 2p_0 = 0 \Rightarrow p_0 = \tfrac{7}{4}$$

$$\therefore \text{GS} \quad y = ae^{-x} + be^{-2x} + \frac{7}{4} - \frac{3}{2}x + \frac{1}{2}x^2$$

(3) $y'' - 5y' - 6y = \cos 3x$

A.E: $\lambda^2 - \lambda - 6 = 0 \Rightarrow \lambda = -1,\ 6 \Rightarrow y_c = \alpha e^{-x} + \beta e^{6x}$

Guided by the rhs, i.e. $g(x)$ is a trigonometric term, we can try $y_p = A\cos 3x + B \sin 3x$,and calculate the coefficients A and B.

How about a more sublime approach? Put $y_p = \mathrm{Re}\, K e^{i3x}$ for the unknown coefficient K.

$\rightarrow y_p' = 3\,\mathrm{Re}\, iKe^{i3x} \rightarrow y_p{}'' = -9\,\mathrm{Re}\, Ke^{i3x}$ and substitute into the DE, dropping Re

$$\begin{aligned} (-9 - 15i - 6)\, Ke^{i3x} &= e^{i3x} \\ -15\,(1+i)\, K &= 1 \\ -15K &= \frac{1}{1+i} \longrightarrow K = \frac{1}{2}(1-i) \end{aligned}$$

Hence $K = -\frac{1}{30}(1-i)$ to give

$$\begin{aligned} y_p &= -\frac{1}{30}\,\mathrm{Re}\,(1-i)(\cos 3x + i\sin 3x) \\ &= -\frac{1}{30}(\cos 3x + i\sin 3x - i\cos 3x + \sin 3x) \end{aligned}$$

so general solution becomes

$$y = \alpha e^{-x} + \beta e^{6x} - \frac{1}{30}(\cos 3x + \sin 3x)$$

3.5.1 Failure Case

Consider the DE $y'' - 5y' + 6y = e^{2x}$, which has a CF given by $y(x) = \alpha e^{2x} + \beta e^{3x}$. To find a PI, if we try $y_p = Ae^{2x}$, we have upon substitution

$$Ae^{2x}[4 - 10 + 6] = e^{2x}$$

so when $k \ (= 2)$ is also a solution of the C.F , then the trial solution $y_p = Ae^{kx}$ fails, so we must seek the existence of an alternative solution.

$Ly = y'' + ay' + b = \alpha e^{kx}$ - trial function is normally $y_p = Ce^{kx}$.

If k is a root of the A.E then $L\left(Ce^{kx}\right) = 0$ so this substitution does not work. In this case, we try $y_p = Cxe^{kx}$ - so 'go one up'.

This works provided k is not a repeated root of the A.E, if so try $y_p = Cx^2e^{kx}$, and so forth ....

3.6 Linear ODE's with Variable Coefficients - Euler Equation

In the previous sections we have looked at various second order DE's with constant coefficients. We now introduce a 2nd order equation in which the coefficients are variable in x. An equation of the form

$$L\ y = ax^2 \frac{d^2 y}{dx^2} + \beta x \frac{dy}{dx} + cy = g(x)$$

is called a Cauchy-Euler equation. Note the relationship between the coefficient and corresponding derivative term, ie $a_n(x) = ax^n$ and $\dfrac{d^{\,n} y}{dx^n}$, i.e. both power and order of derivative are n.

The equation is still linear. To solve the homogeneous part, we look for a solution of the form

$$y = x^{\lambda}$$

So $y' = \lambda x^{\lambda-1} \rightarrow y'' = \lambda\left(\lambda - 1\right)x^{\lambda-2}$, which upon substitution yields the quadratic, A.E.

$$a\lambda^2 + b\lambda + c = 0$$

[where $b = (\beta - a)$] which can be solved in the usual way - there are 3 cases to consider, depending upon the nature of $b^2 - 4ac$.

$\boxed{\text{Case 1:}}$ $b^2 - 4ac > 0 \to \lambda_1,\ \lambda_2 \in \mathbb{R}$ - 2 real distinct roots

$$\text{GS } y = Ax^{\lambda_1} + Bx^{\lambda_2}$$

$\boxed{\text{Case 2:}}$ $b^2 - 4ac = 0 \to \lambda = \lambda_1 = \lambda_2 \in \mathbb{R}$ - 1 real (double fold) root

$$\text{GS } y = x^{\lambda}\left(A + B\ln x\right)$$

$\boxed{\text{Case 3:}}$ $b^2 - 4ac < 0 \to \lambda = \alpha \pm i\beta \in \mathbb{C}$ - pair of complex conjugate roots

$$\text{GS } y = x^{\alpha}\left(A\cos\left(\beta\ln x\right) + B\sin\left(\beta\ln x\right)\right)$$

$\boxed{\text{Example 1}}$ Solve $\quad x^2y'' - 2xy' - 4y = 0$

Put $y = x^\lambda \Rightarrow y' = \lambda x^{\lambda-1} \Rightarrow y'' = \lambda(\lambda-1)x^{\lambda-2}$ and substitute in DE to obtain (upon simplification) the A.E. $\quad \lambda^2 - 3\lambda - 4 = 0 \rightarrow (\lambda-4)(\lambda+1) = 0$

$\Rightarrow \lambda = 4$ & -1 : 2 distinct $\mathbb{R}$ roots. So GS is

$$y(x) = Ax^4 + Bx^{-1}$$

$\boxed{\text{Example 2}}$ Solve $\quad x^2y'' - 7xy' + 16y = 0$

So assume $y = x^\lambda$

A.E $\quad \lambda^2 - 8\lambda + 16 = 0 \Rightarrow \lambda = 4$, 4 (2 fold root)

'go up one', i.e. instead of $y = x^\lambda$, take $y = x^\lambda \ln x$ to give

$$y(x) = x^4(A + B\ln x)$$

$\boxed{\text{Example 3}}$ Solve $\quad x^2y'' - 3xy' + 13y = 0$

Assume existence of solution of the form $y = x^\lambda$

A.E becomes $\quad \lambda^2 - 4\lambda + 13 = 0 \to \lambda = \dfrac{4 \pm \sqrt{16-52}}{2} = \dfrac{4 \pm 6i}{2}$

$\lambda_1 = 2 + 3i, \;\; \lambda_2 = 2 - 3i \;\; \equiv \alpha \pm i\beta \qquad (\alpha = 2, \;\; \beta = 3)$

$$y = x^2 \left(A \cos\left(3 \ln x\right) + B \sin\left(3 \ln x\right)\right)$$

3.6.1 Reduction to constant coefficient

The Euler equation considered above can be reduced to the constant coefficient problem discussed earlier by use of a suitable transform. To illustrate this simple technique we use a specific example.

Solve $\quad x^2y'' - xy' + y = \ln x$

Use the substitution $x = e^t$ i.e. $t = \ln x$. We now rewrite the the equation in terms of the variable t, so require new expressions for the derivatives (chain rule):

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\begin{aligned}\frac{d^{2}y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} \\ &= \frac{1}{x}\frac{dt}{dx}\frac{d}{dt}\frac{dy}{dt} - \frac{1}{x^2}\frac{dy}{dt} = \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\end{aligned}$$

$\therefore$ the Euler equation becomes

$$x^2\left(\frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}\right) - x\left(\frac{1}{x}\frac{dy}{dt}\right) + y = t \qquad \rightarrow$$

$$y''(t) - 2y'(t) + y = t$$

The solution of the homogeneous part , ie C.F. is $y_c = e^t(A + Bt)$.

The particular integral (P.I.) is obtained by using $y_p = p_0 + p_1 t$ to give $y_p = 2 + t$

The GS of this equation becomes

$$y(t) = e^t (A + Bt) + 2 + t$$

which is a function of t . The original problem was $y = y(x)$, so we use our transformation $t = \ln x$ to get the GS

$$y = x(A + B \ln x) + 2 + \ln x.$$