Calibration of volatility surfaces

Christoph Reisinger

christoph.reisinger@maths.ox.ac.uk.

OCIAM, Mathematical Institute

Oxford University

Overview

- Implied and local volatility
  - Black-Scholes model and implied volatility
  - smile modelling, (local) volatility and other models
- Dicrete time formulae: Derman-Kani trees
- Continuous time formulae
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  - Inverse and ill-posed problems
  - Regularisation techniques
  - Calibration examples for local volatility

Implied and local volatility

Black-Scholes model

Assumes geometric Brownian motion

$$\frac{\mathrm{d}S}{S} = \mu \, \mathrm{d}t + \sigma \, \mathrm{d}W$$

with constant coefficients.

In the simplest case of a vanilla call the PDE for the option price is

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + r\, S \frac{\partial u}{\partial S} - r\, u = 0,$$

and the final condition is

$$u(S,T) = \max(S-K, 0).$$

Black-Scholes equation

This problem can be solved analytically and the solution is

$$u(S,t) = S\,N(d_1) - e^{-r(T-t)}K\,N(d_2),$$

where

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sqrt{\sigma^2(T-t)}}, \qquad d_2 = d_1 - |\sigma|\,\sqrt{T-t}.$$

This gives us a mapping

$$u = u(S,t;K,T;\sigma,r)$$

from the asset price S and the parameters to the option price.

Input parameters

Now consider which of the quantities in

$$u = u(S, t; K, T; \sigma, r)$$

are directly observable, and which are not:

quantity	observable?	
u	yes	it is a quoted price
S	yes	it is today's spot price
t	yes	it is today's time/date
T	yes	it is the contracted expiry date
K	yes	it is the contracted strike price
r	yes	it is today's interest rate
σ	**NO!**	

Identification of σ

- u increases/decreases strictly (and continuously) in σ
- $\rightarrow$ one-to-one correspondence between option price and volatility (given S, $T-t$, K, $\ldots$)
- in other words, the option's vega,

$$\varphi(S,t) = \frac{\partial u}{\partial \sigma}(S,t;K,T;\sigma,r),$$

  does not change sign

$\rightarrow$ The volatility can be identified from u and is called *implied volatility*.

Traders quote prices in terms of implied volatility.

Implied volatility

Assume that u, S, t, T, K and r are observed.

For the simplest vanilla call option, we define the implied volatility to be the value of the (unobservable) σ which, when substituted into the Black-Scholes formula (for given S, t, T, K and r) gives the observed value of the option, u.

More generally, the (Black-Scholes) implied volatility for an option is the (constant) volatility which when substituted into the Black-Scholes model (with all other observables fixed) gives the observed market price of the option. (It may be necessary to solve a more complex Black-Scholes model numerically, but the principle is still the same.)

Rebonato: „A smiley implied volatility ist the wrong number to put in the wrong formula to obtain the right price“.

Uniqueness: maximum principle

Which products have a unique implied volatility?

Consider

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2 \frac{\partial^2 u}{\partial S^2} + r\, S \frac{\partial u}{\partial S} - r\, u = 0, \quad u(S,T) = g(S).$$

Differentiating this equation with respect to σ gives

$$\frac{\partial \varphi}{\partial t} + \tfrac{1}{2}\sigma^2\, S^2 \frac{\partial^2 \varphi}{\partial S^2} + r\, S \frac{\partial \varphi}{\partial S} - r\, \varphi = -\sigma\, S^2 \frac{\partial^2 u}{\partial S^2}, \quad \varphi(S,T) = 0.$$

Provided that the “source term” $\partial^2 u/\partial S^2$ is either strictly positive or strictly negative for $t < T$, φ will be strictly negative or strictly positive, respectively (maximum principle for parabolic PDEs).

Uniqueness: financial argument

Consider an option which expires worthless, but which pays the holder a continuous cash flow of $C(S,t)\,\mathrm{d}t$ in the interval $(t, t+\,\mathrm{d}t)$.

The value u of the option satisfies

$$\frac{\partial u}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} + r\,S\frac{\partial u}{\partial S} - r\,u = -C(S,t), \quad u(S,T) = 0.$$

Clearly if the cash flow is strictly positive, $C(S,t) > 0$, then the option has a positive value for the holder prior to expiry, so $u(S,t) > 0$.

If the cash flow is stricly negative, $C(S,t) < 0$, then the option has a negative value for the holder prior to expiry, hence $u(S,t) < 0$.

Remarks on implied vola

- The (Black-Scholes) implied volatility is model independent.
- It is a way of quoting a price, and does not (necessarily) resemble any model parameter.
- It is the same for put and call (put-call-parity).

According to the assumptions of the Black-Scholes model, the implied volatility should be the same for all K, T. In practice, however, it depends on both. This relation $\sigma(K,T)$ is called a volatility surface.

- dependence on K: *smile* (eg FX markets), *skew* (eg stock markets)
- dependence on T: *term structure*

Stylised facts are empirical observation valid for a large class of markets as agreed upon by a vast majority of practitioners.

Interpretation 1

In an *equilibrim* model, we regard our assumptions as correct and the observed data as potentially incorrect and potentially arbitragable.

If we seriously believe in our equilibrium model, fit it in whatever way we deam best, and we find that the prices predicted by the equilibrium model are different from the prices observed in the market then we have an arbritage. This arbitrage is clearly contingent on our model; it is *model* based *arbitrage.*

Interpretation 2

As an alternative, suppose that we regard the observed prices of a given class of options as correct, in the sense that they can not be arbitraged. For instance we might regard the prices of **traded** call options as correct.

This leads to the concept of **no-arbitage pricing;** we try to find a model that is sufficently general that it can be "calibrated" to reproduce *all* the observed prices for our particular class of options.

In a no-arbitrage model, we regard the observed prices of, say, traded vanilla call options as correct. We do not attempt to predict their prices.

Rather we calibrate our model for the underlying to reproduce these prices and then use this model to price more complicated contracts, for instance barrier options, or lookbacks, etc.

Smile modelling

Models that admit smiles:

- local volatility models

$$\mathrm{d}S = \mu\,\mathrm{d}t + \sigma(S,t)S\,\mathrm{d}W$$

- stochastic volatility models, eg

$$\begin{aligned} \mathrm{d}S &= \mu\,\mathrm{d}t + \sigma(V)S\,\mathrm{d}W \\ \mathrm{d}V &= \alpha(m-V)\,\mathrm{d}t + \sigma_V\,\mathrm{d}W_V \end{aligned}$$

  with mean m and reversion parameter α

- jump-diffusion models, eg

$$\mathrm{d}S = \mu\,\mathrm{d}t + \sigma(S,t)S\,\mathrm{d}W + A(S)\,\mathrm{d}N$$

  with Poisson process N and random amplitude A

Local volatility

σ is a parameter in the model for $S \rightarrow$ should be independent of option details

$\rightarrow$ introduce a *local volatility* function $(S, t) \rightarrow \sigma(S, t)$ and assume

$$\mathrm{d}S = \mu \,\mathrm{d}t + \sigma(S, t) S \,\mathrm{d}W$$

Special cases:

- $\sigma = \sigma(t)$: replace σ^2 in the Black-Scholes formulae by $\frac{1}{T-t} \int_t^T \sigma(s)^2 ds$
  $\rightarrow$ analytical solution, but cannot possibly explain smile
- $\sigma = \sigma(S)$, eg a *constant elasticity of variance (CEV) model*

$$\frac{\mathrm{d}S}{S} = \mu \,\mathrm{d}t + \sigma \, S^{\alpha} \, \mathrm{d}W$$

Calibration

For a European option, we obtain

$$\frac{\partial u}{\partial t} + \frac{1}{2} S^2 \sigma^2(S,t) \frac{\partial^2 u}{\partial S^2} + r\left(S\frac{\partial u}{\partial S} - u\right) = 0$$

with suitable initial and boundary conditions.

In general this PDE has to be solved numerically. The solution

$$u = u(S,t;K,T;\sigma(.,.),r)$$

depends on the (unknown) local volatility function.

$\longrightarrow$ calibrate model for vanilla options, then use it to price other (exotic) options

+ hits quoted prices 'exactly'
- predictability?

Inverse problem

Inverse problem: given quoted prices

$$u(S_0, 0; K_i, T_i; \sigma), \quad i = 1, \ldots, N,$$

(spot price $S_0 = S(0)$), find σ!

Problems:

1. observations for different K, T, but PDE in S, t
2. inverse problem is *ill-posed*

Remedy:

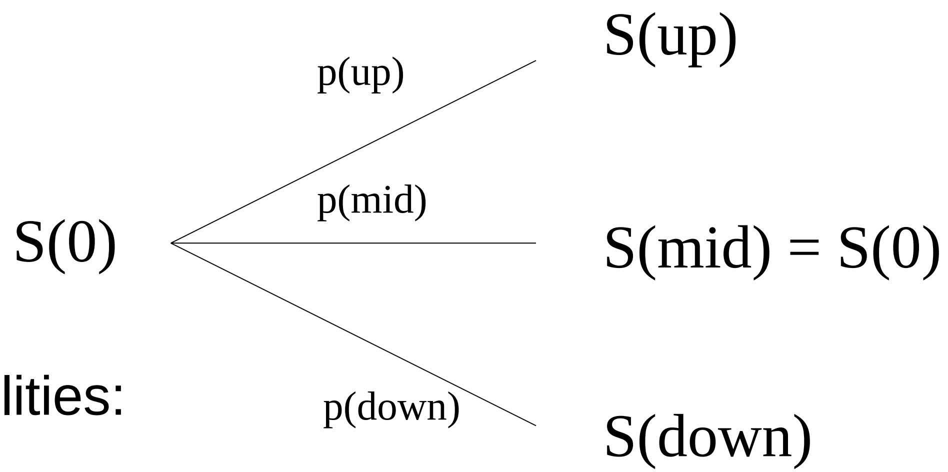
1. translate equation into (K, T)-space via (risk neutral) *probability density function*
$$p(S, t; S', T)$$
2. 'regularisation' and numerical solution

Discrete time formulae

Derman and Kani trees

One period model

Consider one period with three possible states:



Associated risk-neutral probabilities:

$$p(up) + p(mid) + p(down) = 1$$

Martingale condition (Arbitrage argument):

$$S(up)p(up) + S(mid)p(mid) + S(down)p(down) = S(mid)\exp(r\tau)$$

- Now usually: moment matching to local vol.
- Here: extract information from quoted option prices.

One period model

Consider a call with strike $S(mid)$, expiring at 1:

$$payoff = \begin{cases} S(up) - S(mid) & \text{up} \\ 0 & \text{mid} \\ 0 & \text{down} \end{cases}$$

Model option price is discounted expected pay-off

$$\begin{aligned} call(model) &= \exp(-r\tau)[p(up)\cdot(S(up)-S(mid)) + p(mid)\cdot 0 + p(down)\cdot 0] \\ &= \exp(-r\tau)\cdot p(up)\cdot(S(up)-S(mid)) \end{aligned}$$

Now set $call(model)$ equal to observed market price for the call and obtain

$$p(up) = \frac{call(market)\exp(rt)}{S(up)-S(mid)}.$$

$p(mid)$ and $p(down)$ from the other two conditions.

Tree reminder

We consider a trinomial tree with

- root $(t_0, S_1(t_0))$
- at time t_n nodes $(t_n, S_i(t_n))$, $i = 1, \ldots, 2n+1$
- from $(t_n, S_i(t_n))$ the stock
  - moves up to $(t_{n+1}, S_{i+2}(t_{n+1}))$ with probability $p_i(t_n)$,
  - moves down to $(t_{n+1}, S_i(t_{n+1}))$ with probability $q_i(t_n)$
  - remains at $S_i(t_n) = S_{i+1}(t_{n+1})$ with probability $1 - p_i - q_i$
- define $\lambda_i(t_n)$ the probability that the stock has value S_i at t_n
- $\lambda_i(t_n)$ can also be viewed as the price of an option with expiry t_n that pays 1 if the stock is $S_i(t_n)$ and 0 otherwise
- $\lambda_i(t_n)$ is the sum of products of probabilities along all paths ending in $(t_n, S_i(t_n))$
- $\lambda_i(t_n)$ is a discrete Greens function

Calibration

- In the forward problem the probabilities are chosen to match the moments of the approximated process.
- In this setting the local volatility is not known and the probabilities are chosen such that vanillia calls and puts are priced correctly over all maturities and strikes.

Freedom of Arbitrage gives the Martingale condition

$$p_i S_{i+2} + (1 - p_i - q_i) S_{i+1} + q_i S_i = F_i,$$

where actually $S_i = S_i(n)$ and $F_i = F_i(n) = \exp(r\tau) S_i(n)$ is the forward price.

If $C(K, t_{n+1})$ is today's market price of a call struck at K and expiring at t_{n+1}

$$C(K, t_{n+1}) = \exp(-r\tau) \sum_{j=1}^{2n+1} \left[\lambda_{j-2} p_{j-2} + \lambda_{j-1}(1 - p_{j-1} - q_{j-1}) + \lambda_j q_j\right] \max(S_j - K, 0)$$

Fit to options

We choose an option with strike S_{i+1} expiring at t_{n+1}

$$\begin{aligned}
\exp(r\tau)C(S_{i+1}, t_{n+1}) \quad &= \\
&= \sum_{j=i+2} \left[\lambda_{j-2}p_{j-2} + \lambda_{j-1}(1 - p_{j-1} - q_{j-1}) + \lambda_j q_j\right](S_j - S_{i+1}) \\
&= \lambda_i p_i (S_{i+2} - S_{i+1}) + \sum_{j=i+1} \lambda_j p_j (S_{j+2} - S_{i+1}) + \lambda_j (1 - p_j - q_j)(S_{j+1} - S_{i+1}) + \\
&= \lambda_i p_i (S_{i+2} - S_{i+1}) + \sum_{j=i+1} \lambda_j \left[p_j S_{j+2} + (1 - p_j - q_j) S_{j+1} + q_j S_j - S_{i+1}\right] \\
&= \lambda_i p_i (S_{i+2} - S_{i+1}) + \sum_{j=i+1} \lambda_j (F_j - S_{i+1})
\end{aligned}$$

and obtain

$$p_i = \frac{\exp(r\tau)C(S_{i+1}, t_{n+1}) - \sum_{j=i+1}^{2n} \lambda_j (F_j - S_{i+1})}{\lambda_i (S_{i+2} - S_{i+1})}$$

Calls and Puts

From the Martingale condition

$$q_i = \frac{F_i - p_i(S_{i+2} - S_{i+1}) - S_{i+1}}{S_i - S_{i+1}}.$$

The transition densities $\lambda_i(t_{n+1})$ are then updated accordingly.

Call options have been chosen for the ease of recursive solution from the top of the time slice. Alternatively puts can be used to solve from below (put-call parity!) with the result

$$q_i = \frac{\exp(r\tau)P(S_{i+1}, t_{n+1}) - \sum_{j=0}^{i-1} \lambda_j (S_{i+1} - F_j)}{\lambda_i (S_{i+1} - S_i)}, \quad p_i = \frac{F_i + q_i(S_{i+1} - S_i) - S_{i+1}}{S_{i+2} - S_{i+1}}$$

In practise it is advisible to switch between puts and calls at the center node $(n+1, n)$ to avoid far in-the-money options that contain no volatility information.

Practical issues

Approximation error:

- To offset numerical errors due to the tree approximation, sometimes the market option price is replaced by the value obtained from the same tree assuming a constant volatility equal to the implied volatility.
- There seems no theoretical justification why the errors should be strongly correlated, but in practise results improve crucially.

Arbitrage-freeness:

- One may encounter negative probabilities during the algorithm (arbitrage).
- This effect can often be overcome by a skewed state-space with more nodes in regions with fast varying implied vol. Otherwise option prices are modified to ensure positivity (at the expense of a worse fit).

Binomial trees:

- Reduced flexibility due to fewer degrees of freedom $\longrightarrow$ fixed state-space.

Continuous time: Dupire formula

Probability density function

Like in the discrete case, the transition probabilities play a central role. $p(S, t; S', T)$ can be viewed in two ways:

- If S and t are fixed (today's spot price and date) then we can regard $p(S, t; S', T)$ as the probability density that at time $T > t$ the spot price will be S'. This is a conditional probability density for future values, S' and T, given that the present values are S and $t < T$.

- If S' and T are fixed (some given value of the spot price and date, say) then $p(S, t; S', T)$ is the probability density that at time $t < T$ the spot price *was* S; again this is a conditional probability; the probability that the spot price was S at time t given that the spot price is S' at time T.

Kolmogorov equations

p is governed by the Kolmogorov equations

$$\begin{aligned}\frac{\partial p}{\partial T} &= \tfrac{1}{2}\frac{\partial^2}{\partial {S'}^2}\left(\sigma(S',T)^2 {S'}^2 p\right) - \frac{\partial}{\partial S'}\left(rS'p\right),\\ -\frac{\partial p}{\partial t} &= \tfrac{1}{2}\sigma(S,t)^2 S^2 \frac{\partial^2 p}{\partial S^2} + rS\frac{\partial p}{\partial S}.\end{aligned}$$

The price u of a European call option has the representation

$$\begin{aligned}u(S,t;K,T) &= e^{-r(T-t)}\int_0^\infty p(S,t;S',T)u(S',T;K,T)\,\mathrm{d}S'\\ &= e^{-r(T-t)}\int_K^\infty p(S,t;S',T)(S'-K)\,\mathrm{d}S'.\end{aligned}$$

Shimko's method

Use this to derive a PDE for u in K and T with S and t as parameters:

- $$\frac{\partial u}{\partial K} = e^{-r(T-t)} \int_K^\infty p \, \mathrm{d}S'$$
- $$\frac{\partial^2 u}{\partial K^2} = e^{-r(T-t)} p(S, t; K, T)$$

Now use p directly to predict future values of u.

Problems:

- gives only todays price (p is conditional on S, t)
- requires continuum of strikes
- differentiation very unstable

Dupire's formula

To obtain σ, employ

$$\frac{\partial u}{\partial T} = -ru + e^{-r(T-t)} \int_K^\infty \frac{\partial}{\partial T} p(S,t;S',T)(S'-K)\,\mathrm{d}S'$$

and the Kolmogorov equation to express $\frac{\partial p}{\partial T}$:

$$\begin{aligned}
&\int_K^\infty \frac{\partial^2}{\partial S'^2}\left(\sigma^2(S',T)S'^2 p\right)(S'-K)\,\mathrm{d}S' \\
&\quad = \underbrace{\left[\frac{\partial}{\partial S'}\left(\sigma^2(S',T)S'^2 p\right)(S'-K)\right]_K^\infty}_{0} - \int_K^\infty \frac{\partial}{\partial S'}\left(\sigma^2(S',T)S'^2 p\right)\,\mathrm{d}S' \\
&\quad = \sigma^2(K,T)K^2 p(S,t;K,T)
\end{aligned}$$

and the second term is

$$\int_K^\infty \frac{\partial}{\partial S'}(rSp)\,(S'-K)\,\mathsf{d}S' \quad = \quad \underbrace{\left[rS'p(S'-K)\right]_K^\infty}_{0} - \int_K^\infty rS'p\,\mathsf{d}S'.$$

Express

$$\begin{aligned}\int_K^\infty rS'p\,\mathsf{d}S' &= r\int_K^\infty (S'-K)p\,\mathsf{d}S' + rK\int_K^\infty p\,\mathsf{d}S' \\ &= re^{r(T-t)}\left(u + rK\frac{\partial u}{\partial K}\right).\end{aligned}$$

Dupire’s formula

Insert to obtain

$$
\begin{aligned}
\frac{\partial u}{\partial T} &= -ru + e^{-r(T-t)}\left(\frac{1}{2}\sigma^2 K^2 p + r\int_K^\infty S' p \, \mathsf{d}S'\right) \\
&= rK\frac{\partial u}{\partial K} + e^{-r(T-t)}\frac{1}{2}\sigma^2 K^2 p,
\end{aligned}
$$

so

$$
\frac{\partial u}{\partial T} = \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 u}{\partial K^2} - rK\frac{\partial u}{\partial K}.
$$

This is a PDE for u in the parameters K and T, the adjoint equation to the Black-Scholes PDE.

Dupire's formula

'Symbolically', we can compute

$$\sigma^2(K,T) = \frac{\partial u/\partial T + rK(\partial u/\partial K)}{\frac{1}{2}K^2(\partial^2 u/\partial K^2).}$$

If we now retrace our calculations we find that, because the call value u is a function of today's spot, S, today's date, t, the call's strike K and the call's maturity, T,

$$u = u(S,t;K,T),$$

what we have called $\sigma(K,T)$ is actually

$$\sigma^2(S,t;K,T) = \frac{(\partial u/\partial T)(S,t;K,T) + rK(\partial u/\partial K)(S,t;K,T)}{\frac{1}{2}K^2(\partial^2 u/\partial K^2)(S,t;K,T)}.$$

Dupire’s formula

That is, we have found a *local volatility surface*

$$\sigma(K,T) \text{ or more correctly } \sigma(S,t;K,T)$$

as it is conditional on today’s spot price S and date t.

Given today’s spot price S and date t, we can vary the strike K and the maturity T but not S and t. Thus S and t are parameters, whilst K and T are variables.

It is not unreasonable to write

$$\sigma(K,T)$$

in the same way we write $u(S,t)$ in a Black–Scholes model instead of $u(S,t;K,T;\sigma,r)$.

Dupire's formula

Adopting this notation (as is common in the literature), however obscures the rather important fact that "$\sigma(K,T)$" **is conditional on the value of S at the time t we computed** "$\sigma(K,T)$". The notation $\sigma(S,t;K,T)$ makes this dependence clear.

Suppose that today's date is t_1 and today's spot price is S_1. For some given K and $T > t_1$ we compute $\sigma(S_1, t_1; K, T)$.

Now consider tomorrow, t_2, and say the spot price is S_2. For the same values of K and $T > t_2 > t_1$ we can compute $\sigma(S_2, t_2; K, T)$.

In general we would find that

$$\sigma(S_1, t_1; K, T) \neq \sigma(S_2, t_2; K, T).$$

Dupire's formula

By definition, the time, t_1 and spot price, S_1, at which we computed "$\sigma(K,T)$" are fixed; for a given spot price S and time t, we can only only vary the strike K and expiry T of our option.

Now for fixed S_1 and t_1, replace K by S and T by t to arrive at

$$\text{“}\sigma(S,t)\text{”} = \sigma(S_1, t_1; S, t).$$

This gives us a volatility surface "$\sigma(S,t)$" which is conditional on the values t_1 and S_1, ie, conditional on the date and spot price on that date that we computed it.

For this reason, we would expect the local volatility surface "$\sigma(S,t)$" to change from day to day; what we write as "$\sigma(S,t)$" is in fact $\sigma(S_1, t_1; K, T)$ with K replaced by S and T replaced by t.

Dupire's formula

Note that the nominator is $\partial^2 u/\partial K^2$, which tends to zero for $K \to \infty$.

$$\sigma(K,T)^2 = \frac{\hat\sigma^2 + 2\hat\sigma\,(T-t)\frac{\partial\hat\sigma}{\partial T} + 2r\,\hat\sigma\,K(T-t)\frac{\partial\hat\sigma}{\partial K}}{\left(1 + K\,d_1\sqrt{T-t}\frac{\partial\hat\sigma}{\partial K}\right)^2 + \hat\sigma\,(T-t)\,K^2\left(\frac{\partial^2\hat\sigma}{\partial K^2} - d_1\left(\frac{\partial\hat\sigma}{\partial K}\right)^2\sqrt{(T-t)}\right)}$$

in terms of implied vola $\hat\sigma$ (due to Malz) is more stable where, as usual,

$$d_1 = \frac{\log(S/K) + (r + \frac{1}{2}\hat\sigma^2)(T-t)}{\hat\sigma\sqrt{T-t}}.$$

Further problems:

- requires continuum of strikes and maturities (interpolation?)
- numerical differentiation is very unstable

Regularisation of the inverse problem

Well- and ill-posedness

A problem

$$F(x) = y, x \in X, y \in Y$$

is called *well-posed*, if the following is given:

1. existence of solution
2. uniqueness of solution
3. continuous dependence on data

Otherwise it is called *ill-posed*.

In our setting well-posedness means:

1. the model is compatible with the observed prices
2. the model is not ‘over-fitted’
3. the parameters can be identified in a stable way

Ill-posedness

Remedies, if the problem

$$F(x) = y, x \in X, y \in Y$$

is ill-posed:

1. non-existence: find best fit, ie replace equation by an optimisation problem
$$\|F(x) - y\|^2 \to \min$$
$\to$ *least squares solution*
2. non-uniqueness: choose a particular solution, eg
$$\|x - x_0\| \to \min$$
for an initial guess x_0
3. continuous dependence on data: regularisation schemes

Regularisation

Approximate the problem

$$F(x) = y, x \in X, y \in Y$$

by

$$\|F(x) - y\|_Y^2 + \lambda |x|_X^2 \to \min,$$

where $|.|_X$ is eg a seminorm in X. $\lambda \ldots$ regularisation parameter.

In our calibration setting:

$$\sum_{i=1}^{N} |u(S_0, 0; K_i, T_i; \sigma) - V_i|^2 + \lambda |\sigma|^2 \to \min,$$

where V_i are the observed prices at $t = 0$ for spot price S_0, K_i and T_i.

Works in this area differ essentially in the choice of the regularisation term.

Lagnado-Osher

Lagnado and Osher, *Risk*, 1997:

- $$|\sigma|^2 := \|\nabla\sigma\|_2^2 = \int_0^T \int_0^\infty \left(\frac{\partial\sigma}{\partial S}\right)^2 + \left(\frac{\partial\sigma}{\partial t}\right)^2 \,\mathrm{d}S\,\mathrm{d}t$$
- solve “direct problem” by implicit finite differences
- pointwise representation of σ
- variational derivative as solution of inhomogeneous PDE
- observation: volatility values far in- and far out-of-the-money uneffected
- solve for $\bar{V} = (V^a + V^b)/2 \rightarrow$ prices well in bid-ask for S&P 500 calls

Jackson-Süli-Howison

Jackson, Süli, Howison, *Journal of Computational Finance*, 1998:

- same penalty term as Lagnado and Osher
- consider objective function

$$\sum_{i=1}^{N} w_i |u(S_0, 0; K_i, T_i; \sigma) - V_i|^2$$

  $\rightarrow$ greater weight to options that are heavily traded, short dated, close to the money

- representation of σ by cubic splines in S and piecewise linear in t
- more points around the money, constant extrapolation in the far range
- solution of direct problem by adaptive finite elements

Achdou-Pironneau

Achdou and Pironneau, 2004:

- analysis of variational direct and inverse problem for American options
- $|\sigma|^2 =$

$$\int_0^T \int_0^{\bar{S}} \left[a \left(S \frac{\partial \sigma}{\partial S} \right)^2 + b \left(\frac{\partial \sigma}{\partial t} \right)^2 + c \left(S \frac{\partial^2 \sigma}{\partial S \partial t} \right)^2 + d \left(S^2 \frac{\partial^2 \sigma}{\partial S^2} \right)^2 + e(\sigma - \sigma^\star)^2 \right] \mathrm{d}S \, \mathrm{d}t$$

with constants a, b, c, d, e and an *a priori* guess $\sigma^\star$

Egger

Egger, MSc Thesis, Linz, 2001:

- uses adjoint equation

$$\begin{aligned}\frac{\partial u}{\partial T} &= \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 u}{\partial K^2} - rK\frac{\partial u}{\partial K}\\ u(S,t;K,t) &= (K-S)_+\end{aligned}$$

- $$|\sigma|^2 = \int_0^\infty \left(\frac{\partial \sigma}{\partial S}\right)^2 + \left(\frac{\partial^2 \sigma}{\partial S^2}\right)^2 \,\mathrm{d}S$$

- finite element solution and spline representation of σ
- proof of stability and convergence rates
- quasi-Newton BSGF algorithm as non-linear solver

Related work

- Implied binomial trees. Rubinstein, *Journal of Finance*, 1994.
- Implied Trinomial Trees of the Volatility Smile. Derman, Kani and Chriss, *Goldman Sachs Research Notes*, 1996.
- Pricing with a Smile. Dupire, *RISK*, 1994.
- Determining volatility surfaces and option values from an implied volatility smile. Carr and Madan, 1998.
- Volatility and Correlation. Rebonato, *Wiley*, 1999.
- Dynamics of implied volatility surfaces. Cont and da Fonseca, *Quantitative Finance*, 2001.
- Regularization of inverse problems. Engl, Hanke and Neubauer, *Kluwer Academic Publishers*, 1996.