Cubic spline interpolation

Position of the problem

Given n points with ordered abscissas $x_1, \cdots, x_n$ with their ordinates $y_1, \cdots, y_n$, the cubic spline interpolation consists in finding 3^{rd} degree polynomials for each segment $[x_j, x_{j+1}]$. These polynomials must be continuous at each x_j, and so must their first and second order derivatives.

A general piecewise 3^{rd} degree polynomial satisfying these continuity constraints can be written, for each x such that $x_j \leq x < x_{j+1}$:

$$y(x) \quad = \quad \alpha_{1j} y_j + \alpha_{2j} y_{j+1} + \alpha_{3j} d_j + \alpha_{4j} d_{j+1}$$

where :

$$\begin{aligned}
\xi_j &= x_{j+1} - x_j, 1 \leq j \leq n-1 \\
\alpha_{1j} &= \frac{x_{j+1} - x}{\xi_j} \\
\alpha_{2j} &= \frac{x - x_j}{\xi_j} \\
\alpha_{3j} &= \frac{1}{6}(\alpha_{1j}^3 - \alpha_{1j})\xi_j^2 \\
\alpha_{4j} &= \frac{1}{6}(\alpha_{2j}^3 - \alpha_{2j})\xi_j^2
\end{aligned}$$

The n quantities d_j are the second derivatives at each abscissa x_j. They are the solution of the following $n-2$ equations expressing the continuity of the first derivative (for $2 \leq j \leq n-1$) :

$$\frac{\xi_{j-1}}{6} d_{j-1} + \frac{\xi_j + \xi_{j-1}}{3} d_j + \frac{\xi_j}{6} d_{j+1} = \frac{1}{\xi_j} y_{j+1} - \left(\frac{1}{\xi_j} + \frac{1}{\xi_{j-1}} \right) y_j + \frac{1}{\xi_{j-1}} y_{j-1}$$

Since there are only $n-2$ equations while there are n unknowns d_j, two of the latter, say d_1 and d_n, must be arbitrarily fixed. There are thus a double infinity of possible solutions. Traditionally, d_1 and d_n are set to zero ("natural splines") by analogy with the physical problem which consists in setting a beam with zero torque at each extremity. However, this gives rise sometimes to oscillating or overshooting phenomena in the resulting interpolation spline, which is not acceptable in finance or physics.

The idea is therefore to leave the quantities d_1 and d_n variable, and minimize a certain objective function so as to deduce their values instead of imposing them arbitrarily.

Matricial notations

The above problem can be recast in matricial form, by defining the two following $[n-2 \times n]$ matrices :

$$
\boldsymbol{T} = \begin{pmatrix}
\frac{\xi_1}{6} & \frac{\xi_1+\xi_2}{3} & \frac{\xi_2}{6} & 0 & 0 & \cdots & 0 \\
0 & \frac{\xi_2}{6} & \frac{\xi_2+\xi_3}{3} & \frac{\xi_3}{6} & 0 & \cdots & \vdots \\
0 & 0 & \frac{\xi_3}{6} & \frac{\xi_3+\xi_4}{3} & \frac{\xi_4}{6} & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \cdots & \cdots & \frac{\xi_{n-3}}{6} & \frac{\xi_{n-3}+\xi_{n-2}}{3} & \frac{\xi_{n-2}}{6} & 0 \\
0 & \cdots & \cdots & 0 & \frac{\xi_{n-2}}{6} & \frac{\xi_{n-2}+\xi_{n-1}}{3} & \frac{\xi_{n-1}}{6}
\end{pmatrix}
$$

$$
\boldsymbol{S} = \begin{pmatrix}
\frac{1}{\xi_1} & -\left(\frac{1}{\xi_1}+\frac{1}{\xi_2}\right) & \frac{1}{\xi_2} & 0 & 0 & \cdots & 0 \\
0 & \frac{1}{\xi_2} & -\left(\frac{1}{\xi_2}+\frac{1}{\xi_3}\right) & \frac{1}{\xi_3} & 0 & \cdots & \vdots \\
0 & 0 & \frac{1}{\xi_3} & -\left(\frac{1}{\xi_3}+\frac{1}{\xi_4}\right) & \frac{1}{\xi_4} & \ddots & \vdots \\
0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & 0 & 0 \\
\vdots & \cdots & \cdots & \frac{1}{\xi_{n-3}} & -\left(\frac{1}{\xi_{n-3}}+\frac{1}{\xi_{n-2}}\right) & \frac{1}{\xi_{n-2}} & 0 \\
0 & \cdots & \cdots & 0 & \frac{1}{\xi_{n-2}} & -\left(\frac{1}{\xi_{n-2}}+\frac{1}{\xi_{n-1}}\right) & \frac{1}{\xi_{n-1}}
\end{pmatrix}
$$

and the following $[n \times 1]$ vectors:

$$
\boldsymbol{D} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}
$$

$$
\boldsymbol{Y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \\ y_n \end{pmatrix}
$$

The $n-2$ continuity equations can thus be written in compact form :

$$
\boldsymbol{TD} \quad = \quad \boldsymbol{SY}
$$

In order to separate the actual unknowns $d_2, \cdots, d_{n-1}$ from the parameters d_1 and d_n, we further define the following matrices (with dimensions respectively $[2 \times 1]$, $[n-2 \times 1]$, $[n \times 2]$ and $[n \times n-2]$) :

$$\boldsymbol{D'} = \begin{pmatrix} d_1 \\ d_n \end{pmatrix}$$

$$\boldsymbol{D''} = \begin{pmatrix} d_2 \\ d_3 \\ \vdots \\ d_{n-2} \\ d_{n-1} \end{pmatrix}$$

$$\boldsymbol{U'} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\boldsymbol{U''} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus, one can write in compact form :

$$\boldsymbol{D} = \boldsymbol{U'D'} + \boldsymbol{U''D''}$$

and hence, one finds by elimination that $\boldsymbol{D}$ can be expressed linearly with respect to $\boldsymbol{D'}$ and $\boldsymbol{Y}$ as follows :

$$\boldsymbol{D} = \boldsymbol{VD'} + \boldsymbol{WY}$$

where:

$$\begin{aligned} \boldsymbol{V} &= (\boldsymbol{Id} - \boldsymbol{ZT})\boldsymbol{U'} \\ \boldsymbol{W} &= \boldsymbol{ZS} \\ \boldsymbol{Z} &= \boldsymbol{U''}(\boldsymbol{TU''})^{-1} \end{aligned}$$

Minimizing overshoot

The idea to control overshooting is to find that vector $\boldsymbol{D'}$ which will minimize the integral over $[x_1, x_n]$ of the square of the derivative of $y(x)$:

$$\mathcal{I} = \int_{x_1}^{x_n} y'(x)^2 dx$$

More generally, one can compute the following weighted integral :

$$\mathcal{I} = \sum_{j=1}^{n-1} \varepsilon_j \int_{x_j}^{x_{j+1}} y'(x)^2 dx$$

where:

$$\int_{x_j}^{x_{j+1}} y'(x)^2 dx \quad = \quad \frac{1}{45}\xi_j^3 \left(d_j^2 + \frac{7}{4} d_j d_{j+1} + d_{j+1}^2 \right) + \frac{(y_{j+1} - y_j)^2}{\xi_j}$$

Minimizing $\mathcal{I}$ is thus equivalent to minimizing the following quadratic form in the variable $\boldsymbol{D}$:

$$\frac{1}{2}\boldsymbol{D^t Q D}$$

where $\boldsymbol{Q}$ is the following $[n \times n]$ symmetric positive matrix :

$$\boldsymbol{Q} = \begin{pmatrix} \varepsilon_1\xi_1^3 & \frac{7}{8}\varepsilon_1\xi_1^3 & 0 & 0 & 0 & \cdots & 0 \\ \frac{7}{8}\varepsilon_1\xi_1^3 & \varepsilon_1\xi_1^3 + \varepsilon_2\xi_2^3 & \frac{7}{8}\varepsilon_2\xi_2^3 & 0 & 0 & \cdots & \vdots \\ 0 & \frac{7}{8}\varepsilon_2\xi_2^3 & \varepsilon_2\xi_2^3 + \varepsilon_3\xi_3^3 & \frac{7}{8}\varepsilon_3\xi_3^3 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \frac{7}{8}\varepsilon_{n-2}\xi_{n-2}^3 & 0 \\ \vdots & \cdots & \cdots & 0 & \frac{7}{8}\varepsilon_{n-2}\xi_{n-2}^3 & \varepsilon_{n-2}\xi_{n-2}^3 + \varepsilon_{n-1}\xi_{n-1}^3 & \frac{7}{8}\varepsilon_{n-1}\xi_{n-1}^3 \\ 0 & \cdots & \cdots & 0 & 0 & \frac{7}{8}\varepsilon_{n-1}\xi_{n-1}^3 & \varepsilon_{n-1}\xi_{n-1}^3 \end{pmatrix}$$

Expanding $\boldsymbol{D}$ in terms of $\boldsymbol{D'}$ and minimizing on the latter, it can be shown that the optimal solution is given by :

$$\boldsymbol{D'} \quad = \quad -(\boldsymbol{V^t Q V})^{-1} \boldsymbol{V^t Q W Y}$$

and thus finally :

$$\boldsymbol{D} \quad = \quad \boldsymbol{JY}$$

where:

$$\boldsymbol{J} \quad = \quad [\boldsymbol{Id} - \boldsymbol{V}(\boldsymbol{V^t Q V})^{-1}\boldsymbol{V^t Q}]\boldsymbol{W}$$

Once $\boldsymbol{D}$ is known, one can proceed with the interpolation using the above formula for $y(x)$.

Other minimization possibility

Alternatively one may want to minimize the total curvature and thus use as a proxy of the latter for each interval :

$$\int_{x_j}^{x_{j+1}} y''(x)^2 dx \quad = \quad \frac{1}{3}\xi_j \left(d_j^2 + d_j d_{j+1} + d_{j+1}^2\right)$$

In which case the above reasoning still holds with the following $\boldsymbol{Q}$ matrix :

$$\boldsymbol{Q} = \begin{pmatrix} \varepsilon_1\xi_1 & \frac{1}{2}\varepsilon_1\xi_1 & 0 & 0 & 0 & \cdots & 0 \\ \frac{1}{2}\varepsilon_1\xi_1 & \varepsilon_1\xi_1 + \varepsilon_2\xi_2 & \frac{1}{2}\varepsilon_2\xi_2 & 0 & 0 & \cdots & \vdots \\ 0 & \frac{1}{2}\varepsilon_2\xi_2 & \varepsilon_2\xi_2 + \varepsilon_3\xi_3 & \frac{1}{2}\varepsilon_3\xi_3 & 0 & \ddots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \frac{1}{2}\varepsilon_{n-2}\xi_{n-2} & 0 \\ \vdots & \cdots & \cdots & 0 & \frac{1}{2}\varepsilon_{n-2}\xi_{n-2} & \varepsilon_{n-2}\xi_{n-2} + \varepsilon_{n-1}\xi_{n-1} & \frac{1}{2}\varepsilon_{n-1}\xi_{n-1} \\ 0 & \cdots & \cdots & 0 & 0 & \frac{1}{2}\varepsilon_{n-1}\xi_{n-1} & \varepsilon_{n-1}\xi_{n-1} \end{pmatrix}$$

Other spline interpolation formulas

Exponential splines

Given n points with ordered abscissas $x_1, \cdots, x_n$ with their ordinates $y_1, \cdots, y_n$, one exponential spline interpolation technique consists in finding piecewise functions which are linear combinations of 1, x, $e^{\eta x}$ and $e^{-\eta x}$ where η is a constant (generally known as "tension").These functions must be continuous at each x_j, and so must their first and second order derivatives.

The general such piecewise function satisfying the continuity constraints for the function and its second derivative can be written as follows (for each x such that $x_j \leq x < x_{j+1}$) :

$$y(x) \quad = \quad \alpha_{1j} y_j + \alpha_{2j} y_{j+1} + \alpha_{3j} d_j + \alpha_{4j} d_{j+1}$$

where :

$$\begin{aligned} \xi_j &= x_{j+1} - x_j, 1 \leq j \leq n-1 \\ \alpha_{1j} &= \frac{x_{j+1} - x}{\xi_j} \\ \alpha_{2j} &= \frac{x - x_j}{\xi_j} \\ \alpha_{3j} &= \frac{\xi_j^2}{\eta^2}\left(\frac{\sinh(\eta\alpha_{1j})}{\sinh(\eta)} - \alpha_{1j}\right) \\ \alpha_{4j} &= \frac{\xi_j^2}{\eta^2}\left(\frac{\sinh(\eta\alpha_{2j})}{\sinh(\eta)} - \alpha_{2j}\right) \end{aligned}$$

The n quantities d_j are the second derivatives at each abscissa x_j. They are the solution of the following $n-2$ equations expressing the continuity of the first derivative (for $2 \leq j \leq n-1$) :

$$\frac{\xi_{j-1}}{\eta}\left(\frac{1}{\eta}-\frac{1}{\sinh(\eta)}\right)d_{j-1}+\frac{\xi_{j-1}+\xi_j}{\eta}\left(\frac{1}{\tanh(\eta)}-\frac{1}{\eta}\right)d_j+\frac{\xi_j}{\eta}\left(\frac{1}{\eta}-\frac{1}{\sinh(\eta)}\right)d_{j+1}=$$
$$\frac{1}{\xi_j}y_{j+1}-\left(\frac{1}{\xi_j}+\frac{1}{\xi_{j-1}}\right)y_j+\frac{1}{\xi_{j-1}}y_{j-1}$$

Since there are only $n-2$ equations while there are n unknowns d_j, two of the latter, say d_1 and d_n, must be arbitrarily fixed. There are thus a double infinity of possible solutions as was the case for cubic splines.

It can be shown that when $\eta \longrightarrow 0$, the exponential spline method gives the same results as the cubic spline method, while when $\eta \longrightarrow \infty$, it gives the same results as the linear interpolation.

Note also the following equations which are useful to write the first and last equations when it is the first derivative that must be set at an end (rather than the second derivative) :

$$\begin{aligned}
\frac{\xi_j}{\eta}\left(\frac{1}{\tanh(\eta)}-\frac{1}{\eta}\right)d_j+\frac{\xi_j}{\eta}\left(\frac{1}{\eta}-\frac{1}{\sinh(\eta)}\right)d_{j+1} &= \frac{y_{j+1}-y_j}{\xi_j}-y'_j\\
\frac{\xi_j}{\eta}\left(\frac{1}{\eta}-\frac{1}{\sinh(\eta)}\right)d_j+\frac{\xi_j}{\eta}\left(\frac{1}{\tanh(\eta)}-\frac{1}{\eta}\right)d_{j+1} &= y'_{j+1}-\frac{y_{j+1}-y_j}{\xi_j}
\end{aligned}$$

as well as the following integrals (for $x \in [x_j, x_{j+1}]$):

$$\begin{aligned}
\int_{s=x_j}^{x}\alpha_{1j}(s)ds &= \frac{1}{2}(x-x_j)\left(1+\frac{x_{j+1}-x}{\xi_j}\right)\\
\int_{s=x_j}^{x}\alpha_{2j}(s)ds &= \frac{1}{2}\frac{(x-x_j)^2}{\xi_j}\\
\int_{s=x_j}^{x}\alpha_{3j}(s)ds &= \frac{\xi_j^2}{\eta^2}\left[\frac{\xi_j}{\eta\sinh(\eta)}\left(\cosh(\eta)-\cosh\frac{\eta(x_{j+1}-x)}{\xi_j}\right)-\frac{1}{2}(x-x_j)\left(1+\frac{x_{j+1}-x}{\xi_j}\right)\right]\\
\int_{s=x_j}^{x}\alpha_{4j}(s)ds &= \frac{\xi_j^2}{\eta^2}\left[\frac{\xi_j}{\eta\sinh(\eta)}\left(\cosh\frac{\eta(x-x_j)}{\xi_j}-1\right)-\frac{1}{2}\frac{(x-x_j)^2}{\xi_j}\right]
\end{aligned}$$

When $\eta = 0$ the latter two integrals boil down to:

$$\begin{aligned}
\int_{s=x_j}^{x}\alpha_{3j}(s)ds &= -\frac{1}{24}\frac{(x-x_j)^2(x-x_j-2\xi_j)^2}{\xi_j}\\
\int_{s=x_j}^{x}\alpha_{4j}(s)ds &= -\frac{1}{24}\frac{(x-x_j)^2(2x_j^2+2xx_j-x_j^2-x^2)}{\xi_j}
\end{aligned}$$

Lastly, it is possible to refine the above algorithm by attaching a tension coefficient η_j to each interval $[x_j, x_{j+1}]$, in which case the continuity equations become :

$$s_{j-1}\xi_{j-1}d_{j-1}+(t_{j-1}\xi_{j-1}+t_j\xi_j)d_j+s_j\xi_jd_{j+1}=\frac{1}{\xi_j}y_{j+1}-\left(\frac{1}{\xi_j}+\frac{1}{\xi_{j-1}}\right)y_j+\frac{1}{\xi_{j-1}}y_{j-1}$$

where :

$$
\begin{aligned}
s_j &= \frac{1}{\eta_j}\left(\frac{1}{\eta_j} - \frac{1}{\sinh(\eta_j)}\right) \\
t_j &= \frac{1}{\eta_j}\left(\frac{1}{\tanh(\eta_j)} - \frac{1}{\eta_j}\right)
\end{aligned}
$$

Trigonometric splines

Upon replacing η with $\eta\sqrt{-1}$, another similar interpolation algorithm can be found :

$$
y(x) = \alpha_{1j} y_j + \alpha_{2j} y_{j+1} + \alpha_{3j} d_j + \alpha_{4j} d_{j+1}
$$

where :

$$
\begin{aligned}
\xi_j &= x_{j+1} - x_j, 1 \le j \le n-1 \\
\alpha_{1j} &= \frac{x_{j+1} - x}{\xi_j} \\
\alpha_{2j} &= \frac{x - x_j}{\xi_j} \\
\alpha_{3j} &= \frac{\xi_j^2}{\eta^2}\left(\alpha_{1j} - \frac{\sin(\eta\alpha_{1j})}{\sin(\eta)}\right) \\
\alpha_{4j} &= \frac{\xi_j^2}{\eta^2}\left(\alpha_{2j} - \frac{\sin(\eta\alpha_{2j})}{\sin(\eta)}\right)
\end{aligned}
$$

The n quantities d_j being the solution of the following $n-2$ equations (for $2 \le j \le n-1$) :

$$
\frac{\xi_{j-1}}{\eta}\left(\frac{1}{\sin(\eta)} - \frac{1}{\eta}\right) d_{j-1} + \frac{\xi_{j-1} + \xi_j}{\eta}\left(\frac{1}{\eta} - \frac{1}{\tan(\eta)}\right) d_j + \frac{\xi_j}{\eta}\left(\frac{1}{\sin(\eta)} - \frac{1}{\eta}\right) d_{j+1} = \frac{1}{\xi_j} y_{j+1} - \left(\frac{1}{\xi_j} + \frac{1}{\xi_{j-1}}\right) y_j + \frac{1}{\xi_{j-1}} y_{j-1}
$$

Care should be exercised however regarding η in this case, as it should be taken different from $k\pi$ (k being a non-zero positive or negative integer), so as to avoid indefinite values.

Note also the following equations which are useful to write the first and last equations when it is the the first derivative that must be set at an end (rather than the second derivative) :

$$
\begin{aligned}
\frac{\xi_j}{\eta}\left(\frac{1}{\eta} - \frac{1}{\tan(\eta)}\right) d_j + \frac{\xi_j}{\eta}\left(\frac{1}{\sin(\eta)} - \frac{1}{\eta}\right) d_{j+1} &= \frac{y_{j+1} - y_j}{\xi_j} - y'_j \\
\frac{\xi_j}{\eta}\left(\frac{1}{\sin(\eta)} - \frac{1}{\eta}\right) d_j + \frac{\xi_j}{\eta}\left(\frac{1}{\eta} - \frac{1}{\tan(\eta)}\right) d_{j+1} &= y'_{j+1} - \frac{y_{j+1} - y_j}{\xi_j}
\end{aligned}
$$

Here also, it is possible to refine the above algorithm by attaching a tension coefficient η_j to each interval $[x_j, x_{j+1}]$, in which case the continuity equations become :

$$
s_{j-1}\xi_{j-1} d_{j-1} + (t_{j-1}\xi_{j-1} + t_j \xi_j) d_j + s_j \xi_j d_{j+1} = \frac{1}{\xi_j} y_{j+1} - \left(\frac{1}{\xi_j} + \frac{1}{\xi_{j-1}}\right) y_j + \frac{1}{\xi_{j-1}} y_{j-1}
$$

where :

$$\begin{aligned} s_j &= \frac{1}{\eta_j}\left(\frac{1}{\sin(\eta_j)} - \frac{1}{\eta_j}\right) \\ t_j &= \frac{1}{\eta_j}\left(\frac{1}{\eta_j} - \frac{1}{\tan(\eta_j)}\right) \end{aligned}$$