A Guide to FX Options Quoting Conventions

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The foreign-exchange options market is one of the largest and most liquid OTC derivatives markets in the world. The market has developed its own way to quote options, which differs significantly from other markets. A fact that is often ignored in the academic literature is that there are a number of different delta and at-the-money conventions. The authors introduce common FX market quoting conventions and describe in detail the subtleties embedded in these quotations.

It is common market practice to summarize the information of the vanilla options market in a volatility smile table that includes Black–Scholes implied volatilities for different maturities and moneyness levels. The degree of moneyness of an option can be represented by the strike or by any linear or nonlinear transformation of the strike (forward moneyness, log moneyness, and delta). The implied volatility as a function of moneyness for a single time to maturity is generally referred to as the *smile*. To be more precise, the volatility smile is a mapping,

$$X \mapsto \sigma(X) \in [0, \infty)$$

with X being the moneyness variable. The function value $\sigma(\overline{X})$ for a given moneyness $\overline{X}$ and time to maturity $\overline{T}$ represents the implied volatility, which is the "wrong" variable to plug into the "wrong" formula (Black and Scholes [1973]) to get the correct price. The volatility smile is the crucial input into pricing and risk management procedures because it is used to price vanilla, as well as exotic, option books. In the FX OTC derivatives market the volatility versus strike version of the smile is not directly observable, as opposed to the equity markets, where strike-price or strike-volatility pairs are quoted. By contrast, the FX market has two common ways to quote vanilla option prices:

1. a quotation of implied volatility versus the corresponding delta (as opposed to a strike), and
2. volatility quotes for risk reversal, strangle, and at-the-money positions.

This shows that FX OTC market participants are using the delta to represent the moneyness of an option, which is opposite to a strike. A market sample of the first quotation type is shown in Exhibit 1.

An example of the second quotation type is shown in Exhibit 2.

The risk reversal and strangle quotes are assigned to a delta such as 0.25, which is incorporated in the notation in Exhibit 2.

Given the two types of data, one is confronted with the task of transforming these quotes to standard vanilla option prices. This is not a trivial task. The procedure to construct a smile given three quotes (at-the-money, risk reversal, and strangle) requires a calibration procedure that is discussed by Reiswich and

EXHIBIT 1
USD-JPY Volatility Market Data, April 13, 2010

Maturity	10ΔP	15ΔP	20ΔP	25ΔP	30ΔP	35ΔP	40ΔP	55ΔP	ATM	45ΔC	40ΔC	35ΔC	30ΔC	25ΔC	20ΔC	15ΔC	10ΔC
1W	11.700	11.317	10.962	10.663	10.438	10.278	10.163	10.077	10.000	9.920	9.847	9.799	9.790	9.838	9.951	10.113	10.300
1M	12.400	11.993	11.616	11.300	11.067	10.906	10.796	10.717	10.650	10.579	10.515	10.470	10.460	10.500	10.598	10.738	10.900

Note: Numbers represent percentages, $\Delta C/P$ call and put deltas, respectively.
Source: Reuters database.

EXHIBIT 2
FX Market Data for One-Month Maturity, January 20, 2009

	EUR-USD%	USD-JPY%
σ_{ATM}	21.6215	21.00
σ_{25-RR}	−0.5	−5.3
σ_{25-S-Q}	0.7375	0.184

Source: Bloomberg database.

Wystup [2009], Castagna [2010], and Clark [forthcoming]. We focus on the discussion of the first option quotation type and related quotation topics. In the first step, we will show that there are four different types of FX option premium used by market participants. We will then show that delta may not refer to the standard Black–Scholes delta. This is a very common misconception when dealing with this type of data. Also, the at-the-money convention does not usually correspond to the at-the-money spot. Finally, we will show how to transform the data in Exhibit 1 to vanilla option prices.

SPOT, FORWARD, AND VANILLA OPTIONS

FX Spot Rate S_t

The FX spot rate, S_t = FOR-DOM, represents the number of units of domestic currency needed to buy one unit of foreign currency at time t. For example, EUR-USD = 1.3900 means that one EUR is worth 1.3900 USD. In this case, EUR is the foreign currency and USD is the domestic one. The EUR-USD = 1.3900 quote is equivalent to USD-EUR 0.7194. A notional of N units of foreign currency is equal to NS_t units of domestic currency (see also Wystup [2006]). "Domestic" does *not* refer to any geographical region, but merely represents a particular side of the deal. The domestic currency is also referred to as the numeraire currency and the foreign one as the base currency (see Castagna [2010]).

FX Outright Forward Rate $f(t,T)$

A popular and liquid hedge contract for a corporate treasurer is the outright forward contract. This contract trades at time t at zero cost and leads to an exchange of notionals at time t at the pre-specified outright forward rate $f(t,T)$. At time t, the foreign notional amount N is exchanged against an amount of $Nf(t,T)$ domestic currency units. For example, 1,000,000 EUR may be exchanged against 1,390,000 USD assuming an outright forward rate of 1.3900 EUR-USD. The outright forward is related to the FX spot rate via the spot-interest rate parity,

$$f(t,T) = S_t \cdot e^{(r_d - r_f)\tau} \tag{1}$$

where

- r_f is the foreign interest rate (continuously compounded)
- r_d is the domestic interest rate (continuously compounded)
- τ is the time to maturity, equal to $T - t$

FX Forward Value

At inception, an outright forward contract has a value of zero. Thereafter, when foreign exchange rates and/or interest rates change, the value of the forward contract is no longer zero, but is worth

$$v_f(t,T) = e^{-r_d\tau}(f(t,T) - K) = S_t e^{-r_f\tau} - Ke^{-r_d\tau} \tag{2}$$

for a pre-specified exchange rate K. This is the forward contract value in domestic currency units, marked to the market at time t. As stated previously, setting $K = f(t, T)$ yields a zero-cost contract.

FX Vanilla Options

In foreign exchange markets, options are usually physically settled (i.e., the buyer of a EUR-USD vanilla call receives a EUR notional amount N at maturity and pays $N \times K$ USD, where K is the strike). The value of such a vanilla contract is computed with the standard Black–Scholes formula,[1]

$$v(S_t, K, \sigma, \phi) = \phi e^{-r_d \tau}[f(t,T)N(\phi d_+) - K\,N(\phi d_-)] \quad (3)$$

where

$$d_\pm = \frac{\ln\left(\frac{f(t,T)}{K}\right) \pm \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}$$

K is the strike of the FX option

σ is the volatility

$\phi = +1$ for a call, $\phi = -1$ for a put

$N(x)$ is the cumulative normal distribution function

We may drop some of the variables of the function v depending on the context. The formula returns a value v in domestic currency. An equivalent value of the position in foreign currency is v/S_t. The accounting currency (the currency in which the option value is measured) is also called the premium currency. The notional is the amount of currency that the holder of an option is entitled to exchange. The value formula applies by default to one unit of foreign notional (corresponding to one share of stock in equity markets), with a value in units of domestic currency. An example that illustrates these terms follows. Consider a EUR-USD call with a spot of $S_t = 1.3900$, a strike of $K = 1.3500$, and a price of 0.1024 USD. If a notional of 1,000,000 EUR is specified, the holder of the option will receive 1,000,000 EUR and pay 1,350,000 USD at maturity, and the option's current price is 102,400 USD (73,669 EUR). The next section provides a more detailed discussion of premium conventions.

A CLOSER LOOK AT PREMIUM QUOTING CONVENTIONS

In FX markets, there are various ways to express an option premium when the deal is closed. We follow Wystup [2006], Castagna [2010], and Clark [forthcoming] and explain the common ones using the example data provided in the previous section. The most common premium convention is the standard Black–Scholes quotation described earlier. The corresponding premium is called foreign/domestic price and will be denoted with $v^{f\text{-}d}$. Alternatively, the price is called domestic pips price after a multiplication of a currency-dependent factor (such as 10,000 for EUR-USD). In our example, we have $v^{\text{EUR-USD}} = 0.1024$ USD. If a notional of N_f foreign currency units is specified, the actual amount paid/received will be $N_f v^{f\text{-}d}$. In the preceding example, this amount is 102,400 USD. The corresponding domestic pips price is $v^{\text{EUR-USD}} \times 10{,}000 = 1024$ USD pips.

The exchange of one unit of foreign currency versus K units of domestic can be analyzed from the foreign investor's point of view. In our example this is the party that pays foreign currency units and receives domestic ones. The premium $v^{f\text{-}d}/S_t$ for the foreign investor will not be denoted as $v^{d\text{-}f}$ due to an otherwise embedded inconsistency, as will be explained subsequently. The standard case is that the domestic investor receives one foreign currency unit and pays K domestic ones. This shows that the foreign investor receives K units of the other currency, not one. For consistency, the notation $v^{x\text{-}f}$ is used for an option with a premium in currency y and a notional of one in currency X. To obtain $v^{d\text{-}f}$, we need to adjust the exchange of amounts such that the foreign investor receives one domestic currency unit and not K. This can be achieved by a payment of $1/K$ foreign currency units, instead of one, which is equivalent to adjusting the foreign notional ($1/K$ instead of one). As $1/K$ of a foreign currency unit is received by the domestic party, the equivalent domestic amount paid is $(1/K) \times K = 1$. This shows that, after the adjustment, the foreign investor receives one domestic currency unit as desired. The price $v^{d\text{-}f}$ can then be expressed by resetting the foreign notional to $N_f = 1/K$ and converting the premium to the foreign currency. This yields

$$v^{d-f} = \frac{1}{K}\frac{v^{f-d}}{S_t} \quad (4)$$

The price is referred to as the domestic/foreign (or foreign pips price after multiplication with a suitable factor); in our example, we calculate

$$v^{\text{USD-EUR}} = 0.1024/(1.3500 \times 1.3900) = 0.054570$$

This quotation enhances a specification of the notional in the domestic, instead of the foreign, currency. The total premium amount in foreign currency units will be $\nu^{d\text{-}f} N_d$ with N_d being the notional in domestic currency units. For example, one can specify a notional of 1,000,000 USD such that the premium in foreign currency is 54,570 EUR.

Finally, one can ignore the adjustment of the notional and simply express the price $\nu^{f\text{-}d}$ in foreign currency units. The price $\nu^{f\text{-}d}/S_t$ is called the foreign currency percentage price. We define

$$\nu^{f\%} = \frac{\nu^{f-d}}{S_t} \tag{5}$$

The foreign currency percentage price in our example is 0.073669 EUR. This is the standard way some vanilla options and barrier options are quoted in the interbank market (see Castagna [2010]). We will clarify why this name is used. The premium $\nu^{f\%}$ is expressed in foreign currency units for an option where the holder receives 1 = 100% of a foreign currency unit. This shows that $\nu^{f\%}$ is a percentage number compared to the notional in the foreign currency: A number of 0.073669 EUR can then be interpreted in the sense that the option premium in foreign currency units represents 7.3669% of the notional in foreign currency. This interpretation still holds if the notional is changed to any number. Once this number is known, it is easy to multiply the factor with any foreign notional to receive the total premium in foreign currency.

Alternatively, the same can be achieved with $\nu^{d\text{-}f}$. This premium needs to be multiplied by S_t to calculate the price of this option in the domestic currency. Consequently, one can specify a domestic currency percentage price as $\nu^{d\text{-}f} \times S_t$. We define

$$\nu^{d\%} = \nu^{d-f} S_t = \frac{\nu^{f-d}}{K} \tag{6}$$

where we have used Equation (4) to calculate the last equation. In our example, this yields 0.075852 USD. The domestic currency percentage quotation is the standard way in which premiums are quoted for exotic options (one-touch, double-no-touch) where the payoff is in domestic currency units (see Castagna [2010]).

Premium conventions for selected currency pairs are provided in Exhibit 3. For example, the USD-JPY spot is traditionally quoted with JPY as the domestic currency because a number such as 87.00 USD-JPY is easier to quote than 0.01149 JPY-USD, however, the premium will usually be expressed in USD. The market standard is to quote the more commonly traded currency as the premium currency.[2] Virtually all currency pairs involving the USD will express the USD as the premium currency (Clark [forthcoming]). Similarly, contracts on a currency pair including the EUR—and not the USD—will be expressed in EUR. A basic premium currency hierarchy is given as (Clark)[3]

$$\begin{aligned} &\text{USD} \succ \text{EUR} \succ \text{GBP} \succ \text{AUD} \succ \text{NZD} \succ \text{CAD} \succ \text{CHF} \\ &\quad \succ \text{NOK}, \text{SEK}, \text{DKK} \\ &\quad \succ \text{CZK}, \text{PLN}, \text{TRY}, \text{MXN} \succ \text{JPY} \succ \ldots \end{aligned} \tag{7}$$

To summarize, there are always two sides of the currency option deal. Depending on the details of the deal, the investor can be interested in a standardized notional of one, either in the foreign or domestic currency. A domestic investor calculates the premium in domestic currency and can be interested in either a foreign or domestic notional of one. The corresponding premium conventions are $\nu^{f\text{-}d}$ and $\nu^{d\%}$, respectively. The foreign investor expresses the premium in the foreign currency and may also be interestedin either a domestic or foreign notional of one. The corresponding premium conventions are $\nu^{d\text{-}f}$ and $\nu^{f\%}$, respectively.

EXHIBIT 3
Premium Conventions for Selected Currency Pairs

Currency Pair	Convention
EUR-USD	$\nu^{\text{EUR-USD}}$ Pips
EUR-CAD	$\nu^{\text{EUR-CAD}}$ Pips
EUR-CHF	$\nu^{\text{EUR}\%}$
EUR-GBP	$\nu^{\text{EUR-GBP}}$ Pips
EUR-JPY	$\nu^{\text{EUR}\%}$
EUR-ZAR	$\nu^{\text{EUR}\%}$
GBP-CHF	$\nu^{\text{GBP}\%}$
GBP-JPY	$\nu^{\text{GBP}\%}$
GBP-USD	$\nu^{\text{GBP-USD}}$ Pips
USD-CAD	$\nu^{\text{USD}\%}$
USD-CHF	$\nu^{\text{USD}\%}$
USD-JPY	$\nu^{\text{USD}\%}$
USD-ZAR	$\nu^{\text{USD}\%}$

Source: Castagna [2010].

DELTA AND AT-THE-MONEY TERMS

Delta Types

The delta of an option is the percentage of the foreign notional one must buy when selling the option to hold a hedged position (equivalent to buying stock). For instance, a delta of 0.35 = 35% indicates that buying 35% of the foreign notional is necessary to delta hedge a short option. In foreign exchange markets, we distinguish between *spot delta* for a hedge in the spot market and *forward delta* for a hedge in the FX forward market. Furthermore, the standard delta is a quantity in percent of foreign currency. The actual hedge quantity must be changed if the premium is paid in foreign currency, which would be equivalent to paying for stock options in shares of stock. We call this type of delta the *premium-adjusted delta*. In the previous example, the foreign/domestic premium of an option with a notional of 1,000,000 EUR was calculated as 73,669 EUR. Assuming a short vanilla option position with a delta of 60% means that buying 600,000 EUR is necessary to hedge. This is the standard spot hedge, however, the final hedge quantity can also be 526,331 EUR, which is the delta quantity reduced by the received premium in EUR. Consequently, the premium-adjusted delta would be 52.63%. The following sections will introduce the formulas for the different delta types. A detailed introduction on at-the-money and delta conventions has been provided by Beier and Renner [2010] can be used as an orientation. Related work, which is worth reading and describes the standard conventions, is that of Beneder and Elkenbracht-Huizing [2003], Bossens et al. [2009], Castagna [2010], and Clark [forthcoming]. In the rest of this article, v will denote the foreign/domestic premium $v^{f\text{-}d}$, unless otherwise stated.

Unadjusted Deltas

The non-adjusted deltas do not take the premium payment in foreign currency into account. There are two types of unadjusted deltas.

Spot delta. The standard sensitivity of the vanilla option with respect to the spot rate S_t is given as

$$\Delta_S(K,\sigma,\phi) = \frac{\partial v}{\partial S} = v_S \tag{8}$$

where v is the Black-Scholes price in Equation (3). Standard calculus yields

$$\Delta_S(K,\sigma,\phi) = \phi e^{-r_f \tau} N(\phi d_+) \tag{9}$$

Put-call delta parity:

$$\Delta_S(K,\sigma,+1) - \Delta_S(K,\sigma,-1) = e^{-r_f \tau} \tag{10}$$

This delta is also called the pips spot delta, which reflects the fact that an option is hedged using the domestic pips (or foreign/domestic) quotation. In equity markets, one would buy Δ_S units of the stock to hedge a short vanilla option position. In FX markets, this is equivalent to buying Δ_S times the foreign notional N_f. This is equivalent to selling of $\Delta_S \times N_f \times S_t$ units of domestic currency. Note that the absolute value of delta is a number between zero and a discount factor $e^{-r_f \tau} < 100\%$. Therefore, 50% is not the center value for the delta range.

Forward delta. An alternative to the spot hedge is a hedge with a forward contract from Equation (2). The number of forward contracts one would buy in this case differs from the number of units in a spot hedge. The forward hedge amount is given by

$$\begin{aligned}\Delta_f(K,\sigma,\phi) &= \frac{\partial v}{\partial v_f} = \frac{\partial v}{\partial S}\frac{\partial S}{\partial v_f} = \frac{\partial v}{\partial S}\left(\frac{\partial v_f}{\partial S}\right)^{-1} \\ &= \phi N(\phi d_+)\end{aligned} \tag{11}$$

Put-call delta parity:

$$\Delta_f(K,\sigma,+1) - \Delta_f(K,\sigma,-1) = 100\% \tag{12}$$

In the hedge, one would enter $\Delta_f \times N_f$ forward contracts to forward hedge a short vanilla option position. The forward delta is often used in FX option smile tables because the delta of a call and the (absolute value of the) delta of the corresponding put add to 100% (i.e., a 25-delta call must have the same volatility as a 75-delta put). This symmetry only works for forward deltas. For the spot delta, the foreign discount factor must be known in order to convert call to put deltas. If a delta of 100 is observed in a smile, one can conclude that the data correspond to a forwarddelta.

Premium-Adjusted Deltas

The premium-adjusted deltas are the equivalents of the deltas in the preceding subsection, but in this case the premium payment in foreign currency is taken into account.

Premium-adjusted spot delta. The premium-adjusted spot delta takes care of the correction induced by payment of the premium in foreign currency, which is the amount by which the delta hedge in foreign currency has to be corrected. The delta can be represented as

$$\Delta_{S,pa} = \Delta_S - \frac{v}{S_t} = \Delta_S - v^{f\%} \tag{13}$$

with Δ_S being the spot delta from Equation (8). In this hedge scenario, the domestic investor would buy $(\Delta_S - \frac{v}{S_t})$ foreign currency units to hedge a short vanilla position (the notional can be applied easily). The equivalent number of domestic currency units to sell is $(S_t\Delta_S - v)$. This delta has an equivalent foreign point of view. Consider again the foreign percentage premium quotation from Equation (5), which is

$$v^{f\%} = \frac{v^{f-d}}{S_t}$$

Assume as previously that the domestic investor has sold the position and the foreign investor is the buyer. The domestic investor marks the position to market in domestic currency units, but the foreign investor values his position in foreign currencyunits, which is $v^{f\%}$. The spot risk factor for the foreign investor is $1/S_t$ where S_t represents FOR-DOM. The foreign investor has a positive option delta and thus has to sell domestic currency units to be hedged. The corresponding amounts can be calculated as

$$\begin{aligned}
\frac{\partial v^{f\%}}{\partial \frac{1}{S_t}} &= \frac{\partial \frac{v}{S_t}}{\partial \frac{1}{S_t}} \\
&= \frac{\partial \frac{v}{S_t}}{\partial S_t}\frac{\partial S_t}{\partial \frac{1}{S_t}} = \frac{S_t\Delta_S - v}{S_t^2}\left(\frac{\partial \frac{1}{S_t}}{\partial S_t}\right)^{-1} = \frac{S_t\Delta_S - v}{S_t^2}\left(-\frac{1}{S_t^2}\right)^{-1} \\
&= -(S_t\Delta_S - v) \text{ DOM to sell} \\
&= S_t\Delta_S - v \text{ DOM to buy} = \Delta_S - \frac{v}{S_t} \text{ FOR to sell.}
\end{aligned}$$

This calculation shows that the foreign investor has to sell $\Delta_S - \frac{v}{S}$ foreign currency units, which is exactly the amount the domestic investor has to buy after adjusting the delta for the premium. We find

$$\Delta_{S,pa}(K,\sigma,\phi) = \phi e^{-r_f\tau}\frac{K}{f}N(\phi d_-) \tag{14}$$

Put-call delta parity:

$$\Delta_{S,pa}(K,\sigma,+1) - \Delta_{S,pa}(K,\sigma,-1) = e^{-r_f\tau}\frac{K}{f} \tag{15}$$

For the premium-adjusted call delta, the relationship strike versus delta is not monotonic; for a given delta there might exist more than one corresponding strike. This is shown in Panel A of Exhibit 4 where the premium-adjusted delta is compared to the standard delta. The premium-adjusted put delta does not have this property, as shown in Panel B of Exhibit 4.

Premium-adjusted forward delta. As in the case of a spot delta, a premium payment in foreign currency leads to an adjustment of the forward delta. The resulting hedge quantity is given by

$$\Delta_{f,pa}(K,\sigma,\phi) = \phi\frac{K}{f}N(\phi d_-) \tag{16}$$

Put-call delta parity:

$$\Delta_{f,pa}(K,\sigma,+1) - \Delta_{f,pa}(K,\sigma,-1) = \frac{K}{f} \tag{17}$$

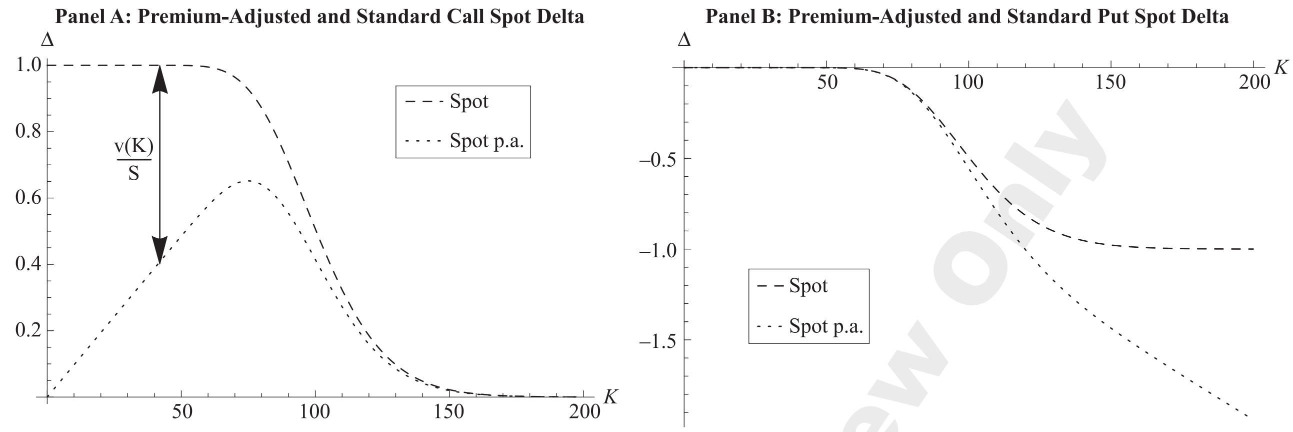
Note again that the premium-adjusted forward delta of a call is not a monotone function of the strike.

Delta Conventions for Selected Currency Pairs

This subsection is based on Clark's [forthcoming] summary of current FX market conventions. If an implied EUR-USD one-month volatility of 12% is mapped to a delta of 0.25, as in Exhibit 1, the question arises which of the four possible definitions of deltas apply. The answer depends on the chosen hedging strategy and premium currency of the option. As shown in Exhibit 3, the premium currency for some currency pairs will not be a domestic, but foreign, percentage. If the option premium is expressed in domestic currency units, the spot delta is used. In case of a quotation in foreign currency units, a premium adjustment is applied. For example, the delta of

Exhibit 4

Premium-Adjusted and Standard Call and Put Spot Delta



Note: $S_t = 100$, $\tau = 1.0$, $r_d = 0.03$, $r_f = 0.0$, $\sigma = 0.2$.

a USD-JPY option will always be premium adjusted. Exhibit 5 provides examples. Whether a spot or forward delta should be used is discussed later.

When do we have to apply forward instead of spot deltas? Generally, forward hedges are popular to capture interest rate risk in addition to spot risk. So, naturally, forward hedges come up for long-term options or currency pairs that have a large difference in interest rates (as in emerging market countries). In this case, the standard forward delta in Equation (11) or its premium-adjusted version in Equation (16) is used. For example, the market default for emerging market currencies is the forward delta, as is the case for options containing USD, EUR, JPY, GBP, AUD, NZD, CAD, CHF, NOK, SEK, or DKK with maturities greater than one year. With maturities less than one year, spot deltas are used for these currencies. For example, the NZD-JPY uses spot deltas for maturities less than one year and forward deltas for maturities greater than one year. A very important last remark: Clark [forthcoming] notes that the forward delta quotation became increasingly popular for all currency pairs after the recent credit crunch crisis. The reason for this is that it became difficult for the trading parties to agree on the discount factor, which is not present in the forward delta. This can be assigned to the general difficulties regarding consistent post–credit crisis yield curve constructions (see Ametrano and Bianchetti [2009], Chibane and Sheldon [2009], and Fujii, Shimada, and Takahashi [2010]).

Exhibit 5

Selected Currency Pairs and the Default Premium Currency Determining the Delta Type

Currency Pair	Premium Currency	Delta Convention
EUR-USD	USD	regular
USD-JPY	USD	premium-adjusted
EUR-JPY	EUR	premium-adjusted
USD-CHF	USD	premium-adjusted
EUR-CHF	EUR	premium-adjusted
GBP-USD	USD	regular
EUR-GBP	EUR	premium-adjusted
AUD-USD	USD	regular
AUD-JPY	AUD	premium-adjusted
USD-CAD	USD	premium-adjusted
USD-BRL	USD	premium-adjusted
USD-MXN	USD	premium-adjusted

Source: Clark [forthcoming].

AT–THE–MONEY TYPES

Defining at the money (ATM) is by no means as obvious as one might think when first studying options. We can think of

$$\begin{aligned} \text{ATM spot} \quad & K = S_t \\ \text{ATM forward} \quad & K = f \\ \text{ATM value - neutral} \quad & K, \text{ such that call value} = \text{put value} \\ \text{ATM } \Delta \text{ - neutral} \quad & K, \text{ such that call delta} = -\text{ put delta} \end{aligned}$$

ATM spot is often used in text books or on term sheets for retail investors, because the majority of market participants are familiar with it. ATM forward takes into account that the risk-neutral expectation of the future spot is the forward as expressed in Equation (1), which is a natural way of specifying the midpoint of the risk-neutral distribution. ATM value-neutral is equivalent to ATM forward because of put-call parity,

$$v(S_t, K, \sigma, 1) - v(S_t, K, \sigma, -1) = S_t e^{-r_f \tau} - K e^{-r_d \tau} = 0$$
$$\Leftrightarrow K = S_t \frac{e^{-r_f \tau}}{e^{-r_d \tau}} = f$$

The notion of ATM Δ-neutral has sub-categories depending on which delta convention is used. Choosing the strike in this sense ensures that a straddle with this strike has a zero spot exposure.[4] The major residual risk is the vega risk. Consequently, a delta-neutral straddle is an option position that accounts for the traders' vega-hedging needs. This ATM convention is considered the default ATM notion for short-dated FX options. For example, the spot delta-neutral strike is derived as follows:

$$\Delta_S(K, \sigma, 1) = -\Delta_S(K, \sigma, -1) \Leftrightarrow e^{-r_f \tau} N(d_+)$$
$$= e^{-r_f \tau} N(-d_+) \Leftrightarrow d_+ = 0$$

The last equation can be solved for K,

$$d_+ = 0 \Leftrightarrow \ln\left(\frac{f}{K}\right) = -\frac{1}{2}\sigma^2 \tau \Leftrightarrow K = f e^{\frac{1}{2}\sigma^2 \tau}$$

Note that the same result can be obtained using forward deltas. For example, if one observes a quoted implied at-the-money volatility of 10%, it is necessary to find out which at-the-money type this volatility corresponds to. A spot delta-neutral quotation implies a strike of

$$f e^{\frac{1}{2}\sigma^2 \tau}$$

Given a volatility of 10%, a time to maturity of 0.5, and a forward of 1.21 would result in

$$K = 1.21 e^{\frac{1}{2} 0.10^2 0.5} = 1.2130$$

as the corresponding strike. The strike-volatility pair (1.2130, 0.10) can then be transferred as one input into the volatility smile in strike space. We summarize the various at-the-money definitions and the relations between all relevant quantities in Exhibit 6. Again, it is advisable to ask the market data provider for the at-the-money type associated with the volatility quote. An at-the-money delta of 50 is very likely to be part of a forward delta smile, as can be seen in Exhibit 6.

DELTA–STRIKE CONVERSION

Many customers on the buy side receive implied volatility-delta pairs from their market data provider. A representative market sample was given in Exhibit 1. These data are usually the result of a suitable calibration and transformation output. The calibration is based on data of the type shown in Exhibit 2. The market participant is then confronted with the task of transforming delta-volatility to strike-volatility or strike-price pairs respecting FX-specific at-the-money and delta definitions. In this section, we outline the algorithms that can be used to accomplish this. For example, assume that implied volatilities are quoted for a regular call delta grid, such as

$$0.10, 0.15, \ldots, \Delta_{ATM}, \ldots, 0.85, 0.90$$

EXHIBIT 6

At-the-Money (ATM) Strike Values and Delta Values for Different Delta Conventions

	ATM Δ-Neutral Strike	ATM Fwd Strike	ATM Δ-Neutral Delta	ATM Fwd Delta
Spot Delta	$f e^{\frac{1}{2}\sigma^2 \tau}$	f	$\frac{1}{2}\phi e^{-r_f \tau}$	$\phi e^{-r_f \tau} N(\phi \frac{1}{2}\sigma \sqrt{\tau})$
Forward Delta	$f e^{\frac{1}{2}\sigma^2 \tau}$	f	$\frac{1}{2}\phi$	$\phi N(\phi \frac{1}{2}\sigma \sqrt{\tau})$
Spot Delta p.a.	$f e^{-\frac{1}{2}\sigma^2 \tau}$	f	$\frac{1}{2}\phi e^{-r_f \tau} e^{-\frac{1}{2}\sigma^2 \tau}$	$\phi e^{-r_f \tau} N(-\phi \frac{1}{2}\sigma \sqrt{\tau})$
Forward Delta p.a.	$f e^{-\frac{1}{2}\sigma^2 \tau}$	f	$\frac{1}{2}\phi e^{-\frac{1}{2}\sigma^2 \tau}$	$\phi N(-\phi \frac{1}{2}\sigma \sqrt{\tau})$

Source: Beier and Renner [2010].

with corresponding implied volatilities

$$\sigma_{0.10\Delta}, \sigma_{0.15\Delta}, \ldots, \sigma_{\Delta_{ATM}}, \ldots, \sigma_{0.85\Delta}, \sigma_{0.90\Delta}$$

The delta type could be one of the four deltas described before (assuming that the grid is within the delta domain). Two typical requests regarding the data are

- the calculation of the strike corresponding to a delta of 0.62, and
- the calculation of the implied volatility corresponding to a strike of 1.4000.

The first request is typically representative of a hedging operation. A trader may have a remaining delta of –30% that needs to be hedged and requests a volatility quote for a delta of 30%. Given the quote, the transaction can be closed by calculating the strike which—given the quoted implied volatility—yields the required delta. The second request is typically representative of a buy-side operation, where a treasurer is interested in a concrete implied volatility for a specific strike.

The first case can be handled by interpolating the smile in the delta space by a suitable procedure, such as cubic or kernel interpolation (see Hakala and Wystup [2002]). The result in the example is a delta-volatility pair,

$$(0.62, \sigma_{0.62\Delta})$$

Both variables can then be used to extract the corresponding strike. The following sections will outline the algorithms that can be used to that end.

Conversion of a Spot Delta-Volatility Pair to a Strike

The conversion of a non-premium-adjusted spot delta to a strike is relatively straightforward. With Δ_S and σ given, we can directly solve (see also Wystup [2006]) Equation (8) for the strike K. We get

$$K = fe^{-\phi N^{-1}(\phi e^{r_f \tau}\Delta_S)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau} \tag{18}$$

with N^{-1} being the inverse of the normal cumulative distribution function.

Conversion of a Forward Delta-Volatility Pair to a Strike

A similar calculation for the forward delta yields

$$K = fe^{-\phi N^{-1}(\phi\Delta_f)\sigma\sqrt{\tau}+\frac{1}{2}\sigma^2\tau} \tag{19}$$

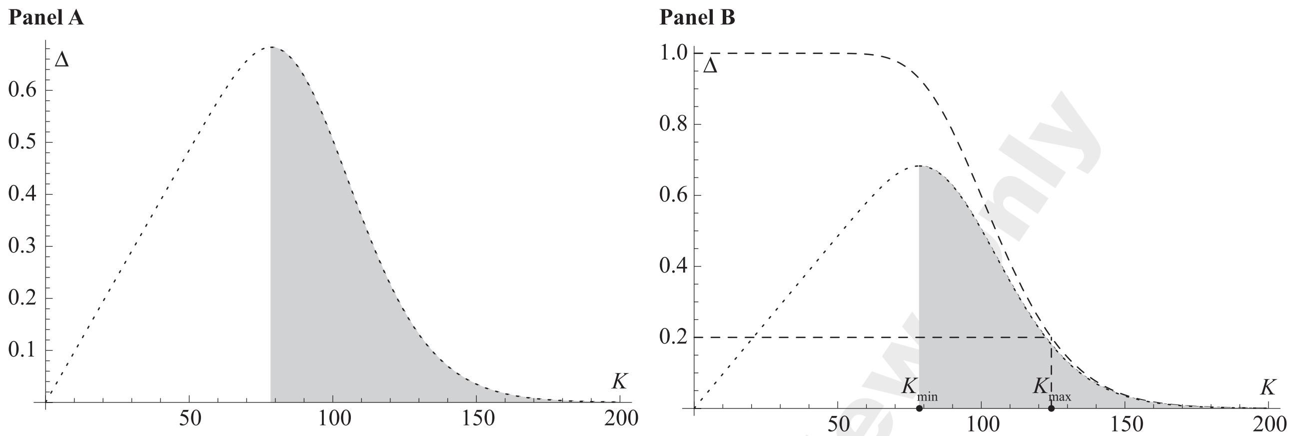
Conversion of a Premium-Adjusted Forward Delta-Volatility Pair to a Strike

The conversion of the premium-adjusted spot delta is analogous to the one described in this subsection and will not be discussed. For a premium-adjusted forward delta, the relationship between delta and strike,

$$\Delta_{f,pa}(K,\sigma,\phi) = \phi\frac{K}{f}N\left(\phi\frac{\ln\left(\frac{f}{K}\right)-\frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}\right)$$

cannot be solved for the strike in closed form. Some numerical procedure has to be used. This is relatively trivial for the put delta because the put delta is monotone in strike. However, this is not the case for the premium-adjusted call delta, as illustrated in Panel A of Exhibit 4. Due to non-monotonicity, two strikes can be obtained for a given premium-adjusted call delta (for example, for $\Delta S_{,pa} = 0.2$). One common solution to this problem is to search for strikes corresponding to deltas that are on the right side of the delta maximum. This is illustrated as a shaded area in Panel A of Exhibit 7. Brent [2002] proposed a root searcher, which can be used to search for $K \in [K_{min}, K_{max}]$ and yields the given delta.[5] The right limit K_{max} can be chosen as the strike corresponding to the non-premium-adjusted delta, because the premium-adjusted delta for a strike K is always smaller than the non-adjusted delta corresponding to the same strike. In our example, if we are looking for a strike corresponding to a premium-adjusted forward delta of 0.20, we can choose K_{max} to be the strike corresponding to a simple forward delta of 0.20. The last strike can be calculated analytically using Equation (19). It is easy to see that the premium-adjusted delta is always

EXHIBIT 7
Strike Region for Given Premium-Adjusted Delta, $S_t = 100$



below the non-zpremium-adjusted one. This follows from

$$\begin{aligned}&\Delta_S(K,\sigma,\phi)-\Delta_{S,pa}(K,\sigma,\phi)\\&\quad= e^{-r_f\tau}\phi N(\phi d_+)-\phi e^{-r_f\tau}\frac{K}{f}N(\phi d_-)\geq 0\\&\quad\Leftrightarrow \phi fN(\phi d_+)-\phi KN(\phi d_-)\geq 0\end{aligned}$$

for the spot delta case; the forward delta case is equivalent. Discounting the last inequality yields the Black–Scholes formula, which is always positive. The maximum for both the premium-adjusted spot and the premium-adjusted forward delta is given implicitly by the equation

$$\sigma\sqrt{\tau}N(d_-)=n(d_-)$$

with $n(x)$ being the normal density at X. One can solve this implicit equation numerically for $K_{\min}$ and then use Brent's method to search for the strike in $[K_{\min}, K_{\max}]$, which yields the input delta. The resulting interval is illustrated in Panel B of Exhibit 7.

CALCULATING A STRIKE-VOLATILITY PAIR FROM A DELTA-VOLATILITY SMILE

Now consider the case when the implied volatility corresponding to a fixed strike of 1.4000 is required. To calculate the pair, we need to have a continuous mapping of deltas to implied volatilities via the function

$$\sigma(\Delta)$$

This function can be obtained from a discrete set of quotes by using an interpolation procedure. The calculation of a strike-volatility pair can then be obtained by using an iterative process as shown by Wystup [2006, p. 11].

At this point, we would like to note how important it is to know which delta type the volatilities correspond to before interpolating. Different delta types will produce different volatility-strike functions, despite the same implied input volatilities. This issue is almost always ignored in academic work, where the standard Black–Scholes delta is typically used. This may potentially lead to non-market-consistent smiles. One example where interpolation of FX volatility data is performed in delta space is the parabolic formula used bt Malz [1997],

$$\sigma(\Delta)=\sigma_{ATM}+c_1(\Delta-0.5)+c_2(\Delta-0.5)^2$$

with suitable parameters c_i and σ_{ATM} being the at-the-money volatility. This formula relates to a forward delta, although this is not explicitly stated in the publication. A common misconception is to use volatility quotes, which correspond to a premium adjusted delta in the formula. This will yield prices that are not consistent with the market. Also, the formula reveals that $\sigma(0.5) = \sigma_{ATM}$. The at-the-money delta is 0.5 and corresponds to a forward delta-neutral at-the-money quotation. The formula cannot be used in case other at-the-money types are used (i.e., spot). Other issues are the misinterpretation of risk reversal and strangle quotes, which are discussed in detail by Reiswich and Wystup [2009].

CONCLUSION

After presenting four ways to express the FX option premium, we show that this results in two different delta definitions: the spot and premium-adjusted delta. The premium-adjusted delta is FX specific and does not have a straightforward equivalent in other markets. Adding the forward as an additional hedging instrument yields two additional deltas. The delta definitions are important for the delta-neutral at-the-money quotation, which we discussed, in addition to alternative at-the-money conventions. Finally, we described tools to convert a delta-volatility smile to a strike-volatility smile. This article's aim is to sensitize academic researchers to the subtle quotation issues in FX markets because they are commonly ignored in research leading to potentially non-market-consistent volatility smiles.

ENDNOTES

We would like to thank Travis Fisher, Iain Clark, Boris Borowski, Andreas Weber, Jürgen Hakala, and an anonymous referee for their helpful comments.

[1]It should be noted that the stated formula is not the one originally derived by Black and Scholes. The original work for currency options is Garman and Kohlhagen [1983]. However, it is common practice to refer to the formula as the Black–Scholes formula because the only difference in the two formulas is the foreign interest rate. Consequently, the term Black–Scholes formula will be used in the rest of this article.

[2]This does not apply to the JPY, which is usually not taken as the premium currency.

[3]Exceptions may occur! In case of doubt, it is advisable to check.

[4]A straddle is equivalent to a long call and put position, where both options have the same strike.

[5]Of course, other root search routines such as Newton's algorithm can be used instead.

REFERENCES

Ametrano, F., and M. Bianchetti. "Bootstrapping the Illiquidity: Multiple Yield Curves Construction for Market Coherent Forward Rates Estimation." In *Modeling Interest Rates* edited by Fabio Mercurio. London, U.K.: Risk Books, 2009.

Beier, C., and C. Renner. "Foreign Exchange Options: Delta and At-the-Money Conventions." In *Encyclopedia of Quantitative Finance* edited by R. Cont and P. Tankov. Chichester, U.K.: Wiley, 2010.

Beneder, R., and M. Elkenbracht-Huizing. "Foreign Exchange Options and the Volatility Smile." *Medium Econometrische Toepassingen*, 2 (2003), pp. 30-36.

Black, F., and M. Scholes. "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy*, Vol. 81, No. 3 (1973), pp. 637-654.

Bossens, F., G. Rayée, N. Skantzos, and G. Deelstra. "Vanna-Volga Methods Applied to FX Derivatives: From Theory to Market Practice." Working Paper, Université Libre de Bruxelles, Solvay Brussels School of Economics and Management, and Centre Emile Bernheim, 2009.

Brent, R. *Algorithms for Minimization without Derivatives.* Courier Dover Publications, 2002.

Castagna, A. *FX Options and Smile Risk*. Chichester, U.K.: Wiley, 2010.

Chibane, M., and G. Sheldon. "Building Curves on a Good Basis." Working Paper, SSRN eLibrary, 2009.

Clark, I. *Foreign Exchange Option Pricing: A Practitioner's Guide*. Wiley, forthcoming.

Fujii, M., Y. Shimada, and A. Takahashi. "A Note on Construction of Multiple Swap Curves with and without Collateral." Working Paper, SSRN eLibrary, 2010.

Garman, M., and Sw. Kohlhagen. "Foreign Currency Option Values." *Journal of International Money and Finance*, Vol. 2, No. 3 (1983), pp. 231-237.

Hakala, J., and U. Wystrup. *Foreign Exchange Risk*. London, U.K.: Risk Books, 2002.

Malz, A. "Option-Implied Probability Distributions and Currency Excess Returns." Working Paper, SSRN eLibrary, 1997.

Reiswich, D., and U. Wystup. "FX Volatility Smile Construction." Working Paper No. 20, Frankfurt School of Finance and Management, 2009.

Wystup, U. *FX Options and Structured Products*. Chichester, U.K.: Wiley, 2006.