

## THE RELATION BETWEEN FORWARD PRICES AND FUTURES PRICES\*

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This paper consolidates the results of some recent work on the relation between forward prices and futures prices. It develops a number of propositions characterizing the two prices. These propositions contain several testable implications about the difference between forward and futures prices. Many of the propositions show that equilibrium forward and futures prices are equal to the values of particular assets, even though they are not in themselves asset prices. The paper then illustrates these results in the context of two valuation models and discusses the effects of taxes and other institutional factors.

### 1. Introduction

Forward markets and futures markets have long played an important role in economic affairs. In spite of the attention that they have collectively received, virtually no consideration has been given to the differences between the two types of markets. Indeed, most of the academic literature has treated them as if they were synonymous. Similarly, most practitioners have viewed the differences as irrelevant administrative details and acted as if the two served exactly the same economic functions. Given the similarity of the two markets, such conclusions are quite understandable, but they are nevertheless incorrect. Forward markets and futures markets differ in fundamental ways.

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An individual who takes a long position in a forward contract agrees to buy a designated good or asset on a specified future date, the maturity date, for the forward price prevailing at the time the contract is initiated. On the maturity date, then, the forward price must equal the spot price of the underlying good or asset. No money changes hands initially or during the lifetime of the contract, only on the maturity date. The equilibrium forward price must thus continually change over time in a way such that newly created forward contracts will always have a zero value when they are initiated.

A futures contract is similar in many ways, but there is an important difference. An individual who takes a long position in a futures contract nominally agrees to buy a designated good or asset on the maturity date, for the futures price prevailing at the time the contract is initiated. Hence, the futures price must also equal the spot price on the maturity date. Again, no money changes hands initially. Subsequently, however, as the futures price changes, the party in whose favor the price change occurred must immediately be paid the full amount of the change by the losing party. As a result, the payment required on the maturity date to buy the underlying good or asset is simply its spot price at that time. The difference between that amount and the initial futures price has been paid (or received) in installments throughout the life of the contract. Like the forward price, the equilibrium futures price must also continually change over time. It must do so in such a way that the remaining stream of future payments described above always has a value of zero.

The difference in the payment schedules is clearly explained in a seminal article by Black (1976). While Black's discussion is completely correct, it is unfortunately presented in the context of a constant interest rate. As it turns out, this obscures a basic economic difference between the two types of markets. With a constant interest rate, the two are essentially equivalent and forward prices are equal to futures prices, but in general this is not true.

Several studies in addition to ours have independently noted the critical role of stochastic interest rates. To our knowledge, the first to do so was Margrabe (1976). Working in a continuous-time framework, Margrabe shows that if forward and futures prices are equal there will be an arbitrage opportunity unless a certain special condition is satisfied; constant interest rates are sufficient but not necessary for this condition to be met. Merton (1979) uses a discrete-time arbitrage argument to derive a way to sign the difference between forward and futures prices. Although Merton considers only forward and futures contracts on Treasury bills, his approach is such that this involves no loss of generality. Jarrow and Oldfield (1981) provide a perspicuous discussion of the contractual differences and use an arbitrage argument to show the importance of stochastic interest rates. They also show the connection between forward contracts and options. Richard and

Sundaresan (1981) derive a continuous-time equilibrium model and use it to analyze forward and futures contracts. Sundaresan (1980) employs the Richard and Sundaresan model to develop and test a number of explicit formulas for forward and futures prices. French (1981) examines a discrete-time utility-based model of forward and futures pricing and undertakes several empirical tests of his results.

One purpose of our paper is to consolidate some of the results of these studies. In so doing, we hope to help clarify the relation between forward prices and futures prices. Section 2 develops a number of propositions characterizing the two prices. These propositions contain several testable implications about the difference between forward and futures prices. Many of the propositions show that equilibrium forward and futures prices are equal to the values of particular assets. This allows one to apply any framework for valuing assets to the determination of forward and futures prices, even though they are not in themselves asset prices. Section 3 illustrates some of these results in a simple two-period framework with a complete set of state prices. Section 4 then uses the propositions developed in section 2 to examine forward and futures prices in the context of a continuous-time valuation model. This model gives a basis for obtaining explicit formulas for equilibrium forward and futures prices and hence provides further opportunities for empirical testing. In section 5, we conclude the paper with some comments and conjectures about the effects of taxes and other institutional factors.

## 2. Some fundamental propositions about forward prices and futures prices

For the most part, our results in this section are based on arbitrage arguments and are thus quite general. They are consequences of what is sometimes called the law of one price: investment strategies which have the same payoffs must have the same current value. To concentrate on the basic issues, we assume perfect frictionless markets. Hence, we shall ignore both taxes and transactions costs until section 5.

We shall use the following notation:

- $s$  = maturity date of the forward and futures contracts,
- $V(s)$  = price at time  $s$  of the good or asset on which the contracts are written,
- $P(t)$  = price at time  $t$  of a default-free discount bond paying one dollar at time  $s$ ,
- $G(t)$  = forward price at time  $t$ ,
- $H(t)$  = futures price at time  $t$ ,
- $R_t$  = one plus the spot interest rate prevailing from time  $t$  to time  $t + 1$ .

Our first two propositions express forward prices and futures prices in terms of assets making particular payments on the maturity date:

*Proposition 1.* The forward price  $G(t)$  is the value at time  $t$  of a contract which will pay at time  $s$  the amount

$$V(s)/P(t). \quad (1)$$

*Proof.* Consider the following strategy: take a long position in  $1/P(t)$  forward contracts and place the amount  $G(t)$  in bonds maturing at time  $s$ . The current investment required is  $G(t)$ . There are no interim payoffs, and the payoff at time  $s$  is

$$\frac{1}{P(t)}[G(s) - G(t)] + \frac{G(t)}{P(t)} = \frac{V(s)}{P(t)}. \quad \blacksquare \quad (2)$$

*Proposition 2.* The futures price  $H(t)$  is the value at time  $t$  of a contract which will pay at time  $s$  the amount

$$V(s) \prod_{k=t}^{s-1} R_k. \quad (3)$$

*Proof.* Consider the following strategy: at time  $t$ , take the amount  $H(t)$  and continually reinvest it and the accumulated interest in one-period bonds until time  $s$ . At each time  $j$ ,  $j=t, t+1, \dots, s-1$ , take a long position in  $\prod_{k=t}^j R_k$  futures contracts. Liquidate each contract after one period and continually reinvest the (possibly negative) proceeds and accumulated interest in one period bonds until time  $s$ . The current investment required for this strategy is  $H(t)$ . The payoff at time  $s$  is

$$\begin{aligned} & H(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} \left( \prod_{k=t}^j R_k \right) [H(j+1) - H(j)] \left( \prod_{k=j+1}^{s-1} R_k \right) \\ &= H(s) \prod_{k=t}^{s-1} R_k = V(s) \prod_{k=t}^{s-1} R_k. \quad \blacksquare \end{aligned} \quad (4)$$

Propositions 1 and 2 show that the distinction between forward prices and futures prices is very much like the distinction between 'going long' and 'rolling over shorts' in the bond market. Each price is equal to the value of a claim which will pay a particular number of units of the underlying good or asset on the maturity date. For the forward price, this number is the total return which will be earned on an investment in a discount bond maturing at time  $s$ . For the futures price, it is the total return which will be earned from a policy of continual reinvestment in one-period bonds. This characterization draws attention to an important difference between futures

prices and forward prices: futures prices will depend on the correlation of spot prices and interest rates, while forward prices will not.

Jarrow and Oldfield (1981) show that forward and futures contracts can be used to create a portfolio which will give a sure return on the maturity date if interest rates are constant, but not if they are random. Propositions 1 and 2, taken together, give essentially the same conclusion. Proposition 2 is identical to a result derived by Richard and Sundaresan (1981) using their equilibrium model and by French (1981) using a discrete-time arbitrage approach. Our next three propositions follow immediately from Propositions 1 and 2.

*Proposition 3 [Black (1976)]. If interest rates are non-stochastic, then  $G(t) = H(t)$ .*

*Proof.* If interest rates are non-stochastic, then

$$\frac{1}{P(t)} = \prod_{k=t}^{s-1} R_k. \quad \blacksquare \quad (5)$$

If there is only one period remaining before the maturity date,  $1/P(t)$  will always equal  $R_t$ . Consequently, there will be no difference between forward and futures contracts in any one-period model or in any two-period model where all goods are consumed in the final period.

*Proposition 4. If  $V(s)$  is non-stochastic, then  $G(t) = H(t) = V(s)$ .*

*Proof.* If  $V(s)$  is non-stochastic, then a current investment of the amount  $V(s)$  in bonds maturing at time  $s$  produces a payoff at that time of  $V(s)/P(t)$ , so  $G(t) = V(s)$ . Similarly, a current investment of  $V(s)$  in a strategy of rolling over one-period bonds gives a payoff at time  $s$  of  $V(s) \prod_{k=t}^{s-1} R_k$ , so  $H(t) = V(s)$ .  $\blacksquare$

*Proposition 5. Let  $h_i(t)$  be the futures price and  $g_i(t)$  be the forward price at time  $t$  of a good  $i$  whose spot price at time  $s$  is  $v_i(s)$ . If*

$$V(s) = \sum_i a_i v_i(s) \quad \text{for some constants } a_i,$$

*then*

$$H(t) = \sum_i a_i h_i(t) \quad \text{and} \quad G(t) = \sum_i a_i g_i(t).$$

*Proof.* This follows immediately from the linearity in  $V(s)$  of the right-hand side of (1) and (3). ■

This result states that the futures price of a portfolio is equal to a corresponding portfolio of futures prices and the same is true for forward prices. If the payoffs were not linear, then this conclusion would not hold; for example, it is well known that an option on a portfolio is not the same as a portfolio of options. While Proposition 5 is quite obvious, it is nevertheless very useful. One example is provided by the forward and futures prices of non-callable government bonds. These bonds can be thought of as portfolios of discount bonds. Consequently, any method for finding the futures price of a discount bond will also give the futures prices for all coupon bonds.

Our next proposition expresses the difference between forward prices and futures prices in terms of the relation between futures prices and bond prices. It is very similar to a result of Merton (1979). Loosely stated, it says that if futures prices and bond prices are positively correlated, then the futures price is less than the forward price; if they are negatively correlated, then the futures price is greater than the forward price. In this and the following propositions, we shall occasionally refer to a continuous-time, continuous-state economy. By this, we mean an economy in which trading takes place continuously and in which all variables relevant to the equilibrium follow diffusion processes.

*Proposition 6.*  $G(t) - H(t)$  is the value at time  $t$  of a payment of

$$-\sum_{j=t}^{s-1} [H(j+1) - H(j)] \left[ \frac{P(j)}{P(j+1)} - 1 \right] / P(t), \quad (6)$$

to be received at time  $s$ . For a continuous-time, continuous-state economy, this sum becomes

$$\int_t^s H(u) [\text{cov } H(u), P(u)] du / P(t), \quad (7)$$

where  $[\text{cov } H(u), P(u)]$  stands for the local covariance of the percentage changes in  $H$  with the percentage changes in  $P$ . Hence,  $[\text{cov } H(u), P(u)] > 0$  for all  $u$  implies  $G(t) > H(t)$  and  $[\text{cov } H(u), P(u)] < 0$  for all  $u$  implies  $G(t) < H(t)$ .

*Proof.* Consider the following strategy, which requires no investment. Take a short position in a forward contract at time  $t$ . In each period  $j$ ,  $j=t, \dots, s-1$ , take a long position in  $P(j)$  futures contracts, liquidate them after one

period, and place the (possibly negative) proceeds in bonds with maturity date  $s$ . At time  $s$ , the payoff to this strategy is

$$\begin{aligned}
 & G(t) - G(s) + \sum_{j=t}^{s-1} P(j)[H(j+1) - H(j)] \left( \frac{1}{P(j+1)} \right) \\
 &= G(t) - G(s) + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \\
 &\quad + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \left[ \frac{P(j)}{P(j+1)} - 1 \right] \\
 &= H(s) - G(s) + G(t) - H(t) + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \left[ \frac{P(j)}{P(j+1)} - 1 \right].
 \end{aligned} \tag{8}$$

Since this strategy requires no investment, the current values of this payoff must be zero. Now note that  $G(s) = H(s)$  and that the current value of a certain payment of  $G(t) - H(t)$  at time  $s$  is  $P(t)[G(t) - H(t)]$ . Consequently,  $G(t) - H(t)$  is the current value of a payment at time  $s$  of

$$- \sum_{j=1}^{s-1} [H(j+1) - H(j)] \left[ \frac{P(j)}{P(j+1)} - 1 \right] / P(t).$$

Hence, in a continuous-time, continuous-state economy, if the local covariance of the percentage changes in  $H$  and  $P$  will always have one sign from  $t$  to  $s$ , then  $G(t) - H(t)$  has the same sign. ■

Like Proposition 2, the following proposition equates the futures price, which is not itself the value of an asset, with another quantity which is the value of an asset. This allows us to apply any equilibrium framework for valuing assets to the determination of equilibrium futures prices. Proposition 8 establishes an analogous result for forward prices.

*Proposition 7.*  $H(t)$  is the value at time  $t$  of a contract which gives a payment of  $V(s)$  at time  $s$  and a flow from time  $t$  to time  $s$  of the prevailing spot rate times the prevailing futures price. That is,  $H(t)$  is the value at time  $t$  of a contract which pays  $V(s)$  at time  $s$  and  $(R_u - 1)H(u)$  at each time  $u+1$  for  $u = t, t+1, \dots, s-1$ .

*Proof.* Consider strategy A: continually reinvest the payouts received from

this security and the accumulated interest in one-period bonds. At time  $s$ , the proceeds will be

$$\sum_{j=t}^{s-1} (R_j - 1)H(j) \prod_{k=j+1}^{s-1} R_k + V(s). \quad (9)$$

But since  $V(s) = H(s)$ , this can be rewritten as

$$H(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \prod_{k=j+1}^{s-1} R_k. \quad (10)$$

Now consider strategy B: invest  $H(t)$  in a one-period bond at time  $t$  and then continually reinvest the proceeds in one-period bonds. Take a long position in one futures contract and continually reinvest the (possibly negative) proceeds received at the end of each period in one-period bonds. At time  $s$ , the proceeds of strategy B will be

$$H(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} [H(j+1) - H(j)] \prod_{k=j+1}^{s-1} R_k. \quad (11)$$

This is the same as the proceeds of strategy A. Since the current value of B is  $H(t)$ , the current value of A must also be  $H(t)$ . ■

*Proposition 8.*  $G(t)$  is the value at time  $t$  of a contract which pays  $V(s)$  at time  $s$  and the flow

$$(R_u - 1)G(u) + [G(u+1) - G(u)] \left[ \frac{P(u+1)}{P(u)} - 1 \right], \quad (12)$$

at each time  $u+1$  for  $u = t, \dots, s-1$ .

*Proof.* Let strategy A be the following: continually reinvest the payouts received from this security and the accumulated interest in one-period bonds. At time  $s$  the proceeds will be

$$\begin{aligned} & \sum_{j=t}^{s-1} (R_j - 1)G(j) \prod_{k=j+1}^{s-1} R_k \\ & + \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \prod_{k=j+1}^{s-1} R_k + V(s). \end{aligned} \quad (13)$$



Since  $G(s) = V(s)$ , this can be rewritten as

$$\begin{aligned}
 G(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} [G(j+1) - G(j)] \prod_{k=j+1}^{s-1} R_k \\
 + \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \prod_{k=j+1}^{s-1} R_k \\
 = G(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} \right] \prod_{k=j+1}^{s-1} R_k. \quad (14)
 \end{aligned}$$

Now consider strategy B: invest  $G(t)$  in a one-period bond at time  $t$  and then continually reinvest the proceeds in one-period bonds. At each time  $j$ ,  $j=t, \dots, s-1$ , take a long position in  $1/P(j)$  forward contracts. Close out each contract after one period, thereby locking in the amount  $G(j+1) - G(j)$  to be received at time  $s$ . Obtain the present value of this amount,  $P(j+1)[G(j+1) - G(j)]$ , and invest it in one-period bonds. Continually reinvest the proceeds in one-period bonds thereafter. At time  $s$ , the proceeds of strategy B will be

$$G(t) \prod_{k=t}^{s-1} R_k + \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} \right] \prod_{k=j+1}^{s-1} R_k. \quad (15)$$

This is identical to the proceeds of strategy A. Since the current value of B is  $G(t)$ , the current value of A must also be  $G(t)$ . ■

Propositions 7 and 8 are useful not only in their own right, but also for obtaining Proposition 9. This proposition shows for forward prices a result analogous to Proposition 6 for futures prices. It expresses the difference between forward prices and futures prices in terms of the relation between forward prices and bond prices.

*Proposition 9.*  $G(t) - H(t)$  is the value at time  $t$  of a payment of

$$\left( \prod_{k=t}^{s-1} R_k \right) \left[ \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \right], \quad (16)$$

to be received at time  $s$ . For a continuous-time, continuous-state economy, the above expression becomes

$$\left[ \exp \left( \int_t^s \log R(u) du \right) \right] \int_t^s G(u) [\text{cov } G(u), P(u)] du. \quad (17)$$

Hence,  $[\text{cov } G(u), P(u)] > 0$  for all  $u$  implies  $G(t) > H(t)$  and  $[\text{cov } G(u), P(u)] < 0$  for all  $u$  implies  $G(t) < H(t)$ .

*Proof.* Propositions 7 and 8 imply that  $H(t) - G(t)$  is the current value of a contract which pays

$$(R_u - 1)[H(u) - G(u)] - [G(u+1) - G(u)] \left[ \frac{P(u+1)}{P(u)} - 1 \right], \quad (18)$$

at each time  $u+1$  for  $u=t, \dots, s-1$ . Consider the following strategy.

Over each period  $j$ , take a long position in  $\prod_{k=t}^j R_k$  of these contracts. Do this with no net investment in the following way. If the current value of the contract is positive, borrow the amount and use the first component of the payout, which will be positive, to repay the borrowing; if the current value is negative, lend the amount and use the proceeds of the lending to make restitution for the first component of the payout, which will be negative. After doing this, the remaining proceeds at time  $j+1$  from the position taken at time  $j$  will be

$$\begin{aligned} & \left[ [H(j+1) - G(j+1)] - [H(j) - G(j)] \right. \\ & \quad \left. - [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \right] \prod_{k=t}^j R_k. \end{aligned} \quad (19)$$

Invest this amount in one period bonds at time  $j+1$  and then continually reinvest it and the accumulated interest in one-period bonds. At time  $s$  the total proceeds from all positions taken from  $t$  to  $s$  will be

$$\begin{aligned} & \left[ \sum_{j=t}^{s-1} [H(j+1) - G(j+1)] - [H(j) - G(j)] \right. \\ & \quad \left. - \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \right] \prod_{k=t}^{s-1} R_k. \end{aligned} \quad (20)$$

Note that since  $G(s) = H(s)$ , then

$$\sum_{j=t}^{s-1} [[H(j+1) - G(j+1)] - [H(j) - G(j)]] = G(t) - H(t). \quad (21)$$

Since the entire position requires no net investment, its current value must be zero. The current value of the amount  $[G(t) - H(t)] \prod_{k=t}^{s-1} R_k$  received at time

$s$  is  $G(t) - H(t)$ . Consequently,  $G(t) - H(t)$  is the current value of a payment at time  $s$  of

$$\left( \prod_{k=t}^{s-1} R_k \right) \left[ \sum_{j=t}^{s-1} [G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \right]. \quad (22)$$

Hence, for a continuous-time, continuous-state economy, if  $[\text{cov } G(u), P(u)]$  always has one sign, then  $G(t) - H(t)$  has the same sign. ■

Note that Propositions 6 and 9 show the relation between forward and futures prices when both forward and futures markets exist simultaneously. If we find, for example, that  $G(t) > H(t)$ , this does not imply that replacing a forward market with a futures market will result in a lower price. Such a change could conceivably affect the equilibrium valuation of all assets and lead instead to a higher price. At the present time, simultaneous forward and futures markets are available for certain U.S. Treasury and Government National Mortgage Association securities, some foreign currencies, and a number of commodities. However, the forward contracts are typically traded with standardized times to maturity rather than standardized maturity dates. In these cases, corresponding forward and futures contracts exist simultaneously only on the days for which a standardized time to maturity in the forward market coincides with a standardized maturity date in the futures market.

Up to this point, nothing that we have said has depended on the existence of a spot market or on the characteristics of the underlying good or asset. Indeed, this good or asset need not even exist at the current time. This could be the case, for example, with a perishable commodity before the next crop is harvested. However, if there is a spot market, or an options market, then we can express our results in terms of spot prices and option prices.

If  $V(s)$  is the price at time  $s$  of a currently traded good or asset, then

$$G(t) = O(t)/P(t), \quad (23)$$

where  $O(t)$  is the current value of a European call option with maturity date  $s$  and exercise price zero. [The strategy of buying  $1/P(t)$  options gives a payoff at time  $s$  of  $V(s)/P(t)$ .] In that case

$$[G(j+1) - G(j)] \left[ \frac{P(j+1)}{P(j)} - 1 \right] \quad (24)$$

$$= \frac{O(j)}{P(j)} \left[ \frac{O(j+1)}{O(j)} - 1 \right] \left[ \frac{P(j+1)}{P(j)} - 1 \right] - \frac{O(j+1)}{P(j+1)} \left[ \frac{P(j+1)}{P(j)} - 1 \right]^2,$$

and for continuous-time, continuous-state economies this becomes

$$G[\text{cov } G, P] = (O/P)[\text{cov } O, P - \text{var } P]. \quad (25)$$

If the asset makes no payouts between  $t$  and  $s$ , then  $O(t) = V(t)$ . This immediately leads to a result which was obtained in a different way by Margrabe (1976):

- (i)  $\text{cov } V, P > \text{var } P$  implies  $G(t) > H(t)$ ,
  - (ii)  $\text{cov } V, P < \text{var } P$  implies  $G(t) < H(t)$ .
- (26)

For Treasury bills,  $V$  is itself a discount bond maturing at some time after  $s$ . We would thus expect  $V$  and  $P$  to be highly correlated and  $\text{var } V > \text{var } P$ . Hence, we would expect  $\text{cov } V, P > \text{var } P$  and  $G(t) > H(t)$ . For an asset which is a hedge against bond price fluctuations (i.e., is negatively correlated with bond prices), we would have  $\text{cov } V, P < \text{var } P$  and  $G(t) < H(t)$ .

With the existence of a spot market, we can also obtain another result somewhat similar to Proposition 6, but involving payouts depending on the spot price rather than the futures price. This gives an additional way to determine the futures price. It also allows us to express the relation between futures prices and spot prices in terms of the relation between the interest rate and the spot rental rate on the good or asset. By the spot rental rate at time  $u$ , we mean the fraction of the beginning-of-period spot price which would have to be paid at the end of the period to obtain the full use of the good or asset during the period, including the right to receive any payouts such as dividends.

*Proposition 10.* Let  $Y_u$  be the spot rental rate at time  $u$ . Then  $H(t) - V(t)$  is equal to the value of a contract which gives a payment of

$$[(R_u - 1 - Y_u)V(u)] \prod_{k=t}^{u-1} R_k, \quad (27)$$

at each time  $u+1$  for  $u=t, t+1, \dots, s-1$ . Consequently, if the spot interest rate is always greater (less) than the spot rental rate, then the futures price is greater (less) than the spot price.

*Proof.* Let  $Z(t)$  be the value at time  $t$  of the contract described. Consider the following strategy. At time  $t$ , take a long position in one contract and buy one unit of the good or asset in the spot market for  $V(t)$ . Finance the spot purchase by rolling over one-period loans. The total investment

required is thus  $Z(t)$ . At each time  $j$ , for  $j=t+1, \dots, s-1$ , use the payment received from the contract and the proceeds from spot rental over the previous period to increase the number of units of the spot good or asset held from  $\prod_{k=t}^{j-1} R_k$  to  $\prod_{k=t}^j R_k$ . The total value of the position at time  $s$  is

$$V(s) \prod_{k=t}^{s-1} R_k - V(t) \prod_{k=t}^{s-1} R_k. \quad (28)$$

Using Proposition 2, the current value of this amount is  $H(t) - V(t)$ . Consequently,  $Z(t) = H(t) - V(t)$ . If all of the payments given by the contract are positive (negative), then its current value must be positive (negative), so  $H(t) > V(t)$  ( $H(t) < V(t)$ ). ■

Our final proposition relates our results to the continuous-time capital asset pricing model (CAPM). It is stated in terms of the CAPM in consumption form as derived by Breeden (1979), but the same conclusions hold for the original multi-factor model of Merton (1973).

*Proposition 11.* *Futures prices will satisfy the capital asset pricing model for arbitrary  $V(s)$ , but forward prices will do so only if interest rates are non-stochastic.*

*Proof.* Consider the dollar return from holding over one period a long position in  $1/P(t)$  forward contracts and the amount  $G(t)$  in one-period bonds. The dollar return is

$$\frac{1}{P(t)} [P(t+1)[G(t+1) - G(t)]] + R_t G(t). \quad (29)$$

Denote the expected value of the dollar return as  $\mu G$ . The CAPM in consumption form says that

$$\mu G - R_t G = \beta_{C,K} \left( \frac{\mu_M M - R_t M}{\beta_{C,M}} \right), \quad (30)$$

where

$$\beta_{C,K} = \text{cov} \left( \frac{P(t+1)}{P(t)} [G(t+1) - G(t)], C(t+1) - C(t) \right) / \sigma_C^2,$$

$C(t)$  is aggregate consumption at time  $t$ ,  $M(t)$  is the value of the market

portfolio at time  $t$ , and  $K$  is the portfolio described above. Consequently, we can write

$$\begin{aligned} & E\left(\frac{P(t+1)}{P(t)}[G(t+1)-G(t)]\right) \\ &= \text{cov}\left(\frac{P(t+1)}{P(t)}[G(t+1)-G(t)], C(t+1)-C(t)\right) \\ & \quad \times \left(\frac{\mu_M M - R_t M}{\sigma_C^2 \beta_{C,M}}\right), \end{aligned} \quad (31)$$

where  $E$  indicates expectation. Also, we have

$$\begin{aligned} & \frac{P(t+1)}{P(t)}[G(t+1)-G(t)] \\ &= [G(t+1)-G(t)] + \left(\frac{P(t+1)}{P(t)} - 1\right)[G(t+1)-G(t)]. \end{aligned} \quad (32)$$

Thus, in the limit for continuous-time, continuous-state economies,

$$\begin{aligned} & \text{cov}\left(\frac{P(t+1)}{P(t)}[G(t+1)-G(t)], C(t+1)-C(t)\right) \\ &= \text{cov}[G(t+1)-G(t), C(t+1)-C(t)], \end{aligned} \quad (33)$$

so forward prices can satisfy the CAPM only if

$$E\left(\left(\frac{P(t+1)}{P(t)} - 1\right)[G(t+1)-G(t)]\right) / [(t+1)-t] \rightarrow 0 \quad (34)$$

in the limit, which will be true for arbitrary  $V$  only if interest rates are non-stochastic.

Consequently, any attempt to apply the CAPM to a series of forward prices will be misdirected. However, a slight modification of this line of reasoning shows that changes in futures prices, when combined with a portfolio as described above, will satisfy the CAPM in consumption form, as is discussed in Breeden (1980). ■

This concludes our series of propositions relating forward prices and futures prices. Although we hope that our list contains the most important propositions, it is not meant to be exhaustive; we have not found a general way to characterize all possible relations between the two prices.

In the remainder of this section, we discuss how some of the features of forward and futures contracts could be combined. Forward contracts provide an easy way for an individual to lock in at time  $t$  the amount he will have to pay at time  $s$  for one unit of the underlying good or asset. By taking a long position in a forward contract, the individual can arrange today to buy the good on the maturity date for a price of  $G(t)$ . An important implication of our results is that futures contracts cannot in general provide exactly the same service. An exact hedging strategy using only (a finite number of) futures contracts may not be possible, and even if possible, it would typically require more information than is needed when employing forward contracts.

It may appear that this is a necessary consequence of the resettlement feature of futures contracts. This would be unfortunate, since resettlement may provide certain advantages. With forward contracts significant implicit or explicit collateral may be necessary; with futures contracts the requirements would be much smaller. Futures markets thus to a large extent separate the actual transactions in the good from the issues of collateralization and financing, while forward markets do not. However, it is easy to specify a contract which will meet the dual requirements of providing a simple exact hedging procedure and requiring only minimal collateral. This is in fact exactly what would be accomplished with a forward contract which had to be settled and rewritten continually.

To make this more precise, we introduce a quasi-futures contract, which is exactly the same as a regular futures contract, except that at the end of each period the person in whose favor the price change occurred is paid not the full amount of the change, but instead the present value that this full amount would have if it were paid on the maturity date. If we denote the quasi-futures price at time  $t$  as  $Q(t)$ , then an individual having a long position in such a contract receives at each time  $j+1$ , for  $j=t, \dots, s-1$ , the amount  $P(j+1)[Q(j+1)-Q(j)]$ . If the individual invests the (possibly negative) proceeds received at each time  $j+1$  in bonds maturing at time  $s$ , then the value of his position at time  $s$  will be

$$\sum_{j=t}^{s-1} P(j+1)[Q(j+1)-Q(j)] \left( \frac{1}{P(j+1)} \right) = V(s) - Q(t). \quad (35)$$

This strategy allows the individual to arrange today to buy the good on the maturity date for a designated price  $Q(t)$ . Since the strategy requires no net investment, it is equivalent to a forward contract, and hence  $Q(t) = G(t)$ .

### 3. A two-period example

In this section, we give a simple example in which forward and futures prices can be found directly and use it to illustrate some of our propositions. In the next section, we shall reverse this procedure and use the propositions to determine forward and futures prices in a more complex setting.

For our first example, we consider a two-period model with a complete system of state prices. We shall supplement our earlier notation in the following way:

$p_i$  = price at time  $t$  of a claim which will pay one dollar at time  $t+1$  if the economy is in state  $i$  at time  $t+1$ ,

$p_{ij}$  = price at time  $t+1$  of a claim which will pay one dollar at time  $t+2$  if the economy is in state  $i$  at time  $t+1$  and state  $j$  at time  $t+2$ ,

$V_{ij}$  = price of the underlying good or asset at time  $t+2$  if the economy is in state  $i$  at time  $t+1$  and state  $j$  at time  $t+2$ ,

$H_i$  = futures price at time  $t+1$  if the economy is in state  $i$  at time  $t+1$ .

As before,  $G(t)$  and  $H(t)$  stand for the current forward price and futures price, respectively.

At time  $t+2$ , the value of a forward contract written at time  $t$  will be

$$V_{ij} - G(t), \quad (36)$$

and the current value of this amount is

$$\sum_{i,j} p_i p_{ij} [V_{ij} - G(t)]. \quad (37)$$

Since no money changes hands initially, both parties will be willing to enter into the contract only if its current value is zero. Consequently,

$$G(t) = \sum_{i,j} p_i p_{ij} V_{ij} / \sum_{i,j} p_i p_{ij}. \quad (38)$$

The current value of a bond paying one dollar at time  $t+2$  is  $\sum_{i,j} p_i p_{ij}$ , so this verifies that  $G(t)$  can be found as shown in Proposition 1.

Now we turn to determining the current futures price. Note that at time  $t+1$  the futures contract is the same as a forward contract, so

$$H_i = \sum_j p_{ij} V_{ij} / \sum_j p_{ij}. \quad (39)$$



At time  $t$ , the holder of a futures contract knows that he will receive at time  $t+1$  the amount

$$H_i - H(t), \quad (40)$$

the current value of which is

$$\sum_i p_i [H_i - H(t)]. \quad (41)$$

Again, since no money changes hands when the contract is initiated, this current value must be zero, so

$$\begin{aligned} H(t) &= \sum_i p_i H_i / \sum_i p_i \\ &= \sum_i p_i \left[ \sum_j p_{ij} V_{ij} / \sum_j p_{ij} \right] / \sum_i p_i \\ &= \sum_{i,j} p_i p_{ij} \left( 1 / \sum_i p_i \right) \left( 1 / \sum_j p_{ij} \right) V_{ij}. \end{aligned} \quad (42)$$

Since  $R_t = 1 / \sum_i p_i$  and  $R_{t+1} = 1 / \sum_j p_{ij}$ , this result illustrates Proposition 2. The current futures price is the same as the value of a claim which will pay at time  $t+2$  the amount  $V_{ij}$  times the total return from rolling over one-period bonds.

If interest rates are non-stochastic, then  $\sum_j p_{ij}$  is the same for all  $i$ , and  $\sum_{i,j} p_i p_{ij} = (\sum_i p_i)(\sum_j p_{ij})$ . Hence, it is apparent by inspection that  $G(t) = H(t)$ . Similarly, if  $V_{ij}$  is a constant, then it is obvious that  $G(t) = H(t)$ .

#### 4. Futures prices and forward prices in continuous-time, continuous-state economies

Propositions 2, 7 and 10 show how to construct assets whose current value must be equal to the current futures price. Propositions 1 and 8 do the same for forward prices. These results enable us to apply any intertemporal valuation model to the determination of forward and futures prices.

For our second example, we shall use a valuation framework which has become standard in finance. It has been shown by various arguments that in a continuous-time, continuous-state economy the value of any contingent

claim  $F$  will satisfy the fundamental partial differential equation

$$\frac{1}{2} \sum_{i,j} (\text{cov } X_i, X_j) F_{X_i X_j} + \sum_i (\mu_i - \phi_i) F_{X_i} + F_t - r(X, t)F + \delta(X, t) = 0, \quad (43)$$

where subscripts on  $F$  indicate partial derivatives and  $X$  is a vector containing all variables necessary to describe the current state of the economy. The remaining symbols are as follows:  $\mu_i$  is the local mean of changes in  $X_i$ ,  $\text{cov } X_i, X_j$  is the local covariance of the changes in  $X_i$  with the changes in  $X_j$ ,  $r(X, t)$  is the spot interest rate,  $\delta(X, t)$  is the continuous payment flow (if any) received by the claim, and  $\phi_i$  is the factor risk premium associated with  $X_i$ .

A number of studies have derived equations similar to (43) based on arbitrage arguments. For example, see Brennan and Schwartz (1979), Garman (1977), and Richard (1978). In these models, the factor risk premiums and the processes driving the state variables are determined exogenously or remain unspecified.

A somewhat different approach leading to the same type of equation is taken in Cox, Ingersoll and Ross (1978). In that paper, an intertemporal equilibrium model is developed in which all economic variables, including the interest rate and the factor risk premiums, are endogenously determined and explicitly identified in terms of individual preferences and production possibilities. Richard and Sundaresan (1981) extend this model to include multiple goods and use it to examine forward and futures contracts. In that setting, Sundaresan (1980) develops several explicit formulas for forward and futures prices. Most of our results in this section are special cases of their results.

Proposition 7 states that the futures price is equal to the value of an asset which receives a continual payout flow of  $(R_u - 1)H(u)$  and the amount  $V(s)$  at time  $s$ . In the present application, this would correspond to  $\delta(X, t) = r(X, t)H(X, t)$  and  $H(X, s) = V(X, s)$ . The futures price must thus satisfy the partial differential equation

$$\frac{1}{2} \sum_{i,j} (\text{cov } X_i, X_j) H_{X_i X_j} + \sum_i (\mu_i - \phi_i) H_{X_i} + H_t = 0, \quad (44)$$

with terminal condition  $H(X, s) = V(X, s)$ .

Two results from Cox, Ingersoll and Ross (1978) will be useful in characterizing futures prices. Lemma 4 of that paper shows that with  $\delta = 0$  the solution of (43) for a claim paying  $\theta(X(s))$  at time  $s$  can be written as

$$\hat{E} \left[ \theta(X(s)) \left[ \exp \left( - \int_t^s r(X(u)) du \right) \right] \right], \quad (45)$$

where  $\hat{E}$  indicates expectation taken with respect to a risk-adjusted process for the state variables. The risk adjustment is accomplished by reducing the local mean of each underlying variable by the corresponding factor risk premium. Proposition 2 states that the futures price is the same as the current value of an asset which will receive a single payment of  $\theta(X(s)) = V(X(s))[\exp(\int_t^s r(X(u))du)]$  at time  $s$ . Consequently, we can write the futures price as the risk-adjusted expected spot price at time  $s$ ,

$$H(X, t) = \hat{E}[V(X(s))]. \quad (46)$$

An immediate application of theorem 4 of Cox, Ingersoll and Ross (1978) shows that the futures price can be written in yet another way as

$$H(X, t) = E\left[V(X(s))\left[\exp\left(\int_t^s r(X(u))du\right)\right]\left(\frac{J_w(s)}{J_w(t)}\right)\right], \quad (47)$$

where  $E$  indicates expectation with respect to the actual process (with no risk-adjustment) for the state variables and  $J_w(\cdot)$  is the marginal utility of wealth of the representative individual. Given proposition 2, this is an intuitively sensible result. Since  $H$  is the value of a security which will pay  $V(X(s))\exp[\int_t^s r(X(u))du]$  at time  $s$ , (47) simply says that the value of this security is the expectation of its marginal-utility-weighted payoffs.

Forward prices can be obtained in a very straightforward way. From Proposition 1,  $G(X, t)$  will equal  $(1/P(t))$  times the solution to (43) with  $\delta=0$  and  $F(X, s) = V(X, s)$ . Similarly, we can write

$$\begin{aligned} G(X, t) &= \hat{E}\left[V(X(s))\left[\exp\left(-\int_t^s r(X(u))du\right)\right]\right]/P(t) \\ &= E\left[V(X(s))\left(\frac{J_w(s)}{J_w(t)}\right)\right]/P(t). \end{aligned} \quad (48)$$

As we have noted, an important historical role of forward and futures markets has been to provide a mechanism by which individuals can lock in today the price which they will have to pay for a good or asset on a future date. The simple strategy of taking a long position in one forward contract will accomplish exactly that, but the corresponding strategy of taking a long position in one futures contract will not. However, this does not rule out the possibility of achieving the same outcome by using futures contracts in a more complicated strategy. In the present context, with say  $n$  state variables,

the results of Black (1976) indicate that we should be able to find a controlled hedging portfolio, along the lines of Merton (1977), containing  $n$  futures contracts and borrowing or lending at the spot interest rate  $r$  which will require no subsequent investment and will duplicate the payoff to a forward contract on the maturity date.

To pursue this without unnecessary complications, we shall consider the case of  $n=1$ ; generalization to an arbitrary number of state variables is straightforward. Let  $\pi$  be the value of the hedging portfolio, and let  $\alpha$  be the number of futures contracts held in this portfolio. Further, let  $D(t)$  be the value at time  $t$  of a forward contract written at time  $q$ ; if  $q$  is the current time  $t$ , then  $D(t)=0$ . Consider the following strategy. At time  $t$ , make an investment of  $D(t)$  in the hedging portfolio. Place this amount in spot lending (rolling over shorts). At each time  $\tau$ , take a long position in  $D_X(\tau)/H_X(\tau)$  futures contracts, using (41) and (42) to find  $D$  and  $H$  in terms of  $X$  and  $t$ . Invest all money received from the futures position in spot lending and finance all money due by spot borrowing. If it is always possible to trade at equilibrium futures prices and interest rates, then this hedging portfolio will have the same value as the forward contract on the maturity date. To see this, consider the following argument. Let  $w(t)$  be the Wiener process driving the state variable, and let  $LH$  denote the differential generator of  $H$ ,  $LH \equiv \frac{1}{2}\sigma^2(X)H_{XX} + \mu_X H_X + H_t$ , where  $\sigma^2(X)$  is the local variance of the changes in  $X$ . From Ito's formula, the value of the hedging portfolio will follow the stochastic differential equation

$$d\pi(t) = r(X, t)\pi(t)dt + \alpha(X, t)dH(t). \quad (49)$$

Hence, the value of the portfolio at time  $s$  is

$$\begin{aligned} \pi(s) = & \left[ \exp\left(\int_t^s r(u)du\right) \right] \\ & \times \left[ \pi(t) + \int_t^s \left[ \exp\left(-\int_t^z r(u)du\right) \right] \alpha(z)LH(z)dz \right. \\ & \left. + \int_t^s \left[ \exp\left(-\int_t^z r(u)du\right) \right] \alpha(z)H_X(z)\sigma(z)dw(z) \right]. \end{aligned} \quad (50)$$

Now note that (43) implies that  $LD = \phi_X D_X + rD$  and (44) implies that  $LH = \phi_X H_X$ , so  $LH = (LD - rD)/(D_X/H_X)$ . Substituting this expression for  $LH$  into (50) and letting  $\alpha = D_X/H_X$  gives

$$\begin{aligned}
\pi(s) &= \exp\left(\int_t^s r(u) du\right) \\
&\times \left[ \pi(t) \int_t^s \left[ \exp\left(-\int_t^z r(u) du\right) \right] (LD(z) - r(z)D(z)) dz \right. \\
&\quad \left. + \int_t^s \left[ \exp\left(-\int_t^z r(u) du\right) \right] D_X(z) \sigma(z) dw(z) \right] \\
&= D(s) + \left[ \exp\left(\int_t^s r(u) du\right) \right] [\pi(t) - D(t)].
\end{aligned} \tag{51}$$

Since  $\pi(t) = D(t)$ , then  $\pi(s) = D(s)$ , so the hedging portfolio will have the same value as a forward contract on the maturity date. The particular nature of the payoff received by a forward contract played no role in the argument, so there is no problem in specifying a more general payoff. Similarly, with multiple state variables, both traded assets and futures contracts can be included in the hedging portfolio. However, readers should be aware that our discussion has not gone into certain technical difficulties connected with continuous trading [see Harrison and Kreps (1979)].

An important advantage of the framework used in this example is that it can easily be specialized to produce testable explicit formulas. An illustration of this is its application to the term structure of interest rates. For instance, under the additional assumptions of logarithmic utility and a technology which leads to a spot interest rate following the stochastic differential equation

$$dr = \kappa(\mu - r)dt + \sigma\sqrt{r}dw, \tag{52}$$

it is shown in Cox, Ingersoll and Ross (1978) that the prices of discount bonds will satisfy the partial differential equation

$$\frac{1}{2}\sigma^2 r P_{rr} + [\kappa\mu - (\kappa + \lambda)r]P_r + P_t - rP = 0, \tag{53}$$

where  $\kappa$ ,  $\mu$  and  $\sigma$  are the parameters of the interest rate process and  $\lambda r$  is the local covariance of changes in the interest rate with percentage changes in aggregate wealth (the market portfolio).

Now consider the forward and futures prices for contracts with maturity date  $s$  on a discount bond paying one dollar at time  $T$ , with  $T > s$ . Straightforward application of the methods discussed earlier shows that

$$G(t) = \left( \frac{A(T-t)}{A(s-t)} \right) \exp[-r(B(T-t) - B(s-t))], \tag{54}$$

and

$$H(t) = A(T-s) \left[ \frac{\eta}{B(T-s) + \eta} \right]^{2\kappa\mu/\sigma^2} \times \exp \left[ -r \left( \frac{\eta B(T-s) e^{-(\kappa+\lambda)(s-t)}}{B(T-s) + \eta} \right) \right], \quad (55)$$

where

$$A(T-t) = \left[ \frac{2\gamma e^{[(\kappa+\lambda+\gamma)(T-t)]/2}}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1) + 2\gamma} \right]^{2\kappa\mu/\sigma^2},$$

$$B(T-t) = \frac{2(e^{\gamma(T-t)}-1)}{(\gamma+\kappa+\lambda)(e^{\gamma(T-t)}-1) + 2\gamma},$$

$$\gamma = [(\kappa+\lambda)^2 + 2\sigma^2]^{\frac{1}{2}}, \quad \eta = \frac{2(\kappa+\lambda)}{\sigma^2(1 - e^{-(\kappa+\lambda)(s-t)}}.$$

Note that since  $A(0)=1$  and  $B(0)=0$ ,  $G(s)=H(s)$ , as of course it must. For all  $t < s$ ,  $G(t) > H(t)$ , confirming the observation made about Treasury bill futures in the discussion following Proposition 9. It is apparent by inspection that forward and futures prices are decreasing convex functions of the interest rate, as is also true of bill prices in this model. However, unlike the bill prices, the forward prices and futures prices can be increasing functions of the time to maturity for sufficiently high interest rates.

This approach can be generalized in a number of ways. For example, the simple mean-reverting drift for the interest rate in (52) can be replaced with exponentially weighted extrapolative and regressive components, as in the De Leeuw-Malkiel term structure hypotheses [see Cox, Ingersoll and Ross (1981)]. Although the resulting forward and futures prices are more complicated than (54) and (55), they still retain the simple exponential form. Furthermore, Proposition 5 shows that our results for discount bonds can be immediately applied to coupon bonds.

Formulas such as (54) and (55) make predictions about simultaneous prices in different markets, and hence offer interesting opportunities for empirical testing. However, the empirical magnitude of the effect introduced by the continual resettlement feature of futures contracts remains an open question. Capozza and Cornell (1979) analyzed futures prices and implicit forward prices in the Treasury bill market and found that, except for very

short maturities, forward prices exceeded futures prices and that the difference increased with time to maturity. Rendleman and Carabini (1979) independently reached a similar but less definitive conclusion. These findings are generally consistent with the qualitative predictions of (26). However, Rendleman and Carabini have examined (54) and (55) for a range of parameter values and have concluded that the implied differences do not fully explain the observed differences between forward prices and futures prices in the Treasury bill market. This may indicate that one of the generalizations mentioned above will be more appropriate. Another possible explanation for the observed discrepancies lies in various tax effects which we have thus far ignored but shall consider in the next section.

### 5. The effects of taxes and other institutional factors

We shall postpone a complete discussion of taxes until another occasion, but some informal comments and conjectures may still be worthwhile. The simplest way to introduce taxes into the setting of section 4 is as follows. Taxes are collected continuously at constant rates which are the same for all individuals. Capital gains are taxed as they accrue, rather than when realized, with full loss offsets. The dollar receipts from futures price changes are taxed as capital gains.

In such a world, investors will be concerned with their after-tax returns and will value contingent claims accordingly. For any given claim  $F$ , let the tax rate for capital gains be  $c$  and the tax rate for payouts and interest income be  $d$ . It can then be shown that the fundamental valuation equation becomes

$$\begin{aligned} & \frac{1}{2} \sum_{i,j} (\text{cov } X_i, X_j) F_{X_i X_j} + \sum_i (\mu_i - \phi_i) F_{X_i} \\ & + F_t - \left( \frac{1-d}{1-c} \right) r(X, t) F + \left( \frac{1-d}{1-c} \right) \delta(X, t) = 0. \end{aligned} \quad (56)$$

Since the last two terms will not apply for a futures price, taxes of this type will have no direct effect upon its value. Of course, there will be indirect effects, since taxes will in general affect the factor risk premiums and the current values and stochastic processes of all endogenously determined state variables. However, if we are considering the comparative effect of a change only in the tax rate applicable to futures markets, then these general equilibrium effects would presumably be negligible or non-existent, and the futures prices would remain unchanged. Notice, too, that for Treasury bills  $\delta=0$  and  $c=d$ , since their price changes are taxed at the rate for ordinary income. Hence, their valuation equation would remain the same as well.

The actual tax law is of course more complex than this, particularly with regard to Treasury bill futures. It currently appears that a gain on a long position in a Treasury bill futures contract will be taxed as a long-term or short-term capital gain, depending on whether the holding period is longer or shorter than six months. On the other hand, if the position shows a loss, by taking delivery and selling the Treasury bills, the basis can be taken to be the original futures price and the loss will be considered as an ordinary loss. If the taxes are collected on the maturity date and all individuals are taxed at the same rate, then a modification of the above analysis can be used to find the futures price. With this type of tax, the terminal condition will depend on the initial value, so a recursive procedure is necessary. The problem is first solved with an arbitrary parameter replacing the initial value in the terminal condition. This parameter is then varied until the initial value and the parameter value are equal. As one would expect, other things equal, this tax option results in a higher futures price.

We have not gone through this analysis explicitly only because we are not convinced of its relevance. Additional considerations may bring us full circle. Although we cannot provide a formal model which includes both differential taxes and transactions costs, it seems likely to us that the agents with transactions costs low enough to be able to conduct arbitrage operations are likely to be professionals who are taxed at the same rate for all trading income and who consequently derive no benefit from the special tax option. The actions of arbitrageurs would thus tend to keep futures prices near the levels we have predicted; simultaneously, individuals who cannot conduct arbitrage operations could nevertheless obtain tax advantages at these prices. Cornell (1980) has persuasively advanced this point of view and has provided some interesting empirical support. If the actions of arbitrageurs did not effectively determine the Treasury bill futures price, then one would expect a discontinuous change in the price (though not in the after-tax returns to the marginal investor) when a contract changes from long-term to short-term tax treatment, but Cornell found no evidence of this.

Some additional institutional factors may have an effect on the futures price in particular markets. In the basic futures contract we described, the seller of the contract can on the maturity date close out his position either by taking an offsetting long position or by delivering the specified amount of the underlying good or asset. In many markets, the seller has somewhat more flexibility than this. He may have one or more of three additional alternatives which we will refer to as a quality option, a quantity option, and a timing option.

The quality option allows the seller some discretion in the good which can be delivered. For example, several different types of a particular grain may all be acceptable. If the spot price of one of the types would always be less



than the others, then this is the one the seller would choose, and the contract would in effect become an ordinary futures on that type. The situation is only slightly more complicated when the price ordering is not always the same. In that case, all of our results would hold, or would require only minor modification, when  $V(s)$  is replaced by the minimum of the spot prices of the acceptable goods on the maturity date.

The quantity option allows the seller some choice in the amount of the good which is to be delivered. In this case, the futures price is quoted on a per unit basis, so the choice concerns only the scale of the contract. In perfect, competitive markets, as we have assumed, the quantity option will be a matter of indifference and will have no effect on the futures price.

The timing option gives the seller some flexibility in the delivery date of the good. In this case, delivery can be made at any time during a designated period beginning on the maturity date. Typically, the designated period is one month or less. Since delivery can be postponed, the futures price will not necessarily be equal to the spot price on the maturity date. Clearly, the futures price cannot be greater than the spot price during the designated period. If it were, then it would be possible to make an arbitrage profit by simultaneously selling a futures contract, purchasing the good in the spot market, and making delivery. Consequently, we must append to any valuation framework the arbitrage condition  $H(\tau) \leq V(\tau)$  for all  $\tau$  such that  $s \leq \tau \leq s'$  and  $H(s') = V(s')$ , where  $s'$  is the end of the designated period. Readers familiar with option pricing theory will note the similarity to the arbitrage condition for an American option. We can now use Proposition 10 to give a sufficient condition for the effective delivery date to be  $s$  or  $s'$ . According to this proposition, if the spot rental rate is always greater than the spot interest rate, then the futures price with maturity date  $s'$  is always less than the spot price. Consequently, the arbitrage condition will always be satisfied and the futures price can be determined as if the maturity date were  $s'$ . On the other hand, if the rental rate is always less than the interest rate, then the arbitrage condition cannot be satisfied for any maturity date later than  $s$  and all deliveries would be made at that time. In this case, the futures price can thus be determined as if the maturity date were  $s$ .

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