Appendix E

The Normal Model

Here is a heuristic sketch of the derivation of the value of swaptions under the assumption of a normal distribution, as compared to the standard Black-Scholes assumption of lognormality. We employ risk-neutrality. In particular, let us consider pricing a T into n payer swaption struck at X, under the assumption that the underlying swap rate $R \sim N(F, \sigma_N^2 T)$, where F denotes the n-year swap rate, T years forward. In this formulation, we refer to σ_N as *normal vol.* For ease of notation, define $\tau = \sigma_N \sqrt{T}$.

For simplicity, assume the swap pays annually on the fixed side and consider now valuing the cash flow at time t_i associated with exercise at time T, where $t_i = T+1, T+2, \ldots, T+n$. The future value of this cash flow at time t_i is equal to

$$\begin{aligned}
&\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\tau} \exp\left[-\frac{(R-F)^2}{2\tau^2}\right] \max(0, R-X)dR \\
&= \int_{-\infty}^{X} \frac{1}{\sqrt{2\pi}\tau} \exp\left[-\frac{(R-F)^2}{2\tau^2}\right] \max(0, R-X)dR \\
&\quad + \int_{X}^{\infty} \frac{1}{\sqrt{2\pi}\tau} \exp\left[-\frac{(R-F)^2}{2\tau^2}\right] \max(0, R-X)dR \\
&= \int_{X}^{\infty} \frac{1}{\sqrt{2\pi}\tau} \exp\left[-\frac{(R-F)^2}{2\tau^2}\right] (R-X)dR. \qquad \text{(E.1)}
\end{aligned}$$

Let us now make a change of variables and let $x = (R-F)/\tau$. Notice that with this substitution we have $dR = \tau dx$. Substituting into Equation E.1 gives

$$\int_{\frac{X-F}{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)(\tau x + F - X)dx$$
$$= \tau \int_{\frac{X-F}{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} x \exp\left(-\frac{x^2}{2}\right) dx$$
$$+(F-X)\int_{\frac{X-F}{\tau}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx. \qquad \text{(E.2)}$$

The first integral on the right-hand side of Equation E.2 is evaluated using integration by parts. The second integral on the right-hand side of Equation E.2 is equal to one less the cumulative normal distribution function evaluated at $(X-F)/\tau$. We have, then, that Equation E.2 is equivalent to

$$\frac{\tau}{\sqrt{2\pi}}\left[-\exp\left(-\frac{x^2}{2}\right)\right]_{\frac{X-F}{\tau}}^{\infty} + (F-X)\left[1 - N\left(\frac{X-F}{\tau}\right)\right]$$
$$= \frac{\tau}{\sqrt{2\pi}} \exp\left[-\frac{(X-F)^2}{2\tau^2}\right] + (F-X)\left[1 - N\left(\frac{X-F}{\tau}\right)\right], \qquad \text{(E.3)}$$

where $N(\cdot)$ denotes the cumulative normal distribution function, that is,

$$N(d) = \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx. \qquad \text{(E.4)}$$

Thus, the full value of the T into n payer swaption struck at X is

$$A\sigma_N\sqrt{T}\left[\frac{1}{\sqrt{2\pi}}e^{-d^2/2} + dN(d)\right], \qquad \text{(E.5)}$$

where

$$d = \frac{F-X}{\sigma_N\sqrt{T}} \qquad \text{(E.6)}$$

and

$$A = \sum_{i=1}^{n} \frac{1}{[1+zc(t_i)]^{t_i}} \qquad \text{(E.7)}$$

is the annuity factor (where $t_i = T + i$, for i = 1, 2, ..., n).

Using swaption parity, it follows immediately that the value of a T into n receiver swaption struck at X is

$$A\sigma_N\sqrt{T}\left[\frac{1}{\sqrt{2\pi}}e^{-d^2/2} - dN(-d)\right], \qquad \text{(E.8)}$$

with d and A as defined in Equations E.6 and E.7, respectively.

The value of a swaption straddle struck at-the-money forward is found by setting $X = F$ in each of Equations E.5 and E.8, then adding the two together. Thus, the value of a T into n swaption straddle struck at-the-money forward is given by

$$\sqrt{T} \times \sqrt{\frac{2}{\pi}} \times \sigma_N \times A. \qquad \text{(E.9)}$$

E.1 The Relationship Between σ_{LN} and σ_N for Swaptions that Are Struck At-the-Money Forward

Under the assumption of lognormality, the value of a payer swaption and a receiver swaption are given by Black's formula:

$$A \times [FN(d_1) - XN(d_2)] \qquad \text{(E.10)}$$

and

$$A \times [XN(-d_2) - FN(-d_1)], \qquad \text{(E.11)}$$

respectively, where

$$d_1 = \frac{\ln(F/X) + \sigma_{LN}^2 T/2}{\sigma_{LN}\sqrt{T}} \qquad \text{(E.12)}$$

and

$$d_2 = d_1 - \sigma_{LN}\sqrt{T}, \qquad \text{(E.13)}$$

where σ_{LN} denotes lognormal vol.

Consider now the value of a swaption straddle struck at-the-money forward in the lognormal model. Setting $X = F$ and using Equations E.10–E.13, we have the value of the straddle under lognormality

$$2 \times A \times F \times \left[N\left(\frac{\sigma_{LN}\sqrt{T}}{2}\right) - N\left(-\frac{\sigma_{LN}\sqrt{T}}{2}\right)\right]. \qquad \text{(E.14)}$$

We will now determine the relationship between σ_N (normal implied vol in the normal model) and σ_{LN} (lognormal implied vol in the lognormal model) for at-the-money swaptions. In order to do this, we will set the two expressions for the price of swaption straddles as given in Equations E.9 and E.14 equal to one another and deduce the restrictions that this places on the volatilities. To this end, let us first do some work to analyze the expression

$$\left[N\left(\frac{\sigma_{LN}\sqrt{T}}{2}\right) - N\left(-\frac{\sigma_{LN}\sqrt{T}}{2}\right)\right], \qquad \text{(E.15)}$$

which appears as a factor in the pricing formula for the swaption straddle under lognormality in Equation E.14. Although the expression in Equation E.15 is not analytically tractable—the $N(\cdot)$ function refers to an integral that cannot be further simplified—we can use a Taylor series expansion of the cumulative normal distribution function to approximate the expression arbitrarily well.

The *error function*, denoted $\mathrm{erf}(x)$, is well known in mathematics and is defined by

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)} = \frac{2}{\sqrt{\pi}} \left(x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots \right). \tag{E.16}$$

The error function and the cumulative normal distribution function are closely related by the identity

$$N(x) = \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{x}{\sqrt{2}}\right)\right]. \tag{E.17}$$

Referring now to Equation E.15, let $\epsilon = \sigma_{LN}\sqrt{T}/2$ and notice that

$$\begin{aligned} N(\epsilon) - N(-\epsilon) &= \frac{1}{2}\left[1 + \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right)\right] - \frac{1}{2}\left[1 + \mathrm{erf}\left(-\frac{\epsilon}{\sqrt{2}}\right)\right] \\ &= \frac{1}{2}\left[\mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right) - \mathrm{erf}\left(-\frac{\epsilon}{\sqrt{2}}\right)\right] \\ &= \frac{1}{2}\left[\mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right) + \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right)\right] \\ &= \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right), \end{aligned} \tag{E.18}$$

where we have made use of the identity

$$\mathrm{erf}\left(-\frac{\epsilon}{\sqrt{2}}\right) = -\mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right).$$

Substituting the results of Equation E.18 into Equation E.14 gives us the price of swaption straddle under the lognormal model as

$$2 \times A \times F \times \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right). \tag{E.19}$$

Equating this price with the price of the swaption straddle under the normal model implies that we must have

$$\sqrt{T} \times \sqrt{\frac{2}{\pi}} \times \sigma_N \times A = 2 \times A \times F \times \mathrm{erf}\left(\frac{\epsilon}{\sqrt{2}}\right). \tag{E.20}$$

Let $\beta = \epsilon/\sqrt{2}$, and substituting the definition of $\mathrm{erf}(\cdot)$ from Equation E.16 into Equation E.20 implies that we must have

$$\sqrt{T} \times \sqrt{\frac{2}{\pi}} \times \sigma_N = 2 \times F \times \frac{2}{\sqrt{\pi}} \times \left(\beta - \frac{\beta^3}{3} + \text{error}\right), \tag{E.21}$$

where "error" denotes a (small) error term. From Equation E.21 we can write

$$\sqrt{T} \times \sqrt{\frac{2}{\pi}} \times \sigma_N \approx 2 \times F \times \frac{2}{\sqrt{\pi}} \times \left(\beta - \frac{\beta^3}{3}\right). \tag{E.22}$$

Recall that $\beta = \sigma_{LN}\sqrt{T}/(2\sqrt{2})$ and substituting into Equation E.22 gives

$$\sigma_N \approx F \times \sigma_{LN} \times \left(1 - \frac{\sigma_{LN}^2\, T}{24}\right). \tag{E.23}$$

Equation E.23 shows why we often approximate normal vol with the quantity $F \times \sigma_{LN}$: the term in parentheses is often close to one, provided that σ_{LN} and T are small. However, when σ_{LN} is large and/or time to expiration T is large, the quantity in Equation E.23 provides a meaningfully better approximation.

E.2 The Relationship Between σ_{LN} and σ_N for Off-the-Money Swaptions

The relationship between lognormal implied vol, σ_{LN}, and normal implied vol, σ_N, for swaptions with an arbitrary strike can be determined with a numerical procedure that we will outline below.[1] Before we begin, recall that under the normal model the value of a T into n receiver swaption with strike X is equal to

$$A\sigma_N\sqrt{T}\left[\frac{1}{\sqrt{2\pi}}e^{-d^2/2} - dN(-d)\right], \tag{E.24}$$

with d and A as defined in Equations E.6 and E.7, respectively.

[1] We can regard lognormal implied vol and normal implied vol as functions of X, the strike of the swaption. Denote these functions by $\sigma_{LN}(X)$ and $\sigma_N(X)$, respectively. Both Zhou (2003) and Sadr (2009) suggest that

$$\sigma_N(X) \approx \sigma_{LN}(X) \times \sqrt{FX},$$

which reduces to Equation 5.27 for the case of $X = F$ (that is, for the case in which the swaption is struck at-the-money forward). The numerical procedure developed in this section is an alternative to this approximation formula.

Sensitivity	Partial	Payer	Receiver
Delta	$\frac{\partial}{\partial F}$	$AN(d)$	$-AN(-d)$
Gamma	$\frac{\partial^2}{\partial F^2}$	$A\frac{n(d)}{\sigma_N\sqrt{T}}$	$A\frac{n(d)}{\sigma_N\sqrt{T}}$
Vega	$\frac{\partial}{\partial \sigma_N}$	$A\sqrt{T}n(d)$	$A\sqrt{T}n(d)$

Table E.1: Option Sensitivities Under the Normal Model

Recall that under the lognormal model the value of a T into n receiver swaption with strike X is given in Equation E.11 and is

$$A \times [XN(-d_2) - FN(-d_1)], \tag{E.25}$$

where d_1 and d_2 are as defined in Equations E.12 and E.13, respectively.

We assume that the normal model holds (in which case σ_N does not vary as we allow the strike of swaptions of a particular expiry and tail to vary) and seek to derive restrictions on σ_{LN}. For a given expiry T, a tail n, and a given strike X:

1. Price up the receiver swaption under the normal model for a given level of normal implied vol, σ_N, using Equation E.24. Denote by price$_N$ the price we obtain.
2. Set the value of a T into n receiver swaption struck at X under the lognormal model in Equation E.25 equal to the price we just solved for in Step 1. That is, set
$$A \times [XN(-d_2) - FN(-d_1)] = \text{price}_N. \tag{E.26}$$
3. Determine the level of lognormal implied vol, σ_{LN}, that satisfies Equation E.26.

Following this procedure for a variety of strikes X allows us to (numerically) deduce the relationship between σ_N and σ_{LN} under the assumption of the normal model. This process was used to create Figure 5.15.[2]

[2] This procedure could be modified to utilize payer swaptions instead of receiver swaptions or even swaption straddles. The point here is to require that the value of an option is the same regardless of which model we use, then to derive restrictions on the implied vols that must be used in each model.

Sensitivity	Partial	Payer	Receiver
Delta	$\frac{\partial}{\partial F}$	$AN(d_1)$	$-AN(-d_1)$
Gamma	$\frac{\partial^2}{\partial F^2}$	$A\frac{n(d_1)}{F\sigma_{LN}\sqrt{T}}$	$A\frac{n(d_1)}{F\sigma_{LN}\sqrt{T}}$
Vega	$\frac{\partial}{\partial \sigma_{LN}}$	$AF\sqrt{T}n(d_1)$	$AF\sqrt{T}n(d_1)$

Table E.2: Option Sensitivities Under the Lognormal Model

E.3 Option Sensitivities Under the Normal Model

The option sensitivities for payer swaptions and receiver swaptions in the normal model are found by partially differentiating the expressions in Equations E.5 and E.8, respectively. Table E.1 lists some of these commonly used sensitivities. In Table E.1, d and $N(d)$ are as defined in Equations E.6 and E.4, respectively, and

$$n(d) = \frac{1}{\sqrt{2\pi}}e^{-d^2/2}. \tag{E.27}$$

By comparison, the sensitivities for payer swaptions and receiver swaptions under the lognormal (Black) model are shown in Table E.2. These sensitivities are found by partially differentiating the expressions in Equations E.10 and E.11, respectively. In Table E.2, d_1 is as given in Equation E.12.