Curve Sensitivites

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Aug 2, 2011

1 Vanilla IRS

A vanilla IRS is priced as,

$$NPV(t) = -NPV(t)_{fix} + NPV(t)_{flt}, \tag{1}$$

where,

$$NPV(t)_{fix} = \sum_{i=1}^{m} N \cdot R_{fix} \tau(T_i^S, T_i^E) P_d(t, T_i^P), \tag{2}$$

and

$$NPV(t)_{flt} = \sum_{j=1}^{n} N \cdot \left(F_f(t, T_j^S, T_j^E) + S \right) \tau(T_i^S, T_i^E) P_d(t, T_j^P). \tag{3}$$

Compunding

$$NPV(t)_{comp} = \sum_{i=1}^{n} A_i P_d(t, T_I^P). \tag{4}$$

Where

$$A_i = \sum_{s=1}^{subPeriods} AI_{sp}. \tag{5}$$

1.1 Compounding Swaps

Compunding swaps come in two different flavors, Flat Compunding ,

$$AI_s^{flat} = \tag{6}$$

$$= N\left(F(t,T_s^S,T_s^E)+S\right)\tau(T_s^S,T_s^E)+F(t,T_s^S,T_s^E)\tau(T_s^S,T_s^E)\sum_{k=1}^{s-1}AI_k^{flat}. \tag{7}$$

and Straight Compunding

$$AI_s^{Str} = \tag{8}$$

$$= N\left(F(t,T_s^S,T_s^E)+S\right)\tau(T_s^S,T_s^E)+\left(F(t,T_s^S,T_s^E)+S\right)\tau(T_s^S,T_s^E)\sum_{k=1}^{s-1}AI_k^{Str}. \tag{9}$$

1.1.1 OIS

$$NPV(t)_{OIS} = \sum_{j=1}^{n} N\cdot\left(F_{OIS}(t,T_j^S,T_j^E)+S\right)\tau(T_j^S,T_j^E)P_{OIS}(t,T_j^P)/(\tag{10}$$

1.2 Notation

Where,
T_t^S Time when period i start
T_t^E Time when period i end
T_t^P Time when payment is made for period i
R_{fix} Fixed rate
$\tau(T_1,T_2)$ Duration between T_1 and T_2
$P_d(t,T)$ The discount factor, on curve d, from time T to time t
$F_f(t,T_1,T_2)$ Forward rate, on curve f, from time T_1 to time T_2 as viewed from time t
S The spread of the floating leg

Observe that the discount factors P used are for the discounting curve d and the forward rate F is from the forward curve f. The forward rate can be expressed in discount factors, in the general case,

$$F(t,T_1,T_2)=\left(\frac{P(t,T_1)}{P(t,T_2)}-1\right)/\tau(T_1,T_2). \tag{11}$$

Hence in a slight twist of notation, the forward rate on the forward curve can be expressed as discount factors on the forward curve.

1.3 Sensitivities

Sensitivities are calculated on the zero coupong curve. A zero coupong curve can be transformed from a discount curve according to,

$$z(t,T) = -\frac{\ln P(t,T)}{\tau(t,T)}, \tag{12}$$

or,

$$P(t,T) = \exp\left(-z(t,T)\tau(t,T)\right), \tag{13}$$

The first order sensitivity of the forward rate towards knot point j can be expressed as,

$$\frac{\partial F_f(t,T_1,T_2)}{\partial z_{f,T_j}} = \frac{1}{\tau(T_1,T_2)} \frac{\partial \frac{P(T_1)}{P(T_2)}}{\partial z_{f,T_j}} = \tag{14}$$

$$= \frac{1}{\tau(T_1,T_2)} \left(\frac{1}{P(T_2)} \frac{\partial P(T_1)}{\partial z_{f,T_j}} + P(T_1) \frac{\partial \frac{1}{P(T_2)}}{\partial z_{f,T_j}} \right) = \tag{15}$$

$$= \frac{1}{\tau(T_1,T_2)} \left(\frac{1}{P(T_2)} \frac{\partial P(T_1)}{\partial z_{f,T_1}} \frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + P(T_1) \frac{\partial \frac{1}{P(T_2)}}{\partial z_{f,T_2}} \frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} \right) = \tag{16}$$

$$= \frac{1}{\tau(T_1,T_2)} \frac{P(T_1)}{P(T_2)} \left(-\tau(T_1) \frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2) \frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} \right) = \tag{17}$$

$$\tag{18}$$

The second order sensitivity of the forward rate towards knot point j and G can be expressed as,

$$\frac{\partial^2 F_f(t,T_1,T_2)}{\partial z_{f,T_j} \partial z_{f,T_g}} = \frac{1}{\tau(T_1,T_2)} \frac{\partial}{\partial z_{f,T_g}} \frac{\partial \frac{P(T_1)}{P(T_2)}}{\partial z_{f,T_j}} = \tag{19}$$

$$= \frac{1}{\tau(T_1,T_2)} \frac{\partial}{\partial z_{f,T_g}} \left\{ \frac{P(T_1)}{P(T_2)} \left(-\tau(T_1) \frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2) \frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} \right) \right\} = \tag{20}$$

$$= \frac{1}{\tau(T_1,T_2)} \Big\{ \frac{\partial \frac{P(T_1)}{P(T_2)}}{\partial z_{f,T_g}} \left(-\tau(T_1) \frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2) \frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} \right) + \tag{21}$$

$$+ \frac{P(T_1)}{P(T_2)} \frac{\partial}{\partial z_{f,T_g}} \left(-\tau(T_1) \frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2) \frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} \right) \Big\} = \tag{22}$$

$$= \frac{1}{\tau(T_1,T_2)}\Big\{\frac{P(T_1)}{P(T_2)}\left(-\tau(T_1)\frac{\partial z_{f,T_1}}{\partial z_{f,T_g}} + \tau(T_2)\frac{\partial z_{f,T_2}}{\partial z_{f,T_g}}\right)\cdot \tag{23}$$

$$\cdot\left(-\tau(T_1)\frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2)\frac{\partial z_{f,T_2}}{\partial z_{f,T_j}}\right) + \tag{24}$$

$$+\frac{P(T_1)}{P(T_2)}\frac{\partial}{\partial z_{f,T_g}}\left(-\tau(T_1)\frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} + \tau(T_2)\frac{\partial z_{f,T_2}}{\partial z_{f,T_j}}\right)\Big\} = \tag{25}$$

$$= \frac{1}{\tau(T_1,T_2)}\frac{P(T_1)}{P(T_2)}\Big\{\tau^2(T_1)\frac{\partial z_{f,T_1}}{\partial z_{f,T_g}}\frac{\partial z_{f,T_1}}{\partial z_{f,T_j}} - \tau(T_1)\tau(T_2)\frac{\partial z_{f,T_1}}{\partial z_{f,T_g}}\frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} + \tag{26}$$

$$-\tau(T_1)\tau(T_2)\frac{\partial z_{f,T_1}}{\partial z_{f,T_j}}\frac{\partial z_{f,T_2}}{\partial z_{f,T_g}} + \tau^2(T_2)\frac{\partial z_{f,T_2}}{\partial z_{f,T_g}}\frac{\partial z_{f,T_2}}{\partial z_{f,T_j}} + \tag{27}$$

$$-\tau(T_1)\frac{\partial^2 z_{f,T_1}}{\partial z_{f,T_g}\partial z_{f,T_j}} + \tau(T_2)\frac{\partial^2 z_{f,T_2}}{\partial z_{f,T_g}\partial z_{f,T_j}}\Big\} \tag{28}$$

The sensitivity of the NPV with regards to nodes T_i and T_j on a zero rate curve is expressed as,

$$\frac{\partial NPV}{\partial z(t,T_i)}, \tag{29}$$

and

$$\frac{\partial^2 NPV}{\partial z(t,T_i)\partial z(t,T_j)}. \tag{30}$$

1.3.1 Delta

The fixed leg only has sensitivity towards the j:th node on the discounting curve,

$$\frac{\partial NPV_{fix}}{\partial z_d(t,T_j)} = \sum_{i=1}^{m} N\cdot R_{fix}\tau(T_i^S,T_i^E)\frac{\partial}{\partial z_{d,T_j}}P_d(t,T_i^P) \tag{31}$$

$$= -\sum_{i=1}^{m} N\cdot R_{fix}\tau(T_i^S,T_i^E)\tau(t,T_i^P)P_d(t,T_i^P)\cdot\frac{\partial z_{d,T^P}}{\partial z_{d,T_j}} \tag{32}$$

$$\tag{33}$$

While the floating leg has sensitivity toward both the discounting and forward curve,

$$\frac{\partial NPV_{float}}{\partial z_d(t,T_j)} = \sum_{i=1}^{m} N\cdot\Big(F_f(t,T_j^S,T_j^E)+S\Big)\tau(T_i^S,T_i^E)\frac{\partial}{\partial z_{d,T_j}}P_d(t,T_i^P) \tag{34}$$

$$= -\sum_{i=1}^{m} N \cdot \Big(F_f(t, T_j^S, T_j^E) + S \Big) \tau(T_i^S, T_i^E) \tau(t, T_i^P) P_d(t, T_i^P) \cdot \frac{\partial z_{d,T^P}}{\partial z_{d,T_j}} \tag{35}$$

$$\tag{36}$$

here we have ignored that the forward rates has a sensitivity towards changes in the discounting curve (through bootstraping). And expressing the sensitivity towards node j on the forward curve,

$$\frac{\partial NPV_{float}}{\partial z_f(t, T_j)} = \sum_{i=1}^{m} N \cdot \frac{\partial}{\partial z_{f,T_j}} \Big(F_f(t, T_j^S, T_j^E) + S \Big) \tau(T_i^S, T_i^E) P_d(t, T_i^P) \tag{37}$$

$$\sum_{i=1}^{m} N \cdot \frac{\partial F_f(t, T_j^S, T_j^E)}{\partial z_{f,T_j}} \tau(T_i^S, T_i^E) P_d(t, T_i^P) \tag{38}$$

1.3.2 Gamma

A leg has sensitivity towards the j:th and the g:th node on the discounting curve,

$$\frac{\partial^2 NPV_{fix}}{\partial z_j^d \partial z_g^d} = \sum_{i=1}^{m} \cdot A_i \frac{\partial^2}{\partial z_j^d \partial z_g^d} P_d(t, T_i^P) = \tag{39}$$

$$= \sum_{i=1}^{m} A_i \frac{\partial}{\partial z_j^d} \left\{ \frac{\partial P_d(t, T_i^P)}{\partial z^P(T_i)} \frac{\partial z^P(T_i)}{\partial z_g^d} \right\} = \tag{40}$$

$$= \sum_{i=1}^{m} -A_i \tau(T_i^P) \frac{\partial}{\partial z_j^d} \left\{ P_d(t, T_i^P) \frac{\partial z^P(T_i)}{\partial z_g^d} \right\} \tag{41}$$

$$= \sum_{i=1}^{m} -A_i \tau(T_i^P) P_d(t, T_i^P) \cdot \tag{42}$$

$$\cdot \left\{ -\tau(T_i^P) \frac{\partial z^P(T_i)}{\partial z_j^d} \frac{\partial z^P(T_i)}{\partial z_g^d} + \frac{\partial^2 z^P(T_i)}{\partial z_g^d \partial z_j^d} \right\} \tag{43}$$

$$\tag{44}$$

Where A_i is the amount of each cashflow i.

The forward curve sensitivity is,

$$\frac{\partial^2 NPV(t)_{flt}}{\partial z_j^f \partial z_g^f} = \sum_{j=1}^{n} N \cdot \frac{\partial^2 F_f(t, T_j^S, T_j^E)}{\partial z_j^f \partial z_g^f} \tau(T_i^S, T_i^E) P_d(t, T_j^P). \tag{45}$$

and the cross gamma between the curves are,

$$\frac{\partial^2 NPV_{fix}}{\partial z_j^d \partial z_g^f} = -\sum_{i=1}^{m} \tau(T_i^P) P_d(t, T_i^P) \tau(T_i^S, T_i^E) \frac{\partial F_f(t, T_j^S, T_j^E)}{\partial z_g^f} \frac{\partial z^d(T_i)}{\partial z_g^d} \tag{46}$$

$$\tag{47}$$

1.3.3 Compounding legs

And for compunding swaps,

$$\frac{\partial NPV(t)_{comp}}{\partial z_j^f} = \sum_{i=1}^{n} \frac{\partial A_i}{\partial z_j^f} P_d(t, T_I^P). \tag{48}$$

Where

$$\frac{\partial A_i}{\partial z_j^f} = \sum_{s=1}^{subPeriods} \frac{\partial AI_{sp}}{\partial z_j^f}. \tag{49}$$

$$\frac{\partial AI_s}{\partial z_j^f} = N \frac{\partial F(t, T_s^S, T_s^E)}{\partial z_j^f} \tau(T_s^S, T_s^E) + \tag{50}$$

$$+ \frac{\partial F(t, T_s^S, T_s^E)}{\partial z_j^f} \tau(T_s^S, T_s^E) \sum_{k=1}^{s-1} AI_k^{Str} + \tag{51}$$

$$+ \left(F(t, T_s^S, T_s^E) + S^{flat/str} \right) \tau(T_s^S, T_s^E) \sum_{k=1}^{s-1} \frac{\partial AI_k^{Str}}{\partial z_j^f} \tag{52}$$

And the second order sensitivity,

$$\frac{\partial^2 A_i}{\partial z_j^f \partial z_g^f} = \sum_{s=1}^{subPeriods} \frac{\partial^2 AI_{sp}}{\partial z_j^f \partial z_g^f}. \tag{53}$$

Where

$$\frac{\partial^2 AI_s}{\partial z_j^f \partial z_g^f} = N \frac{\partial^2 F(t, T_s^S, T_s^E)}{\partial z_j^f \partial z_g^f} \tau(T_s^S, T_s^E) + \tag{54}$$

$$+ \frac{\partial^2 F(t, T_s^S, T_s^E)}{\partial z_j^f \partial z_g^f} \tau(T_s^S, T_s^E) \sum_{k=1}^{s-1} AI_k^{Str} + \tag{55}$$

$$+\frac{\partial F(t,T_s^S,T_s^E)}{\partial z_j^f}\tau(T_s^S,T_s^E)\sum_{k=1}^{s-1}\frac{\partial AI_k^{Str}}{\partial z_g^f}+ \tag{56}$$

$$+\frac{\partial F(t,T_s^S,T_s^E)}{\partial z_g^f}\tau(T_s^S,T_s^E)\sum_{k=1}^{s-1}\frac{\partial AI_k^{Str}}{\partial z_j^f} \tag{57}$$

$$+\left(F(t,T_s^S,T_s^E)+S^{flat/str}\right)\tau(T_s^S,T_s^E)\sum_{k=1}^{s-1}\frac{\partial^2 AI_k^{Str}}{\partial z_j^f\partial z_j^f} \tag{58}$$

The cross curve gammas can be expressed as,

$$\frac{\partial^2 NPV(t)_{comp}}{\partial z_j^d \partial z_g^f} = \sum_{i=1}^{n}\frac{\partial A_i}{\partial z_g^f}\frac{\partial P_d(t,T_I^P)}{\partial z_j^d} \tag{59}$$

$$= -\sum_{i=1}^{n}\tau(t,T_i^P)P_d(t,T_i^P)\frac{\partial A_i}{\partial z_g^f}\cdot\frac{\partial z_{d,T^P}}{\partial z_{d,T_j}} \tag{60}$$

1.3.4 OIS

$$NPV(t)_{OIS}=\sum_{j=1}^{n}N\cdot\left(F_{OIS}(t,T_j^S,T_j^E)+S\right)\tau(T_j^S,T_j^E)P_{OIS}(t,T_j^P) \tag{61}$$

If the valuation date, T_v, is in $\left[T_j^S,T_j^E\right]$.

$$F_{OIS}(t,T_j^S,T_j^E)= \tag{62}$$

$$=\left(\prod_{t_{ON}=T^S}^{T^E}\{1+R(t_{ON})\tau(t_{ON},t_{ON}+1)\}-1\right)/\tau(T_j^S,T_j^E)= \tag{63}$$

$$=\left(\prod_{t_{ON}=T^S}^{T^{v-1}}\{1+R_{fixed}(t_{ON})\tau(t_{ON},t_{ON}+1)\}\cdot \right. \tag{64}$$

$$\left.\cdot\prod_{t_{ON}=T^v}^{T^E}\{1+F(t_{ON})\tau(t_{ON},t_{ON}+1)\}-1\right)/\tau(T_j^S,T_j^E)= \tag{65}$$

$$=\left(\frac{P(T^{T^{v-1}})}{P(T^E)}\prod_{t_{ON}=T^S}^{T^{v-1}}\{1+R_{fixed}(t_{ON})\tau(t_{ON},t_{ON}+1)\}-1\right)/\tau(T_j^S,T_j^E)= \tag{66}$$

$$\tag{67}$$

$$NPV(t)_{OIS} = \sum_{j=1}^{n} N \cdot \left(F_{OIS}(t,T_j^S,T_j^E)+S\right)\tau(T_j^S,T_j^E)P_{OIS}(t,T_j^P) = \tag{68}$$

$$= \sum_{j=1}^{n} N \cdot \left(\left(\frac{P_{OIS}(t,T_j^S)}{P_{OIS}(t,T_j^E)}-1\right)/\tau(T_j^S,T_j^E)+S\right)\tau(T_j^S,T_j^E)P_{OIS}(t,T_j^P) = \tag{69}$$

$$= \sum_{j=1}^{n} N \cdot \left(\frac{P_{OIS}(t,T_j^S)}{P_{OIS}(t,T_j^E)}-1+\tau(T_j^S,T_j^E)S\right)P_{OIS}(t,T_j^P) \tag{70}$$

2 FRA

Two different kind of discounting methods are implimented,

$$NPV = N \cdot P_d(t,T^P)\frac{\left(F_{strike}-F_f(t,T^S,T^E)\right)\tau(T^S,T^E)}{1+F_f(t,T^S,T^E)\tau(T^S,T^E)}. \tag{71}$$

and

$$NPV = N \cdot P_d(t,T^P)\frac{\left(F_{strike}-F_f(t,T^S,T^E)\right)\tau(T^S,T^E)}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))(1+F_{strike})\tau(T^S,T^E))}. \tag{72}$$

2.0.5 Delta discount

$$\frac{\partial NPV}{\partial z_d(t,T_j)} = -\tau(t,T^P)\frac{\partial z_{d,T^P}}{\partial z_{d,T_j}} \cdot NPV \tag{73}$$

2.0.6 Gamma discount

$$\frac{\partial^2 NPV}{\partial z_j^d \partial z_g^d} = -NPV\tau(T^P)\cdot\left\{-\tau(T^P)\frac{\partial z^d(T^P)}{\partial z_j^d}\frac{\partial z^d(T^P)}{\partial z_g^d}+\frac{\partial^2 z^d(T^P)}{\partial z_g^d \partial z_j^d}\right\} \tag{74}$$

$$\tag{75}$$

2.0.7 Delta forward

$$\tfrac{\partial NPV}{\partial z_f(t,T_j)} = N \cdot P_d(t,T^P)\tfrac{\left(F_{strike}-F_f(t,T^S,T^E)\right)\tau(T^S,T^E)}{1+F_f(t,T^S,T^E)\tau(T^S,T^E)} = \tag{76}$$

$$= N \cdot P_d(t,T^P)\tau(T^S,T^E)\frac{\partial}{\partial z_f(t,T_j)}\left\{\frac{\left(F_{strike}-F_f(t,T^S,T^E)\right)}{1+F_f(t,T^S,T^E)\tau(T^S,T^E)}\right\} = \tag{77}$$

$$= NP_d(t,T^P)\tau(T^S,T^E) \tag{78}$$

$$\left[\frac{\partial}{\partial z_f(t,T_j)}\left\{F_{strike} - F_f(t,T^S,T^E)\right\}(1+F_f(t,T^S,T^E)\tau(T^S,T^E)) + \right.$$
$$\left. -\left(F_{strike} - F_f(t,T^S,T^E)\right)\frac{\partial}{\partial z_f(t,T_j)}(1+F_f(t,T^S,T^E)\tau(T^S,T^E))\right]$$
$$/(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2$$
$$= NP_d(t,T^P)\tau(T^S,T^E)$$
$$\left[-\frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)}(1+F_f(t,T^S,T^E)\tau(T^S,T^E)) + \right. \tag{79}$$
$$\left. -\left(F_{strike} - F_f(t,T^S,T^E)\right)\frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)}\tau(T^S,T^E)\right]$$
$$/(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2$$
$$= -NP_d(t,T^P)\tau(T^S,T^E)\frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)}\frac{(1+F_{strike}\tau(T^S,T^E))}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2}$$

2.0.8 Gamma forward

$$\frac{\partial^2 NPV}{\partial z_f(t,T_i)\partial z_f(t,T_j)} = -NP_d(t,T^P)\tau(T^S,T^E)\cdot \tag{80}$$
$$\cdot\frac{\partial}{\partial z_f(t,T_i)}\left\{\frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)}\frac{(1+F_{strike}\tau(T^S,T^E))}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2}\right\}$$

Before we attac this expression we calculate,

$$\frac{\partial}{\partial z_f(t,T_i)}\left\{\frac{(1+F_{strike}\tau(T^S,T^E))}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2}\right\} = \tag{81}$$

$$= -2\cdot\frac{(1+F_{strike}\tau(T^S,T^E))\cdot\tau(T^S,T^E)}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^3}\frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_i)} \tag{82}$$

$$\tag{83}$$

Continuing,

$$\frac{\partial^2 NPV}{\partial z_f(t,T_i)\partial z_f(t,T_j)} = -NP_d(t,T^P)\tau(T^S,T^E)\cdot \tag{84}$$
$$\cdot\left\{\frac{\partial^2 F_f(t,T^S,T^E)}{\partial z_f(t,T_j)\partial z_f(t,T_i)}\frac{(1+F_{strike}\tau(T^S,T^E))}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2} + \right.$$

$$-2 \cdot \frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)} \frac{(1+F_{strike}\tau(T^S,T^E)) \cdot \tau(T^S,T^E)}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^3} \frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_i)} \Big\}$$

2.0.9 Cross gammas

$$\frac{\partial^2 NPV}{\partial z_f(t,T_j) \partial z_d(t,T_i)} = \tag{85}$$

$$= N \frac{\partial z_{d,T^P}}{\partial z_{d,T_j}} \cdot P_d(t,T^P)\tau(t,T^P)\tau(T^S,T^E) \frac{\partial F_f(t,T^S,T^E)}{\partial z_f(t,T_j)} \frac{(1+F_{strike}\tau(T^S,T^E))}{(1+F_f(t,T^S,T^E)\tau(T^S,T^E))^2}$$