Calculating Delta Risks and Hedges via Waves

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CALCULATING DELTA RISKS AND HEDGES VIA WAVES

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Abstract. A book's delta risks are usually calculated by bumping the rate of each stripping instrument, re-stripping the discount curve, and re-valuing the book using the new discount curve. The difference between the new and old value of the book is the book's bucket delta risk with respect to that stripping instrument. After finding the bucket delta risks with respect to all the stripping instruments, the hedges are obtained by calculating the combination of stripping instruments needed to offset this vector of bucket risks.

This methodology leads to several problems: risks bleeding between risk buckets, non-intuitive risk scenarios, over-specification of risks, and needless intertwining of the risk and stripping processes.

Here we present a simpler alternative method for calculating delta risks, the "wave" or "scenario" method, which is largely free of these problems.

1. Conventional approach. The delta risks of a book are usually calculated by bumping the rate of each stripping instrument, re-stripping the discount curve, and re-valuing the book using the new discount curve. The difference between the new and old values of the book is the book's bucket delta risk with respect to that stripping instrument. This methodology leads to several problems: bleeding, non-intuitive risk profiles, over-specification of risks, and needless intertwining of the risk and stripping methodologies. Here we present an alternative, more effective, method of calculating delta risks and hedges, the "wave" or "scenario" method, which has been developed to overcome these problems. Since the wave method directly manipulates the forward curve itself, it can be used for any other curves (rate projection curves – like Libor or Euribor projection curves, basis spread curves, survival curves for credit linked deals, . . .) without modification.

1.1. Stripping. Let us briefly consider curve stripping. In depth accounts can be found in, *e.g.*, [1], [2]. Consider a discount curve

$$D(T) = e^{-\int_0^T f^0(t)dt}, \tag{1.1}$$

where $f^0(t)$ is the corresponding instantaneous forward rate curve. Discount curves are obtained by stripping a set of fixed income instruments – typically deposits, futures, and swaps. Let $S_1, S_2, \dots, S_m$ be the stripping instruments used to obtain this discount curve. Parsing the market value of each instrument j yields a relation of the form

$$V_j^S = \sum_{i=1}^{n_j} (a_{j,i} R_j + b_{j,i})\, D(t_{j,i}) \qquad \text{for } j = 1, \dots, m, \tag{1.2}$$

where V_j^S is the instrument's current value (often 0), and R_j is the instrument's rate (deposit rate, futures rate, swap rate, . . .) in the current market. The paydates $t_{j,i}$ and coefficients $a_{j,i}$ and $b_{j,i}$ are determined by the particulars of instrument j. Let us arrange these instruments in order of increasing maturity, so that

$$0 < T_1 < T_2 < \ldots < T_m, \tag{1.3}$$

where $T_j = t_{i_{n_j}}$ is the final paydate of instrument j.

The art of stripping curves is to create a "good looking" instantaneous forward rate curve $f^0(t)$ which satisfies all m financial constraints in 1.2. The simplest method is to use a piecewise constant forward curve,

$$f^0(t) \equiv f_j^0 \qquad \text{for } T_{j-1} < t < T_j, \qquad j = 1, 2, ..., m-1, \tag{1.4a}$$

$$f^0(t) \equiv f_m^0 \qquad \text{for } T_{m-1} < t < \infty, \tag{1.4b}$$

where $T_0 \equiv 0$ is today. For the first inteval $T_0 < t < T_1$, the constant forward rate f_0^1 is chosen so that the $j = 1$ constraint in 1.2 is satisfied. For the interval $T_1 < t < T_2$, the constant f_0^2 is chosen to satisfy the $j = 2$ constraint; this doesn't upset the first constraint since the first constraint only involves dates $t < T_1$. Continuing this bootstrap procedure through all m steps yields a piecewise constant forward curve $f^0(t)$ which satisfies all m contraints.

Although the piecewise constant forward curve satisfies all financial constraints, the jumps in the forward curve at the nodes T_j are clearly artifacts of our stripping methodology, and are unlikely to be present in real discount curves. I.e., we could get picked off. To create more realistic *continuous* forward curves, more sophisticated interpolation methods have been developed. Piecewise linear and cubic splines turn out to be unstable, so to strip discount curves, one turns to using cubic splines under tension, smart quadratic or smart quartic interpolations, or to monotone convex splines for the instantaneous forward curve. The most common interpolation methods used for stripping are examined in depth in [1], [2], where the relative merits and drawbacks of the schemes are discussed.

1.2. Coventional delta risks. After stripping to obtain the discount curve $D(T)$, one values the book using this "base" curve to find the book's current value, its mark-to-market. Under the conventional approach, one then bumps the rate of the first stripping instrument,

$$R_1 \to R_1 + \delta, \tag{1.5}$$

usually by 1 bp, and re-strips the discount curve, obtaining a new "bumped" curve $\hat{D}(T)$. The book is re-valued using the new curve, and the difference between the book's new value and its base value is the book's "bucket delta risk" with respect to the first instrument, ΔV_1^{book}. Putting R_1 back to its original value, one then bumps the second rate R_2, re-strips the curve, re-values the book, and subtracts the new value from the base value to obtain the bucket delta risk with respect to the second instrument, ΔV_2^{book}. Repeating the procedure for all the stripping instruments yields the vector of bucket delta risks,

$$\mathbf{\Delta V}^{book} \equiv \left(\Delta V_1^{book}, \Delta V_2^{book}, \ldots, \Delta V_m^{book}\right)^T. \tag{1.6}$$

The delta hedge is the linear combination of stripping instruments

$$\sum_{j=1}^{m} a_j S_j \tag{1.7}$$

which cancels out the vector of bucket delta risks. It can be obtained by calculating the bucket delta risks of the stripping instruments. So, after bumping each rate R_k, re-stripping the curve, and re-valuing the book with the bumped curve $\hat{D}(T)$, one also re-values each of the stripping instruments. Taking the difference between the new and old values yields the bucket delta risks of the stripping instruments with respect to the k^{th} instrument,

$$\Delta V_{jk}^S, \qquad \text{for } j = 1, 2, \ldots, m. \tag{1.8}$$

Consider the porfolio

$$V^{book} + \sum_{j=1}^{m} a_j S_j. \tag{1.9}$$

For this portfolio to be hedged, we need all the bucket delta risks of this porfolio to be zero. That is, we need

$$\Delta V_k^{book} + \sum_{j=1}^{m} a_j \Delta V_{jk}^S = 0 \qquad \text{for all } k = 1, \ldots, m. \tag{1.10}$$

Solving yields

$$a_j = \sum_{k=1}^{m} \Delta V_k^{book} \left(M^{-1}\right)_{kj} \tag{1.11a}$$

where M^{-1} is the inverse of the sensitivity matrix,

$$M_{jk} = \Delta V_{jk}^{S}. \tag{1.11b}$$

1.3. Disadvantages of the conventional approach. An advantage of the conventional approach is that the software for re-stripping the curve and re-valuing the books is the same software used in the base case to obtain the book's mark-to-market.

The first problem with the conventional approach is the *bleeding* of risks into different risk buckets. Requiring forward curves $f^0(T)$ to be continuous means that bumping the rate R_j affects not only the bucket $T_{j-1} < t < T_j$, but must also affect the rates in neighboring buckets so that continuity can be preserved. In turn, these affect other buckets, which affect other buckets, until all the intervals are involved. Consequently, a 2.5y swap may have significant delta risks in the 5y, 10y, and even the 30y buckets, and thus its hedges would include 5y, 10y, and 30y swaps. Clearly this is inappropriate since instruments cannot depend on events after maturity.

A second problem is that bucket delta risks are highly non-intuitive. Suppose we were to use piecewise constant forwards to strip the 1y, 2y, ..., 9y, 10y swaps. Consider what happens when we bump, say, the 4y swap rate. Since the first three swap rates have not changed, the instantaneous forward curve $f^0(t)$ does not change for $t \leq 3$y. For 3y $< t <$ 4y, the instantaneous forward rate $f^0(t)$ must go up by roughly 4 bps in order that the 4y swap rate goes up by 1bp. For 4y $< t <$ 5y, the instantaneous forward rate $f^0(t)$ must *decrease* by roughly 4 bps to leave the 5y swap rate unchanged. Beyond five years, the instantaneous rate curve $f^0(t)$ should exhibit only minor changes arising from the swap coupons. Thus, the delta risk in the 4y bucket measures *payments received during year 3 minus the payments received during year 4*. Worse, using more sophisticated interpolation schemes, or stripping more complicated sets of instruments, leads to even less intuitive risks.

A third issue is over-specfication of risks. Stripping a discount curve may involve 30 to 40 different stripping instruments, which would lead to 30 to 40 different delta risk buckets and hedges. For some purposes, such as hedging risks to a major currency swap book, this may be appropriate. But for risk managers trying to understand an organization's interest rate risks over twenty or more different currencies, some method of systematically consolidating and simplifying the risk is needed.

A final problem is the coupling of risk calculations to the stripping process. Whenever the stripping procedure changes, whether it's the interpolation method or the choice of stripping instruments, this changes the meaning of the bucket delta risks. So one cannot pick the best stripping method without considering its effect on the downstream risk processes.

2. The wave method. The "wave method," is a scenario based method of calculating delta risks and hedges which has been developed to overcome these problems. Abstractly, by bumping the rates and re-stripping the curve, we have been generating m distinct, more-or-less reasonable scenarios, each of which is a small perturbation from the base case. Since our hedged book is flat to these perturbations, our book should be hedged to the extent that the real curve movements can be approximated by a linear combination of these perturbations, at least until the changes become large enough for nonlinear effects to set in. The wave method follows the same approach, except that we directly specify the scenarios, instead of obtaining them by bumping and re-stripping.

Suppose one strips a curve $D(T)$ using one's best stripping methodology, and let $f^0(T)$ be the resulting instantaneous forward rate curve. One selects a set of hedging instruments, $H_1, H_2, \ldots, H_K$. Usually these hedging instruments are either the stripping instruments, $S_1, \ldots, S_m$, or are a subset of these instruments. Let us arrange these instruments in order of increasing maturity, and let

$$0 < T_1 < T_2 < \cdots < T_K \tag{2.1}$$

be their final maturity dates. For each maturity date T_k (except the last one), we define a new instantaneous forward rate curve,

$$f_k(t) = f^0(t) + \delta \begin{cases} 0 & \text{for } t < T_{k-1} \\ 1 & \text{for } T_{k-1} < t < T_k \\ 0 & \text{for } T_k < t \end{cases} \tag{2.2a}$$

for $k = 1, 2, \ldots, K-1$. Here $f^0(T)$ is the instantaneous forward rate curve for the base case, and δ is typically 1 bp. For the last curve, we define

$$f_K(t) = f^0(t) + \delta \begin{cases} 0 & \text{for } t < T_{K-1} \\ 1 & \text{for } T_{K-1} < t < T_K \\ 1 & \text{for } T_K < t \end{cases}, \tag{2.2b}$$

so that its bump has flat extrapolation beyond the final date T_K.Thus, for each *scenario* k, we have a new discount curve

$$\tilde{D}_k(t) = D(t) \cdot \begin{cases} 1 & \text{for } t < T_{k-1} \\ e^{-\delta(t-T_{k-1})} & \text{for } T_{k-1} < t < T_k \\ e^{-\delta(T_k-T_{k-1})} & \text{for } T_k < t \end{cases} \qquad k = 1, 2, \ldots, K-1, \tag{2.3a}$$

and

$$\tilde{D}_K(t) = D(t) \cdot \begin{cases} 1 & \text{for } t < T_{K-1} \\ e^{-\delta(t-T_{K-1})} & \text{for } T_{K-1} < t \end{cases} \qquad k = K. \tag{2.3b}$$

For each k, one values the book in scenario k; that is, one values it using the new discount curve $\tilde{D}_k(t)$. The difference between the book's value in scenario k and the base scenario is the bucket delta risk with repect to the k^{th} scenario, ΔV_k^{book}. Repeating for all k yields the vector of bucket deltas,

$$\mathbf{\Delta V}^{book} \equiv \left(\Delta V_1^{book}, \Delta V_2^{book}, \ldots, \Delta V_K^{book}\right)^T. \tag{2.4}$$

The delta hedge is the linear combination of hedging instruments which cancels out the vector of bucket delta risks. Consider the hedged portfolio

$$V^{book} + \sum_{j=1}^{K} a_j H_j, \tag{2.5}$$

and let

$$\Delta H_{jk}, \qquad \text{for } j, k = 1, 2, \ldots, K \tag{2.6}$$

be the k^{th} bucket delta of hedging instrument j; that is, the difference in the value of hedging instrument j under scenario k and under the base case. Setting the bucket delta risks of the portfolio to zero for all scenarios k requires that

$$\Delta V_k^{book} + \sum_{j=1}^{K} a_j \Delta H_{jk} = 0 \qquad \text{for } k = 1, \ldots, K. \tag{2.7}$$

Solving these equations yields the hedges,

$$a_j = \sum_{k=1}^{K} \Delta V_k^{book} \left(\Delta H^{-1}\right)_{kj}, \tag{2.8}$$

where ΔH^{-1} is the inverse of the sensitivity matrix ΔH_{jk}.

For the wave method, the sensitivity matrix ΔH_{jk} is lower triangular. To see this, recall that the final maturity date of hedging instrument H_j is T_j, and note that scenario k does not change the discount curve for dates $t < T_{k-1}$. So any scenario k with $k > j$ cannot affect instrument j. Thus,

$$\Delta H_{jk} = 0 \qquad \text{for scenarios } k > j. \tag{2.9a}$$

The fact that ΔH_{jk} is a lower triangular matrix makes inverting the matrix trivial, and also makes the inverse of ΔH_{jk} lower triangular,

$$\left(\Delta H^{-1}\right)_{kj} = 0 \qquad \text{for } k < j, \tag{2.9b}$$

Thus

$$a_j = \sum_{k=j}^{K} \Delta V_k^{book} \left(\Delta H^{-1}\right)_{kj}, \tag{2.10}$$

so the j^{th} hedge is constructed only from the bucket delta risks from scenarios $k \geq j$.

3. Conclusions. The wave method does not suffer from bleeding. To see this, consider a deal whose last paydate $T^* < T_{j^*}$ for some j^*. Since scenario k only changes the discount curve for dates $t > T_{k-1}$, clearly the deal's bucket delta risk is zero for all scenarios $k > j^*$,

$$\Delta V_k^{deal} = 0 \qquad \text{for } k > j^*. \tag{3.1}$$

With 2.10, this means that

$$a_j = \sum_{k=j}^{j^*} \Delta V_k^{deal} \left(\Delta H^{-1}\right)_{kj} \qquad \text{for } j \leq j^*, \tag{3.2a}$$

$$a_j = 0 \qquad \text{for } j > j^*. \tag{3.2b}$$

Thus, the hedges of a deal are exactly zero for all buckets which occur after the last pay date of the deal.

The scenarios used to calculate risks in the wave method also clear. Since these scenarios are completely independent of the stripping process, one can develop and use the best stripping methods without considering it's effect on downstream risk processes. Similarly, one can choose the best risk buckets to use without regards to the stripping process. For some purposes, such as hedging a large swap book, it is appropriate to use all the stripping instruments as the hedging instruments; one then gets very detailed delta risks and hedges. For other processes, it may be more appropriate to choose, say, the 1y, 5y, and 10y swaps as the hedging instruments. This then expresses the risks as short term (under 1y), medium term (from 1y to 5y), and long term (over 5y) risks. Or one can choose a single instrument as the hedging instrument, so that the delta risk is the risk to a parallel shift.

The wave method can be used to find the delta risks for any curves, not just discount curves. The risks for projection curves, like Libor projection curves, the risks for basis spread curves, and the risks for survival curves in credit problems can be handled in an identical manner.

3.1. Compound IDs. An effective way of implementing the wave method is to replace curve names with compound curve names, and curve IDs with compound curve IDs. At the interface level, curves are usually identified by name, such as "USDSwap" or "EONIA." The software identifies these curves as, say, curve 6 and curve 19, by comparing the names against the list of curves that have been created. Internally, curves are usually specified by these integers, the curve IDs.

To use compound IDs, one defines sets of scenarios by specifying a set of dates,

$$0 < T_1 < T_2 < \cdots < T_K,$$

and stores and names these scenarios, as in "USDRisks," or "EMRiskSet." Instead of using simple curve names like "USDSwap" at the interface level, one uses compound names like

USDSwap | USDRisks | 7 | 1.5

This identifies the curve as the base curve "USDSwap" with a shift of $\delta = 1.5$bps in the seventh scenario ($k = 7$) from the set of scenarios named "USDRisks." See eqs. 2.2a, 2.3b. At the interface level, the compound name is parsed into a compound ID (a curve ID, an ID specifying the set of scenarios, the scenario number, and the shift amount). This compound ID threads through the system, with the scenario information being ignored, until reaching the "discount factor" function. Only at this level does the scenario information get used to modify the discount factors.

Compound names are also effective for other market objects, such as vol cubes. Using compound names for all market objects enables one to quickly create standard and customized risk scenarios, and then use scripting to automate the downstream processes.

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