PnL prediction under extreme scenarios

Julien Pantz

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Abstract

We study the PnL prediction of an option from its Greeks under extreme shocks. In this situation the classical delta gamma approximation fails and adding higher order Greeks does not improve significantly the situation due to the slow convergence of the Taylor's expansion (which even diverges in the Black-Scholes case). One obvious situation involves a far out of the money option under a scenario pushing the option in the money. Since the Greeks were almost zero initially, the delta-gamma PnL will be almost zero and off the real PnL. Another drawback of expanding further the Taylor's expansion is the need of high order derivatives which are costly and not reliable especially at the level of the book of a large investment bank derivatives desk. Consequently we find a simple alternative solution involving only 3 deltas which we found by re-interpreting the PnL prediction as a numerical integration problem. More precisely we suggest using the Simpson's method which predicts correctly the PnL under extreme stressed scenarios.

This paper is organized as follows. In the first section we derive the different PnL prediction schemes. In the second section we compare their performance empirically. In the third section we illustrate the drawbacks of the Taylor's expansion. In the fourth section we derive analytically the unexplained PnL under the different methods.

Keywords: PnL predict, numerical integration, delta-gamma approximation, Trapezoid's method, Simpson's method, stress testing, scenario hedging

1 PnL predictors

The delta-gamma approximation explains the PnL with:

$$\pi(f+s)-\pi(f)\approx \Delta_f s+\frac{1}{2}\Gamma_f s^2$$

Usually, the Gamma is replaced by its finite difference approximation which leads to the following formula:

$$\pi(f+s)-\pi(f)\approx \Delta_f s+\frac{1}{2}\left(\frac{\Delta_{f+s}-\Delta_f}{s}\right)s^2=\frac{s}{2}\left(\Delta_f+\Delta_{f+s}\right)$$

Writing the PnL as the integral of the delta brings us to the world of numerical integration and we recognize the trapezoid rule since:

$$\pi(f+s)-\pi(f)=\int_f^{f+s}\Delta_x dx\approx\frac{s}{2}\left(\Delta_f+\Delta_{f+s}\right)$$

The trapezoid's method approximates the integrand linearly. This is a first order method. For large shocks s the approximation error (hence the unexplained PnL) becomes material which suggests using instead the Simpson's rule:

$$\pi(f+s)-\pi(f)=\int_f^{f+s}\Delta_x dx\approx\frac{s}{6}\left(\Delta_f+4\Delta_{f+s/2}+\Delta_{f+s}\right)$$

Since we focus on the underlying, we can neglect the Theta, Vega and cross-gammas for our PnL prediction so we work under the Bachelier model where the price of a Call option with initial forward f, strike k, volatility σ and maturity T has the following expression.

$$\pi(f)=(f-k)\mathcal{N}\left(\frac{f-k}{\sigma\sqrt{T}}\right)+\sigma\sqrt{T}\,n\left(\frac{f-k}{\sigma\sqrt{T}}\right)$$

We denoted $\mathcal{N}$ and n the standard Gaussian CDF and PDF.

2 Empirical results

We consider a 50bps forward with 1% volatility and compare the 3 methods when predicting the PnL of a 1 year maturity Call option with 2% strike, with shocks from 10bps to 3%. Note that in our Bachelier world the only quantities that matter are the relative strike and the standard deviation. We choose the parameters to be in the relevant situation where the delta are close to zero initial and converge to 1 when the shock increases.

Parameters			
forward	**strike**	**volatility**	**maturity**
0.005	0.020	0.010	1.000

Table 2.1 *Option parameters*

shock	delta-gamma	trapezoid	Simpson	shock	delta-gamma	trapezoid	Simpson
0.001	0.000	0.000	0.000	**0.016**	0.330	0.141	0.004
0.002	0.001	0.000	0.000	**0.017**	0.349	0.141	0.005
0.003	0.006	0.003	0.000	**0.018**	0.366	0.139	0.005
0.004	0.016	0.009	0.000	**0.019**	0.381	0.135	0.006
0.005	0.032	0.017	0.000	**0.020**	0.395	0.129	0.006
0.006	0.053	0.028	0.000	**0.021**	0.408	0.121	0.007
0.007	0.077	0.041	0.000	**0.022**	0.419	0.111	0.007
0.008	0.103	0.055	0.000	**0.023**	0.429	0.100	0.007
0.009	0.131	0.070	0.000	**0.024**	0.438	0.088	0.007
0.010	0.159	0.084	0.001	**0.025**	0.445	0.075	0.006
0.011	0.187	0.097	0.001	**0.026**	0.452	0.061	0.006
0.012	0.214	0.109	0.001	**0.027**	0.457	0.046	0.005
0.013	0.240	0.120	0.002	**0.028**	0.462	0.031	0.003
0.014	0.265	0.128	0.002	**0.029**	0.466	0.016	0.002
0.015	0.288	0.135	0.003	**0.030**	0.469	0.000	0.000

Table 2.2 *Unexplained PnL (absolute)*

For small shocks the delta-gamma method is reasonable but still inferior to the numerical integration schemes. When the shocks increase, both the delta-gamma and the trapezoid method diverge while the unexplained PnL stays negligible under the Simpson method.

3 Taylor's series

The normal distribution function is composed of entire functions so the Taylor's series of the PnL converges everywhere under the Bachelier model. However the slow convergence makes it impractical as demonstrated here where we illustrate the superiority of our method compared to a Taylor's expansion with up to 10^{th} order Greeks. The option parameters are the same as in the previous section.

shock	delta-gamma	trapezoid	Simpson	3th	4th	5th	6th	7th	8th	9th	10th
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
0.003	0.096	0.005	0.000	0.106	0.006	0.000	0.106	0.006	0.000	0.106	0.006
0.005	0.204	0.028	0.000	0.256	0.036	0.000	0.255	0.036	0.000	0.255	0.036
0.008	0.251	0.062	0.000	0.361	0.087	0.001	0.359	0.086	0.001	0.359	0.086
0.010	0.247	0.097	0.001	0.420	0.148	0.003	0.413	0.147	0.003	0.413	0.146
0.013	0.216	0.124	0.002	0.448	0.210	0.007	0.430	0.205	0.005	0.431	0.205
0.015	0.174	0.139	0.003	0.462	0.267	0.012	0.423	0.254	0.007	0.427	0.254
0.018	0.132	0.140	0.005	0.473	0.317	0.019	0.399	0.288	0.006	0.413	0.287
0.020	0.097	0.129	0.006	0.488	0.360	0.028	0.360	0.303	0.002	0.396	0.300
0.023	0.069	0.106	0.007	0.510	0.400	0.037	0.305	0.296	0.023	0.387	0.290
0.025	0.050	0.075	0.006	0.542	0.439	0.048	0.228	0.263	0.066	0.400	0.247
0.028	0.039	0.039	0.004	0.585	0.483	0.061	0.122	0.197	0.143	0.460	0.163
0.030	0.036	0.000	0.000	0.639	0.535	0.074	0.025	0.087	0.273	0.604	0.020
0.033	0.040	0.039	0.006	0.703	0.600	0.090	0.226	0.080	0.480	0.894	0.210
0.035	0.049	0.076	0.013	0.778	0.679	0.109	0.497	0.324	0.799	1.422	0.565
0.038	0.063	0.110	0.022	0.861	0.777	0.132	0.856	0.673	1.272	2.324	1.100
0.040	0.081	0.140	0.031	0.954	0.894	0.160	1.325	1.158	1.960	3.797	1.892
0.043	0.103	0.168	0.041	1.054	1.031	0.195	1.928	1.821	2.937	6.112	3.046
0.045	0.126	0.191	0.049	1.162	1.190	0.238	2.691	2.712	4.298	9.639	4.700
0.048	0.152	0.212	0.057	1.277	1.371	0.291	3.644	3.888	6.163	14.873	7.040
0.050	0.179	0.230	0.064	1.398	1.575	0.354	4.818	5.418	8.680	22.464	10.307

Table 3.1 *Unexplained PnL (absolute)*

We observe that the Taylor's convergence is quite slow and does not beat the Simpson's method even at the 10^{th} order. When the shock increases further the Simpson's method starts to diverge as well as expected so if more accuracy is needed we can use higher order Gaussian numerical quadratures.

The slow convergence of the Taylor's series is well illustrated using:

$$\mathrm{e}^{-\mathbf{x}^2} = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^{2n}$$

The next figure shows the convergence around $\mathbf{x} = \mathbf{2}$ with oscillations up to the 7th order expansion. In particular the convergence is not monotonic and every time we need to reduce the tracking error we need to add two terms.

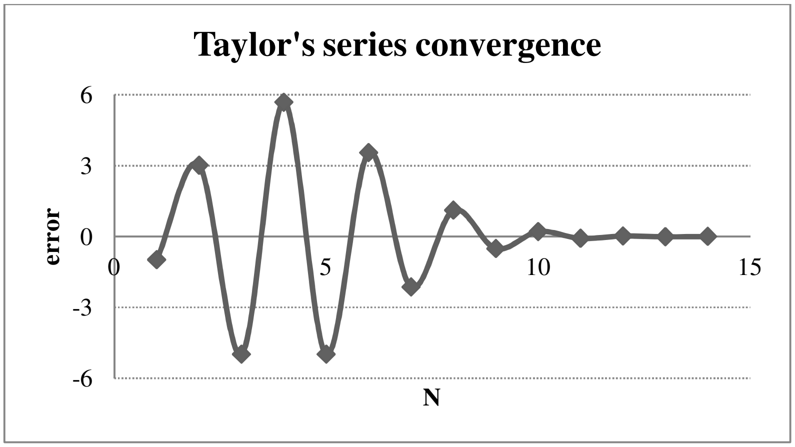

Figure 3.1 *Taylor series convergence of* e^{-x^2} *at* $x = 2$

The Taylor's series diverge for the Black-Scholes model when $s > f$ as noticed in [1]. This comes from the logarithm which radius of convergence is 1 and we illustrate the divergence using the 4th order expansion around $\mathbf{x} = \mathbf{e^2}$ (consistently with the previous example):

$$\mathrm{e}^{-\ln(\mathrm{x})^2} = \frac{1}{2} - \frac{4}{3}x + 5x^2 - 4x^3 + \frac{5}{6}x^4 + o((x-1)^4)$$

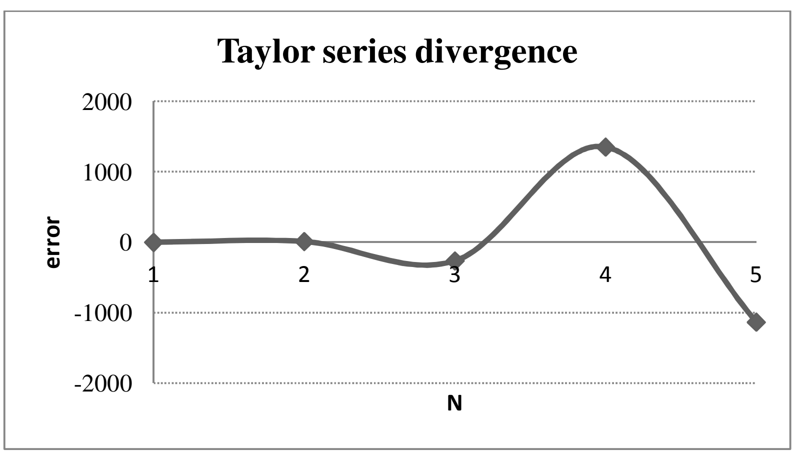

Figure 3.2 *Taylor series divergence of* $e^{-\ln(x)^2}$ *at* $x = e^2$

4 Error analysis

The unexplained PnL can be solved analytically under the Bachelier model. For the delta-gamma method the error follows from the Taylor's integral remainder:

$$\varepsilon^{delta-gamma} = \frac{1}{2}\int_f^{f+s} (f+s-t)^2 \pi_3(t)dt = -\frac{1}{2}\int_f^{f+s} (f+s-t)^2 \frac{t-k}{v^3} n\left(\frac{t-k}{v}\right) dt$$

Concerning numerical integration schemes, the errors involve the Peano's kernels of order 1 (Trapezoid's method) and 3 (Simpson's method) which after some computations give:

$$\varepsilon^{Trapezoid} = \frac{1}{2}\int_f^{f+s} \left((f+s-t)^2 - s\big((f+s-t)\big)\right) \pi_3(t)dt$$

$$\begin{aligned}\varepsilon^{Simpson} &= \frac{1}{8}\int_f^{f+s} (f+s-t)^4 \pi_5(t)dt \\ &\quad - \frac{s}{3}\left(\int_f^{f+s/2} (f+s/2-t)\pi_5(t)dt - \frac{1}{4}\int_f^{f+s} (f+s-t)\pi_5(t)dt\right)\end{aligned}$$

These expressions can be solved exactly in functions of the Gaussian PDF and CDF which might be useful to perform further comparison analysis on the unexplained PnL.

5 Conclusion

We gave a simple method to predict the PnL under stressed scenarios. A first application is the generation of PnL prediction under stress scenarios at the level of a large derivative book where a full re-evaluation is not feasible. In particular the method allows isolating the PnL explained by a large move of the underlying from other PnL components. Other potential applications which will be considered in further studies are scenario hedging schemes under crash market conditions.

6 Reference

[1] A. Estrella 1995 *Taylor, Black and Scholes: Series approximations and risk management pitfalls* Federal bank of New York research paper

7 Appendix

The next table regroups the Greeks we used for the Taylor expansion with $s = f - k$ and $v = \sigma\sqrt{T}$ such that the reader can easily check the experimental results.

Greek	**formula**
$\Delta = \pi_1$	$\mathcal{N}\left(\frac{s}{v}\right)$
$\Gamma = \pi_2$	$\frac{1}{v} n\left(\frac{s}{v}\right)$
π_3	$-\frac{s}{v^3} n\left(\frac{s}{v}\right)$
π_4	$\left(\frac{s^2}{v^2} - 1\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_5	$\left(\frac{3s}{v^2} - \frac{s^3}{v^4}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_6	$\left(\frac{s^4}{v^6} - 6\frac{s^2}{v^4} + \frac{3}{v^2}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_7	$\left(-\frac{s^5}{v^8} + 10\frac{s^3}{v^6} - 15\frac{s}{v^4}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_8	$\left(\frac{s^6}{v^{10}} - 15\frac{s^4}{v^8} + 45\frac{s^2}{v^6} - 15\frac{1}{v^4}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_9	$\left(-\frac{s^7}{v^{12}} + 21\frac{s^5}{v^{10}} - 105\frac{s^3}{v^8} + 105\frac{s}{v^6}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$
π_{10}	$\left(\frac{s^8}{v^{14}} - 30\frac{s^6}{v^{12}} + 210\frac{s^4}{v^{10}} - 420\frac{s^2}{v^8} + 105\frac{1}{v^6}\right)\frac{1}{v^3} n\left(\frac{s}{v}\right)$

Table 7.1 *Greeks up to the 10th order*