THE VIXHAL-1 CONJECTURE: ON THE COMPOSITENESS OF A PRIME–FIBONACCI SEQUENCE

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Abstract. This paper introduces a new conjecture at the intersection of two fundamental areas of number theory: prime numbers and the Fibonacci sequence. Let p_n denote the n-th prime number and F_n denote the n-th Fibonacci number, with the sequence standardized as $F_0 = 0, F_1 = 1, F_2 = 1, \ldots$. We define a sequence a_n for $n \geq 1$ as:

$$a_n = p_n^{F_{n-1}} + F_{n-1}$$

Based on initial computational evidence, we propose the **Vixhal-1 Conjecture**, which states that a_n is composite for all $n > 1$. We present a formal proof that this conjecture holds for two-thirds of all cases, specifically for all integers $n > 1$ where $n - 1$ is not a multiple of 3. The proof relies on a straightforward parity analysis. For the remaining cases ($n - 1 \equiv 0 \pmod 3$), where the parity argument does not apply, we analyze specific examples using modular arithmetic, demonstrating their compositeness and highlighting the challenges in formulating a general proof. This conjecture, if proven true, would establish an infinite family of guaranteed composite numbers of extraordinary size, generated from a simple and elegant formula.

1. Introduction

The study of prime numbers, from Euclid's proof of their infinitude to the unresolved Riemann Hypothesis, has been a cornerstone of mathematics for millennia. Similarly, the Fibonacci sequence, originating from a simple model of population growth, has revealed surprising connections across various scientific disciplines, famously linking to the golden ratio and appearing in natural phenomena [7].

Conjectures that unite these disparate fields are rare and often lead to new avenues of inquiry. The prime numbers, defined by their lack of non-trivial divisors, and the Fibonacci numbers, defined by a simple recurrence relation, appear structurally distinct. Yet, the history of mathematics shows that combining fundamental objects can reveal unforeseen patterns. In this paper, we explore such a pattern, discovered through computational experimentation—a modern approach to number theory that allows for the identification of regularities that might otherwise remain hidden.

We construct a sequence of integers, a_n, derived directly from the n-th prime and the $(n-1)$-th Fibonacci number. The structure of this sequence,

$$a_n = p_n^{F_{n-1}} + F_{n-1},$$

combines exponential and additive elements, resulting in numbers that grow at a remarkable rate. Our investigation began with computational exploration, which revealed a striking regularity: for every integer n tested from 2 to 24, the resulting number a_n was found to be composite. This empirical evidence led to the formulation of the central conjecture of this paper, hereafter referred to as the Vixhal-1 Conjecture.

This paper is structured as follows:

- **Section 2:** Provides formal definitions for the core mathematical objects.
- **Section 3:** States the Vixhal-1 Conjecture and presents supporting data.
- **Section 4:** Presents a partial proof of the conjecture using parity analysis.
- **Section 5:** Analyzes the unresolved cases using modular arithmetic.
- **Section 6:** Discusses the computational methods and tools used.
- **Section 7:** Concludes with the significance and future research directions.
- **Appendix:** Provides the Python source code for verification.

2. Preliminaries and Definitions

To ensure clarity and rigor, we first establish the fundamental definitions that form the basis of our conjecture.

Definition 2.1 (The Prime Numbers). *The sequence of prime numbers, denoted by $(p_n)_{n\geq 1}$, is the ordered sequence of positive integers greater than 1 that have no positive divisors other than 1 and themselves. The sequence begins: $p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, \ldots$. A key property for our analysis is that $p_1 = 2$ is the only even prime. For all $n > 1$, p_n is an odd integer.*

Definition 2.2 (The Fibonacci Sequence). *The Fibonacci sequence, denoted by $(F_n)_{n\geq 0}$, is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$, with the standard initial values $F_0 = 0$ and $F_1 = 1$. The sequence begins: $0, 1, 1, 2, 3, 5, 8, \ldots$.*

Lemma 2.3 (Parity of Fibonacci Numbers). *The Fibonacci number F_n is even if and only if its index n is a multiple of 3 (i.e., $n \equiv 0 \pmod 3$).*

Proof. This follows from observing the sequence of parities modulo 2: $F_0 \equiv 0$, $F_1 \equiv 1$, $F_2 \equiv 1$, $F_3 \equiv 0$, $F_4 \equiv 1$, $F_5 \equiv 1, \ldots$. The pattern of (Even, Odd, Odd) repeats with a period of 3. Thus, F_n is even precisely when n is a multiple of 3. □

Definition 2.4 (The Sequence a_n). *The primary object of study in this paper is the sequence $(a_n)_{n\geq 1}$, defined by combining the n-th prime and the $(n-1)$-th Fibonacci number:*

$$a_n = p_n^{F_{n-1}} + F_{n-1}$$

3. The Conjecture and Computational Evidence

Based on direct computation for $1 < n \leq 24$, we propose the following conjecture, named in honor of its discoverer.

Conjecture 3.1 (The Vixhal-1 Conjecture). *The integer given by the formula*

$$a_n = p_n^{F_{n-1}} + F_{n-1}$$

is composite for all integers $n > 1$.

Table 1 presents the computational results that support this conjecture. The values of a_n grow extremely rapidly; for instance, a_{24} is a number with over 55,000 digits, making primality testing by simple trial division infeasible and requiring sophisticated algorithms [6].

Table 1. Computational evidence for the Vixhal-1 Conjecture.

n	p_n	F_{n-1}	$a_n = p_n^{F_{n-1}} + F_{n-1}$	**Status**
1	2	0	$2^0 + 0 = 1$	Unit
2	3	1	$3^1 + 1 = 4$	Composite
3	5	1	$5^1 + 1 = 6$	Composite
4	7	2	$7^2 + 2 = 51$	Composite
5	11	3	$11^3 + 3 = 1334$	Composite
6	13	5	$13^5 + 5 = 371298$	Composite
7	17	8	$17^8 + 8$	Composite
...	...	...	...	...
24	89	28657	$89^{28657} + 28657$	Composite

4. Main Result: A Partial Proof of the Conjecture

While a complete proof remains elusive, we can prove the conjecture for a substantial majority of cases. The proof hinges on a simple parity analysis.

Theorem 4.1. *For all integers $n > 1$ such that $n-1$ is not a multiple of 3, the integer $a_n = p_n^{F_{n-1}} + F_{n-1}$ is an even composite number.*

Proof. Let $n > 1$ be an integer such that $n - 1 \not\equiv 0 \pmod 3$. We analyze the parity of the two terms constituting a_n: $p_n^{F_{n-1}}$ and F_{n-1}.

1. **Parity of p_n:** Since $n > 1$, the prime p_n must be odd ($p_1 = 2$ is excluded).
2. **Parity of F_{n-1}:** By the condition of the theorem, the index $n-1$ is not divisible by 3. From Lemma 2.3, this implies that F_{n-1} is an **odd** integer.
3. **Parity of $p_n^{F_{n-1}}$:** The first term is an odd number (p_n) raised to a positive odd integer power (F_{n-1}, since $n > 1 \implies n-1 \geq 1 \implies F_{n-1} \geq 1$). An odd number raised to any positive integer power remains odd. Thus, $p_n^{F_{n-1}}$ is an **odd** integer.
4. **Parity of a_n:** We determine the parity of a_n by summing the parities of its terms:
$$a_n = p_n^{F_{n-1}} + F_{n-1} = (\text{odd}) + (\text{odd}) = \text{even}$$
Therefore, a_n is an even integer.
5. **Establishing Compositeness:** To show that a_n is composite, we must also show that $a_n > 2$. For $n = 2$, $a_2 = p_2^{F_1} + F_1 = 3^1 + 1 = 4$. Since $4 > 2$, it is composite. For any $n > 2$, we have $p_n \geq 5$ and $F_{n-1} \geq F_2 = 1$. Thus $p_n^{F_{n-1}} + F_{n-1} \geq 5^1 + 1 = 6$. In all cases, a_n is an even integer strictly greater than 2.

Any even integer greater than 2 is, by definition, composite. Thus, for all $n > 1$ where $n - 1 \not\equiv 0 \pmod 3$, a_n is composite. □

Corollary 4.2. *The Vixhal-1 Conjecture is true for two-thirds of all integers $n > 1$.*

5. The Unresolved Case: $n - 1 \equiv 0 \pmod 3$

The theorem does not apply when $n-1$ is a multiple of 3. This corresponds to the cases $n = 4, 7, 10, 13, \ldots$. In these instances, the parity argument fails to establish compositeness. When $n - 1 \equiv 0 \pmod 3$, F_{n-1} is an **even** number by Lemma 2.3. The parity of a_n becomes:

$$a_n = p_n^{F_{n-1}} + F_{n-1} = (\text{odd})^{\text{even}} + (\text{even}) = (\text{odd}) + (\text{even}) = \text{odd}$$

An odd number may be prime or composite, so this analysis is inconclusive. We investigate the first few such cases using modular arithmetic to find a non-trivial factor.

Case n=4: Here, $n - 1 = 3$. The calculation is as follows:

$$\begin{aligned} a_4 &= p_4^{F_3} + F_3 \\ &= 7^2 + 2 \\ &= 49 + 2 = 51. \end{aligned}$$

We see that $51 = 3 \times 17$, which is composite.

Case n=7: Here, $n - 1 = 6$. For $a_7 = p_7^{F_6} + F_6 = 17^8 + 8$, we check for divisibility by 3. Since $17 \equiv 2 \pmod 3$ and $8 \equiv 2 \pmod 3$, we have:

$$\begin{aligned} a_7 &\equiv 2^8 + 2 \pmod 3 \\ &\equiv (2^2)^4 + 2 \pmod 3 \\ &\equiv 1^4 + 2 \pmod 3 \\ &\equiv 1 + 2 \equiv 0 \pmod 3. \end{aligned}$$

This shows that a_7 is divisible by 3 and is composite.

Case n=10: Here, $n-1=9$. For $a_{10}=p_{10}^{F_9}+F_9=29^{34}+34$, we check for divisibility by 5. Since $29\equiv -1 \pmod 5$ and $34\equiv 4 \pmod 5$, we have:

$$\begin{aligned} a_{10} &\equiv (-1)^{34}+4 \pmod 5 \\ &\equiv 1+4 \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

This shows that a_{10} is divisible by 5 and is composite.
Case n=13: Here, $n-1=12$. For $a_{13}=p_{13}^{F_{12}}+F_{12}=41^{144}+144$, we check modulo 5. Since $41\equiv 1 \pmod 5$ and $144\equiv 4 \pmod 5$, we have:

$$\begin{aligned} a_{13} &\equiv 1^{144}+4 \pmod 5 \\ &\equiv 1+4 \pmod 5 \\ &\equiv 0 \pmod 5. \end{aligned}$$

This shows that a_{13} is divisible by 5 and is composite.

Remark 5.1. These examples suggest that even when a_n is odd, it possesses a small prime factor. However, the identity of this factor (3, 5, etc.) changes, indicating that a general proof for this case would require more advanced techniques than a simple congruence argument with a fixed modulus.

6. Computational Methods

The initial discovery and verification of this conjecture were performed using the Python programming language. The `sympy` library, a powerful tool for symbolic mathematics in Python, was used for its robust functions for generating prime numbers (`sympy.prime`) and for primality testing (`sympy.isprime`) [8]. The code, provided in Appendix A, is capable of verifying the conjecture for small n. For larger n (approx. $n>20$), the arbitrary-precision integers involved become extremely large, and the probabilistic primality tests used by libraries like `sympy` are essential. All computations up to $n=24$ confirmed the conjecture. For a rigorous proof of compositeness for such large numbers, one would rely on sophisticated probabilistic (e.g., Miller-Rabin) and deterministic (e.g., AKS) algorithms [6, 4, 1].

7. Discussion and Future Work

We have presented a compelling conjecture stating that $a_n=p_n^{F_{n-1}}+F_{n-1}$ is composite for all $n>1$. We have proven this conjecture for the two-thirds of cases where $n-1\not\equiv 0 \pmod 3$.

The remaining one-third of cases, where $n-1\equiv 0 \pmod 3$, present the primary challenge. A full proof of the Vixhal-1 Conjecture now rests on demonstrating compositeness for this specific subset of n. Several avenues for future research are apparent:

(1) **Systematic Search for Divisors:** A more general modular arithmetic argument may succeed. One might prove that for every n such that $n-1\equiv 0 \pmod 3$, there exists a prime q dividing a_n. The choice of q may depend on the properties of p_n and F_{n-1} modulo small integers.
(2) **Pisano Periods:** The properties of Fibonacci numbers modulo a prime k, known as Pisano periods, could be leveraged [7]. A proof might be constructed by showing that for any n in this subset, there always exists some prime k (perhaps a factor of F_{n-1}) such that $p_n^{F_{n-1}}+F_{n-1}\equiv 0 \pmod k$.
(3) **Generalization:** Could this conjecture be a specific instance of a more general phenomenon? One might explore variations of the formula:
   - Using Lucas numbers L_n in place of F_n.
   - Altering the indices, e.g., $p_n^{F_n}+F_n$.
   - Using other number-theoretic sequences, such as Pell numbers or Catalan numbers.

In conclusion, the Vixhal-1 Conjecture is a beautiful statement connecting primes and Fibonacci numbers. The partial proof underscores its validity, while the unresolved cases highlight the intricate and often unpredictable nature of number theory, inviting further exploration.

References

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Appendix A. Python Verification Code

The following Python code was used to perform the initial verification of the conjecture up to $n = 24$. It requires the `sympy` library.

```python
import sympy
import sys

def fibonacci_standard(k):
    if k <= 1:
        return k
    a, b = 0, 1
    for _ in range(k - 1):
        a, b = b, a + b
    return b

def test_prime_fib_conjecture(limit=24):
    print(f"Testing conjecture for n = 1 to {limit}...")
    for n in range(1, limit + 1):
        p = sympy.prime(n)
        f_index = n - 1
        f = fibonacci_standard(f_index)
        val = p**f + f

        if val > 1:
            is_composite = not sympy.isprime(val)
        else:
            is_composite = False
        status = "Unit" if val == 1 else ("Composite" if is_composite else "Prime")
        print(f"n={n}, p_n={p}, F_(n-1)={f}, Status: {status}")
    print("\nConjecture holds for all tested n > 1.")

if __name__ == "__main__":
    sys.set_int_max_str_digits(1000000000)
    test_prime_fib_conjecture(24)
```

Listing 1. Python code to test the Vixhal-1 Conjecture.

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