Optimization of a functional

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April 23, 2018

1 Introduction

Aim

To derive the necessary conditions for a functional to have an extremum.

Strategy

- Define a Functional
- Calculate the variation in the functional
- Use Taylor Series approximation to find the condition.

Assumptions

- All the functions are continuous and differentiable in the domain of interest.

Derivation

Let, I be the defined as,

$$I = \int_{x_1}^{x_2} F\Big(x, y, \frac{dy}{dx}\Big) dx \tag{1}$$

Let us assume that there exists $y(x)$ such that I is optimized.
Let us define $y_1(x)$ and I_1 as,

$$y_1(x) = y(x) + \epsilon\ \eta(x),$$
$$I_1 = \int_{x_1}^{x_2} F\Big(x, y_1, \frac{dy_1}{dx}\Big) dx$$

Where, $\eta(x)$ is an arbitrary function. The only condition we are imposing on $\eta(x)$ is that

$$\eta(x_1) = \eta(x_2) = 0$$

,

Thus,

$$I_1 = \int_{x_1}^{x_2} F\Big(x, y(x) + \epsilon\ \eta(x), \frac{dy}{dx} + \epsilon\frac{d\eta}{dx}\Big) dx \tag{2}$$

For I_1 to be minimum with respect to ϵ, we know that $\frac{\partial I_1}{\partial \epsilon}$ must be zero when $\epsilon = 0$. That implies,

$$\frac{\partial I_1}{\partial \epsilon}_{\epsilon=0} = 0,$$
$$\int_{x_1}^{x_2} \Big[\frac{\partial F}{\partial y_1}\frac{\partial y_1}{\partial \epsilon} + \frac{\partial F}{\partial y_1'}\frac{\partial y_1'}{\partial \epsilon}\Big]_{\epsilon=0} dx = 0,$$

$$\int_{x_1}^{x_2} \Big[\frac{\partial F}{\partial y}\eta + \frac{\partial F}{\partial y'}\eta'\Big] dx = 0 \tag{3}$$

Equation 3 can be integrated by parts,

$$\int_{x_1}^{x_2} \Big[\frac{\partial F}{\partial y}\eta - \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big)\eta\Big]dx + \Big[\frac{\partial F}{\partial y}\eta\Big]_{x_1}^{x_2} = 0$$

The second term in the above equation goes to zero because at the boundaries $\eta(x) = 0$.

This leads to the necessary condition, which is

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\Big(\frac{\partial F}{\partial y'}\Big) = 0 \tag{4}$$

A similar approach has been implemented in the least action principle.

Least Action Formulation

In determining the path followed by an object, the position(x) becomes a function of time. The Action is defined as

$$L = \int_{t_1}^{t_2} (T(x') - V(x))dt,$$
$$L = \int_{t_1}^{t_2} F(t, x, \frac{dx}{dt}),$$

where,

$$T(x') = K.E = \frac{1}{2}mx'^2,$$
$$V(x) = P.E,$$

Thus, trajectory is given by,

$$\frac{\partial F}{\partial x} - \frac{d}{dt}\Big(\frac{\partial F}{\partial x'}\Big) = 0 \tag{5}$$