3. If $\beta_n = \dfrac{\alpha}{\bar{O}_n}$ is used

where $\bar{O}_n = \bar{O}_{n-1} + \alpha(1 - \bar{O}_{n-1})$

then, show that, where $\bar{O}_0 = 0$

Q_n is an exponential recency weighted average without bias,

$\Rightarrow$ Q_n will be independant of Q_0

$$\bar{O}_n = \bar{O}_{n-1} + \alpha(1 - \bar{O}_{n-1})$$

$$= (1-\alpha)\bar{O}_{n-1} + \alpha$$

$$\Rightarrow \bar{O}_n = (1-\alpha)\left[\bar{O}_{n-2}(1-\alpha) + \alpha\right] + \alpha$$

$$= (1-\alpha)^2 \bar{O}_{n-2} + \alpha + \alpha(1-\alpha)$$

$$\vdots$$

$$\Rightarrow \bar{O}_n = \cancel{(1-\alpha)^n \bar{O}_0} \xrightarrow{} 0 + \alpha + \alpha(1-\alpha) + \cdots$$

$$\Rightarrow \bar{O}_n = \alpha\left[\frac{1-(1-\alpha)^n}{1-1+\alpha}\right]$$

$$\Rightarrow \bar{O}_n = 1 - (1-\alpha)^n$$

$$\Rightarrow \beta_n = \frac{\alpha}{1-(1-\alpha)^n} \quad \text{——— (1)}$$

$$Q_{n+1} = Q_n + \beta_n (R_n - Q_n)$$

$$= (1-\beta_n) Q_n + \beta_n R_n$$

$$= (1-\beta_n)\left[Q_{n-1}(1-\beta_{n-1}) + \beta_{n-1} R_{n-1}\right] + \beta_n R_n$$

$$= (1-\beta_n)(1-\beta_{n-1}) Q_{n-1} + \beta_n R_n + (1-\beta_n)\beta_{n-1} R_{n-1}$$

$$\vdots$$

$$\Rightarrow Q_{n+1} = \prod_{i=0}^{n} (1-\beta_i)\, Q_0 + \beta_n R_n + (1-\beta_n)\beta_{n-1} R_{n-1} + \ldots$$

$\Rightarrow$ Coefficient of Q_0 is $\prod_{i=0}^{n} (1-\beta_i)$

$$1-\beta_n = 1 - \frac{\alpha}{\bar{O}_n}$$

$$= \frac{1-(1-\alpha)^n - \alpha}{1-(1-\alpha)^n} \quad \text{using (1),}$$

$$1-\beta_n = (1-\alpha)\frac{\bar{O}_{n-1}}{\bar{O}_n} \quad \text{—— (2)}$$

$$\prod_{i=0}^{n} (1-\beta_i) = \prod_{i=0}^{n} (1-\alpha)\frac{\bar{O}_{i-1}}{\bar{O}_i}$$

$$= (1-\alpha)^n \cancel{\frac{\bar{o}_{n-1}}{\bar{o}_n}} \cdot \cancel{\frac{\bar{o}_{n-2}}{\bar{o}_{n-1}}} \cdots \cancel{\cdot} \frac{\bar{o}_0}{\cancel{\bar{o}_1}}$$

$$\prod_{i=0}^{n} (1-\beta_i) = 0 \qquad \because \bar{o}_0 = 0$$

Since, coeff of $Q_0 = 0$

$\Rightarrow$ Q_n is an exponentially weighted recency average without initial bias.