Problem Formulation:
Robust Economic Dispatch with Uncertain Demand

Team OptiMinds

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1 Problem Definition

We address the problem of **Economic Dispatch** under uncertainty. The objective is to allocate production loads to a set of n parallel machines to minimize total quadratic operating costs while satisfying stochastic demand.

1.1 Notation

Symbol	**Description**
$k \in \{1, \ldots, n\}$	Index for machines
x_k	Decision variable: Production quantity of machine k
a_k, b_k, c_k	Cost coefficients ($a_k > 0$ implies convexity)
ℓ_k, u_k	Minimum and maximum capacity limits for machine k
D	Random variable representing Demand (Mean μ, Variance σ^2)
α	Maximum allowable risk (failure probability)

2 Mathematical Model

2.1 Objective Function

The operating cost of each machine is modeled as a strictly convex quadratic function. We seek to minimize the total system cost:

$$\min_{\mathbf{x}} f(\mathbf{x}) = \sum_{k=1}^{n} \left(a_k x_k^2 + b_k x_k + c_k \right) \tag{1}$$

2.2 Constraints

The system is subject to physical capacity limits and a probabilistic demand requirement:

$$\textbf{Capacity:} \quad \ell_k \leq x_k \leq u_k \quad \forall k = 1, \ldots, n \tag{2}$$

$$\textbf{Reliability:} \quad \mathbb{P}\left(\sum_{k=1}^{n} x_k \geq D \right) \geq 1 - \alpha \tag{3}$$

3 Deterministic Reformulation

To solve the problem using convex optimization, the probabilistic constraint (3) is converted into a deterministic linear constraint:

$$\sum_{k=1}^{n} x_k \geq D_{\text{eff}} \tag{4}$$

where D_{eff} is the effective demand target. We analyze two formulations for D_{eff}:

3.1 Model 1: Gaussian Approximation (Standard)

Assuming $D \sim \mathcal{N}(\mu, \sigma^2)$, we use the inverse CDF of the standard normal distribution (Φ^{-1}):

$$D_{\text{eff}}^{\text{Normal}} = \mu + \Phi^{-1}(1-\alpha)\sigma \tag{5}$$

3.2 Model 2: Distributionally Robust (Proposed Innovation)

Assuming only the mean and variance are known (distributional ambiguity), we use the **One-Sided Chebyshev Inequality** to guarantee reliability for *any* underlying distribution.

$$D_{\text{eff}}^{\text{Robust}} = \mu + \sqrt{\frac{1-\alpha}{\alpha}}\sigma \tag{6}$$

Note: Model 2 imposes a strictly higher production target than Model 1, reflecting the "Price of Robustness."

4 KKT Conditions (Standard Form Analysis)

We analyze the optimality conditions for the deterministic problem. First, we convert all constraints into the standard form $g_i(\mathbf{x}) \leq 0$.

4.1 Standard Form Transformation

1. **Demand Constraint:** $\sum x_k \geq D_{\text{eff}} \implies D_{\text{eff}} - \sum_{k=1}^{n} x_k \leq 0$
2. **Max Capacity:** $x_k \leq u_k \implies x_k - u_k \leq 0$
3. **Min Capacity:** $x_k \geq \ell_k \implies \ell_k - x_k \leq 0$

Let the Lagrange multipliers be μ_0 (demand), $\mu_{u,k}$ (upper bounds), and $\mu_{\ell,k}$ (lower bounds).

4.2 The Lagrangian

$$\mathcal{L}(\mathbf{x}, \boldsymbol{\mu}) = \sum_{k=1}^{n}(a_k x_k^2 + b_k x_k) + \mu_0 \left(D_{\text{eff}} - \sum_{k=1}^{n} x_k \right) + \sum_{k=1}^{n} \mu_{u,k}(x_k - u_k) + \sum_{k=1}^{n} \mu_{\ell,k}(\ell_k - x_k) \tag{7}$$

4.3 Necessary and Sufficient Conditions

Since the problem is convex, the KKT conditions are necessary and sufficient for the global optimum $\mathbf{x}^*$.

1. Primal Feasibility ($g_i(\mathbf{x}^*) \leq 0$)

$$D_{\text{eff}} - \sum_{k=1}^{n} x_k^* \leq 0 \tag{8}$$

$$x_k^* - u_k \leq 0 \quad \forall k \tag{9}$$

$$\ell_k - x_k^* \leq 0 \quad \forall k \tag{10}$$

2. Dual Feasibility ($\mu \geq 0$)

$$\mu_0 \geq 0, \quad \mu_{u,k} \geq 0, \quad \mu_{\ell,k} \geq 0 \quad \forall k \tag{11}$$

3. Stationarity ($\nabla_{\mathbf{x}} \mathcal{L} = 0$) For each machine k, the gradient vanishes:

$$(2a_k x_k^* + b_k) - \mu_0 + \mu_{u,k} - \mu_{\ell,k} = 0 \tag{12}$$

Interpretation: For unconstrained machines ($\mu_{u,k} = \mu_{\ell,k} = 0$), the marginal cost $2a_k x_k^* + b_k$ must equal the system shadow price μ_0.

4. Complementary Slackness ($\mu_i g_i(\mathbf{x}^*) = 0$)

$$\mu_0 \left(D_{\text{eff}} - \sum_{k=1}^{n} x_k^* \right) = 0 \tag{13}$$

$$\mu_{u,k}(x_k^* - u_k) = 0 \quad \forall k \tag{14}$$

$$\mu_{\ell,k}(\ell_k - x_k^*) = 0 \quad \forall k \tag{15}$$