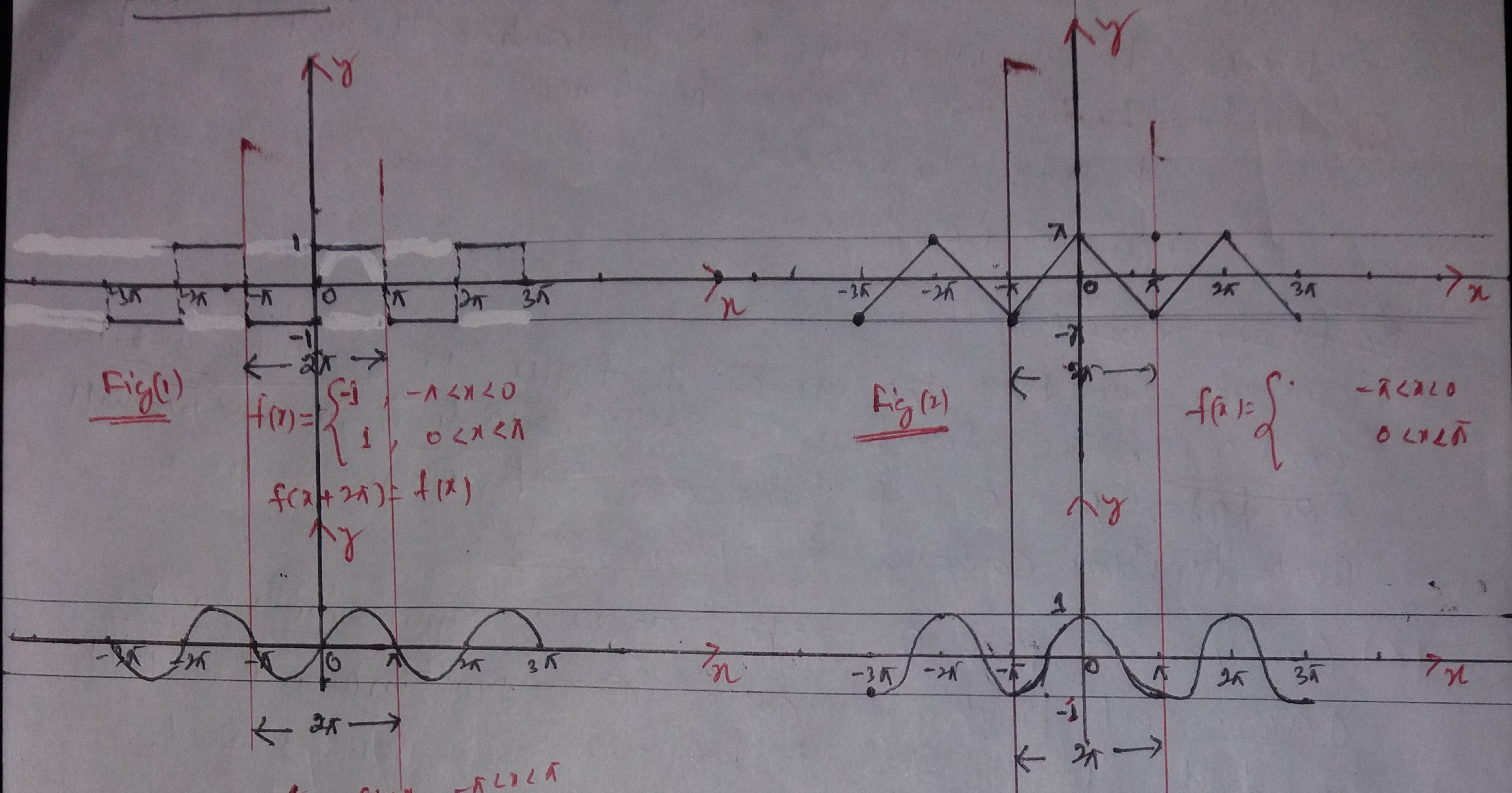
L23: Fourier Series of a Periodic function f(x) with Period $T=2\pi$

Introduction: What is Fourier Series?



Fig(1)

$$f(x)=\begin{cases}-1, & -\pi<x<0\\ 1, & 0<x<\pi\end{cases}$$

$$f(x+2\pi)=f(x)$$

Fig(2)

$$f(x)=\begin{cases} & -\pi<x<0\\ & 0<x<\pi\end{cases}$$

$$f(x)=\sin x,\quad -\pi<x<\pi$$

$$f(x+2\pi)=f(x)$$

$$f(x)=\cos x,\quad -\pi<x<\pi$$

$$f(x+2\pi)=f(x)$$

Discussion:

(1) Can you see graph of all these functions $\boxed{\text{repeating}}$ (wave functions) in every interval of length $T=2\pi$. (Periodic functions)

(2) Can you see that the pattern of rise and fall of graph in fig(1) and that of $\sin x$ is similar. Can be represent a complicated function f(x) in fig(1) terms of simple function $\sin x$ or a combination of ~~Sin~~ scaled ~~Sine~~ Sin function?

$$b_1 \sin x + b_2 \sin 2x + \cdots,$$ with b_1 ~~and~~ $b_2 \ldots$ etc. unknown constants (weights) depending upon the complexity of function f(x)

(3) Can you see the pattern of rise and fall of graph in figure (2) is similar to graph of $\cos x$. Can we represent a complicated function by a simple function $\cos x$ or its variants

$$a_1 \cos x + a_2 \cos 2x + \cdots$$

~~(4) Can you see that In fig(1) graph is Symmetric w.r.t origin. Such functions are called odd functions~~

(2)

(4) ~~Can~~ Q. What about some other complicated functions $f(x)$ whose graph does not follow the pattern either of $\sin x$ or $\cos x$, but is periodic with period 2π?

We can think in terms of representing them in terms of combination of Sin and Cosine functions

$$b_1 \sin x + b_2 \sin 2x + \cdots$$
$$+ a_1 \cos x + a_2 \cos 2x + \cdots$$

~~So fourier series of a~~

(5) Can you see that the graph of fig(1) is symmetric w.r.t. origin $(0,0)$. Such functions are called odd functions and we will see later that ~~such functions are get~~ fourier series of such functions contains Sine terms only.

(6) Can you see that the graph of fig(2) is symmetric w.r.t. y axis. Such functions are called even functions and we will see later that ~~such function are~~ fouries series of such functions contains Cosine terms only

Conclusion:

1. Periodic functions: A function which is defined for all $x \in (-\infty, \infty)$ (except possibly at a countably many points) and whose graph repeats ~~at~~ in every interval of length T is called a periodic function with period $T > 0$. The smallest such T is called fundamental period of the function.

~~If this the fundament~~

~~In symbols~~, if $f(x+T) = f(x)$ for all $x \Rightarrow f(x)$ is periodic function with period T

- If T is the fundamental period of $f(x)$ then $T, 2T, 3T, \cdots$ are also periods of $f(x)$
- If $f(x)$ has fundamental period T, then $f(ax)$ will have fundamental period as ~~$f($~~ $\frac{T}{a}$

Eg • $f(x) = \sin x$. $f(x+2\pi) = \sin(x+2\pi) = \sin x$. $T = 2\pi$

• also $T = 2\pi$ is the smaller one $\Rightarrow T = 2\pi$ fundamental period • $T = 2\pi, 4\pi, 6\pi, \cdots$ are all period

(3)

(2) If $f(x)$ is a Periodic function with Period $T=2\pi$, then

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

constant of adjustment

Next is: How to find these constants a_n's and b_n's & follow on?

* Consider the sequence:

$$\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x, \sin 3x, \cdots\}$$

This sequence is two properties

(i) Every function in this sequence has period $T=2\pi$ (not necessarily fundamental period)

1 (constant function)	$\cos x$	$\sin x$	$\cos 2x$	$\sin 2x$
			π	π
	$\boxed{2\pi}$	$\boxed{2\pi}$		
			$\boxed{2\pi}$	$\boxed{2\pi}$
	4π	4π		
Period any $T>0$	6π	6π	3π	3π
So, $\boxed{2\pi}$ also	$\vdots$	$\vdots$	$\vdots$	

etc.

(ii) Integral of Product of any two ~~functions~~ different functions over an interval of length 2π is always zero. If we take the same function twice the value of integral will be non zero (in fact π)

i.e. $\displaystyle\int_{\alpha}^{\alpha+2\pi} \underbrace{\sin x \cos 2x}_{\text{different}}\, dx = \int_{\alpha}^{\alpha+2\pi} \underbrace{\sin x \sin 2x}_{\text{different}}\, dx = \int_{\alpha}^{\alpha+2\pi} \underbrace{1 . \cos 50x}_{\text{different}}\, dx = \cdots = 0$

$$\int_{\alpha}^{\alpha+2\pi} \underbrace{\sin x . \sin x}_{\text{same}}\, dx \neq 0 \ (=\pi)$$

(In Mathematics such systems are called orthogonal system and this property is called orthogonal property)

N.B: We will use (i) and (ii) above to find a_0, a_n, b_n

Statement of Fourier series (4) of $f(x)$ with Period $T=2\pi$ in Interval $(\alpha, \alpha+2\pi)$

Sol: Let $f(x)$ be defined in interval $(\alpha, \alpha+2\pi)$ and outside it $f(x+2\pi)=f(x)$, that is, $f(x)$ is a periodic function with Period $T=2\pi$

Then fourier series of $f(x)$ in interval $(\alpha, \alpha+2\pi)$ is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$$

(OR)

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos nx + \sum_{n=1}^{\infty} \sin nx}\quad x \in [\alpha, \alpha+2\pi]$$

To be memorize

Where $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$

$a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$

To be memorize.

a_0, a_n, b_n are Called Fourier Coefficients

Proof for a_0, a_n and b_n

If $\frac{a_0}{2}$ we get $a_0 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx$

Let $f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots$ (1)

To find a_0:

Multiplying the expression (1) by 1 and Integrating between $\alpha, \alpha+2\pi$

$$\int_{\alpha}^{\alpha+2\pi} 1.f(x)\,dx = a_0\int_{\alpha}^{\alpha+2\pi} 1\,dx + a_1\int_{\alpha}^{\alpha+2\pi} 1.\cos x\,dx + b_1\int_{\alpha}^{\alpha+2\pi} 1.\sin x\,dx + a_2\int_{\alpha}^{\alpha+2\pi} 1.\cos 2x\,dx + \cdots$$

one term, 2nd term of sequence $1, \cos x, \sin x, \cos 2x$ etc.

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\,dx = a_0[x]_{\alpha}^{\alpha+2\pi} + a_1(0) + b_1(0) + a_2(0) + 0$$

($\because$ By Property (ii) on page (2) $\int_{\alpha}^{\alpha+2\pi}$ two different terms $dx = 0$)

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\,dx = a_0 \times 2\pi$$

$$\Rightarrow \boxed{a_0 = \frac{1}{2\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\,dx}$$

Note: In Fourier series we take a_0 as $\frac{a_0}{2}$, so that all formulas have uniformity $\frac{1}{\pi}$

To find, say a_2,

multiplying (1) by $\cos 2x$ and integrating

$$\int_{\alpha}^{\alpha+2\pi} f(x)\cos 2x\,dx = a_0\int_{\alpha}^{\alpha+2\pi} 1\cdot\cos 2x\,dx + a_1\int_{\alpha}^{\alpha+2\pi}\cos x\cdot\cos 2x\,dx + b_1\int_{\alpha}^{\alpha+2\pi}\sin x\cos 2x$$

$$+ a_2\int_{\alpha}^{\alpha+2\pi}\cos^2 2x + b_2\int_{\alpha}^{\alpha+2\pi}\sin 2x\cos 2x\,dx + \cdots$$

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\cos 2x\,dx = a_0(0) + a_1(0) + b_1(0) + a_2\int_{\alpha}^{\alpha+2\pi}\frac{1+\cos 2x}{2}\,dx + b_2(0) + 0\cdots$$

(by orthogonality property of sequence $1, \cos x, \sin x, \ldots$)

$\sin(2\pi+\theta) = \sin(4\pi+\theta) = \sin(6\pi+\theta) = \cdots = \sin\theta$

$$\Rightarrow \int_{\alpha}^{\alpha+2\pi} f(x)\cos 2x\,dx = a_2\cdot\frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_{\alpha}^{\alpha+2\pi}$$

$$= a_2\cdot\frac{1}{2}\left[\left(\alpha+2\pi + \frac{\sin(2\alpha+4\pi)}{2}\right) - \left(\alpha + \frac{\sin 2\alpha}{2}\right)\right]$$

$$= a_2\cdot\frac{1}{2}\left[\alpha+2\pi + \cancel{\frac{1}{2}\sin 2\alpha} - \alpha - \cancel{\frac{1}{2}\sin 2\alpha}\right]$$

$$= \pi a_2$$

$$\Rightarrow \boxed{a_2 = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos 2x\,dx}$$

In general $a_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\cos nx\,dx$

Similarly, we can prove

$$b_n = \frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi} f(x)\sin nx\,dx$$

(6)

Q(1) find fourier series of: ~~$f(x)=x$ in $f(x+2\pi)=f(x)$~~

$$f(x)=x,\quad 0<x<2\pi$$

$$f(x+2\pi)=f(x)$$

Sol^n ~~we f~~ Given function

$$f(x)=x$$

- Interval $(0,2\pi) \Rightarrow$ Time Period $=2\pi-0=2\pi$

  $(\alpha,\alpha+2\pi)$

- fourier series of $f(x)$ in $(0,2\pi)$

$$f(x)=\frac{a_0}{2}+a_1\cos x+b_1\sin x+a_2\cos 2x+b_2\sin 2x+\cdots$$

or $$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx} \quad \text{— (A)}$$

$$a_0=\frac{1}{\pi}\int_{\alpha}^{\alpha+2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}x\,dx=\frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}=\frac{1}{2\pi}\left[4\pi^2-0\right]=2\pi$$

$$\boxed{a_0=2\pi}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\int_0^{2\pi}\underset{\mathrm{I}}{x}\,\underset{\mathrm{II}}{\cos nx}\,dx \quad \text{(By Parts Short cut)}$$

$$=\frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right)-(1)\left(-\frac{\cos nx}{n^2}\right)\right]_{x=0}^{2\pi}$$

$$=\frac{1}{\pi}\left[\frac{x}{n}\sin nx+\frac{\cos nx}{n^2}\right]_{x=0}^{2\pi}$$

$$=\frac{1}{\pi}\left[\underset{\text{Upper limit}}{\left(0+\cancel{\frac{1}{n^2}}\right)}-\underset{\text{Lower limit}}{\left(0+\cancel{\frac{1}{n^2}}\right)}\right]=0$$

$$\boxed{a_n=0}\quad n=1,2,3,\ldots$$

Note: $\sin n\pi=\sin 2n\pi=0$

$\sin 0=0$

$\sin\pi=\sin 2\pi=\cdots=\sin n\pi=\cdots=0$

$\cos 2\pi=\cos 4\pi=\cos 6\pi=\cdots=1$

$\cos\pi=\cos 3\pi=\cdots=-1$

$\cos 2n\pi=1$

$\cos n\pi=(-1)^n$

$\cos 0=1$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nx=\frac{1}{\pi}\int_0^{2\pi}x\sin nx\,dx=\frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right)-(1)\left(-\frac{\sin nx}{n^2}\right)\right]_{x=0}^{2\pi}$$

$$=\frac{1}{\pi}\left[-\frac{x}{n}\cos nx+\frac{\sin nx}{n^2}\right]_{x=0}^{2\pi}=\frac{1}{\pi}\left[\left\{\left(-\frac{2\pi}{n}\right)(1)+0\right\}-\{0+0\}\right]$$

$$\Rightarrow \boxed{b_n=-\frac{2}{n}}\quad n=1,2,3,\ldots$$

Put a_0, a_n, b_n in (1)

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$f(x) = \frac{2\pi}{2} + \sum_{n=1}^{\infty} \underbrace{(0)\cos nx}_{=0} + \sum_{n=1}^{\infty}\left(\frac{-2}{n}\right)\sin nx$

$$\boxed{x = \pi - 2\sum_{n=1}^{\infty} \frac{1}{n}\sin nx}$$ Ans

(*) $$\boxed{x = \pi - 2\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \cdots\right]}$$

Ex (2) find fourier series of $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

Solⁿ

(*) function $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$

(*) Interval $(-\pi, \pi)$. ∴ So Time Period $T = \pi - (-\pi) = 2\pi$

(*) fourier series

$$\boxed{f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx}$$ (1)

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)dx + \int_{0}^{\pi} f(x)dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right]$

$= \frac{1}{\pi}\left[0 + \left(\frac{x^2}{2}\right)_0^{\pi}\right] = \frac{1}{\pi}\left[\frac{1}{2}(\pi^2 - 0)\right] = \frac{\pi}{2} \Rightarrow \boxed{a_0 = \frac{\pi}{2}}$

(*) $a_n = \frac{1}{\pi}\left[\int_{-\pi}^{\pi} f(x)\cos nx\,dx\right] = \frac{1}{\pi}\left[\int_{-\pi}^{0} \overset{\to 0}{f(x)}\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right]$

$= \frac{1}{\pi}\left[0 + \int_{0}^{\pi} \underset{L}{x} \underset{I}{\cos nx}\,dx\right]$ (By Parts Shortcut)

$= \frac{1}{\pi}\left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{x=0}^{\pi} = \frac{1}{\pi}\left[\frac{x}{n}\sin nx + \frac{\cos nx}{n^2}\right]_{x=0}^{\pi}$

$= \frac{1}{\pi}\left[\underbrace{\left\{0 + \frac{(-1)^n}{n^2}\right\}}_{\text{upper limit}} - \underbrace{\left\{0 + \frac{1}{n^2}\right\}}_{\text{lower limit}}\right]$ [∵ $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\cos 0 = 1$]

$$\boxed{a_n = \frac{1}{\pi}\left[\frac{(-1)^n - 1}{n^2}\right]}$$ $n = 1, 2, 3, \ldots$

(8)

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right]$$

0 in $-\pi$ to 0

$$= \frac{1}{\pi}\left[0 + \int_0^{\pi} x\sin nx\,dx\right] = \frac{1}{\pi}\left[(x)\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_{x=0}^{\pi} = \frac{1}{\cancel{\pi}}\left[\left(-\frac{\cancel{\pi}(-1)^n}{n} + 0\right) - (0+0)\right]$$

$$\Rightarrow \boxed{b_n = -\frac{1}{n}(-1)^n} \quad n = 1, 2, 3, \ldots$$

Put a_0, a_n, b_n in (1)

$$f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1}{\pi n^2}\left[(-1)^n - 1\right]\cos nx + \sum_{n=1}^{\infty}\left(\frac{-1}{n}\right)(-1)^n \sin nx$$

put $n = 1, 2, 3\ldots$ put $n = 1, 2, 3$

(8) $$f(x) = \frac{\pi}{4} + \frac{1}{\pi}\left[-\frac{2}{1^2}\cos x + 0 - \frac{2}{3^2}\cos 3x + 0 - \frac{2}{5^2}\cos 5x + \cdots\right] + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)$$

$$\Rightarrow \boxed{f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right] + \left(\frac{\sin x}{1} - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \cdots\right)}$$ Ans

Ex(3) $f(x) = x$ in $(-\pi, \pi)$

Ans $a_0 = 0$, $a_n = 0$, $b_n = -\frac{2}{n}(-1)^n$ It is an odd function so we

Note we will see later But

use a simpler formula

Ex(4) $f(x) = \begin{cases} -x & -\pi < x < 0 \\ x & 0 < x < \pi \end{cases}$ Ans $a_0 = \pi$, $a_n = \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$, $b_n = 0$

(Note: we will see later But it is an even function. So we will use a simpler formula

Ex(5) $f(x) = \begin{cases} -1 & -\pi \le x < 0 \\ 1 & 0 < x \le \pi \end{cases}$ Ans $a_0 = 0$, $a_n = 0$, $b_n = \frac{2}{\pi}[\ ]$

Note:→ This is an odd function. We will solve it using a simpler formula