Assignment 8

Vishal Ashok

***Abstract*—This document uses the concepts of singular value decomposition in solving a problem.**

Download Python code from

https://github.com/vishalashok98/AI5006/tree/master/Assignment3

Download latex-tikz codes from

https://github.com/vishalashok98/AI5006/tree/master/Assignment3

1 Problem

Find the foot of the perpendicular from the point $\begin{pmatrix}-1\\2\\4\end{pmatrix}$ to the plane

$$2x + 3y - 4z + 5 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of a plane is given by

$$px + by + cz = d \tag{2.0.1}$$

and can be expressed as

$$\mathbf{n}^T\mathbf{x} = d \tag{2.0.2}$$

where

$$\mathbf{n} = \begin{pmatrix}p\\q\\r\end{pmatrix} \tag{2.0.3}$$

The equation of a line passing through the point a and having direction vector $\mathbf{b}$ is given by

$$\mathbf{x} = \mathbf{a} + \lambda\mathbf{b} \tag{2.0.4}$$

3 Solution

Writing the equation of given plane in (2.0.2) form, we get

$$\begin{pmatrix}2 & 3 & -4\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = -5 \tag{3.0.1}$$

where : $\mathbf{n} = \begin{pmatrix}2\\3\\-4\end{pmatrix}$, $\mathbf{x} = \begin{pmatrix}x\\y\\z\end{pmatrix}$ and $c = -5$

We need to represent equation of plane in parametric form,

$$\mathbf{x} = \mathbf{p} + \lambda_1\mathbf{q} + \lambda_2\mathbf{r} \tag{3.0.2}$$

Here $\mathbf{p}$ is any point on plane and $\mathbf{q}, \mathbf{r}$ are two vectors parallel to plane and hence $\perp$ to $\mathbf{n}$. Find two vectors that are $\perp$ to $\mathbf{n}$

$$\implies \begin{pmatrix}2 & 3 & -4\end{pmatrix}\begin{pmatrix}a\\b\\c\end{pmatrix} = 0 \tag{3.0.3}$$

Put $a = 1$ and $b = 0$ in (3.0.3), $\implies c = \frac{1}{2}$
Put $a = 0$ and $b = 1$ in (3.0.3), $\implies c = \frac{3}{4}$

Hence $\mathbf{q} = \begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix}, \mathbf{r} = \begin{pmatrix}0\\1\\\frac{3}{4}\end{pmatrix}$

Let us find point $\mathbf{p}$ on the plane. Put $x = 1, y = 1$ in (3.0.1), we get $\mathbf{p} = \begin{pmatrix}0\\1\\\frac{5}{2}\end{pmatrix}$

Since given plane is perpendicular to given line, if we take $\mathbf{P} = \begin{pmatrix}-1\\2\\4\end{pmatrix}$ it will give us the shortest distance to the plane.

Let $\mathbf{Q}$ be the point on plane with shortest distance to $\mathbf{P}$. $\mathbf{Q}$ can be expressed in (3.0.2) form as

$$\mathbf{Q} = \begin{pmatrix}1\\1\\\frac{5}{2}\end{pmatrix} + \lambda_1\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\\frac{3}{4}\end{pmatrix} \tag{3.0.4}$$

Equate $\mathbf{P}$ and $\mathbf{Q}$, and compute λ_1, λ_2 using *pseudoinverse* from *SVD*

$$\begin{pmatrix}1\\1\\\frac{5}{2}\end{pmatrix} + \lambda_1\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\\frac{3}{4}\end{pmatrix} = \begin{pmatrix}-1\\2\\4\end{pmatrix} \tag{3.0.5}$$

$$\lambda_1\begin{pmatrix}1\\0\\\frac{1}{2}\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\\frac{3}{4}\end{pmatrix} = \begin{pmatrix}-2\\1\\\frac{3}{2}\end{pmatrix} \tag{3.0.6}$$

$$\begin{pmatrix}1 & 0\\0 & 1\\\frac{1}{2} & \frac{3}{4}\end{pmatrix}\begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} = \begin{pmatrix}-2\\1\\\frac{3}{2}\end{pmatrix} \tag{3.0.7}$$

$$\mathbf{M}\mathbf{x} = \mathbf{b} \tag{3.0.8}$$

$$\implies \mathbf{x} = \mathbf{M}^{+}\mathbf{b} \tag{3.0.9}$$

where $\mathbf{M} = \begin{pmatrix}1 & 0\\0 & 1\\\frac{1}{2} & \frac{3}{4}\end{pmatrix}$, $\mathbf{x} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix}-2\\1\\\frac{3}{2}\end{pmatrix}$

Diagonalize $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix}1 & 0\\0 & 1\\\frac{1}{2} & \frac{3}{4}\end{pmatrix}\begin{pmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{3}{4}\end{pmatrix} = \begin{pmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{3}{4}\\\frac{1}{2} & \frac{3}{4} & \frac{13}{16}\end{pmatrix} \tag{3.0.10}$$

We get three eigen values as 1.81, 1 and 0. Normalizing the eigen vectors, $\mathbf{U}$ is calculated as :

$$\mathbf{U} = \begin{pmatrix}\frac{-3}{\sqrt{13}} & \frac{8}{\sqrt{377}} & \frac{-2}{\sqrt{29}}\\\frac{2}{\sqrt{13}} & \frac{12}{\sqrt{377}} & \frac{-3}{\sqrt{29}}\\0 & \frac{13}{\sqrt{377}} & \frac{4}{\sqrt{29}}\end{pmatrix} \tag{3.0.11}$$

Diagonalize $\mathbf{M}^T\mathbf{M}$

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix}1 & 0 & \frac{1}{2}\\0 & 1 & \frac{3}{4}\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1\\\frac{1}{2} & \frac{3}{4}\end{pmatrix} = \begin{pmatrix}\frac{5}{4} & \frac{3}{8}\\\frac{3}{8} & \frac{25}{16}\end{pmatrix} \tag{3.0.12}$$

We get two eigen values as 1.81, 1. Normalizing the eigen vectors, $\mathbf{V}$ is calculated as :

$$\mathbf{V} = \begin{pmatrix}\frac{-3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\\\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{pmatrix} \tag{3.0.13}$$

Taking square root of eigen values, We get $\mathbf{\Sigma}$ as :

$$\mathbf{\Sigma} = \begin{pmatrix}1 & 0\\0 & \frac{29}{16}\\0 & 0\end{pmatrix} \tag{3.0.14}$$

Thus, we performed Singular Value Decomposition for $\mathbf{M}$. It is easy to check that $\mathbf{U\Sigma V}^T = \mathbf{M}$.

For calculating the psuedoinverse, we know

$$\mathbf{M}^{+} = \mathbf{V\Sigma}^{-1}\mathbf{U}^T \tag{3.0.15}$$

$$= \begin{pmatrix}\frac{-3}{\sqrt{13}} & \frac{2}{\sqrt{13}}\\\frac{2}{\sqrt{13}} & \frac{3}{\sqrt{13}}\end{pmatrix}\begin{pmatrix}1 & 0 & 0\\0 & \frac{16}{29} & 0\end{pmatrix}\begin{pmatrix}\frac{-3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0\\\frac{8}{\sqrt{377}} & \frac{12}{\sqrt{377}} & \frac{13}{\sqrt{377}}\\\frac{-2}{\sqrt{29}} & \frac{-3}{\sqrt{29}} & \frac{4}{\sqrt{29}}\end{pmatrix} \tag{3.0.16}$$

$$= \begin{pmatrix}\frac{86206}{100000} & \frac{-20689}{100000} & \frac{275862}{1000000}\\\frac{-20689}{100000} & \frac{6896}{10000} & \frac{413793}{1000000}\end{pmatrix} \tag{3.0.17}$$

Substitute (3.0.17) in (3.0.9),

$$\mathbf{x} = \begin{pmatrix}\frac{86206}{100000} & \frac{-20689}{100000} & \frac{275862}{1000000}\\\frac{-20689}{100000} & \frac{6896}{10000} & \frac{413793}{1000000}\end{pmatrix}\begin{pmatrix}-2\\1\\\frac{3}{2}\end{pmatrix} = \begin{pmatrix}\frac{-151721}{100000}\\\frac{17240695}{10000000}\end{pmatrix} = \begin{pmatrix}\lambda_1\\\lambda_2\end{pmatrix} \tag{3.0.18}$$

Substituting λ_1, λ_2 in (3.0.4)

$$\mathbf{Q} = \begin{pmatrix}\frac{-1}{4}\\\frac{25}{8}\\\frac{111}{32}\end{pmatrix} \tag{3.0.19}$$

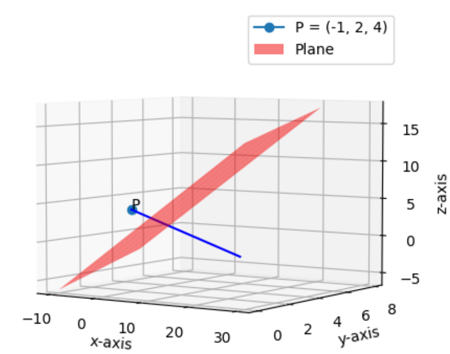

Fig. 0: Foot of perpendicular from $\begin{pmatrix}-1 & 2 & 4\end{pmatrix}$ to the plane $2x + 3y - 4z + 5 = 0$.

Verifying solution to (3.0.8) with *least squares* method

$$\mathbf{M}^T(\mathbf{b} - \mathbf{M}\mathbf{x}) = 0 \tag{3.0.20}$$
$$\implies \mathbf{M}^T\mathbf{M}\mathbf{x} = \mathbf{M}^T\mathbf{b} \tag{3.0.21}$$

Substituting $\mathbf{M}, \mathbf{b}$ from (3.0.7) in (3.0.21)

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & 2 \\ 0 & 1 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ \frac{3}{2} \end{pmatrix} \tag{3.0.22}$$

$$\begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \frac{-5}{4} \\ \frac{17}{8} \end{pmatrix} \tag{3.0.23}$$

$$\implies 5\lambda_1 + 4\lambda_2 = \frac{-5}{4} \tag{3.0.24}$$

$$4\lambda_1 + 5\lambda_2 = \frac{17}{8} \tag{3.0.25}$$

$$\lambda_1 = \frac{-12499}{10000} \tag{3.0.26}$$

$$\text{and } \lambda_2 = \frac{21249}{10000} \tag{3.0.27}$$

$$\implies \mathbf{x} = \begin{pmatrix} \frac{-12499}{10000} \\ \frac{21249}{10000} \end{pmatrix} \tag{3.0.28}$$

Comparing (3.0.18) and (3.0.28) solution is verified.