Assignment 2

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***Abstract*—This document explains how to trace a curve given its equation.**

Download Python code from

https://github.com/vishalashok98/AI5106/tree/main/Assignment_1

Download the latex-tikz codes from

https://github.com/vishalashok98/AI5106/tree/main/Assignment_1

1 Problem

Trace the curve

$$35x^2 + 30y^2 + 32x - 108y - 12xy + 59 = 0 \tag{1.0.1}$$

2 Explanation

The general equation of second degree is given by

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0 \tag{2.0.1}$$

and can be expressed as

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

where

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{2.0.3}$$

$$\mathbf{u}^T = \begin{pmatrix} d & e \end{pmatrix} \tag{2.0.4}$$

3 Solution

Comparing (1.0.1) with (2.0.1), we get

$$\mathbf{V} = \begin{pmatrix} 35 & -6 \\ -6 & 30 \end{pmatrix} \tag{3.0.1}$$

$$\mathbf{u}^T = \begin{pmatrix} 16 & -54 \end{pmatrix} \tag{3.0.2}$$

If $|\mathbf{V}| > 0$, then (2.0.2) is an ellipse.

$$|V| = \begin{vmatrix} 35 & -6 \\ -6 & 30 \end{vmatrix} = 1014 > 0 \tag{3.0.3}$$

(2.0.2) can be expressed as

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \qquad |V| \neq 0 \tag{3.0.4}$$

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = -\eta\begin{pmatrix} 1 & 0 \end{pmatrix}\mathbf{y} \qquad |V| = 0 \tag{3.0.5}$$

with center as

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \qquad |V| \neq 0 \tag{3.0.6}$$

Calculating the center for given curve we get,

$$\mathbf{c} = -\frac{1}{\left|35 * 30 - 6 * 6\right|}\begin{pmatrix} 30 & 6 \\ 6 & 35 \end{pmatrix}\begin{pmatrix} 16 \\ -54 \end{pmatrix} \tag{3.0.7}$$

$$= \frac{1}{1014}\begin{pmatrix} 156 \\ -1794 \end{pmatrix} \tag{3.0.8}$$

$$= \begin{pmatrix} \frac{2}{13} \\ \frac{-23}{13} \end{pmatrix} \tag{3.0.9}$$

For

$$|\mathbf{V}| > 0, \quad \text{or, } \lambda_1 > 0, \lambda_2 > 0 \tag{3.0.10}$$

(3.0.4) becomes

$$\lambda_1 y_1^2 + \lambda_2 y_1^2 = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{3.0.11}$$

which is the equation of an ellipse with major and minor axes parameters

$$\sqrt{\frac{\lambda_1}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}}, \sqrt{\frac{\lambda_2}{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}} \tag{3.0.12}$$

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left|\lambda\mathbf{I} - \mathbf{V}\right| = \begin{vmatrix} \lambda - 35 & 6 \\ 6 & \lambda - 30 \end{vmatrix} = 0 \tag{3.0.13}$$

$$\implies \lambda^2 - 65\lambda + 1014 = 0 \tag{3.0.14}$$

The eigenvalues are the roots of (3.0.14) given by

$$\lambda_1 = 39, \lambda_2 = 26 \tag{3.0.15}$$

Calculating the major and minor axes lengths using (3.0.12), we get

$$\begin{aligned}
\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} &= \\
&= \begin{pmatrix}16 - 54\end{pmatrix}\frac{1}{1014}\begin{pmatrix}30 & 6\\ 6 & 35\end{pmatrix}\begin{pmatrix}16\\ -54\end{pmatrix}\\
&= \frac{1}{1014}\begin{pmatrix}16 & -54\end{pmatrix}\begin{pmatrix}156\\ -1794\end{pmatrix}\\
&= 98
\end{aligned}$$

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 98 - 59 = 39 \tag{3.0.16}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{39}{39}} = 1 \tag{3.0.17}$$

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\lambda_2}} = \sqrt{\frac{39}{26}} = \frac{\sqrt{6}}{2} \tag{3.0.18}$$

Assignment/Assignment7.png

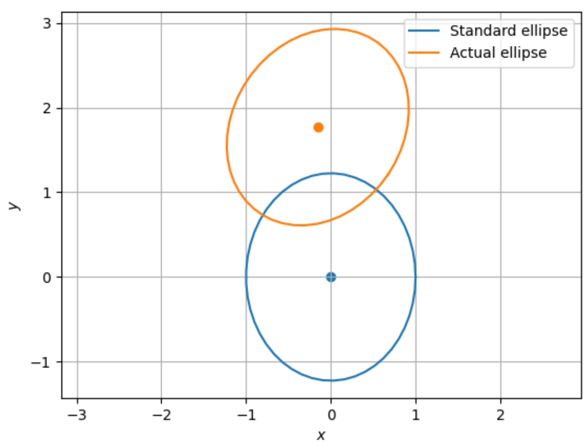

Fig. 0: Ellipse with center $\begin{pmatrix}\frac{2}{13} & \frac{-23}{13}\end{pmatrix}$ and having the axes lengths as 1 and $\frac{\sqrt{6}}{2}$