Assignment 2

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***Abstract*—This document explains how to find the point of intersection of a line and a plane.**

Download Python code from

https://github.com/vishalashok98/AI5106/tree/main/Assignment_1

Download the latex-tikz codes from

https://github.com/vishalashok98/AI5106/tree/main/Assignment_1

1 Problem

Find the equations of tangents to the circle

$$\mathbf{x}^T\mathbf{x} - \begin{pmatrix} 4 & 3 \end{pmatrix}\mathbf{x} + 5 = 0 \tag{1.0.1}$$

that are parallel to the line

$$\begin{pmatrix} 1 & 1 \end{pmatrix}\mathbf{x} = 0 \tag{1.0.2}$$

2 Explanation

The vector equation of a line can be expressed as

$$\mathbf{x} = \mathbf{q} + \mu\mathbf{m} \tag{2.0.1}$$

The general equation of a second degree can be expressed as :

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.2}$$

3 Solution

Comparing (1.0.1) with (2.0.2)

$$\mathbf{u} = \begin{pmatrix} -2 \\ \frac{-3}{2} \end{pmatrix}, f = 5 \tag{3.0.1}$$

If $\mathbf{n}$ is the normal vector of a line, equation of that line can be written as

$$\mathbf{n}^T\mathbf{x} = c \tag{3.0.2}$$

Comparing (1.0.2) with (3.0.20)

$$\mathbf{n} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \tag{3.0.3}$$

Since it is mentioned that the tangent is parallel to given line it will have same normal vector.
The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (2.0.2) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{3.0.4}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.0.5}$$

The point of contact $\mathbf{q}$, of a line with a normal vector $\mathbf{n}$ to the conic in (2.0.2) is given by:

$$\mathbf{q} = \mathbf{V}^{-1}\left(\kappa\mathbf{n} - \mathbf{u}\right) \tag{3.0.6}$$

$$\kappa = \pm\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f}{\mathbf{n}^T\mathbf{V}^{-1}\mathbf{n}}} \tag{3.0.7}$$

We know that, for a circle,

$$\mathbf{V} = \mathbf{I} \tag{3.0.8}$$

and from the properties of an Identity matrix,

$$\mathbf{I}^{-1} = \mathbf{I} \tag{3.0.9}$$

$$\mathbf{I}\mathbf{X} = \mathbf{X} \tag{3.0.10}$$

Solving for the point of contact using the above equations we get,

$$\kappa = \pm\sqrt{\frac{\begin{pmatrix} -2 & -1.5 \end{pmatrix}\begin{pmatrix} -2 \\ -1.5 \end{pmatrix} - 5}{\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}} \tag{3.0.11}$$

$$= \pm\sqrt{\frac{6.25 - 5}{2}} \tag{3.0.12}$$

$$= \pm\sqrt{\frac{5}{8}} \tag{3.0.13}$$

$$\tag{3.0.14}$$

So there are two tangents to a circle which are parallel to given line which touch circle at two different points $\mathbf{q_1}$ and $\mathbf{q_2}$

$$\mathbf{q_1} = \begin{pmatrix} \sqrt{\frac{5}{8}} \\ \sqrt{\frac{5}{8}} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} \tag{3.0.15}$$

$$= \begin{pmatrix} \frac{279}{100} \\ \frac{229}{100} \end{pmatrix} \tag{3.0.16}$$

$$\mathbf{q_2} = \begin{pmatrix} -\sqrt{\frac{5}{8}} \\ \sqrt{\frac{5}{8}} \end{pmatrix} + \begin{pmatrix} 2 \\ \frac{3}{2} \end{pmatrix} \tag{3.0.17}$$

$$= \begin{pmatrix} \frac{121}{100} \\ \frac{71}{100} \end{pmatrix} \tag{3.0.18}$$

Since points $\mathbf{q_1}$ and $\mathbf{q_2}$ lie on tangent they satisfy the line equation of tangents, there are two different tangents with same normal vector

$$\mathbf{n}^T\mathbf{q_1} = c_1 \tag{3.0.19}$$

$$\mathbf{n}^T\mathbf{q_2} = c_2 \tag{3.0.20}$$

c_1 and c_2 are some constants

$$c_1 = \begin{pmatrix} 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{279}{100} \\ \frac{229}{100} \end{pmatrix} = \frac{127}{25} \tag{3.0.21}$$

$$c_2 = \begin{pmatrix} \frac{121}{100} \\ \frac{71}{100} \end{pmatrix} = \frac{48}{25} \tag{3.0.22}$$

So line equations of tangents to the given circle which are parallel to line are

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{127}{25} \tag{3.0.23}$$

and

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = \frac{48}{25} \tag{3.0.24}$$

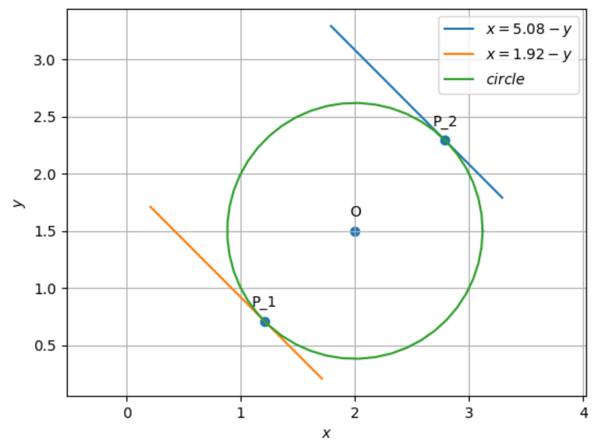

Fig. 0: Circle with center (2 1.5) and having the lines (1 1)x =5.08 and (1 1)x =1.92 as tangents with (2.79 2.29) and (1.21 0.71) as point of contact.