# Optimal Execution

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At first, fix a probability space $(\Omega, \mathbb{F}, \mathcal{F} = (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$, where $\mathbb{F}$ is generated by $(M_t)_{0\le t\le T}$ and $(S_t)_{0\le t\le T}$.

the Basic Model:
a) MidPrice: the MidPrice $M_t$ satisfies $dM_t = \mu dt + \sigma dW_t$, where $dW_t$ is a standard Brownian motion.
b) Spread: the Spread $S_t$ is a discrete Markov chain with probability transition matrix $\rho_{ij}$.
As our test in foreign exchange high frequency data, the transition matrix is:

| Spread | 0.00001 | 0.00002 | 0.00003 | 0.00004 | 0.00005 | 0.00006 | 0.00007 | 0.00008 | 0.00009 |
|---|---|---|---|---|---|---|---|---|---|
| 0.00001 | 0.806 | 0.132 | 0.054 | 0.008 | 0 | 0 | 0 | 0 | 0 |
| 0.00002 | 0.072 | 0.831 | 0.084 | 0.012 | 0.001 | 0 | 0 | 0 | 0 |
| 0.00003 | 0.037 | 0.118 | 0.813 | 0.030 | 0.002 | 0 | 0 | 0 | 0 |
| 0.00004 | 0.019 | 0.056 | 0.114 | 0.792 | 0.015 | 0.003 | 0.001 | 0 | 0 |
| 0.00005 | 0.004 | 0.017 | 0.028 | 0.061 | 0.851 | 0.028 | 0.005 | 0.002 | 0.002 |
| 0.00006 | 0.002 | 0.004 | 0.009 | 0.020 | 0.034 | 0.897 | 0.020 | 0.006 | 0.007 |
| 0.00007 | 0.002 | 0.001 | 0.003 | 0.009 | 0.016 | 0.033 | 0.888 | 0.035 | 0.013 |
| 0.00008 | 0.001 | 0 | 0.002 | 0.004 | 0.005 | 0.014 | 0.040 | 0.885 | 0.050 |
| 0.00009 | 0 | 0 | 0 | 0 | 0.001 | 0.001 | 0.004 | 0.008 | 0.984 |

The best ask price is $P^a_t = M_t + \frac{S_t}{2}$, and the best bid price is $P^b_t = M_t - \frac{S_t}{2}$.
The market maker quotes an ask price at $P^+_t = M_t + \sigma^+$, quotes at $P^-_t = M_t - \sigma^-$ with distance $\sigma^+$ and $\sigma^-$.
If large market orders are coming with size $\mathcal{Q}$, then market maker will meet a price impact and get an executed price $P^{\mathcal{Q}}$, if $P^{\mathcal{Q}} \le P^+$, market marker's ask order will be fully filled, similarly if $P^{\mathcal{Q}} \ge P^-$, market maker's bid order will be fully filled.
Also we can define these two rate parameters $\lambda^a(\sigma^+)$ and $\lambda^b(\sigma^-)$, based on which we could define two Poisson process $N^a_t$ and $N^b_t$ to measure the process of market maker's orders filled by market orders. Obviously, $\lambda^a \propto -\sigma^+$ and $\lambda^b \propto -\sigma^-$.
Now we need to measure the market orders' size $\mathcal{Q}$ and the executed price $P^{\mathcal{Q}}$, according to Maslow and Mills(2001) we know that the density function of $\mathcal{Q}$ satisfies $f(x) \propto x^{-1-\alpha}$ where $\alpha$ is contingent. Thus, we can assume that

$$f(x) = C_1 x^{-1-\alpha} \tag{1}$$

and according to Potters and Bouchaud(2003), we can know the relationship between $\mathcal{Q}$ and the distance between executed price and MidPrice, $\Delta P = |P^{\mathcal{Q}} - M_t|$ is $\Delta P \propto ln(\mathcal{Q})$. Hence, we assume that $\Delta P = C_2 ln(\mathcal{Q})$.
The Spread is a important loquidity factor, obviously the frequency of market orders $\mu$(arriving rate) is related to spread, so $\mu = g(S_t)$ and $g(\cdot)$ is a decreasing function. Hence we could assume that

$$\mu = Ce^{-S_t}. \tag{2}$$

From all the above, we can calculate the intensity of $N_t^a$ and $N_t^b$, $\lambda^a(\sigma^+)$ and $\lambda^b(\sigma^-)$ which measure the arrival of the buy orders and sell orders:

$$\begin{align}
\lambda(\sigma) &= \mu P(\Delta P > \sigma) \tag{3}\\
&= \mu P(ln(\mathcal{Q}) > C_2\sigma) \tag{4}\\
&= \mu P(\mathcal{Q} > e^{C_2\sigma}) \tag{5}\\
&= \mu C_1 \int_{e^{C_2\sigma}}^{+\infty} x^{-1-\alpha} dx \tag{6}\\
&= \mu \frac{C_1}{\alpha} e^{-C_2\alpha\sigma} \tag{7}
\end{align}$$

It is easy to know that market maker's order would be executed at $P^+$ or $P^-$, then we could define a wealth process as following:

$$dX_t = P_t^+ dN_t^a - P_t^- dN_t^b \tag{8}$$

Then the market maker aims to maximize his wealth, including $X_t$ and his positions.

$$H(t, S, M, x, q) = \sup_{\sigma^+,\sigma^-} \mathbb{E}[X_T + q_T(M_T - l(q_T))] \tag{9}$$

where, $q_t = N_t^b - N_t^a$ is the number of stock held at time t; $l(q)$ is liquid penalty function increasing in q, which inconsists of trading fees and the price impact and $l(0) = 0$, so $l = e^{-\theta q^2}$ is a proper choise with good properties. Also, market maker won't hold a huge position, for simplicity we assume that $q_T \leq q$, $q \in \mathbb{R}^+$ and it is very small.
From Equation(8), we know that $X_T$ could be expressed as this:

$$\begin{align}
X_T &= \sum (M_t + \sigma^+)(N_{t+1}^a - N_t^a) - (M_t - \sigma^-)(N_{t+1}^b - N_t^b) \tag{10}\\
&= \sum (M_t + \sigma^+) N_1^a - (M_t - \sigma^-) N_1^b \tag{11}\\
&= \sum \beta_t^a \lambda^a(\sigma^+)(M_t + \sigma^+) - \sum \beta_t^b \lambda^b(\sigma^-)(M_t - \sigma^-) \tag{12}
\end{align}$$

where, $\beta_i^a$ and $\beta_i^b$ are constant.
Similarly, we can obtain:

$$q_T = \sum (N_{t+1}^b - N_t^b) - (N_{t+1}^a - N_t^a) \tag{13}$$
$$= \sum N_1^b - N_1^a \tag{14}$$
$$= \sum \beta_t^b \lambda^b(\sigma^+) - \sum \beta_t^a \lambda^a(\sigma^-) \tag{15}$$

In order to solve this problem, it seems we could use Recurrent Neural Newwork framwork to find the optimal parameters, $\sigma^+$ and $\sigma^-$.There are two questions we need to solve, the first one is that we should find a proper function to measure the market maker's wealth, the second one is that the wealth function could converge while the parameters can be updated through gradient descending method. As for the training set, we can use Monte-Carlo method to generate enough pathes to train the neural network.
Firt, we have to define a utility function $\rho(\cdot)$. And $\rho(\cdot)$ must satisfy:
(1)monotonically decreasing: if $X_1 > X_2$, then we get $\rho(X_1) < \rho(X_2)$
(2)convex: $\rho(\alpha_1 X_1 + (1-\alpha_1)X_2) \leq \rho(\alpha_1 X_1) + \rho((1-\alpha_1)X_2)$
then our target is as following:

$$\pi(X) = \inf_{\sigma^+,\sigma^-} \rho(\mathbb{E}\{X_T + q_T(M_T - l(q_T))\}) \tag{16}$$

then $\pi$ must be a convex measure.
$Proof$ : we assume that $\alpha_1 \in [0,1]$ and $\alpha_1 + \alpha_2 = 1$, because of the properties of $\rho(\cdot)$, $q_T$ and the convexity of $l(\cdot)$, we can define a penalty function $l(x)$ which satisfies that if $x \in [0,q]$, $xl(x)$ is a concave function.

$$\pi(\alpha_1 X_1 + \alpha_2 X_2) \tag{17}$$
$$= \rho(\alpha_1 X_1 + \alpha_2 X_2 + (\alpha_1 q_1 + \alpha_2 q_2)\{M - l(\alpha_1 q_1 + \alpha_2 q_2)\}) \tag{18}$$
$$= \rho(\alpha_1 X_1 + \alpha_2 X_2 + (\alpha_1 q_1 + \alpha_2 q_2)M - (\alpha_1 q_1 + \alpha_2 q_2)l(\alpha_1 q_1 + \alpha_2 q_2)) \tag{19}$$
$$\leq \rho(\alpha_1 X_1 + \alpha_2 X_2 + (\alpha_1 q_1 + \alpha_2 q_2)M - \alpha_1 q_1 l(q_1) - \alpha_2 q_2 l(q_2) \tag{20}$$
$$= \rho(\alpha_1(X_1 + q_1(M - l(q_1))) + \alpha_2(X_2 + q_2(M - l(q_2)))) \tag{21}$$
$$\leq \alpha_1 \rho(X_1 + q_1(M - l(q_1))) + \alpha_2 \rho(X_2 + q_2(M - l(q_2))) \tag{22}$$
$$= \alpha_1 \pi(X_1) + \alpha_2 \pi(X_2) \tag{23}$$

Then we can prove that $\pi(\cdot)$ is convex.
In the next step, we will build a Recurrent Neural Network framework to solve such a optimization problem. The recurrent nueral network framework is defiined as following:
**Nueral Network**: $L, N_0, N_1, \ldots, N_L \in \mathbb{N}$, $\kappa : \mathbb{R} \to \mathbb{R}$ and for $\ell = 1, \ldots, L$, $W_\ell : \mathbb{R}^{N_{\ell-1}} \to \mathbb{R}^{N_\ell}$ and the function $F : \mathbb{R}^{N_0} \to \mathbb{R}^{N_L}$ is defined as:

$$F(x) = W_L \circ F_{L-1} \circ \cdots \circ F_1 \tag{24}$$

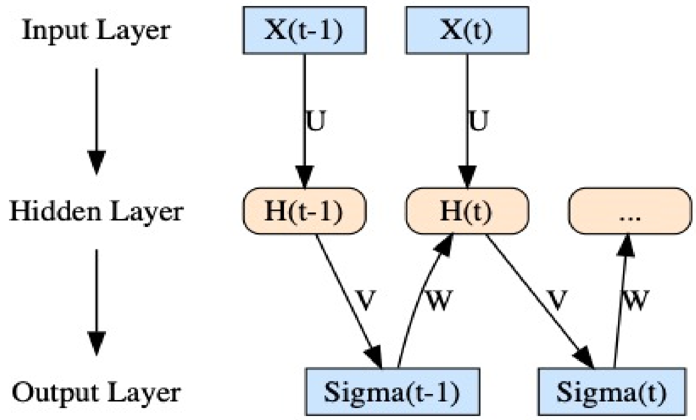


Figure 1: Recurrent Neural Network

where, with $F_\ell = \kappa \circ W_\ell$ for $\ell = 1, \ldots, L-1$, $\kappa$ is an activation function. And $L$ denotes the number of layers, $N_1 \ldots N_{L-1}$ denote the dimensions of layers and $N_0, N_L$ are the dimensions of input and output layers. As for the function $W_\ell(x)$, we define it as $W_\ell(x) = A^\ell x + b^\ell$, obviously $A^\ell \in \mathbb{R}^{N_\ell \times N_{\ell-1}}$ and $b^\ell \in \mathbb{R}^{N_\ell}$. For any $i = 1, \ldots, L-1$ and $j = 1, \ldots, L$ the $A^\ell_{ij}$ is the weight of the edge connecting node i of layer $\ell - 1$ to node j of layer $\ell$. Figure 1 is the RNN structure.

And for the neural network framework $\mathcal{NN}^\kappa_{L,N_0,N_L}$, we need to prove that through $\mathcal{NN}^\kappa_{L,N_0,N_L}$, the utility $\rho(X)$ could converge to the minimum utility $\pi(X)$.

*Proof* : firstly for the given function $\rho(\cdot)$ is convex and monotonically decreasing, then we just need to prove that $\lim\limits_{M\to\infty} \mathcal{X}^M = \mathcal{X}$, where:

$$\mathcal{X}^M = X_T(\sigma^M) + q_T(\sigma^M)M_T - q_T(\sigma^M)l(q_T(\sigma^M))), \sigma^M \in \mathcal{H}^M \tag{25}$$

$$\mathcal{H}^M = \{(\sigma^j_k)_{k=0,1,\ldots,l-1}, \sigma^j_k = F^j_k(M_t, S_t, \sigma^j_{k-1}), F^j_k \in \mathcal{NN}^\kappa_{L,N_0,N_L}\} \tag{26}$$

And since $\mathcal{H}^M \subset \mathcal{H}^{M+1} \subset \mathcal{H}$ for all $M \in \mathbb{R}$, we can know that $\mathcal{X} \geq \mathcal{X}^{M+1} \geq \mathcal{X}^M$. And obviously, for any $\epsilon > 0$, there exists a $\sigma \in \mathcal{H}$ which satisfies:

$$X_T(\sigma) + q_T(\sigma)M_T - q_T(\sigma)l(q_T(\sigma)) \geq \mathcal{X} - \frac{\epsilon}{2} \tag{27}$$

And here $\sigma_k = F(M_t, S_t, \sigma_{k-1})$ and according to K. Hornik(1991), $F_n$ will converge to $F$ when n is large enough. Also because $\sigma^n_k = F^n_k(M_t, S_t, \sigma^n_{k-1})$, we could obtain that $\lim\limits_{n\to\infty} \sigma^n_k = \sigma_k$. Then we get that:

$$\lim_{n\to\infty} \sup_{\sigma^n} X_T(\sigma^n) + q_T(\sigma^n)M_T - q_T(\sigma^n)l(q_T(\sigma^n)) \geq X_T(\sigma) + q_T(\sigma)M_T - \lim_{n\to\infty} \inf_{\sigma^n} q_T(\sigma^n)l(q_T(\sigma^n)) \tag{28}$$

$$\geq X_T(\sigma) + q_T(\sigma)M_T - q_T\sigma q_T(\sigma)l(q_T(\sigma)) \tag{29}$$

Hence, for any $\epsilon > 0$ and a large enough number $M$, when $n \geq M$, we have $\mathcal{X}^n \geq \mathcal{X} - \epsilon$ and $\rho(X)$ could converge to $\pi(X)$.

Therefore, the wealth function's local optimum is its global optimum as well. Then we can define the cost function and rewrite the optimization problem as:

$$\pi(X) = \inf_{\sigma^+,\sigma^-} \rho(\mathbb{E}\{X_T + q_T(M_T - l(q_T))\}) \tag{30}$$

$$= \inf_{\sigma} J(\sigma) \tag{31}$$

Obviously, there is a minimum price movement $\tau$, which means $\sigma^+ = \theta^+\tau$ and $\sigma^- = \theta^-\tau$, then we can initialize the $\sigma+$ and $\sigma-$ with $r\tau$, where r is genereated randomly and $r \in \mathbb{N}^+$, we rewrite Equation(31) like this:

$$J(\theta) := \rho(\mathbb{E}\{X_T + q_T(M_T - l(q_T))\}) \tag{32}$$

where, $\rho(\cdot)$ could be a exponential utility function $\rho(x) = e^{-\theta x}$ and to find a local optimum of cost function J, we can use gradient descent algorithm with initial value $\sigma_0$:

$$\theta^{(j+1)} := \theta^{(j)} - \eta_j \nabla J_j(\theta^{(j)}) \tag{33}$$