ILP Formulation of MLRC and MRLC Scheduling

Dr. Chandan Karfa

Department of Computer Science and Engineering

भारतीय प्रौद्योगिकी संस्थान गुवाहाटी
INDIAN INSTITUTE OF TECHNOLOGY GUWAHATI

Integer Linear Programming (ILP)

- Given:
  - integer-valued matrix $A_{m \times n}$
  - *variables:* $\boldsymbol{x} = (x_1, x_2, \ldots , x_n)^T$
  - constants: $\boldsymbol{b} = (b_1, b_2, \ldots , b_m)^T$ and $\boldsymbol{c} = (c_1, c_2, \ldots , c_n)^T$

- Minimize: $c^T \boldsymbol{x}$

subject to:

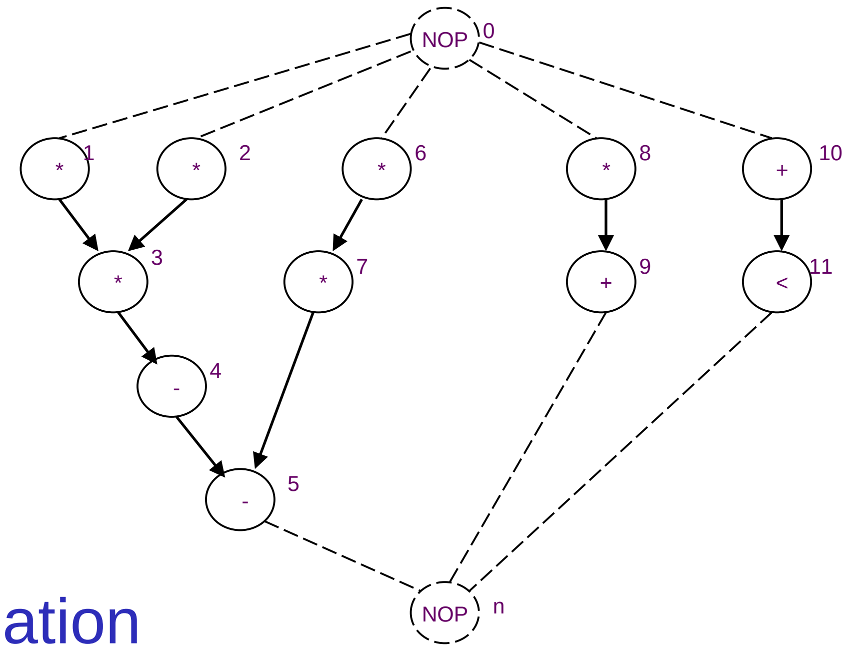
$$\begin{cases} A\,\boldsymbol{x} \le \boldsymbol{b} \\ \boldsymbol{x} = (x_1, x_2, \ldots , x_n) \text{ is an integer-valued vector} \end{cases}$$

- If all variables are *continuous*, the problem is called linear (LP)
- Problem is called *Integer* LP (ILP) if some variables *x* are integer
  - special case: 0,1 (binary) ILP

ILP Model of Scheduling

- Binary decision variables x_{il}

$x_{il} = 1$ if operation v_i starts in step l,
otherwise $x_{il} = 0$
$i = 0, 1, \ldots, n$ (operations)
$l = 1, 2, \ldots \lambda+1$ (steps, with limit λ)



Q1: How many Binary variable do we need?
Q2: Can we use ASAP and ALAP scheduling information reduce the decision variable?

ILP Model of Scheduling - Constraints

- Start time of each operation v_i is unique:

$$\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$$

Note: $\sum_l x_{il} = \sum_{l=t_i^S}^{l=t_i^L} x_{il}$

where:

t_i^S = time of operation I computed with *ASAP*

t_i^L = time of operation I computed with *ALAP*

Start time for v_i:

$$t_i = \sum_l l \cdot x_{il}$$

ILP Model of Scheduling - constraints

- Precedence relationships must be satisfied

$$\sum_l l \cdot x_{il} \geq \sum_l l \cdot x_{jl} + d_j, \quad i, j = 0, 1, \ldots, n \quad : (v_j, v_i) \in E$$

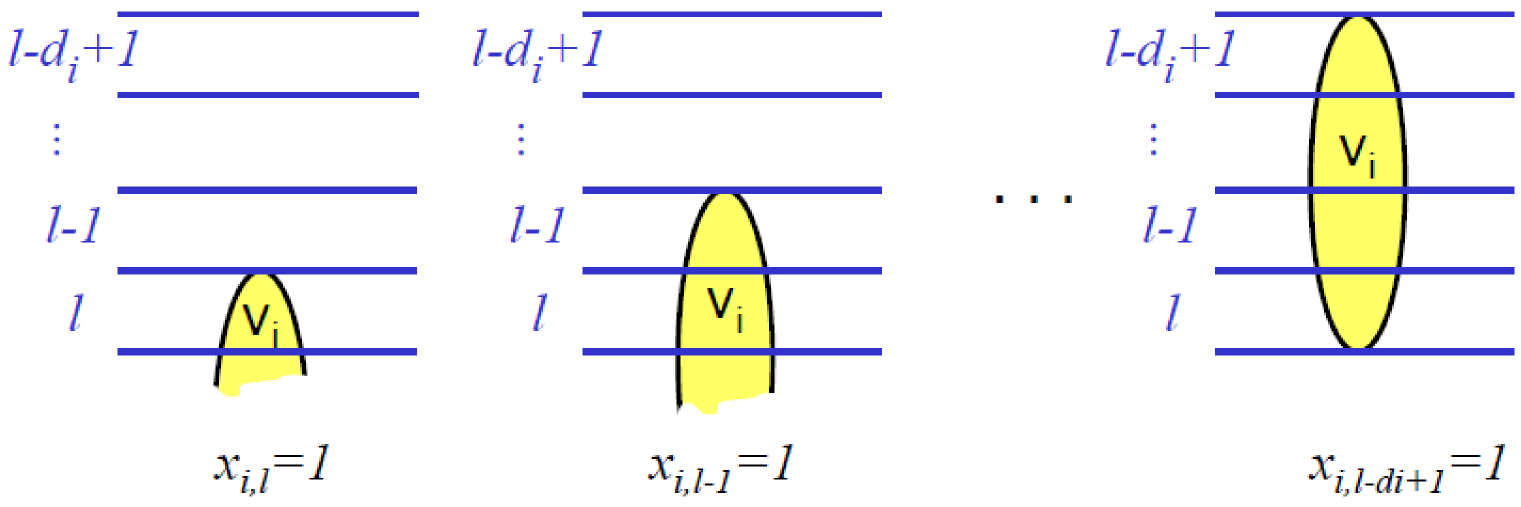
ILP Model of Scheduling - constraints

- Resource constraints must be met
  - let upper bound on number of resources of type k be a_k

$$\sum_{i:\mathcal{T}(v_i)=k} \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

Operation v_i Still Running at Step l ?

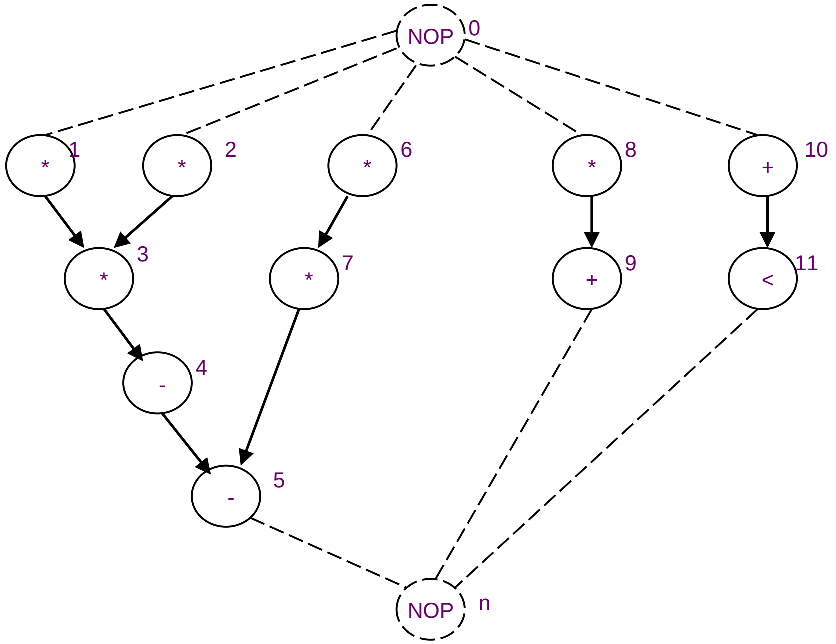
- Is v_i running at step l ?
  - Is $x_{i,l} + x_{i,l-1} + \ldots + x_{i,l-di+1} = 1$?



ILP Model of Scheduling - Constraints

– Latency bound must be satisfied

– $\Sigma_l \, l \, x_{nl} \leq \lambda + 1$



Scheduling Problems

- Minimum Latency Unconstrained minimum-latency scheduling problem (Unconstraint) - ASAP
- Minimum latency under resource constraints (MLRC)
- Minimum resource under latency constraints (MRLC)

Minimum latency under resource constraints (MLRC)

- Given a set of ops V with integer delays D, a partial order on the operations E, and upper bounds $\{ a_k ; k = 1, 2, \ldots, n_{res} \}$ on resource usage:
- Find an integer labeling of the operation $\varphi : V \rightarrow z^+$ such that :

$$t_i = \varphi(v_i),$$

$$t_i \geq t_j + d_j \quad \text{for all } i,j \text{ s.t. } (v_j, v_i) \in E,$$

$$| \{v_i \mid T(v_i) = k \text{ and } t_i \leq l < t_j + d_j \} | \leq a_k \quad \text{for all types } k = 1,2,\ldots,n_{res} \text{ and steps } l$$

and t_n *is minimum* → minimize

Minimum Resource under latency constraints (MRLC)

- Additional constraint: Latency
  - Latency bound must be satisfied
- Resource usage is unknown in the constraints (a_k is unknown)
- Objective is to minimize resource usage such that

  L
    - Dependency constraints are satisfied
    - Latency constraint is satisfied

Minimum-Latency Scheduling under Resource Constraints (ML-RC)

- Let ***t*** be the vector whose entries are *start times*

$t = [\, t_0, t_1, \ldots, t_n \,]$

$v_j \rightarrow v_i$

- Formal ILP model

$$\text{minimize } \mathbf{c}^T\mathbf{t} \text{ such that}$$

C = [0, 0, …, 1]

$$\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$$

v_i v_j

$$\sum_l l \cdot x_{il} - \sum_l l \cdot x_{jl} + d_j \geq 0, \quad i, j = 0, 1, \ldots, n \; : (v_j, v_i) \in E$$

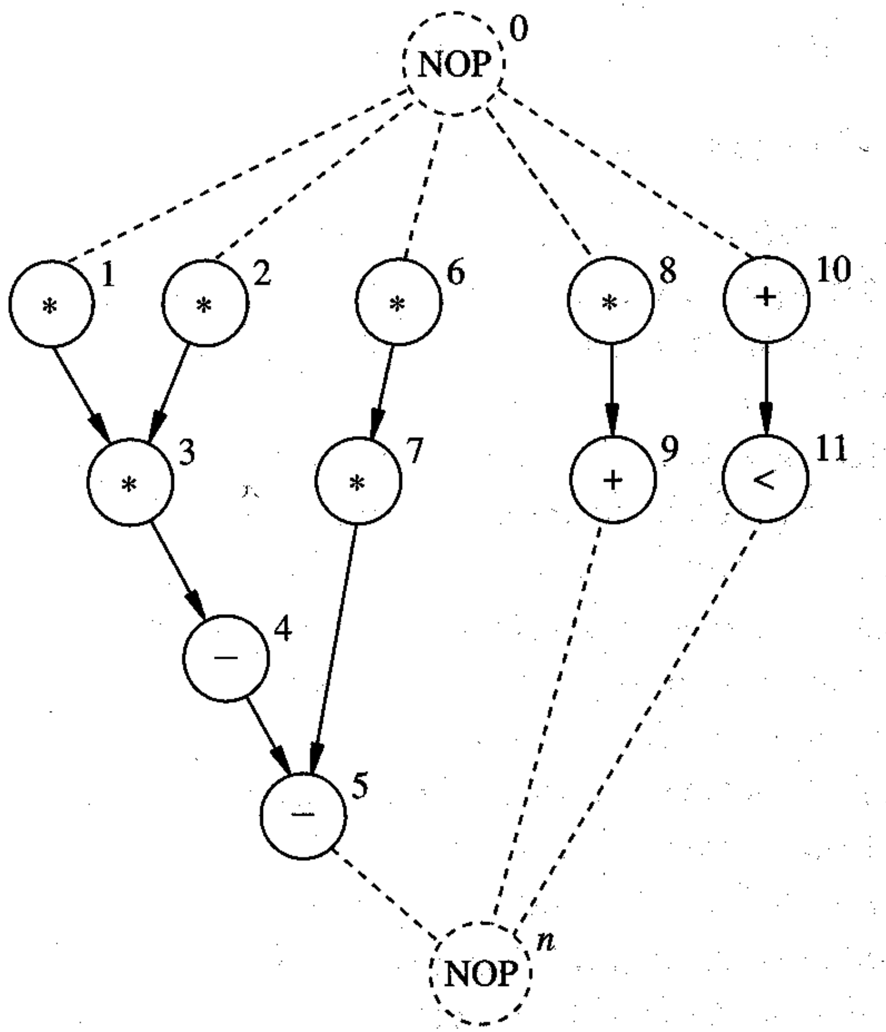
$$\sum_{i:\mathcal{T}(v_i)=k} \; \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda}+1$$

$$x_{il} \in \{0, 1\}, \quad i = 0, 1, \ldots, n, \quad l = 1, 2, \ldots, \bar{\lambda}+1$$

Example 1

- Resource Constraints
  - MULT: 2
  - ALU: 2
    - Adder, Subtraction
    - Comparator
- Each take 1 cycle of execution time

- Assume upper bound on latency, L = 4
- Use ALAP and ASAP to derive bounds on start times for each operator



Example 1 (cont'd.)

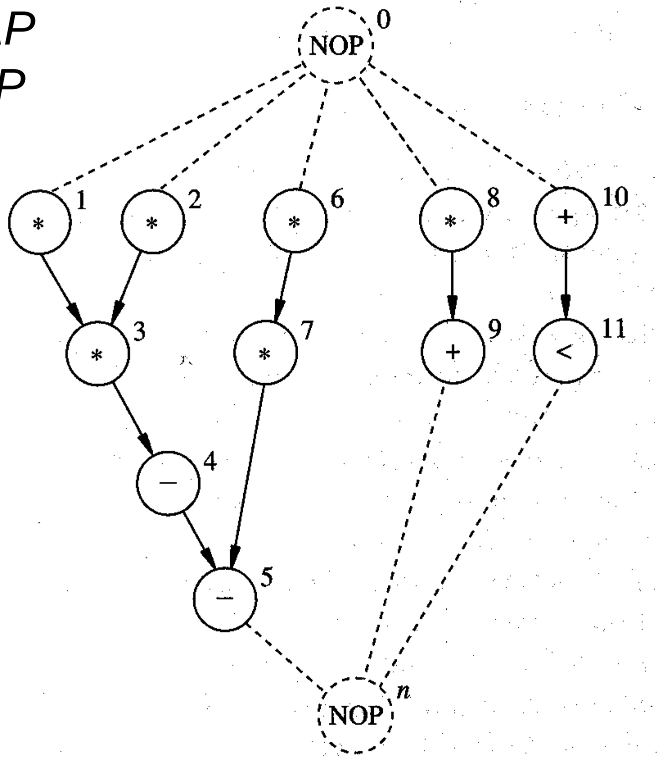
- Start time must be unique

Recall: $\sum_l x_{il} = \sum_{l=t_i^S}^{l=t_i^L} x_{il}$

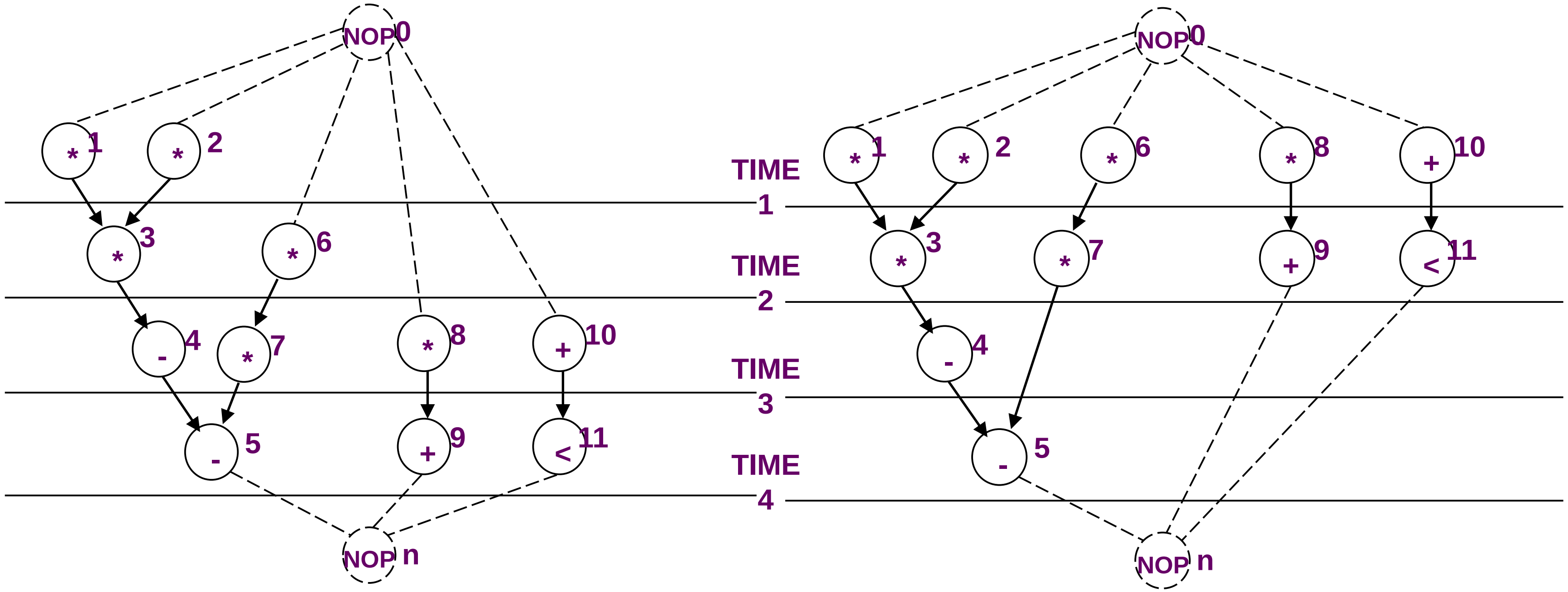
where:

$t_i^S = t_i$ computed with ASAP

$t_i^L = t_i$ computed with ALAP

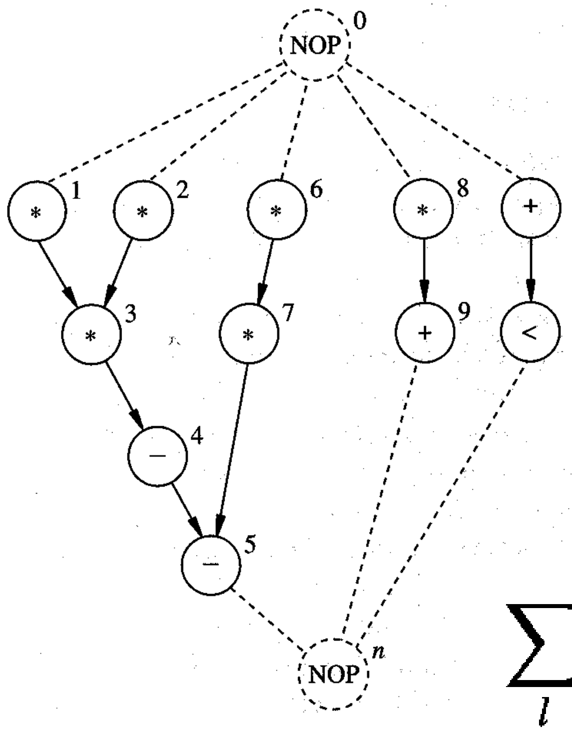


$$
\begin{aligned}
x_{0,1} &= 1\\
x_{1,1} &= 1\\
x_{2,1} &= 1\\
x_{3,2} &= 1\\
x_{4,3} &= 1\\
x_{5,4} &= 1\\
x_{6,1} + x_{6,2} &= 1\\
x_{7,2} + x_{7,3} &= 1\\
x_{8,1} + x_{8,2} + x_{8,3} &= 1\\
x_{9,2} + x_{9,3} + x_{9,4} &= 1\\
x_{10,1} + x_{10,2} + x_{10,3} &= 1\\
x_{11,2} + x_{11,3} + x_{11,4} &= 1\\
x_{n,5} &= 1
\end{aligned}
$$


NOP 0
1
2
3
4
5
6
7
8
9
10
11
NOP n
TIME 1
TIME 2
TIME 3
TIME 4
NOP 0
1
2
3
4
5
6
7
8
9
10
11
NOP n

Example 1 (cont'd.)

- Precedence constraints
  - Note: only non-trivial ones listed

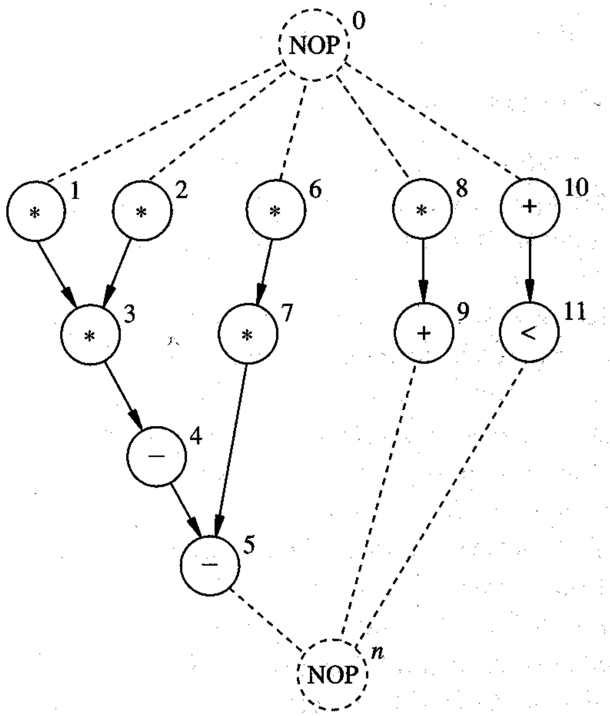


$$
\begin{aligned}
2x_{7,2} + 3x_{7,3} - x_{6,1} - 2x_{6,2} - 1 &\geq 0 \\
2x_{9,2} + 3x_{9,3} + 4x_{9,4} - x_{8,1} - 2x_{8,2} - 3x_{8,3} - 1 &\geq 0 \\
2x_{11,2} + 3x_{11,3} + 4x_{11,4} - x_{10,1} - 2x_{10,2} - 3x_{10,3} - 1 &\geq 0 \\
4x_{5,4} - 2x_{7,2} - 3x_{7,3} - 1 &\geq 0 \\
5x_{n,5} - 2x_{9,2} - 3x_{9,3} - 4x_{9,4} - 1 &\geq 0 \\
5x_{n,5} - 2x_{11,2} - 3x_{11,3} - 4x_{11,4} - 1 &\geq 0
\end{aligned}
$$

$$\sum_l l \cdot x_{il} \geq \sum_l l \cdot x_{jl} + d_j, \quad i, j = 0, 1, \ldots, n \quad : (v_j, v_i) \in E$$

Example 1 (cont'd.)

- Resource constraints



MULT
a1=2

$$\begin{cases} x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} \leq 2 \\ x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} \leq 2 \\ x_{7,3} + x_{8,3} \leq 2 \end{cases}$$

ALU
a2=2

$$\begin{cases} x_{10,1} \leq 2 \\ x_{9,2} + x_{10,2} + x_{11,2} \leq 2 \\ x_{4,3} + x_{9,3} + x_{10,3} + x_{11,3} \leq 2 \\ x_{5,4} + x_{9,4} + x_{11,4} \leq 2 \end{cases}$$

$$\sum_{i:\mathcal{T}(v_i)=k} \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

Example 1 (cont'd.)

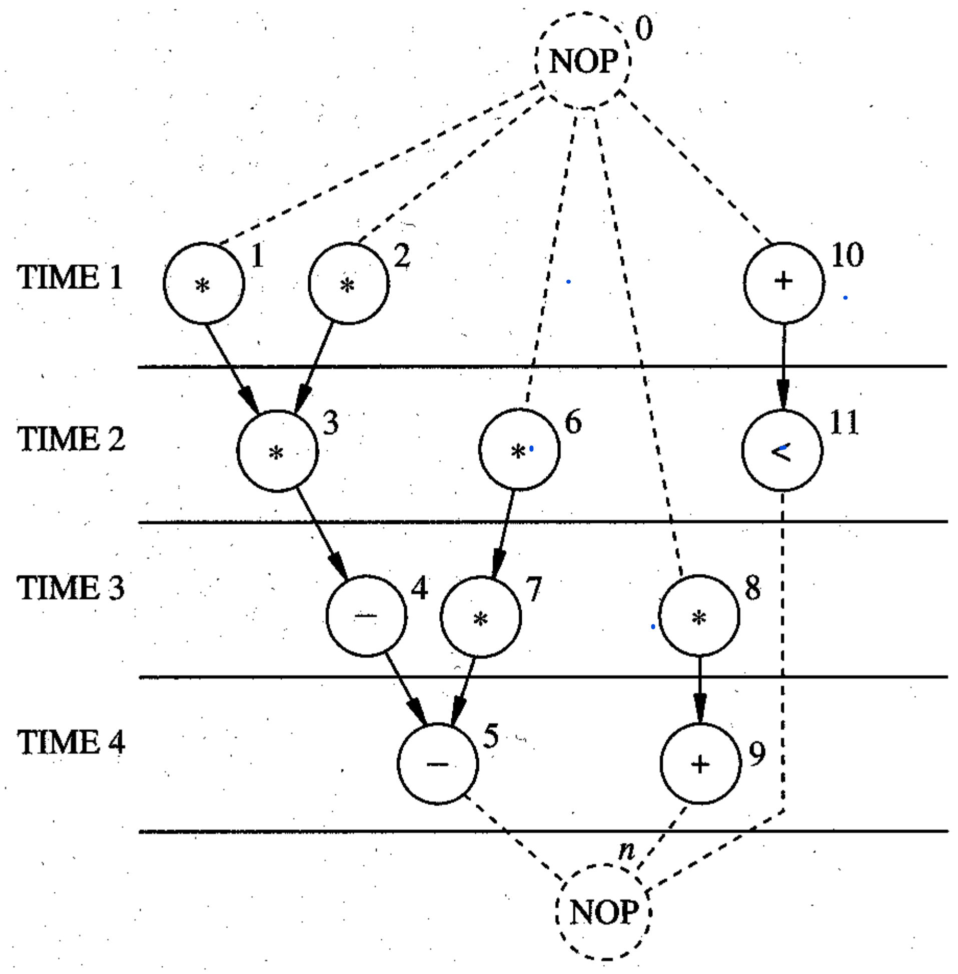
- Objective function for MLRC: $F = c^T t$
- $F1$: $\mathbf{c} = [0, 0, \ldots, 1]^T$
  - Minimum latency schedule
  - When $\lambda = 4$, since sink has no mobility ($x_{n,5} = 1$), any feasible schedule is optimum.

Example Solution 1:

Min. Latency ~~Schedule~~ Under Resource Constraint

ALU = 2

MUL = 2

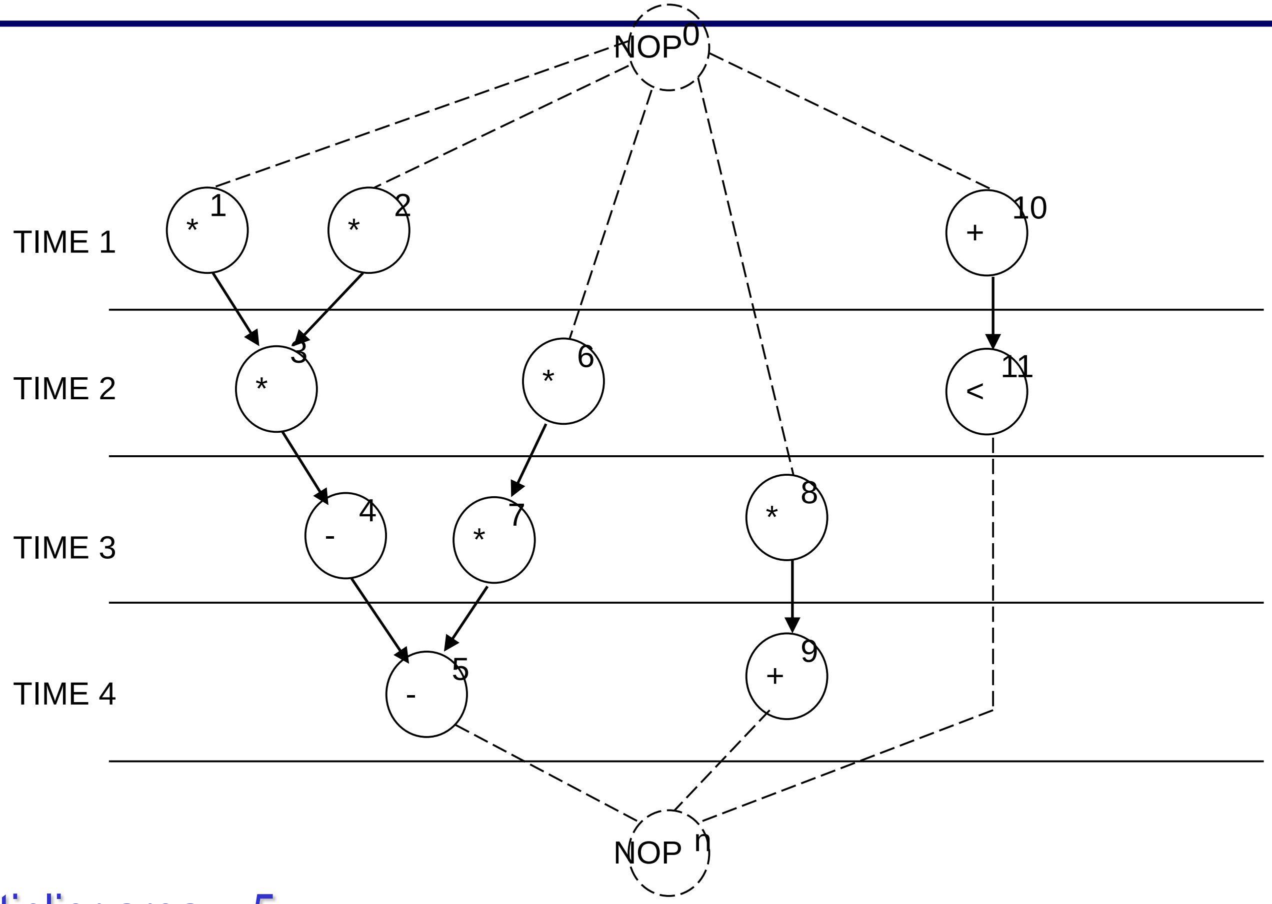


ILP formulation of MRLC Scheduling

- Minimize resource usage under latency constraint
- Dual of th MLRC problem.
- Resource usage is unknown here.
- The optimization goal is a weighted sum of the resource usage represented by vector **a**.
- Hence the objective function can be expressed as scalar product between a vector whose entries are the individual resource (area) costs and the vector **a**.
  - Minimize $C^T a$,

C in R^{nres} is a vector which entry is individual resource cost.

Example



- Multiplier area = 5 .
- ALU area = 1.
- Objective function: $5a_1 + a_2$

ILP formulation of MRLC Scheduling - Constraints

- Minimize $C^T a$ such that
- Unique start time $\sum_l x_{il} = 1, \quad i = 0, 1, \ldots, n$
- Precedence constraints

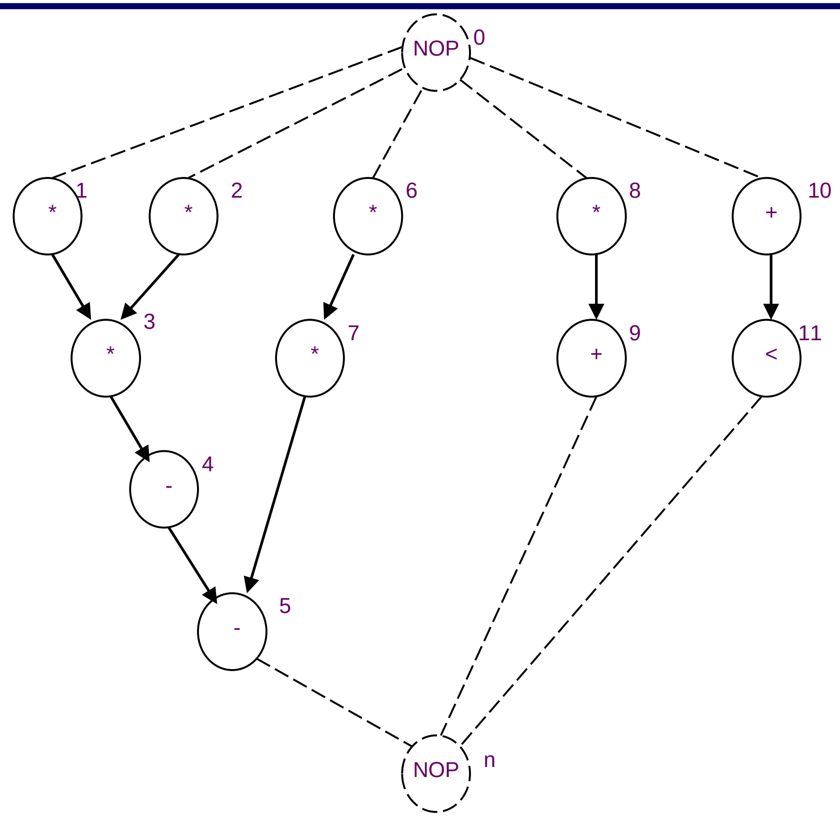
$$\sum_l l \cdot x_{il} \geq \sum_l l \cdot x_{jl} + d_j, \quad i, j = 0, 1, \ldots, n \quad : (v_j, v_i) \in E$$

- Resource constraints (a_k is also unknown here)

$$\sum_{i:\mathcal{T}(v_i)=k} \sum_{m=l-d_i+1}^{l} x_{im} \leq a_k, \quad k = 1, 2, \ldots, n_{res}, \quad l = 1, 2, \ldots, \bar{\lambda} + 1$$

- Latency bound must be satisfied
  - $\Sigma_l \, l \, x_{nl} \leq \lambda + 1$

Example

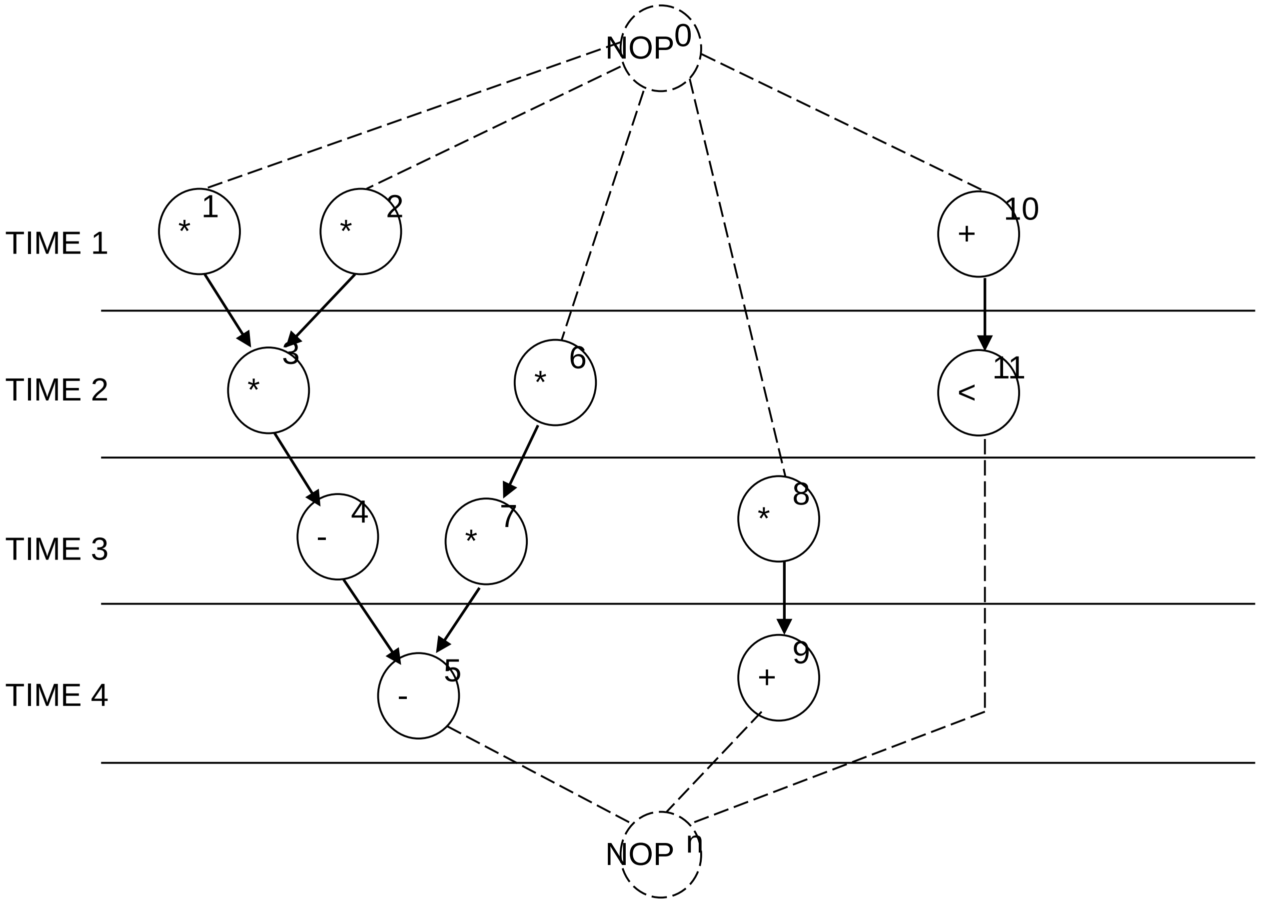


- Multiplier area = 5
- ALU area = 1.
- Latency = 4
- Objective function: $5a_1 + a_2$

Example

The objective function to minimize is $\mathbf{c}^T\mathbf{a} = 5 \cdot a_1 + 1 \cdot a_2$.

$$
\begin{aligned}
x_{1,1} + x_{2,1} + x_{6,1} + x_{8,1} - a_1 &\leq 0 \\
x_{3,2} + x_{6,2} + x_{7,2} + x_{8,2} - a_1 &\leq 0 \\
x_{7,3} + x_{8,3} - a_1 &\leq 0 \\
x_{10,1} - a_2 &\leq 0 \\
x_{9,2} + x_{10,2} + x_{11,2} - a_2 &\leq 0 \\
x_{4,3} + x_{9,3} + x_{10,3} + x_{11,3} - a_2 &\leq 0 \\
x_{5,4} + x_{9,4} + x_{11,4} - a_2 &\leq 0
\end{aligned}
$$


NOP 0
TIME 1
* 1
* 2
+ 10
TIME 2
* 3
* 6
< 11
TIME 3
- 4
* 7
* 8
TIME 4
- 5
+ 9
NOP n

ILP Solution

- The ILP formulation of the constrained scheduling problems is attractive for three major reasons.
- First it provides an exact solution to the scheduling problems.
- Second, general-purpose software packages can be used to solve the ILP.
- Last, additional constraints and problem extensions (e.g. scheduling pipelined circuits) can be easily incorporated.
- The disadvantage of the ILP formulation is the computational complexity of the problem.
- The number of variables, the number of inequalities and their tightness affect the ability of computer programs to find a solution.

Thank You