Functions and Potency of Sets (infinity and beyond . . .)

Apurba Sarkar

Indian Institute of Engineering Science and Technology, Shibpur

September 29, 2015

Introduction

- If f is a relation between A and B then an element x of A may be related to one element or no element or many elements of B by f.
- A relation with the property that **each** element x of A is related to exactly one element y of B is said to be *mapping from* A to B.
- A relation f is a subset of $A \times B$. f is a mapping if each element x of A appears exactly once as the first element of the ordered pairs of f.

Definitions

Definition 1

Let A *and* B *be two non-empty sets. A mapping* f *from* A *to* B *is a rule that assigns to each element* x *of* A *a definite element* y *in* B.
A *is said to be the domain of* f *and* B *is said to be the co-domain of* f *and the mapping from* A *to* B *is denoted by* $f : \mathrm{A} \to \mathrm{B}$.

Definitions

Definition 1

Let A *and* B *be two non-empty sets. A mapping f from* A *to* B *is a rule that assigns to each element x of* A *a definite element y in* B.
A *is said to be the domain of f and* B *is said to be the co-domain of f and the mapping from* A *to* B *is denoted by* $f : \mathrm{A} \to \mathrm{B}$.

Definition 2

Let $f : \mathrm{A} \to \mathrm{B}$ be mapping, Then the unique element y of B *that corresponds to x is called the f-image of x.*
The set of all f-images, i.e., the set $\{f(x) \mid x \in \mathrm{A}\}$ is denoted by $f(\mathrm{A})$ and is said to be the image set or the range set of f.

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
   2. $f_2 = \{(1, a), (2, b), (3, c)\}$

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
   2. $f_2 = \{(1, a), (2, b), (3, c)\}$
   3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
   2. $f_2 = \{(1, a), (2, b), (3, c)\}$
   3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$
   4. $f_4 = \{(1, b), (2, c), (3, d), (4, a)\}$

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
   2. $f_2 = \{(1, a), (2, b), (3, c)\}$
   3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$
   4. $f_4 = \{(1, b), (2, c), (3, d), (4, a)\}$
2. Let f is a relation between $\mathbb{R}$ and $\mathbb{R}$ such that x is related to y iff $y = \frac{1}{x}$, $x, y \in \mathbb{R}$.

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
    1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
    2. $f_2 = \{(1, a), (2, b), (3, c)\}$
    3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$
    4. $f_4 = \{(1, b), (2, c), (3, d), (4, a)\}$
2. Let f is a relation between $\mathbb{R}$ and $\mathbb{R}$ such that x is related to y iff $y = \frac{1}{x}$, $x, y \in \mathbb{R}$.
   Every element x, other than 0 is related to a unique element y of $\mathbb{R}$.

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
   1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
   2. $f_2 = \{(1, a), (2, b), (3, c)\}$
   3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$
   4. $f_4 = \{(1, b), (2, c), (3, d), (4, a)\}$
2. Let f is a relation between $\mathbb{R}$ and $\mathbb{R}$ such that x is related to y iff $y = \frac{1}{x}$, $x, y \in \mathbb{R}$.
   Every element x, other than 0 is related to a unique element y of $\mathbb{R}$.
   $\therefore f$ is not a function.

Examples

1. Let $S = \{1, 2, 3, 4\}$, $T\{a, b, c, d\}$. Which all of the following are function?
    1. $f_1 = \{(1, a), (1, b), (2, c), (3, c), (4, d)\}$
    2. $f_2 = \{(1, a), (2, b), (3, c)\}$
    3. $f_3 = \{(1, b), (2, b), (3, c), (4, d)\}$
    4. $f_4 = \{(1, b), (2, c), (3, d), (4, a)\}$
2. Let f is a relation between $\mathbb{R}$ and $\mathbb{R}$ such that x is related to y iff $y = \frac{1}{x}$, $x, y \in \mathbb{R}$.
   Every element x, other than 0 is related to a unique element y of $\mathbb{R}$.
   $\therefore$ f is not a function.
   What about $\mathbb{R} - \{0\}$?

Relation vs function

- Let A and B be two sets, then
  1. A relation ρ between A and B is a subset of cartesian product of A and B, i.e. $\rho \subseteq A \times B$.
  2. A function f between A and B is also a subset of cartesian product of A and B. i.e. $f : A \to B \subseteq A \times B$.

Relation vs function

- Let A and B be two sets, then
  1. A relation ρ between A and B is a subset of cartesian product of A and B, i.e. $\rho \subseteq A \times B$.
  2. A function f between A and B is also a subset of cartesian product of A and B. i.e. $f : A \to B \subseteq A \times B$.

Whats the big deal then?

Relation vs function

- Let A and B be two sets, then
  1. A relation ρ between A and B is a subset of cartesian product of A and B, i.e. $\rho \subseteq A \times B$.
  2. A function f between A and B is also a subset of cartesian product of A and B. i.e. $f : A \to B \subseteq A \times B$.

Whats the big deal then?

All functions are relation but not all relations are function.

Definitions

Definition 3

A mapping $f : \mathrm{A} \to \mathrm{B}$ *is said to be an into mapping if* $f(\mathrm{A})$ *is a proper subset of* B*. In this case we say* A *maps into* B

Definitions

Definition 3

A mapping $f : \mathrm{A} \to \mathrm{B}$ *is said to be an into mapping if* $f(\mathrm{A})$ *is a proper subset of* B. *In this case we say* A *maps into* B

Definition 4

A mapping $f : \mathrm{A} \to \mathrm{B}$ *is said to be an onto mapping if* $f(\mathrm{A}) = \mathrm{B}$. *In this case we say* A *maps onto* B

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$. Then $f(\mathbb{Z})$ (the set of all even integers) is a proper subset of $\mathbb{Z}$.

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$. Then $f(\mathbb{Z})$ (the set of all even integers) is a proper subset of $\mathbb{Z}$.

2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) =| x |$, $x \in \mathbb{Z}$ Then $f(\mathbb{Z})$ (the set of all non-negative integers) is a proper subset of $\mathbb{Z}$.

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$. Then $f(\mathbb{Z})$ (the set of all even integers) is a proper subset of $\mathbb{Z}$.
2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = \mid x \mid$, $x \in \mathbb{Z}$ Then $f(\mathbb{Z})$ (the set of all non-negative integers) is a proper subset of $\mathbb{Z}$.
3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$, $x \in \mathbb{Z}$. Then every element y in the co-domain set $\mathbb{Z}$ has a pre-image $y - 1$ in the domain set $\mathbb{Z}$.

Observations

1. Let $f : A \rightarrow B$ be a mapping. It may happen that an element $y \in B$ has *one* pre-image, *no* pre-image or *many* pre-images in A

Observations

1. Let $f : \mathrm{A} \to \mathrm{B}$ be a mapping. It may happen that an element $y \in \mathrm{B}$ has *one* pre-image, *no* pre-image or *many* pre-images in A
2. In Example 2, 0 in the co-domain set $\mathbb{Z}$ has only one pre-image in the domain set; 1 in the co-domain set $\mathbb{Z}$ has two pre-images in the domain set. -2 in the co-domain set has no pre-image in the domain set.

Observations

1. Let $f : A \to B$ be a mapping. It may happen that an element $y \in B$ has *one* pre-image, *no* pre-image or *many* pre-images in A
2. In Example 2, 0 in the co-domain set $\mathbb{Z}$ has only one pre-image in the domain set; 1 in the co-domain set $\mathbb{Z}$ has two pre-images in the domain set. -2 in the co-domain set has no pre-image in the domain set.
3. Thus the pre-images of an element y in B form a subset of B, which may be the null set, or a singleton set, or a set containing more than one elements.

Definitions

Definition 5

A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be injective (or one-to-one) if for each pair of distinct elements of A, their f-images are distinct.

Definitions

Definition 5

A mapping $f : \mathrm{A} \to \mathrm{B}$ *is said to be injective (or one-to-one) if for each pair of distinct elements of* A*, their* f*-images are distinct.*

Definition 6

A mapping $f : \mathrm{A} \to \mathrm{B}$ *is said to be surjective (or onto)if* $f(\mathrm{A}) = \mathrm{B}$.

Definitions

Definition 5

A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be injective (or one-to-one) if for each pair of distinct elements of A, their f-images are distinct.

Definition 6

A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be surjective (or onto)if $f(\mathrm{A}) = \mathrm{B}$.

Definition 7

A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be bijective if f is both injective and surjective.

Observations

1. The mapping $f : \mathrm{A} \to \mathrm{B}$ is *injective* if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B. So, if f is injective, each element of B has *at most* one pre-image. It also means that $|\mathrm{A}| \leq |\mathrm{B}|$.

Observations

1. The mapping $f : \mathrm{A} \to \mathrm{B}$ is *injective* if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B. So, if f is injective, each element of B has *at most* one pre-image. It also means that $|\mathrm{A}| \leq |\mathrm{B}|$.
2. If $f : \mathrm{A} \to \mathrm{B}$ is *surjective* each element of B has *at least* one pre-image. So, it implies $|\mathrm{A}| \geq |\mathrm{B}|$.

Observations

1. The mapping $f : \mathrm{A} \to \mathrm{B}$ is *injective* if $x_1 \neq x_2$ in A implies $f(x_1) \neq f(x_2)$ in B. So, if f is injective, each element of B has *at most* one pre-image. It also means that $|\mathrm{A}| \leq |\mathrm{B}|$.
2. If $f : \mathrm{A} \to \mathrm{B}$ is *surjective* each element of B has *at least* one pre-image. So, it implies $|\mathrm{A}| \geq |\mathrm{B}|$.
3. If $f : \mathrm{A} \to \mathrm{B}$ is *bijective* each element of B has *exactly* one pre-image. So, it implies $|\mathrm{A}| = |\mathrm{B}|$.

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$.

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$.
2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = | x |$, $x \in \mathbb{Z}$.

Examples

1. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$, $x \in \mathbb{Z}$.
2. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = | x |$, $x \in \mathbb{Z}$.
3. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x + 1$, $x \in \mathbb{Z}$.

Special mappings

1. A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be a *constant mapping* if f maps each element of A to one and the same element of B i.e., $f(\mathrm{A})$ is a singleton set. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$, $x \in \mathbb{R}$.

Special mappings

1. A mapping $f : \mathrm{A} \to \mathrm{B}$ is said to be a *constant mapping* if f maps each element of A to one and the same element of B i.e., $f(\mathrm{A})$ is a singleton set. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2$, $x \in \mathbb{R}$.

2. A mapping $f : \mathrm{A} \to \mathrm{A}$ is said to be the *identity mapping* on A if $f(x) = x$, $x \in \mathrm{A}$. The identity mapping on A is denoted by i_A. The mapping i_A is obviously a bijective mapping from A to B.

Equality of mappings

Two mappings $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{A} \to \mathrm{C}$ are said to be equal if $f(x) = g(x)$ for all $x \in \mathrm{A}$. For the equality of two mappings f and g the following conditions must hold.

1. f and g must have the same domain D.

Equality of mappings

Two mappings $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{A} \to \mathrm{C}$ are said to be equal if $f(x) = g(x)$ for all $x \in \mathrm{A}$. For the equality of two mappings f and g the following conditions must hold.

1. f and g must have the same domain D.
2. for all $x \in \mathrm{D}$, $f(x) = g(x)$.

Restriction of mappings

- Let $f : \mathrm{A} \to \mathrm{B}$ be mapping and let D be a non-empty subset of A. Then the mapping $g : \mathrm{D} \to \mathrm{B}$ defined by $g(x) = f(x)$, $x \in \mathrm{D}$ is said to be the restriction of f to D.
- f is said to be an *extension* of g to A. An extension $f : \mathrm{A} \to \mathrm{B}$ of the mapping $f : \mathrm{D} \to \mathrm{B}$ is not unique. Since the f – *images* of the elements of $\mathrm{A} - \mathrm{D}$ may be arbitrarily chosen.
- it is possible for a non-bijective mapping f to have bijective restriction of f.

Composition of mappings

- Let $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{C} \to \mathrm{D}$ be two mappings such that $f(\mathrm{A})$ is a subset of C. Let $x \in \mathrm{A}$. Then f maps x to an element $y \in f(\mathrm{A}) \subset \mathrm{B}$ and since $y \in f(\mathrm{A}) \subset \mathrm{C}$, g maps y to an element z in D.
- A mapping $h : \mathrm{A} \to \mathrm{D}$ can be defined by $h(x) = g(f(x))$, $x \in \mathrm{A}$. The mapping $h : \mathrm{A} \to \mathrm{D}$ is said to be *composite* (or the *product*) of f and g and is denoted by $g \circ f$.
- The *composite* $g \circ f : \mathrm{A} \to \mathrm{D}$ is defined only if $f(\mathrm{A})$ is a subset of the domain of g.

Composition of mappings

- For the mapping $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ the composite $g \circ f : \mathrm{A} \to \mathrm{C}$ is defined.
- For the mapping $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{A}$ both the composites $g \circ f : \mathrm{A} \to \mathrm{C}$ and $f \circ g : \mathrm{B} \to \mathrm{B}$ are defined.

Examples

1. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g(x) = 3x$, $x \in \mathbb{R}$. What about $g \circ f$ and $f \circ g$?

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x + 1$, $x \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x + 5$, $x \in \mathbb{R}$. What about $g \circ f$ and $f \circ g$?

Observation

1. The composition of mappings is not commutative. That is for two mappings f and g their composites $g \circ f$ and $f \circ g$ are, in general, not equal.

2. One of them may be defined the other may bot be defined at all.

3. The composition of mappings is associative. That is for three mappings f, g and h, $h \circ (g \circ f) = (h \circ g) \circ f$, when both sides are defined mappings.

Few theorems on compositions

Theorem 1

Let $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$ *and* $h : \mathrm{C} \to \mathrm{D}$ *be three mappings. Then* $h \circ (g \circ f) = (h \circ g) \circ f$.

Few theorems on compositions

Theorem 1

Let $f : \mathrm{A} \to \mathrm{B}$, $g : \mathrm{B} \to \mathrm{C}$ and $h : \mathrm{C} \to \mathrm{D}$ be three mappings. Then $h \circ (g \circ f) = (h \circ g) \circ f$.

Proof.

Here the composite mappings $g \circ f$, $h \circ g$ are defined.(why?...)
The composite mappings $h \circ (g \circ f)$ and $(h \circ g) \circ f$ are also defined (why? ...) $\square$

Few theorems on compositions

Theorem 2

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ are both injective mappings then the composite mapping $g \circ f : \mathrm{A} \to \mathrm{C}$ is injective.

Few theorems on compositions

Theorem 2

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ are both injective mappings then the composite mapping $g \circ f : \mathrm{A} \to \mathrm{C}$ is injective.

Proof.

Let x_1 and x_2 be two distinct elements of A
Let $f(x_1) = y_1$, $f(x_2) = y_2$. Since f is injective, y_1 and y_2 are distinct elements of B.
Let $g(y_1) = z_1$, $g(y_2) = z_2$. Since g is injective, z_1 and z_2 are distinct elements of C.
Now $g \circ f(x_1) = z_1$, $g \circ f(x_2) = z_2$ and $x_1 \neq x_2 \Rightarrow z_1 \neq z_2$.
$\therefore g \circ f$ is injective. $\square$

What about the converse of the theorem?

Few theorems on compositions

Theorem 3

If $f : \mathrm{A} \to \mathrm{B}$ *and* $g : \mathrm{B} \to \mathrm{C}$ *be two mappings such that* $g \circ f : \mathrm{A} \to \mathrm{C}$ *is injective then* f *is injective.*

Few theorems on compositions

Theorem 3

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be two mappings such that $g \circ f : \mathrm{A} \to \mathrm{C}$ is injective then f is injective.

Proof.

If possible, let f be not injective.
Then there exist two distinct elements x_1, x_2 in A such that $f(x_1) = f(x_2)$. $\therefore g \circ f(x_1) = g \circ f(x_2)$ and this contradicts that $g \circ f$ is injective.
$\therefore f$ is injective. $\square$

Few theorems on compositions

Example 1

Let $f : \mathbb{R} \to \mathbb{R}$ *be defined by* $f(x) = e^x$, $x \in \mathbb{R}$ *and* $g : \mathbb{R} \to \mathbb{R}$ *be defined by* $g(x) = x^2$, $x \in \mathbb{R}$.
Clearly g *is not injective as* $g(2) = g(-2) = 4$.
Here $g \circ f : \mathbb{R} \to \mathbb{R}$ *is defined by* $g \circ f(x) = e^{2x}$, $x \in \mathbb{R}$
$g \circ f$ *is injective but* g *is not injective.*

Few theorems on compositions

Theorem 4

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be both surjective then the composite mappings $g \circ f : \mathrm{A} \to \mathrm{C}$ is surjecctive.

Proof.

Let z be an element of C. Since g is surjective, there is atleast one pre-image of z in B. Let one such pre-image be y. Then $y \in \mathrm{B}$ and $g(y) = z$.
Since f is surjective and $y \in \mathrm{B}$, there is atleast one pre-image of y in A. Let one such be x. Then $x \in \mathrm{A}$ and $f(x) = y$.
$g \circ f(x) = g(y) = z$
This implies that z has a pre-image in A under the mapping $g \circ f$.
Since z is arbitrary, $g \circ f$ is surjective. □

What about the converse of the theorem?

Few theorems on compositions

Theorem 5

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be two mappings such that $g \circ f : \mathrm{A} \to \mathrm{C}$ is surjective then g is surjective.

Theorem 5

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be two mappings such that $g \circ f : \mathrm{A} \to \mathrm{C}$ is surjective then g is surjective.

Proof.

Let z be an element of C. Since $g \circ f$ is surjective, there is an element element x in A such that $g \circ f(x) = z$.
$\therefore gf(x) = z$.
This shows that z has a pre-image $f(x)$ in B under the mapping g.
Since z is arbitrary, g is surjective. □

Note: In order that $g \circ f$ may be surjective it is not necessary that f is surjective.

Few theorems on compositions

Example 2

Let $f : \mathbb{Z} \to \mathbb{Z}$ *be defined by* $f(x) = 2x$, $x \in \mathbb{Z}$ *and* $g : \mathbb{Z} \to \mathbb{Z}$ *be defined by* $g(x) = \lfloor \frac{x}{2} \rfloor$, $x \in \mathbb{Z}$. $\lfloor \frac{x}{2} \rfloor$ *denotes the greatest integer* $\leq x$

Then $g \circ f : \mathbb{Z} \to \mathbb{Z}$ *is defined by* $g \circ f(x) = x$, $x \in \mathbb{Z}$

$g \circ f$ *is the identity mapping on* $\mathbb{Z}$ *and is, therefore, surjective; but* f *is not surjective.*

Few theorems on compositions

Theorem 6

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be be both bijective then the composite mapping $g \circ f : \mathrm{A} \to \mathrm{C}$ is bijective.

Proof.

Since both f and g are bijective, they both are injective as well as surjective.
Now, as f and g are each injective by theorem 2, the composite mapping $g \circ f$ is injective. similarly, as f and g are each surjective by theorem 4, the composite mapping $g \circ f$ is also surjective.
Since the composite mapping $g \circ f$ is injective as well as surjective, it is bijective. $\square$

What about the converse of the theorem?

Few theorems on compositions

Theorem 7

If $f : \mathrm{A} \to \mathrm{B}$ and $g : \mathrm{B} \to \mathrm{C}$ be two mappings such that $g \circ f : \mathrm{A} \to \mathrm{C}$ is bijective then f is injective and g is surjective.

Proof.

Since $g \circ f$ is bijective, it is injective as well as injective.
Now, for $g \circ f$ to be injective, f has to be injective (by theorem 3) and for $g \circ f$ to be surjective, g has to be surjective (by theorem 5).
Hence the theorem. □

Note: In order that $g \circ f$ may be bijective it is neither necessary that f is surjective nor necessary that g is injective.

Definitions

Definition 8

Let $f : \mathrm{A} \to \mathrm{B}$ be a mapping. If there exists a mapping $g : \mathrm{B} \to \mathrm{A}$ such that $g \circ f = i_A$ then g is said to be a left inverse of f. If there exists a mapping $h : \mathrm{B} \to \mathrm{A}$ such that $f \circ h = i_B$ such then h is said to be a right inverse of f

Definitions

Definition 8

Let $f : \mathrm{A} \to \mathrm{B}$ *be a mapping. If there exists a mapping* $g : \mathrm{B} \to \mathrm{A}$ *such that* $g \circ f = i_A$ *then* g *is said to be a left inverse of* f*. If there exists a mapping* $h : \mathrm{B} \to \mathrm{A}$ *such that* $f \circ h = i_B$ *such then* h *is said to be a right inverse of* f

Definition 9

Let $f : \mathrm{A} \to \mathrm{B}$ *be a mapping.* f *is said to be invertible if there exist a mapping* $g : \mathrm{B} \to \mathrm{A}$ *such that* $g \circ f = i_A$ *and* $f \circ g = i_B$ *and* g *is said to be an inverse of* f*.*

Few theorems on compositions

Theorem 8

If $f : \mathrm{A} \to \mathrm{B}$ be an invertible mapping then its inverse is unique.

Proof.

Since $f : \mathrm{A} \to \mathrm{B}$ is invertible, there exist a mapping $g : \mathrm{B} \to \mathrm{A}$ such that $g \circ f = i_A$ and $f \circ g = i_B$. If possible, let there exist another mapping $h : \mathrm{B} \to \mathrm{A}$ such that $h \circ f = i_A$ and $f \circ h = i_B$.
Since composition of mapping is associative,
$h \circ (f \circ g) = (h \circ f) \circ g$.
$\therefore h \circ i_B = i_A \circ g$.
or $h = g$ and hence the theorem. $\square$

Few theorems on compositions

Theorem 9

If $f : \mathrm{A} \to \mathrm{B}$ is an invertible mapping if and only if f is a bijection.

Proof.

$\Longrightarrow$
Let $f : \mathrm{A} \to \mathrm{B}$ be invertible, then there exist a mapping $g : \mathrm{B} \to \mathrm{A}$ such that $g \circ f = i_A$ and $f \circ g = i_B$.
Since i_A is is injective and $g \circ f = i_A$, f is injective.
Since i_B is is surjective and $f \circ g = i_B$, f is surjective. $\therefore f$ is bijective.
$\Longleftarrow$
Let $y \in B$. Since f is a bijection, y has one and only one pre-image x in A.
Define a mapping $g : \mathrm{B} \to \mathrm{A}$ by $g(y) = x$(the pre-image of y under f), $y \in \mathrm{B}$.
Then $g \circ f(x) = g(y) = x, x \in \mathrm{A}$
and $f \circ g(y) = f(x) = y, y \in \mathrm{B}$
$\therefore g \circ f = i_A$ and $f \circ g = i_B$.

$\therefore f$ is invertible. $\square$

Equipotent Set

Definition 10

A set A *is said to be equipotent with a set* B *if there exist a bijective mapping* f *from* A *to* B *and we write* $\mathrm{A} \sim \mathrm{B}$. *If* A *is equipotent with* B, *then* B *is equipotent with* A, *since the existence of a bijective mapping* $f : \mathrm{A} \to \mathrm{B}$ *implies the existence of the inverse mapping* $f^{-1} : \mathrm{B} \to \mathrm{A}$ *which is also bijective. Thus the two sets* A *and* B *are equipotent with each other.*

An equivalence relation involving potency

Consider the set S of all subsets of a universal set U.

1. Let $\mathrm{A} \in S$. $\mathrm{A} \sim \mathrm{A}$ because the identity mapping $i : \mathrm{A} \to \mathrm{A}$ is a bijective mapping. reflexive

2. Let $\mathrm{A}, \mathrm{B} \in S$ and $\mathrm{A} \sim \mathrm{B}$. Then there exist a bijective mapping $f : \mathrm{A} \to \mathrm{B}$. The inverse mapping $f^{-1} : \mathrm{B} \to \mathrm{A}$ is also a bijective mapping.
   $\therefore \mathrm{A} \sim \mathrm{B} \Rightarrow \mathrm{B} \sim \mathrm{A}$. symmetric

3. Let $\mathrm{A}, \mathrm{BC} \in S$ and $\mathrm{A} \sim \mathrm{B}$, $\mathrm{B} \sim \mathrm{C}$. Then there exist a bijective mapping $f : \mathrm{A} \to \mathrm{B}$ and another bijective mapping $g : \mathrm{B} \to \mathrm{C}$. The composite mapping $g \circ f : \mathrm{A} \to \mathrm{C}$ is also a bijective mapping. $\therefore \mathrm{A} \sim \mathrm{B}$ and $\mathrm{B} \sim \mathrm{C} \Rightarrow \mathrm{A} \sim \mathrm{C}$. transitive

$\therefore$ equipotence of subsets is an *equivalence* relation on S and so S is partitioned into classes of *equipotent* sets.

Few Observations

1. Two equipotent sets belonging to the same class are said to have the same potency or *cardinal number*.
2. The cardinal number of a set A is denoted by $card(\mathrm{A})$ $\therefore$ $card(\mathrm{A}) = card(\mathrm{B})$ if and only if A and B belong to the same class.
3. Two non-empty finite sets A and B belong to the same class if and only if they have same number of elements.
4. The cardinal number assigned assigned to the equipotence class of finite sets each with n elements is n.
5. The cardinal number assigned to the null set ϕ is 0.
6. The cardinal number of and infinite set is said to be a *transfinite cardinal number*
7. The cardinal number of set $\mathbb{N}$ is denoted by $\aleph_0$ (read aleph null)

Examples

1. The set $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is equipotent with the set $\mathbb{N}$, because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = \frac{1}{n}, n \in \mathbb{N}$ is a bijection.

2. The set $2\mathbb{Z}$ of all even integers is equipotent with the set $\mathbb{Z}$, because of the mapping $f : \mathbb{Z} \to 2\mathbb{Z}$ defined by $f(x) = 2x, x \in \mathbb{Z}$ is a bijection.

3. The set $\mathrm{A} = \{x \in \mathbb{R}/0 \leq x \leq 1\}$ is equipotent with the set $\mathrm{B} = \{x \in \mathbb{R}/0 \leq x \leq 3\}$ because the mapping $f : \mathrm{A} \to 2\mathrm{B}$ defined by $f(x) = 3x, x \in \mathrm{A}$ is a bijection.

Definitions

Definition 11

A set A is said to be enumerable (or denumerable) if A is equipotent with the set $\mathbb{N}$.
A set which is either finite or enumerable is said to be countable. Sometimes enumerable sets are called countably infinite sets.
A set which is not countable is said to be uncountable.

Observations

1. When a set is finite and contains n elements, its elements can described as $a_1, a_2, \ldots a_n$, the elements being indexed by the finite set $\{1, 2, \ldots n\}$.
2. When a set A is enumerable, there is a bijective mapping $f : \mathbb{N} \to \mathrm{A}$ and f assigns to each $n \in \mathbb{N}$ an element $f(n)$ in A. Thus the elements of A can be described as $f(1), f(2), f(3), \ldots f(n), \ldots$ or as $a_1, a_2, a_3, \ldots a_n, \ldots$ showing that the elements are indexed by the set $\mathbb{N}$.

Examples

1. The set $\mathbb{N}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = n, n \in \mathbb{N}$ is a bijection.

2. The set $S = \{2, 4, 6, 8, 10, \ldots\}$ is enumerable because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = 2n, n \in \mathbb{N}$ is a bijection.

3. The set The set $S = \{1^2, 2^2, 3^2, \ldots\}$ is enumerable because the mapping $f : \mathbb{N} \to S$ defined by $f(n) = n^2, n \in \mathbb{N}$ is a bijection.

4. The set $\mathbb{Z}$ is enumerable, because the mapping $f : \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = \begin{cases} \frac{1}{2}n & \text{if n is even} \\ \frac{1}{2}(1-n) & \text{if n is odd} \end{cases}$$

is a bijection.

Few theorems on Equipotent Sets

Theorem 10

Every infinite set contains an enumerable subset.

Proof.

Let A be an infinite set. Let us define a mapping $h : \mathbb{N} \to \mathrm{A}$ by
$h(1) =$ one element of A, say a_1.
$h(2) =$ one element of $\mathrm{A} - \{a_1\}$, say a_2.
$h(2) =$ one element of $\mathrm{A} - [\{a_1\} \cup \{a_2\}]$, say a_3.
$\ldots \qquad \ldots \qquad \ldots$
$h(n) =$ one element of $\mathrm{A} - [\{a_1\} \cup \{a_2\} \cup \ldots \{a_{n-1}\}]$, say a_n
Since A is infinite, $\mathrm{A} - [\{a_1\} \cup \{a_2\} \cup \ldots \{a_{n-1}\}]$ is infinite for all $n > 1$.
So $h(n)$ is well defined for all $n \in \mathbb{N}$.
Also for $p, q \in \mathbb{N}, p \neq q \Rightarrow h(p) \neq h(q)$
$\therefore h$ is an injective mapping from $\mathbb{N}$ into A.
Let B be the subset $\{h(1), h(2), \ldots, h(n), \ldots\}$ then $setB \subset \mathrm{A}$ and $h : \mathbb{N} \to \mathrm{B}$ is a bijection. Therefore, B is enumerable and hence the theorem. □

Theorem 11

An infinite subset of an enumerable set is enumerable.

Proof.

Let A be an enumerable set and B be an infinite subset of A. Since A is enumerable its elements can be described as $a_1, a_2, \dots, a_n, \dots$. B contains infinite number of a's and the suffixes of the elements of B form an infinite subset P of the set of all natural numbers.
P, being a subset of $\mathbb{N}$ contains a least element, say μ_1. Let $\mathrm{B}_1 = \mathrm{B} - \{a_{\mu_1}\}$. Then B_1 is an infinite set and the suffixes of the elements of B_1 form an infinite subset P_1 of $\mathbb{N}$. Therefore P_1 contains a least element, say μ_2. Let $\mathrm{B}_2 = \mathrm{B} - \{a_{\mu_1}, a_{\mu_2}\}$. Following the same argument with B_2, $\mathrm{B}_3, \dots$, we get the elements $a_{\mu_3}, \dots a_{\mu_n}, \dots$
Let us define a mapping $f : \mathbb{N} \to \mathrm{B}$ by
$f(1) = a_{\mu_1}, f(2) = a_{\mu_2}, \dots, f(n) = a_{\mu_n}, \dots$
Let $p, q \in \mathbb{N}$ and let $p < q$. Then $f(p) \in \{a_{\mu_1}, a_{\mu_2} \dots a_{\mu_{q-1}}\}$ and
$f(q) \in \mathrm{B}_{q-1} = \mathrm{B} - \{a_{\mu_1}, a_{\mu_2} \dots a_{\mu_{q-1}}\}$.
$\therefore f(q) \neq f(p)$. So f is injective.
Let us take an element $a_r \in \mathrm{B}$. Since r is a positive integer, there are at most $r-1$ elements in B whose suffixes are less than r. So, a_r is one of $f(1), f(2), \dots . f(r-1), f(r)$. So f is surjective.

Corollary

Corollary 12

A non-empty subset of an enumerable set is either finite or enumerable.

Corollary 13

A non-empty subset of a countable set is countable.

Few theorems on Equipotent Sets

Theorem 14

The union of a finite set and and an enumerable set is enumerable.

Proof.

Let A be an enumerable set and B be a finite set of containing ma elements $b_1, b_2, \ldots, b_m$.The elements of A can be described as $a_1, a_2, a_3, \ldots\ldots$

- **Case1** $\mathrm{A} \cap \mathrm{B} = \phi$
  Let us define a mapping $f : \mathbb{N} \to \mathrm{A} \cup \mathrm{B}$ by
  $$f(i) = b_i, i = 1, 2, \ldots, m$$
  $$f(m+i) = a_i, i = 1, 2, 3, \ldots$$
- **Case2** $\mathrm{A} \cap \mathrm{B} \neq \phi$
  Let $\mathrm{A}_1 = \mathrm{A} - \mathrm{B}$ Then $\mathrm{A}_1 \cup \mathrm{B} = \mathrm{A} \cup \mathrm{B}$ and $\mathrm{A}_1 \cap \mathrm{B} = \phi$
  Now A_1 is infinite subset of A and therefore A_1 is enumerable.
  $\therefore$ by case 1, $\mathrm{A}_1 \cup \mathrm{B}$ is enumerable and therefore $\mathrm{A} \cup \mathrm{B}$ is enumerable.

$\square$

Few theorems on Equipotent Sets

Theorem 15

The union of two enumerable sets is enumerable.

Proof.

Let $A = \{a_1, a_2, \ldots, a_n, \ldots\}$, $B = \{b_1, b_2, \ldots, a_n, \ldots\}$ be two enumerable sets.

- **Case1** $A \cap B = \phi$
  Let us define a mapping $f : \mathbb{N} \to A \cup B$ by
  $$f(n) = a_{(n+1)/2}, \text{ if } n \text{ is odd}$$
  $$= b_{n/2} \text{ if } n \text{ is even.}$$
  Then f is a bijection and so $A \cup B$ is enumerable.
- **Case2** $A \cap B \neq \phi$
  Let $A_1 = A$, $B_1 = B - A$ Then $A_1 \cup B_1 = A \cup B$ and $A_1 \cap B_1 = \phi$
  Now B_1 is non-empty subset of B and so it is either finite or enumerable.
  if B_1 is enumerable, $A_1 \cup B_1$ is enumerable, by case 1.
  If B_1 is finite, $A_1 \cup B_1$ is enumerable by theorem 14.
  $\therefore$ $A \cup B$ is enumerable.

□

Few theorems on Equipotent Sets

Theorem 16

The union of an enumerable number of enumerable sets is enumerable.

Proof.

Let $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n, \ldots$ be an enumerable family of enumerable sets.

$$
\begin{array}{rl}
\text{Let } \mathrm{A}_1 & = \{a_{11}, a_{12}, \ldots, a_{1n}, \ldots\} \\
\mathrm{A}_2 & = \{a_{21}, a_{22}, \ldots, a_{2n}, \ldots\} \\
\ldots & \quad \ldots \qquad \ldots \\
\mathrm{A}_n & = \{a_{n1}, a_{n2}, \ldots, a_{nn}, \ldots\} \\
\ldots & \quad \ldots \qquad \ldots
\end{array}
$$

Case1. Let $\mathrm{A}_i \cap \mathrm{A}_j = \phi$ for all i, j. Let

$$\mathrm{B} = \bigcup_{k=1}^{\infty} \mathrm{A}_k.$$

Each element of B is of type a_{mn} where $m, n \in \mathbb{N}$.

$\square$

Theorem 17

The union of an enumerable number of enumerable sets is enumerable.

continued.

Let us define a mapping $f : \mathrm{B} \to \mathbb{N}$ by $f(m, n) = 2^m 3^n$. f is injective.
Because for any two elements $a_{mn}, a_{pq} \in \mathrm{B}$,
$a_{mn} \neq a_{pq} \Rightarrow (m, n) \neq (p, q) \Rightarrow 2^m 3^n \neq 2^p 3^q$, $f(\mathrm{B})$ is a proper subset of $\mathbb{N}$, because there are elements in $\mathbb{N}$ (e.g. $5, 7, 13, \ldots$) which have no pre-image in B. Let $f(\mathrm{B}) = \mathbb{N}_1$.
Then $f : \mathrm{B} \to \mathrm{N}_1$ is a bijection. Sine N_1 is infinite subset of $\mathbb{N}$, it is enumerable. Thus B is equipotent with an enumerable set $\mathbb{N}$ and so B. is enumerable.
Case2. Let the sets $\{\mathrm{A}_i\}$ be not pairwise disjoint.
Let us define sets $\{\mathrm{B}_i\}$ such that
$\mathrm{B}_1 = \mathrm{A}_1, \mathrm{B}_2 = \mathrm{A}_2 - \mathrm{A}_1, \mathrm{B}_3 = \mathrm{A}_3 - (\mathrm{A}_1 \cup \mathrm{A}_2), \ldots$
$\mathrm{B}_k = \mathrm{A}_k - (\mathrm{A}_1 \cup \mathrm{A}_2 \cup \ldots \cup \mathrm{A}_{k-1}), \ldots\ldots$

$\square$

Theorem 18

The union of an enumerable number of enumerable sets is enumerable.

continued.

Then $\mathrm{B}_k \subset \mathrm{A}_k$ for all k,

$$\bigcup_{i=1}^{\infty} \mathrm{B}_i = \bigcup_{i=1}^{\infty} \mathrm{A}_i$$

and $\mathrm{B}_i \cap \mathrm{B}_j = \phi$ for all i, j.
Since $B_k \subset \mathrm{A}_k$, B_k is either empty or finite or enumerable.
$\therefore$

$$\bigcup_{i=1}^{\infty} \mathrm{A}_k$$

is enumerable. $\square$

Examples

1. The set of all positive rational number is enumerable. Prove it
2. The set of all rational number is enumerable. Prove it

Few theorems on Equipotent Sets

Theorem 19

The open interval $(0,1) = \{x \in \mathbb{R}/0 < x < 1\}$ *is not enumerable.*

continued.

$\square$