Solved Problems

1. You toss a fair coin three times. What is the probability of three heads?

Solution: Let us assume that the coin tosses are independent. $P(HHH) = P(H) \cdot P(H) \cdot P(H) = (0.5)^3 = 1/8$

2. A bag contains red and blue marbles. Two marbles are drawn without replacement. The probability of selecting a red marble and then a blue marble is 0.28. The probability of selecting a red marble on the first draw is 0.5. What is the probability of selecting a blue marble on the second draw, given that the first marble drawn was red?

Solution: P(Blue | Red) = P(Blue and Red) / P(Red) = 0.28/0.5 = 0.56

3. Suppose that five good fuses and two defective ones have been mixed up. To find the defective fuses, we test them one-by-one, at random and without replacement. What is the probability that we are lucky and find both of the defective fuses in the first two tests?

Solution: $P(D1 \cap D2) = P(D1) . P(D2|D1) = 2/7 * 1/6 = 1/21$

4. If two fair dice are rolled, what is the conditional probability that the first one lands on 6 given that the sum of the dice is i? (Compute for all values of i between 2 and 12.)

Solution:

P{6 |sumof7} = P{(6,1)}1/6 =1/6
P{6 |sumof8} = P{(6,2)} 5/36 =1/5
P{6 |sumof9} = P{(6,3)} 4/36 =1/4
P{6 |sumof10} = P{(6,4)} 3/36 =1/3
P{6 |sumof11} = P{(6,5)} 2/36 =1/2
P{6 |sum of 12} = 1

5. Consider an urn containing 12 balls, of which 8 are white. A sample of size 4 is to be drawn with replacement (without replacement). What is the conditional probability (in each case) that the first and third balls drawn will be white given that the sample drawn contains exactly 3 white balls?

Solution: In both cases the one black ball is equally likely to be in either of the 4 positions. Hence the answer is 1/2.

6. A total of 48 percent of the women and 37 percent of the men that took a certain "quit smoking" class remained nonsmokers for at least one year after completing the class. These people then attended a success party at the end of a year. If 62 percent of the original class was male, what percentage of the original class attended the party?

Solution:

Choose a random member of the class.
Let A be the event that this person attends the party and let W be the event that this person is a woman.
P(A) = .48(.38) + .37(.62) = .4118
Therefore, 41.18 percent of the class attended.

7. An investor owns shares in a stock whose present value is 25. She has decided that she must sell her stock if it goes either down to 10 or up to 40. If each change of price is either up 1 point with probability .55 or down 1 point with probability .45, and the successive changes are independent, what is the probability that the investor retires a winner?

Solution:

Let P_n be the probability that the investor retires a winner given that she owns stock whose present value is n, $10 \leq n \leq 40$. Note $P_{10} = 0$ and $P_{40} = 1$. By Bayes' formula,

$$\begin{aligned} P_n &= P(\text{winner}|\text{price goes up to } n+1)P(\text{price goes up}) \\ &\quad + P(\text{winner}|\text{price goes down to } n-1)P(\text{price goes down}) \\ &= P_{n+1} \cdot .55 + P_{n-1} \cdot .45. \end{aligned}$$

Let's try to solve for the difference $P_{n+1} - P_n$. From the equation

$$P_n = P_{n+1} \cdot .55 + P_{n-1} \cdot .45$$

we get that

$$.45(P_n - P_{n-1}) = .55(P_{n+1} - P_n).$$

Rearranging, we get

$$P_{n+1} - P_n = (.45/.55)(P_n - P_{n-1}).$$

Hence

$$P_{n+1} - P_n = (.45/.55)(P_n - P_{n-1}) = (.45/.55)^{n-10}(P_{11} - P_{10}) = (.45/.55)^{n-10} P_{11}$$

since $P_{10} = 0$.

To compute P_{11}, note that

$$1 = P_{40} - P_{10} = \sum_{n=10}^{39} (P_{n+1} - P_n) = \sum_{n=10}^{39} (.45/.55)^{n-10} P_{11} = \frac{1 - (.45/.55)^{30}}{1 - .45/.55} P_{11},$$

where the last equality comes from the sum of a geometric series. So

$$P_{11} = \frac{1 - .45/.55}{1 - (.45/.55)^{30}}.$$

Thus P_{25} is given by

$$\begin{aligned} P_{25} &= P_{25} - P_{10} \\ &= \sum_{n=10}^{24} (P_{n+1} - P_n) \\ &= \sum_{n=10}^{24} (.45/.55)^{n-10} P_{11} \\ &= \frac{1 - (.45/.55)^{15}}{1 - .45/.55} P_{11} \\ &= \frac{1 - (.45/.55)^{15}}{1 - .45/.55} \cdot \frac{1 - .45/.55}{1 - (.45/.55)^{30}} \\ &= \frac{1 - (.45/.55)^{15}}{1 - (.45/.55)^{30}} \\ &\approx 0.9530. \end{aligned}$$

8. Three cooks, A, B, and C, bake a special kind of cake, and with respective probabilities .02, .03, and .05, it fails to rise. In the restaurant where they work, A bakes 50 percent of these cakes, B 30 percent, and C 20 percent. What proportion of "failures" is caused by A?

Solution:

$$P\{A \mid \text{failure}\} = \frac{(.02)(.5)}{(.02)(.5) + (.03)(.3) + (.05)(.2)} = \frac{10}{29}$$

9. A deck of cards is shuffled and then divided into two halves of 26 cards each. A card is drawn from one of the halves; it turns out to be an ace. The ace is then placed in the second half-deck. The half is then shuffled, and a card is drawn from it. Compute the probability that this drawn card is an ace. Hint: Condition on whether or not the inter- changed card is selected.

Solution: Let

$E = \{\text{first card drawn is an } A\}$
$F = \{\text{second card drawn is the original } A \text{ drawn in the first round}\}$
$G = \{\text{second card drawn is an } A\}$

We must compute

$$
\begin{aligned}
P(G|E) &= P(G|F \cap E) \times P(F|E) + P(G|F^c \cap E) \times P(F^c|E) \\
&= 1 \times \frac{1}{27} + \frac{3}{51} \times \frac{26}{27}.
\end{aligned}
$$

Alternate solution:

Number the A's 1 through 4 and let A_i be the event that the i-th ace was drawn on the first hand. You don't have to number the A's but it helps later. Keeping G and E as above, for $i = 1, \ldots, 4$

$$P(G|E) = P(G|A_i).$$

That is, the (conditional) probability the second card drawn is an A, given the first card was an A is the same as the (conditional) probability that second card drawn is an A, given the first card drawn was A_i.

Let $F_j, j = 0, 1, 2, 3, 4$ be the event that the second half of the deck has j aces after the initial split.

Then, choosing A_1 as our first draw

$$
\begin{aligned}
P(G|E) &= P(G|A_1) \\
&= \sum_{j=0}^{4} P(G|A_1 \cap F_j)P(F_j|A_1) \\
&= \sum_{j=0}^{3} P(G|A_1 \cap F_j)P(F_j|A_1)
\end{aligned}
$$

because $P(F_4|A_1) = 0$.

Now,

$$P(G|A_1 \cap F_j) = \frac{j+1}{27}$$

and

$$P(F_j|A_1) = \frac{\binom{3}{j}\binom{48}{26-j}}{\binom{51}{3}}.$$

Therefore,

$$
\begin{aligned}
P(G|E) = &\frac{1}{27}\frac{\binom{3}{0}\binom{48}{26}}{\binom{51}{3}} + \frac{2}{27}\frac{\binom{3}{1}\binom{48}{25}}{\binom{51}{3}} \\
&\frac{3}{27}\frac{\binom{3}{2}\binom{48}{24}}{\binom{51}{3}} + \frac{4}{27}\frac{\binom{3}{3}\binom{48}{23}}{\binom{51}{3}},
\end{aligned}
$$

which agrees with the first solution.

10. Consider 3 urns. Urn A contains 2 white and 4 red balls, urn B contains 8 white and 4 red balls, and urn C contains 1 white and 3 red balls. If 1 ball is selected from each urn, what is the probability that the ball chosen from urn A was white given that exactly 2 white balls were selected?

Solution:

Let $P(A = w)$ be the probability that the ball chosen from urn A was white.

Let $P(2\text{w})$ be the probability that exactly 2 white balls were selected.

We wish to know $P(A = w \mid 2w)$.

From Bayes rule:

$P(A = w \mid 2w) = \frac{P(A = w,\ 2w)}{P(2w)}$

Which can be rewritten as:

$$P(A = w \mid 2w) = \frac{P\{A=w, B=w, C \neq w\} + P\{A=w, B \neq w, C=w\}}{P\{2w\}}$$

$$= \frac{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\}}{P\{A = w, B = w, C \neq w\} + P\{A = w, B \neq w, C = w\} + P\{A \neq w, B = w, C = w\}}$$

$$= \frac{\frac{1}{3}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4}}{\frac{1}{2}\frac{2}{3}\frac{3}{4} + \frac{1}{3}\frac{1}{3}\frac{1}{4} + \frac{2}{3}\frac{2}{3}\frac{1}{4}} = \frac{7}{11}$$

11. An ectopic pregnancy is twice as likely to develop when the pregnant woman is a smoker as it is when she is a non-smoker. If 32 percent of women of childbearing age are smokers, what percentage of women having ectopic pregnancies are smokers?

Solution:

Let E be the event that a randomly chosen pregnant women has an ectopic pregnancy and S the event that the chosen person is a smoker.

Then the problem states that $P(E|S) = 2\,P(E|S^C)$, and $P(S) = 0.32$

From Bayes rule:

$$P(S|E) = \frac{P\{E|S\}\,P\{S\}}{P\{E\}}$$

$$= \frac{P\{E|S\}\,P(S)}{P\{E|S\}\,P(S) + P\{E|S^C\}\,P(S^C)}$$

$$= \frac{2\,P(S)}{2\,P\{S\} + P(S^C)}$$

= 64/132

= 0.455

12. You go to see the doctor about excessive weight gain. The doctor selects you at random to have a blood test for hypothyroidism, which is currently suspected to occur 1 in 10,000 people in India. The test is 99% accurate, that is, the probability of a false positive is 1%. The probability of a false negative is zero. You test positive. What is the new probability that you have hypothyroidism?

Solution:

Let P(H) be the probability you have hypothyroidism.
Let P(T) be the probability of a positive test.
We wish to know P(H|T).

From Bayes rule:

$$P(H|T) = \frac{P(T|H) * P(H)}{P(T)}$$

Which can be rewritten as:

$$P(H|T) = \frac{P(T|H) * P(H)}{\mathrm{P(T|H)P(H)} + \mathrm{P(T|NH)P(NH)}}$$

where P(NH) means the probability of not having hypothyroidism.

We have P(H) = 0.0001, P(NH) = 0.9999 P(T|H) = 1 (if you have hypothyroidism the test is always positive). P(T|NH) = 0.01 (1% chance of a false positive).
Plugging these numbers in we get P(H|T) = 1 × 0.0001 1 × 0.0001 + 0.01 × 0.9999 ≈ 0.01
That is, even though the test was positive your chance of having hypothyroidism is only 1%.

13. Stores A, B, and C have 50, 75, and 100 employees, respectively, and 50, 60, and 70 percent of them respectively are women. Resignations are equally likely among all employees, regardless of sex. One woman employee resigns. What is the probability that she works in store C?

Solution:

$$P\{C|\text{ woman}\} = \frac{P\{\text{women }|C\}P\{C\}}{P\{\text{women }|A\}P\{A\} + P\{\text{women }|B\}P\{B\} + P\{\text{women }|C\}P\{C\}}$$

$$= \frac{.7\frac{100}{225}}{.5\frac{50}{225} + .6\frac{75}{225} + .7\frac{100}{225}} = \frac{1}{2}$$

Unsolved Problems

1. In a class, 30% of the students study math and science while 70% of the students study math. What is the probability of a student studying science given he/she is already studying math?

2. If four cards are selected at random (without replacement) from a standard deck of 52 cards, what is the probability there will be no pairs? (two cards of the same denomination)

3. I am taking a probability class and in each week, I can be either up-to-date or I may have fallen behind. If I am up-to-date in a given week, the probability that I will be up-to-date the following week is 0.8. If I am behind in the given week, the probability that I will be up-to-date the following week is 0.4. Given that I am up-to-date this week, what is the probability that I will be up-to-date three weeks later?
4. Three cards are randomly selected, without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the first card selected is a spade given that the second and third cards are spades.

5. An urn contains 12 balls, of which 4 are white. Three players—A, B, and C—successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the probability of winning for each player if each ball is replaced after it is drawn;

6. Two chefs A and B are challenged to separately create a new dish in 2 hours. From past experience, we know that:
   a. The probability that chef A's dish is a hit is 2/3
   b. The probability that chef B's dish is a hit is 1/2
   c. The probability that at least one of their dishes is a hit is 3/4

   Assuming that only one of the dishes can be labelled a hit, what is the probability that it was created by chef A?