Series solution of II^{nd} order ordinary differential equation.

Power Series - An infinite series of the form

$$\sum_{n=0}^{n=\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

is called power series.

Eg
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$$

$$= \sum_{n=0}^{n=\infty} \frac{x^n}{n!} \qquad a_n = \frac{1}{n!}$$

Analytic Functions -

Let $f : I \to R$: $x_0 \in I$ $\quad I \to$ interval

then f is analytic at $x = x_0$, if

(i) Taylor expansion of $f(x)$ at $x = x_0$ exists

(ii) it converges to $f(x)$ for all x in the interval of convergence.

in one variable Taylor's series expansion at $x = x_0$

$$f(x) = f(x_0) + (x-x_0) f'(x_0) + \frac{1}{2!}(x-x_0)^2 f''(x_0) + \frac{1}{3!}(x-x_0)^3 f'''(x_0) + \cdots$$

★ **Note** Rational functions are not analytic at the point where the denominator is equal to zero.

$f(x) = \frac{1}{x+1} \longrightarrow$ not analytic

Eg $f(x) = \frac{x+1}{x-2}$

it is analytic at $x = 2$? No

Consider II order ODE :

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x) y = 0$$

① **ordinary point** – A point $x = x_0$ is called an ordinary point if both $P(x)$ & $Q(x)$ are analytic at $x = x_0$

Eg $\frac{d^2y}{dx^2} + \frac{1}{x+1}\frac{dy}{dx} + \frac{x+1}{x+2} y = 0$.

$x = 0$; is $x = 0$ an ordinary points ?

Yes

2.) **Singular point** - The point which is not an ordinary point called as singular point.

Types of singular point-

a) Regular singular point

b) irregular singular point.

a) **Regular singular point** - A singular point $x = x_0$ is called regular singular point, if both $(x-x_0) P(x)$ & $(x-x_0)^2 Q(x)$ are analytic at $x = x_0$.

Eg
$$\frac{d^2y}{dx^2} + \frac{1}{(x-1)}\frac{dy}{dx} + \frac{x+2}{(x-1)^2}y = 0$$

$x = 1$ it is singular point.

$$(x-x_0) P(x) = (x-1)\frac{1}{(x-1)} = 1$$

$$(x-x_0)^2 Q(x) = \cancel{(x-1)^2}\frac{x+2}{\cancel{(x-1)^2}} = x+2$$

Both $(x-x_0) P(x)$ & $(x-x_0)^2 Q(x)$ are analytic at $x = x_0 = 1$

$x=1$ is a regular singular point.

h) irregular singular point – A singular point $x = x_0$ is called irregular singular point, if it is not regular

Eg $$\frac{d^2y}{dx^2} + \frac{1}{(x-1)}\frac{dy}{dx} + \frac{(x+2)}{(x-1)^3}y = 0$$

$x_0 = 1$ singular point.

$$(x-x_0)P(x) = \cancel{(x-1)}\frac{1}{\cancel{(x-1)}} = 1$$

$$(x-x_0)^2 Q(x) = \cancel{(x-1)^2}\frac{x+2}{(x-1)^{\cancel{3}}} = \frac{x+2}{x-1}$$

$(x-x_0)^2 Q(x)$ is not analytic. at $x = 1$

$x = 1$ is an irregular singular point.

I Power series solution, where $x = 0$ is an ordinary point.

Qn $$\frac{d^2y}{dx^2} + xy = 0 \quad \text{——— (1)}$$

Comparing Eqn (1) with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

(I) $P(x) = 0$, $Q(x) = x$.

Both $P(x)$ & $Q(x)$ are analytic at $x=0 \Rightarrow x=0$ is an ordinary point

Let $$y = \sum_{n=0}^{\infty} a_n x^n$$

$$\frac{dy}{dx} = \sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

Putting in eq ①

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + x\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

[in summation I, replace n by $n+2$ & in summation II replace n by $n-1$]

$$\sum_{n=2}^{\infty} a_{n+2}(n+2)(n+2-1) x^{n+2-2} + \sum_{n=0}^{\infty} a_{n-1} x^{n-1+1} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$n=0$ $\quad a_2(2)(1) = 0 \quad \boxed{a_2 = 0}$

for $n \geq 1$

$$a_{n+2}(n+2)(n+1) + a_{n-1} = 0$$

$$\boxed{a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}}$$

$n=1$: $a_3 = \frac{-a_0}{3\times 2} = \frac{-a_0}{6}$

$n=2$: $a_4 = \frac{-a_1}{4\times 3}$

$n=3$: $a_5 = \frac{-a_2}{5\times 4} = 0$

$n=4$: $a_6 = \frac{-a_3}{6\times 5} = \frac{a_0}{6\times 5} \cdot \frac{1}{6} =$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \ldots$$

$$= a_0 + a_1 x + \left(-\frac{a_0}{6}\right) x^3 + \left(\frac{-a_1}{12}\right) x^4 + \frac{a_0}{180} x^6 + \ldots$$

Que $(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$

comparing with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

$$P(x) = \frac{x}{1+x^2}, \quad Q(x) = -\frac{1}{1+x^2}$$

Both $P(x)$ & $Q(x)$ are analytic at $x=0$
$x=0$ is an ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$

$$\frac{dy}{dx} = \sum_{n=1}^{n=\infty} a_n n x^{n-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

putting in Eq^n ①

$$(1+x^2)\sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} a_n n x^{n-1}$$

$$-\sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^n +$$

$$\sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+2-1)x^{n+2-2} + \sum_{n=2}^{\infty} a_n n(n-1)x^n$$

$$+ \sum_{n=1}^{\infty} a_n n x^n - \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n + \sum_{n=2}^{\infty} a_n n(n-1)x^n +$$

$$\sum_{n=1}^{\infty} a_n n x^{n} - \sum_{n=0}^{\infty} a_n x^n = 0.$$

$\underline{\underline{n=0}}$ $a_2 \cdot 2 \cdot 1 - a_0 = 0.$ $\quad 2a_2 = a_0$ $\quad \boxed{a_2 = \frac{a_0}{2}}$

$\underline{\underline{n=1}}$ $a_3 \cdot 3 \cdot 2 + \cancel{a_1} - \cancel{a_1} = 0$ $\quad \boxed{a_3 = 0}$

$n \geq 2$ $\quad a_{n+2}(n+2)(n+1) + a_n n(n-1) + a_n n - a_n = 0.$

$$a_{n+2} = -\frac{a_n [n^2 - 1]}{(n+2)(n+1)} = \frac{-(n-1)}{(n+2)} a_n$$

$\underline{\underline{n=2}}$ $\quad a_4 = -\frac{1}{4} a_2 = -\frac{1}{4} \frac{a_0}{2} = -\frac{a_0}{8}.$

$n=3$ $\quad a_5 = -\frac{2}{5} a_3 = 0$

$n=4$ $\quad a_6 = -\cancel{\frac{3}{6}}\frac{1}{2} a_4 = +\frac{3a_0}{16}$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_0}{8} x^4 + \frac{a_0}{16} x^6 + \cdots$$

Frobenius Method for series solution when $x=0$ is a regular singular point

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

Case I :- when roots of indicial equation are distinct & do not differ by an integer.

Eg $1, \frac{1}{2}$

$$y = c_1 y_1 + c_2 y_2$$

Que $2x(1-x)\frac{d^2y}{dx^2} + [5-7x]\frac{dy}{dx} - 3y = 0$

Sol $P(x) = \dfrac{5-7x}{2x(1-x)}$ $\quad Q(x) = \dfrac{-3}{2x(1-x)}$

Both $xP(x)$ & $x^2Q(x)$ are analytic at $x=0$

$x=0$ is a regular singular point.

let $y = \sum_{n=0}^{\infty} a_n x^{n+k}$

$$\frac{dy}{dx} = \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1}$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)\, x^{n+k-2}$$

$x_0 = 0$

$(x-x_0)P(x)$

$x(P(x))$

$\dfrac{5-7x}{2(1-x)}$

$(x-x_0)^2 Q(x)$

$x^2\,Q(x)$

$= \dfrac{-3x}{2(1-x)}$

$$\Rightarrow (2x-2x^2)\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)\, x^{n+k-2} +$$

$$(5-7x)\sum_{n=0}^{\infty} a_n(n+k)\, x^{n+k-1} - 3\sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} 2a_n(n+k)(n+k-1)\, x^{n+k-1} - \sum_{n=0}^{\infty} 2a_n(n+k)$$

$$(n+k-1)\, x^{n+k} + \sum_{n=0}^{\infty} 5a_n(n+k)\, x^{n+k-1} -$$

$$\sum 7a_n(n+k)\, x^{n+k} - \sum_{n=0}^{\infty} 3a_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+k)\{2(n+k-1)+5\}\, x^{n+k-1} -$$

$$\sum_{n=0}^{\infty} a_n\{2(n+k)^2 - 2(n+k) - 7(n+k)+3\}\, x^{n}$$

$$\Rightarrow \quad 2(n+k)^2 + 5(n+k) + 3$$

$$= 2(n+k)^2 + 2(n+k) + 3(n+k) + 3$$

$$= 2(n+k)(n+k+1) + 3(n+k+1)$$

$$= (n+k+1)(2n+2k+3)$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+k)(2n+2k+3)\, x^{n+k-1} - \sum_{n=0} a_n(n+k+1)$$

$$(2n+2k+3)\, x^{n+k} = 0$$

{In second Σ, replace n by $n-1$}

$$\sum_{n=0}^{\infty} a_n (n+k)(2n+2k+3) x^{n+k-1} - \sum_{n=1}^{\infty} a_{n-1} (n-\cancel{1}+k+\cancel{1})(2n-2+2k+3) x^{n-1} = 0$$

$n=0$: $a_0 k(2k+3) = 0$ $\quad a_0 \neq 0$

$k(2k+3) = 0$ Indicial Eq^n

$k = 0, -\frac{3}{2}$

$n \geq 1$: $a_n (n+k)(2n+2k+3) - a_{n-1}(n+k)(2n+2k+1) = 0$

$$a_n = \frac{(2n+2k+1)\, a_{n-1}}{(2n+2k+3)}$$

$n=1$ $\quad a_1 = \frac{(2k+3)}{(2k+5)} a_0$

$$a_2 = \frac{(2k+5)}{(2k+7)} a_1 = \frac{(2k+5)}{(2k+7)} \frac{(2k+3)a_0}{(2k+5)}$$

$$= \frac{(2k+3)}{(2k+7)} a_0$$

$$a_3 = \frac{(2k+3)}{(2k+9)} a_0$$

$k = 0$	$k = -3/2$
$a_1 = \frac{3}{5} a_0$	$a_1 = 0$
$a_2 = \frac{3}{7} a_0$	$a_2 = 0$
$a_3 = \frac{3}{9} a_0$	$a_3 = 0$

$$y_1 = y\big|_{at\ k=k_1} = y\big|_{k=0} \qquad k_1 = 0$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+k_1} = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \frac{3}{5} a_0 x + \frac{3}{7} a_0 x^2 + \frac{3}{9} a_0 x^3 + \cdots$$

$$y_2 = \sum_{n=0}^{\infty} a_n x^{n+k_2} = \sum_{n=0}^{\infty} a_n x^{n-3/2}$$

$$= a_0 x^{-3/2} + 0 + 0 + \cdots$$

$$= a_0 x^{-3/2}$$

complete solution

$$y = C_1 y_1 + C_2 y_2$$

$$C_1 a_0 \left[1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \cdots\right] + C_2 a_0 x^{-3/2}$$

Qu $2x^2y'' + (2x^2 - x)y' + y = 0$

Sol $$y'' + \frac{2x^2 - x}{2x^2} y' + \frac{y}{2x^2} = 0$$

$$y'' + \left(1 - \frac{1}{2x}\right) y' + \frac{y}{2x^2} = 0$$

$$P(x) = 1 - \frac{1}{2x}, \quad Q(x) = \frac{1}{2x^2}$$

$x = 0$ is a singular point
because both $p(x)$ & $Q(x)$ are not analytic

$$A = (x - x_0) P(x) = x\left(1 - \frac{1}{2x}\right) = \frac{\cancel{x}(2x - 1)}{2\cancel{x}}$$

$$B = (x - x_0)^2 Q(x) = \cancel{x^2}\left(\frac{1}{2\cancel{x^2}}\right) = \frac{1}{2}$$

$x = 0$ is a regular singular point

let $$y = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$y' = \sum_{n=0}^{\infty} a_n (n+k) x^{n+k-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2}$$

$2x^2 \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-2} + (2x^2 - x) \sum_{n=0}^{\infty}$

$a_n (n+k) x^{n+k-1} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$

$\Rightarrow \sum_{n=0}^{\infty} 2a_n (n+k)(n+k-1) x^{n+k} + \sum_{n=0}^{\infty} 2a_n (n+k)$

$x^{n+k+1} - \sum_{n=0}^{\infty} a_n (n+k) x^{n+k} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$

$\Rightarrow 2(n+k)^2 - 2(n+k) - (n+k) + 1$

$= 2(n+k)\{n+k-1\} - 1(n+k-1)$

$= (n+k-1)(2n+2k-1)$

from ①

$\sum_{n=0}^{\infty} 2a_n (n+k) x^{n+k+1} + \sum_{n=0}^{\infty} a_n (n+k-1)(2n+2k-1)$

$x^{n+k} = 0.$

Put $n=0$ in the coeff. of least power of x + Equate to zero to find indicial Equation

$(k-1)(2k-1) = 0 \qquad k = 1, \frac{1}{2}$

[Replace n by $n-1$ in Σ Ist]

$$\sum_{n=1}^{\infty} 2a_{n-1}(n+k-1)\, x^{n-\cancel{1}+k+\cancel{1}} + \sum_{n=0}^{\infty} a_n (n+k-1)$$

$$(2n+2k-1)\, x^{n+k} = 0.$$

$n \geq 1$: $\quad 2a_{n-1}(n+k-1) + a_n(n+k-1)(2n+2k-1) = 0$

$$a_n = \frac{-2a_{n-1}(n+k-1)}{(n+k-1)(2n+2k-1)}$$

$$\boxed{a_n = \frac{-2a_{n-1}}{(2n+2k-1)}}$$

		$k=1$	$k=1/2$
$n=1$	$a_1 = \frac{-2a_0}{1+2k}$	$a_1 = -\frac{2a_0}{3}$	$a_1 = -a_0$
$n=2$	$a_2 = \frac{-2a_1}{(3+2k)}$	$a_2 = \frac{4a_0}{3\cdot 5}$	$a_2 = \frac{\cancel{4}a_0}{2\cdot\cancel{4}}$
$n=3$	$a_3 = \frac{-6a_0}{(1+2k)(3+2k)(5+2k)}$	$a_3 = \frac{-6a_0}{3\cdot 5\cdot 7}$	$a_3 = \frac{-\cancel{6}a_0}{2\cdot 4\cdot\cancel{6}}$

$$(y)_{k=1} = \sum_{n=0}^{\infty} a_n x^{n+1} = a_0x + a_1x^2 + a_2x^3 + a_3x^4 + \cdots$$

$$= a_0x - \frac{2a_0x^2}{3} + \frac{4a_0x^3}{15} - \frac{2a_0x^4}{35} + \cdots$$

$$(y)_{k=\frac{1}{2}} = \sum_{n=0}^{\infty} a_n x^{n+1/2}$$

$$= a_0 x^{1/2} + a_1 x^{3/2} + a_2 x^{5/2} + a_3 x^{7/2} + \cdots$$

$$= a_0 x^{1/2} - a_0 x^{3/2} + \frac{a_0}{2} x^{5/2} - \frac{a_0}{8} x^{7/2} + \cdots$$

$$y = C_1 y_1 + C_2 y_2.$$

Case I if roots are distinct and do not differ by an integer

$$y = C_1 (y)_{K_1} + C_2 (y)_{K_2} \qquad K_1, K_2 \quad K \neq K_2$$

Case II if roots are Equal : $K_1 = K_2$.

$$y = C_1 (y)_{K_1} + C_2 \left(\frac{\delta y}{\delta K} \right)_{K=K_1}$$

Que $x(x-1)y'' + (3x-1)y' + y = 0$

$x=0$ is a regular singular point

let, $y = \sum_{n=0}^{\infty} a_n x^{n+k}$

$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k} - \sum_{n=0}^{\infty} a_n (n+k)$

$(n+k-1) x^{n+k-1} + \sum_{n=0}^{\infty} a_n x^{n+k} -$

$\sum_{n=0}^{\infty} a_n x^{n+k-1} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$

indicial Eq^n -

$-k(k-1) - k = 0$

$-k^2 + \cancel{k} - \cancel{k} = 0$

$k^2 = 0 \qquad k = 0, 0$

(simplify coeff. of x^{n+k})

$= (n+k)^2 - (n+k) + 3(n+k) + 1$

$= (n+k)^2 + 2(n+k) + 1$

$= (n+k+1)^2$

(simplify coeff of x^{n+k-1})

$-(n+k)(n+k-1+1) = -(n+k)^2$

$$\sum_{n=0}^{\infty} a_n (n+k+1)^2 x^{n+k} - \sum_{n=0}^{\infty} a_n (n+k)^2 x^{n+k-1} = 0$$

$$= \sum_{n=1}^{\infty} a_{n-1} (n-1+k+1)^2 x^{n+k-1} - \sum_{n=0}^{\infty} a_n (n+k)^2 x^{n+k-1} = 0$$

$$a_n = \frac{a_{n-1} \cancel{(n+k)^2}}{\cancel{(n+k)^2}} \Rightarrow \boxed{a_n = a_{n-1}} \; n \geq 1$$

$n=1 \quad a_1 = a_0$

$n=2 \quad a_2 = a_1 = a_0$

$n=3 \Rightarrow a_3 = a_0$

$$y = x^k (a_0 + a_1 x + a_2 x^2 + \text{-----})$$

$$y = x^k a_0 (1 + x + x^2 + \text{------})$$

$$(y)_{k=0} = a_0 (1 + x + x^2 + \text{-----})$$

$$\frac{\delta y}{\delta k} = x^k \log_e x \, a_0 (1 + x + x^2 + \text{----})$$

$$\left(\frac{\delta y}{\delta k}\right)_{k=0} = \log_e x \, a_0 (1 + x + x^2 + \text{-----})$$

$$y = C_1 (y)_{k=0} + C_2 \left(\frac{\delta y}{\delta k}\right)_{k=0}$$

$$y = a_0 (1 + x + x^2 + \text{---}) (C_1 + C_2 \log_e x)$$

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Ans $xy'' + y' - y = 0$

$x=0$ is a regular singular point

let,

$$y = \sum_{n=0}^{\infty} a_n x^{n+k}.$$

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1)x^{n+k-1} + \sum_{n=0}^{\infty} a_n (n+k) x^{n+k-1}$$

$$- \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

I.E $: k(k-1) + k = 0 \quad k^2 = 0 \quad k = 0, 0$

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1+1) x^{n+k-1} - \sum_{n=1}^{\infty} a_{n-1} x^{n+k-1} \Rightarrow$$

$$\boxed{a_n = \frac{a_{n-1}}{(n+k)^2}} \qquad n \geq 1$$

$n=1 \qquad a_1 = \dfrac{a_0}{(1+k)^2}$

$n=2 \qquad a_2 = \dfrac{a_1}{(2+k)^2} \Rightarrow \dfrac{a_0}{(1+k)^2(2+k)^2}$

$$a_3 = \frac{a_0}{(1+k)^2(2+k)^2(3+k)^2}$$

$$y = x^k (a_0 + a_1 x + a_2 x^2 + \text{-----})$$

$$= a_0 x^k \left[1 + \frac{x}{(1+k)^2} + \frac{x^2}{(1+k)^2(2+k)^2} + \cdots \right]$$

$$(y)_{k=0} \; a_0 \left[1 + \frac{x}{1^2} + \frac{x^2}{1^2 2^2} + \cdots \right]$$

$$\left(\frac{\delta y}{\delta k}\right) = a_0 x^k \log_e x \left[1 + \frac{x}{(1+k)^2} + \frac{x^2}{(1+k)^2(2+k)^2} + \cdots \right]$$

$$+ a_0 x^k \left[\frac{-2x}{(1+k)^3} + x^2 \left\{ \frac{-2}{(1+k)^3(2+k)^2} - \frac{2}{(1+k)^2(2+k)^3} \right\} \right.$$

$$\left(\frac{\delta y}{\delta k}\right)_{k=0} = a_0 \log_e x \left[1 + \frac{x}{1^2} + \frac{x^2}{1^2 2^2} + \cdots \right] -$$

$$2a_0 \left[\frac{x}{1^3} + x^2 \left\{ \frac{1}{1^3 2^2} + \frac{1}{1^2 2^3} \right\} \right.$$

$$y = c_1 (y)_{k=0} + c_2 \left(\frac{\delta y}{\delta k}\right)_{k=0}$$

$$= y_1 [c_1 + c_2 \log_e x] - 2a_0 c_2 \left[\frac{x}{1^3} + x^2 \left\{\frac{1}{4} + \frac{1}{8}\right\} + \cdots \right]$$

Case VIII — When roots are distinct, differ by an integer and making one or more coefficients interminate

let $K = K_1, K_2$

if coeff. of y becomes infinite at $K = K_1$

$\Rightarrow \quad y = (y)_{K_1}$

Case (4) — When roots are distinct differ by an integer and making coeff of y infinite

$$y = C_1 (y)_{K_1} + C_2 \left(\frac{\delta y}{\delta K}\right)_{K_2}$$

* if some coeff. of y becomes infinite at $K = K_2$ first replace a_0 by $b_0(K - K_2)$ then differentiate it.

Qu $x(x-1)\frac{d^2y}{dx^2} + (3x-1)\frac{dy}{dx} + y = 0$

$x=0$ is a regular singular point

Let,

$$y = \sum_{n=0}^{\infty} a_n x^{n+k}, \quad y' = \sum_{n=0} a_n(n+k)x^{n+k-1}$$

$$y'' = \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k-2}$$

$$\Rightarrow (x^2-x)\sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k-2} +$$

$$(3x-1)\sum_{n=0}^{\infty} a_n(n+k)x^{n+k-1} + \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)x^{n+k} - \sum_{n=0}^{\infty} a_n(n+k)(n+k-1)$$

$$x^{n+k-1} + \sum_{n=0}^{\infty} 3a_n(n+k)x^{n+k} - \sum_{n=0}^{\infty} a_n(n+k)x^{n+k-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+k} = 0$$

$-k(k-1) - k = 0$

$k^2 = 0 \qquad k = 0, 0$

$(n+k)^2 - (n+k) + 3(n+k) + 1$

$(n+k)^2 + 2(n+k) + 1$

$(n+k+1)^2$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+k+1)\, x^{n+k} - \sum_{n=0}^{\infty} a_n (n+k)^2 x^{n+k-1} = 0$$

$$\Rightarrow \sum_{n=0}^{\infty} a_{n-1} (n-\cancel{1}+k+\cancel{1})\, x^{n-1+k} - \sum_{n=0}^{\infty} a_n (n+k)^2 x^{n+k-1} = 0$$

$n \geq 1$: $a_{n-1}(n+k)^2 - a_n (n+k)^2 = 0$

$$a_n = a_{n-1}$$

$n=1$: $a_1 = a_0$

$n=2$: $a_2 = a_1 = a_0$.

$n=3$: $a_3 = a_0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+k} = x^k [a_0 + a_1 x + a_2 x^2 + \cdots] \quad -(1)$$

$$(y)_{k=0} = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

$$= a_0 [1 + x + x^2 + x^3 + \cdots]$$

from (1).

$$\frac{\delta y}{\delta k} = x^k \log_e x [a_0 + a_1 x + a_2 x^2 + \cdots]$$

$$\left(\frac{\delta y}{\delta k}\right)_{k=0} = \log e^{n} a_0 [1 + n + n^2 + \text{-----}]$$

$$y = C_1 (y)_{k=0} + C_2 \left(\frac{\delta y}{\delta k}\right)_{k=0}$$

Maths

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Case III : when roots are distinct & differ by an integer & making one or more coefficient of y interminate.

Let $k = k_1, k_2$

if coeff. of y becomes interminate for

$k = k_1$

$$y = (y)_{k_1} \qquad a_1 \left(\frac{0}{0} \right)$$

Que $xy'' + 2y' + xy = 0$

Sol $x = 0$ is a regular singular point

Let,

$$y = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k-1} + \sum_{n=0}^{\infty} 2a_n (n+k) x^{n+k-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^{n+k+1} = 0$$

I.E $k(k-1) + 2k = 0$

$k^2 + k = 0$

$k = 0, -1$

$$\sum_{n=0}^{\infty} a_n (n+k)(n+k-1+2) \, x^{n+k-1} + \sum_{n=2}^{\infty} a_{n-2} \, x^{n+k-1} = 0$$

$n=1$: $a_1(1+k)(k+2) = 0$.

$a_1 \cdot 0 = 0$ ★ Case III (identity)

★ a_1 can have any arbitary value.

$n \geq 2$

$$\boxed{a_n = \frac{-a_{n-2}}{(n+k)(n+k+1)}}$$

$n=2$

$$a_2 = \frac{-a_0}{(2+k)(k+3)} \qquad a_2 = \frac{-a_0}{1\cdot 2} = \frac{-a_0}{2!}$$

$n=3$

$$a_3 = \frac{-a_1}{(k+3)(k+4)} \qquad a_3 = \frac{-a_1}{2\cdot 3} = \frac{-a_1}{3!}$$

$n=4$

$$a_4 = \frac{-a_2}{(k+4)(k+5)} \qquad a_4 = \frac{a_0}{1\cdot 2\cdot 3\cdot 4} = \frac{a_0}{4!}$$

$$= \frac{a_0}{(k+2)(k+3)(k+4)(k+5)}$$

$n=5$

$$a_5 = \frac{a_1}{(k+3)(k+4)(k+5)(k+6)} \qquad a_5 = \frac{a_1}{5!}$$

$$y = (y)_{k_1} = (y)_{k=-1}$$

$= x^{-1}\left[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \cdots\right]$

$= x^{-1}\left[a_0 + a_1 x - \frac{a_0 x^2}{2!} - \frac{a_1 x^3}{3!} - \frac{a_0 x^4}{4!} + \cdots\right]$

$$\boxed{\begin{aligned} \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \\ \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \end{aligned}}$$

$y = x^{-1}\left[a_0 \cos x + a_1 \sin x\right]$

<u>Ques</u> $x^2 y'' + 4xy' + (x^2+2)y = 0$

$x = 0$ is a regular singular point

Let.

$y = \sum_{n=0}^{\infty} a_n x^{n+k}$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+k)(n+k-1) x^{n+k} + \sum_{n=0}^{\infty} 4a_n (n+k) x^{n+k}$

$+ \sum_{n=0}^{\infty} a_n x^{n+k+2} + \sum_{n=0}^{\infty} 2a_n x^{n+k} = 0$

I.E $k(k-1)+4k+2=0$

$k^2+3k+2=0$

$k=-1,-2$

$=(n+k)^2-(n+k)+4(n+k)+2$

$=(n+k)^2+3(n+k)+2$

$=(n+k+1)(n+k+2)$

$\Rightarrow \sum_{n=0}^{\infty} a_n (n+k+1)(n+k+2) x^{n+k} + \sum_{n=2} a_{n-2} x^{n+k}$

$n=1 \quad a_1(k+2)(k+3)=0$

for $k=-2 \quad a_1 \cdot 0 = 0$

Case III

a_1 i.e an arbitary constant

$n \geq 2$

$a_2 = \dfrac{-a_0}{(k+3)(k+4)}$ $\quad a_2 = \dfrac{-a_0}{1 \cdot 2} \Rightarrow \dfrac{-a_0}{2!}$

$a_3 = \dfrac{-a_1}{(k+4)(k+5)}$ $\quad a_3 = \dfrac{-a_1}{2 \cdot 3} = \dfrac{-a_1}{3!}$

$a_4 = \dfrac{-a_2}{(k+5)(k+6)}$ $\quad a_4 = \dfrac{a_0}{1 \cdot 2 \cdot 3 \cdot 4} = \dfrac{a_0}{4!}$

$= \dfrac{a_0}{(k+3)(k+4)(k+5)(k+6)}$

$a_5 = \dfrac{a_1}{(k+4)(k+5)(k+6)(k+7)}$

$a_5 = \dfrac{a_1}{5!}$

$K = -2$

$$y = (y)_{K=-2}$$

$$= x^{-2}\left[a_0 + a_1 x + \frac{a_0 x^2}{2!} - \frac{a_1 x^3}{3!} + \frac{a_0 x^4}{4!} - \frac{a_1 x^5}{5!} + \cdots\right]$$

$$y = x^{-2}\left[a_0 \cos x + a_1 \sin x\right]$$

Case (IV)

when roots are distinct differ by an integer & making one or more coeff. of y as $1/0$

$$y = C_1 (y)_{K_1} + C_2 \left(\frac{\delta y}{\delta K}\right)_{K_2}$$

if for $K = K_2$, some coeff. of y comes in the form of $1/0$, then replace

(i) a_0 by $b_0 (K - K_2)$ in by

(ii) diff. new by w.r.t. K.

Q. $x^2y'' + xy' + (x^2-4)y = 0$

$x=0$ is regular singular point

let,

$$y = \sum_{n=0}^{\infty} a_n x^{n+k}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)x^{n+k} + \sum_{n=0}^{\infty} a_n (n+k) x^{n+k} +$$

$$\sum_{n=0}^{\infty} a_n x^{n+k+2} - \sum_{n=0}^{\infty} 4a_n x^{n+k} = 0$$

Combine Power same

I.E : $k(k-\cancel{1}) + \cancel{k} - 4 = 0$

$$k^2 - 4 = 0$$

$$k = 2, -2$$

$$(n+k)^2 - \cancel{(n+k)} + \cancel{(n+k)} - 4$$

$$(n+k-2)(n+k+2)$$

$$\sum_{n=0}^{\infty} a_n (n+k+2)(n+k-2) x^{n+k} + \sum_{n=2}^{\infty} a_n x^{n+k} = 0$$

$n=1$ $\quad a_1 (k+3)(k-1) = 0$

$\boxed{a_1 = 0}$ $\qquad$ Case IV

$$\boxed{a_n = \frac{-a_{n-2}}{(n+k+2)(n+k-2)}}$$

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$n=2 \quad a_2 = \frac{-a_0}{(k+4)k} \qquad a_2 = \frac{-a_0}{2.6}$

$n=3 \quad a_3 = \frac{-a_1}{(k+1)(k+5)} = 0 \qquad a_3 = 0$

$n=4 \quad a_4 = \frac{-a_2}{(k+6)(k+2)} \qquad a_4 = \frac{a_0}{2.4.6.8}$

$= \frac{a_0}{k(k+2)(k+4)(k+6)}$

$a_5 = 0$

$a_5 = 0$

$$(y)_{k=2} = x^2 \left[a_0 - \frac{a_0 x^2}{2.6} + \frac{a_0}{2.4.6.8} x^4 - \frac{a_0}{2.4.6.8.10} x^6 + \cdots \right]$$

but $a_0 = b_0 (k+2)$
$k = -2$

$$y = x^{-2} \left[b_0 (k+2) - \frac{b_0 (k+2)}{k(k+4)} x^2 + \frac{b_0 x^3}{k(k+4)(k+6)} + \cdots \right]$$

$$\frac{\delta y}{\delta k} = x^k \log_e x \, b_0 \left[(k+2) - \frac{k+2}{k(k+4)} x^2 + \cdots \right.$$

$$\left. \frac{x^4}{k(k+4)(k+6)} \cdots \right] + x^k b_0 \left[1 - \left\{ \frac{k(k+4) \cdot 1 - (k+2)(2k+4)}{k^2 (k+4)^2} \right\} x^2 \right]$$

$$+ \frac{x^4(3k^2+20k+24)}{k^2(k+4)^2(k+6)^2} + \cdots\Big]$$

$$\left(\frac{\delta y}{\delta k}\right)_{k=-2} = x^{-2}\log_e x \, b_0\left[\frac{x^4}{-16} + \cdots\right] +$$

$$x^{-2} b_0\left[1 - \frac{1}{(-2)(2)} x^2\right] + \frac{(-4)x^4}{4 \cdot 4 \cdot 1 \cdot 6} + \cdots\Big]$$

$$y = C_1 (y)_{k=2} + C_2\left(\frac{\delta y}{\delta k}\right)_{k=-2}$$