UNIT 2

Series Solution of Ordinary Differential Equations and Special Functions

2.1. INTRODUCTION

The solution of ordinary linear differential equations of second order with variable coefficients in the form of an infinite convergent series is called *solution in series* or *integration in series.*

The series solution of certain differential equations give rise to *special functions* such as Bessel's function, Legendre's polynomials, Laguerre's polynomial, Hermite's polynomial, Chebyshev polynomials. These special functions have wide applications in engineering.

In this unit, we will discuss methods of solution of second order linear differential equations with variable coefficients in series along with Bessel's function, Legendre's polynomial and their properties.

2.2. DEFINITIONS

2.2.1. Power Series

An infinite series of the form

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots$$

is called a power series in ascending powers of $x - x_0$.

In particular, a power series in ascending powers of x is an infinite series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

e.g.,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

2.2.2. Analytic Function

A function $f(x)$ defined on an interval containing the point $x = x_0$ is called analytic at x_0 if its Taylor series $\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$ exists and converges to $f(x)$ for all x in the interval of convergence of Taylor's series.

Note 1. A rational function is analytic except at those values of x at which its denominator is zero. *e.g.*, Rational function $\dfrac{x}{x^2 - 5x + 6}$ is analytic everywhere except at $x = 2$ and $x = 3$.

Note 2. All polynomial functions e^x, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ are analytic everywhere.

2.2.3. Ordinary Point

A point $x = x_0$ is called an ordinary point of the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0 \qquad \ldots(1)$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

2.2.4. Regular and Irregular Singular Points

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a singular point of equation (1). There are two types of singular points :

(*i*) Regular singular point.

(*ii*) Irregular singular point.

A singular point $x = x_0$ of the differential equation (1) is called a regular singular point of (1) if both $(x - x_0)\,P(x)$ and $(x - x_0)^2\,Q(x)$ are analytic at $x = x_0$.

A singular point which is not regular is called an irregular singular point.

Remark 1. When $x = 0$ is an ordinary point of equation (1), its every solution can be expressed as a series of the form

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots = \sum_{n=0}^{\infty} a_n x^n.$$

Remark 2. When $x = 0$ is a regular singular point of equation (1), at least one of its solution can be expressed as

$$y = a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \ldots = x^m\,(a_0 + a_1x + a_2x^2 + \ldots) = \sum_{n=0}^{\infty} a_n\, x^{m+n}$$

where m may be a positive or negative integer or a fraction.

Remark 3. If $x = 0$ is an irregular singular point of equation (1), then discussion of solution of the equation is beyond the scope of this book.

2.3. SOME IMPORTANT DIFFERENTIAL EQUATIONS

(*i*) **Legendre's differential equation**

The differential equation of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n + 1)\,y = 0$$

or

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dy}{dx}\right\} + n(n + 1)\,y = 0$$

is called Legendre's differential equation, where n is a real number.

(*ii*) **Bessel's differential equation**

The differential equation of the form

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

is called Bessel's differential equation of order n, where n is a positive constant.

(*iii*) **Chebyshev's differential equation**

The differential equation of the form

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + n^2 y = 0$$

is called Chebyshev's differential equation.

2.4. POWER SERIES SOLUTION, WHEN $x = 0$ IS AN ORDINARY POINT OF THE EQUATION $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$

Steps for solution :

1. Assume its solution to be of the form $y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ ...(1)
2. Find $\frac{dy}{dx}$ (or y') and $\frac{d^2y}{dx^2}$ (or y'') from y.
3. Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given differential equation.
4. Equate to zero the coefficients of various powers of x and find $a_2, a_3, a_4, a_5, \ldots$ in terms of a_0 and a_1.
5. Equate to zero, the coefficient of x^n. The relation so obtained is called the *recurrence relation*. It helps us in finding the values of other constants easily.
6. Give different values to n in the recurrence relation to determine various a_i's in terms of a_0 and a_1.
7. Substitute the values of $a_2, a_3, a_4, \ldots$ in assumed solution (1) above to get the series solution of the given equation having a_0 and a_1 as arbitrary constants.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series the differential equation*

$$\frac{d^2y}{dx^2} + xy = 0.$$

[G.B.T.U. (SUM) 2010]

Sol. Comparing the given equation with the form $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$,

we get $\quad P(x) = 0, Q(x) = x$

At $x = 0$, Both $P(x)$ and $Q(x)$ are analytic, hence $x = 0$ is an *ordinary point*.

Assume its solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \qquad \ldots(1)$$

Then, $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + n\,a_nx^{n-1} + \dots$$

and $$\frac{d^2y}{dx^2} = 2.1\,a_2 + 3.2\,a_3x + 4.3\,a_4x^2 + 5.4\,a_5x^3 + \dots + n(n-1)\,a_nx^{n-2} + \dots$$

Substituting these values in the given differential equation, we get

$$[2.1.\,a_2 + 3.2\,.\,a_3x + 4.3\,.\,a_4x^2 + 5.4\,.\,a_5x^3 + \dots + n(n-1)\,a_nx^{n-2} + \dots]$$
$$+ x\,[a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots] = 0$$

$$2.1\,.\,a_2 + (3.2\,a_3 + a_0)\,x + (4.3\,.\,a_4 + a_1)x^2 + (5.4\,.\,a_5 + a_2)x^3 + \dots$$
$$+ \{(n+2)(n+1)\,a_{n+2} + a_{n-1}\}x^n + \dots = 0$$

Equating to zero, the various powers of x as,

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1\,.\,a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$$

Coefficient of $x = 0$

$$\Rightarrow \quad 3.2\,.\,a_3 + a_0 = 0$$

$$\Rightarrow \quad a_3 = -\frac{a_0}{3.2} \quad \Rightarrow \quad \boxed{a_3 = -\frac{a_0}{3!}}$$

Coefficient of $x^2 = 0$

$$\Rightarrow \quad 4.3\,.\,a_4 + a_1 = 0$$

$$\Rightarrow \quad a_4 = -\frac{a_1}{4.3} \quad \text{or} \quad \boxed{a_4 = -\frac{2a_1}{4!}}$$

Coefficient of $x^3 = 0$

$$\Rightarrow \quad 5.4\,.\,a_5 + a_2 = 0$$

$$\Rightarrow \quad a_5 = -\frac{a_2}{5.4} \quad \text{or} \quad \boxed{a_5 = 0}$$

Coefficient of $x^4 = 0$

$$\Rightarrow \quad 6.5\,.\,a_6 + a_3 = 0$$

$$\Rightarrow \quad a_6 = -\frac{a_3}{6.5} = \frac{a_0}{6.5.3!} \quad \text{or} \quad \boxed{a_6 = \frac{4a_0}{6!}}$$

and so on.

Coefficient of $x^n = 0$

$$\Rightarrow \quad (n+2)(n+1)\,a_{n+2} + a_{n-1} = 0$$

$$\Rightarrow \quad \boxed{a_{n+2} = -\frac{a_{n-1}}{(n+2)(n+1)}}$$

which is the recurrence relation.

Putting $n = 5, 6, 7, \dots$, successively in recurrence relation, we obtain

$$a_7 = \frac{5.2a_1}{7!},\ a_8 = 0,\ a_9 = \frac{-7.4}{9!}\,a_0 \text{ and so on.}$$

Substituting these values in (1), we get

$$y = a_0 + a_1x - \frac{a_0}{3!}\,x^3 - \frac{2a_1}{4!}\,x^4 + \frac{4a_0}{6!}\,x^6 + \frac{5.2\,a_1}{7!}\,x^7 - \frac{7.4}{9!}\,a_0\,x^9 + \dots$$

$$\Rightarrow \qquad y = a_0\left[1 - \frac{x^3}{3!} + \frac{1.4}{6!}x^6 - \frac{1.4.7}{9!}x^9 + \dots\right] + a_1\left[x - \frac{2}{4!}x^4 + \frac{2.5}{7!}x^7 - \dots\right]$$

where a_0 and a_1 are constants.

Example 2. *Solve in series the differential equation*

$$(1+x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \text{ about the point } x = 0.$$

Sol. Comparing the given differential equation with the form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{x}{1+x^2} \quad \text{and} \quad Q(x) = \frac{-1}{1+x^2}.$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Assume the solution to be

$$y = a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots \qquad \dots(1)$$

Then, $$\frac{dy}{dx} = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1} + \dots$$

and $$\frac{d^2y}{dx^2} = 2.1.\ a_2 + 3.2.\ a_3x + \dots + n(n-1)a_n\,x^{n-2} + \dots$$

Substituting these values in given equation, we get

$$(1+x^2)\,[2.1.\ a_2 + 3.2.\ a_3x + 4.3.a_4x^2 + \dots + n(n-1)\,a_nx^{n-2} + \dots]$$
$$+ x\,[a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + \dots + n\,a_nx^{n-1} + \dots]$$
$$- [a_0 + a_1x + a_2x^2 + a_3x^3 + \dots + a_nx^n + \dots] = 0$$

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1.\ a_2 - a_0 = 0 \qquad \Rightarrow \qquad \boxed{a_2 = \frac{a_0}{2}}$$

Coefficient of $x = 0$

$$\Rightarrow \quad 3.2\ a_3 + a_1 - a_1 = 0 \qquad \Rightarrow \qquad \boxed{a_3 = 0}$$

Coefficient of $x^2 = 0$

$$\Rightarrow \quad 2.1\,.\ a_2 + 4.3.a_4 + 2a_2 - a_2 = 0$$

$$\Rightarrow \quad 4.3\ a_4 + 3a_2 = 0$$

$$\Rightarrow \quad a_4 = -\frac{a_2}{4} = -\frac{a_0}{8} \quad \text{or} \quad \boxed{a_4 = -\frac{a_0}{8}}$$

Coefficient of $x^3 = 0$

$$\Rightarrow \quad 5.4.\ a_5 + 3.2.\ a_3 + 3a_3 - a_3 = 0$$

$$\Rightarrow \quad 20a_5 + 8a_3 = 0 \qquad \Rightarrow \qquad \boxed{a_5 = 0}$$

Coefficient of $x^4 = 0$

$\Rightarrow \quad 6.5 \,.\, a_6 + 4.3 \,.\, a_4 + 4a_4 - a_4 = 0$

$\Rightarrow \quad 30a_6 + 15a_4 = 0$

$$\Rightarrow \quad a_6 = -\frac{a_4}{2} = \frac{a_0}{16} \quad \text{or} \quad \boxed{a_6 = \frac{a_0}{16}}$$

Similarly, $a_7 = 0, a_9 = 0, a_{11} = 0$ and so on.

Also, coefficient of $x^n = 0$

$$(n + 2)(n + 1)\, a_{n+2} + n(n - 1)a_n + na_n - a_n = 0$$

$$\Rightarrow \quad a_{n+2} = -\left(\frac{n-1}{n+2}\right) a_n \qquad | \because \; n + 1 \neq 0$$

Putting $n = 6, 8, 10, \ldots$, we get

$$a_8 = -\frac{5}{8} a_6 = -\frac{5a_0}{128}$$

$$a_{10} = -\frac{7}{10} a_8 = \frac{7a_0}{256} \text{ and so on.}$$

Substituting these values in (1), we get

$$y = a_0 + a_1 x + \frac{a_0}{2} x^2 - \frac{a_0}{8} x^4 + \frac{a_0}{16} x^6 - \frac{5a_0}{128} x^8 + \frac{7a_0}{256} x^{10} - \ldots$$

$$\Rightarrow \quad y = a_0 \left(1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \frac{5x^8}{128} + \frac{7x^{10}}{256} - \ldots\right) + a_1 x$$

where a_0 and a_1 are constants.

Example 3. *Solve in series Chebyshev's differential equation (when $n = 2$)*

Or

Solve : $(1 - x^2)y'' - xy' + 4y = 0$ in series. [G.B.T.U. 2012; U.P.T.U. 2006]

Sol. Comparing the given differential equation with the form

$y'' + P(x)\, y' + Q(x)\, y = 0$, we get

$$P(x) = \frac{-x}{1 - x^2}, \; Q(x) = \frac{4}{1 - x^2}$$

Since both $P(x)$ and $Q(x)$ are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \qquad \ldots(1)$$

Then, $y' = a_1 + 2.a_2 x + 3.a_3 x^2 + \ldots + na_n x^{n-1} + \ldots$

and $y'' = 2.1.\, a_2 + 3.2.a_3 x + \ldots + n(n - 1)a_n x^{n-2} + \ldots$

Substituting these values in given equation, we get

$$(1 - x^2)\,[2.1.\, a_2 + 3.2.\, a_3 x + 4.3.\, a_4 x^2 + \ldots + n(n - 1)\, a_n x^{n-2} + \ldots]$$
$$- x\,[a_1 + 2a_2 x + 3a_3 x^2 + \ldots + na_n x^{n-1} + \ldots] + 4\,[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots] = 0$$

Coefficient of $x^0 = 0$

$$\Rightarrow \quad 2.1.\, a_2 + 4a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -2a_0}$$

Coefficient of $x = 0$

$\Rightarrow \quad 3.2\, a_3 - a_1 + 4a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = -\dfrac{a_1}{2}}$

Coefficient of $x^2 = 0$

$\Rightarrow \quad 4.3.\, a_4 - 2.1\, a_2 - 2a_2 + 4a_2 = 0 \quad \Rightarrow \quad \boxed{a_4 = 0}$

Coefficient of $x^3 = 0$

$\Rightarrow \quad 5.4\, a_5 - 3.2\, a_3 - 3a_3 + 4a_3 = 0$

$$\Rightarrow \quad a_5 = \frac{a_3}{4} = \frac{1}{4}\left(\frac{-a_1}{2}\right) = -\frac{a_1}{8}$$

$$\Rightarrow \quad \boxed{a_5 = -\frac{a_1}{8}} \text{ and so on.}$$

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1 x - 2a_0 x^2 - \frac{a_1}{2} x^3 - \frac{a_1}{8} x^5 + \ldots$$

$$\Rightarrow \quad y = a_0 (1 - 2x^2) + a_1 x \left(1 - \frac{x^2}{2} - \frac{x^4}{8} - \ldots\right)$$

where a_0 and a_1 are constants.

Example 4. *Find the power series solution of the following differential equation about $x = 0$*

$$(1 - x^2)\frac{d^2 y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0.$$

Sol. Comparing the given differential equation with the form

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{-2x}{1 - x^2},\ Q(x) = \frac{2}{1 - x^2}$$

Since both $P(x)$ and $Q(x)$ are analytic at $x = 0$, hence $x = 0$ is an *ordinary point* of the given equation.

Assume the solution to be

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots \quad \ldots(1)$$

Then,
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n\, a_n\, x^{n-1} + \ldots$$

and
$$y'' = 2.1.\, a_2 + 3.2.\, a_3\, x + 4.3.\, a_4\, x^2 + \ldots + n(n-1)a_n\, x^{n-2} + \ldots$$

Substituting these values in given equation, we get

$$(1 - x^2)\,[2.1.\, a_2 + 3.2.\, a_3 x + 4.3.\, a_4 x^2 + \ldots + n(n-1)\, a_n x^{n-2} + \ldots]$$
$$- 2x\,[a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n\, a_n\, x^{n-1} + \ldots]$$
$$+ 2\,[a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots] = 0$$

Coefficient of $x^0 = 0$

$\Rightarrow \quad 2.1.\, a_2 + 2a_0 = 0 \quad \Rightarrow \quad \boxed{a_2 = -a_0}$

Coefficient of $x = 0$

$\Rightarrow \quad 3.2.\, a_3 - 2a_1 + 2a_1 = 0 \quad \Rightarrow \quad \boxed{a_3 = 0}$