Then, $y' = m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots$

and $y'' = m(m-1) a_0 x^{m-2} + (m+1) m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + (m+3)(m+2) a_3 x^{m+1} + \ldots$

Substituting these values in given equation, we get

$$2x^2 [m(m-1) a_0 x^{m-2} + (m+1)m a_1 x^{m-1} + (m+2)(m+1) a_2 x^m + (m+3)(m+2) a_3 x^{m+1} + \ldots] + (2x^2 - x) [m a_0 x^{m-1} + (m+1) a_1 x^m + (m+2) a_2 x^{m+1} + (m+3) a_3 x^{m+2} + \ldots] + [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, coeff. of lowest power of $x = 0$ *i.e.*, coeff. of $x^m = 0$

$$2m(m-1) a_0 - m a_0 + a_0 = 0$$

$\Rightarrow$ $$(2m^2 - 3m + 1) a_0 = 0$$

$\Rightarrow$ $$(2m-1)(m-1) = 0 \text{ (since } a_0 \neq 0)$$

which is indicial equation.

Its roots are $\boxed{m = 1, \frac{1}{2}}$

Roots are distinct and donot differ by an integer.

Now, coeff. of $x^{m+1} = 0$

$\Rightarrow$ $$2m(m+1) a_1 + 2m a_0 - (m+1) a_1 + a_1 = 0$$

$\Rightarrow$ $$(2m^2 + m) a_1 + 2m a_0 = 0$$

$\Rightarrow$ $$\boxed{a_1 = -\frac{2}{2m+1} a_0} \qquad | \because m \neq 0$$

Coefficient of $x^{m+2} = 0$

$\Rightarrow$ $$2(m+2)(m+1) a_2 + 2(m+1) a_1 - (m+2) a_2 + a_2 = 0$$

$\Rightarrow$ $$(2m^2 + 5m + 3) a_2 + 2(m+1) a_1 = 0$$

$\Rightarrow$ $$(2m+3)(m+1) a_2 + 2(m+1) a_1 = 0$$

$\Rightarrow$ $$a_2 = \frac{-2}{2m+3} a_1 = \frac{(-2)}{2m+3} \cdot \frac{(-2)}{2m+2} a_0$$

$\Rightarrow$ $$\boxed{a_2 = \frac{4}{(2m+1)(2m+3)} a_0}$$

Similarly, we can find

$$a_3 = \frac{-8}{(2m+1)(2m+3)(2m+5)} a_0$$

$$a_4 = \frac{16}{(2m+1)(2m+3)(2m+5)(2m+7)} a_0$$

and so on.

$$\therefore \quad y = a_0 x^m \left[1 - \frac{2}{2m+1} x + \frac{4}{(2m+1)(2m+3)} x^2 - \frac{8}{(2m+1)(2m+3)(2m+5)} x^3 + \ldots\right] \qquad \ldots(2)$$

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Now, $y_1 = (y)_{m=1}$

$$y_1 = a_0\, x\left[1 - \frac{2}{3}x + \frac{4}{3.5}x^2 - \frac{8}{3.5.7}x^3 + \ldots\right]$$

or

$$\boxed{y_1 = a_0\, x\left(1 - \frac{2}{3}x + \frac{2^2}{3.5}x^2 - \frac{2^3}{3.5.7}x^3 + \ldots\right)} \quad \ldots(3)$$

and $y_2 = (y)_{m=1/2}$

$$\boxed{y_2 = a_0\, x^{1/2}\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right]} \quad \ldots(4)$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2$$

$$= c_1 a_0\, x\left(1 - \frac{2}{3}x + \frac{2^2}{3.5}x^2 - \frac{2^3}{3.5.7}x^3 + \ldots\right) + c_2 a_0 \sqrt{x}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)$$

$$\Rightarrow \quad \boxed{y = \mathrm{A}x\left(1 - \frac{2}{3}x + \frac{2^2}{3.5}x^2 - \frac{2^3}{3.5.7}x^3 + \ldots\right) + \mathrm{B}\sqrt{x}\left(1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \ldots\right)}$$

where A and B are constants.

TEST YOUR KNOWLEDGE

Solve in series :

1. $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0$

2. $x(2+x^2)\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 6xy = 0$

3. $3x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$

4. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (1-x^2)y = 0$ *[G.B.T.U. (C.O.) 2011]*

5. $2x^2y'' + xy' - (x+1)y = 0$ *(G.B.T.U. 2011)*

6. $2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0$ *(G.B.T.U. 2010)*

7. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x-5)y = 0$

8. $y'' + \dfrac{1}{4x}y' + \dfrac{1}{8x^2}y = 0$

9. $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (x^2+1)y = 0$

10. $4x\dfrac{d^2y}{dx^2} + 2(1-x)\dfrac{dy}{dx} - y = 0.$

Answers

1. $y = \mathrm{A}\left(1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \ldots\right) + \mathrm{B}x^{7/3}\left(1 + \frac{8}{10}x + \frac{8.11}{10.13}x^2 + \frac{8.11.14}{10.13.16}x^3 + \ldots\right)$

$$\quad + Bx^{3/2}\left(1+\frac{3}{8}x^2-\frac{3.1}{8.16}x^4+\frac{5.3.1}{8.16.24}x^6-...\right)$$

2. $y = A\left(1+3x^2+\frac{3}{5}x^4-\frac{1}{15}x^6+...\right)+Bx^{1/3}\left(1-\frac{x}{4}+\frac{x^2}{56}-\frac{x^3}{1680}+...\right)$

3. $y = A\left(1-\frac{x}{2}+\frac{x^2}{20}-\frac{x^3}{480}+...\right)+Bx^{1/2}\left(1+\frac{x^2}{2.3}+\frac{x^4}{2.4.3.7}+...\right)$

4. $y = Ax\left(1+\frac{x^2}{2.5}+\frac{x^4}{2.4.5.9}+...\right)+Bx^{-1/2}\left(1-x-\frac{1}{2}x^2+...\right)$

5. $y = Ax\left(1+\frac{1}{5}x+\frac{1}{70}x^2+...\right)+B\sqrt{x}\,(1-x)$

6. $y = A\left(1-3x+\frac{3x^2}{1.3}+\frac{3x^3}{3.5}+\frac{3x^4}{5.7}+...\right)$

7. $y = c_1x^{5/2}\left(1-\frac{x}{9}+\frac{x^2}{198}-\frac{x^3}{7722}+...\right)+c_2x^{-1}\left(1+\frac{x}{5}+\frac{x^2}{30}+\frac{x^3}{90}+...\right)$

8. $y = A\sqrt{x}+Bx^{1/4}$

9. $y = Ax\left(1-\frac{x^2}{10}+\frac{x^4}{360}-...\right)+Bx^{1/2}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-...\right)$

10. $y = A\left(1+\frac{x}{2.1!}+\frac{x^2}{2^2.2!}+\frac{x^3}{2^3 3!}+...\right)+B\sqrt{x}\left(1+\frac{x}{1.3}+\frac{x^2}{1.3.5}+\frac{x^3}{1.3.5.7}+...\right).$

2.5.2. Case II. When Roots are Equal e.g., $m_1 = m_2 = 0$

Complete solution is

$$\boxed{y = c_1\,(y)_{m_1}+c_2\left(\frac{\partial y}{\partial m}\right)_{m_1}}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series :*

$$x\,(x-1)\frac{d^2y}{dx^2}+(3x-1)\frac{dy}{dx}+y=0.$$

Sol. Comparing the given equation with

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)\,y=0, \text{ we get}$$

$$P(x)=\frac{3x-1}{x(x-1)} \text{ and } Q(x)=\frac{1}{x\,(x-1)}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now, $\quad x\,P(x)=\frac{3x-1}{x-1}$ and $x^2\,Q(x)=\frac{x}{x-1}$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0x^m + a_1x^{m+1}+a_2x^{m+2}+a_3x^{m+3}+...$$

Then, $\quad y' = m\,a_0\,x^{m-1}+(m+1)\,a_1\,x^m+(m+2)\,a_2\,x^{m+1}+(m+3)\,a_3\,x^{m+2}+...$

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and
$$y'' = m(m-1)a_0 x^{m-2} + (m+1)m\, a_1 x^{m-1} + (m+2)(m+1)\, a_2 x^m + (m+3)(m+2)\, a_3 x^{m+1} + \ldots$$

Substituting these values in given equation, we get

$$x(x-1)\left[m(m-1)a_0 x^{m-2} + (m+1)\, m\, a_1 x^{m-1} + (m+2)(m+1)\, a_0 x^m + (m+3)(m+2)\, a_3 x^{m+1} + \ldots\right]$$
$$+ (3x-1)\left[m\, a_0 x^{m-1} + (m+1)\, a_1 x^m + (m+2)\, a_2 x^{m+1} + (m+3)\, a_3 x^{m+2} + \ldots\right]$$
$$+ \left[a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots\right] = 0$$

Now, coefficient of lowest power of $x = 0$

$\Rightarrow$ coefficient of $x^{m-1} = 0$

$\Rightarrow$ $-m(m-1)a_0 - m\, a_0 = 0 \quad \Rightarrow \quad -m^2 a_0 = 0$

$m^2 = 0 \qquad (\because \; a_0 \neq 0)$

which is *Indicial equation*

Its roots are $\boxed{m = 0, 0}$

Roots are equal.

Now, coefficient of $x^m = 0$

$\Rightarrow$ $m(m-1)a_0 - (m+1)\, m\, a_1 + 3m\, a_0 - (m+1)\, a_1 + a_0 = 0$

$\Rightarrow$ $(m+1)^2 a_0 - (m+1)^2 a_1 = 0$

$\Rightarrow$ $\boxed{a_1 = a_0} \qquad (\because \; m \neq -1)$

Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+1)\, m\, a_1 - (m+2)(m+1)\, a_2 + 3(m+1)\, a_1 - (m+2)a_2 + a_1 = 0$

$\Rightarrow$ $(m+2)^2 a_1 - (m+2)^2 a_2 = 0$

$\Rightarrow$ $a_2 = a_1 \qquad (\because \; m \neq -2)$

$\Rightarrow$ $\boxed{a_2 = a_0}$

Similarly, we can show that

$$a_3 = a_0$$
$$a_4 = a_0 \quad \text{and so on.}$$

$\therefore$ $y = a_0 x^m (1 + x + x^2 + x^3 + \ldots)$ | From (1)

Now, $y_1 = (y)_{m=0} = a_0 x^0 (1 + x + x^2 + x^3 + \ldots) = a_0 (1 + x + x^2 + x^3 + \ldots)$

$$y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = \left[a_0 (1 + x + x^2 + x^3 + \ldots)\, x^m \log x\right]_{m=0}$$
$$= a_0 \log x (1 + x + x^2 + x^3 + \ldots)$$

Hence the complete solution is given by

$$y = c_1 y_1 + c_2 y_2 = c_1 a_0 (1 + x + x^2 + x^3 + \ldots) + c_2 a_0 \log x (1 + x + x^2 + x^3 + \ldots)$$
$$y = (A + B \log x)(1 + x + x^2 + x^3 + \ldots)$$

where A and B are constants.

Example 2. *Solve in series the differential equation :*

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} - y = 0.$$

Sol. Comparing with the equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0, \text{ we get}$$

$$P(x) = \frac{1}{x} \text{ and } Q(x) = -\frac{1}{x}$$

Since at $x = 0$, both $P(x)$ and $Q(x)$ are not analytic $\therefore$ $x = 0$ is a *singular point*.

Also, $x\,P(x) = 1$ and $x^2\,Q(x) = -x$

Both $x\,P(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is a *regular singular point*.

Let us assume

$$y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots \quad \ldots(1)$$

Then, $$y' = m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots$$

and $$y'' = m(m-1)\,a_0 x^{m-2} + (m+1)\,m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots$$

Substituting these values in the given equation, we get

$$x\,[m(m-1)\,a_0 x^{m-2} + (m+1)m\,a_1 x^{m-1} + (m+2)(m+1)\,a_2 x^m + (m+3)(m+2)\,a_3 x^{m+1} + \ldots]$$
$$+ [m\,a_0 x^{m-1} + (m+1)\,a_1 x^m + (m+2)\,a_2 x^{m+1} + (m+3)\,a_3 x^{m+2} + \ldots]$$
$$- [a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + a_3 x^{m+3} + \ldots] = 0$$

Now, coefficient of $x^{m-1} = 0$

$\Rightarrow$ $m(m-1)\,a_0 + m a_0 = 0$

$\Rightarrow$ $m^2 a_0 = 0 \Rightarrow m^2 = 0$ $(\because a_0 \neq 0)$

which is Indicial equation.

Its roots are $\boxed{m = 0, 0}$ *which are equal.*

Coefficient of $x^m = 0$

$\Rightarrow$ $(m+1)\,m a_1 + (m+1)a_1 - a_0 = 0 \Rightarrow (m+1)^2\,a_1 = a_0$

$$\Rightarrow \quad \boxed{a_1 = \frac{a_0}{(m+1)^2}}$$

Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $(m+2)(m+1)\,a_2 + (m+2)\,a_2 - a_1 = 0 \Rightarrow (m+2)^2\,a_2 = a_1$

$$\Rightarrow \quad a_2 = \frac{a_1}{(m+2)^2} \Rightarrow \boxed{a_2 = \frac{a_0}{(m+1)^2\,(m+2)^2}}$$

Similarly,

$$a_3 = \frac{a_0}{(m+1)^2\,(m+2)^2\,(m+3)^2} \text{ and so on.}$$

$$\therefore \text{ From (1), } y = a_0 x^m \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2\,(m+2)^2} + \frac{x^3}{(m+1)^2\,(m+2)^2\,(m+3)^2} + \ldots\right] \quad \ldots(2)$$

Now, $$y_1 = (y)_{m=0} = a_0\left[1 + x + \frac{x^2}{(2!)^2} + \frac{x^3}{(3!)^2} + \ldots\right] \quad \ldots(3)$$

To get the second independent solution, differentiate (1) partially w.r.t. m.

$$\frac{\partial y}{\partial m} = a_0 x^m \log x \left[1 + \frac{x}{(m+1)^2} + \frac{x^2}{(m+1)^2 (m+2)^2} + \frac{x^3}{(m+1)^2 (m+2)^2 (m+3)^2} + \ldots\right]$$

$$+ a_0 x^m \left[-\frac{2x}{(m+1)^3} - \frac{2}{(m+1)^2 (m+2)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2}\right\} x^2\right.$$

$$\left. - \frac{2}{(m+1)^2 (m+2)^2 (m+3)^2}\left\{\frac{1}{m+1} + \frac{1}{m+2} + \frac{1}{m+3}\right\} x^3 - \ldots\right]$$

The second solution is $y_2 = \left(\frac{\partial y}{\partial m}\right)_{m=0} = a_0 \log x \left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$

$$- 2a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

$$= y_1 \log x - 2a_0 \left[x + \frac{1}{(2\,!)^2} + \left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

Hence the complete solution is

$$y = c_1 y_1 + c_2 y_2 = (c_1 a_0 + c_2 a_0 \log x)\left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$$

$$- 2c_2 a_0 \left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

$$\Rightarrow \quad y = (\mathrm{A} + \mathrm{B} \log x)\left[1 + x + \frac{x^2}{(2\,!)^2} + \frac{x^3}{(3\,!)^2} + \ldots\right]$$

$$- 2\mathrm{B}\left[x + \frac{1}{(2\,!)^2}\left(1 + \frac{1}{2}\right) x^2 + \frac{1}{(3\,!)^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) x^3 + \ldots\right]$$

where $c_1 a_0 = \mathrm{A}$, $c_2 a_0 = \mathrm{B}$.

TEST YOUR KNOWLEDGE

Solve in series :

1. (*i*) $xy'' + (1 + x)\, y' + 2y = 0$ (*ii*) $x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - xy = 0$ (*M.T.U. 2012*)

2. $x^2 \dfrac{d^2y}{dx^2} + x(x - 1)\dfrac{dy}{dx} + (1 - x)\, y = 0$ [*M.T.U. 2011 ; U.P.T.U. (SUM) 2008*]

3. $(x - x^2)\dfrac{d^2y}{dx^2} + (1 - 5x)\dfrac{dy}{dx} - 4y = 0$

4. $(x - x^2)\, y'' + (1 - x)\, y' - y = 0$

5. $x^2 y'' - x(1 + x)\, y' + y = 0$

6. $xy'' + y' + x^2 y = 0$ (*U.P.T.U. 2007*)

7. $xy'' + y' + xy = 0$ (*Bessel's equation of order zero*)

Answers

1. (i) $y = A\left(1 - 2x + \frac{3}{2!}x^2 - \frac{4}{3!}x^3 + \ldots\right) + B\left[y_1 \log x + a_0\left(3x - \frac{13}{4}x^2 + \ldots\right)\right]$

   (ii) $y = (A + B\log x)\left(1 + \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} + \ldots\right) - B\left(\frac{x^2}{2^2} + \frac{3x^4}{2.4^3} + \ldots\right)$

2. $y = Ax + B\left[x \log x - x + \frac{x^2}{4} - \ldots\right]$

3. $y = A(1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots) + B\,[y_1 \log x - 2a_0\,(1.2x + 2.3\,x^2 + 3.4\,x^3 + \ldots)]$

4. $y = A\left(1 + x + \frac{2}{4}x^2 + \frac{2.5}{4.9}x^3 + \ldots\right) + B\left[y_1 \log x + a_0\left(-2x - x^2 - \frac{14}{27}x^3 - \ldots\right)\right]$

5. $y = Ax\left(1 + x + \frac{1}{2}x^2 + \frac{1}{2.3}x^3 + \ldots\right) + B\left[y_1 \log x + a_0 x^2\left(-1 - \frac{3}{4}x + \ldots\right)\right]$

6. $y = A\left[1 - \frac{x^3}{3^2} + \frac{x^6}{3^4 (2!)^2} - \frac{x^9}{3^6 (3!)^2} + \ldots\right] + B\left[y_1 \log x + 2a_0\left\{\frac{x^3}{3^3} - \frac{1}{3^5 (2!)^2}\left(1 + \frac{1}{2}\right)x^6 + \ldots\right\}\right]$

7. $y = A\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 . 4^2} - \frac{x^6}{2^2 . 4^2 . 6^2} + \ldots\right) + B\left[y_1 \log x + a_0\left\{\frac{x^2}{2^2} - \frac{1}{2^2 . 4^2}\left(1 + \frac{1}{2}\right)x^4\right.\right.$

   $\left.\left. + \frac{1}{2^2 . 4^2 . 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right)x^6 - \ldots\right\}\right]$

2.5.3. Case III. When Roots are Distinct, Differ by Integer and Making a Coefficient of y Infinite

Let m_1 and m_2 be the roots such that $m_1 > m_2$

In this case, if some of the coefficients of y become infinite when $m = m_2$, we modify the form of y by replacing a_0 by $b_0\,(m - m_2)$.

Complete solution is

$$\boxed{y = c_1\,(y)_{m_1} + c_2\left(\frac{\partial y}{\partial m}\right)_{m_2}}.$$

Remark. We can also obtain two independent solutions by putting $m = m_2$ (value of m for which some coefficients of y become infinite) in modified form of y and $\frac{\partial y}{\partial m}$. The result of putting $m = m_1$ in will give a numerical multiple of that obtained by putting $m = m_2$.

ILLUSTRATIVE EXAMPLES

Example 1. *Obtain the series solution of the Bessel's equation of order two*

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - 4)\,y = 0 \quad near\ x = 0.$$

Sol. Comparing the given equation with the form

$$\frac{d^2 y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$