$$\therefore \quad \text{C.F.} = e^x (c_1 \cos x + c_2 \sin x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 2}(x) + \frac{1}{D^2 - 2D + 2}(e^x \cos x)$$

$$= \frac{1}{2 - 2D + D^2}(x) + e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 2}(\cos x)$$

$$= \frac{1}{2}\left[1 - \left(\frac{2D - D^2}{2}\right)\right]^{-1}(x) + e^x \cdot \frac{1}{D^2 + 1}\cos x$$

$$= \frac{1}{2}\left[1 + \left(\frac{2D - D^2}{2}\right)\right](x) + e^x \cdot x \cdot \frac{1}{2D}\cos x \qquad | \text{ Case of failure}$$

$$= \frac{1}{2}[1 + D](x) + e^x \cdot \frac{x}{2}\sin x \qquad | \text{ Leaving higher powers}$$

$$= \frac{1}{2}(x + 1) + \frac{xe^x}{2}\sin x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x (c_1 \cos x + c_2 \sin x) + \frac{1}{2}(x + 1) + \frac{xe^x}{2}\sin x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve :* $\frac{d^2 y}{dx^2} - 4y = x \sinh x.$

Sol. Given equation is

$$(D^2 - 4)y = x \sinh x$$

Auxiliary equation is

$$m^2 - 4 = 0 \text{ so that } m = \pm 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4} x \sinh x = \frac{1}{D^2 - 4} x \left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4} e^x \cdot x - \frac{1}{D^2 - 4} e^{-x} \cdot x\right] = \frac{1}{2}\left[e^x \frac{1}{(D+1)^2 - 4} x - e^{-x} \frac{1}{(D-1)^2 - 4} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^2 + 2D - 3} x - e^{-x} \frac{1}{D^2 - 2D - 3} x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{-3\left(1 - \frac{2D}{3} - \frac{D^2}{3}\right)} x - e^{-x} \frac{1}{-3\left(1 + \frac{2D}{3} - \frac{D^2}{3}\right)} x\right]$$

$$= -\frac{1}{6}\left[e^x \left\{1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right\}^{-1} x - e^{-x} \left\{1 + \left(\frac{2D}{3} - \frac{D^2}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^x\left(1+\frac{2D}{3}\ldots\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}\ldots\ldots\right)x\right] = -\frac{1}{6}\left[e^x\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right]$$

$$= -\frac{x}{3}\left(\frac{e^x - e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x + e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 6. *Solve :* $\dfrac{d^4 y}{dx^4} - y = \cos x \cosh x.$

Sol. Given equation is $(D^4 - 1)y = \cos x \cosh x$.

Auxiliary equation is

$m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0$ so that $m = \pm 1, \pm i$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{-x} + e^{0x}(c_3 \cos x + c_4 \sin x)$$
$$= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4 - 1}\cos x \cosh x = \frac{1}{D^4 - 1}\cos x\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^4 - 1}e^x \cos x + \frac{1}{D^4 - 1}e^{-x}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(D+1)^4 - 1}\cos x + e^{-x}\frac{1}{(D-1)^4 - 1}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{D^4 + 4D^3 + 6D^2 + 4D}\cos x + e^{-x}\frac{1}{D^4 - 4D^3 + 6D^2 - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-1^2)^2 + 4D(-1^2) + 6(-1^2) + 4D}\cos x\right.$$
$$\left. + e^{-x}\frac{1}{(-1^2)^2 - 4D(-1^2) + 6(-1^2) - 4D}\cos x\right]$$

$$= \frac{1}{2}\left[e^x \frac{1}{(-5)}\cos x + e^{-x}\frac{1}{(-5)}\cos x\right] = -\frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x = -\frac{1}{5}\cosh x \cos x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - \frac{1}{5}\cos x \cosh x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 7. (*i*) *Solve:* $(D^2 - 2D + 1)y = x e^x \sin x$ [M.T.U. (B. Pharm.) 2011]

(*ii*) *Solve:* $\dfrac{d^2 y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \cos x.$ (U.P.T.U. 2009)

Sol. (*i*) Auxiliary equation is

$$m^2 - 2m + 1 = 0$$

$\Rightarrow$ $m = 1, 1$

$\therefore$ $\text{C.F.} = (c_1 + c_2 x)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \sin x) = \frac{1}{(D-1)^2}(x\, e^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \sin x) = e^x \cdot \frac{1}{D^2}(x \sin x)$$

$$= e^x \cdot \frac{1}{D}(-x \cos x + \sin x) = -e^x(x \sin x + 2 \cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)\, e^x - e^x(x \sin x + 2 \cos x)$$

where c_1 and c_2 are arbitrary constants of integration.

(*ii*) Auxiliary equation is

$m^2 - 2m + 1 = 0$ $\Rightarrow$ $m = 1, 1$

$\therefore$ $\text{C.F.} = (c_1 + c_2\, x)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x\, e^x \cos x) = \frac{1}{(D-1)^2}(x\, e^x \cos x)$$

$$= e^x \cdot \frac{1}{(D+1-1)^2}(x \cos x) = e^x \cdot \frac{1}{D^2}(x \cos x)$$

$$= e^x \cdot \frac{1}{D}(x \sin x + \cos x) = e^x(-x \cos x + 2 \sin x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2\, x)\, e^x + e^x(-x \cos x + 2 \sin x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve:* $(D^4 + 6D^3 + 11D^2 + 6D)y = 20\, e^{-2x} \sin x.$

Sol. Auxiliary equation is

$$m^4 + 6m^3 + 11m^2 + 6m = 0$$

$\Rightarrow$ $m(m^3 + 6m^2 + 11m + 6) = 0$

$\Rightarrow$ $m = 0, -1, -2, -3$

$\therefore$ $\text{C.F.} = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4\, e^{-3x}$

$$\text{P.I.} = \frac{1}{D^4 + 6D^3 + 11D^2 + 6D}(20\, e^{-2x} \sin x)$$

$$= \frac{1}{D(D+1)(D+2)(D+3)}(20\, e^{-2x} \sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{(D-2)(D-1)\, D(D+1)}(\sin x)$$

$$= 20\, e^{-2x} \cdot \frac{1}{D^4 - 2D^3 - D^2 + 2D}(\sin x)$$

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$$= 20\, e^{-2x} \cdot \frac{1}{2+4D} \sin x = 10\, e^{-2x} \frac{1-2D}{1-4D^2} \sin x$$

$$= 2\, e^{-2x} (\sin x - 2 \cos x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2\, e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} + 2e^{-2x}(\sin x - 2 \cos x)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 9. *Solve the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = \frac{e^{-x}}{x+2}.$$

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad (m+1)^2 = 0 \quad \Rightarrow \quad m = -1, -1$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\left(\frac{e^{-x}}{x+2}\right) = e^{-x} \cdot \frac{1}{(D-1+1)^2}\left(\frac{1}{x+2}\right)$$

$$= e^{-x} \frac{1}{D^2}\left(\frac{1}{x+2}\right) = e^{-x} \cdot \frac{1}{D} \log(x+2)$$

$$= e^{-x}\left[\log(x+2) \cdot x - \int \frac{1}{x+2} \cdot x\, dx\right] = e^{-x}\left[x \log(x+2) - \int \left(1 - \frac{2}{x+2}\right) dx\right]$$

$$= e^{-x}\,[x \log(x+2) - x + 2 \log(x+2)] = e^{-x}\,[(x+2)\log(x+2) - x]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x)e^{-x} + e^{-x}\,\{(x+2)\log(x+2) - x\}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 10. *Solve* : $(D^2 + 2D + 1)y = x \cos x$.

Sol. Auxiliary equation is

$$m^2 + 2m + 1 = 0$$

$$\Rightarrow \quad m = -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}(x \cos x) = \text{Real part of } \frac{1}{D^2 + 2D + 1}(xe^{ix})$$

$$= \text{R.P. of } e^{ix} \cdot \frac{1}{(D+i)^2 + 2(D+i) + 1}(x) = \text{R.P. of } e^{ix} \cdot \frac{1}{D^2 + 2D(1+i) + 2i}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left[1 + \frac{1+i}{i} D + \frac{D^2}{2i}\right]^{-1}(x)$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(1 - \frac{1+i}{i} D\right) x \qquad \text{| Leaving higher powers}$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(x - \frac{1+i}{i}\right)$$

$$= \text{R.P. of } \frac{1}{2}(\cos x + i \sin x)(-ix + 1 + i) = \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

$\therefore$ The complete solution is given by

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)\, e^{-x} + \frac{1}{2}\cos x + \frac{1}{2}(x-1)\sin x$$

where c_1 and c_2 are the arbitrary constants of integration.

Example 11. *Solve the following differential equation :*

$$(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x.$$

[U.P.T.U. (B. Pharm.) 2009]

Sol. The auxiliary equation is

$$m^2 - 4m + 4 = 0 \;\Rightarrow\; m = 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2}(8x^2 e^{2x} \sin 2x) = 8\, e^{2x} \cdot \frac{1}{(D+2-2)^2}(x^2 \sin 2x)$$

$$= 8\, e^{2x} \cdot \frac{1}{D^2}(x^2 \sin 2x) = 8\, e^{2x} \cdot \frac{1}{D}\int x^2 \sin 2x\, dx$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[x^2 \cdot \left(-\frac{\cos 2x}{2}\right) - \int 2x \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + x \cdot \frac{\sin 2x}{2} - \int 1 \cdot \frac{\sin 2x}{2}\, dx\right]$$

$$= 8\, e^{2x} \cdot \frac{1}{D}\left[\frac{-x^2}{2}\cos 2x + \frac{x \sin 2x}{2} + \frac{\cos 2x}{4}\right]$$

$$= 8\, e^{2x} \cdot \left[\left(\frac{-x^2}{2}\right)\frac{\sin 2x}{2} - \int (-x)\frac{\sin 2x}{2}\, dx + \int x \frac{\sin 2x}{2}\, dx + \frac{\sin 2x}{8}\right]$$

$$= 8\, e^{2x}\left[\frac{-x^2}{4}\sin 2x + \frac{\sin 2x}{8} + \int x \sin 2x\, dx\right]$$

$$= 8\, e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x + x \cdot \left(\frac{-\cos 2x}{2}\right) - \int 1 \cdot \left(\frac{-\cos 2x}{2}\right) dx\right]$$

$$= 8\, e^{2x}\left[\left(\frac{1}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x + \frac{\sin 2x}{4}\right]$$

$$= 8\, e^{2x}\left[\left(\frac{3}{8} - \frac{x^2}{4}\right)\sin 2x - \frac{x}{2}\cos 2x\right]$$

$$= e^{2x}\left[(3 - 2x^2)\sin 2x - 4x\cos 2x\right]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x)\, e^{2x} + e^{2x}[(3 - 2x^2)\sin 2x - 4x \cos 2x]$$

where c_1 and c_2 are arbitrary constants of integration.