Classification

Classification of linear PDE of second order with two independent variables

General form –

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y}$$

$$+ Fu = \phi(x, y) \quad \text{--- (1)}$$

Where A, B, C, D, E & F are constants or functions of x & y

Equation (1) is

(i) Hyperbolic, if $B^2 - 4AC > 0$

(ii) Elliptic, if $B^2 - 4AC < 0$

(iii) Parabolic, if $B^2 - 4AC = 0$

Qu $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0$

$A = 1$, $B = 1$, $C = 1$

$B^2 - 4AC = -3$ PDE is Elliptic

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$$\begin{array}{c|c} - + & + + \\ \hline - - & + - \end{array}$$

Que $x^2 \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u = 0$

$A = x^2, \quad B = 0, \quad C = -1$

$B^2 - 4AC = 0 - 4(x^2)(-1) = 4x^2$

if $x = 0 \Rightarrow B^2 - 4AC = 0$ PDE is parabolic

if $x \neq 0 \Rightarrow B^2 - 4AC > 0$ PDE is hyperbolic

Que $\sqrt{y^2+x^2}\, u_{xx} + 2(x-y) u_{xy} + \sqrt{y^2+x^2}\, u_{yy} = 0$

$A = \sqrt{y^2+x^2}, \quad B = 2(x-y), \quad C = \sqrt{y^2+x^2}$

$B^2 - 4AC = -8xy$

In I^{st} quadrant, $B^2 - 4AC < 0 \rightarrow$ Elliptic

II " $> 0 \rightarrow$ Hyperbolic

III " $< 0 \rightarrow$ Elliptic

IV " $> 0 \rightarrow$ Hyperbolic

$\Rightarrow$ on y-axis or x-axis

$B^2 - 4AC = 0$ parabolic

Que $t\frac{\delta^2 u}{\delta t^2} + 2\frac{\delta^2 u}{\delta x \delta t} + x\frac{\delta^2 u}{\delta x^2} + \frac{\delta u}{\delta x} = 0$

$A = t, \quad B = 2, \quad C = x.$

$B^2 - 4AC = 4(1 - tx)$

if $(1 - tx) > 0 \Rightarrow 1 > tx \Rightarrow tx < 1$

$B^2 - 4AC > 0 \Rightarrow$ Hypebolic

if. $1 - tx < 0 \Rightarrow B^2 - 4AC < 0 \Rightarrow$ Elliptic

if $1 - tx = 0 \Rightarrow tx = 1 \Rightarrow B^2 - 4AC = 0$ Parabolic

Method of separation of variables -

it is used to solve linear PDE

let,

$u = XT \quad , X \equiv X(x) \quad , T \equiv T(t)$

or

$u = XY \quad , X \equiv X(x) \quad , Y \equiv Y(y)$

Que $\frac{\delta u}{\delta x} = \frac{2\delta u}{\delta t} + u$, $u(x, 0) = 6e^{-3x}$

Sol let $u = XT$, where $X \equiv X(x)$
$T \equiv T(t)$

u

x t

$$\frac{\delta u}{\delta x} = T \frac{dX}{dx}.$$

$$\frac{\delta u}{\delta t} = X \frac{dT}{dt}$$

Putting in eq ①

$$T \frac{dX}{dx} = 2X \frac{dT}{dt} + XT$$

dividing by XT on both side

$$\frac{T}{XT} \frac{dX}{dx} = \frac{2X}{XT} \frac{dT}{dt} + \frac{XT}{XT}$$

$$\frac{1}{X} \frac{dX}{dx} = \frac{2}{T} \frac{dT}{dt} + 1 = \lambda \quad (\text{say})$$

$$\frac{1}{X} \frac{dX}{dx} = \lambda \text{ --- } ② \qquad \frac{2}{T} \frac{dT}{dt} + 1 = \lambda \text{ --- } ③$$

$$\int \frac{dX}{X} = \int \lambda dx$$

$$\log_e X = \lambda x + a$$

$$X = e^{(\lambda x + a)}$$

$$\boxed{X = c_1 e^{\lambda x}} \qquad c_1 \equiv e^a$$

$$\int \frac{dT}{T} = \int \frac{(\lambda-1)}{2} dt$$

$$\log_e T = \frac{(\lambda-1)}{2} t + b.$$

$$\boxed{T = C_2 e^{\frac{(\lambda-1)}{2} t}} , \quad C_2 = e^b$$

$$u = XT$$

$$\boxed{u = C_1 C_2 e^{\lambda x + \left(\frac{\lambda-1}{2}\right) t}}$$

General solution is

$$u = \sum_{n=1}^{n=\infty} a_n e^{\lambda_n x + \left(\frac{\lambda_n - 1}{2}\right) t}$$

$$u(x,0) = \sum a_n e^{\lambda_n x}$$

$$6e^{-3x} = a_1 e^{\lambda_1 x} + a_2 e^{\lambda_2 x} + \cdots\cdots\cdots$$

$a_1 = 6, \quad a_2 = 0, \quad a_2 = a_3 = a_4 \cdots\cdots = 0$

$\lambda_1 = -3$

$$u = 6e^{-3x - 2t}$$

Ans $u\frac{\delta u}{\delta t} + \frac{\delta u}{\delta x} = 3u$ $\quad u = 3e^{-x} - e^{-5x}$ at $t=0$

Let $u = XT \quad , \quad X \equiv X(x) \quad , \quad T \equiv T(t)$

$$\frac{\delta u}{\delta x} = T\frac{dX}{dx} \quad , \quad \frac{\delta u}{\delta t} = X\frac{dT}{dt}$$

$$uX\frac{dT}{dt} + T\frac{dX}{dx} = 3XT$$

dividing by XT

$$\frac{u\cancel{X}}{\cancel{X}T}\frac{dT}{dt} + \frac{\cancel{T}}{X\cancel{T}}\frac{dX}{dx} = \frac{3\cancel{XT}}{\cancel{XT}}$$

$$\frac{u}{T}\frac{dT}{dt} + \frac{1}{X}\frac{dX}{dx} = 3$$

$$\frac{u}{T}\frac{dT}{dt} = 3 - \frac{1}{X}\frac{dX}{dx} = \lambda \text{ (say)}$$

$$\frac{u}{T}\frac{dT}{dt} = \lambda \quad \text{—— (2)}$$

$$\int\frac{dT}{T} = \int\frac{\lambda}{u}dt$$

$$\log_e T = \frac{1}{u}\lambda t + a$$

$$\boxed{T = C_1 e^{\lambda t/u}} \qquad C_1 \equiv e^a$$

$$3 - \frac{1}{X}\frac{dX}{dx} = \lambda \quad (3)$$

from Eq^n (3)

$$3 - \lambda = \frac{1}{X}\frac{dX}{dx}$$

$$\int \frac{dX}{X} = \int (3-\lambda)\, dx$$

$$\log_e X = (3-\lambda)x + b$$

$$X = c_2 e^{(3-\lambda)x}$$

$$u = XT$$

$$u = c_1 c_2 e^{\frac{\lambda t}{4} + (3-\lambda)x}$$

General Sol. is

$$u = \sum_{n=1}^{n=\infty} a_n e^{\frac{\lambda_n t}{4} + (3-\lambda_n)x} \quad (4)$$

Put $t = 0$

$$u(x,0) = \sum a_n e^{(3-\lambda_n)x}$$

$$3e^{-x} - e^{-5x} = a_1 e^{(3-\lambda_1)x} + a_2 e^{(3-\lambda_2)x} + a_3 \ldots$$

$a_1 = 3$, $a_2 = -1$, $a_3 = a_4 = \ldots 0$

$-1 = 3 - \lambda_1$ $\qquad$ $3 - \lambda_2 = -5$

$\boxed{\lambda_1 = 4}$ $\qquad$ $\boxed{\lambda_2 = 8}$

$$u = 3e^{y-x} + (-e^{2y-5x})$$

$$\boxed{u = 3e^{y-x} - e^{2y-5x}}$$

Qn $\frac{\delta^2 u}{\delta x^2} - 2\frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = 0$ —— (1)

$u = XY \qquad X \equiv X(x), \quad Y \equiv Y(y)$

$\frac{\delta u}{\delta x} = Y\frac{dX}{dx} \qquad \frac{\delta u}{\delta y} = X\frac{dY}{dy}$

$\frac{\delta^2 u}{\delta x^2} = Y\frac{d^2X}{dx^2}$

$Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$

divide by XY

$\frac{\cancel{Y}d^2X}{X\cancel{Y}dx^2} - \frac{2\cancel{Y}}{X\cancel{Y}}\frac{dX}{dx} + \frac{\cancel{X}}{\cancel{X}Y}\frac{dY}{dy} = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 0$

$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = \frac{1}{Y}\frac{dY}{dy} = \lambda$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = \lambda \quad \text{— (2)}, \quad \frac{1}{Y}\frac{dY}{dy} = \lambda \quad \text{— (3)}$$

$$\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - \lambda x = 0 \qquad \int \frac{1}{Y}\,dY = \int \lambda\,dy$$

$$(D^2 - 2D - \lambda)X = 0 \quad D \equiv \frac{d}{dx} \qquad \log_e Y = \lambda y + a$$

$$Y = e^{\lambda y + a}$$

$$A.E = m^2 - 2m - \lambda = 0 \qquad Y = C_3 e^{\lambda y}$$

$$m = \frac{2 \pm \sqrt{4 + 4\lambda}}{2}$$

$$= \frac{2 \pm 2\sqrt{1+\lambda}}{2}$$

$$m = 1 \pm \sqrt{1+\lambda}$$

$$C.F = C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x}$$

$$P.I = 0$$

$$X = C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x}$$

$$u = XY$$

$$\overbrace{\left\{ C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x} \right\} C_3 e^{\lambda y}}^{C_1 C_2 C_3}$$

$$= C_1 C_3 e^{(1+\sqrt{1+\lambda})x + \lambda y} + C_2 C_3 e^{(1-\sqrt{1+\lambda})x + \lambda y}$$

General solution is.

$$u = \sum a_n e^{(1+\sqrt{1+\lambda_n})x + \lambda_n y} + b_n e^{(1-\sqrt{1+\lambda_n})x + \lambda_n y}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = \lambda \quad \text{— (1)}, \qquad \frac{1}{Y}\frac{dY}{dy} = \lambda \quad \text{— (2)}$$

$$\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - \lambda x = 0 \qquad\qquad \int \frac{1}{Y}\, dY = \int \lambda\, dy$$

$$(D^2 - 2D - \lambda)X = 0 \quad D = \frac{d}{dx} \qquad\qquad \log_e Y = \lambda y + a$$

$$Y = e^{\lambda y + a}$$

$$A.E = m^2 - 2m - \lambda = 0 \qquad\qquad Y = C_3 e^{\lambda y}$$

$$m = \frac{2 \pm \sqrt{4 + 4\lambda}}{2}$$

$$= \frac{2 \pm 2\sqrt{1+\lambda}}{2}$$

$$m = 1 \pm \sqrt{1+\lambda}$$

$$C.F = C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x}$$

$$P.I = 0$$

$$X = C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x}$$

$$u = XY$$

$C_1 C_2 C_3$

$$\left\{ C_1 e^{(1+\sqrt{1+\lambda})x} + C_2 e^{(1-\sqrt{1+\lambda})x} \right\} C_3 e^{\lambda y}$$

$$= C_1 C_3 e^{(1+\sqrt{1+\lambda})x + \lambda y} + C_2 C_3^{(1-\sqrt{1+\lambda})x + \lambda y}$$

General solution is.

$$u = \sum a_n e^{(1+\sqrt{1+\lambda_n})x + \lambda_n y} + b_n e^{(1-\sqrt{1+\lambda_n})x + \lambda_n y}$$

Ans $\frac{\delta^2 u}{\delta x^2} = \frac{\delta u}{\delta y} + 2u$

$$\frac{\delta^2 u}{\delta x^2} - \frac{\delta u}{\delta y} - 2u = 0 \quad \text{——— (1)}$$

$X \equiv X(x) \quad Y \equiv Y(y)$

$u = XY$

$\frac{\delta u}{\delta x} = Y\frac{dX}{dx} \qquad \frac{\delta u}{\delta y} = X\frac{dY}{dy}$

$$\frac{\delta^2 u}{\delta x^2} = Y\frac{d^2X}{dx^2}$$

$$Y\frac{d^2X}{dx^2} - X\frac{dY}{dy} - 2XY = 0$$

~~diviting~~ dividing by XY

$$\frac{\cancel{Y}}{X\cancel{Y}}\frac{d^2X}{dx^2} - \frac{\cancel{X}}{\cancel{X}Y}\frac{dY}{dy} - 2\frac{\cancel{XY}}{\cancel{XY}} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{1}{Y}\frac{dY}{dy} - 2 = 0$$

$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda$ ——— (2)

$m^2 - \lambda = 0$

$m = \pm\sqrt{\lambda}$

$X = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

$\frac{1}{Y}\frac{dY}{dy} + 2 = \lambda$

$\int \frac{1}{Y}\, dY = \int (\lambda - 2)\, dy$

$\log Y = (\lambda - 2)y + a$

$Y = e^{(\lambda-2)y + a}$

$Y = C_1 e^{(\lambda-2)y}$

$$u = XY$$
$$= c_1 c_2 e^{\sqrt{\lambda} x + (\lambda - 2) y} + c_1 c_3 e^{-\sqrt{\lambda} x + (\lambda - 2) y}$$

general solution is -

$$u = \sum a_n e^{\sqrt{\lambda_n} x + (\lambda_n - 2) y} + b_n e^{-\sqrt{\lambda_n} x + (\lambda_n - 2) y}$$

Que $\frac{\delta^2 u}{\delta x \delta t} = e^{-t} \cos x$ —— (1)

$u = 0$, when $t = 0$ & $\frac{\delta u}{\delta t} = 0$ when $x = 0$

Sol $u = XT$ $\quad X \equiv X(x)$, $T \equiv T(t)$

$$\frac{\delta u}{\delta t} = X \frac{dT}{dt}$$

$$\frac{\delta^2 u}{\delta x\, dt} = \frac{dX}{dx} \frac{dT}{dt}$$

Putting in eq^n (1)

$$\frac{dX}{dx} \cdot \frac{dT}{dt} = e^{-t} \cos x$$

$$\cancel{\frac{dX}{dx}}\ \frac{dT}{dt} \cdot e^{t} = \frac{\cos x}{\frac{dX}{dx}} = \lambda \text{ —— (say)}$$

$$\frac{dT}{dt} e^{t} = \lambda \text{ —— (2)} \qquad \frac{\cos x}{\frac{dX}{dx}} = \lambda \text{ —— (3)}$$

$$\int dT = \int \lambda e^{-t}\, dt$$

$$\boxed{T = -\lambda e^{-\lambda} + C_1}$$

$$\cos x = \lambda \frac{dX}{dx}$$

$$\int \frac{\cos x}{\lambda} \, dx = \int dX$$

$$\boxed{X = \frac{1}{\lambda} \sin x + C_2}$$

$$u = XT$$

$$= \left(\frac{1}{\lambda} \sin x + C_2\right)\left(-\lambda e^{-t} + C_1\right) \quad \text{—— (4)}$$

$u = 0$, when $t = 0$

at $t = 0$, from (4)

$$u(x, 0) = \left(\frac{1}{\lambda} \sin x + C_2\right)\left(-\lambda + C_1\right)$$

$$= \left(\frac{1}{\lambda} \sin x + C_2\right)\left(-\lambda + C_1\right)$$

$$-\lambda + C_1 = 0 \Rightarrow \boxed{C_1 = \lambda}$$

Putting value of C_1 in (4).

$$\boxed{u = \left(-\frac{1}{\lambda} \sin x + C_2\right)\left(-\lambda e^{-t} + \lambda\right)} \quad \text{—— (5)}$$

diff. Eq^n (5) w.r.t t

$$\frac{\delta u}{\delta t} = \left(\frac{1}{\lambda} \sin x + C_2\right)\left(\lambda e^{-t}\right)$$

$$0 = \left(\frac{1}{\lambda} \times 0 + c_2\right)\left(\lambda e^{-t}\right)$$

$$0 = c_2 \lambda e^{-t} \Rightarrow \boxed{c_2 = 0}$$

putting value of c_2 in (5)

$$u = \frac{1}{\lambda} \sin x \left(-\lambda e^{-t} + \lambda\right)$$

$$= \frac{1}{\lambda} \sin x \,(\lambda)\left(1 - e^{-t}\right)$$

$$\boxed{u = \sin x \left(1 - e^{-t}\right)}$$

Q. $u_{xx} = u_y + 2u$ ——— (1)

$u(0, y) = 0$, $\frac{\delta}{\delta x} u(0, y) = 1 + e^{-3y}$

Sol $u = XY$ $X \equiv X(x)$, $Y \equiv Y(y)$.

$u_y = X \frac{dY}{dy}$, $u_x = Y \frac{dX}{dx}$, $u_{xx} = Y \frac{d^2X}{dx^2}$

Putting in eq^n (1)

$$\frac{Y}{XY}\frac{d^2X}{dx^2} = \frac{X}{XY}\frac{dY}{dy} + 2\frac{XY}{XY}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \frac{1}{Y}\frac{dY}{dy} + 2 = \lambda \text{ (say)}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda \quad \text{—} ②$$

$$\frac{1}{Y}\frac{dY}{dy} + 2 = \lambda \quad \text{—} ③$$

from eqⁿ ②

$$\frac{d^2X}{dx^2} - \lambda X = 0$$

$$D^2X - \lambda X = 0$$

$$(D^2 - \lambda)X = 0. \qquad D \equiv \frac{d}{dx}$$

$$m^2 - \lambda = 0$$

$$m = \pm\sqrt{\lambda}$$

$$C.F = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$$

$$X = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$$

from eqⁿ ③

$$\int \frac{dY}{Y} = \int (\lambda - 2)\, dy$$

$$\log Y = (\lambda - 2)y + a$$

$$\boxed{Y = C_3 e^{(\lambda-2)y}} \qquad C_3 = e^a$$

$$u = XY$$

$$u = C_1C_3 e^{\{\sqrt{\lambda}x + (\lambda-2)y\}} + C_2C_3 e^{\{-\sqrt{\lambda}x + (\lambda-2)y\}} \quad \text{—} ④$$

$$0 = C_1C_3 e^{(\lambda-2)y} + C_2C_3 e^{(\lambda-2)y}$$

$$0 = e^{(\lambda-2)y}\{c_1c_3 + c_2c_3\}$$

$$c_1c_3 + c_2c_3 = 0 \qquad c_2 = -c_1$$

$$u = c_1c_3\, e^{\sqrt{\lambda}x + (\lambda-2)y} - c_1c_3\, e^{-\sqrt{\lambda}x + (\lambda-2)y} \quad \text{--- (5)}$$

diff. (5) w.r.t ~~y~~ x

$$\frac{\delta u}{\delta y} = c_1c_3(\lambda-2)\, e^{\{\sqrt{\lambda}x + (\lambda-2)y\}} - c_1c_3(\lambda-2)\, e^{\{-\sqrt{\lambda}x + (\lambda-2)y\}}$$

using condition (ii).

$$1 + e^{-3y} = c_1c_3(\lambda-2)\, e^{(\lambda-2)y} - c_1c_3(\lambda-2)\, e^{(\lambda-2)y}$$

$$= e^{(\lambda-2)y}\{c_1c_3 + c_2c_3\}$$

$$\frac{\delta u}{\delta x} = c_1c_3\sqrt{\lambda}\{e^{\sqrt{\lambda}x} + (\lambda-2)y\} - c_1c_3\sqrt{\lambda}\, e^{\{-\sqrt{\lambda}x + (\lambda-2)y\}}$$

using Condition (ii)

$$\cancel{1+e^{-3}}\ \frac{\delta u(0,y)}{\delta x} = c_1c_3\sqrt{\lambda}\, e^{(\lambda-2)y} + c_1c_3\sqrt{\lambda}\, e^{(\lambda-2)y}$$

$$= 2c_1c_3\sqrt{\lambda}\, e^{(\lambda-2)y}$$

$$u = \sum 2a_n\sqrt{\lambda_n}\, e^{(\lambda_n-2)y}$$

$$e^{0\cdot y} + e^{-3y} = 2a_1\sqrt{\lambda_1}\,e^{(\lambda_1-2)y} + 2a_2\sqrt{\lambda_2}\,e^{(\lambda_2-2)y}$$

$\lambda_1 - 2 = 0$ $\qquad$ $\lambda_2 - 2 = -3$

$\boxed{\lambda_1 = 2}$ $\qquad$ $\boxed{\lambda_2 = -1}$

$2a_1\sqrt{\lambda_1} = 1$ $\qquad$ $1 = 2a_2\sqrt{\lambda_2}$

$2a_1\sqrt{2} = 1$ $\qquad$ $1 = 2a_2\sqrt{-1}$

$a_1 = \dfrac{1}{2\sqrt{2}}$ $\qquad$ $a_2 = \dfrac{1}{2i}$

$$u = \cancel{2}\frac{1}{\cancel{2}\cancel{\sqrt{2}}}\cancel{\sqrt{2}}\,e^{(2-2)y} + \cancel{2}\frac{1}{\cancel{2}i}\sqrt{-1}\,e^{(-1-2)y}$$

$$u = 1 + e^{-3y}$$