Hence, the solution is

$$u + F(y) = c$$

$$\Rightarrow \quad x^4 + 2x^3y + x^2y^2 + 3yx^2 + 6xy^2 + 3y^3 = c$$

where c is an arbitrary constant.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{dy}{dx} + 2(x+y)^2 = 1$
2. $(x - y^2)\,dx + 2xy\,dy = 0$
3. $(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$
4. $x\dfrac{dy}{dx} + y\log y = xye^x$
5. $\sin y\dfrac{dy}{dx} = \cos y\,(1 - x\cos y)$
6. $\dfrac{dy}{dx} = \dfrac{e^y}{x^2} - \dfrac{1}{x}$
7. $x\,dx + y\,dy = m(x\,dy - y\,dx)$.

Answers

1. $1 + x + y = ce^{4x}(1 - x - y)$
2. $y^2 + x\log cx = 0$
3. $y^2 = 3x^2 - 6x - x^3 + ce^{-x} + 4$
4. $x\log y = e^x(x-1) + c$
5. $\sec y = x + 1 + ce^x$
6. $cx^2 + 2x\,e^{-y} = 1$
7. $m\tan^{-1}\dfrac{y}{x} - \dfrac{1}{2}\log(x^2 + y^2) = c$.

1.22. LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$a_0\frac{d^n y}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \dots\dots + a_n y = Q \qquad \dots(1)$$

where $a_0, a_1, a_2, \dots\dots, a_n$ are all constants and Q is a function of x alone is called a linear differential equation of n^{th} order with constant coefficients.

1.23. THE OPERATOR D

The part $\dfrac{d}{dx}$ of the symbol $\dfrac{dy}{dx}$ may be regarded as an operator such that when it operates on y, the result is the derivative of y. Similarly, $\dfrac{d^2}{dx^2}, \dfrac{d^3}{dx^3}, \dots\dots, \dfrac{d^n}{dx^n}$ may be regarded as operators.

For brevity, we write $\dfrac{d}{dx} \equiv D, \dfrac{d^2}{dx^2} \equiv D^2, \dots\dots, \dfrac{d^n}{dx^n} \equiv D^n$

Thus, the symbol D is a **differential operator** or simply an **operator.**

Written in symbolic form, equation (1) becomes

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots\dots + a_{n-1}D + a_n)y = Q$$

or $$\boxed{f(D)y = Q}$$

02

The operator D can be treated as an algebraic quantity. Thus,

$$D(u+v) = Du + Dv, \quad D(\lambda u) = \lambda\, Du \quad \text{and} \quad D^p D^q\, u = D^q D^p\, u = D^{p+q}\, u$$

The polynomial $f(D)$ can be factorised by ordinary rules of algebra and the factors may be written in any order.

1.24. THEOREMS

Theorem 1. *If $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are n linearly independent solutions of the differential equation*

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$$

then $u = c_1 y_1 + c_2 y_2 + \ldots + c_n y_n$ is also its solution, where $c_1, c_2, \ldots\ldots, c_n$ are arbitrary constants.

Theorem 2. *If $y = u$ is the complete solution of the equation $f(D)y = 0$ and $y = v$ is a particular solution (containing no arbitrary constants) of the equation $f(D)y = Q$, then the complete solution of the equation*

$$f(D)y = Q \text{ is } y = u + v.$$

Note 1. The part $y = u$ is called the **complementary function (C.F.)** and the part $y = v$ is called the **particular integral (P.I.)** of the equation $f(D)\, y = Q$.

Note 2. The complete solution is $\boxed{y = \text{C.F.} + \text{P.I.}}$

Thus in order to solve the equation $f(D)\, y = Q$, we first find the C.F. i.e., the complete solution of equation $f(D)\, y = 0$ and then the P.I. i.e., a particular integral (solution) of equation $f(D)\, y = Q$.

1.25. COMPLEMENTARY FUNCTION (C.F.)

Consider the differential equation

$$f(D)y = Q \qquad \ldots(1)$$

Complementary function is actually the solution of the given differential equation (1) when its right hand side member *i.e.*, Q is replaced by zero. To find C.F., we first find auxiliary equation.

1.26. AUXILIARY EQUATION (A.E.)

Consider the differential equation $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = 0$ ...(1)

Let $y = e^{mx}$ be a solution of (*i*), then

$$Dy = me^{mx},\ D^2 y = m^2 e^{mx},\ \ldots\ldots,\ D^{n-2}y = m^{n-2}e^{mx},\ D^{n-1}y = m^{n-1}e^{mx},\ D^n y = m^n e^{mx}$$

Substituting the values of y, Dy, D^2y, ..., $D^n y$ in (1), we get

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n)\, e^{mx} = 0$$

or

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0, \text{ since } e^{mx} \neq 0 \qquad \ldots(2)$$

Thus $y = e^{mx}$ will be a solution of equation (1) if m satisfies equation (2).

Equation (2) is called the auxiliary equation for the differential equation (1).

1.26.1. Definition

The equation obtained by equating to zero the symbolic coefficient of y is called the **auxiliary equation**, briefly written as **A.E.**

1.26.2. Steps for Finding Auxiliary Equation

Step 1. Replace y by 1

Step 2. Replace $\dfrac{dy}{dx}$ by m

Step 3. Replace $\dfrac{d^2y}{dx^2}$ by m^2 and so on replace $\dfrac{d^ny}{dx^n}$ by m^n

Step 4. By doing so, we get an algebraic equation in m of degree n called auxiliary equation.

1.27. RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n)y = 0 \quad$ ...(1)

where all the a_i's are constant.

Its auxiliary equation is $\quad m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0 \quad$ ...(2)

It is an algebraic equation in m of degree n. So it will give n values of m on solving.

Let $m = m_1, m_2, m_3, \ldots\ldots, m_n$ be the roots of the A.E. The C.F. of equation (1) depends upon the nature of roots of the A.E. The following cases arise.

Case I. When the roots of auxiliary equation are real and distinct

Equation (1) is equivalent to

$$(D - m_1)(D - m_2)\ldots\ldots(D - m_n)y = 0 \quad \text{...(3)}$$

Equation (3) will be satisfied by the solutions of the equations

$$(D - m_1)y = 0,\ (D - m_2)y = 0,\ \ldots\ldots,\ (D - m_n)y = 0$$

Now, consider the equation $(D - m_1)y = 0$, *i.e.*, $\dfrac{dy}{dx} - m_1 y = 0$

It is a linear equation and $\text{I.F.} = e^{\int -m_1\,dx} = e^{-m_1 x}$

$\therefore$ Its solution is $\quad y \cdot e^{-m_1 x} = \int 0 \cdot e^{-m_1 x}\,dx + c_1 \quad$ or $\quad y = c_1 e^{m_1 x}$

Similarly, the solution of $(D - m_2)y = 0$ is $y = c_2 e^{m_2 x}$

....

the solution of $(D - m_n)y = 0$ is $y = c_n e^{m_n x}$

$$\therefore \quad \boxed{\text{C.F.} = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x}} \quad \text{...(4)}$$

Case II. When the roots of auxiliary equation are equal

(*a*) When two roots of auxiliary equation are equal

Let $\quad m_1 = m_2$

Solution of eqn. (3) is (as in case I)

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= c_1 e^{m_1 x} + c_2 e^{m_2 x} + \ldots\ldots + c_n e^{m_n x} + 0 && \mid \because\ \text{P.I.} = 0 \text{ as } Q = 0 \\ &= (c_1 + c_2) e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} && \mid \text{Here } m_1 = m_2 \\ &= c\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \end{aligned}$$

It contains $(n-1)$ arbitrary constants and is, therefore, not the complete solution of equation (1).

The part of C.F. corresponding to the repeated root is the complete solution of

$$(D - m_1)(D - m_1)y = 0$$

$$\Rightarrow \quad (D - m_1)v = 0 \qquad \text{| Putting } (D - m_1)y = v$$

$$\Rightarrow \quad \frac{dv}{dx} - m_1 v = 0$$

Its solution is

$$v = c_2 e^{m_1 x}$$

$$\therefore \quad (D - m_1)y = c_2 e^{m_1 x}$$

$$\Rightarrow \quad \frac{dy}{dx} - m_1 y = c_2 e^{m_1 x}, \quad \text{which is a linear equation}$$

$$\therefore \quad \text{I.F.} = e^{-m_1 x}$$

Its solution is

$$y e^{-m_1 x} = \int c_2 e^{m_1 x} . e^{-m_1 x}\, dx + c_1 = c_2 x + c_1$$

$$\Rightarrow \quad y = (c_2 x + c_1)\, e^{m_1 x}$$

$\therefore$ Part of C.F. $= (c_1 + c_2 x)\, e^{m_1 x}$

Hence, $\boxed{\text{complete C.F.} = (c_1 + c_2 x) e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}}$

(*b*) If however, three roots of the auxiliary equation are equal say $m_1 = m_2 = m_3$, then proceeding as above,

$$\boxed{\text{C.F.} = (c_1 + c_2 x + c_3 x^2) e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n e^{m_n x}}$$

Case III. When two roots of auxiliary equation are imaginary

Let $\quad m_1 = \alpha + i\beta \quad$ and $\quad m_2 = \alpha - i\beta$, then from (4),

$$\begin{aligned}
\text{C.F.} &= c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
&= e^{\alpha x} [(c_1 + c_2) \cos \beta x + i (c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x} \\
\text{C.F.} &= e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}
\end{aligned}$$

[taking $c_1 + c_2 = C_1$, $i(c_1 - c_2) = C_2$]

Case IV. When roots of auxiliary equation are repeated imaginary

Let $\quad m_1 = m_2 = \alpha + i\beta \quad$ and $\quad m_3 = m_4 = \alpha - i\beta$ then by case II,

$$\text{C.F.} = e^{\alpha x}[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x] + c_5 e^{m_5 x} + \ldots\ldots + c_n e^{m_n x}$$

Case V. When roots of auxiliary equation are irrational

Let $m_1 = \alpha + \sqrt{\beta}$ and $m_2 = \alpha - \sqrt{\beta}$ then

C.F. of eqn. (1) is given by

$$\text{C.F.} = e^{\alpha x}(c_1 \cosh \sqrt{\beta}x + c_2 \sinh \sqrt{\beta}x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

Case VI. When roots of auxiliary equation are repeated irrational

Let $m_1 = m_2 = \alpha + \sqrt{\beta}$ and $m_3 = m_4 = \alpha - \sqrt{\beta}$ then by case II,

$$\text{C.F.} = e^{\alpha x}[(c_1 + c_2 x)\cosh \sqrt{\beta}x + (c_3 + c_4 x)\sinh \sqrt{\beta}x] + c_5 e^{m_5 x} + c_6 e^{m_6 x} + \ldots\ldots + c_n e^{m_n x}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0.$

Sol. The auxiliary equation is

$$m^3 - 7m - 6 = 0$$

$$\Rightarrow \quad (m+1)(m+2)(m-3) = 0 \quad \Rightarrow \quad m = -1, -2, 3$$

The roots are real and distinct

$\therefore$ Complementary Function (C.F.) $= c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$

Particular Integral (P.I.) = 0

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve :* $(D^3 - 3D^2 + 4)\,y = 0$, *where* $D \equiv \dfrac{d}{dx}$.

Sol. The auxiliary equation is

$$m^3 - 3m^2 + 4 = 0$$

$$\Rightarrow \quad (m+1)(m-2)^2 = 0 \quad \Rightarrow \quad m = -1, 2, 2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

$$\text{P.I.} = 0$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-x} + (c_2 + c_3 x)\, e^{2x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 3. *Solve :* $(D^4 - n^4)y = 0$, *where* $D \equiv \dfrac{d}{dx}$.

Sol. The auxiliary equation is

$$m^4 - n^4 = 0$$

$$\Rightarrow \quad (m^2 - n^2)(m^2 + n^2) = 0$$

$$\Rightarrow \quad m = \pm n, \pm ni$$

$$\therefore \quad \text{C.F.} = c_1 e^{nx} + c_2 e^{-nx} + e^{0x}(c_3 \cos nx + c_4 \sin nx)$$

$$= c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{nx} + c_2 e^{-nx} + c_3 \cos nx + c_4 \sin nx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 4. *Solve :* $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0.$

Sol. The auxiliary equation is

$$m^4 + 13m^2 + 36 = 0$$

$$\Rightarrow \quad (m^2 + 9)(m^2 + 4) = 0 \Rightarrow m = \pm 3i, \pm 2i$$

$$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos 3x + c_2 \sin 3x) + e^{0x}(c_3 \cos 2x + c_4 \sin 2x)$$

$$= c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 3x + c_2 \sin 3x + c_3 \cos 2x + c_4 \sin 2x$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve :* $(D^2 - 2D + 4)^2 y = 0 ;\ D \equiv \dfrac{d}{dx}.$

Sol. The auxiliary equation is

$$(m^2 - 2m + 4)^2 = 0$$

$$m = \frac{2 \pm \sqrt{4 - 16}}{2} \text{ (twice)} = 1 \pm \sqrt{3}\, i,\ 1 \pm \sqrt{3}\, i$$

The roots are repeated imaginary

$$\therefore \quad \text{C.F.} = e^x [(c_1 + c_2 x) \cos \sqrt{3}\, x + (c_3 + c_4 x) \sin \sqrt{3}\, x]$$

$$\text{P.I.} = 0$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^x [(c_1 + c_2 x) \cos \sqrt{3}\, x + (c_3 + c_4 x) \sin \sqrt{3}\, x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 6. *Solve :* $\dfrac{d^4y}{dx^4} - 4\dfrac{d^3y}{dx^3} + 8\dfrac{d^2y}{dx^2} - 8\dfrac{dy}{dx} + 4y = 0.$

Sol. The auxiliary equation is

$$m^4 - 4m^3 + 8m^2 - 8m + 4 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)^2 = 0$$

$$\Rightarrow \quad m = \frac{2 \pm \sqrt{4 - 8}}{2} \text{ (twice)} = \frac{2 \pm 2i}{2} \text{ (twice)} = 1 \pm i,\ 1 \pm i$$

$$\therefore \quad \text{C.F.} = e^x [(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x]$$

$$\text{P.I.} = 0$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\Rightarrow \quad y = e^x [(c_1 + c_2 x) \cos x + (c_3 + c_4 x) \sin x]$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Hence the general solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x \quad \text{...(1)}$$

Applying the condition $y(0) = 2$, we get $\quad 2 = c_1$

Applying the condition $y\left(\frac{\pi}{2}\right) = -2$, we get $\quad -2 = c_2$

Hence from (1), the particular solution is

$$y = 2(\cos x - \sin x)$$

TEST YOUR KNOWLEDGE

Solve the differential equations :

1. $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 12y = 0$
2. $\dfrac{d^2y}{dx^2} + (a+b)\dfrac{dy}{dx} + aby = 0$
3. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 11\dfrac{dy}{dx} + 6y = 0$
4. $\dfrac{d^2x}{dt^2} + 6\dfrac{dx}{dt} + 9x = 0$
5. $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = 0$
6. $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = 0$
7. $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$
8. $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$
9. $\dfrac{d^5y}{dx^5} - \dfrac{d^3y}{dx^3} = 0$ *(U.P.T.U. 2009)*
10. $\dfrac{d^3y}{dx^3} - 2\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 8y = 0$
11. $(D^2+1)^2(D-1)\,y = 0$
12. $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$
13. $(D^6 - 1)\,y = 0$
14. $(D^6 + 1)\,y = 0$
15. $\dfrac{d^2x}{dt^2} - 3\dfrac{dx}{dt} + 2x = 0$, given that when $t = 0$, $x = 0$ and $\dfrac{dx}{dt} = 0$
16. $\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} + 12\dfrac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$

[G.B.T.U.(AG) SUM 2010]

Answers

1. $y = c_1e^{3x} + c_2e^{4x}$
2. $y = c_1e^{-ax} + c_2e^{-bx}$
3. $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$
4. $x = (c_1 + c_2t)e^{-3t}$
5. $y = (c_1 + c_2x + c_3x^2)e^x$
6. $y = (c_1 + c_2x)\,e^x + c_3\,e^{-x}$
7. $y = e^{-x}(c_1 + c_2x + c_3x^2) + c_4e^{4x}$
8. $y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$
9. $y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + c_5e^x$
10. $y = c_1e^{2x} + c_2\cos 2x + c_3\sin 2x$
11. $y = (c_1 + c_2x)\cos x + (c_3 + c_4x)\sin x + c_5e^x$
12. $y = e^{2x}(c_1\cosh\sqrt{3}\,x + c_2\sinh\sqrt{3}\,x)$