13. $y = c_1 e^x + c_2 e^{-x} + e^{x/2}\left(c_3 \cos \frac{\sqrt{3}}{2} x + c_4 \sin \frac{\sqrt{3}}{2} x\right) + e^{-x/2}\left(c_5 \cos \frac{\sqrt{3}}{2} x + c_6 \sin \frac{\sqrt{3}}{2} x\right)$

14. $y = c_1 \cos x + c_2 \sin x + e^{\sqrt{3}x/2}\left(c_3 \cos \frac{x}{2} + c_4 \sin \frac{x}{2}\right) + e^{-\sqrt{3}x/2}\left(c_5 \cos \frac{x}{2} + c_6 \sin \frac{x}{2}\right)$

15. $x = 0$

16. $y = x^2 e^{-3x}$

1.28. THE INVERSE OPERATOR $\frac{1}{f(D)}$

$\frac{1}{f(D)}$ Q is that function of x, *free from arbitrary constants*, which when operated upon by $f(D)$ gives Q.

Thus

$$f(D)\left\{\frac{1}{f(D)} Q\right\} = Q$$

$\therefore$ $f(D)$ and $\frac{1}{f(D)}$ are inverse operators.

Note 1. $\frac{1}{D} Q = \int Q \, dx.$

Note 2. $\frac{1}{D - a} Q = e^{ax} \int Q e^{-ax} \, dx.$

1.29. RULES FOR FINDING THE PARTICULAR INTEGRAL (P.I.)

Consider the differential equation, $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_{n-1} D + a_n) y = Q$

It can be written as $f(D) y = Q$

$$\therefore \qquad \text{P.I.} = \frac{1}{f(D)} Q.$$

1.29.1. Case I. When $Q = e^{ax}$ (or e^{ax+b})

Since

$$D\, e^{ax} = a\, e^{ax}$$
$$D^2 e^{ax} = a^2 e^{ax}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$
$$D^{n-1} e^{ax} = a^{n-1} e^{ax}$$
$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore \quad (D^n + a_1 D^{n-1} + \ldots\ldots + a_{n-1} D + a_n) e^{ax} = (a^n + a_1 a^{n-1} + \ldots\ldots + a_{n-1} a + a_n) e^{ax}$$

or

$$f(D)\, e^{ax} = f(a)\, e^{ax}$$

Operating on both sides by $\frac{1}{f(D)}$;

$$\frac{1}{f(D)} [f(D)\, e^{ax}] = \frac{1}{f(D)} [f(a)\, e^{ax}]$$

$$e^{ax} = f(a)\,\frac{1}{f(\mathrm{D})}\,e^{ax}$$

or

Dividing both sides by $f(a)$, $\frac{1}{f(a)}\,e^{ax} = \frac{1}{f(\mathrm{D})}\,e^{ax}$, provided $f(a) \neq 0$

Hence $$\boxed{\frac{1}{f(\mathbf{D})}\,e^{ax} = \frac{1}{f(a)}\,e^{ax}, \text{ provided } f(a) \neq 0.}$$

Case of failure : If $f(a) = 0$, the above method fails.

Since $f(a) = 0$, $\mathrm{D} = a$ is a root of $f(\mathrm{D}) = 0$ ...(1)

$\therefore$ $\mathrm{D} - a$ is a factor of $f(\mathrm{D})$.

Let $f(\mathrm{D}) = (\mathrm{D} - a)\,\phi(\mathrm{D})$, where $\phi(a) \neq 0$

Then
$$\frac{1}{f(\mathrm{D})}\,e^{ax} = \frac{1}{(\mathrm{D}-a)\,\phi(\mathrm{D})}\,e^{ax} = \frac{1}{\mathrm{D}-a}\cdot\frac{1}{\phi(a)}\,e^{ax}$$
$$= \frac{1}{\phi(a)}\cdot\frac{1}{\mathrm{D}-a}\,e^{ax} = \frac{1}{\phi(a)}\,e^{ax}\int e^{ax}\,.\,e^{-ax}\,dx \quad \text{[by Note 2]}$$
$$= \frac{1}{\phi(a)}\,e^{ax}\int 1\,dx = x\,.\,\frac{1}{\phi(a)}\,e^{ax} \quad ...(2)$$

Differentiating both sides of (1) w.r.t. D, we get
$$f'(\mathrm{D}) = (\mathrm{D} - a)\,\phi'(\mathrm{D}) + \phi(\mathrm{D})$$
$$\Rightarrow \quad f'(a) = \phi(a)$$

$\therefore$ From (2), we have $\frac{1}{f(\mathbf{D})}\,e^{ax} = x\,.\,\frac{1}{f'(a)}\,e^{ax}$ **provided** $f'(a) \neq 0$

Another case of failure :

If $f'(a) = 0$, then $\frac{1}{f(\mathrm{D})}\,e^{ax} = x^2\,.\,\frac{1}{f''(a)}\,e^{ax}$, provided $f''(a) \neq 0$ and so on.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the P.I. of* $(4D^2 + 4D - 3)y = e^{2x}$.

Sol.
$$\text{P.I.} = \frac{1}{4\mathrm{D}^2 + 4\mathrm{D} - 3}\,e^{2x} = \frac{1}{4(2)^2 + 4(2) - 3}\,e^{2x} \quad \text{(Replacing D by 2)}$$
$$= \frac{1}{21}\,e^{2x}.$$

Example 2. *Find the P.I. of* $(D^3 - 3D^2 + 4)y = e^{2x}$.

Sol.
$$\text{P.I.} = \frac{1}{\mathrm{D}^3 - 3\mathrm{D}^2 + 4}\,e^{2x}.$$

Here the denominator vanishes when D is replaced by 2. It is a case of failure.
We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$
$$\text{P.I.} = x\,.\,\frac{1}{3\mathrm{D}^2 - 6\mathrm{D}}\,e^{2x}$$

It is again a case of failure. We multiply the numerator by x and differentiate the denominator w.r.t. D.

$\therefore$

$$\text{P.I.} = x^2 \cdot \frac{1}{6D - 6} e^{2x} = x^2 \cdot \frac{1}{6(2) - 6} e^{2x} = \frac{x^2}{6} e^{2x}.$$

Example 3. *Find the P.I. of* $(D + 1)^3 y = e^{-x}$.

Sol.

$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} = x \cdot \frac{1}{3(D+1)^2} e^{-x} \qquad | \text{ Case of failure}$$

$$= x^2 \cdot \frac{1}{3.2(D+1)} e^{-x} \qquad | \text{ Again case of failure}$$

$$= x^3 \cdot \frac{1}{3.2.1} e^{-x} = \frac{x^3}{6} e^{-x}.$$

Example 4. *Solve :*

(*i*) $(D^3 - 2D^2 + 4D - 8)\, y = 8$

(*ii*) $(D - 2)^3\, y = 17\, e^{2x}$. [U.P.T.U. (B.Pharm.) SUM 2009] (M.T.U. 2011)

Sol. (*i*) Auxiliary equation is

$$m^3 - 2m^2 + 4m - 8 = 0$$

$$\Rightarrow \quad (m^2 + 4)(m - 2) = 0$$

$$\Rightarrow \quad m = 2, \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 + 4D - 8} (8\, e^{0x}) \qquad | \because \ e^{0x} = 1$$

$$= \frac{1}{(0)^3 - 2(0)^2 + 4(0) - 8} (8e^{0x}) = -1$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x - 1$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

(*ii*) Auxiliary equation is

$$(m - 2)^3 = 0$$

$$\Rightarrow \quad m = 2, 2, 2$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^3} 17e^{2x} \qquad | \text{ Case of failure}$$

$$= 17x \cdot \left[\frac{1}{3(D-2)^2} e^{2x}\right] \qquad | \text{ Again case of failure}$$

$$= \frac{17}{3} x^2 \cdot \left[\frac{1}{2(D-2)} e^{2x}\right] \qquad | \text{ Again a case of failure}$$

$$= \frac{17}{6} x^3\, e^{2x}$$

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Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2x + c_3x^2)\, e^{2x} + \frac{17}{6} x^3 e^{2x}$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 5. *Solve :* $2\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x$. (U.P.T.U. 2007)

Sol. The auxiliary equation is

$$2m^3 - m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (2m - 1)(m^2 + 2) = 0$$

$$\Rightarrow \quad m = \frac{1}{2}, \pm \sqrt{2}\, i$$

$$\therefore \quad \text{C.F.} = c_1 e^{x/2} + c_2 \cos \sqrt{2}\, x + c_3 \sin \sqrt{2}\, x$$

$$\text{P.I.} = \frac{1}{2D^3 - D^2 + 4D - 2} e^x = \frac{1}{2(1)^3 - (1)^2 + 4(1) - 2} e^x = \frac{1}{3} e^x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{x/2} + c_2 \cos \sqrt{2}\, x + c_3 \sin \sqrt{2}\, x + \frac{1}{3} e^x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 6. *Solve* $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$.

Sol. Auxiliary equation is

$$m^3 - 6m^2 + 11m - 6 = 0$$

or $$(m - 1)(m - 2)(m - 3) = 0$$

whence $$m = 1, 2, 3$$

$$\therefore \quad \text{C.F.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 6D^2 + 11D - 6} (e^{-2x} + e^{-3x})$$

$$= \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-2x} + \frac{1}{D^3 - 6D^2 + 11D - 6} e^{-3x}$$

$$= \frac{1}{(-2)^3 - 6(-2)^2 + 11(-2) - 6} e^{-2x} + \frac{1}{(-3)^3 - 6(-3)^2 + 11(-3) - 6} e^{-3x}$$

$$= -\frac{1}{60} e^{-2x} - \frac{1}{120} e^{-3x} = -\frac{1}{120} (2e^{-2x} + e^{-3x})$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - \frac{1}{120} (2e^{-2x} + e^{-3x})$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 7. *Solve :* $(D^2 - a^2)y = e^{ax} - e^{-ax}$.

Sol. Auxiliary equation is

$$m^2 - a^2 = 0$$

$\Rightarrow$
$\therefore$

$$m = \pm a$$

$$\text{C.F.} = c_1 e^{ax} + c_2 e^{-ax}$$

$$\text{P.I.} = \frac{1}{D^2 - a^2}(e^{ax} - e^{-ax}) = \frac{1}{D^2 - a^2}(e^{ax}) - \frac{1}{D^2 - a^2}(e^{-ax})$$

$$= x \cdot \frac{1}{2D}(e^{ax}) - x \cdot \frac{1}{2D}(e^{-ax}) = \frac{x}{2}\frac{e^{ax}}{a} - \frac{x}{2}\left(\frac{e^{-ax}}{-a}\right)$$

$$= \frac{x}{2}\left(\frac{e^{ax} + e^{-ax}}{a}\right) = \frac{x}{a}\cosh ax$$

The complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{ax} + c_2 e^{-ax} + \frac{x}{a}\cosh ax$$

where c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve : $(D^2 + D + 1)\, y = (1 + e^x)^2$.*

Sol. Auxiliary equation is

$$m^2 + m + 1 = 0$$

$\Rightarrow$

$$m = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}\, i$$

$$\text{C.F.} = e^{-\frac{1}{2}x}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}(1 + e^x)^2 = \frac{1}{D^2 + D + 1}(1 + e^{2x} + 2e^x)$$

$$= \frac{1}{D^2 + D + 1}(e^{0x}) + \frac{1}{D^2 + D + 1}(e^{2x}) + \frac{1}{D^2 + D + 1}(2e^x)$$

$$= \frac{1}{(0)^2 + (0) + 1}e^{0x} + \frac{1}{(2)^2 + (2) + 1}e^{2x} + \frac{2}{(1)^2 + (1) + 1}e^x = 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = e^{-\frac{x}{2}}\left(c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right) + 1 + \frac{1}{7}e^{2x} + \frac{2}{3}e^x$$

re c_1 and c_2 are arbitrary constants of integration.

Example 9. *Solve : $(D + 2)(D - 1)^2\, y = e^{-2x} + 2\sinh x$.*

Sol. Auxiliary equation is

$$(m + 2)(m - 1)^2 = 0 \quad \Rightarrow \quad m = -2, 1, 1$$

$\therefore$

$$\text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)\, e^x$$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}(e^{-2x} + e^x - e^{-x}) \qquad \left[\because\ \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now, $\frac{1}{(D+2)(D-1)^2} e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2} e^{-2x}\right] = \frac{1}{D+2}\left[\frac{1}{(-2-1)^2} e^{-2x}\right]$ | Case of failure

$$= \frac{1}{9} \cdot \frac{1}{D+2} e^{-2x}$$

$$= \frac{x}{9} e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{x} = \frac{1}{(D-1)^2}\left[\frac{1}{D+2} e^{x}\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2} e^{x}\right]$$

| Case of failure

$$= \frac{1}{3} \cdot \frac{1}{(D-1)^2} e^{x}$$

| Case of failure

$$= \frac{1}{3} \cdot x \frac{1}{2(D-1)} e^{x}$$

$$= \frac{1}{3} \cdot x^2 \cdot \frac{1}{2} e^{x} = \frac{1}{6} x^2 e^{x}$$

$$\frac{1}{(D+2)(D-1)^2} e^{-x} = \frac{1}{(-1+2)(-1-1)^2} e^{-x} = \frac{1}{4} e^{-x}$$

$$\therefore \qquad \text{P.I.} = \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-2x} + (c_2 + c_3 x)e^{x} + \frac{x}{9} e^{-2x} + \frac{x^2}{6} e^{x} + \frac{1}{4} e^{-x}.$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 10. *Solve the differential equation*

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = e^x + 2.$$

Sol. The given equation is

$$(D^3 - 3D^2 + 3D - 1)y = e^x + 2$$

or

$$(D-1)^3 y = e^x + 2$$

Auxiliary equation is

$$(m-1)^3 = 0 \quad \Rightarrow \quad m = 1, 1, 1$$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x + c_3 x^2)e^x$$

$$\text{P.I.} = \frac{1}{(D-1)^3}(e^x + 2) = \frac{1}{(D-1)^3} e^x + \frac{1}{(D-1)^3}(2e^{0x})$$

$$= x \cdot \frac{1}{3(D-1)^2} e^x + \frac{1}{(0-1)^3}(2e^{0x}) = x^2 \cdot \frac{1}{3.2(D-1)}(e^x) - 2$$

$$= x^3 \cdot \frac{1}{3.2.1}(e^x) - 2 = \frac{x^3}{6} e^x - 2$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 x + c_3 x^2)e^x + \frac{x^3}{6} e^x - 2$$

where c_1, c_2 and c_3 are arbitrary constants of integration.