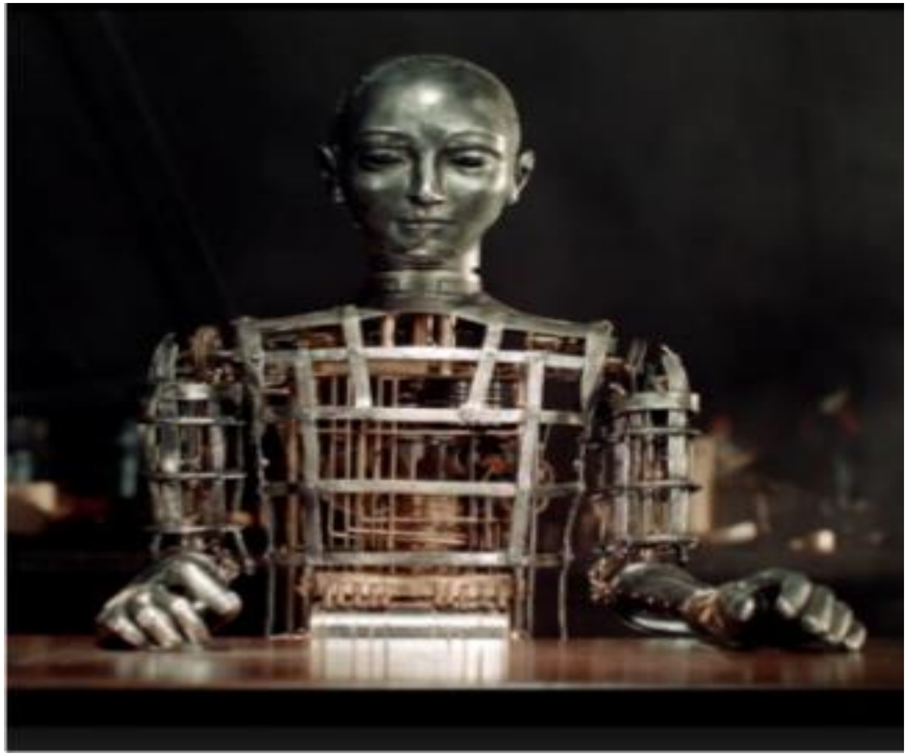
BCSC0011 : THEORY OF AUTOMATA & FORMAL LANGUAGES (TAFL)



Dr. Sandeep Rathor

What is Automata?

- It is the plural of automaton, it means "something that works automatically"
- **A system where energy, materials and information, are transformed for performing some specific task, without direct participation of man.**
- **Example:** Automatic photo printing machine, Packing machine, etc.

Model of Discrete Automaton

- In Computer Science, Automaton = an abstract computing device which process discrete information.

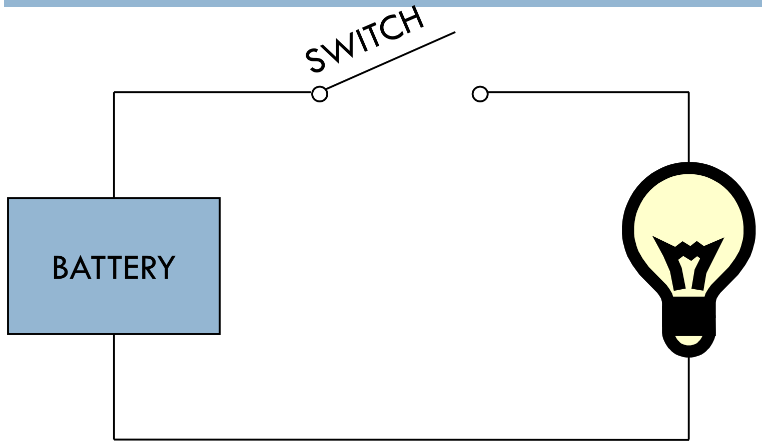
Input: I1, I2,.....Ip
Output: O1,O2,...Oq
States: q1,q2,...qn

Why do we need abstract models?

- Abstract model is **free from "Programming Language"**

- It's **easy to manipulate** these theoretical machines **mathematically** to prove things about their capabilities.

A simple computer



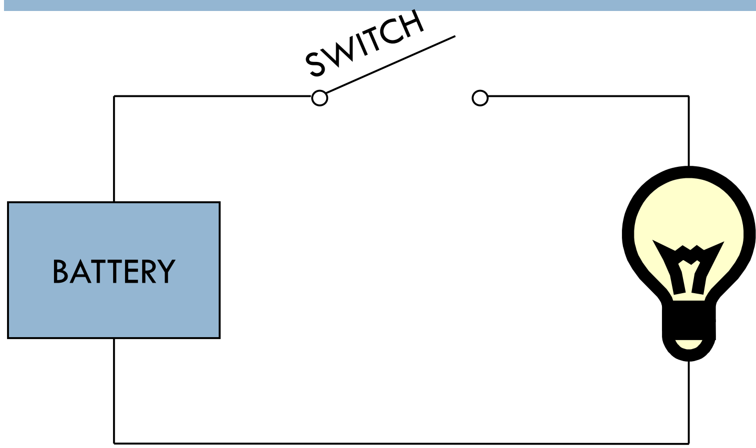
input: switch

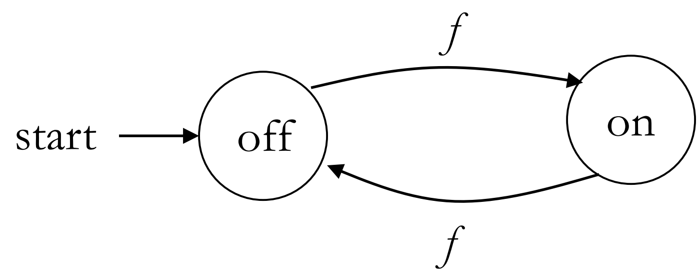
output: light bulb

actions: flip switch

states: on, off

A simple "computer"





input: switch

output: light bulb

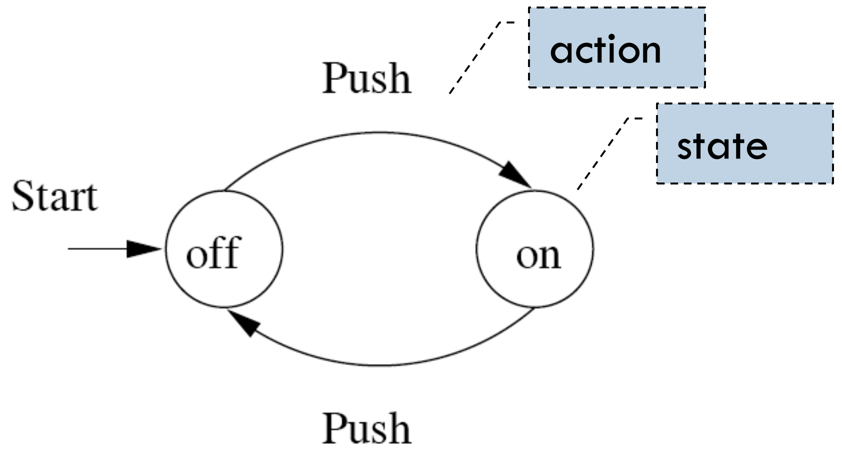
actions: f for "flip switch"

states: on, off

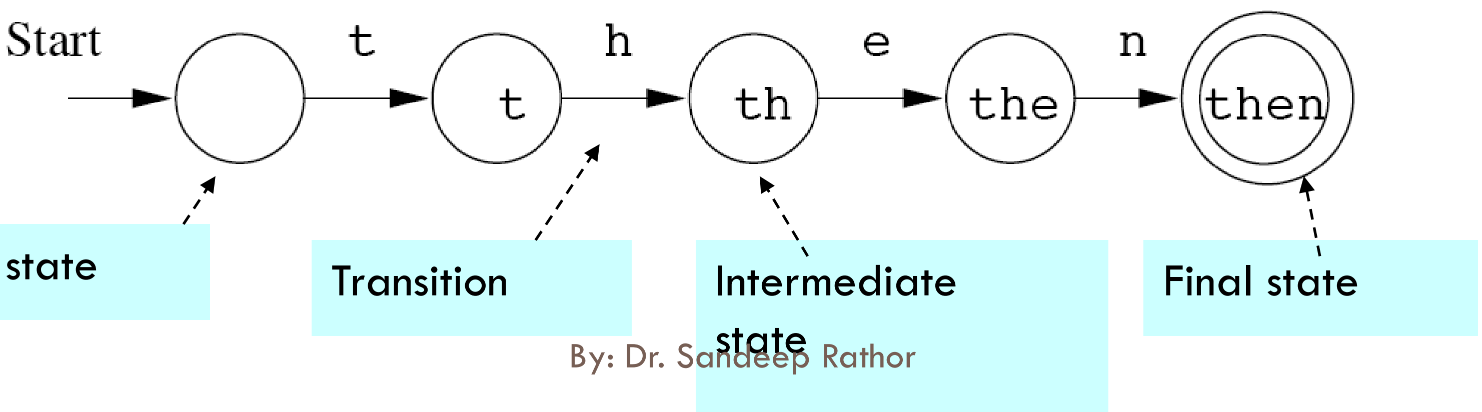
bulb is on if and only if there was an odd number of flips

Finite Automata : Examples

- On/Off switch

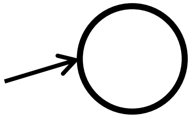


- Modeling recognition of the word "*then*"

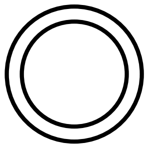


Transition Diagram

- Directed graph consists of set of vertices and edges where vertices represent "states" and edges represent "input/output"
- Circle with an arrow is called initial state



- Two concentric circle represents the final state



Alphabets

- A finite non empty set of symbols.

  Symbol : Σ

  1. English lang. small letter $\Sigma = \{a, b, c \ldots z\}$

  2. Binary number $\Sigma = \{0, 1\}$

  3. Decimal number $\Sigma = \{0, 1, 2, \ldots, 9\}$

  4. Alphanumeric : $\Sigma = \{A\text{-}Z, a\text{-}z, 0\text{-}9\}$

Strings

- A word or a string is a finite sequence of symbols taken from Σ.
- Empty string : ϵ
- Length of string w is denoted by $|w|$ is number of non empty characters in the string.

Ex. $x = 01000$ $|x| = 5$

$x = 01\epsilon01\epsilon00\epsilon$ $|x| = ?$

- xy = concatenation of two string.

Strings

A string over alphabet Σ is a finite sequence of symbols in Σ.

- The empty string will be denoted by ε
- Examples

abfbz is a string over $\Sigma_1 = \{a, b, c, d, \ldots, z\}$

9021 is a string over $\Sigma_2 = \{0, 1, \ldots, 9\}$

ab#bc is a string over $\Sigma_3 = \{a, b, \ldots, z, \#\}$

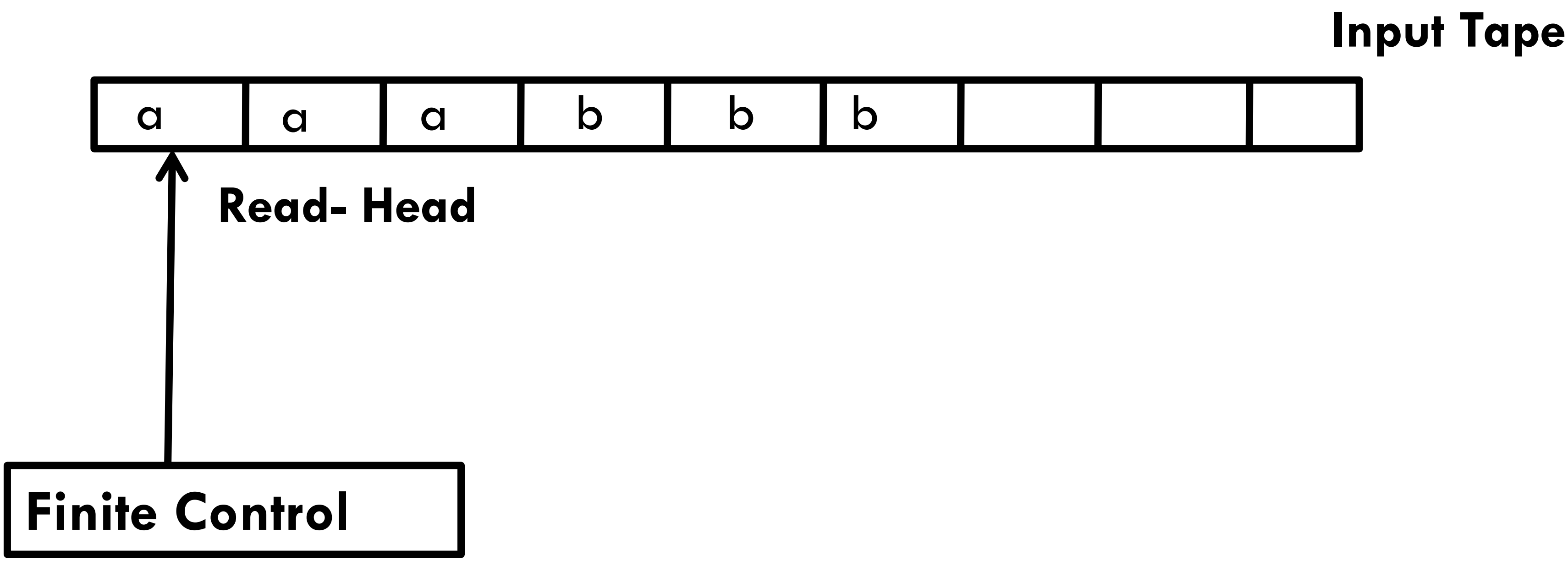
))()(() is a string over $\Sigma_4 = \{ (,) \}$

Languages

- L is said to be a language over alphabet Σ only if L $\subseteq \Sigma^*$.

- Σ^* Set of all string over the Σ

Deterministic Finite Automata (DFA)



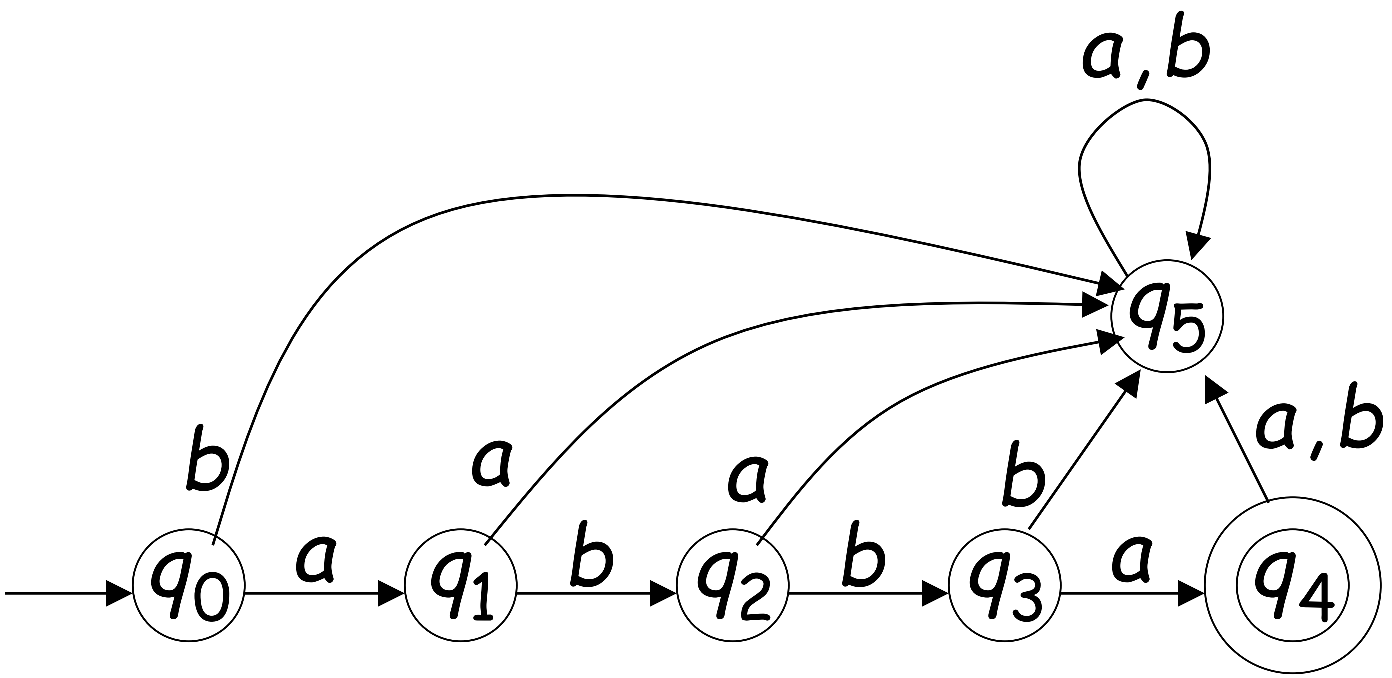
Model Diagram of DFA

Deterministic Finite Automata (DFA)

- A deterministic finite automaton (DFA) is a 5-tuple $(Q, \Sigma, \delta, q_0, F)$ where
  - $\mathcal{Q}$ is a finite set of states
  - Σ is an input alphabet
  - $\delta: \mathcal{Q} \times \Sigma \rightarrow \mathcal{Q}$ is a transition function
  - $q_0 \in \mathcal{Q}$ is the initial state
  - $F \subseteq \mathcal{Q}$ is a set of accepting states (or final states).

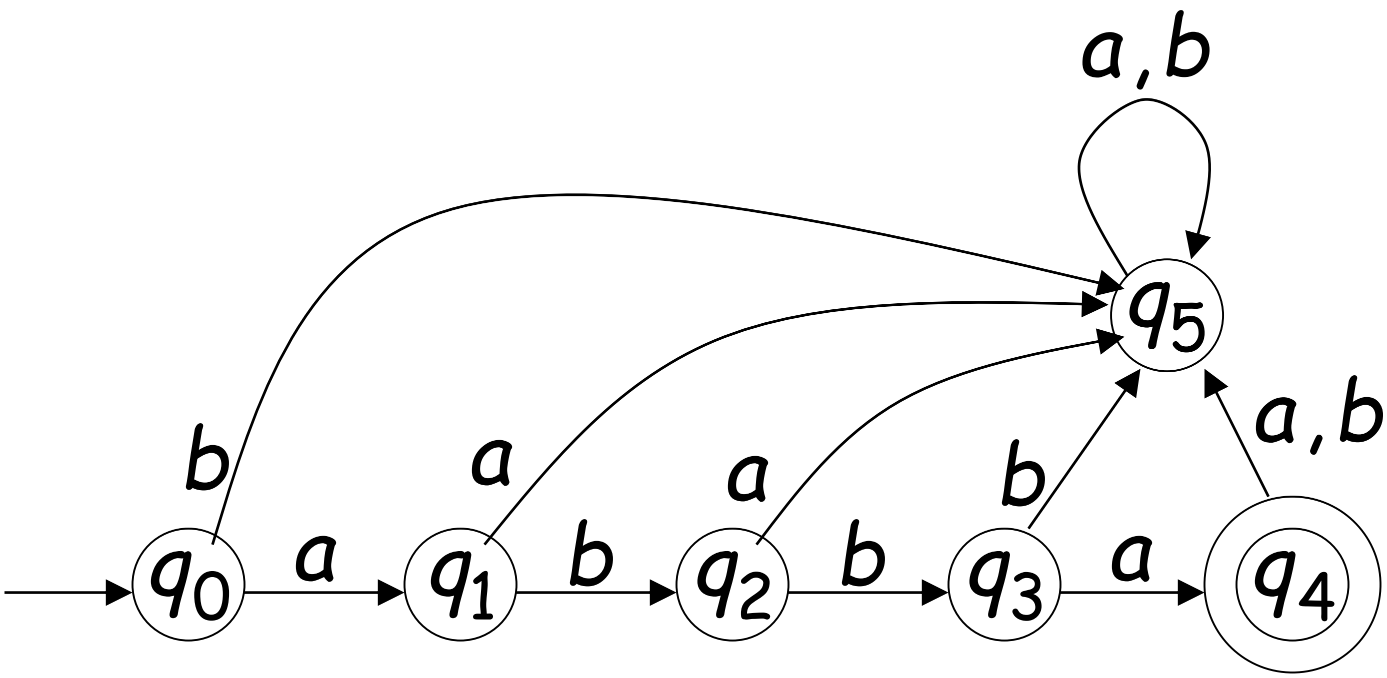
Input Alphabet Σ

- $$\Sigma = \{a, b\}$$

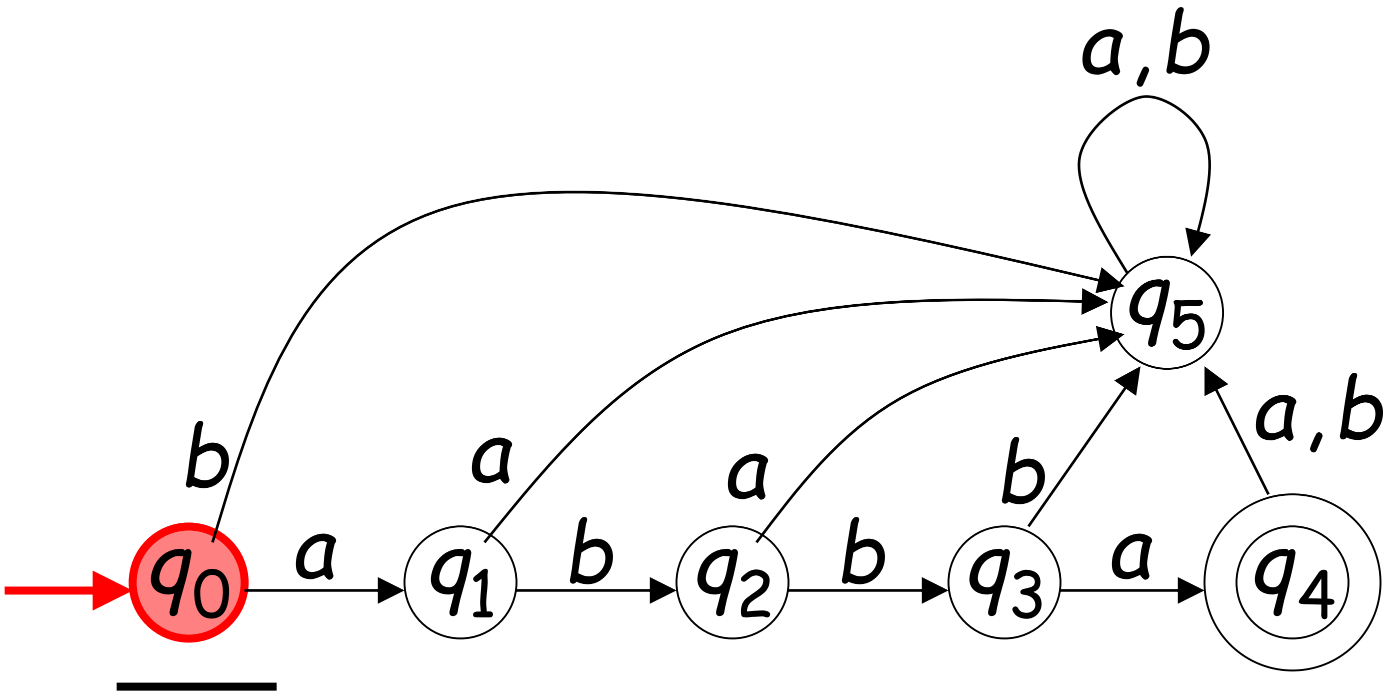


Set of States Q

$$Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$$

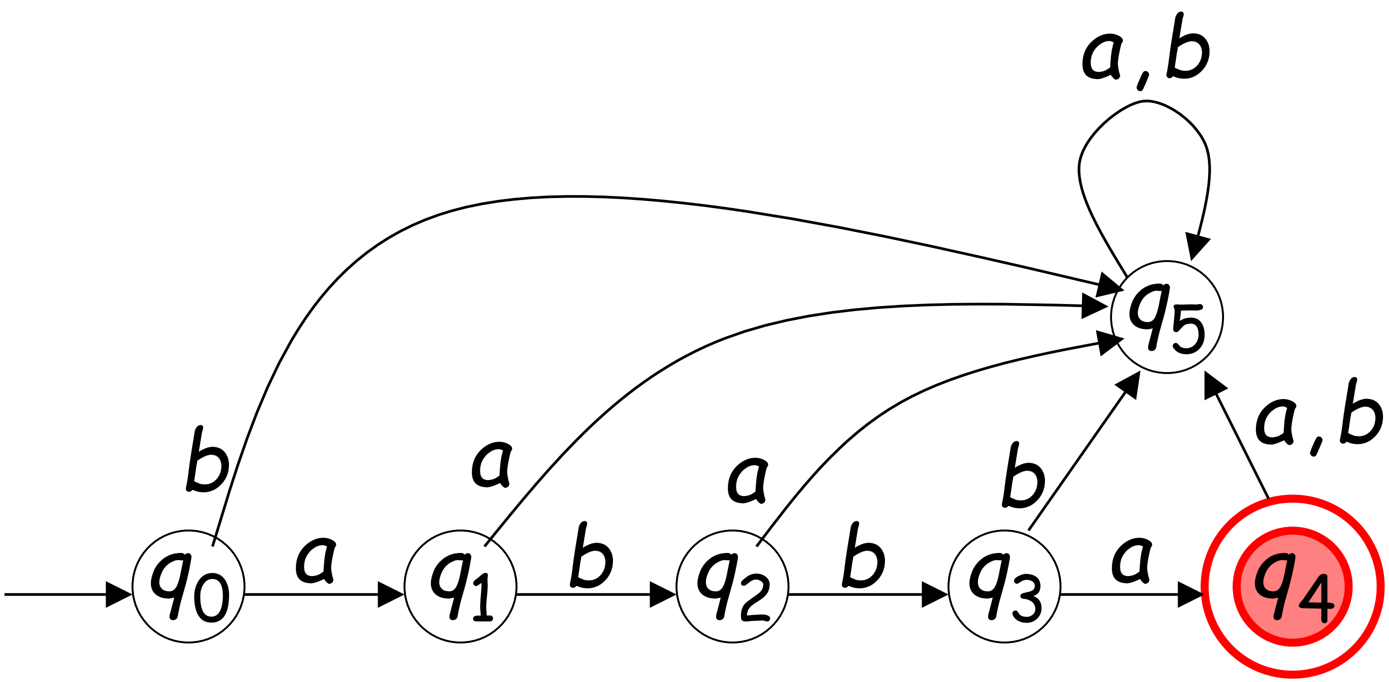


Initial State q_0



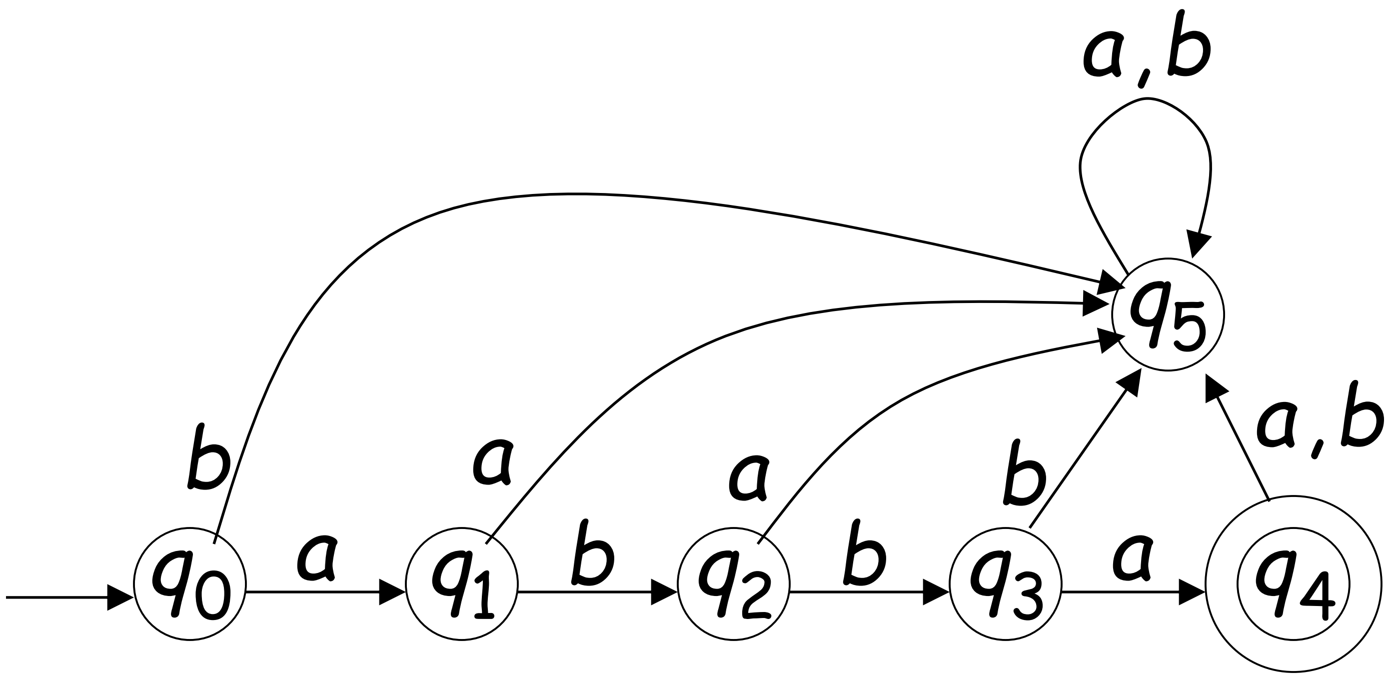
Set of Accepting States or Final State F

- $F = \{q_4\}$

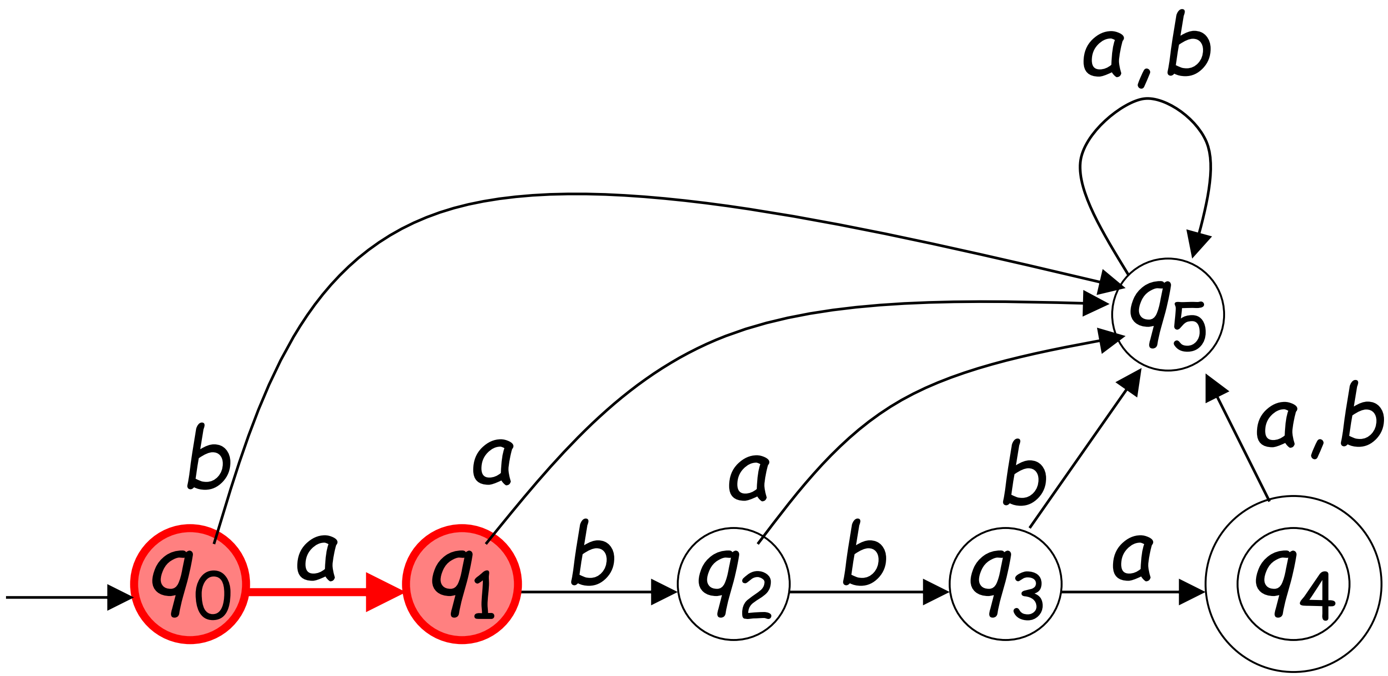


Transition Function δ

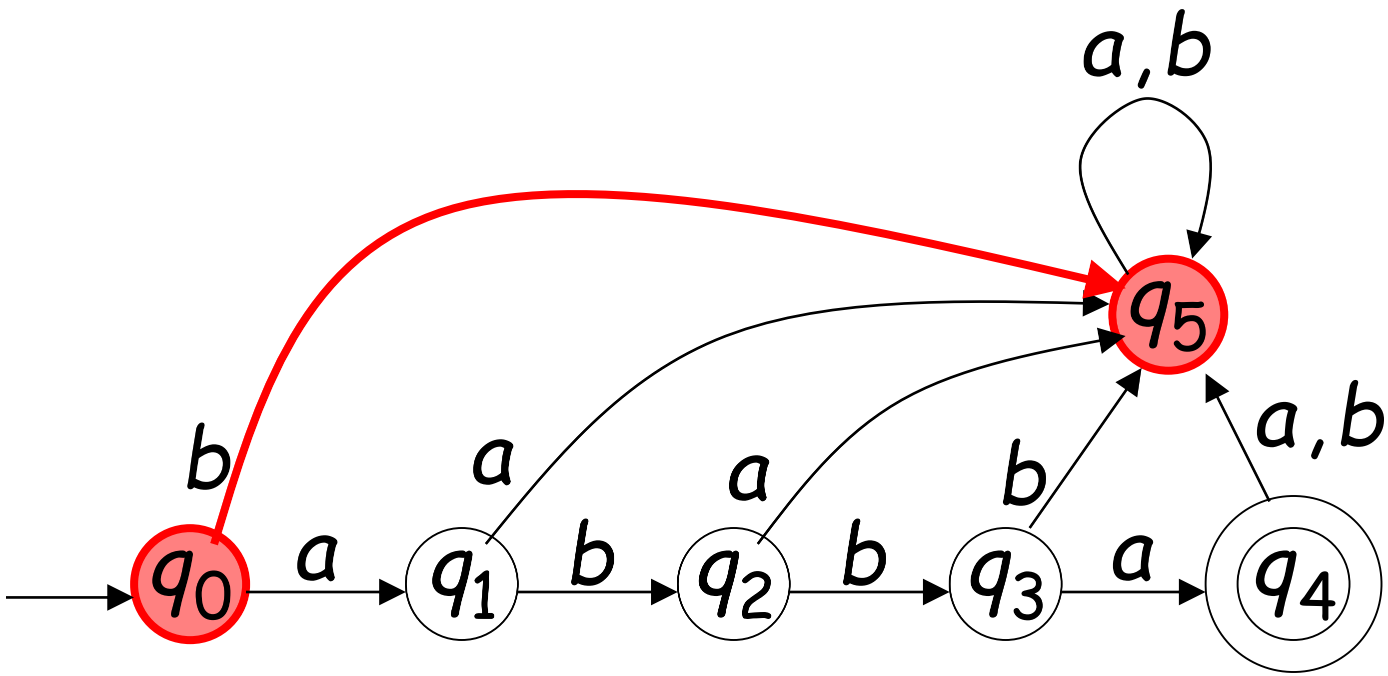
$$\delta : Q \times \Sigma \rightarrow Q$$



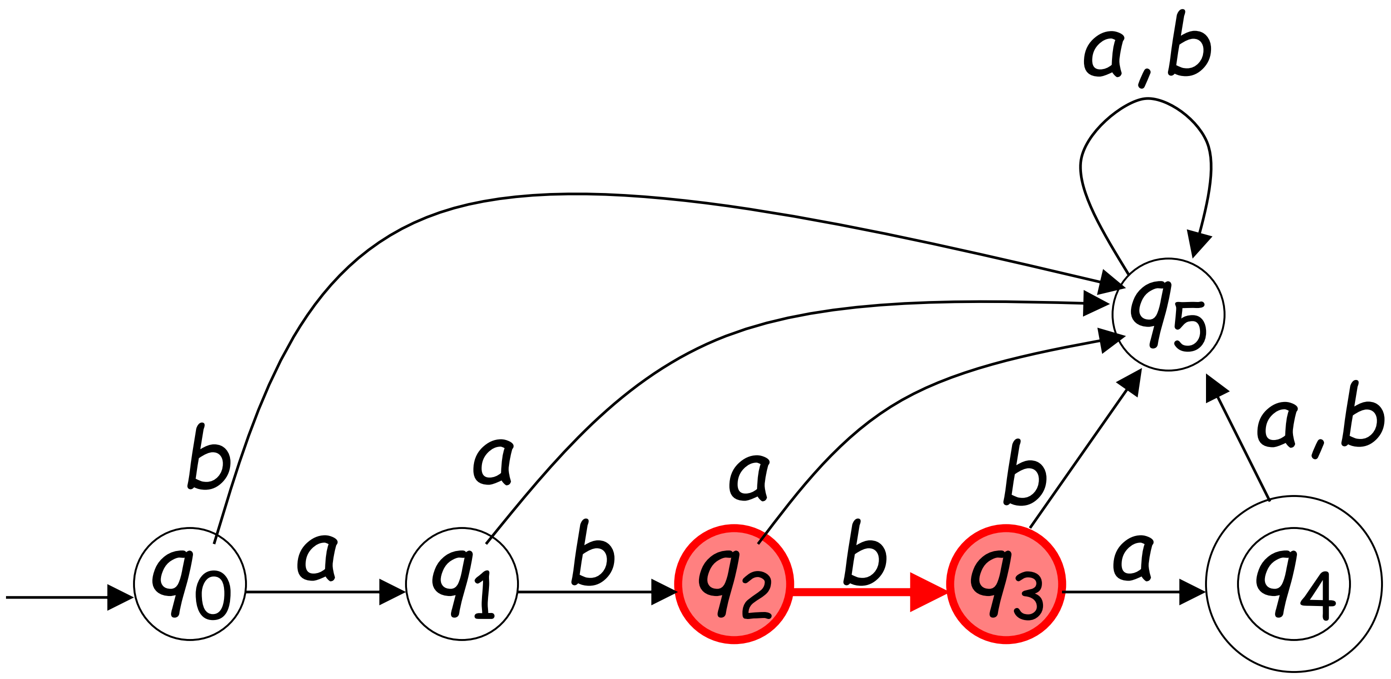
$$\delta(q_0, a) = q_1$$



$$\delta(q_0, b) = q_5$$



$$\delta(q_2, b) = q_3$$



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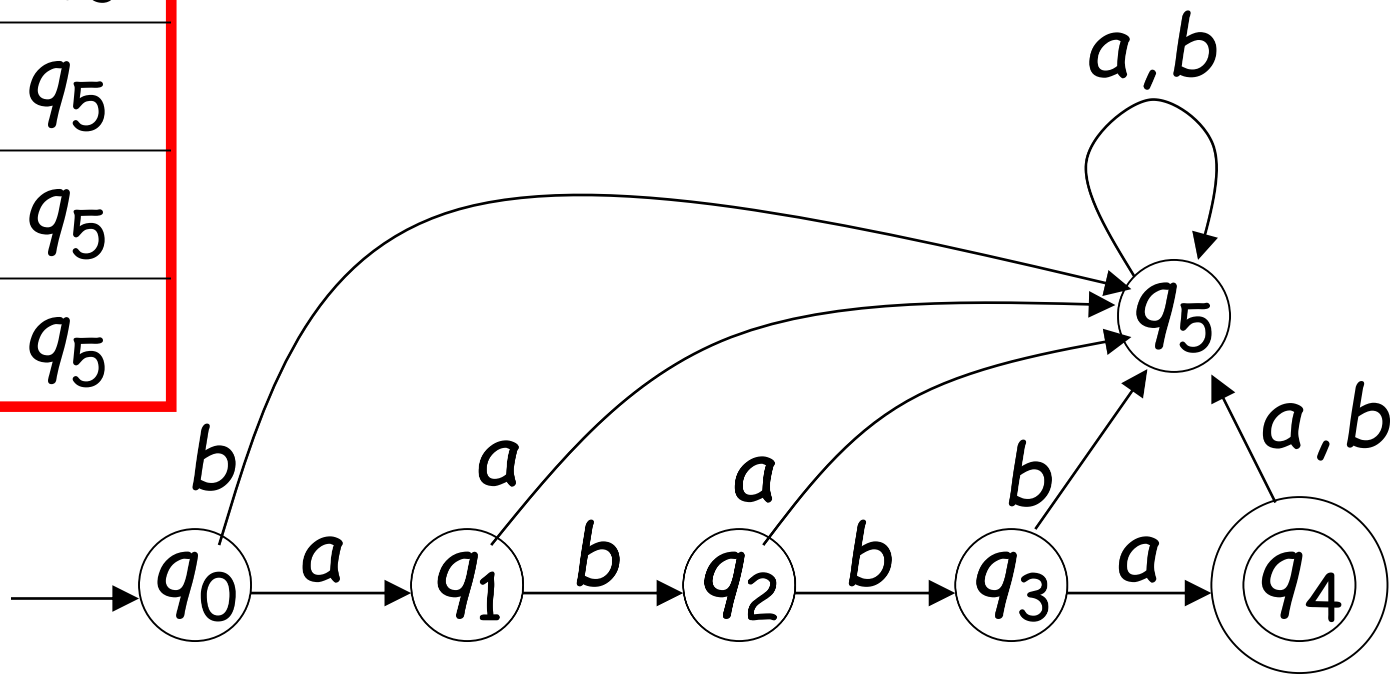
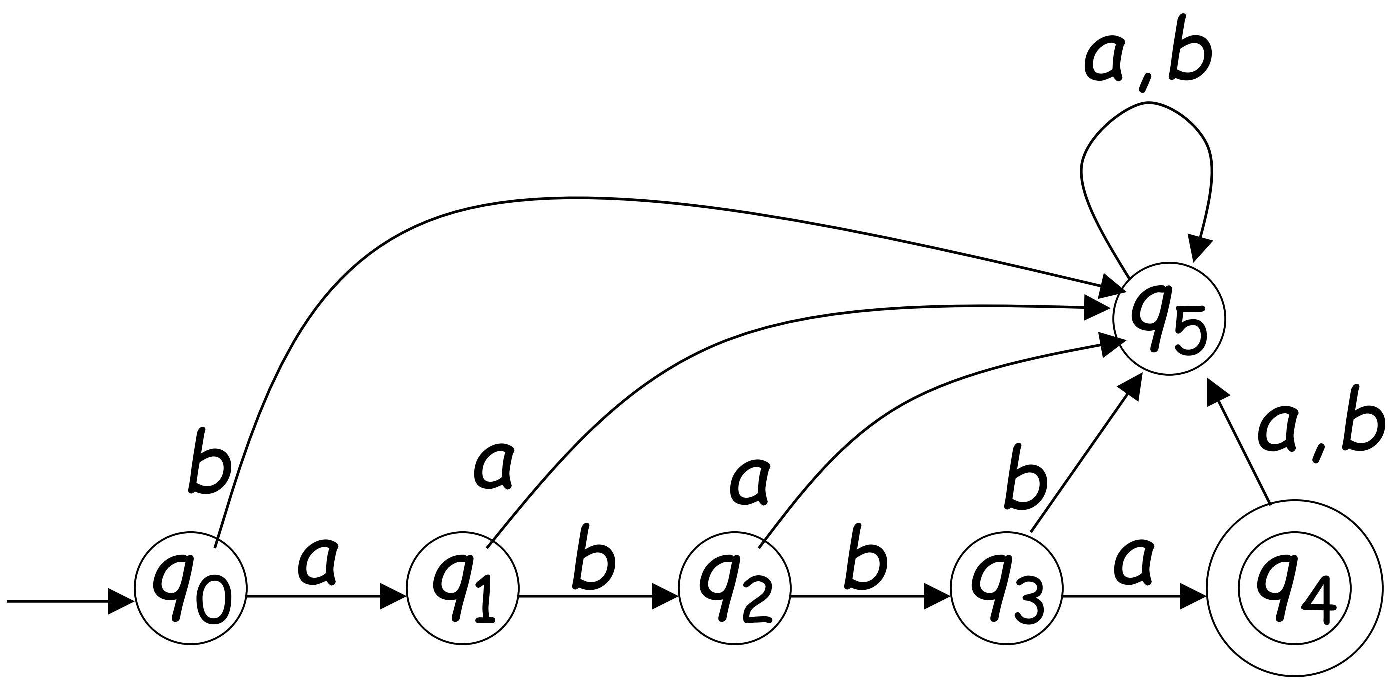
Transition Function (δ) Contd…

δ	a	b
q_0	q_1	q_5
q_1	q_5	q_2
q_2	q_5	q_3
q_3	q_4	q_5
q_4	q_5	q_5
q_5	q_5	q_5

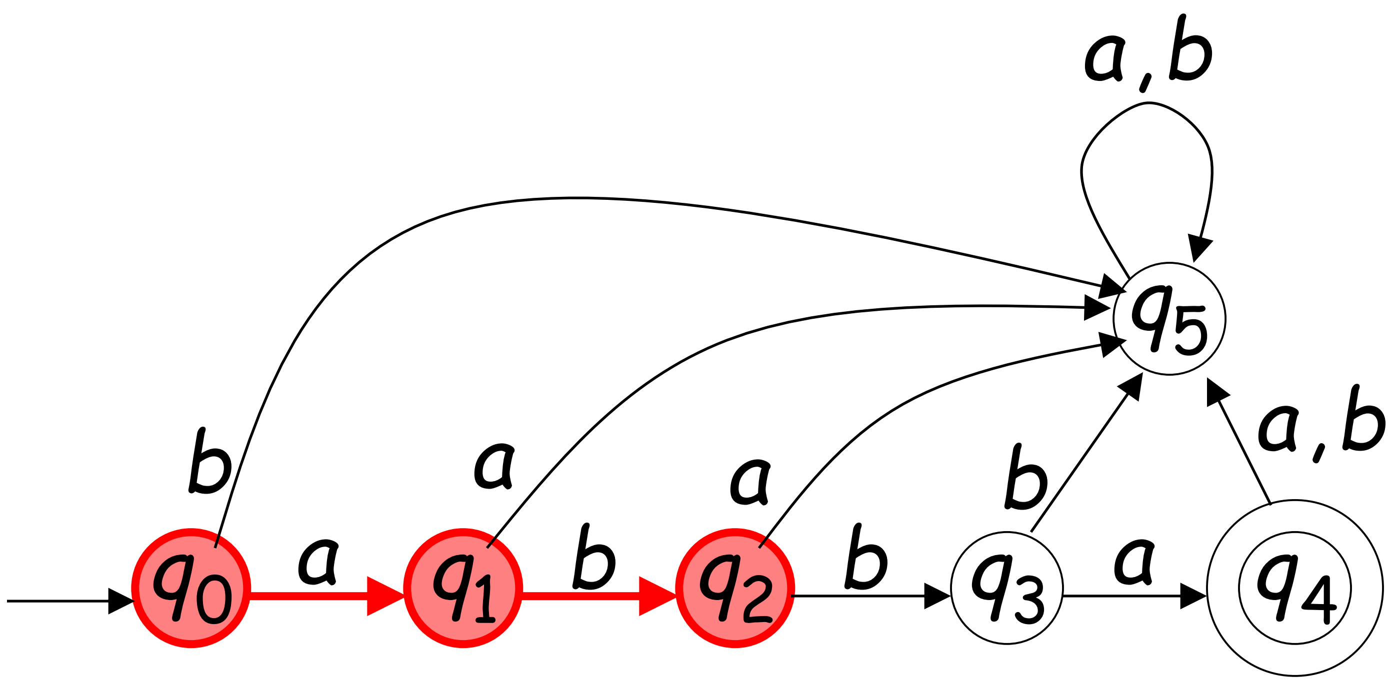


Extended Transition Function δ^*

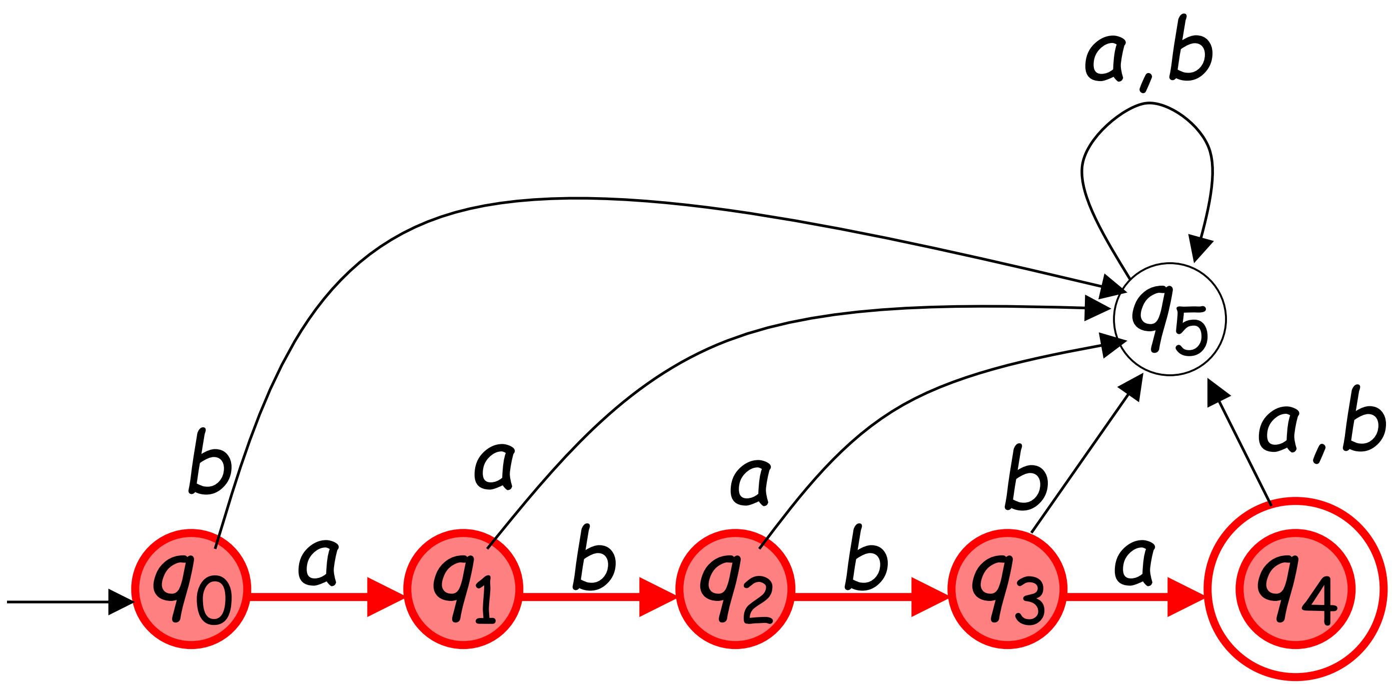
- $$\delta^* : Q \times \Sigma^* \to Q$$



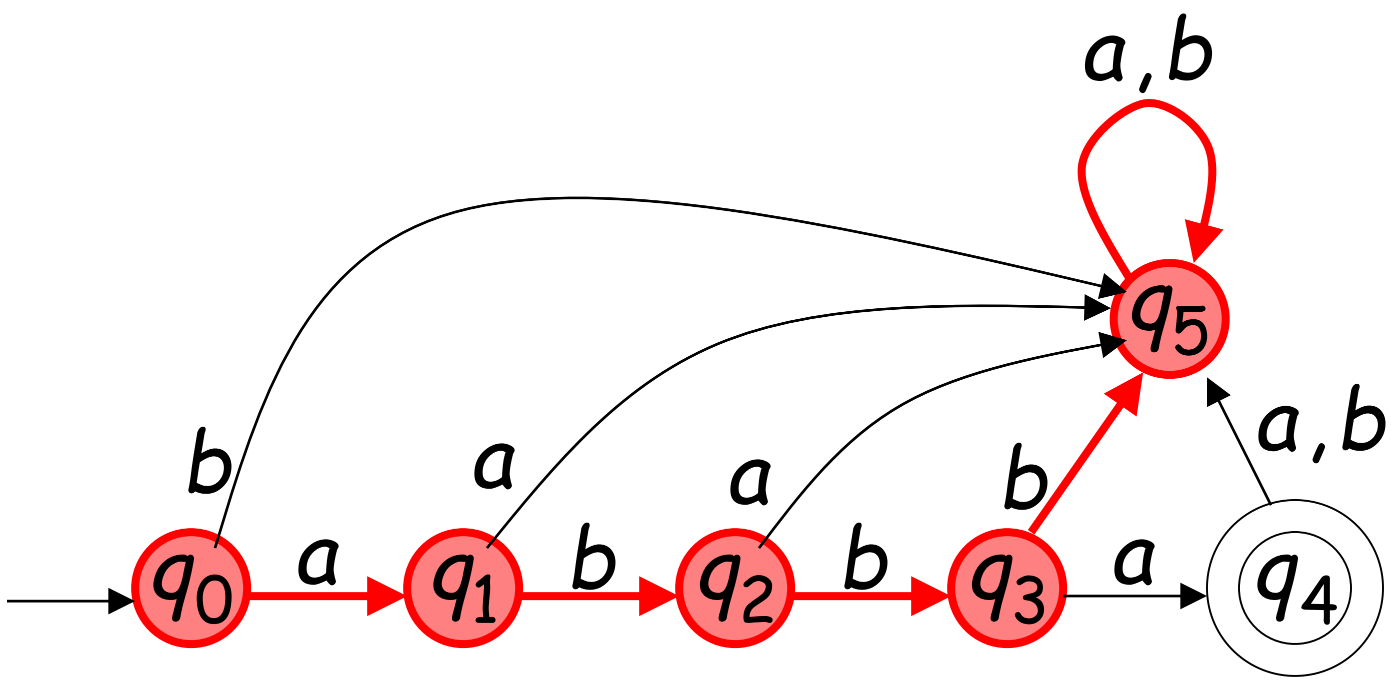
$$\delta^*(q_0, ab) = q_2$$



$$\delta^*(q_0, abba) = q_4$$



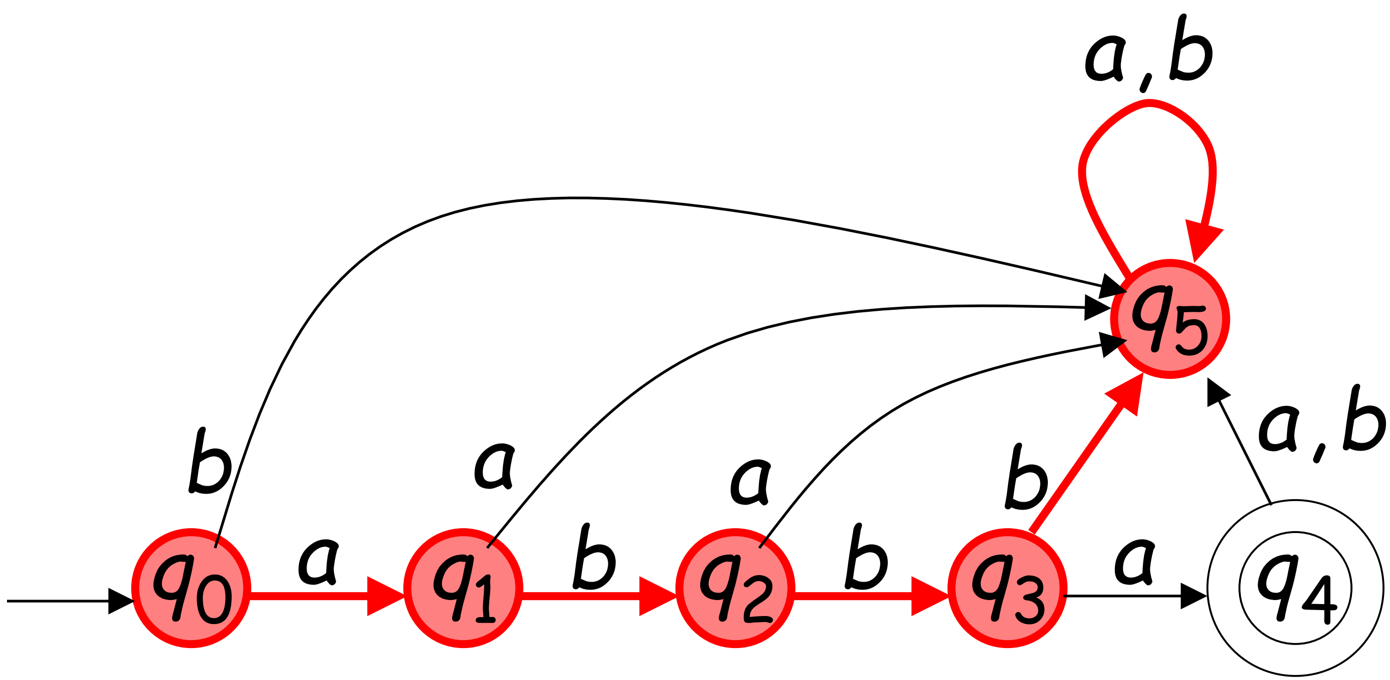
$$\delta^*(q_0, abbbaa) = q_5$$



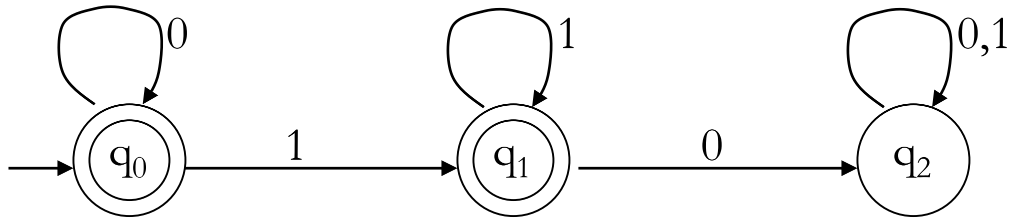
Observation:

Example: There is a walk from q_0 to q_5 with label $abbbaa$

$$\delta^*(q_0, abbbaa) = q_5$$



Example-2

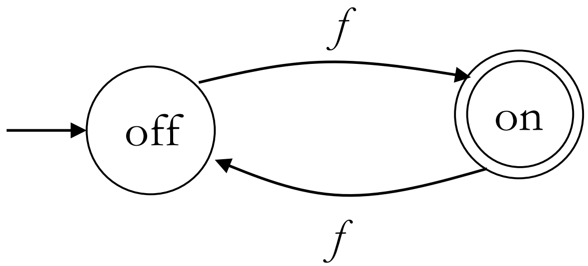


alphabet $\Sigma = \{0, 1\}$
states $Q = \{q_0, q_1, q_2\}$
initial state q_0
Final/ accepting states $F = \{q_0, q_1\}$

transition function δ:

states \ inputs	0	1
q_0	q_0	q_1
q_1	q_2	q_1
q_2	q_2	q_2

Example of a finite automaton

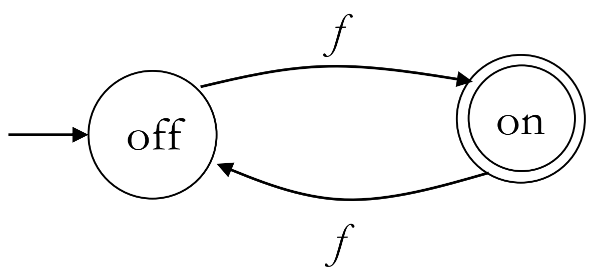


- There are states off and on, the automaton starts in off and tries to reach the "good state" on
- What sequences of fs lead to the final state?
- Answer: $\{f, fff, fffff, \ldots\} = \{f^{\,n}: n \text{ is odd}\}$
- This is an example of a deterministic finite automaton over alphabet $\{f\}$

Language of a DFA

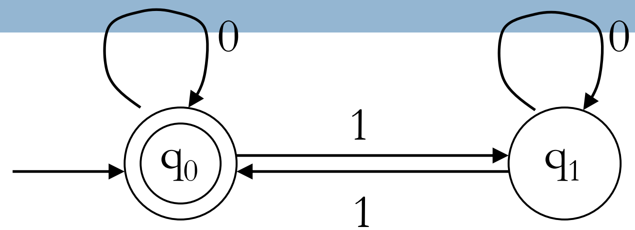
The language of a DFA $(Q, \Sigma, \delta, q_0, F)$ is the set of all strings over Σ that, starting from q_0 and following the transitions as the string is read left to right, will reach some accepting state.

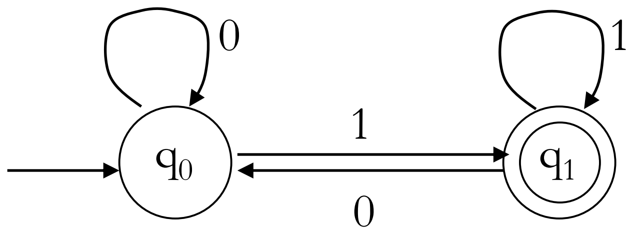
M:

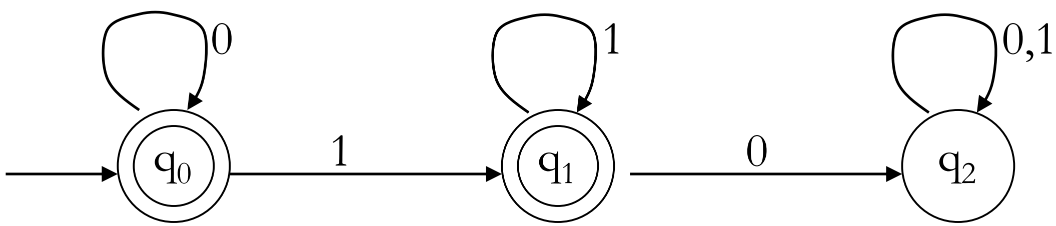


- Language of M is $\{f, fff, fffff, \ldots\} = \{f^n : n \text{ is odd}\}$

Examples

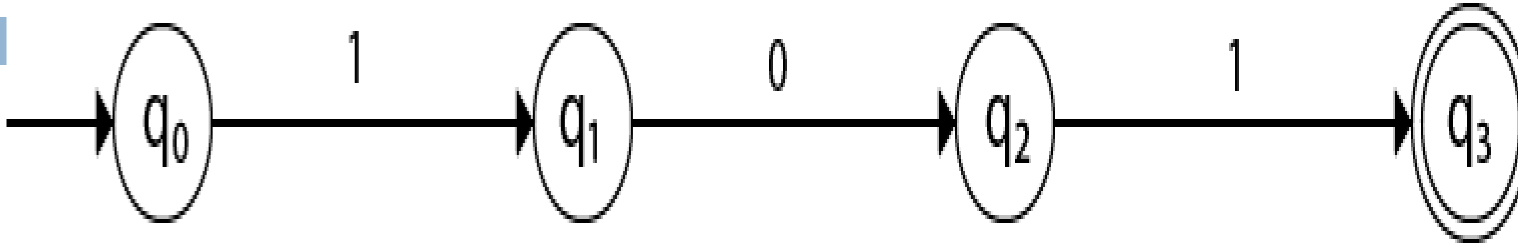




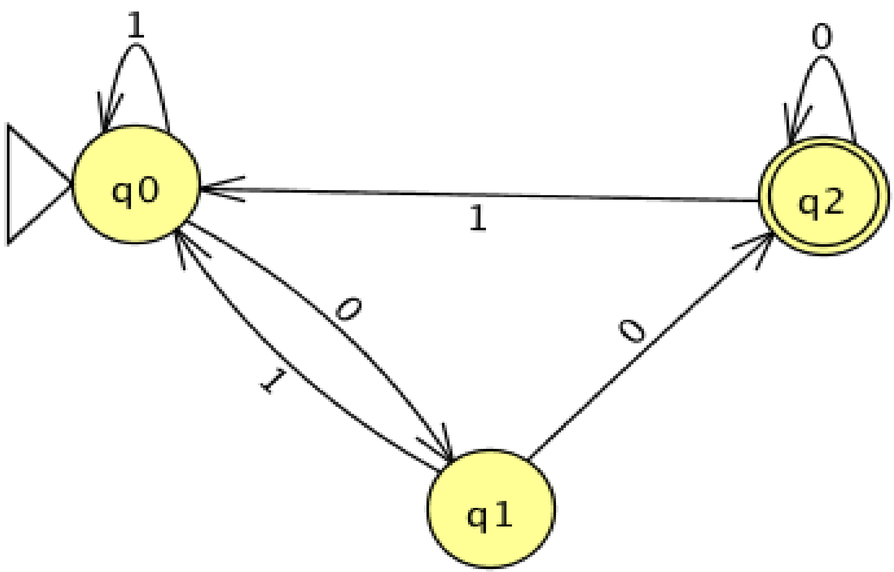


What are the languages of these DFAs?

Example: Design a FA with $\sum = \{0, 1\}$ accepts the only input 101.



Example: The set of all strings $\sum = \{0, 1\}$ ending in 00.

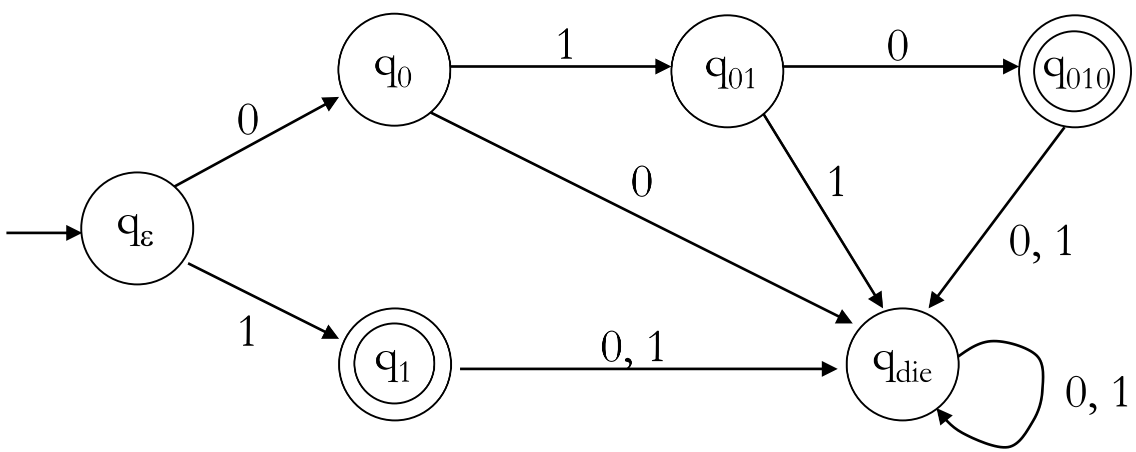


Examples

- Construct a DFA that accepts the language

$$L = \{010, 1\} \qquad (\Sigma = \{0, 1\})$$

- Answer



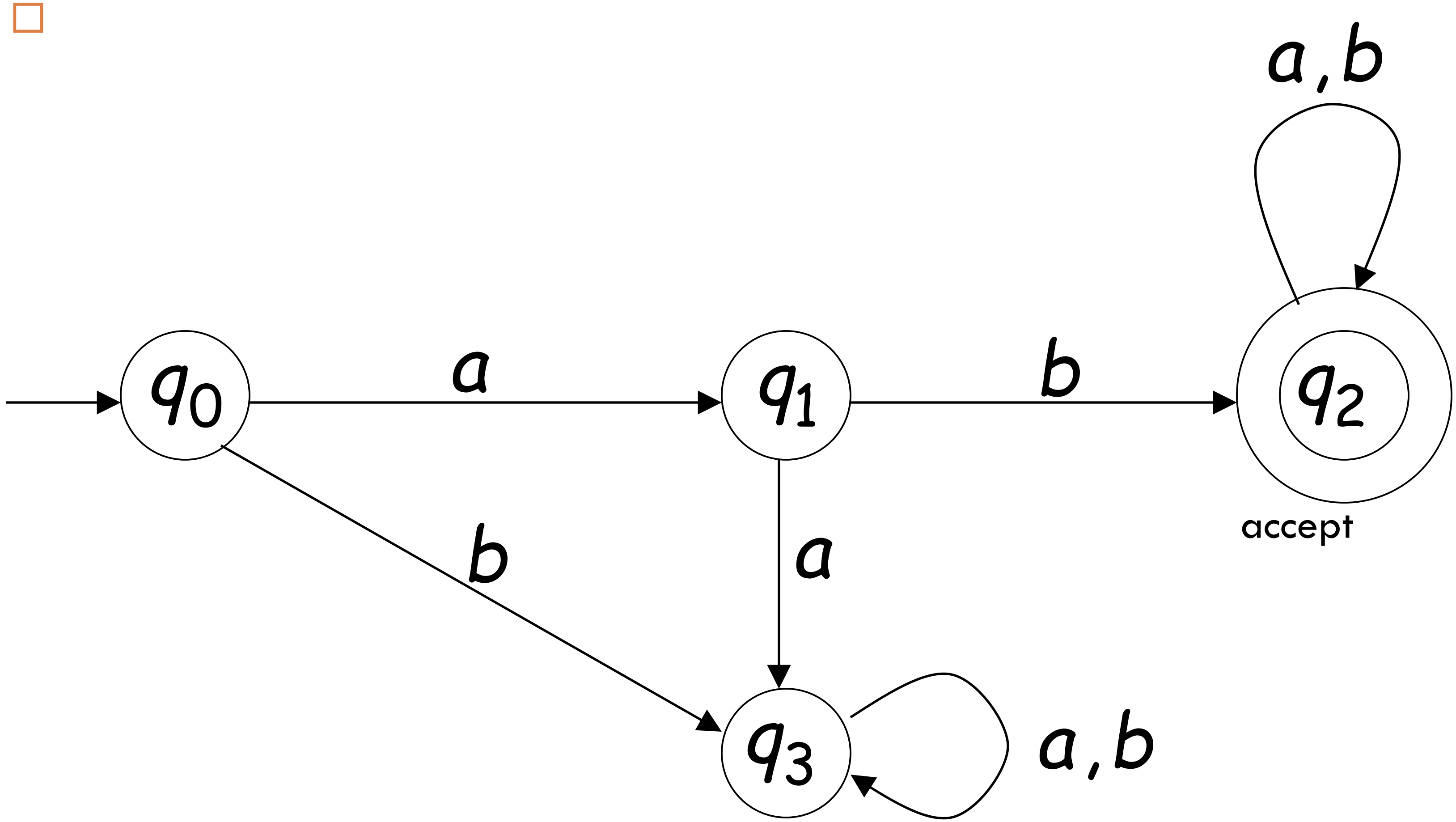
Σ^+ and Σ^*

- Σ is an alphabet , $\Sigma = \{y\}$
- Σ^* is the set of all strings including null or obtained by concatenating zero or more symbols from Σ
- $\Sigma^* = \{\varepsilon, y, yy, yyy, yyyy, \ldots\}$

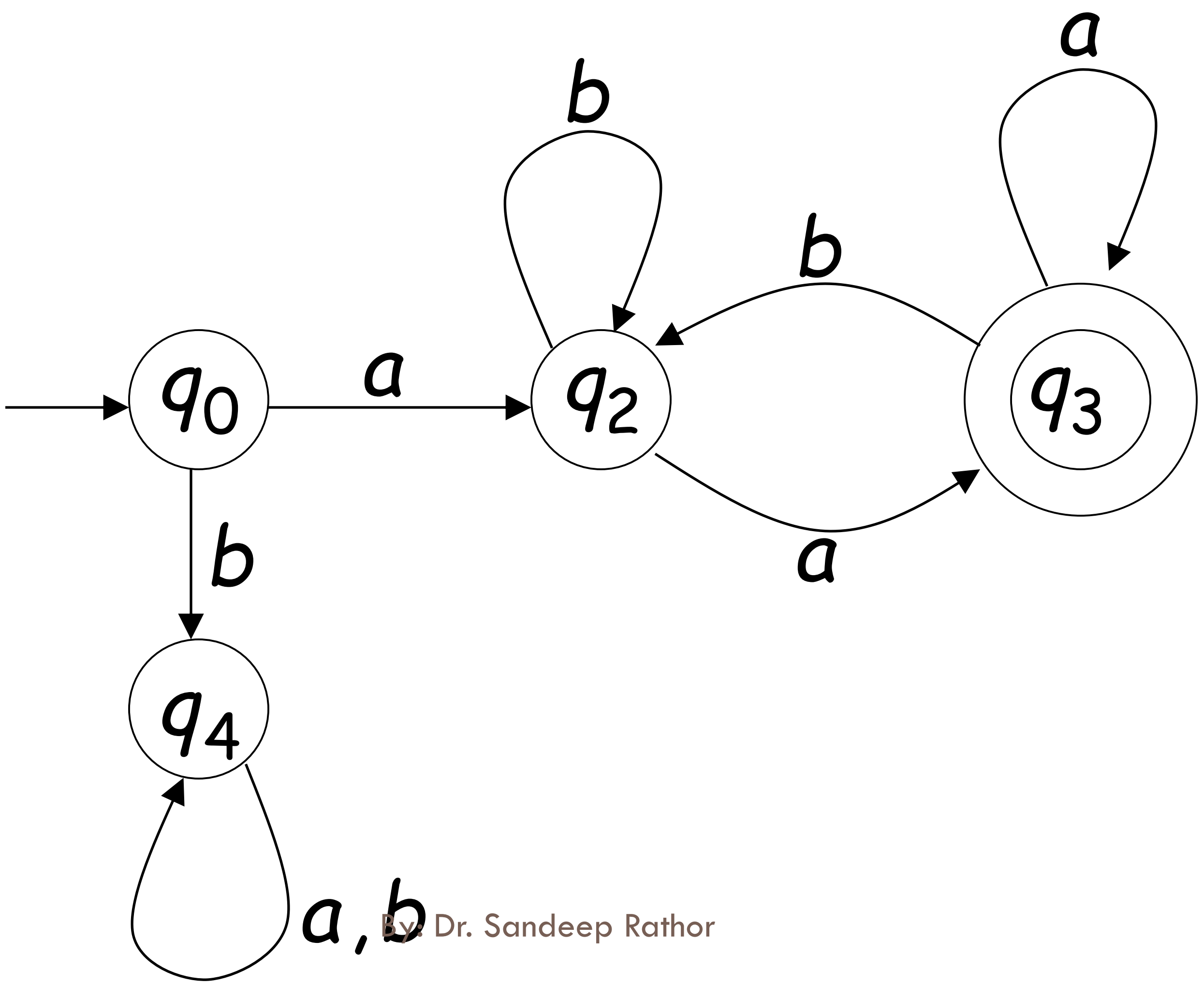
- Σ^+ is the set of all strings excluding null or obtained by concatenating one or more symbols
- $\Sigma = \{y\}$
- $\Sigma^+ = \{y, yy, yyy, yyyy, \ldots\}$

Example-4: Design a DFA for

$L(M) = \{$ all strings with prefix ab $\}$

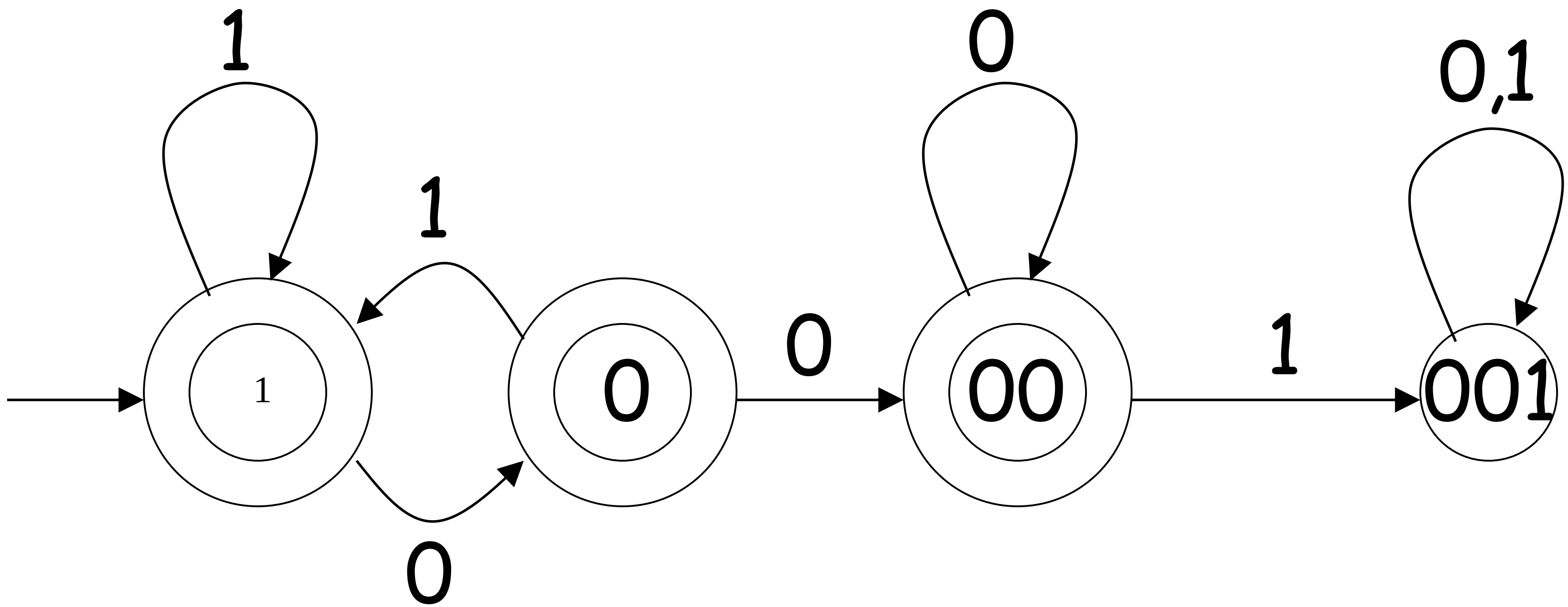


Example-5: DFA for $L(M) = \{awa : w \in \{a,b\}^*\}$



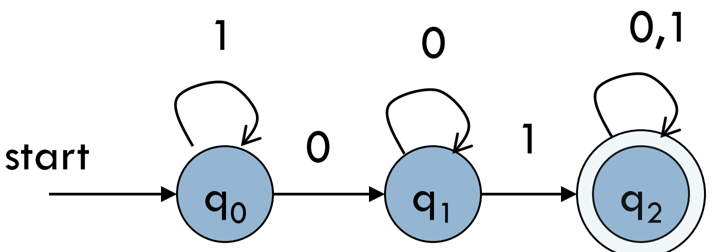
Example-6: DFA for

$L(M) = \{$ all strings without substring 001 $\}$



Ex-7:Design a DFA for language: $L = \{w \mid w$ is a binary string that contains 01 as a substring$\}$

DFA for strings containing 01



Accepting state

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{0,1\}$
- start state = q_0
- $F = \{q_2\}$
- Transition table

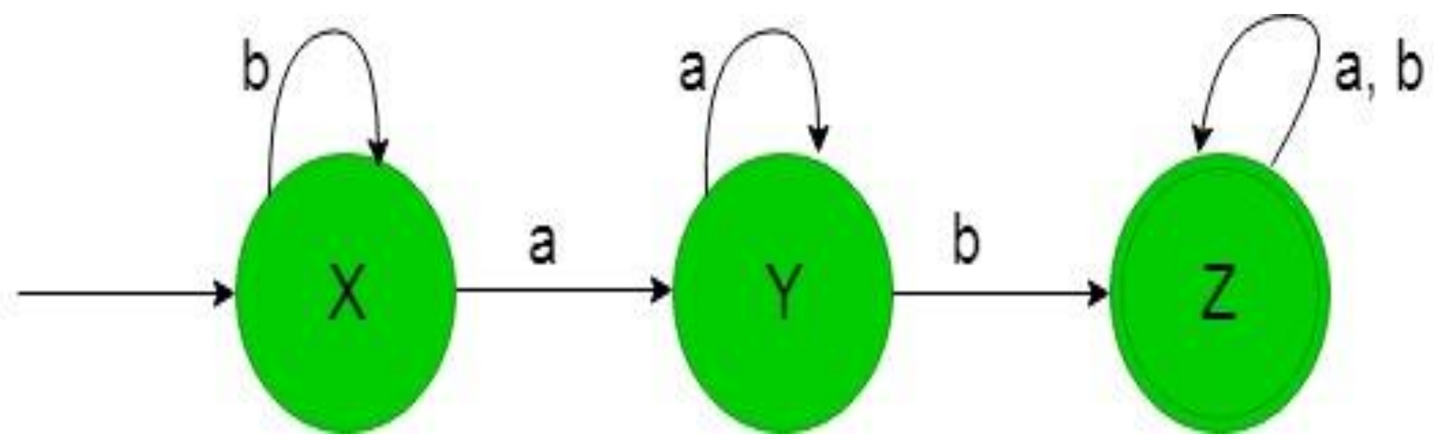
symbols

δ	0	1
→ q_0	q_1	q_0
q_1	q_1	q_2
$*q_2$	q_2	q_2

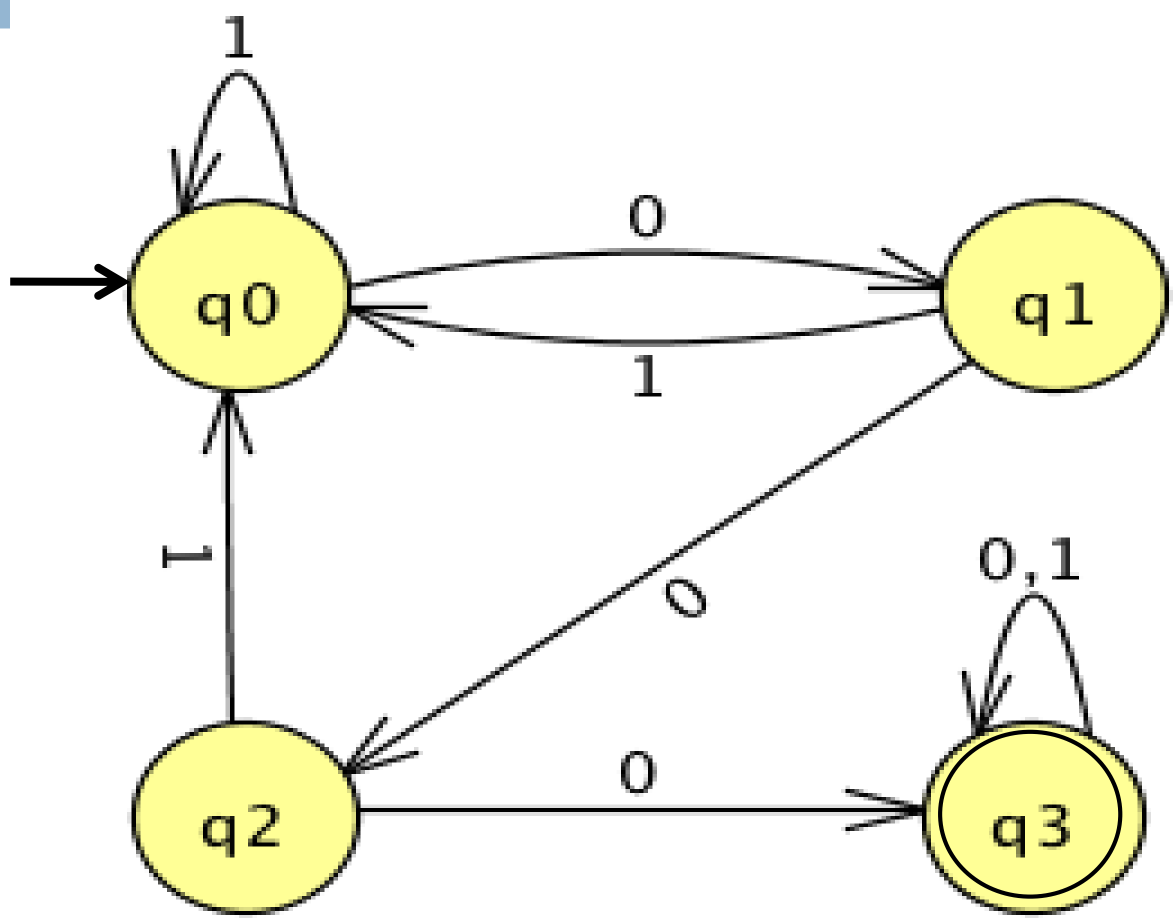
states

- What if the language allows empty strings?

Construction of a DFA accepting set of string over {a, b} where each string containing 'ab' as the substring.

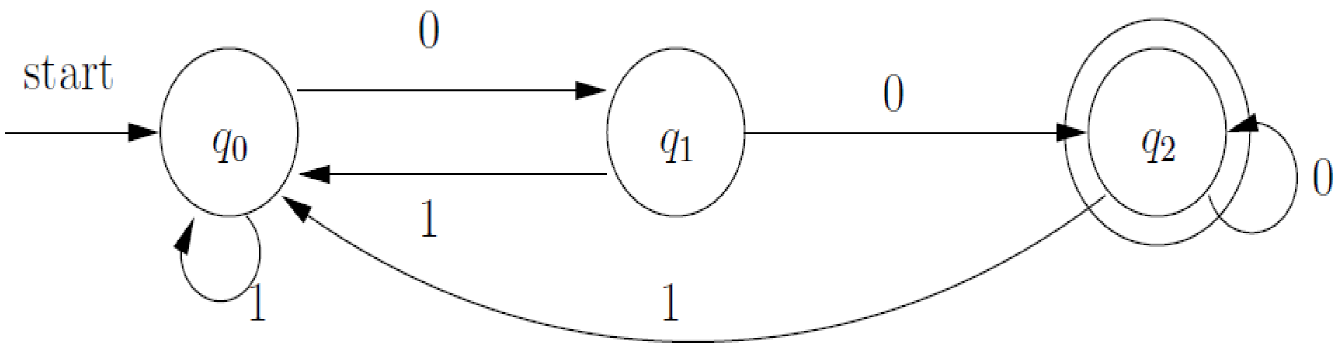


Example: The set of all strings with three consecutive 0's (not necessarily at the end).



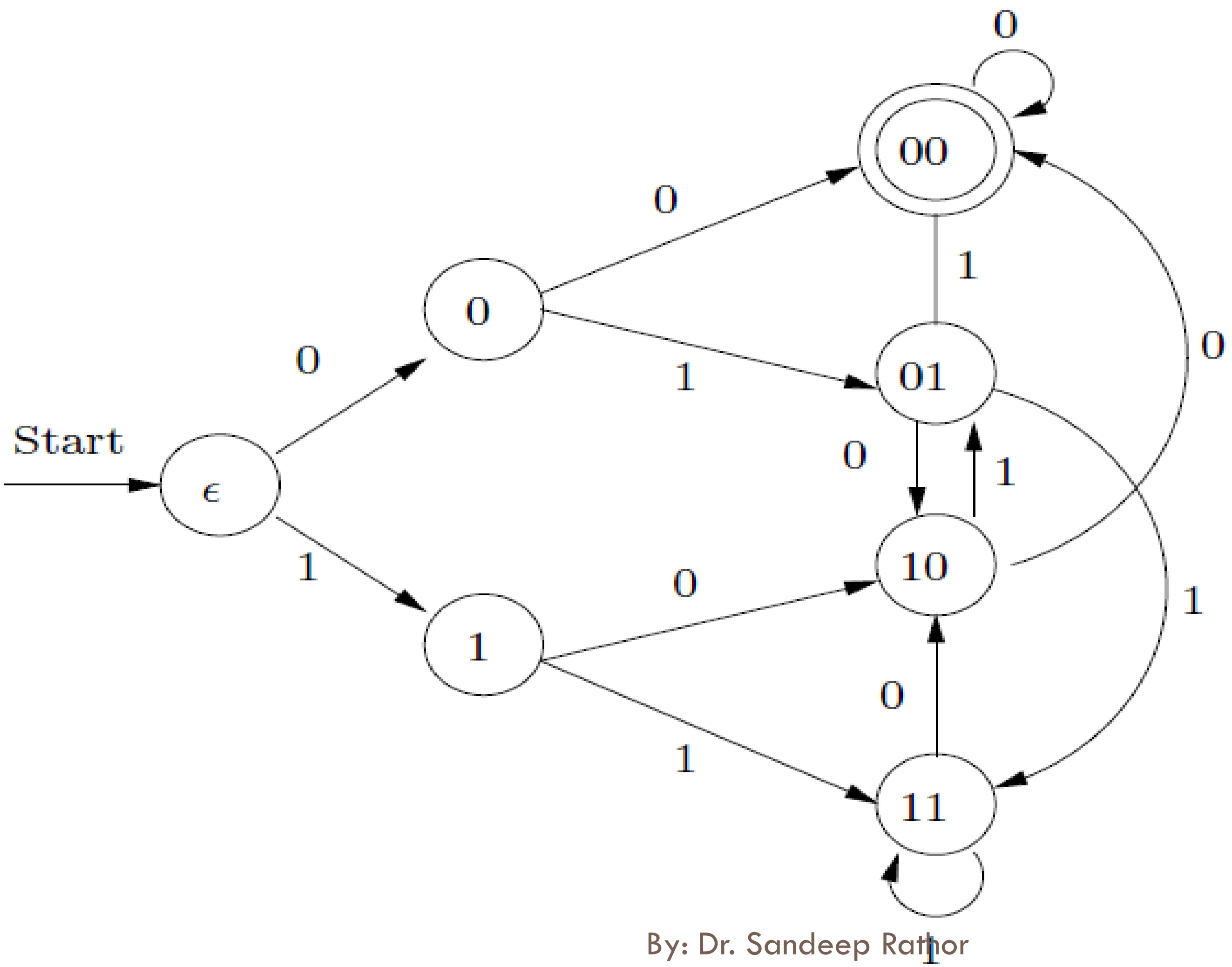
Construct DFA's accepting the following languages over the alphabet {0, 1}.

1. **The set of all strings ending in 00.**

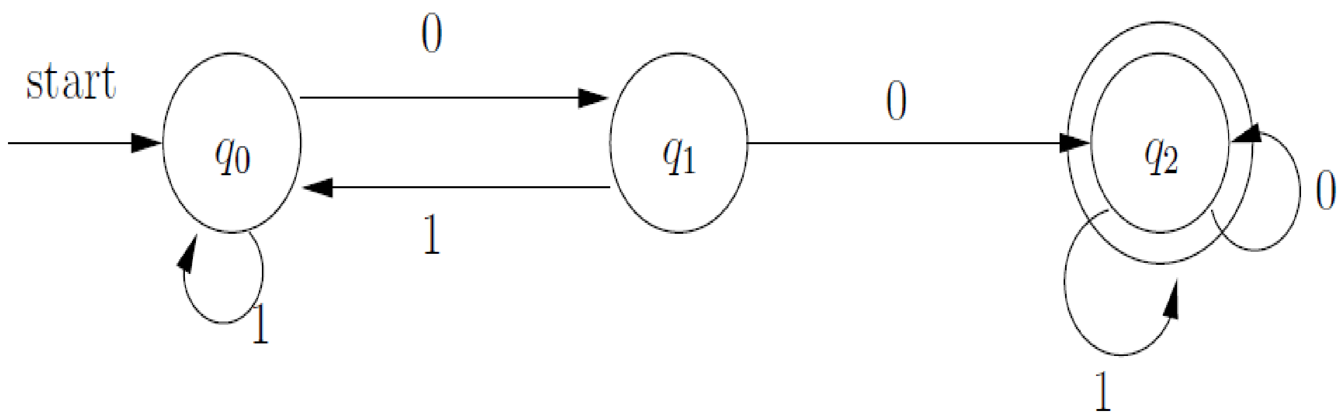


OR

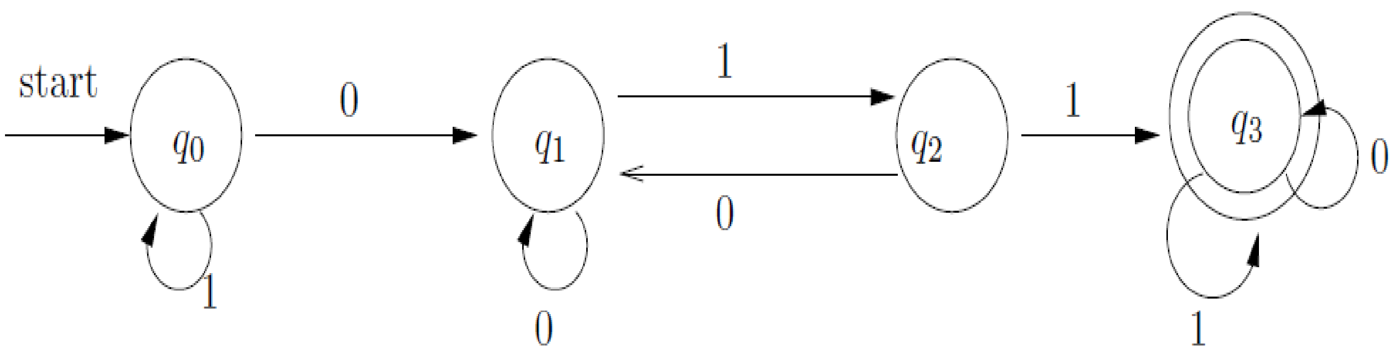
The set of all strings ending in 00.



2. The set of all strings with two consecutive 0's (not necessarily at the end).

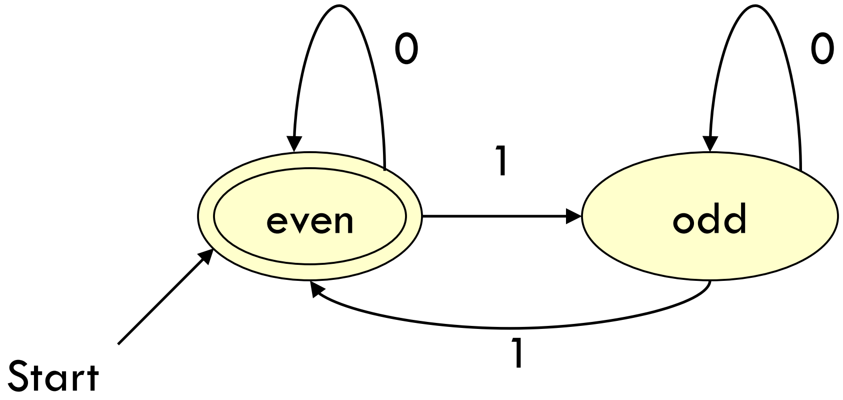


3. The set of strings with 011 as a substring

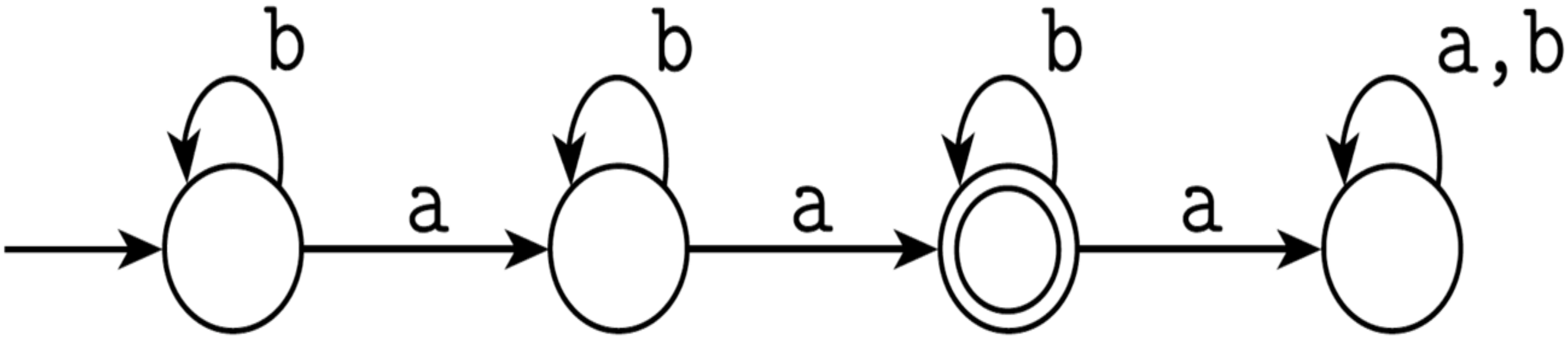


Practice Questions…

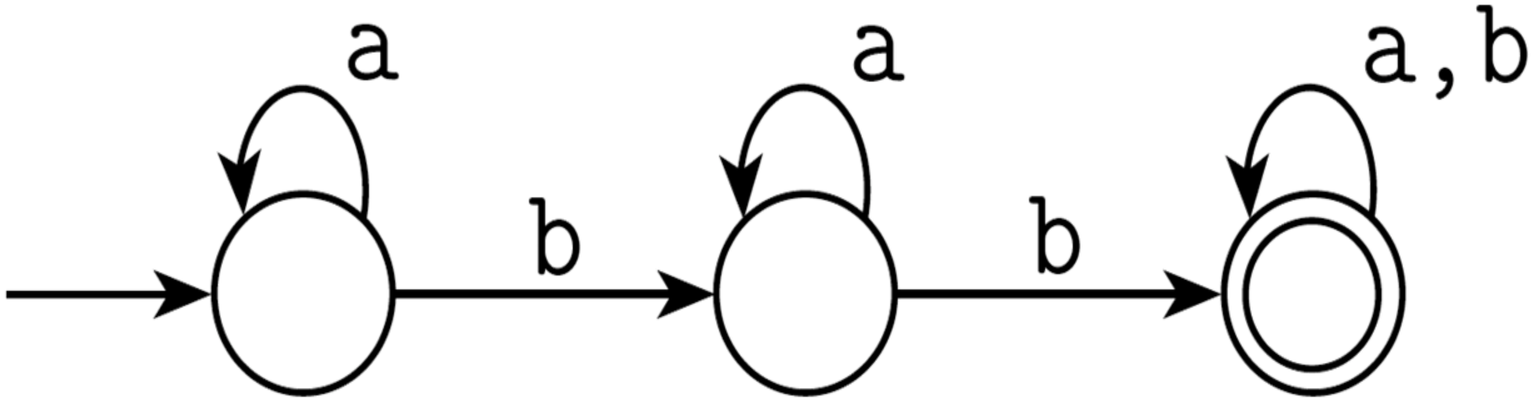
DFA for: An Even Number of 1's



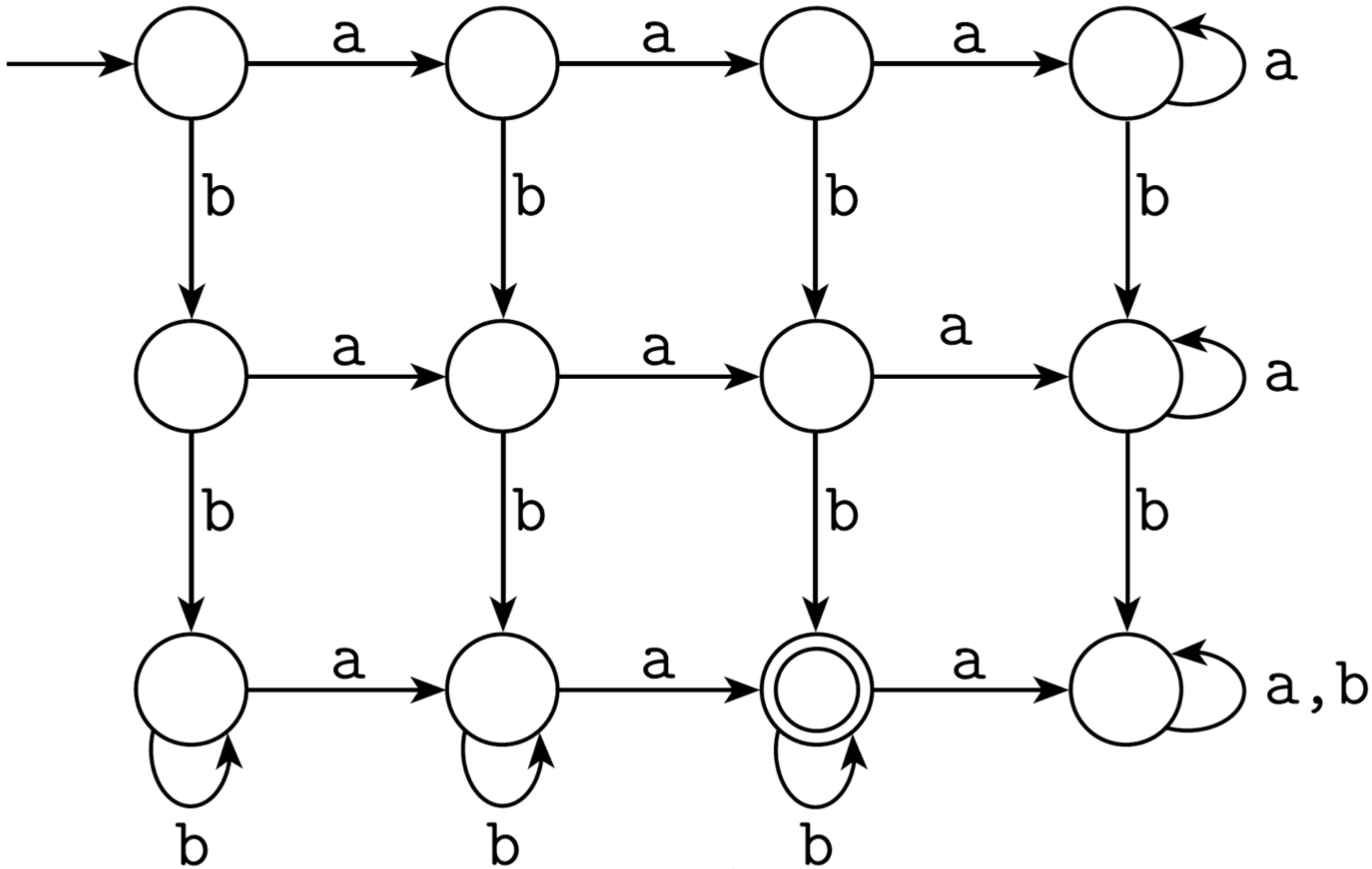
Exactly Two a's



At Least Two b's



Exactly two a's and at least two b's



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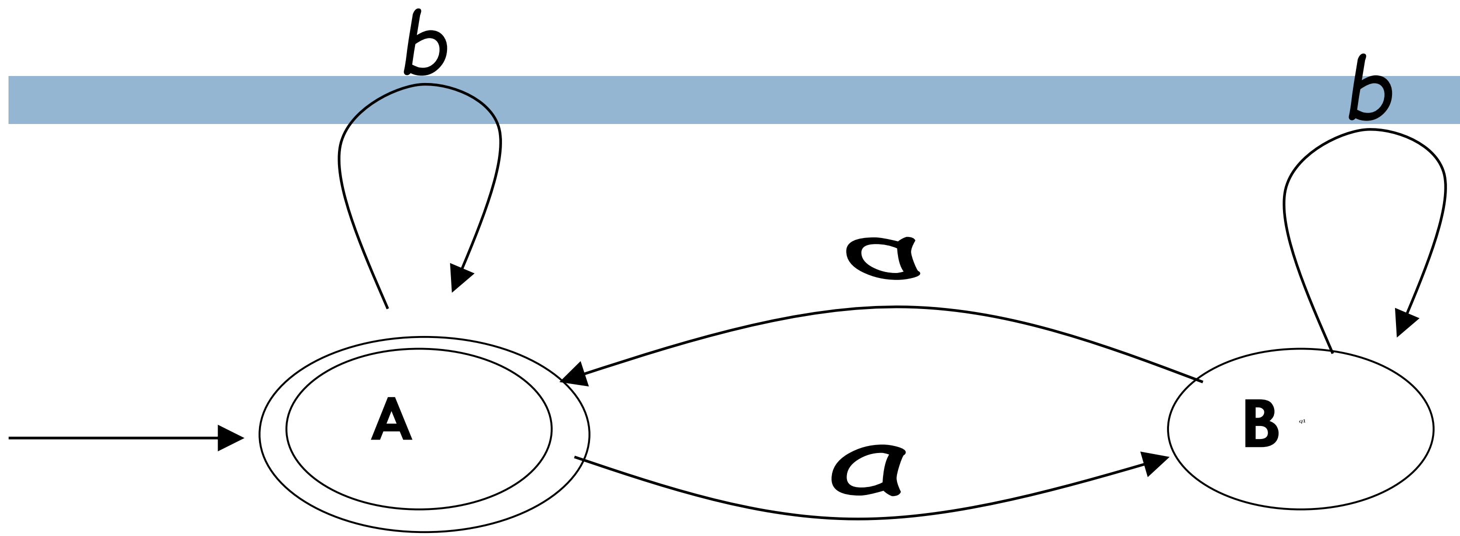
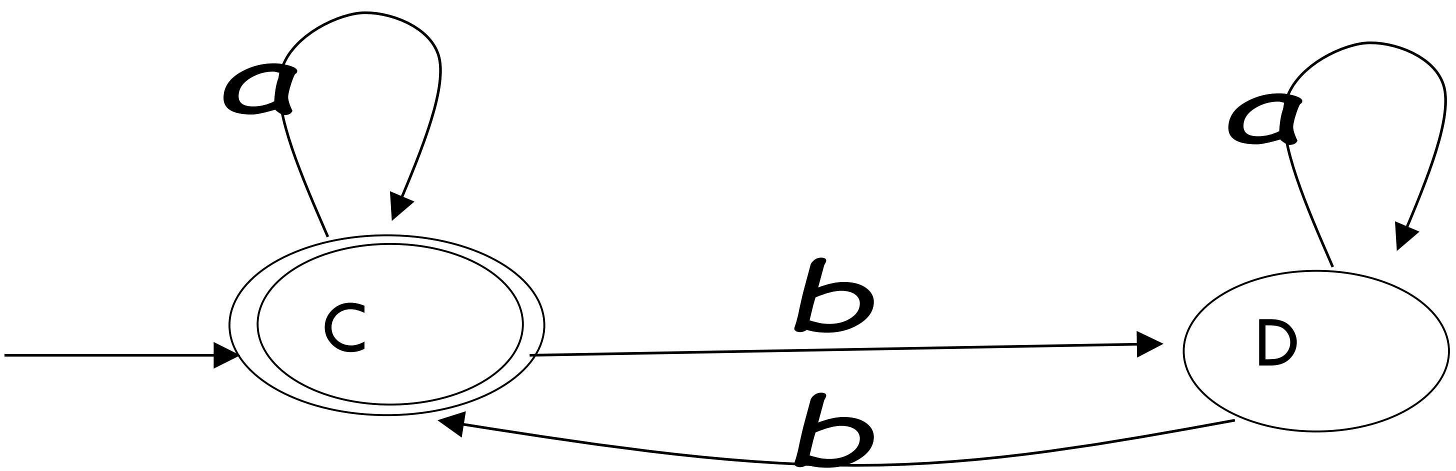
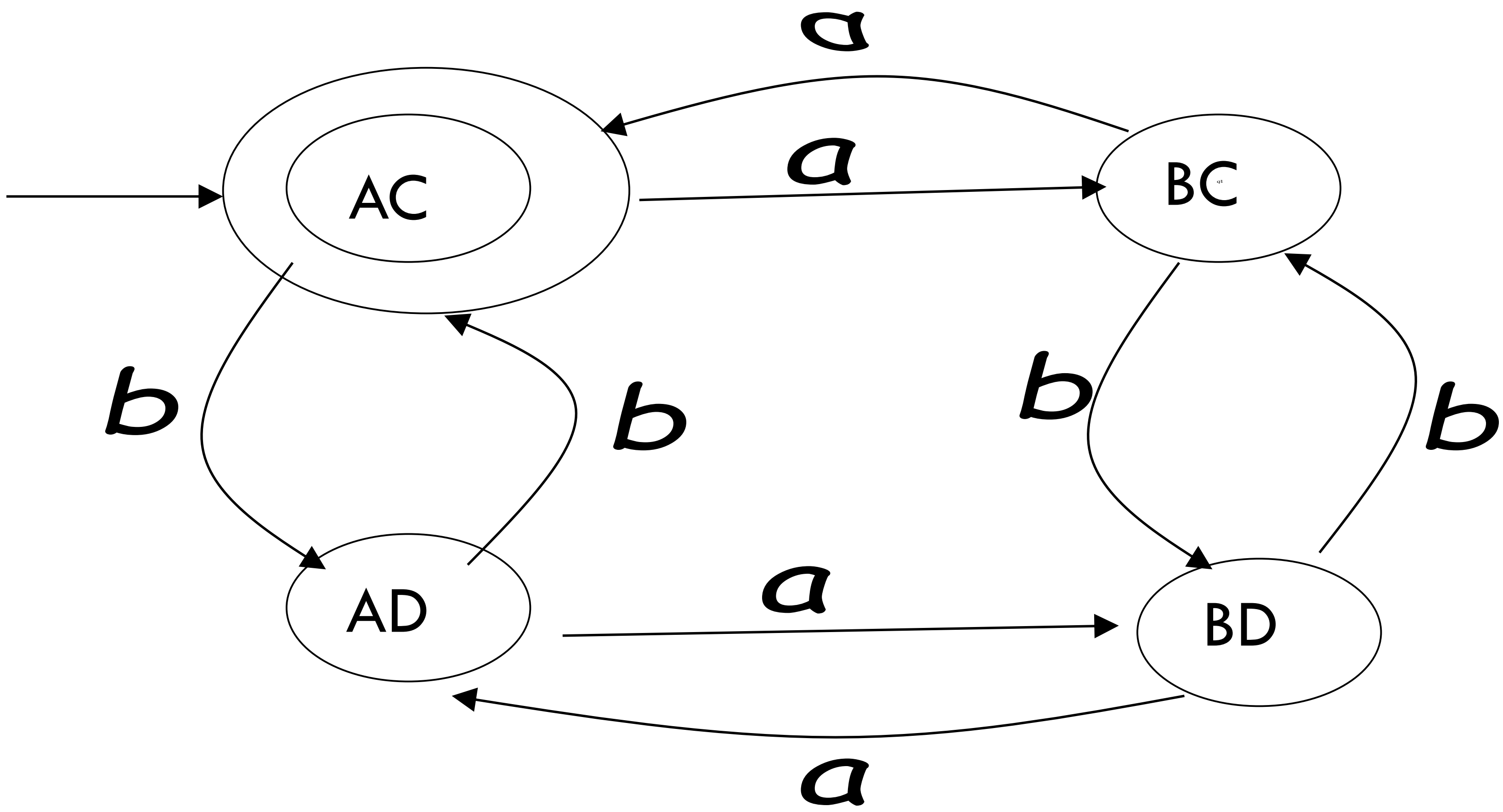
Construct a FA that accepts the strings having no. of 'a' divisible by 2 or even no. of 'a'



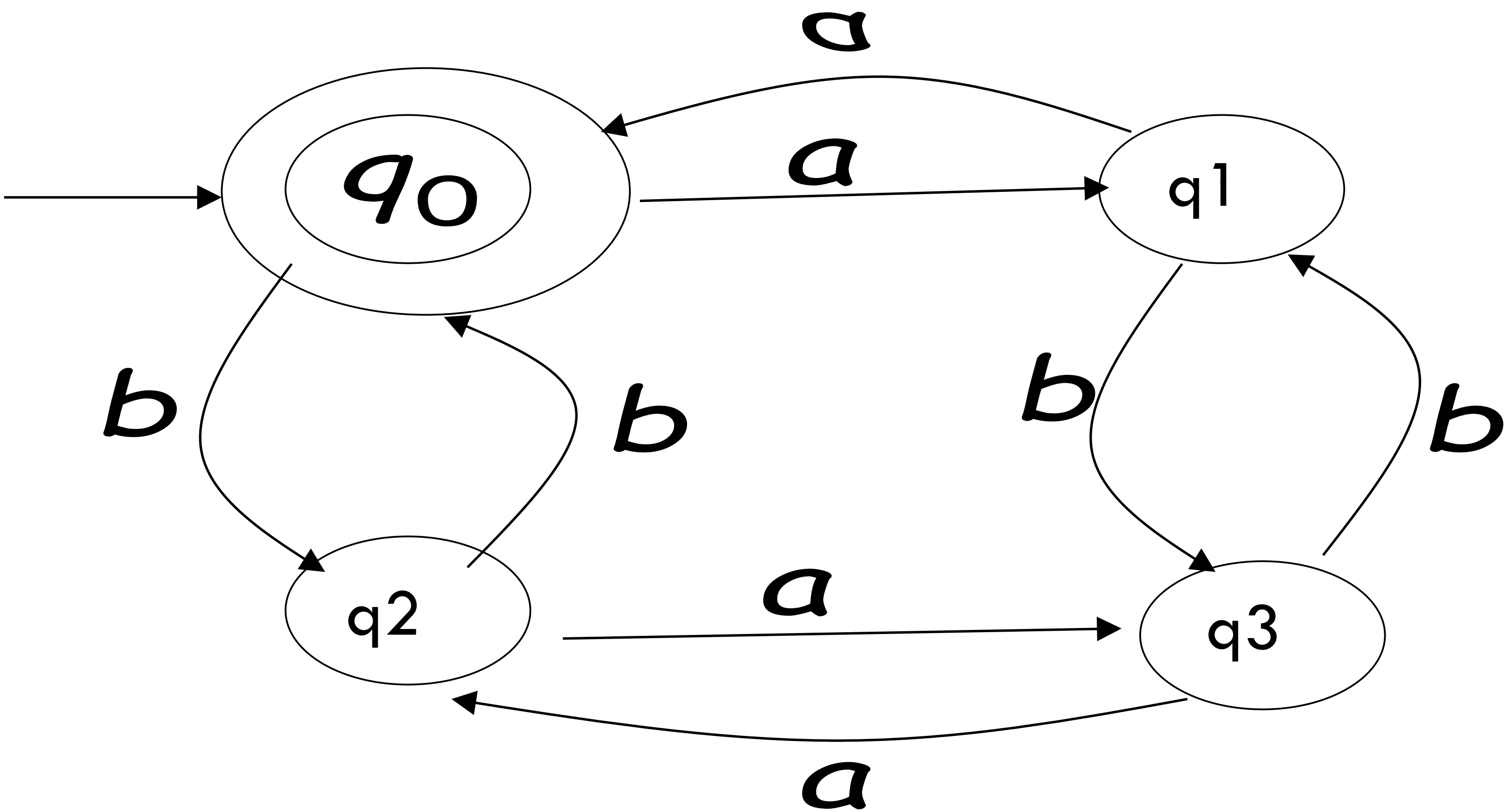
Construct a FA that accepts the strings having no. of 'b' divisible by 2 or even no. of 'b'



Construct a FA that accepts the strings having no. of ‘a’ divisible by 2 and no. of ‘b’ divisible by 2

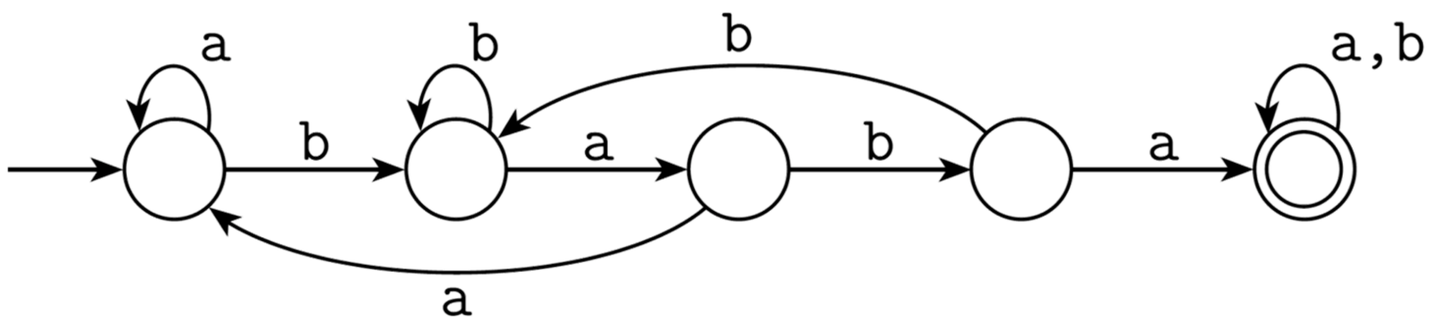


Construct a FA that accepts the strings having no. of 'a' divisible by 2 and no. of 'b' divisible by 2

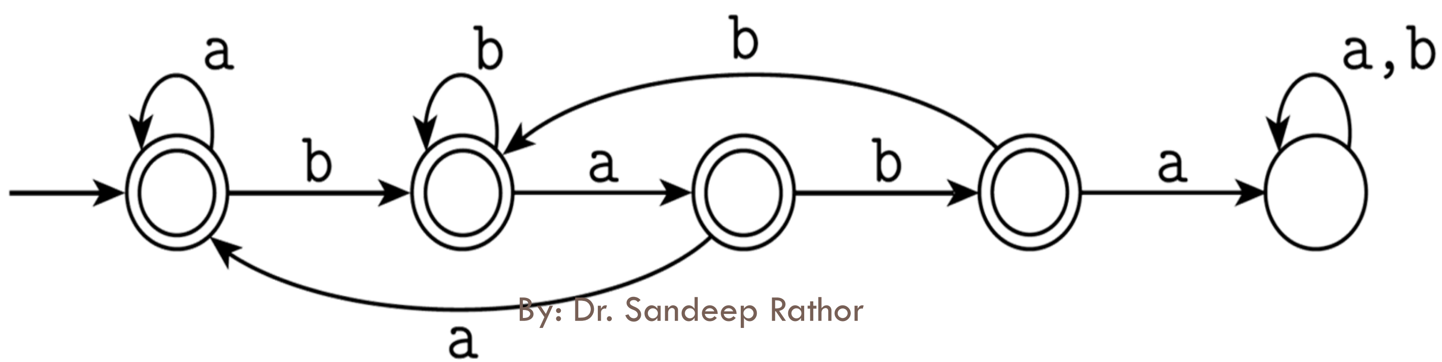


Containing Substrings or Not

- Contains baba:



- Does not contain baba:



Non-Deterministic Finite Automata (NFA)

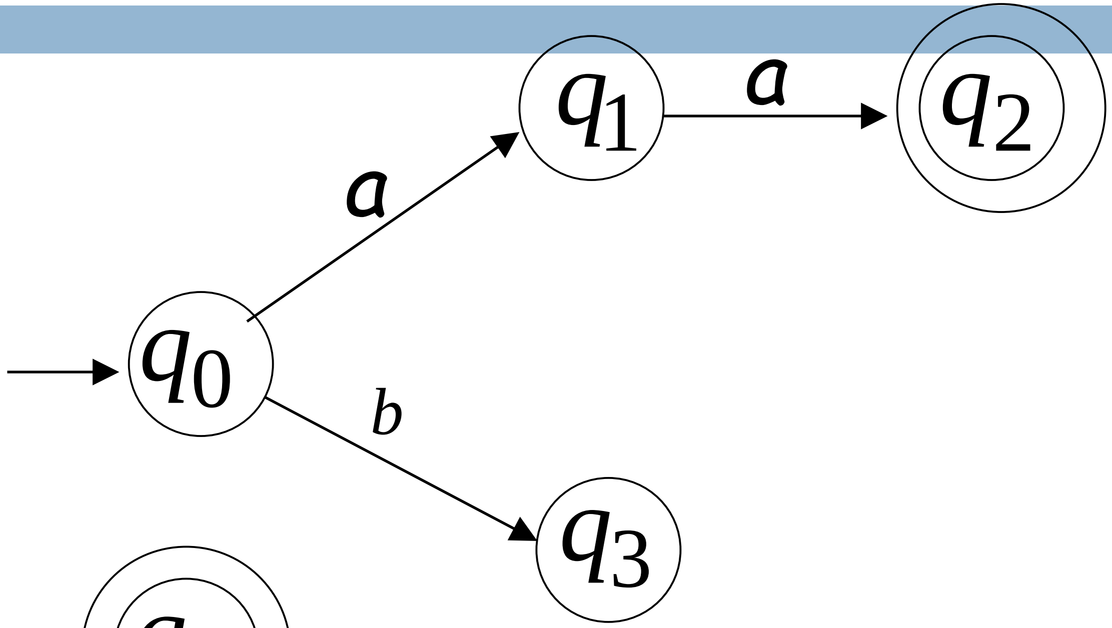
A Non-deterministic Finite Automata (NFA) is a 5-tuple

$(Q, \Sigma, \delta, q_0, F)$ where:

- Q is a finite set of states
- Σ is an input alphabet
- δ: is a transition function $Q \times \Sigma \rightarrow 2^Q$ [power set of Q]
- $q_0 \in Q$ is an initial state
- $F \subseteq Q$ is a set of accepting states (or final states).

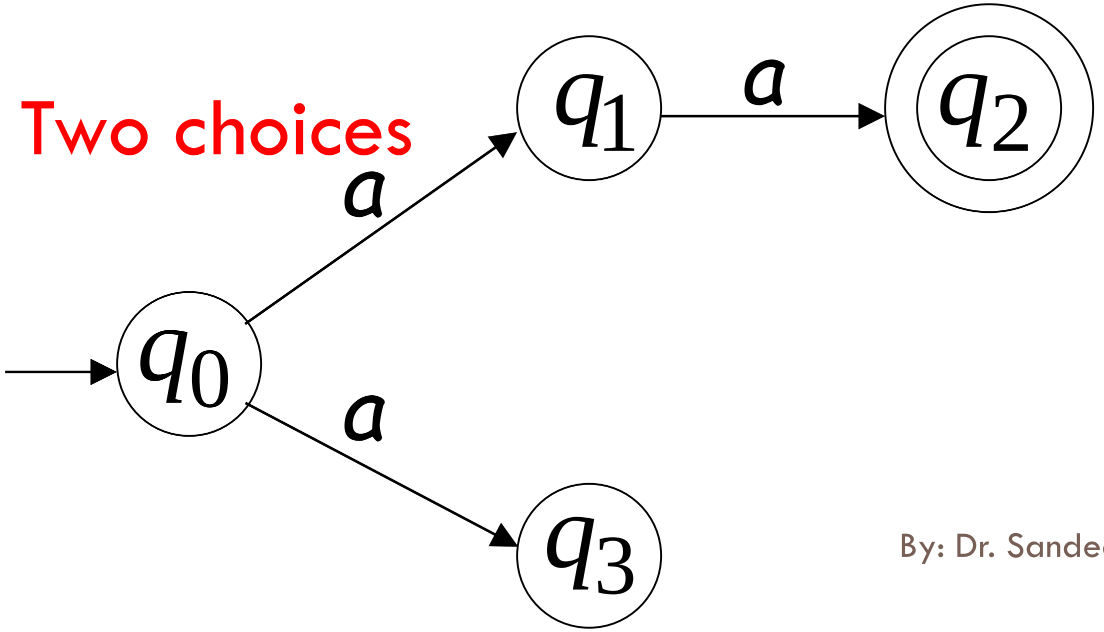
Example of DFA & NFA

DFA

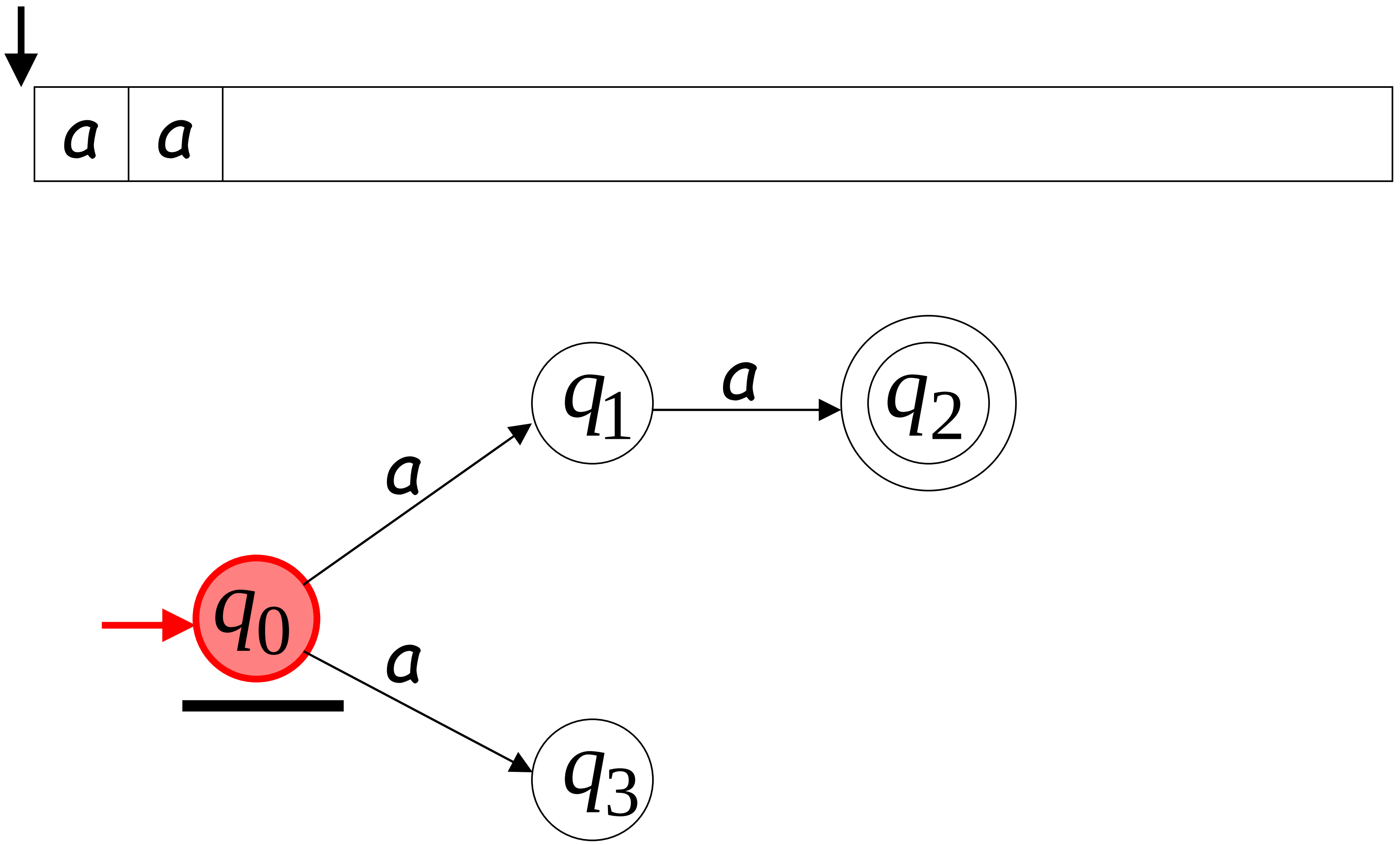


NFA/ NDFA

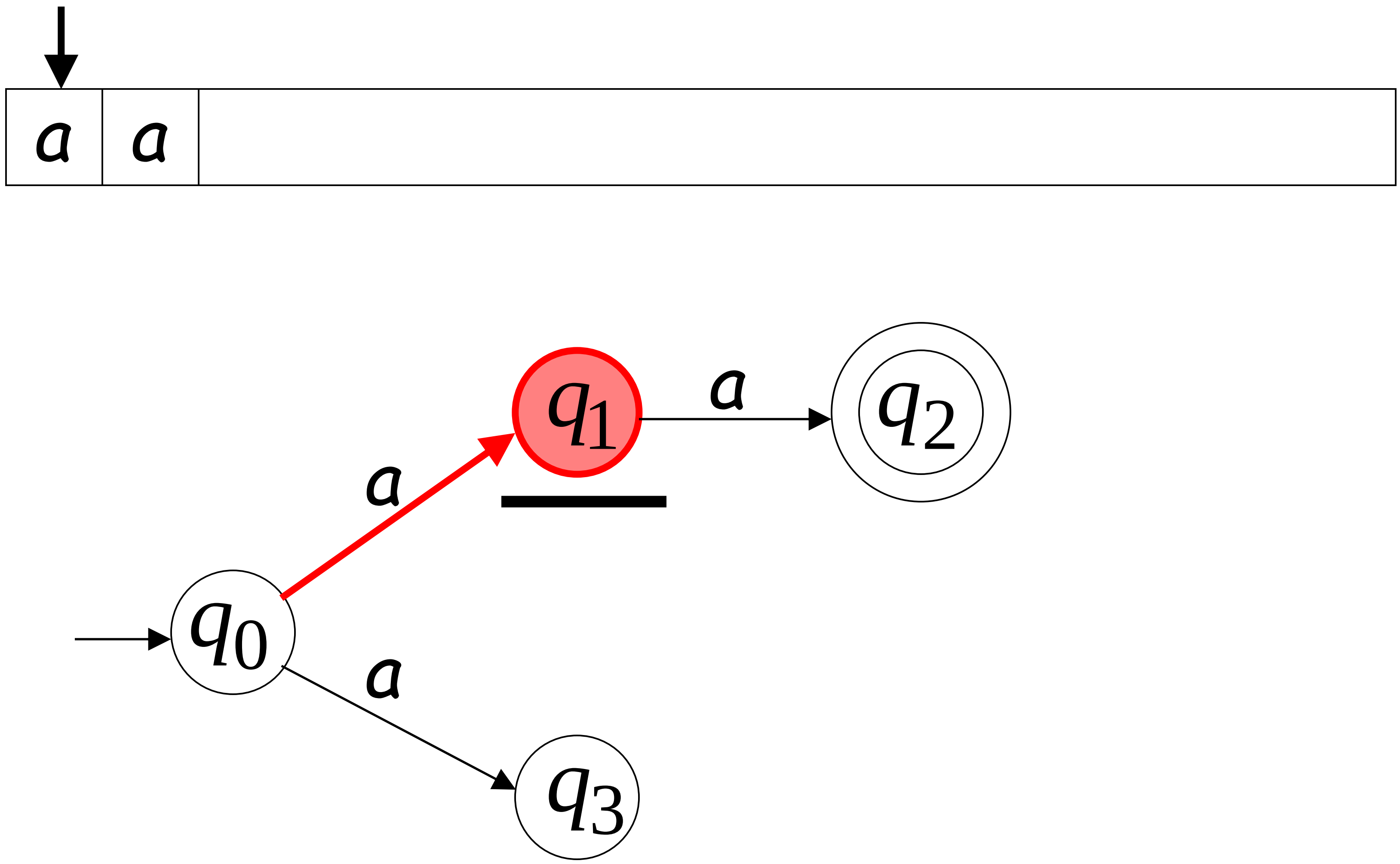
Two choices



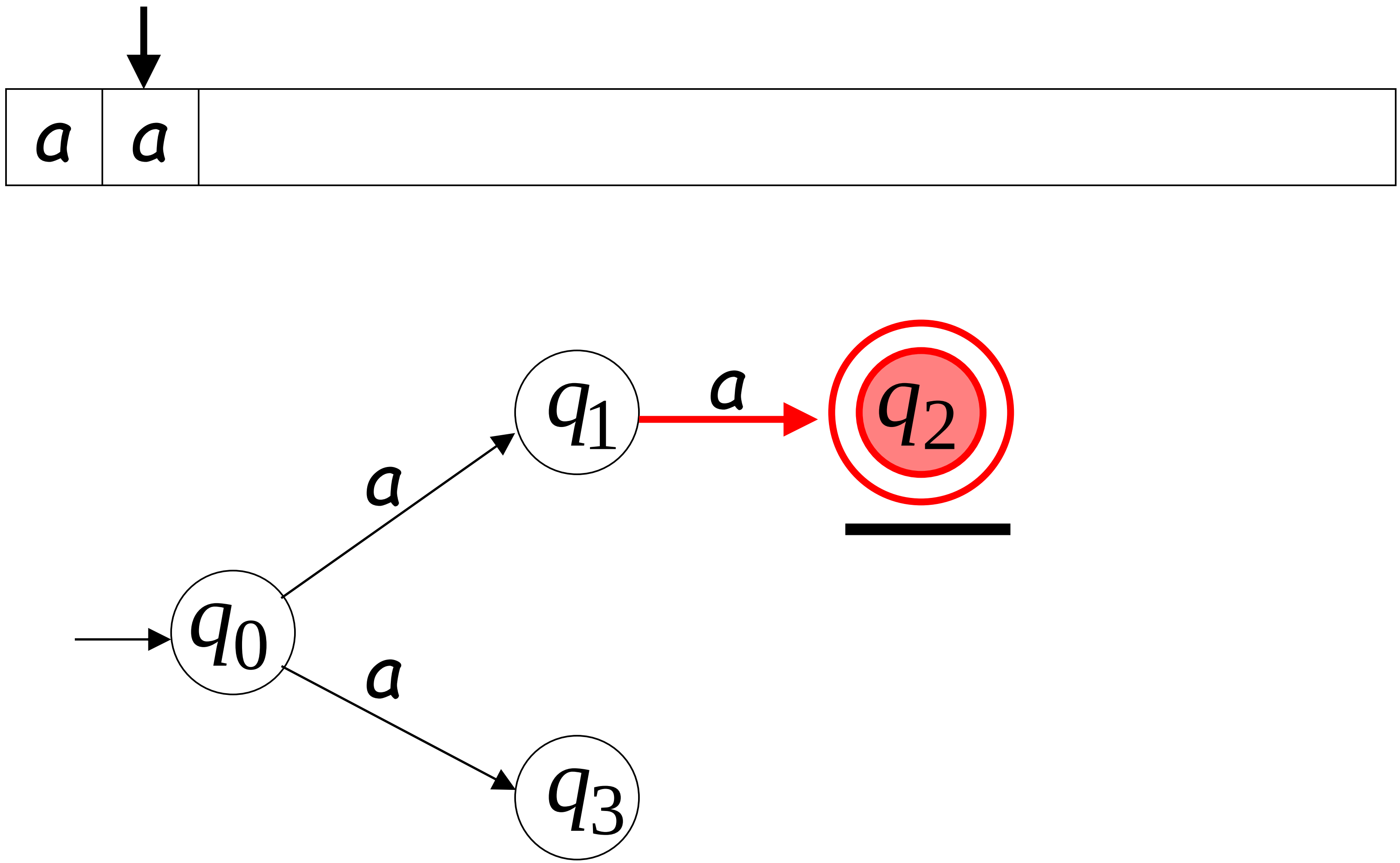
First Choice



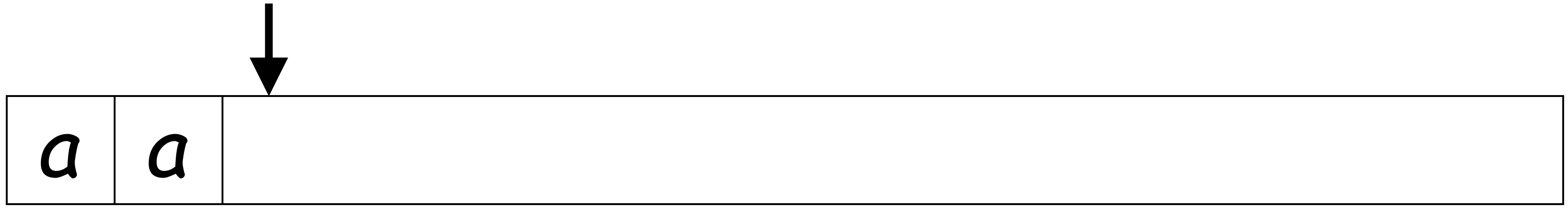
First Choice



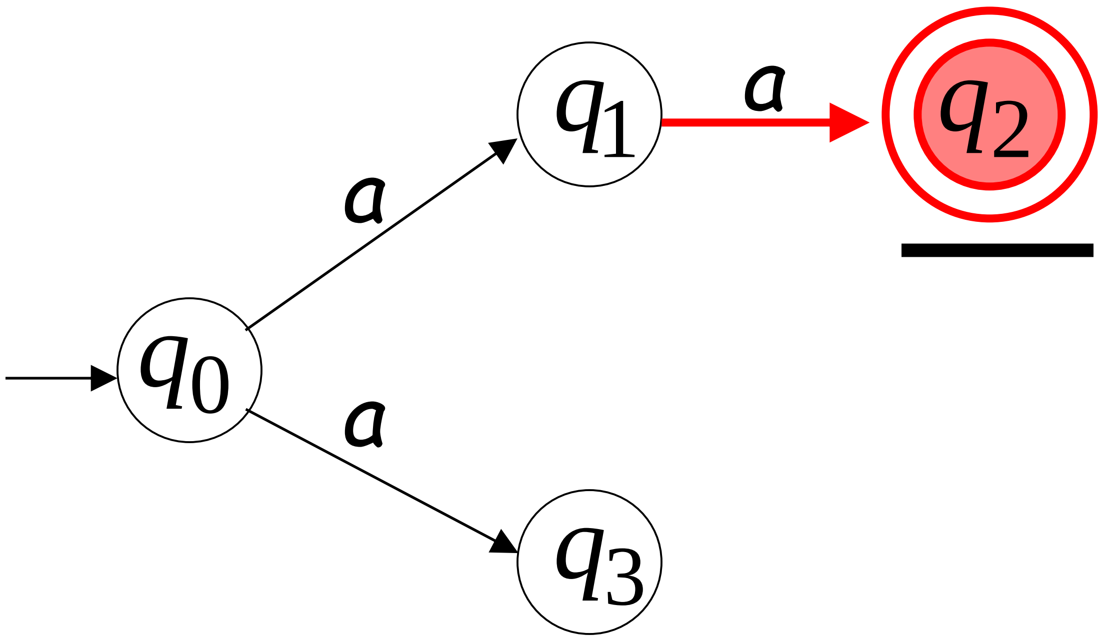
First Choice



First Choice

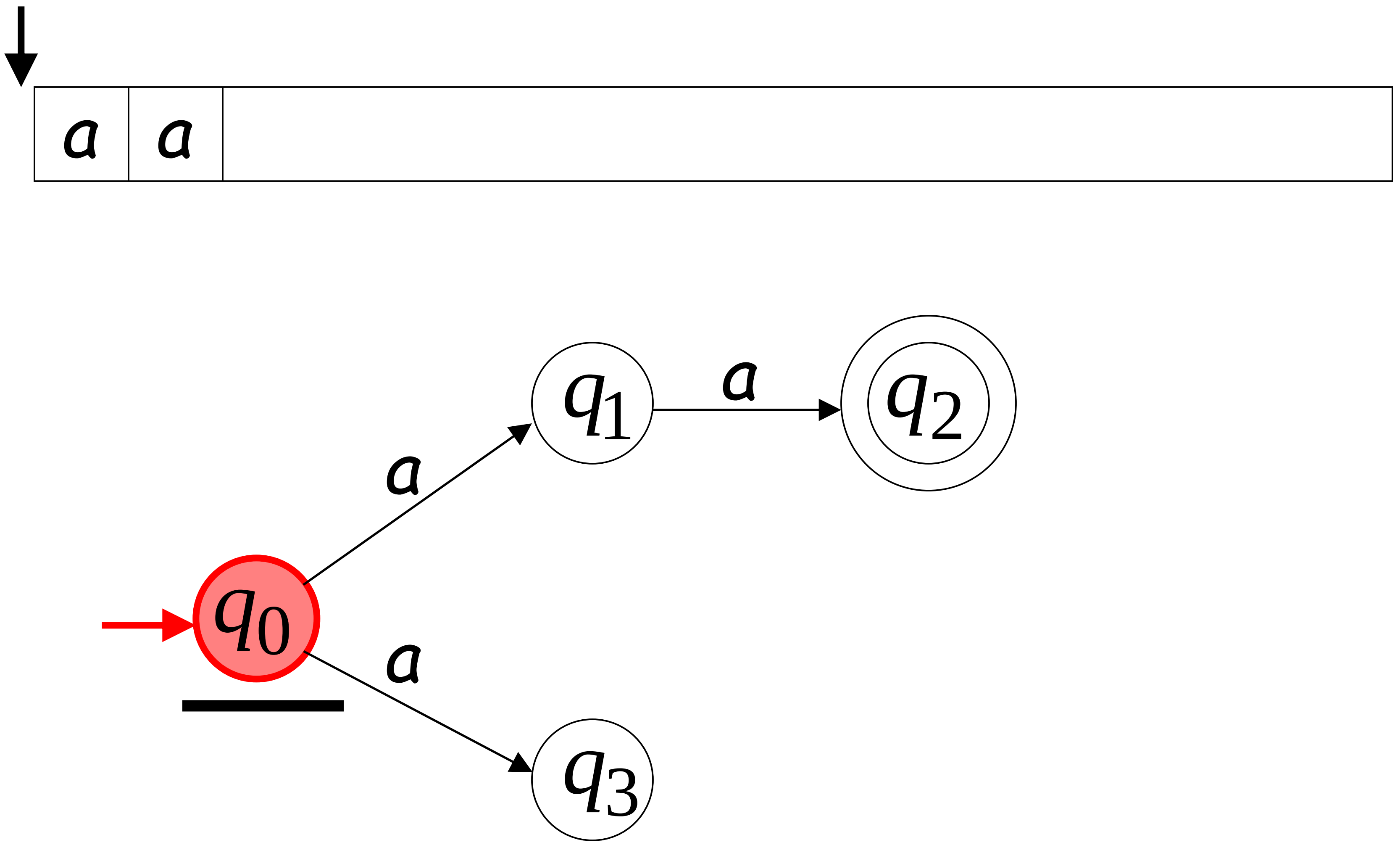


All input is consumed

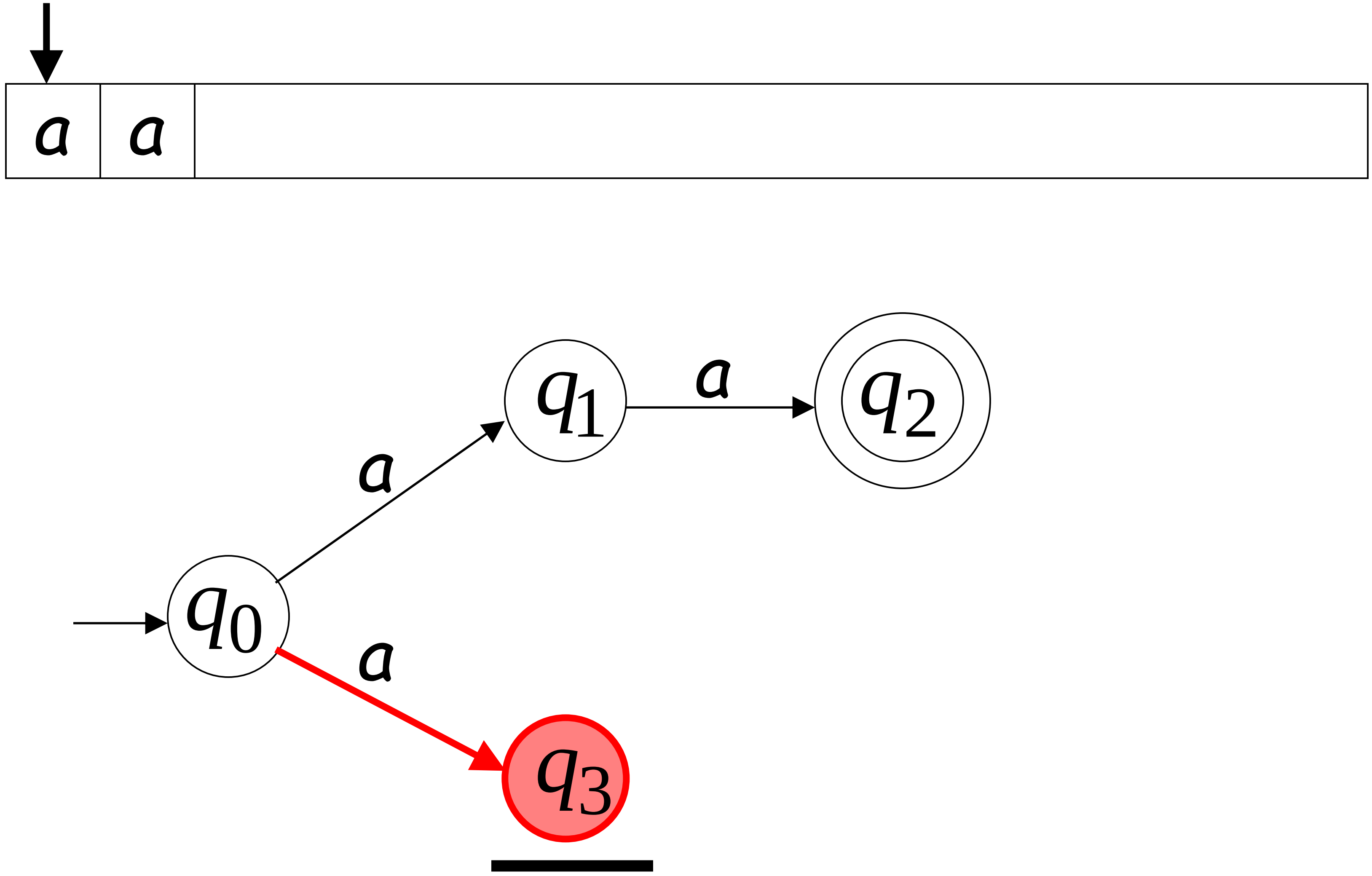


"accept"

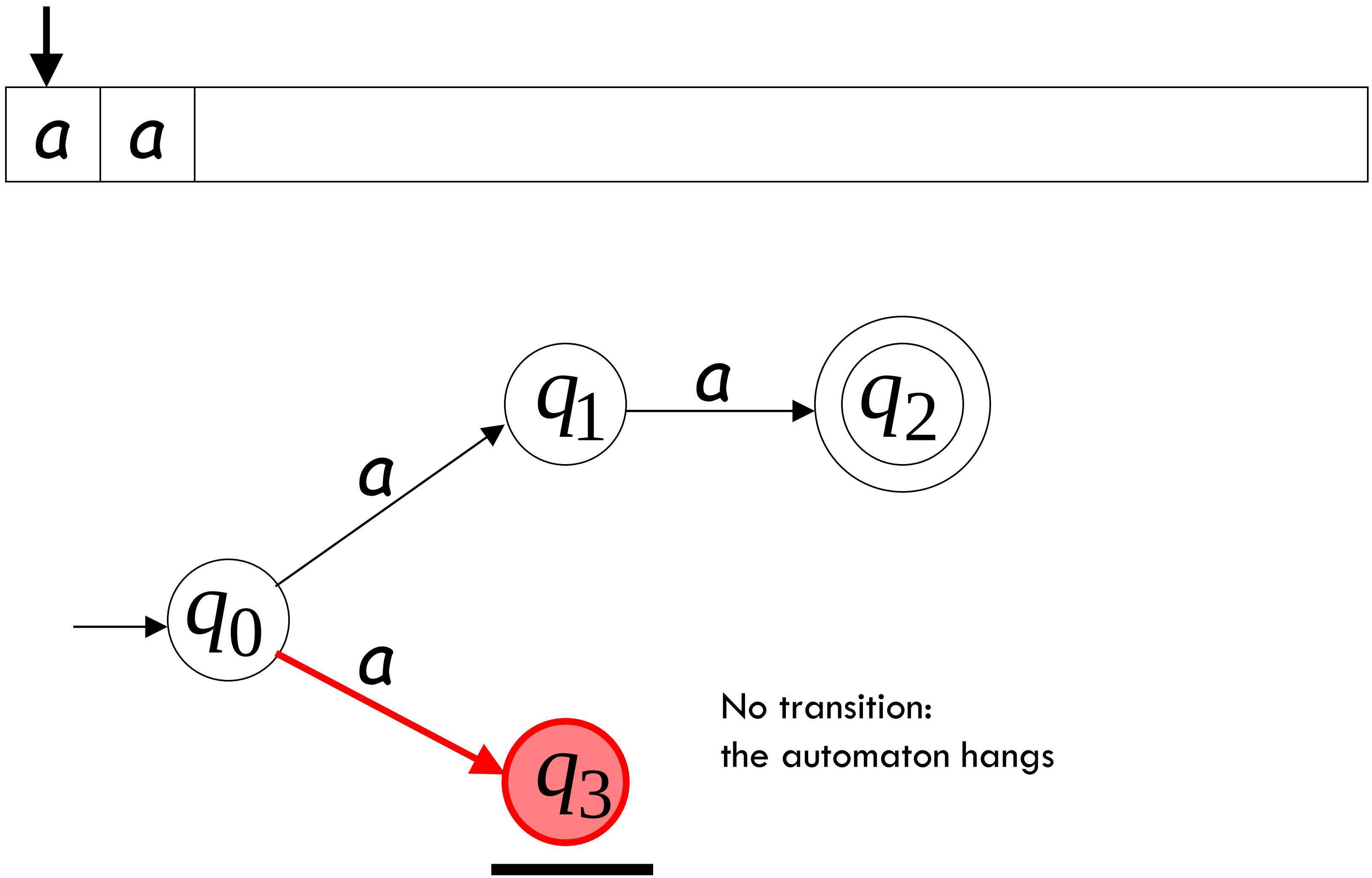
Second Choice



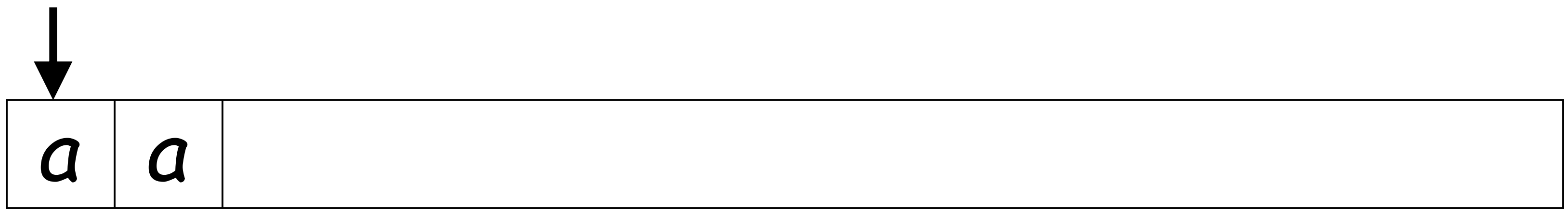
Second Choice



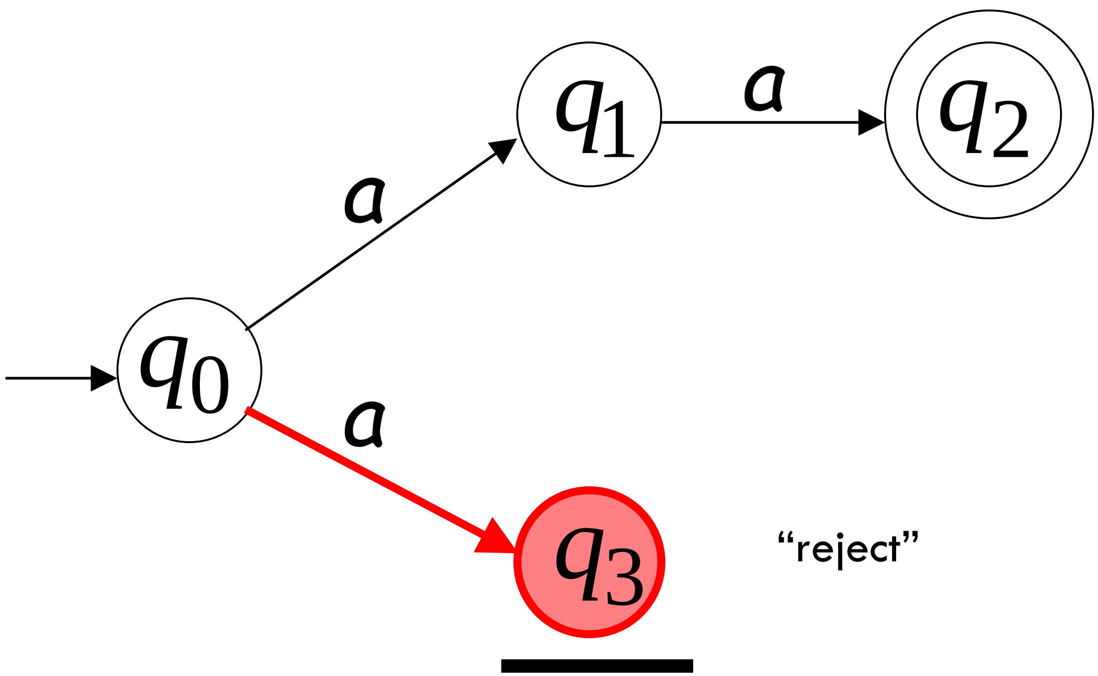
Second Choice



Second Choice



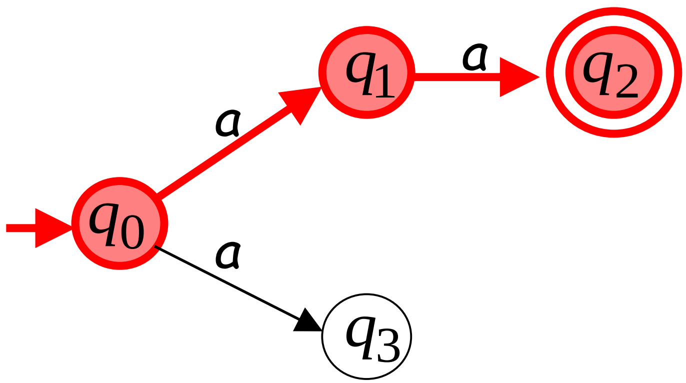
Input cannot be consumed

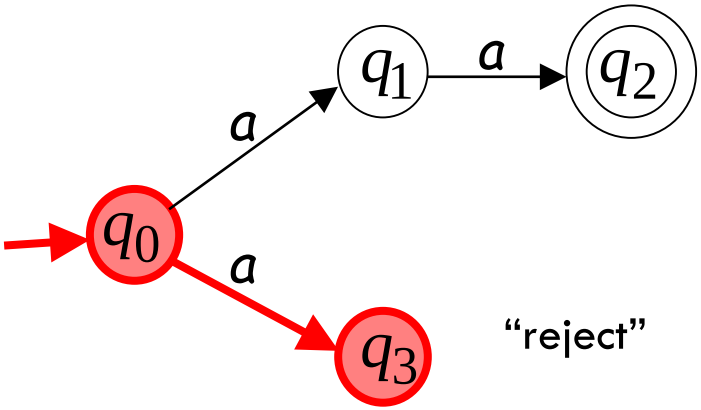


Example

aa is accepted by the NFA:

“accept”





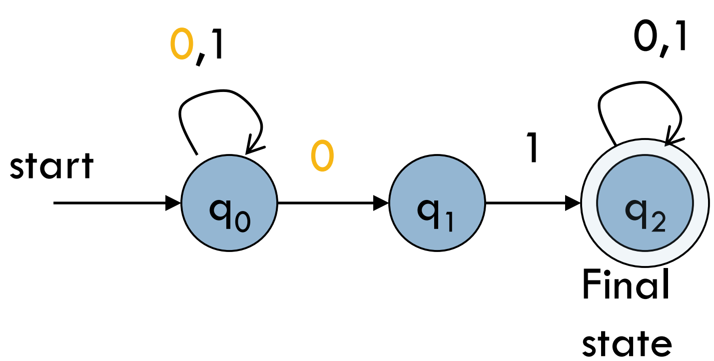
“reject”

because this computation accepts

aa

NFA for strings containing 01

Why is this non-deterministic?



What will happen if at state q_1 an input of 0 is received?

- $Q = \{q_0, q_1, q_2\}$
- $\Sigma = \{0,1\}$
- start state $= q_0$
- $F = \{q_2\}$
- Transition table

symbols

δ (states)	0	1
→ q_0	$\{q_0, q_1\}$	$\{q_0\}$
q_1	Φ	$\{q_2\}$
$^*q_2$	$\{q_2\}$	$\{q_2\}$

Differences: DFA vs. NFA

- **DFA**

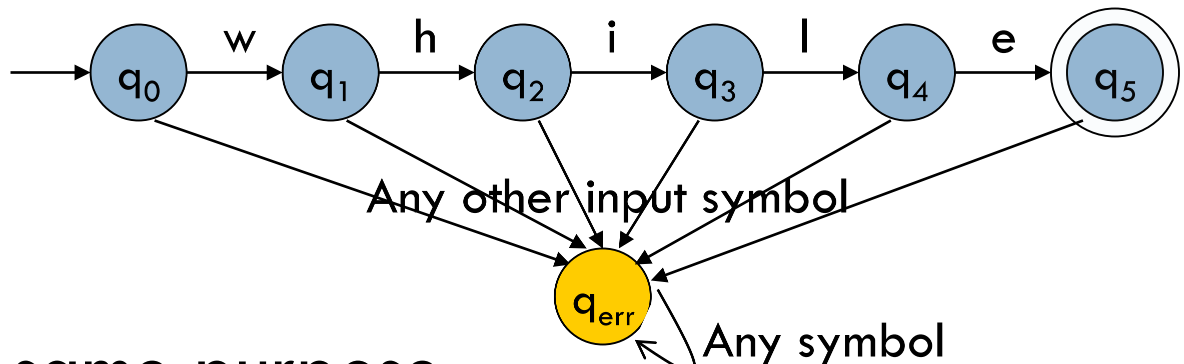
1. All transitions are deterministic
   - Each transition leads to exactly one state
2. For each state, transition on all possible symbols (alphabet) should be defined
3. Accepts input if the last state visited is in F
4. Sometimes harder to construct because of the number of states
5. $\delta: Q \times \Sigma \rightarrow Q$ is a transition function

- **NFA**

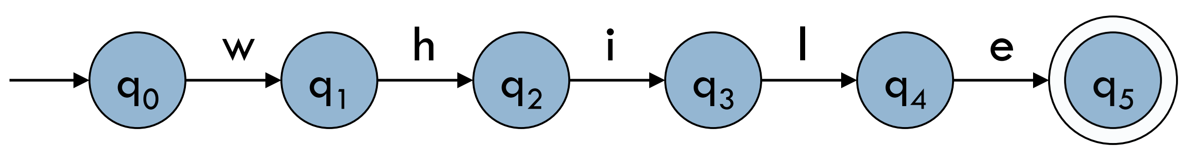
1. Some transitions could be non-deterministic
   - A transition could lead to a subset of states
2. Not all symbol transitions need to be defined explicitly (if undefined will go to an error state – this is just a design convenience, not to be confused with "non-determinism")
3. Accepts input if *one of* the last states is in F
4. Generally easier than a DFA to construct
5. $Q \times \Sigma \rightarrow 2^Q$ [subset of Q]

What is an "error state" or dummy state?

- A DFA for recognizing the key word "*while*"



An NFA for the same purpose:

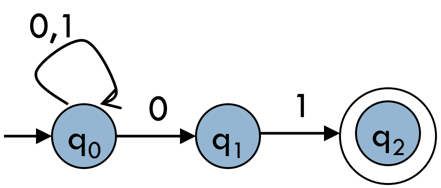


Transitions into a dead state are implicit

NFA to DFA construction: Example

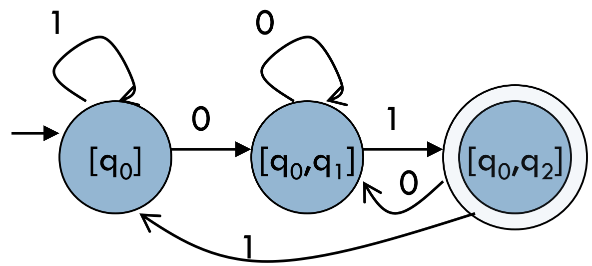
- $L = \{w \mid w \text{ ends in } 01\}$

NFA:



δ_N	0	1
→ q_0	$\{q_0,q_1\}$	$\{q_0\}$
q_1	Ø	$\{q_2\}$
$*q_2$	Ø	Ø

DFA:



δ_D
~~Ø~~
→ $[q_0]$
~~$[q_1]$~~
~~$*[q_2]$~~
$[q_0,q_1]$
$*[q_0,q_2]$
~~$*[q_1,q_2]$~~
~~$*[q_0,q_1,q_2]$~~

⇒

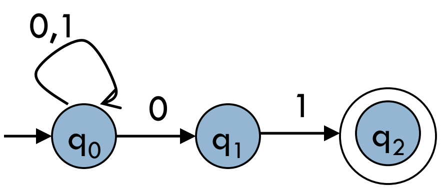
δ_D	0	1
→ $[q_0]$	$[q_0,q_1]$	$[q_0]$
$[q_0,q_1]$	$[q_0,q_1]$	$[q_0,q_2]$
$*[q_0,q_2]$	$[q_0,q_1]$	$[q_0]$

0. Enumerate all possible subsets
1. Determine transitions
2. Retain only those states reachable from $\{q_0\}$

NFA to DFA Contd...

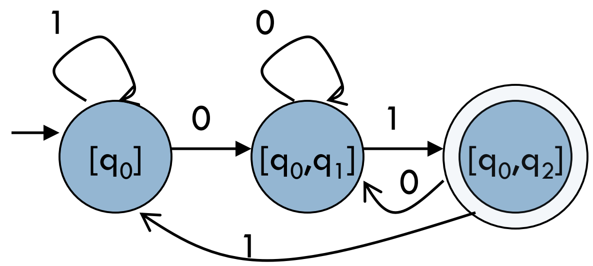
- $L = \{w \mid w \text{ ends in } 01\}$

NFA:



δ_N	0	1
→q_0	$\{q_0,q_1\}$	$\{q_0\}$
q_1	Ø	$\{q_2\}$
$*q_2$	Ø	Ø

DFA:



δ_D	0	1
→$[q_0]$	$[q_0,q_1]$	$[q_0]$

Main Idea:

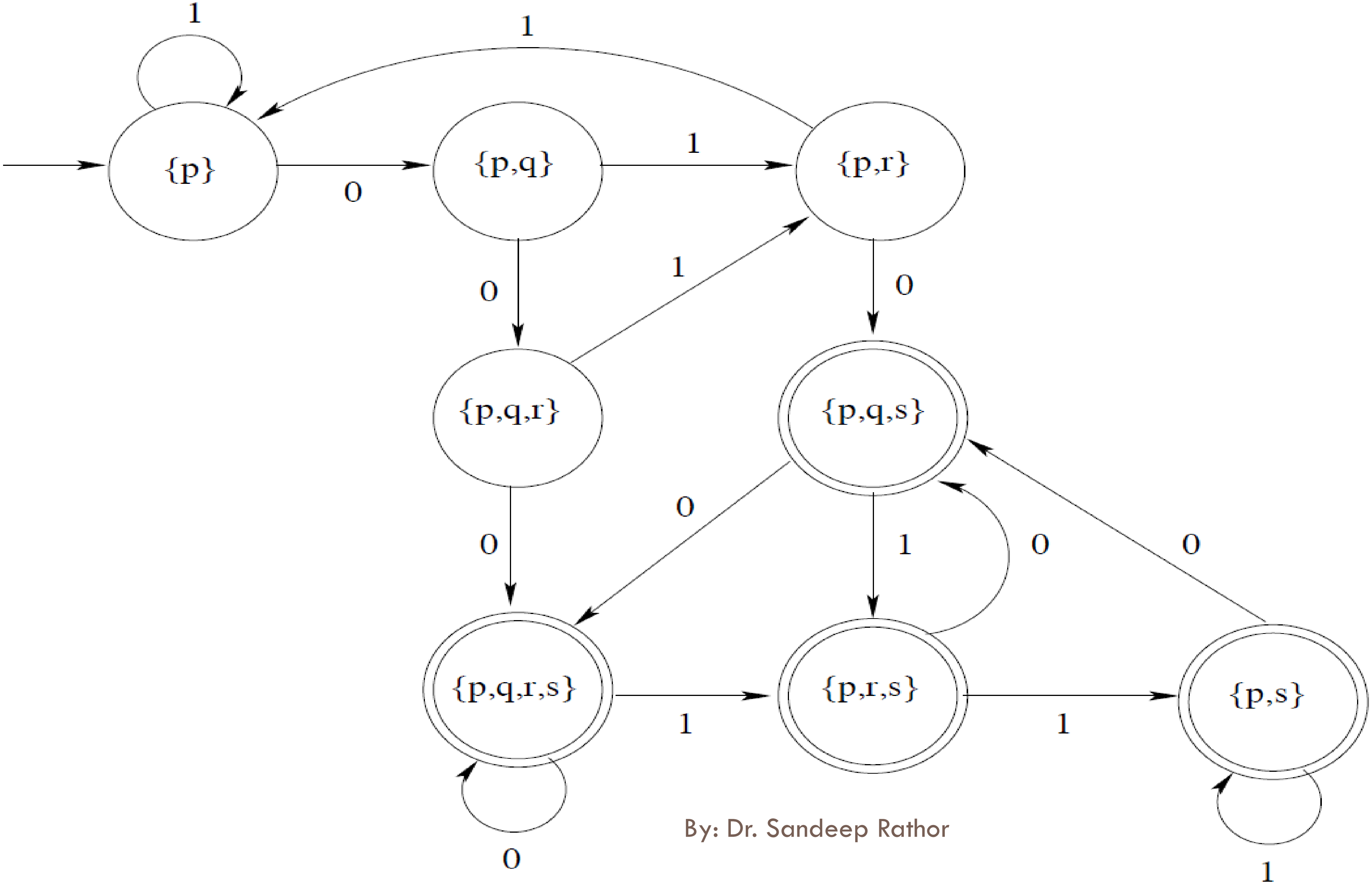
Introduce states as you go (on a need basis)

Convert the given NFA to DFA

	0	1
$\rightarrow p$	$\{p, q\}$	$\{p\}$
q	$\{r\}$	$\{r\}$
r	$\{s\}$	$\{\}$
$*s$	$\{s\}$	$\{s\}$.

States	0	1
->[p]	[p,q]	[p]
[p,q]	[p,q,r]	[p,r]
[p,r]	[p,q,s]	[p]
[p,q,r]	[p,q,r,s]	[p,r]
[p,q,s]+	[p,q,r,s]	[p,r,s]
[p,r,s]+	[p,q,s]	[p,s]
[p,s]+	[p,q,s]	[p,s]
[p,q,r,s]+	[p,q,r,s]	[p,r,s]

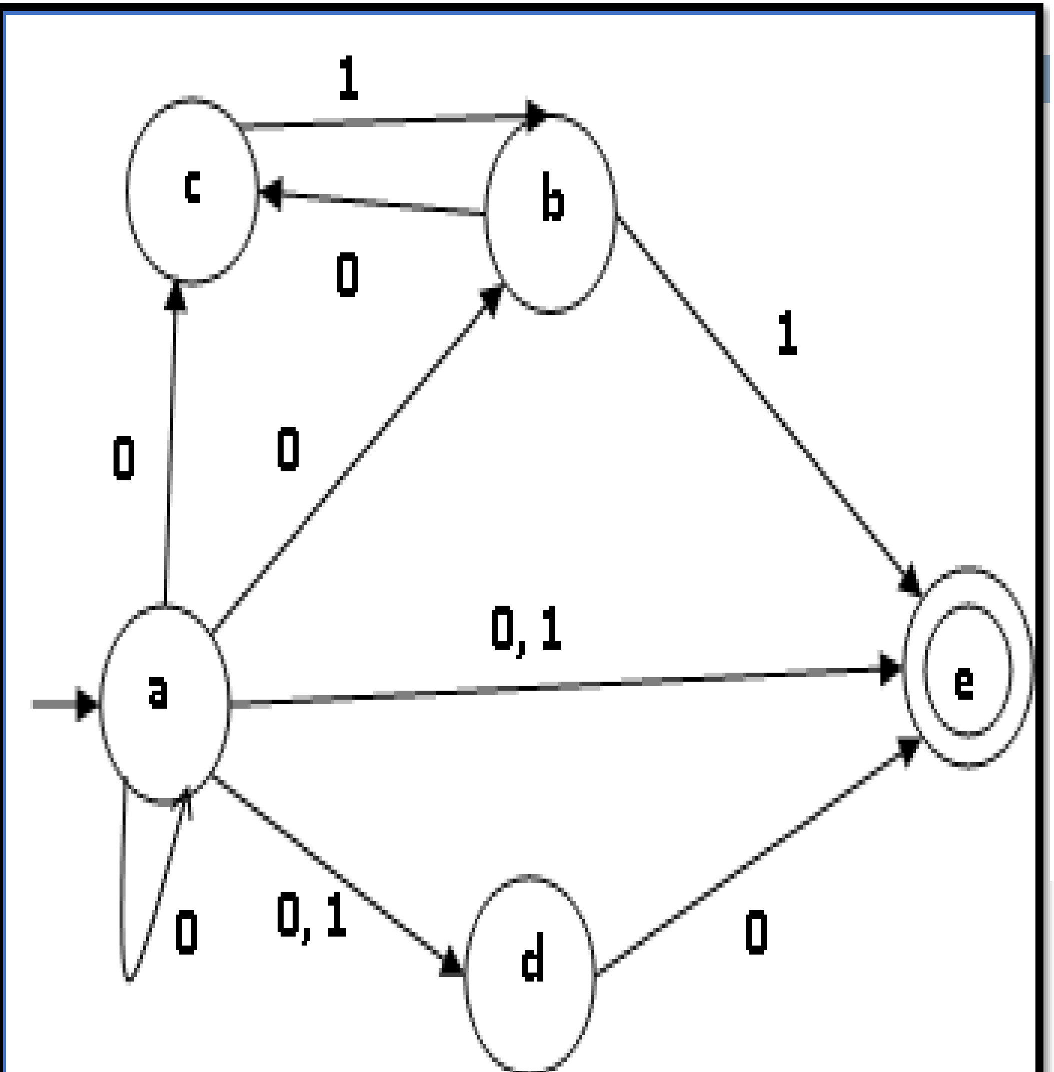
Transition diagram



By: Dr. Sandeep Rathor

For Practice: NFA to DFA

1. Convert NFA to DFA

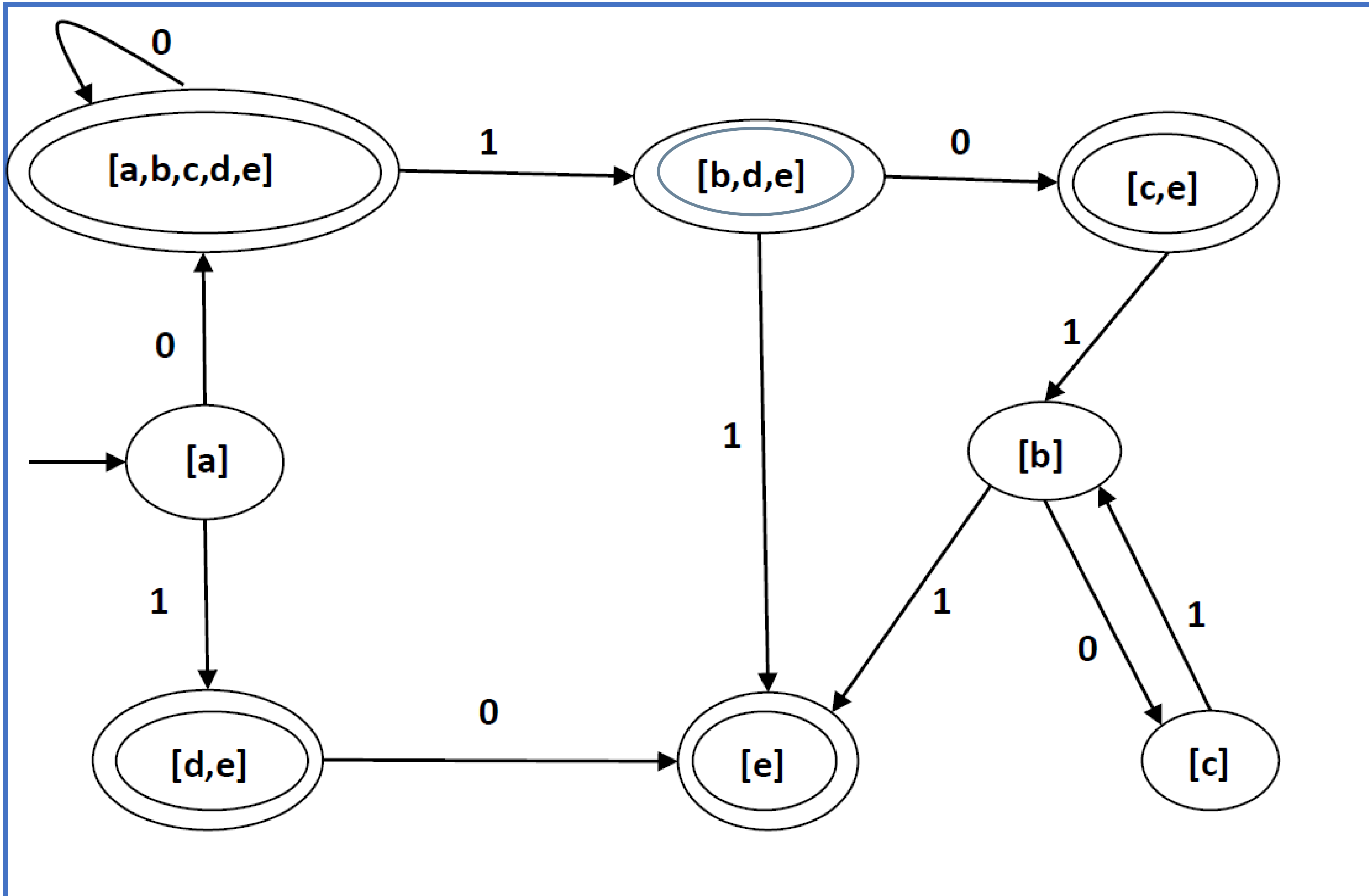


q	δ(q,0)	δ(q,1)
a	{a,b,c,d,e}	{d,e}
b	{c}	{e}
c	∅	{b}
d	{e}	∅
e	∅	∅

q	δ(q,0)	δ(q,1)
→[a]	[a,b,c,d,e]	[d,e]
[a,b,c,d,e]*	[a,b,c,d,e]	[b,d,e]
[d,e]*	[e]	Ø
[b,d,e]*	[c,e]	[e]
[e]*	Ø	Ø
[c,e]*	Ø	[b]
[b]	[c]	[e]
[c]	Ø	[b]

Required DFA

Transition diagram on next page…



[a,b,c,d,e]
[b,d,e]
[c,e]
[a]
[b]
[d,e]
[e]
[c]
0
1
0
0
1
1
1
1
0
1
0

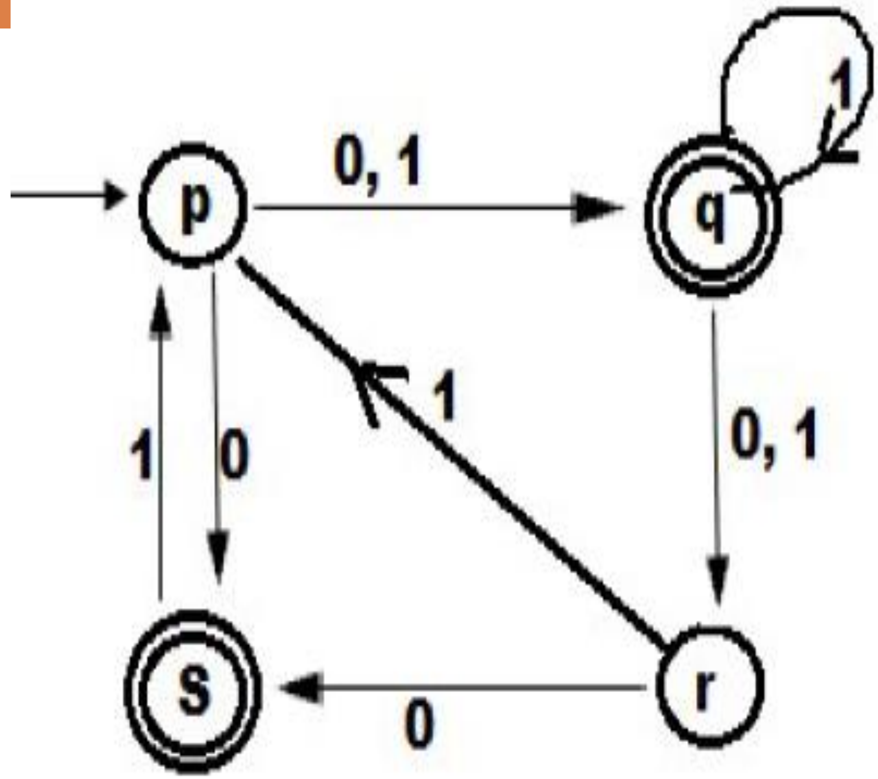
2. Convert into DFA

States	0	1
->p+	p	q
q	q	p,q

DFA equivalent to given NFA

States	0	1
->[p]+	[p]	[q]
[q]	[q]	[p,q]
[p,q] +	[p,q]	[p,q]

3. Convert given NFA to DFA

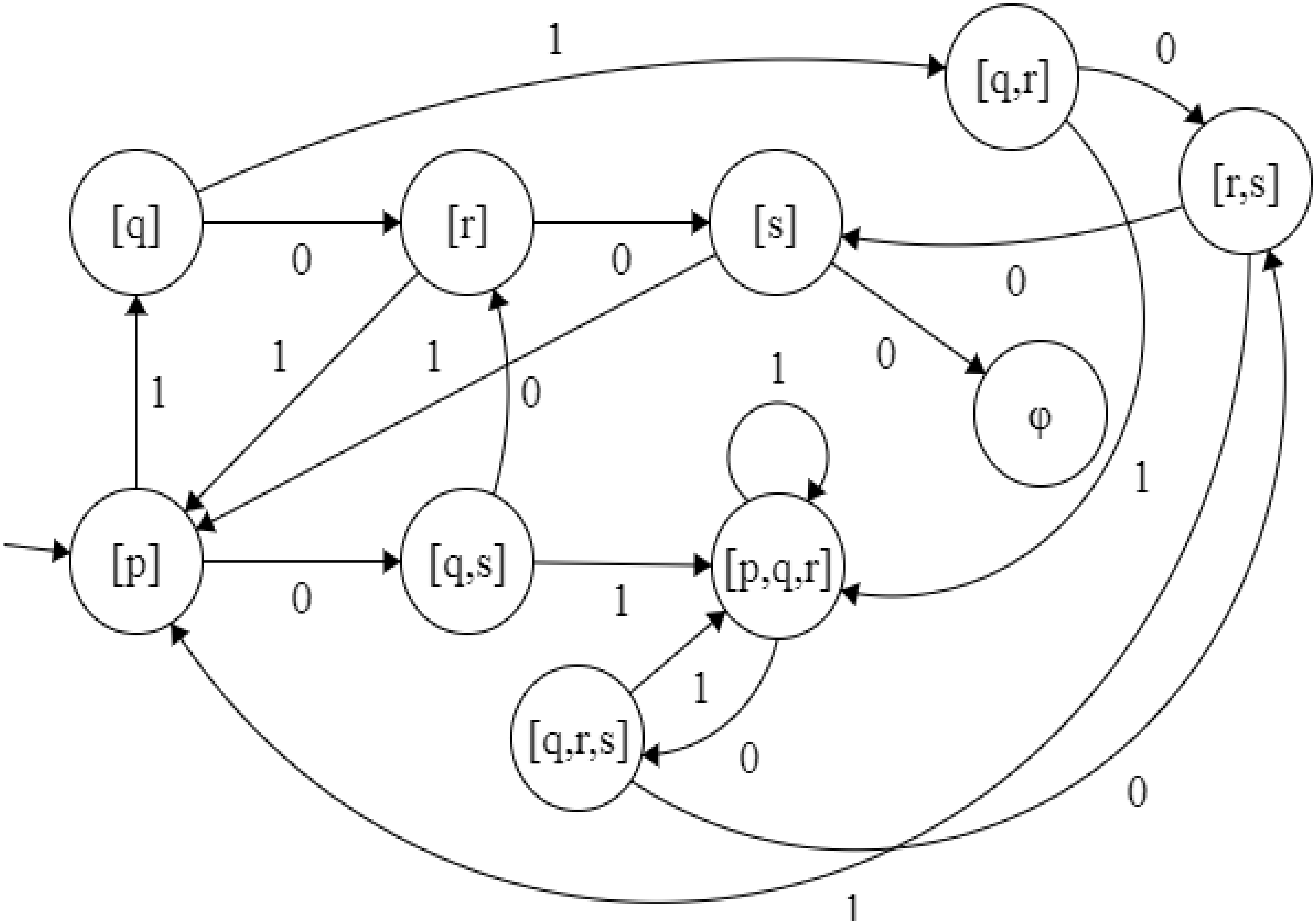


	0	1
→p	q, s	q
*q	r	q, r
r	s	p
*s	-	p

Required DFA

State/ Input	0	1
→[p]	[q, s]	[q]
*[q]	[r]	[q, r]
[r]	[s]	[p]
*[s]	φ	[p]
*[q, r]	[r, s]	[p, q, r]
*[q, s]	[r]	[p, q, r]
*[r, s]	[s]	[p]
*[p, q, r]	[q, r, s]	[p, q, r]
*[q, r, s]	[r, s]	[p, q, r]

Transition Diagram…



[q,r]
[r,s]
[q]
[r]
[s]
φ
[p]
[q,s]
[p,q,r]
[q,r,s]
0
1

Practice Contd…

4. Convert given NFA to DFA

States/input	0	1
->P+ (Final)	Q, S	Q
Q+ (Final)	R	R,Q
R	S	S
S	-	P

Answer: Required DFA

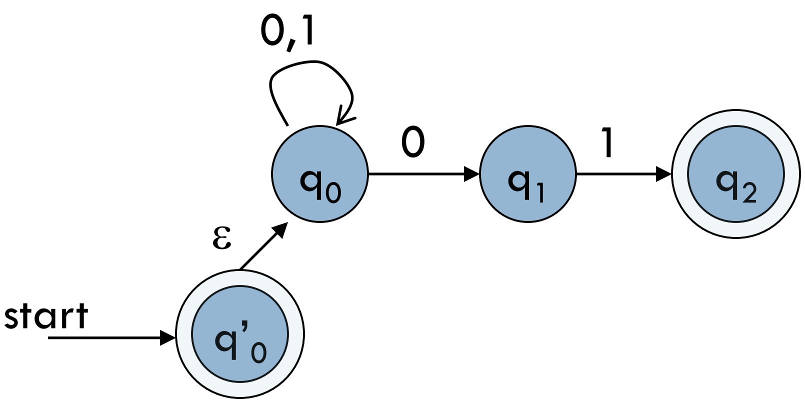
State/ Input	0	1
→[P]+	[Q,S]	[Q]
[Q]+	[R]	[R,Q]
[R]	[S]	[S]
[S]	-	[P]
[Q,S] +	[R]	[P,Q,R]
[R,Q] +	[R,S]	[Q,R,S]
[R,S]	[S]	[P,S]
[P,S]+	[Q,S]	[P,Q]
[P,Q] +	[Q,R,S]	[R,Q]
[P,Q,R] +	[Q,R,S]	[Q,R,S]
[Q,R,S] +	[R,S]	[P,Q,R,S]
[P,Q,R,S] +	[Q,R,S]	[P,Q,R,S]

FA with ε-Transitions

- ***ε -NFAs* are those NFAs with at least one explicit ε-transition defined.**
- Explicit ε-transitions is transition from one state to another state without consuming any additional input symbol

Example of an ε-NFA

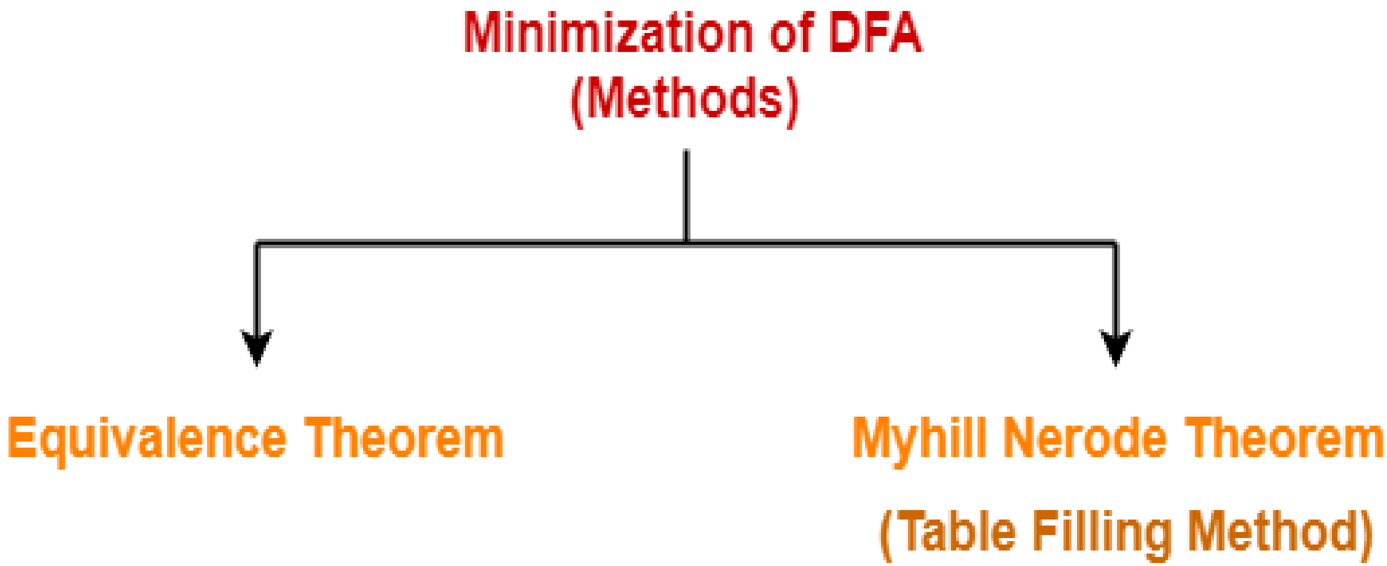
L = {w | w is empty, <u>or</u> if non-empty will end in 01}



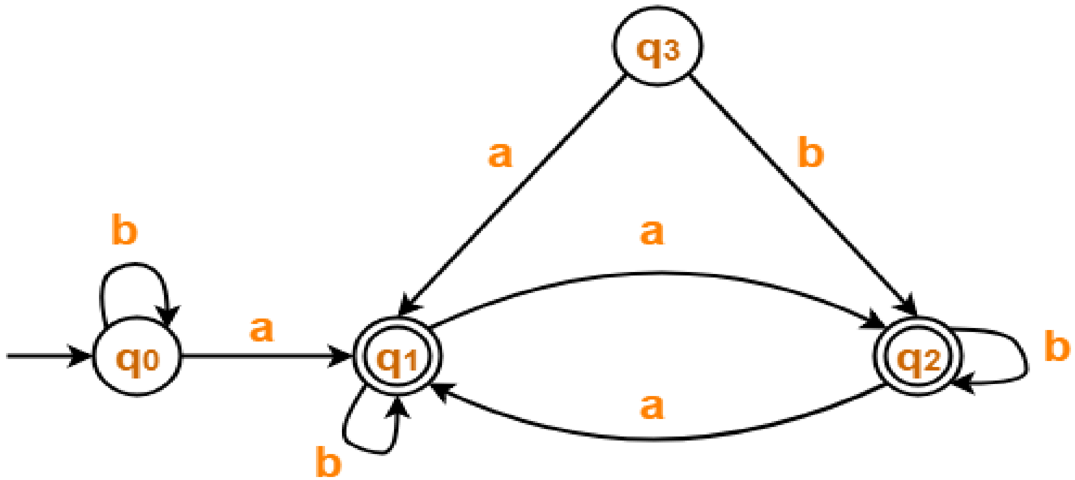
δ_E	0	1	ε
→ $*q'_0$	Ø	Ø	$\{q'_0, q_0\}$
q_0	$\{q_0, q_1\}$	$\{q_0\}$	$\{q_0\}$
q_1	Ø	$\{q_2\}$	$\{q_1\}$
$*q_2$	Ø	Ø	$\{q_2\}$

- ε-closure of a state q, ***ECLOSE(q)***, is the set of all states (including itself) that can be *reached* from q by repeatedly making an arbitrary number of ε-transitions.

Minimization of DFA



Minimization



Transition Table

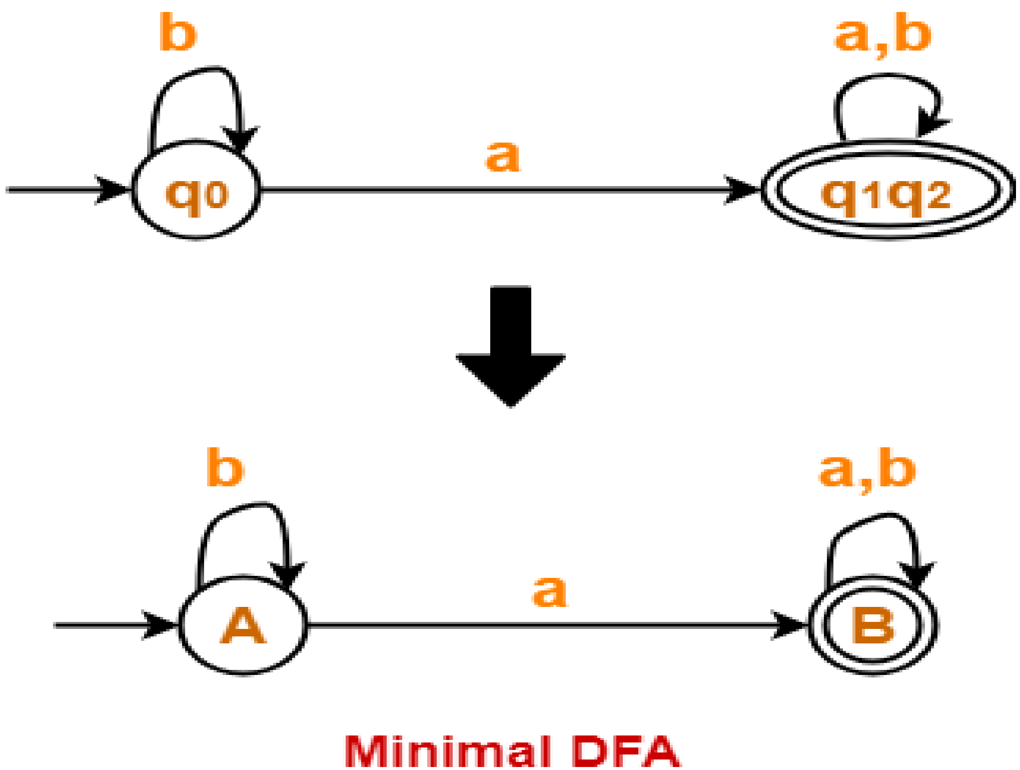
	a	b
→q0	*q1	q0
*q1	*q2	*q1
*q2	*q1	*q2

$P_0 = \{ q_0 \} \{ q_1 , q_2 \}$

$P_1 = \{ q_0 \} \{ q_1 , q_2 \}$

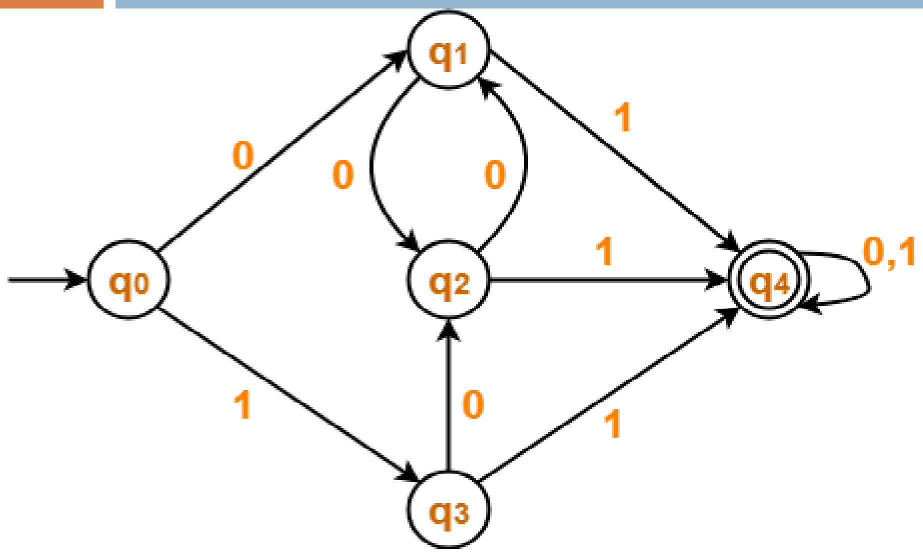
Since $P_1 = P_0$, so we stop.

Minimized automata…



Minimized the following:

Example-3:

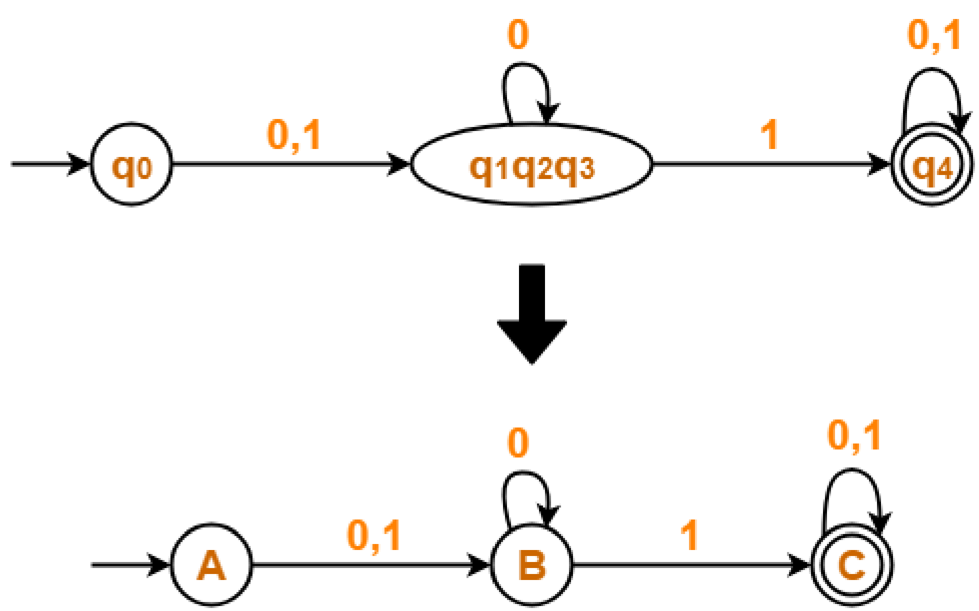


$P_0 = \{ q_0 , q_1 , q_2 , q_3 \} \{ q_4 \}$

$P_1 = \{ q_0 \} \{ q_1 , q_2 , q_3 \} \{ q_4 \}$

$P_2 = \{ q_0 \} \{ q_1 , q_2 , q_3 \} \{ q_4 \}$

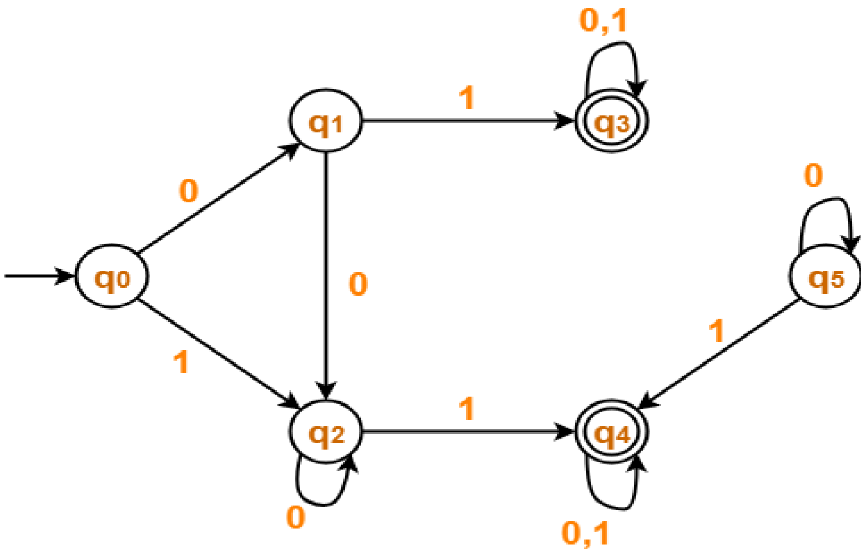
Minimized



By: Dr. Sandeep Rathor

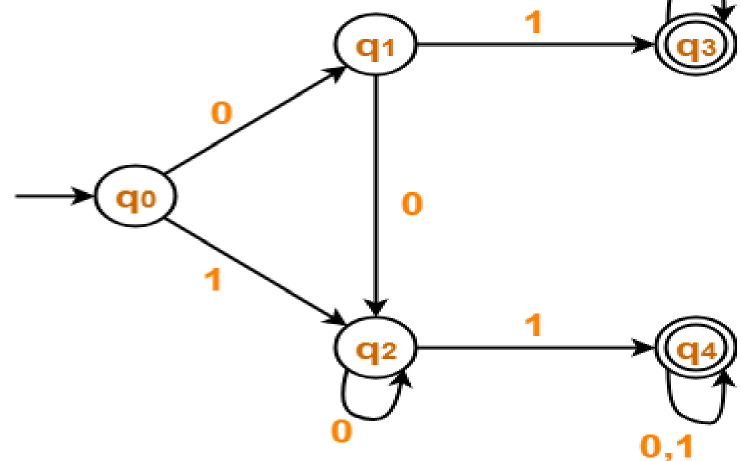
Minimal DFA

Example-4:



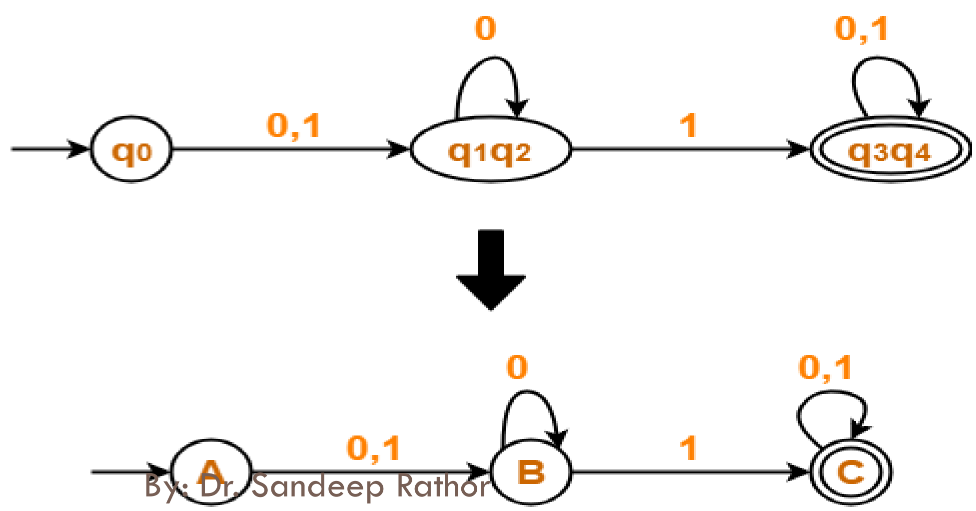
State q_5 is inaccessible from the initial state.
So, we eliminate it and its associated edges from the DFA.

After eliminate q_5 we get



$$P_0 = \{ q_0 , q_1 , q_2 \} \{ q_3 , q_4 \}$$
$$P_1 = \{ q_0 \} \{ q_1 , q_2 \} \{ q_3 , q_4 \}$$
$$P_2 = \{ q_0 \} \{ q_1 , q_2 \} \{ q_3 , q_4 \}$$

Since $P_2 = P_1$, so we st



Minimal DFA

Example-5 for Practice…

Minimize the given Automata

States/input	0	1
->q_0	q1	q5
q1	q6	q2
q2+ (Final)	q0	q2
q3	q2	q6
q4	q7	q5
q5	q2	q6
q6	q6	q4
q7	q6	q2

Minimizing (Using Equivalence theorem)

$$\pi_0 = \{\{q_2\}, \{q_0, q_1, q_3, q_4, q_5, q_6, q_7\}\}$$

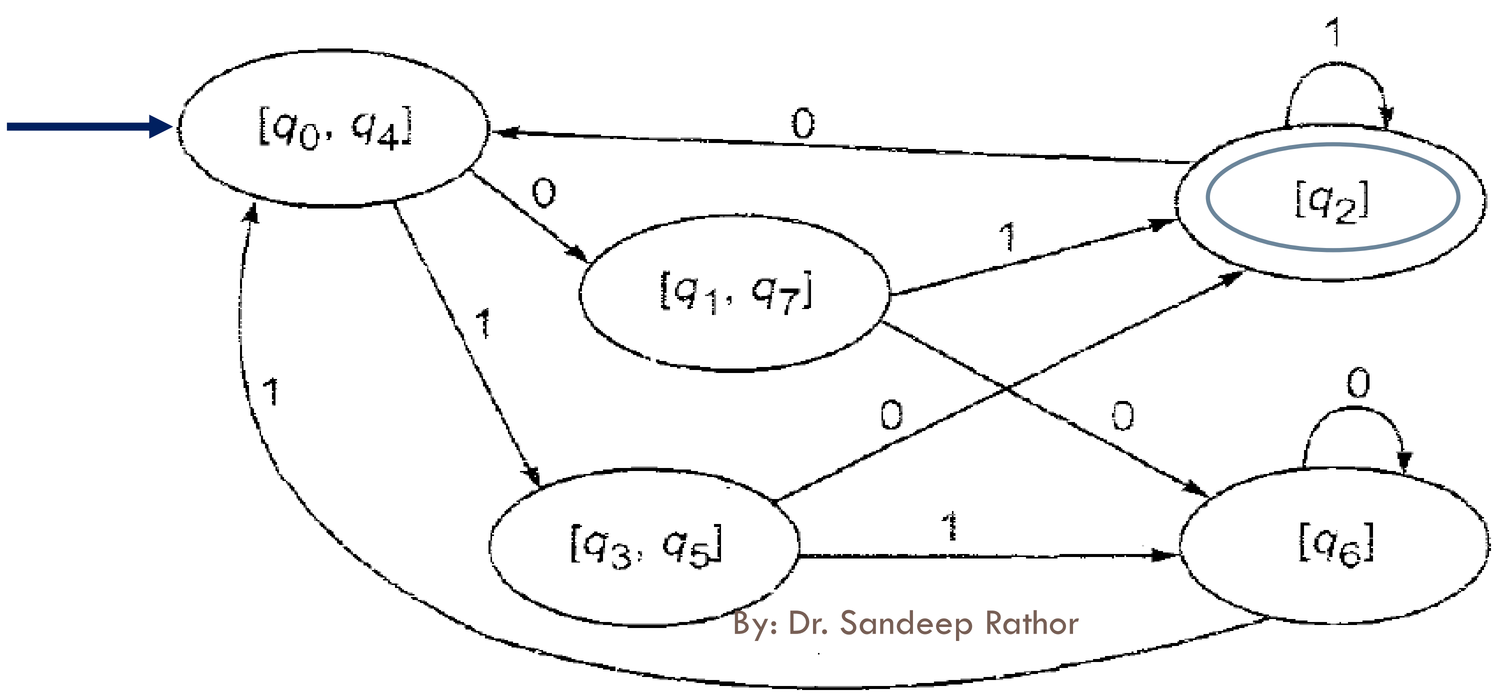
$$\pi_1 = \{\{q_2\}, \{q_0, q_4, q_6\}, \{q_1, q_7\}, \{q_3, q_5\}\}$$

$$\pi_2 = \{\{q_2\}, \{q_0, q_4\}, \{q_6\}, \{q_1, q_7\}, \{q_3, q_5\}\}$$

$$\pi_3 = \{\{q_2\}, \{q_0, q_4\}, \{q_6\}, \{q_1, q_7\}, \{q_3, q_5\}\}$$

$$\pi_2 = \pi_3,$$

State/Σ	0	1
→ $[q_0, q_4]$	$[q_1, q_7]$	$[q_3, q_5]$
$[q_1, q_7]$	$[q_6]$	$[q_2]$
$[q_2]$	$[q_0, q_4]$	$[q_2]$
$[q_3, q_5]$	$[q_2]$	$[q_6]$
$[q_6]$	$[q_6]$	$[q_0, q_4]$



Example-6 for Practice…

Minimize the given Automata

States/input	0	1
->q_0	q1	q0
q1	q0	q2
q2	q3	q1
q3+ (Final)	q3	q0
q4	q3	q5
q5	q6	q4
q6	q5	q6

Minimizing (Using Equivalence theorem)

$$\pi_0 = \{\{q_3\}, \{q_0, q_1, q_2, q_4, q_5, q_6$$

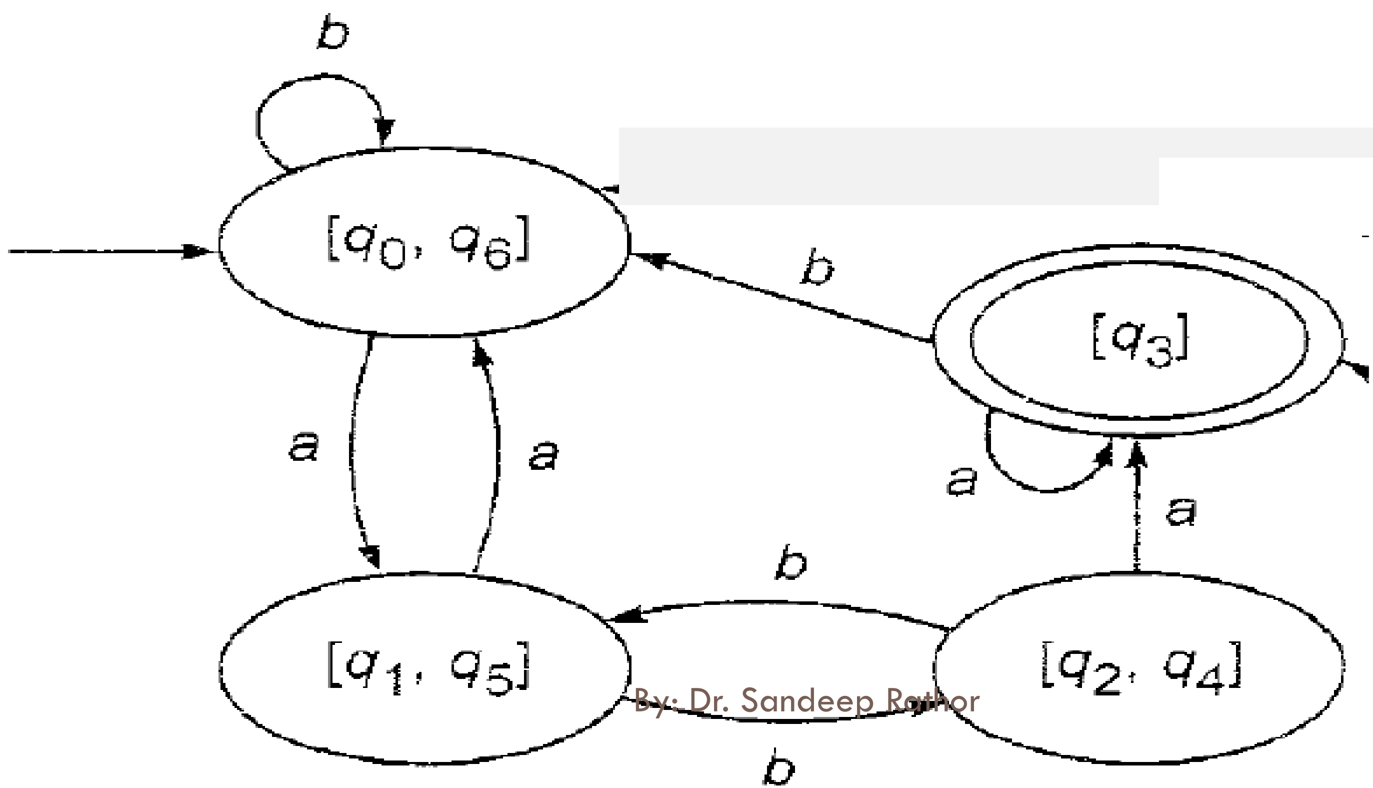
$$\pi_1 = \{\{q_3\}, \{q_0, q_1, q_5, q_6\}, \{q_2, q_4\}$$

$$\pi_2 = \{\{q_3\}, \{q_0, q_6\}, \{q_1, q_5\}, \{q_2, q_4\}$$

$$\pi_3 = \{\{q_3\}, \{q_0, q_6\}, \{q_1, q_5\}, \{q_2, q_4\}$$

$$\pi_3 = \pi_2$$

State/Σ	a	b
→ $[q_0, q_6]$	$[q_1, q_5]$	$[q_0, q_6]$
$[q_1, q_5]$	$[q_0, q_6]$	$[q_2, q_4]$
$[q_2, q_4]$	$[q_3]$	$[q_1, q_5]$
$[q_3]$	$[q_3]$	$[q_0, q_6]$



DFA Minimization using Myhill-Nerode Theorem

Algorithm

Input – DFA

Output – Minimized DFA

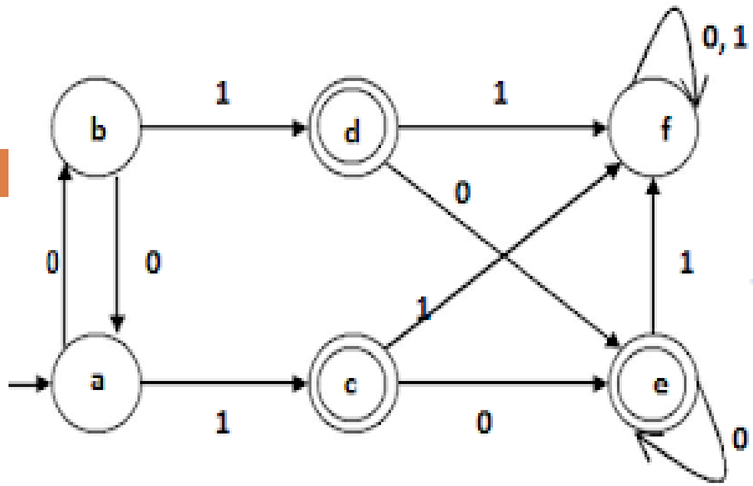
Step 1 – Draw a table for all pairs of states (Q_i, Q_j) not necessarily connected directly [All are unmarked initially]

Step 2 – Consider every state pair (Q_i, Q_j) in the DFA where $Q_i \in F$ and $Q_j \notin F$ or vice versa and mark them. **[Here F is the set of final states]**

Step 3 – Repeat this step until we cannot mark anymore states –
If there is an unmarked pair (Q_i, Q_j), mark it if the pair {δ (Q_i, A), δ (Q_i, A)} is marked for some input alphabet.

Step 4 – Combine all the unmarked pair (Q_i, Q_j) and make them a single state in the reduced DFA.

Step 2 − We mark the state pairs.



State Diagram of DFA

Step 1 : We draw a table for all pair of states.

	a	b	c	d	e	f
a						
b						
c						
d						
e						
f						

Step 2 : We mark the state pairs.

	a	b	c	d	e	f
a						
b						
c	✓	✓				
d	✓	✓				
e	✓	✓				
f			✓	✓	✓	

Step 3 − We will try to mark the state pairs, with green colored check mark, transitively. If we input 1 to state ‘a’ and ‘f’, it will go to state ‘c’ and ‘f’ respectively. (c, f) is already marked, hence we will mark pair (a, f). Now, we input 1 to state ‘b’ and ‘f’; it will go to state ‘d’ and ‘f’ respectively. (d, f) is already marked, hence we will mark pair (b, f).

a	b	c	d	e	f	
a						
b						
c	✓	✓				
d	✓	✓				
e	✓	✓				
f	✓	✓	✓	✓	✓	

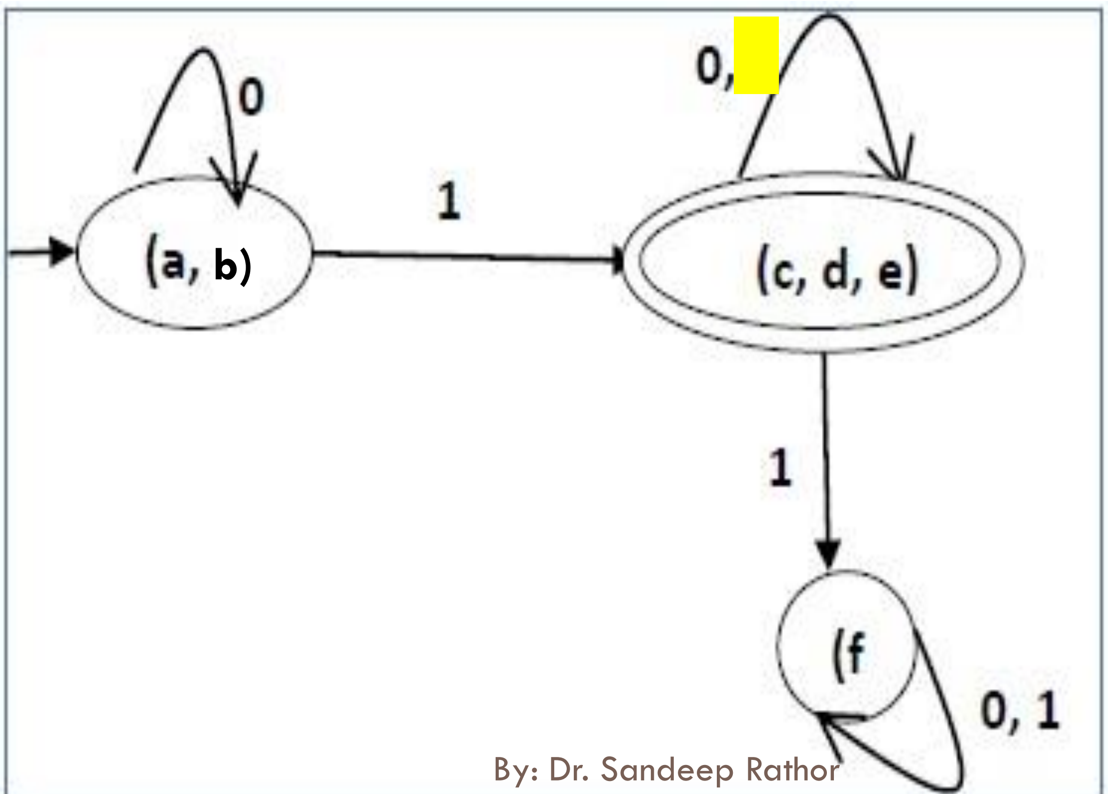
After step 3, we have got state combinations {a, b} {c, d} {c, e} {d, e} that are unmarked.

We can recombine {c, d} {c, e} {d, e} into {c, d, e}

Hence we got two combined states as − **{a, b} and {c, d, e}**

Minimization using Myhill-Nerode Contd…

So the final minimized DFA will contain three states {f}, {a, b} and {c, d, e}



Finite Automata with Output

DFA, NFA, s -NFA are FA without outputs (language acceptors)

Language transducers: Produces output on input

Finite automata may have outputs corresponding to each transition.

There are two types of finite state machines that generate output −

- **Moore Machine**
- **Mealy Machine**

Moore Machine

A Moore machine can be described by a 6 tuple

i.e. $(Q, \sum, \Delta, \delta, \lambda, q_0)$ where −

Q is a finite set of states.

$\sum$ is a finite set of symbols called the input alphabet.

Δ is a finite set of symbols called the output alphabet.

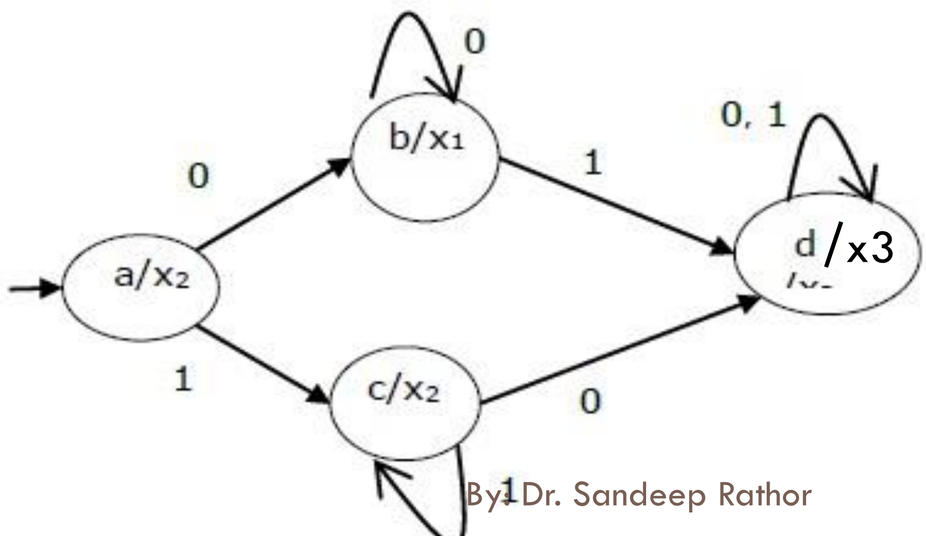
δ is the input transition function where $\delta: Q \times \sum \rightarrow Q$

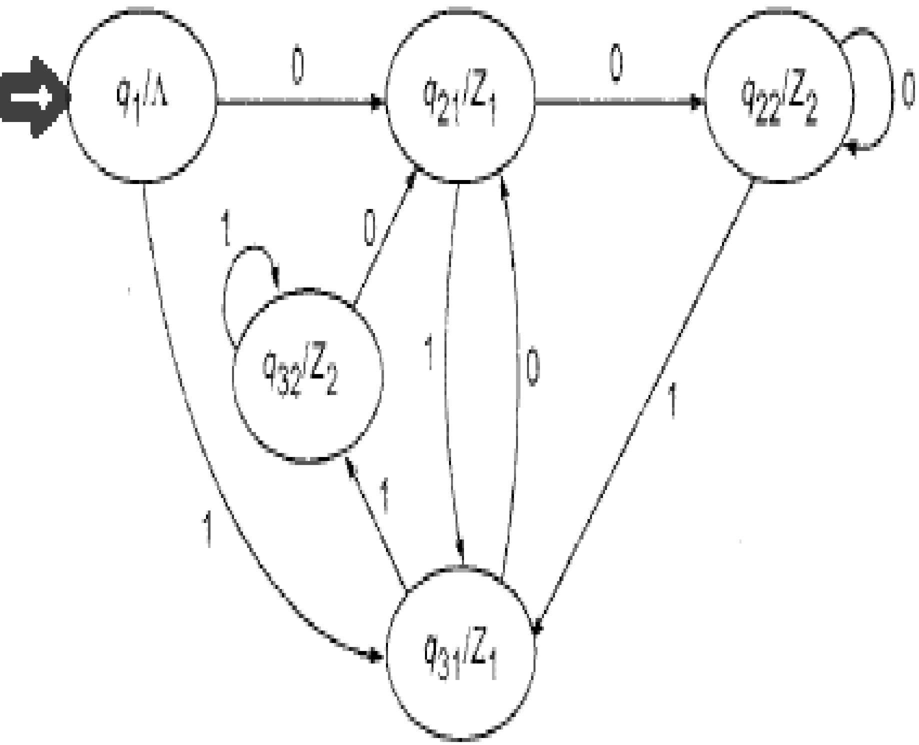
λ is the output function where $\lambda : Q \rightarrow \Delta$

$\mathbf{q_0}$ is the initial state ($q_0 \in Q$).

Example of Moore Machine

Present state	Next State		Output
	Input = 0	Input = 1	
→ a	b	c	x2
b	b	d	x1
c	c	d	x2
d	d	d	x3





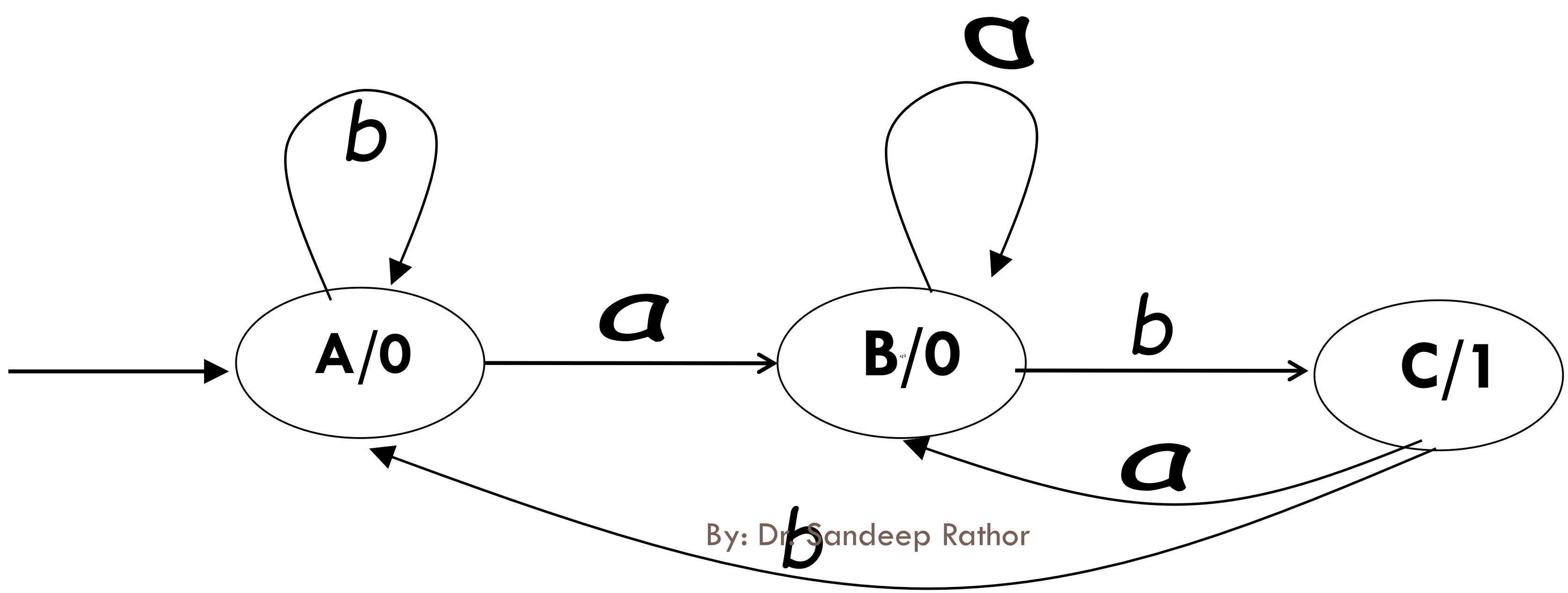
Present State	Next State		Output
	0	1	Δ
→q0	q21	q31	Λ
q21	q22	q31	Z1
q22	q22	q31	Z2
q31	q21	q32	Z1
q32	q21	q32	Z2

Construction of Moore Machine

Question1: Construct a Moore machine that takes set of all strings {a,b} as input and prints **'1'** as output for every occurance of '**ab**' as substring.

Solution: $\sum=\{a,b\}$

$\Delta=\{0,1\}$

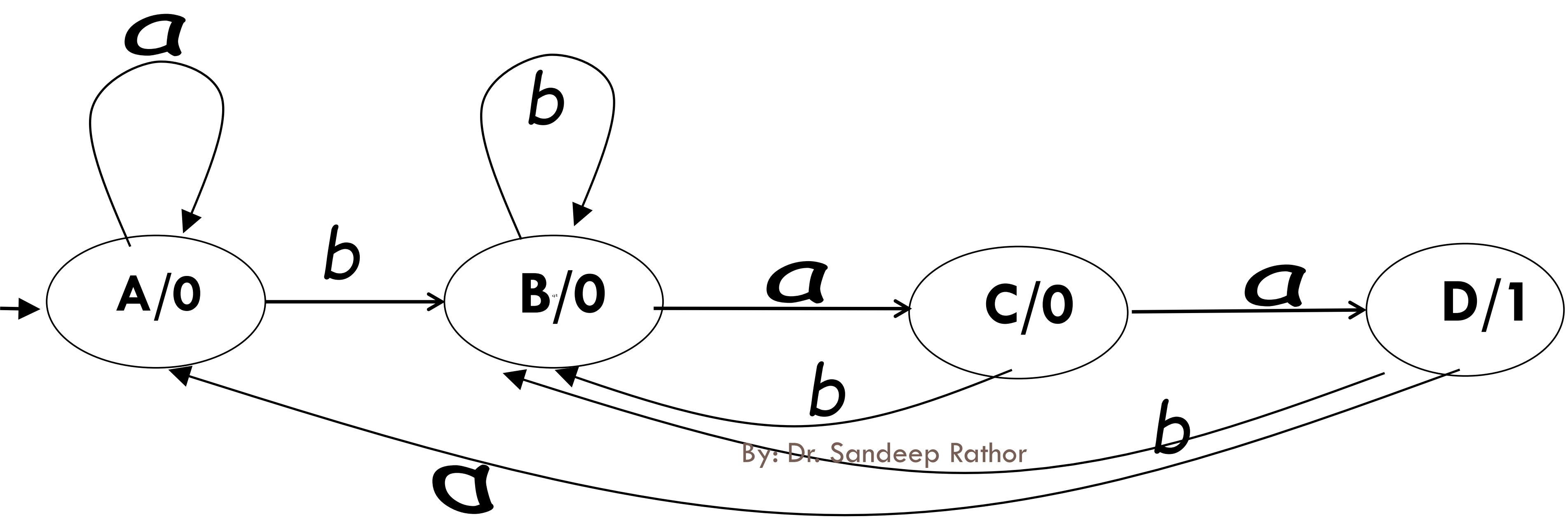


Construction of Moore Machine Contd…

Question2: Construct a Moore machine that takes set of all strings over {a,b} and counts no. of occurrences of substring **'baa'**.

Solution: $\sum = \{a,b\}$

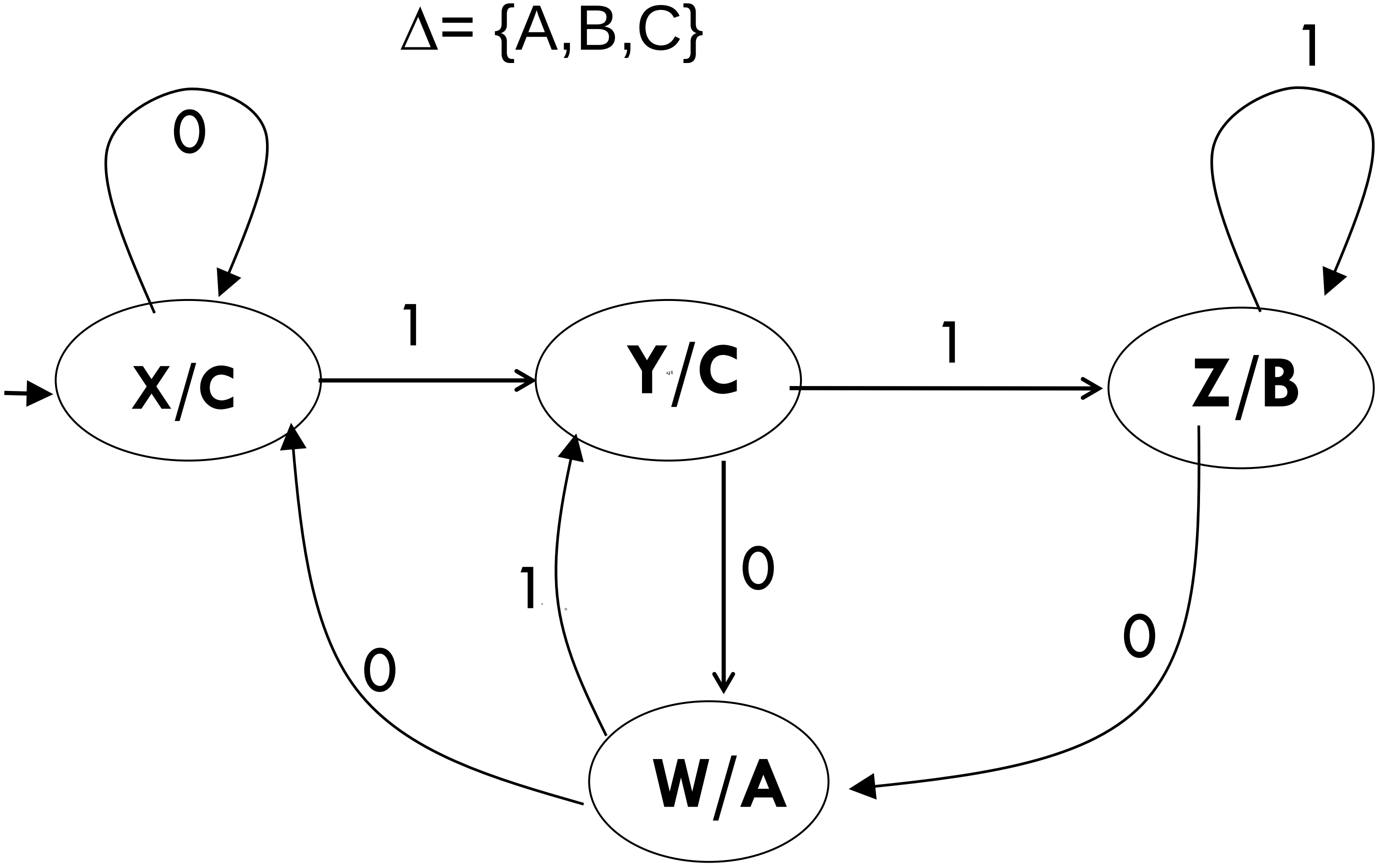
$\Delta = \{0,1\}$



Question3: Construct a Moore machine that takes set of all strings over {0,1} and produce 'A' as output if input ends with '10' or produces 'B' as output if ends with '11' otherwise 'C'.

Solution: $\Sigma=\{0,1\}$

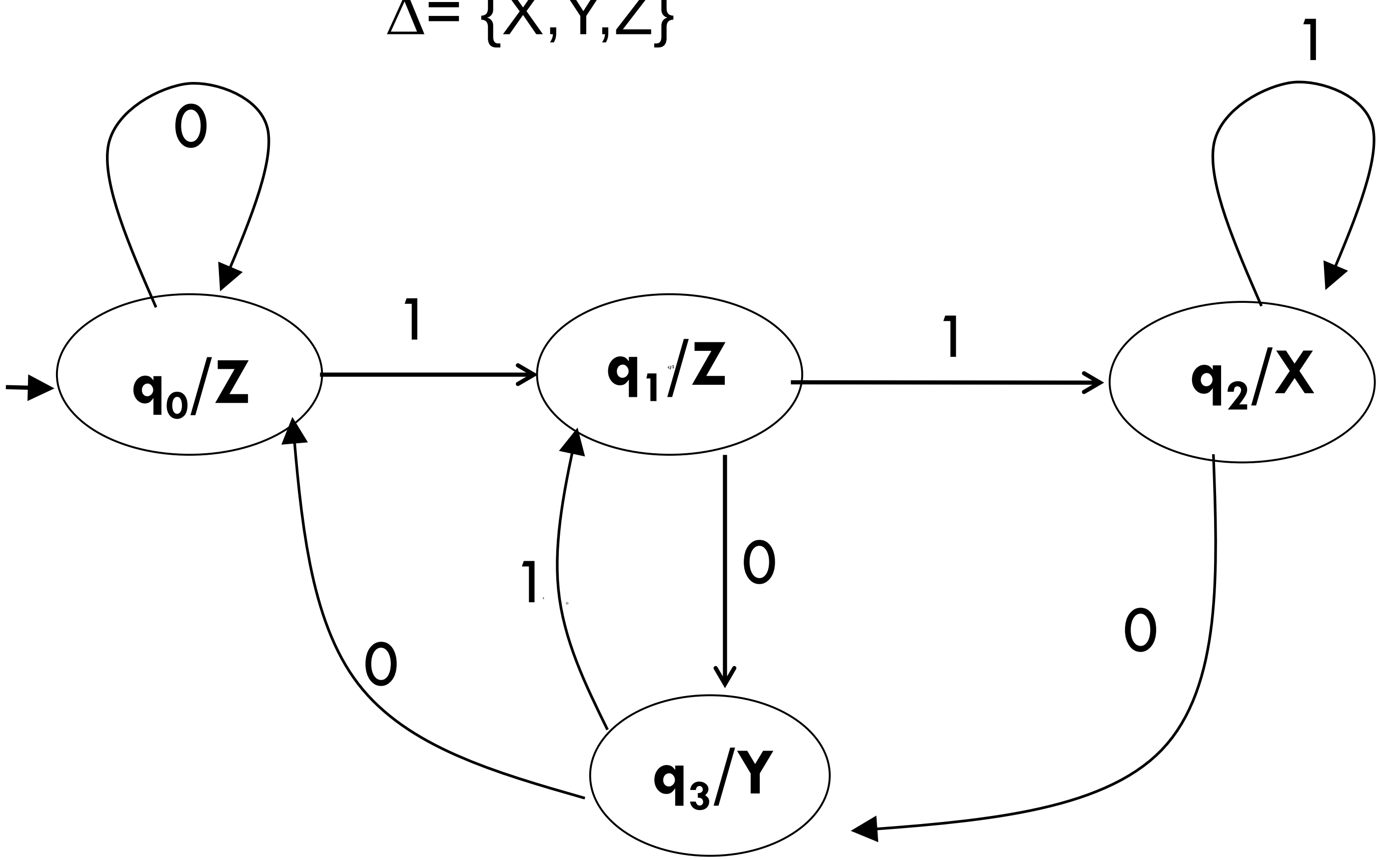
$\Delta=\{A,B,C\}$



Question4: Construct a Moore machine that takes set of all strings over {0,1} and produce 'X' as output if input ends with '11' or produces 'Y' as output if ends with '10' otherwise 'Z'.

Solution: $\Sigma=\{0,1\}$

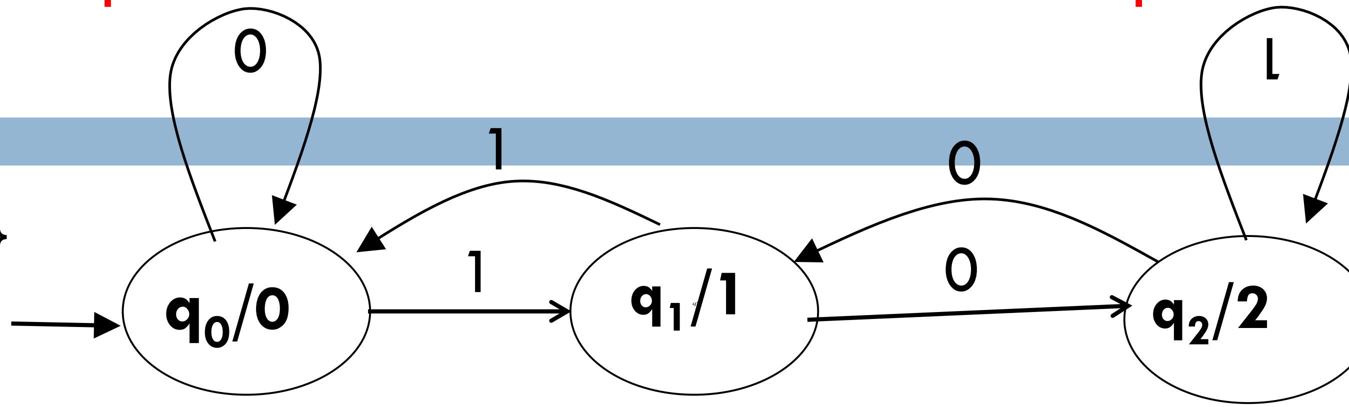
$\Delta= \{X,Y,Z\}$



Question5: Construct a Moore machine that takes binary numbers as input and produces residue modulo '3' as output.

Solution: $\Sigma=\{0,1\}$

$\Delta=\{0,1,2\}$



States	0	1	Δ
$\rightarrow q_0$	q_0	q_1	0
q_1	q_2	q_0	1
q_2	q_1	q_2	2

Mealy Machine

A Mealy machine can be described by a 6 tuple

i.e. $(Q, \sum, \Delta, \delta, \lambda, q_0)$ where −

Q is a finite set of states.

$\sum$ is a finite set of symbols called the input alphabet.

Δ is a finite set of symbols called the output alphabet.

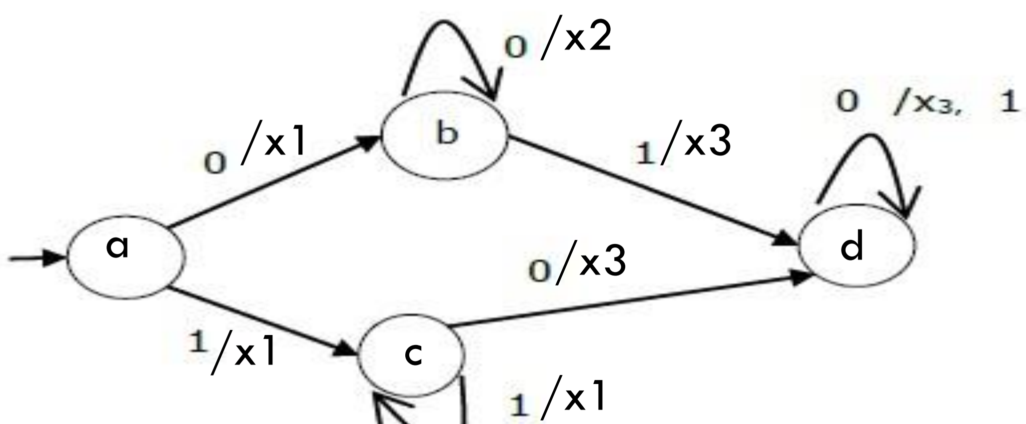
δ is the input transition function where $\delta: Q \times \sum \rightarrow Q$

λ is the output function where λ **: Q ×** $\sum \rightarrow \Delta$

$\mathbf{q_0}$ is the initial state ($q_0 \in Q$).

Example of Mealy Machine

Present state	Next state			
	input = 0		input = 1	
	State	Output	State	Output
→ a	b	x1	c	x1
b	b	x2	d	x3
c	d	x3	c	x1
d	d	x3	d	x2

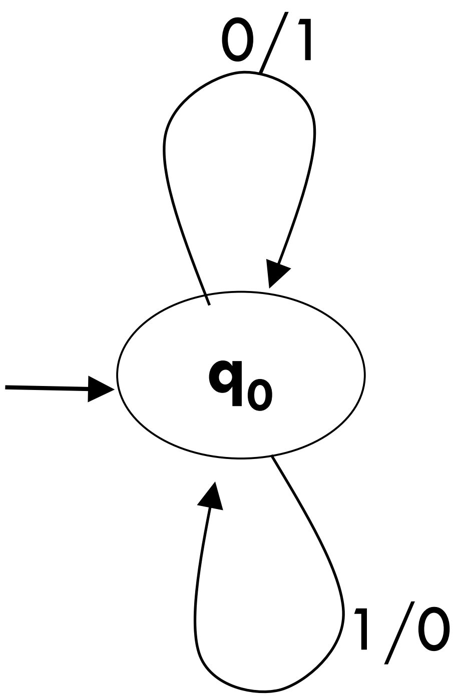


Construction of Mealy Machine

Question1: Construct a mealy machine that takes binary number as input and produces 1's complement of that number as output. Assume that string is read LSB to MSB.

Solution:

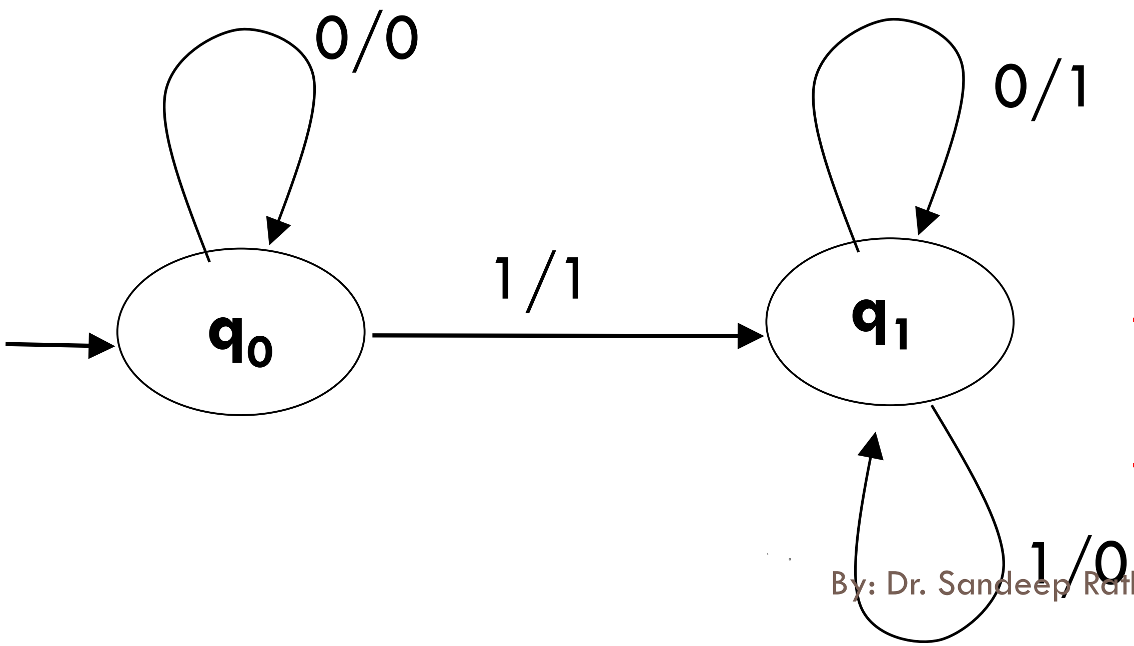
$\Sigma = \{0,1\}$
$\Delta = \{0,1\}$



Construction of Mealy Machine

Question1: Construct a mealy machine that takes binary number as input and produces 2's complement of that number as output. Assume that string is read LSB to MSB and end carry is discarded.

Solution: Hint



MSB							LSB	
1	1	1	0	1	1	0	0	
0	0	0	1	0	0	1	1	1's
							1	2's
0	0	0	1	0	1	0	0	2's

$\underbrace{0\ 0\ 0\ 1\ 0}_{\text{1's Complement}}\ \boxed{1\ 0\ 0}$

Difference b/w Moore & Mealy Machine

Moore Machine	Mealy Machine
Output depends only upon the present state.	Output depends both upon the present state and the present input
Generally, it has more states than Mealy Machine.	Generally, it has fewer states than Moore Machine.
The value of the output function is a function of the current state and the changes at the clock edges, whenever state changes occur.	The value of the output function is a function of the transitions and the changes, when the input logic on the present state is done.
In Moore machines, more logic is required to decode the outputs resulting in more circuit delays. They generally react one clock cycle later.	Mealy machines react faster to inputs. They generally react in the same clock cycle.

Conversion: Moore to Mealy Machine

Algorithm

Input – Moore Machine

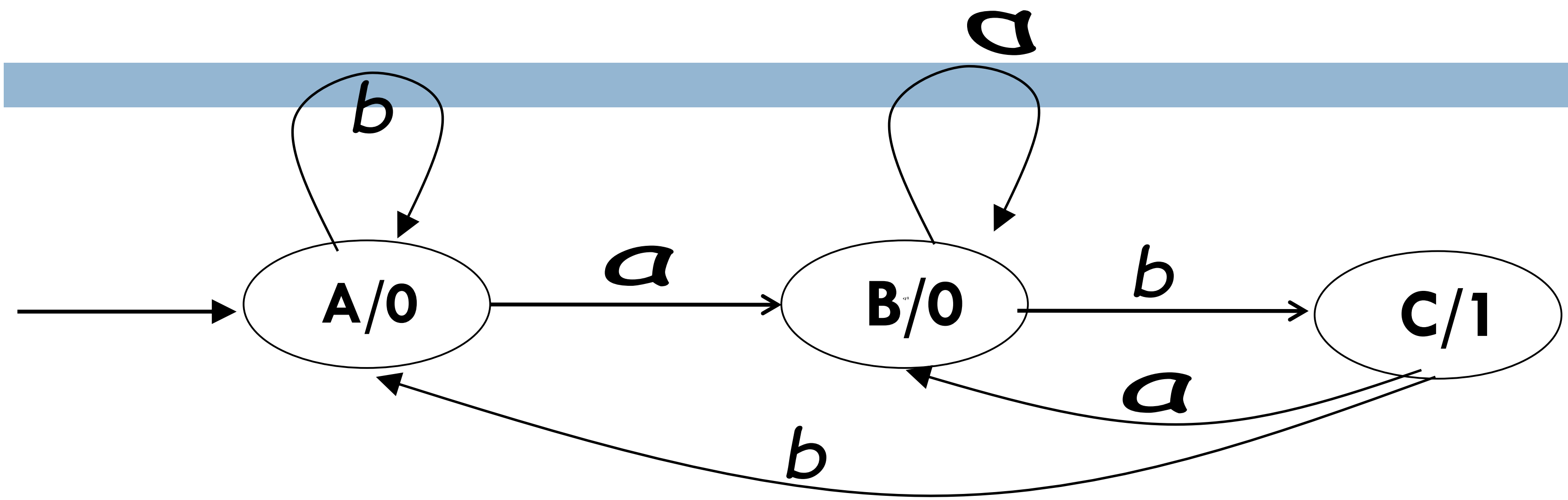
Output – Mealy Machine

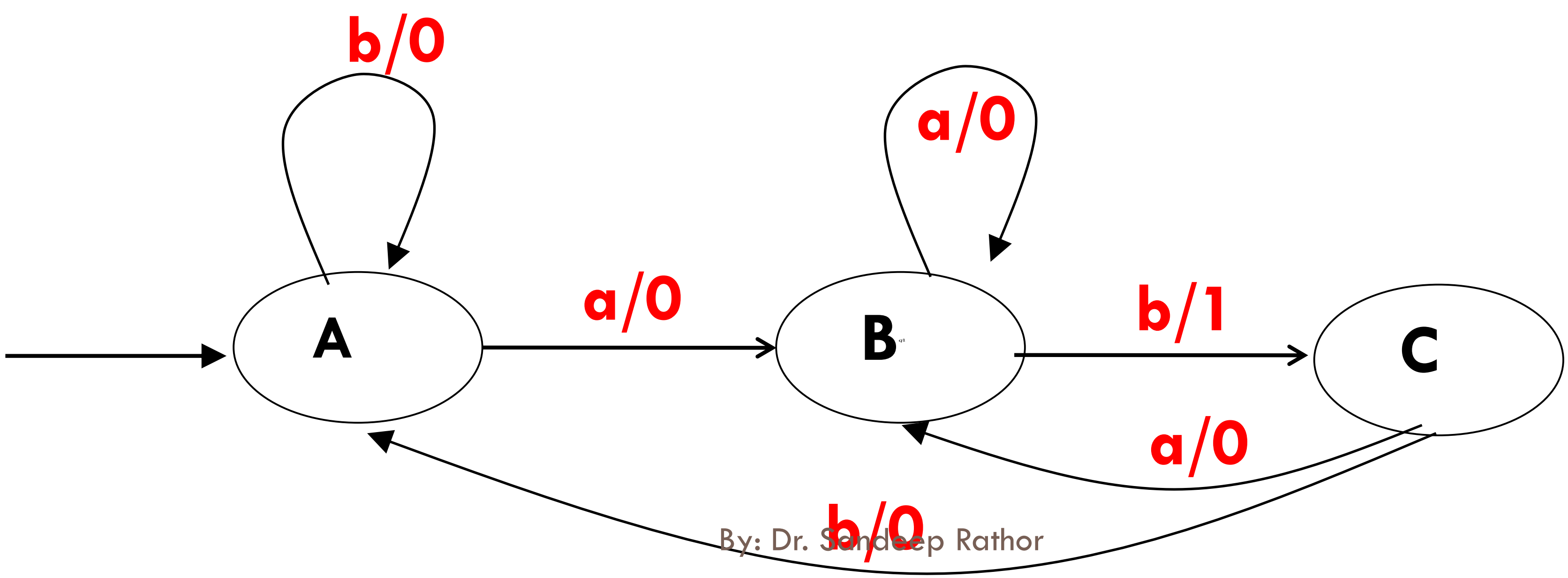
Step 1 – Take a blank Mealy Machine transition table format.

Step 2 – Copy all the Moore Machine transition states into this table format.

Step 3 – Check the present states and their corresponding outputs in the Moore Machine state table; if for a state Q_i output is m, copy it into the output columns of the Mealy Machine state table wherever Q_i appears in the next state.

Convert given Moore M/C to Mealy





Convert given Moore M/C to Mealy

Present State	Next State		Output
	a = 0	a = 1	
→ a	d	b	1
b	a	d	0
c	c	c	0
d	b	a	1

Present State	Next State			
	a = 0		a = 1	
	State	Output	State	Output
=> a	d	1	b	0
b	a	1	d	1
c	c	0	c	0
d	b	0	a	1

Conversion: Mealy Machine to Moore

Algorithm

Input – Mealy Machine

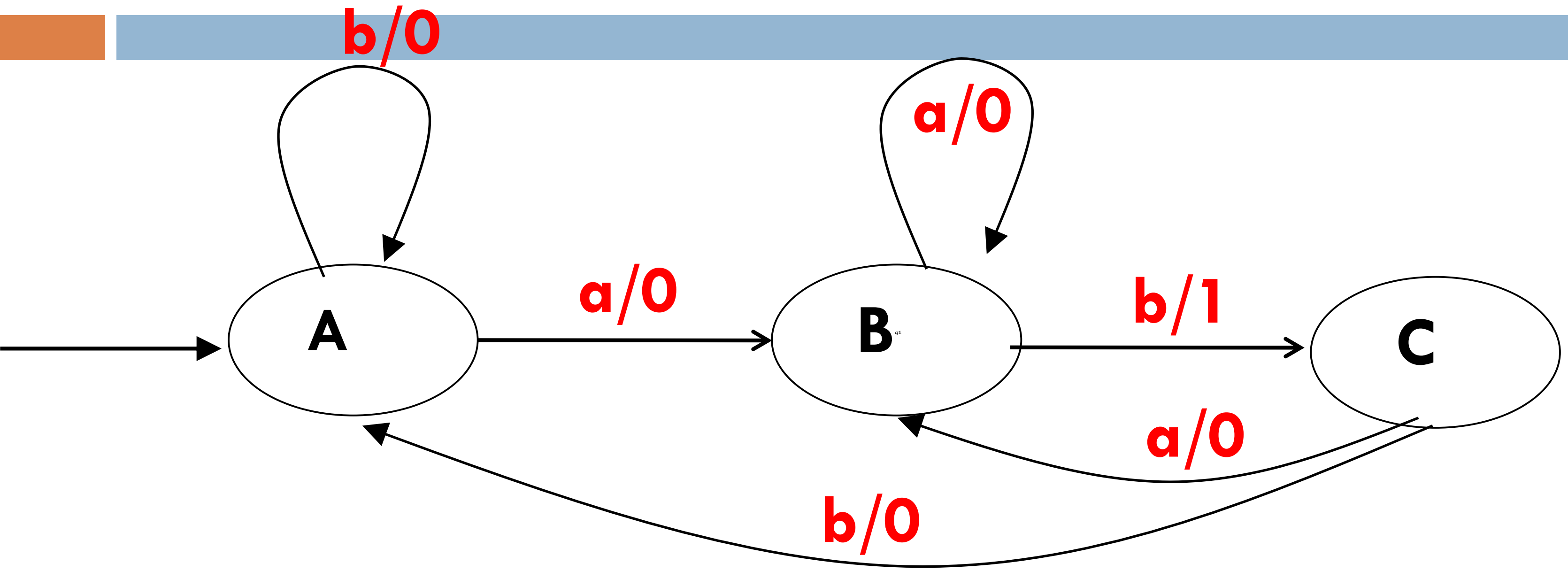
Output – Moore Machine

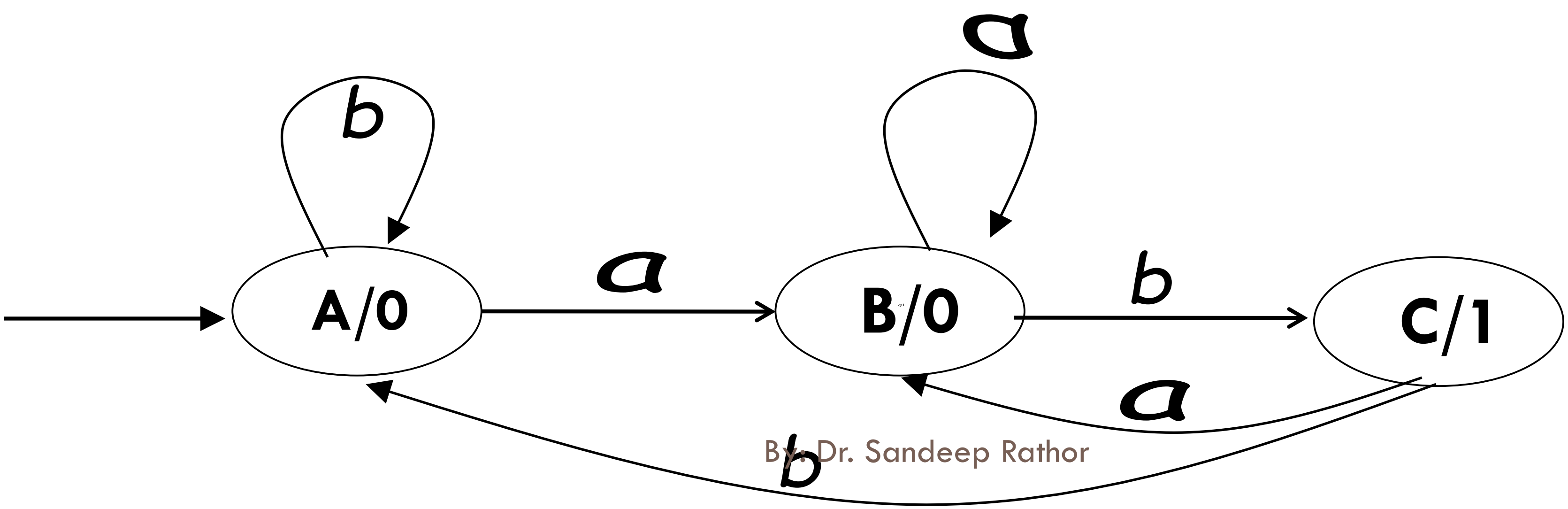
Step 1 – Calculate the number of different outputs for each state (Q_i) that are available in the state table of the Mealy machine.

Step 2 – If all the outputs of Qi are same, copy state Q_i. If it has n distinct outputs, break Q_i into n states as Q_{in} where **n** = 0, 1, 2…….

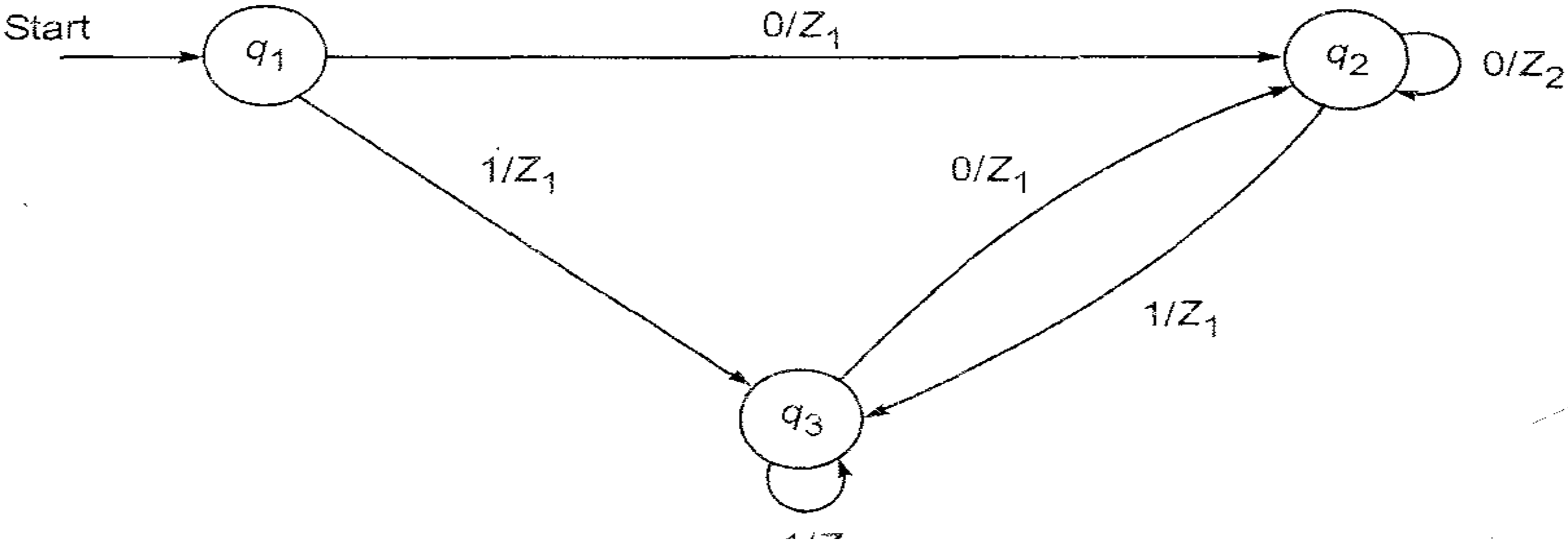
Step 3 – If the output of the initial state is 1, insert a new initial state at the beginning which gives 0 output.

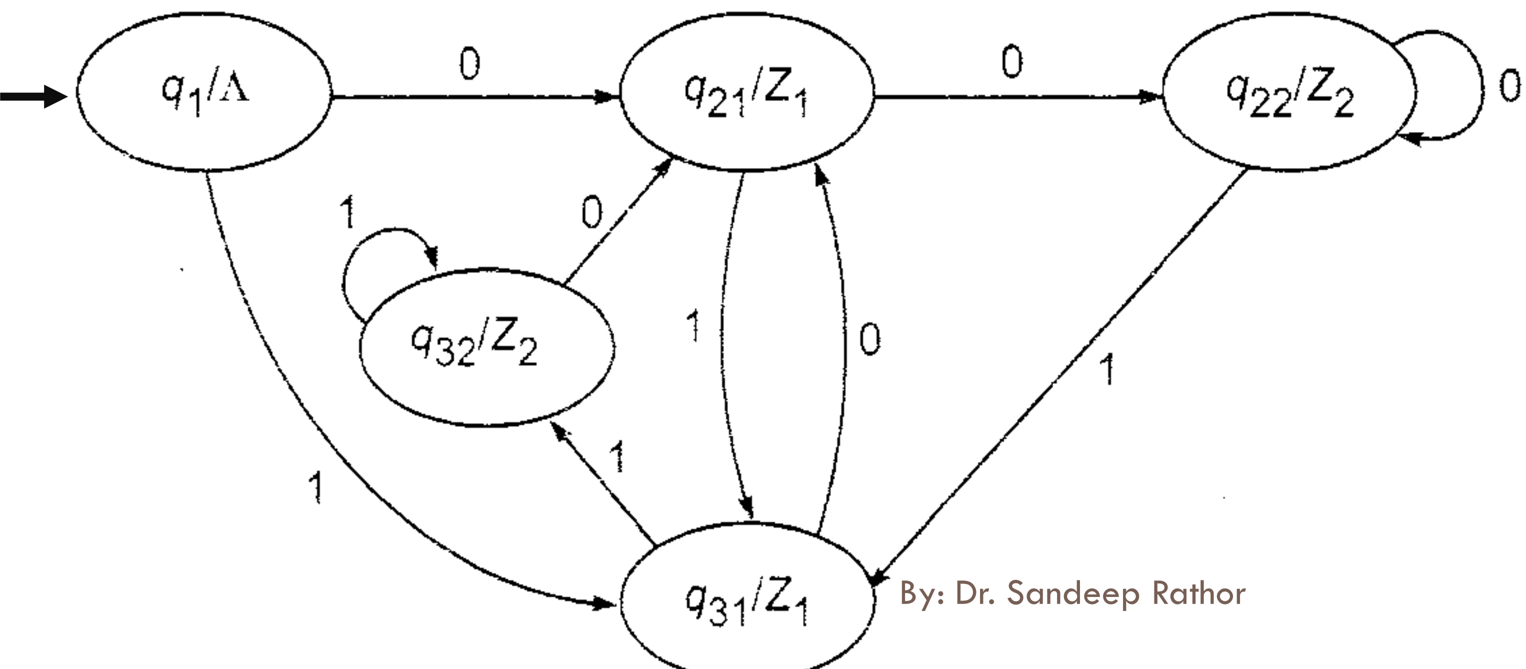
Convert given Mealy M/C to Moore





Convert given Mealy M/C to Moore





Convert given Mealy M/C to Moore

(*No. of states may be increased)

Present State	Next State			
	a = 0		a = 1	
	Next State	Output	Next State	Output
→ a	d	0	b	1
b	a	1	d	0
c	c	1	c	0
d	b	0	a	1

Present State	Next State		Output
	a = 0	a = 1	
→ a	d	b_1	1
b_0	a	d	0
b_1	a	d	1
c_0	c_1	C_0	0
c_1	c_1	C_0	1
d	b_0	a	0

REGULAR EXPRESSION

By: Sandeep Rathor

Regular Expressions

- RE are used for representing certain sets of string in an algebraic form.
- It describe the language that is accepted by **Finite Automata.**
- The symbols that appear in RE are letters of alphabets ∑, symbol for null string ε, parenthesis, star operator and plus sign.

A Regular Expression can be recursively defined as:

- ε is a Regular Expression indicates the language containing an empty string. (L (ε) = {ε})

- φ is a Regular Expression denoting an empty language. (L (φ) = { })

Regular expressions: Rule

- Union of two RE, R1and R2, written as **R1+R2**, is also a RE.
- Concatenation of two RE, R1and R2, written as **R1R2**, is also a RE.
- Iteration of RE, R written as **R***, is also a RE.

RE Examples

Regular Expressions	Regular Set
(0 + 10*)	L = { 0, 1, 10, 100, 1000, 10000, … }
(0*10*)	L = {1, 01, 10, 010, 0010, …}
(0 + ε)(1 + ε)	L = {ε, 0, 1, 01}
(a+b)*	Set of strings of a's and b's of any length including the null string. So L = { ε, a, b, aa , ab , bb , ba, aaa…….}
(a+b)*abb	Set of strings of a's and b's ending with the string abb. So L = {abb, aabb, babb, aaabb, ababb, …………..}
(11)*	Set consisting of even number of 1's including empty string, So L= {ε, 11, 1111, 111111, ……….}
(aa)*(bb)*b	Set of strings consisting of even number of a's followed by odd number of b's , so L = {b, aab, aabbb, aabbbbb, aaaab, aaaabbb, …………..}
(aa + ab + ba + bb)*	String of a's and b's of even length can be obtained by concatenating any combination of the strings aa, ab, ba and bb including null, so L = {aa, ab, ba, bb, aaab, aaba, …………..}

Represent these sets by RE

$\{\varepsilon, 0, 00, 000, 0000, \ldots\}$ Ans: 0^*

$\{\varepsilon, ab\}$ Ans: $\varepsilon + ab$

$\{01, 10\}$ Ans: $01+10$

$\{\varepsilon, 10, 01\}$ Ans: $\varepsilon + 10+01$

Set of all strings ending in b. Ans: $(a+b)^*b$

Set of all strings staring with a and ending with ba Ans: $a(a+b)^*ba$

(a) The set of all strings over {0, 1} with three consecutive 0's. $(0+1)^*000(0+1)^*$

(b) The set of all strings over {0, 1} beginning with 00. $00(0+1)^*$

(c) The set of all strings over {0. 1} ending with 00 and beginning with 1. $1(0+1)^*00$

(d)all the string containing exactly two 0's. $1^*01^*01^*$

(e)

- All strings containing an even number of 0's:

$1^* + (1^*01^*0)^*1^*$

- All strings having at least two occurences of the substring 00:

$(1 + 0)^* 00(1 + 0)^* 00(1 + 0)^* + (1 + 0)^* 000(1 + 0)^*$

Find a regular expression corresponding to the language of strings of even lengths over the alphabet of { a, b }.

(aa + ab + ba + bb)*

Find the Regular Expression of the following:

1. The set of all strings containing exactly 2a's:

   b*ab*ab*

2. The set of all strings containing at least 2a's:

   (a+b)* a (a+b)*a (a+b)*

3. The set of all strings containing at most 2a's:

   b*+b*ab*+b*ab*ab*

4. The set of all strings containing substring aa:

   (a+b)* aa (a+b)*

- Find a regular expression corresponding to the language of all strings over the alphabet { a, b } that do not end with ab.

  $(a + b)^*(a + bb)$

- Find a regular expression corresponding to the language of all strings over the alphabet { a, b } that contain exactly two a's.

- $b^* a\, b^* a\, b^*$

Identities Related to Regular Expressions

Given R, P, L, Q as regular expressions, the following identities hold

$\emptyset^* = \varepsilon$
$\varepsilon^* = \varepsilon$
$RR^* = R^*R$
$R^*R^* = R^*$
$(R^*)^* = R^*$

$(PQ)^*P = P(QP)^*$
$(a+b)^* = (a^*b^*)^* = (a^*+b^*)^* = (a+b^*)^* = a^*(ba^*)^*$

$R + \emptyset = \emptyset + R = R$ (The identity for union)
$R\,\varepsilon = \varepsilon\,R = R$ (The identity for concatenation)
$\emptyset\,L = L\,\emptyset = \emptyset$ (The annihilator for concatenation)
$R + R = R$ (Idempotent law)

$L(M + N) = LM + LN$ (Left distributive law)
$(M + N)L = ML + NL$ (Right distributive law)
$\varepsilon + RR^* = \varepsilon + R^*R = R^*$

Arden's Theorem

By: Dr. Sandeep Rathor

Arden's Theorem

If P and Q are two regular expressions over Σ and if P does not contain ε , then the following equation in R given by $R = Q + RP$ has a unique solution i.e., $R = QP^*$.

The Arden's Theorem is useful for checking the equivalence of two regular expressions as well as in the conversion of DFA to a regular expression.

Proof

Part I: Prove that $R = QP^*$ is the solution of this equation

$R = Q + RP$

Replace R by QP^* on both sides

LHS = QP^*

RHS = $Q + QP^*P$

$= Q(\varepsilon + P^*P)$

$= QP^*$ // (As we know that $\varepsilon + A^*A = A^*$)

$= LHS$

Thus, $R = QP^*$ is the solution of the equation $R = Q + RP$.

Part II : Prove that this is the only solution of this equation.

$R = Q + RP$

Replace R by Q + RP on RHS

$R = Q + (Q + RP)\,P$

$\quad = Q + QP + RP^2$

Keep replacing R by Q + RP

$R = Q + QP + (Q + RP)\,P^2$

$\quad = Q + QP + QP^2 + RP^3$

…

$\quad = Q + QP + QP^2 + QP^3 + \ldots + QP^i + RP^{i+1}$

$R = Q + QP + QP^2 + QP^3 + \ldots + QP^i + RP^{i+1}$

$= Q (\varepsilon + P + P^2 + \ldots + P^i) + RP^{i+1}$ for $i \geq 0$

We claim that any soln of $R = Q + RP$ must be equivalent to QP^*

Let $w \in R$ and $|w| = i$.

then w belongs to set $Q (\varepsilon + P + P^2 + \ldots + P^i) + RP^{i+1}$,

as P does not contain null. RP^{i+1} has no string of length less i+1 so,

w is not in set RP^{i+1}

It means w belongs to the set $Q (\varepsilon + P + P^2 + \ldots + P^i)$

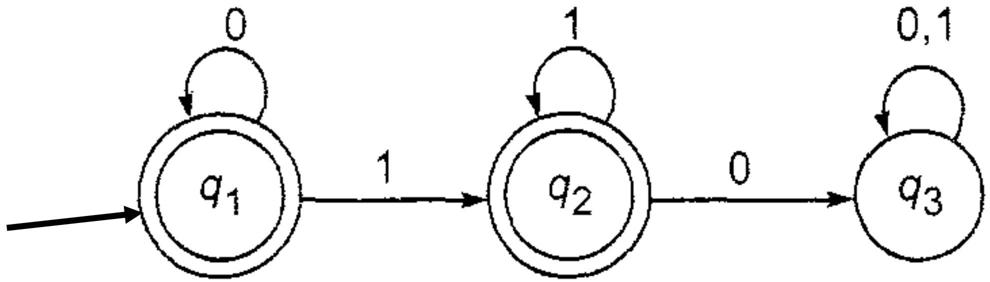
and hence it is equivalent to QP^*

Arden's Theorem
DFA to RE

By: Dr. Sandeep Rathor

Assumptions for Applying Arden's Theorem

- The transition diagram must not have NULL transitions
- It must have only one initial state



Find Regular Expression of the given DFA?

$$q_1 = q_1 0 + \Lambda$$

$$q_2 = q_1 1 + q_2 1$$

$$q_3 = q_2 0 + q_3(0 + 1)$$

Using Arden's Theorem

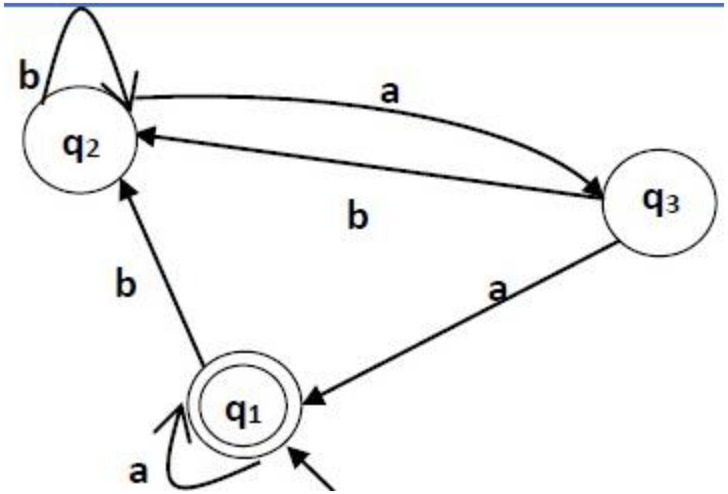
$$q_1 = \Lambda 0^* = 0^*$$

$$q_2 = q_1 1 + q_2 1 = 0^*1 + q_2 1$$

$$q_2 = (0^*1)1^*$$

$$q_1 + q_2 = 0^* + 0^*(11^*) = 0^*(\Lambda + 11^*) = 0^*(1^*)$$

Find the RE



Step 1: Construct the equations

$q_1 = q_1a + q_3a + \varepsilon$

$q_2 = q_1b + q_2b + q_3b$

$q_3 = q_2a$

Step 2: Solve the equations

$$(a + b(b + ab)^*aa)^*$$

$q_1 = q_1a + q_3a + \varepsilon$

$q_2 = q_1b + q_2b + q_3b$

$q_3 = q_2a$

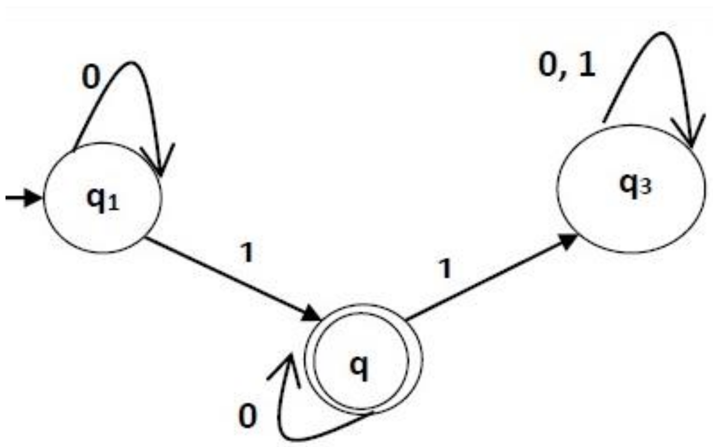
Now, we will solve these three equations –

$q_2 = q_1b + q_2b + q_3b$
$= q_1b + q_2b + (q_2a)b$ (Substituting value of q_3)
$= q_1b + q_2(b + ab)$
$= q_1b\ (b + ab)^*$ (Applying Arden's Theorem)

$q_1 = q_1a + q_3a + \varepsilon$
$= q_1a + q_2aa + \varepsilon$ (Substituting value of q_3)
$= q_1a + q_1b(b + ab^*)aa + \varepsilon$ (Substituting value of q_2)
$= q_1(a + b(b + ab)^*aa) + \varepsilon$
$= \varepsilon\ (a+ b(b + ab)^*aa)^*$
$= (a + b(b + ab)^*aa)^*$

Hence, the regular expression is **$(a + b(b + ab)^*aa)^*$**.

Find the RE



Step 1: Construct the equations

$q_1 = q_1 0 + \varepsilon$

$q_2 = q_1 1 + q_2 0$

$q_3 = q_2 1 + q_3 (0 + 1)$

Step 2: Solve the equations

0*10*

Solution –

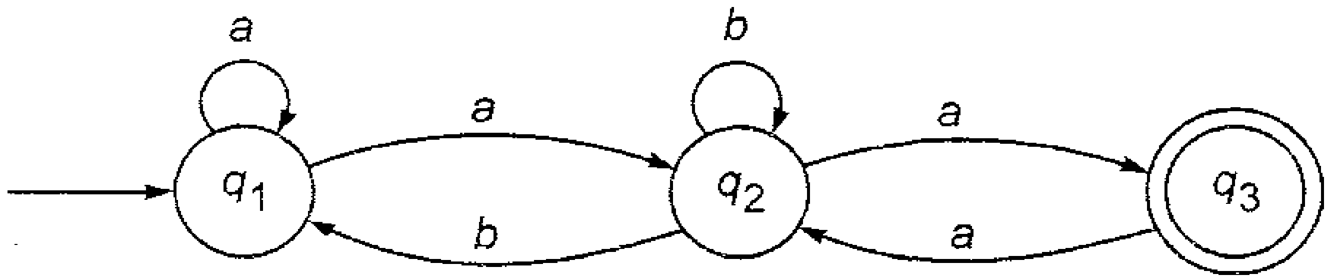
Here the initial state is q_1 and the final state is q_2
Now we write down the equations –

$q_1 = q_1 0 + \varepsilon$
$q_2 = q_1 1 + q_2 0$
$q_3 = q_2 1 + q_3 0 + q_3 1$

Now, we will solve these three equations –
$q_1 = \varepsilon 0^*$ [As, $\varepsilon R = R$]
So, $q_1 = 0^*$
$q_2 = 0^*1 + q_2 0$
So, $q_2 = 0^*1(0)^*$ [By Arden's theorem]

Hence, the regular expression is 0^*10^*.

For Practice



$ql = q_1a + q_2b + \varepsilon$

$q2 = q_1\, a + q_2b + q_3\, a$

$q3 = q_2a$

$q2 = q_1\, a + q_2\, b + q2aa$

$= q_1\, a + q2(b + aa)$

$= q_1\, a(b + aa)^*$

$q_1 = q_1\, a + q_1\, a(b + aa)^*b + \varepsilon$

$= q_1(a + a(b + aa)^*b) + \varepsilon$

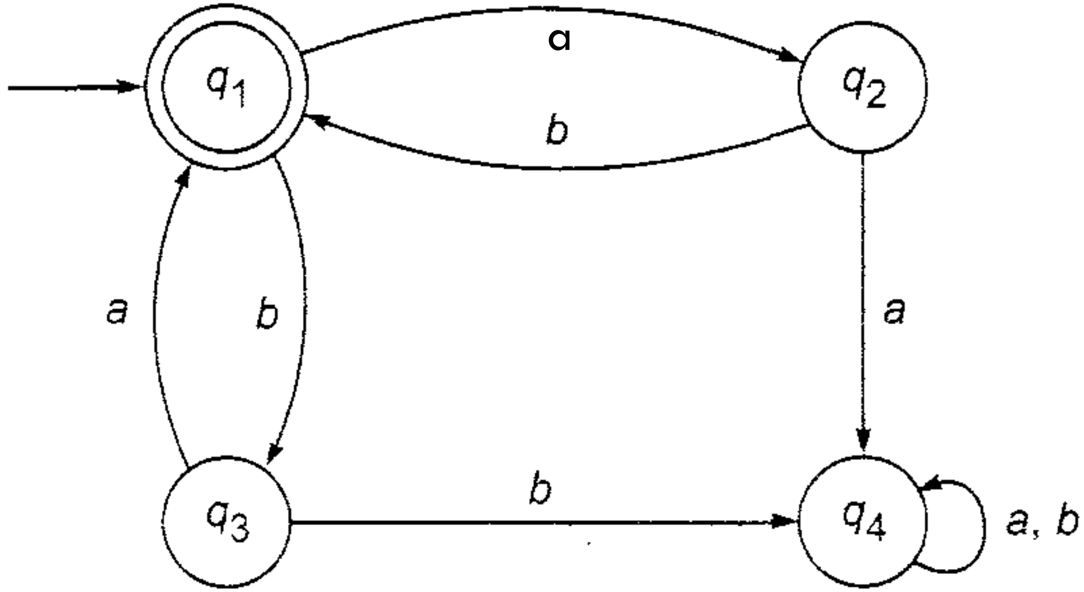
$q_1 = \varepsilon(a + a(b + aa)^*b)^*$

$q2 = (a + a(b + aa)^*b)^*\, a(b + aa)^*$

$q3 = (a + a(b + aa)^*b)^*\, a(b + aa)^*a$

Q3 is the final state. So RE

$(a + a(b + aa)^*b)^*a(b + aa)^*a$



$$q_1 = q_2 b + q_3 a + \Lambda$$

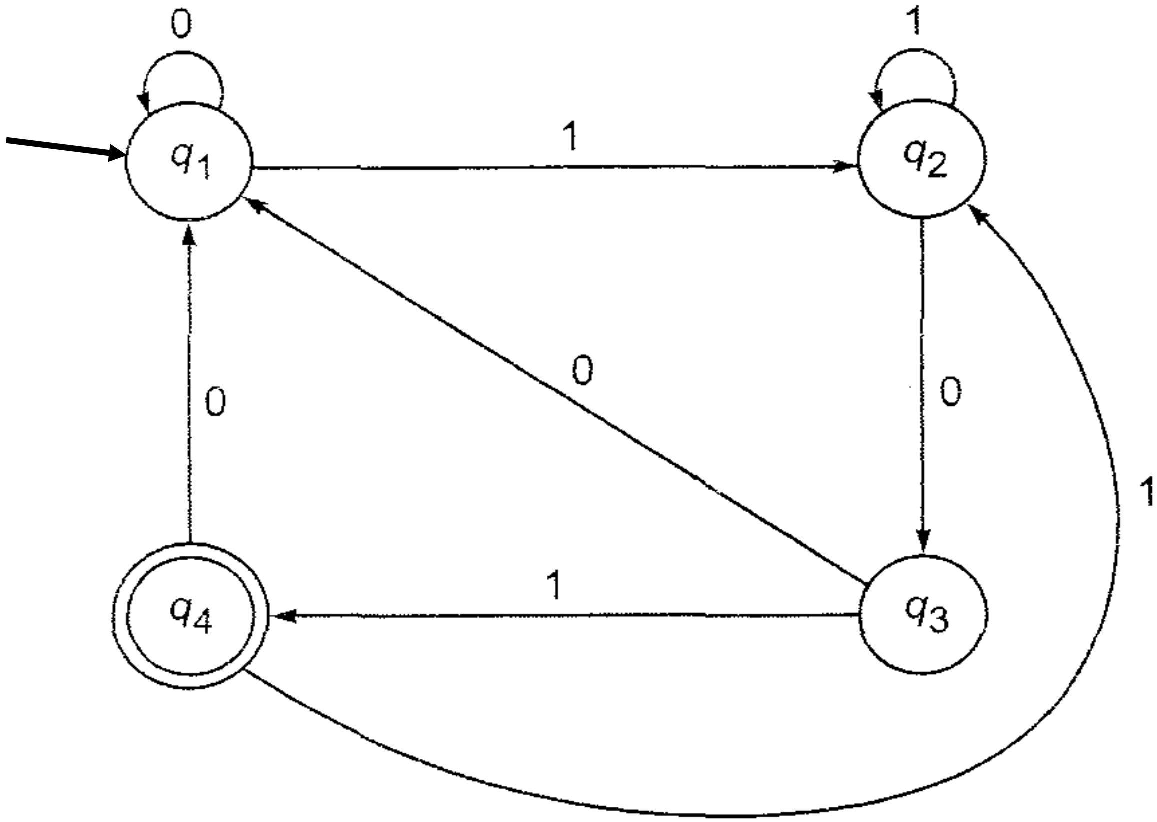
$$q_2 = q_1 a$$

$$q_3 = q_1 b$$

$$q_4 = q_2 a + q_3 b + q_4 a + q_4 b$$

$$q_1 = q_1 ab + q_1 ba + \Lambda = q_1(ab + ba) + \Lambda$$

$$q_1 = \Lambda(ab + ba)^* = (ab + ba)^*$$



$$q_1 = q_1 0 + q_3 0 + q_4 0 + \Lambda$$

$$q_2 = q_1 1 + q_2 1 + q_4 1$$

$$q_3 = q_2 0$$

$$q_4 = q_3 1$$

$$q_4 = q_3 1 = (q_2 0)1 = q_2 01$$

$$q_2 = q_1 1 + q_2 1 + q_2 011 = q_1 1 + q_2(1 + 011)$$

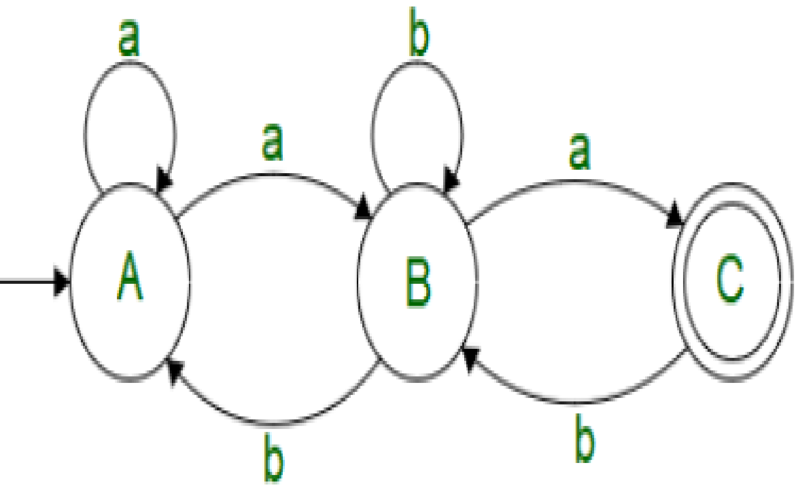
$$q_2 = (q_1 1)(1 + 011)^* = q_1(1(1 + 011)^*)$$

$$q_1 = q_1 0 + q_2 00 + q_2 010 + \Lambda$$
$$= q_1 0 + q_2(00 + 010) + \Lambda$$
$$= q_1 0 + q_1 1(1 + 011)^* (00 + 010) + \Lambda$$

Using Arden's theorem

$$q_1 = \Lambda(0 + 1(1 + 011)^* (00 + 010))^*$$
$$q_4 = q_2 01 = q_1 1(1 + 011)^* 01$$
$$= (0 + 1(1 + 011)^*(00 + 010))^*(1(1 + 011)^* 01)$$

Find the RE



Step 1: Construct the equations

$A = Aa + Bb + \varepsilon$

$B = Aa + Bb + Cb$

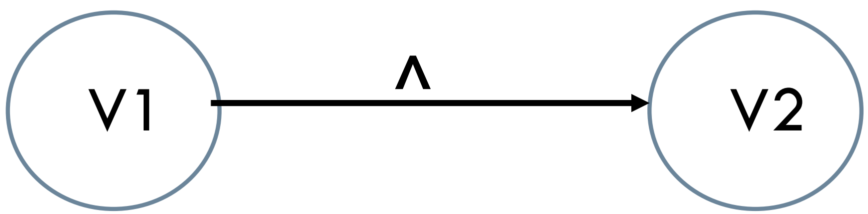
$C = Ba$

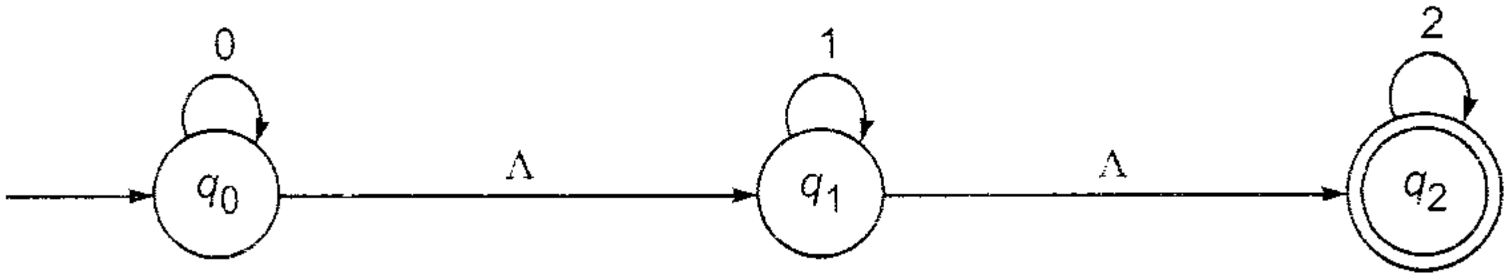
Step 2: Solve the equations

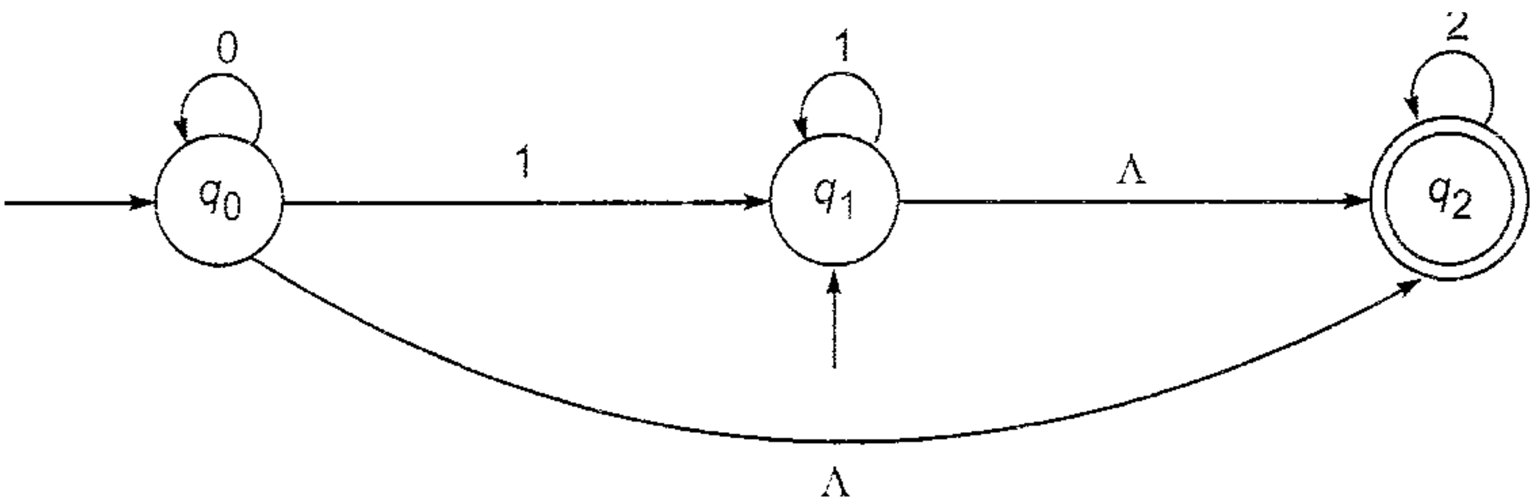
$$(a + a(b + ab)^*b)^* a (b + ab)^* a$$

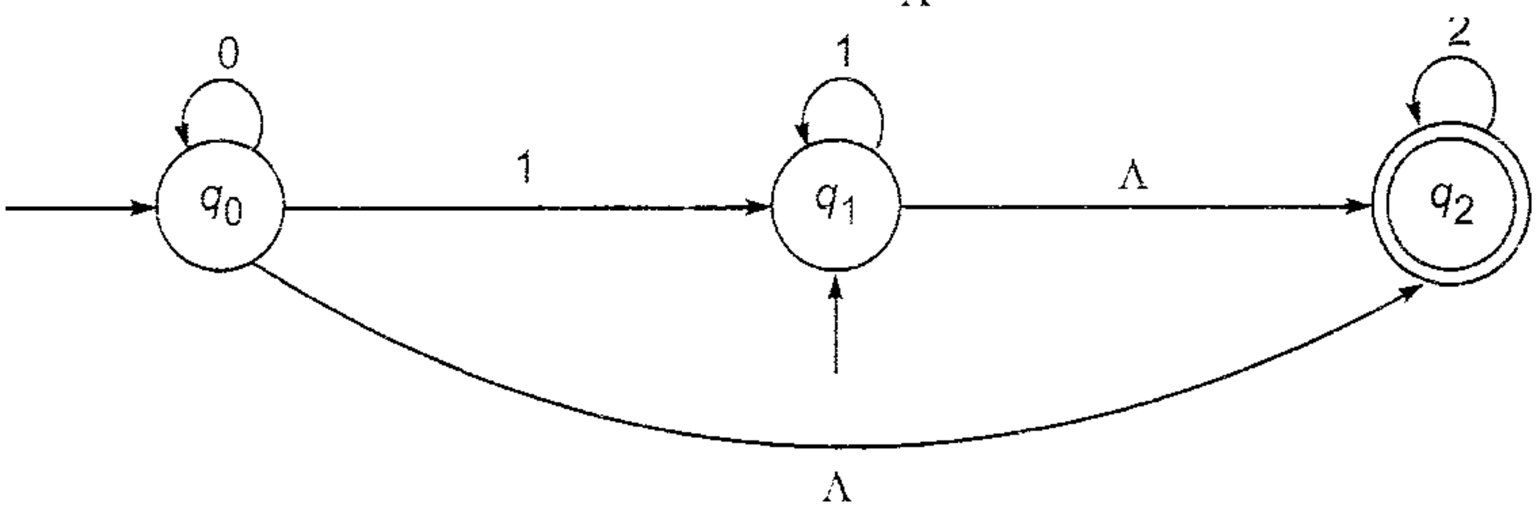
Elimination of Null Moves

- **Step 1** Find all the edges starting from *V2*.
- **Step 2** Duplicate all these edges starting from *V1* without changing the edge labels.
- **Step 3** If V1is an initial state, make V2 also as initial state.
- **Step 4** If V2 is a final state. make V1 also as the final state.

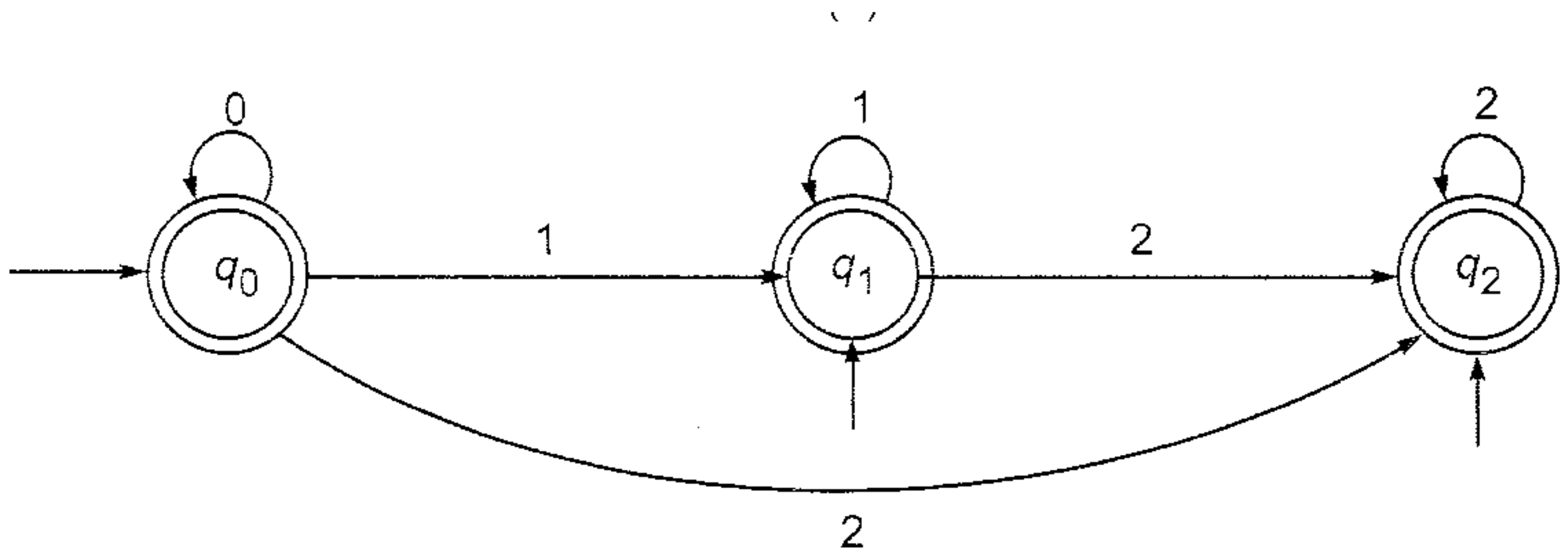



0
1
2
Λ
Λ
q_0
q_1
q_2


0
1
2
1
Λ
q_0
q_1
q_2
Λ

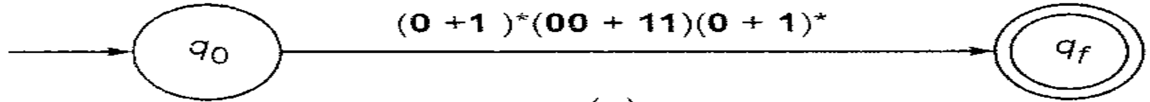

0
1
2
1
Λ
q_0
q_1
q_2
Λ

Elimination of Null Moves

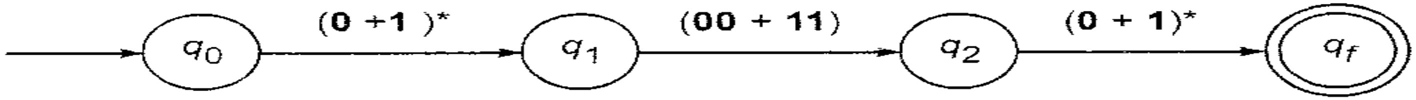


RE to DFA

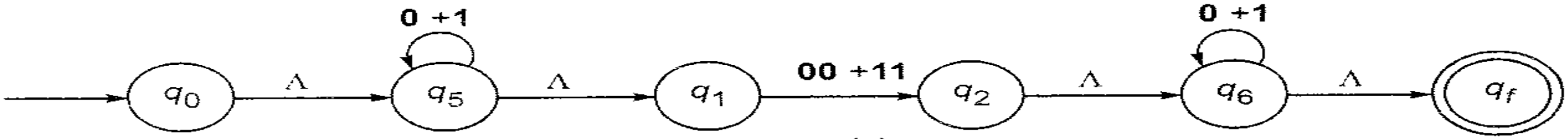
$$(0 + 1)^*(00 + 11)(0 + 1)^*$$


q_0
$(0+1)^*(00+11)(0+1)^*$
q_f

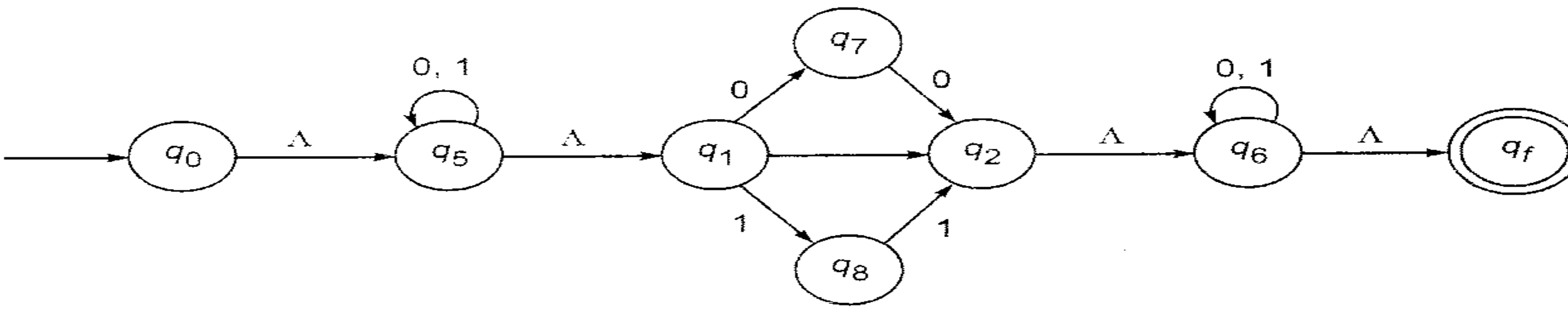
(a)


q_0
$(0+1)^*$
q_1
$(00+11)$
q_2
$(0+1)^*$
q_f

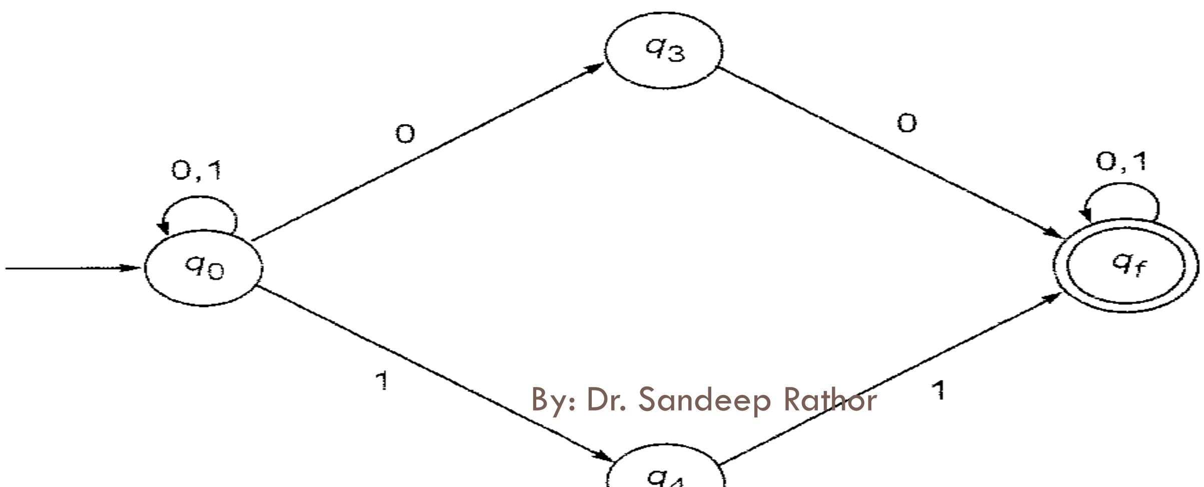
(b)

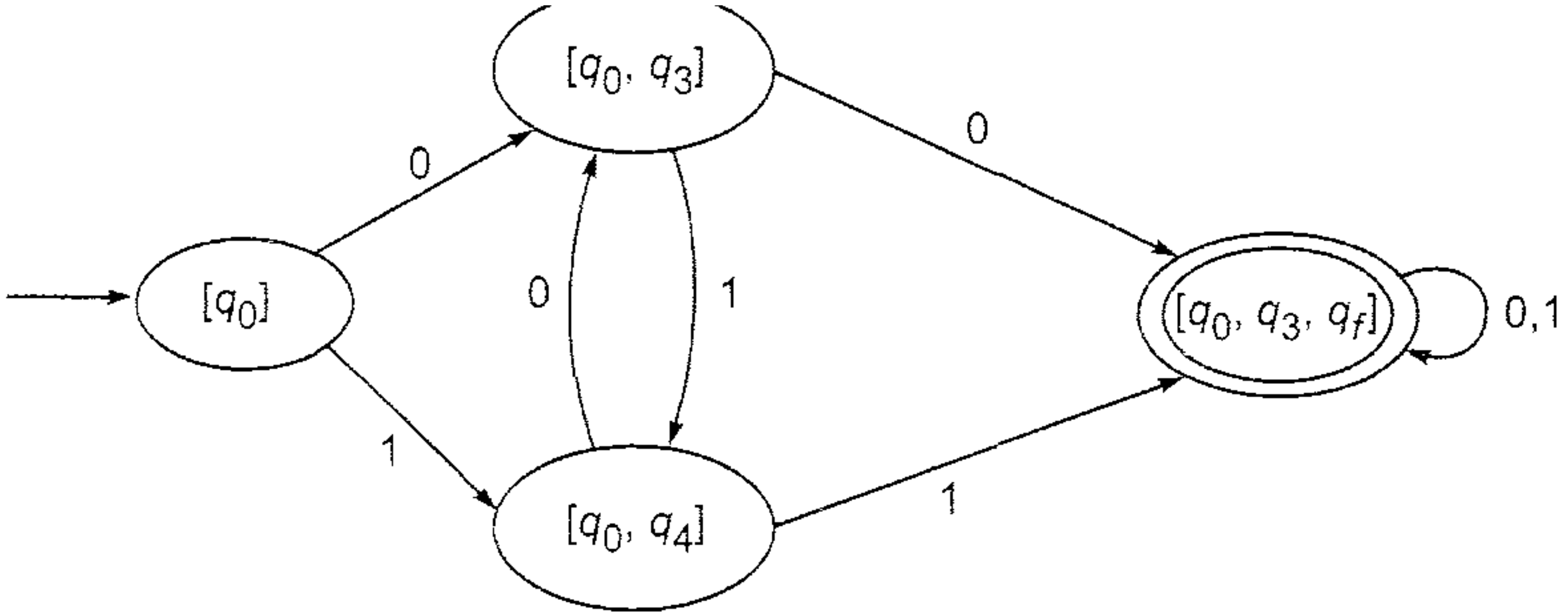

q_0
Λ
$0+1$
q_5
Λ
q_1
$00+11$
q_2
Λ
$0+1$
q_6
Λ
q_f

(c)


q_0
Λ
0, 1
q_5
Λ
q_1
0
q_7
0
1
q_8
1
q_2
Λ
0, 1
q_6
Λ
q_f

(d)

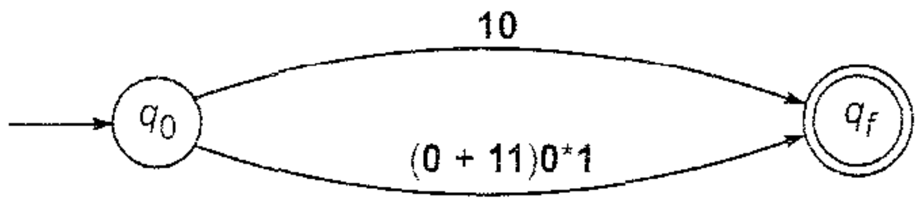

0,1
q_0
0
q_3
0
0,1
q_f
1
1
q_4

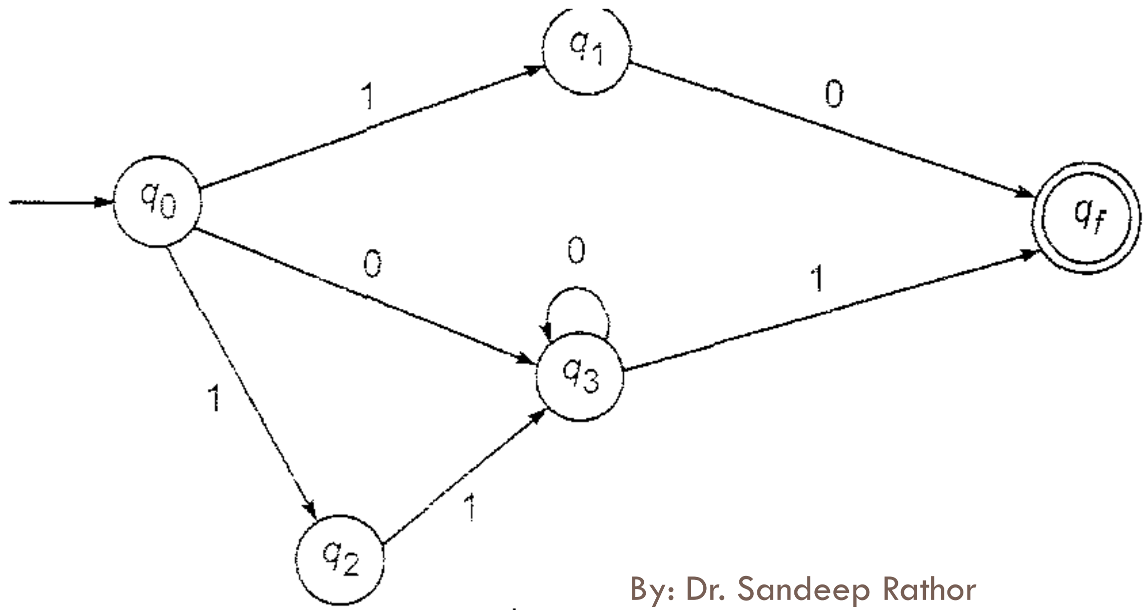


DFA corresponding to given RE

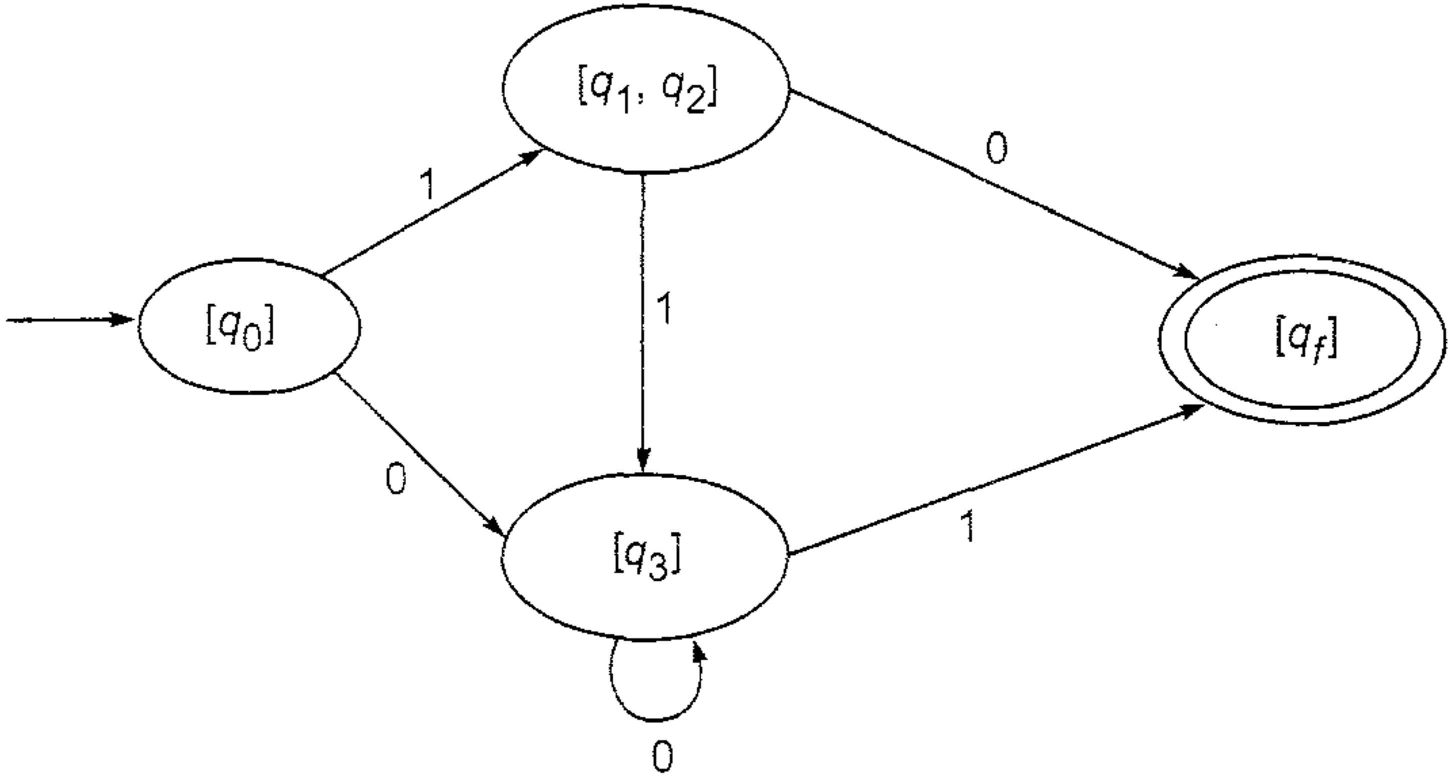
Find DFA from given RE

$10 + (0 + 11)0^*1$





By: Dr. Sandeep Rathor



Final DFA as per the given RE

Pumping lemma for Regular Sets

Statement:

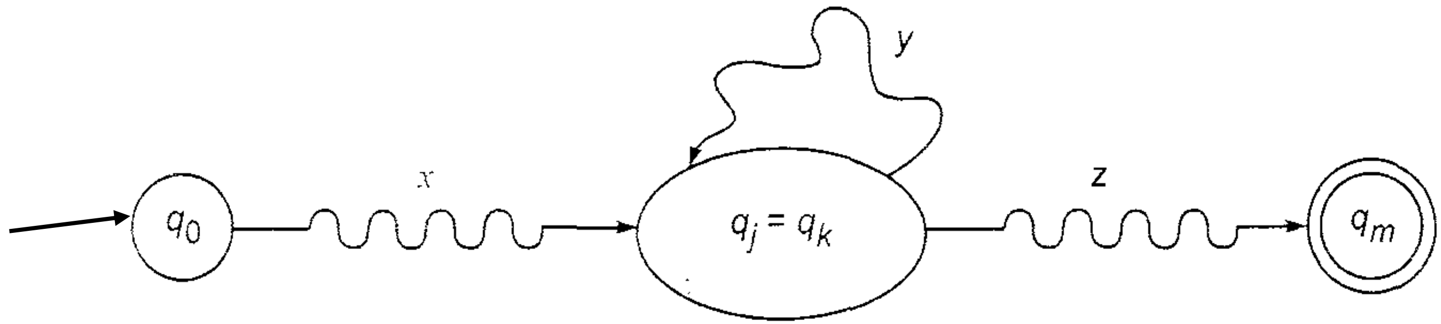
Let $M = (Q, \Sigma, \delta, q_0, F)$ be a finite automaton with n states. Let L be the regular set accepted by M. Let $w \in L$ *and* $|w| \geq m$. If $m \geq n$, then there exists x, y, z such that $w=xyz$, $y \neq \wedge$, $i=$ A and $xy^iz \in L$ for each $i \geq 0$.

*Pumping Repeating a section of the string an arbitrary number of times (≥0), with the resulting string remaining in the language.

Proof:

- Let $w = a_1, a_2, \ldots a_m$, $m \geq n$
- $\delta(q_0, a_1, a_2, a_3 \ldots a_i) = q_i$ **for $i = 1, 2, 3, \ldots, m$**
- $Q = \{q_0, q_1, q_2, \ldots q_m\}$
- Q is the sequence of states in the path with path value $w = a_1, a_2, a_3 \ldots a_m$. As there are only n distinct states, at least two states in Q must coincide. Among various pair of repeated states, we take the first pair. Let take them as q_j and q_k ($q_j = q_k$). J and k satisfy the condition $0 \leq j < k \leq n$.

- String w can be decomposed into 3-substrings,
- x=**a1,a2,.........aj**
- Y=**aj+1...........ak**
- Z= **ak+1........................ am**
- The path with the path value w in the transition diagram of *M* is shown as:



Automaton *M* starts from the initial state *qQ.* On applying the string *x*, it reaches *qj(=qk).* On applying the string *y*, it comes back to *qj(= qk)* So, after application of y' for each $i \geq 0$, the automaton is in the same state *qj.* On applying z, it reaches *qm,* a final state. Hence, *xy'z* ∈ *L*. As every state in Q is obtained by applying an input symbol, *y* ≠ A (null).

Prove that the language $\{a^k b^k \mid k \geq 0\}$ is not regular.

Prove that $L = \{a^k b^k \mid k \geq 0\}$ is not regular.

Step 1:

Suppose L is regular & is accepted by a FA having n states.

Step 2:

- Let $w = a^k b^k$ where $k > n$
- $w \in L$
- We can write $w = xyz$ where
  - $|xy| \leq n$
  - $|y| > 0$
- Since $k > n$, we have
  - $x = a^p$
  - $y = a^q$
  - $z = a^r b^n$
  - where $p + q + r = n$ & $q \neq 0$

Step 3:

- Let $i = 2$
- $w' = xy^i z$ where
  - $w' = a^p a^{2q} a^r b^n$
  - $w' = a^{p+2q+r} b^n$
  - $w' = a^{n+q} b^n$
  - Since $q \neq 0$, w' has more a's than b's
- Hence $w' \notin L$
- This is a contradiction, so L is not Regular

Prove that $\mathbf{L = \{a^i b^i \mid i \geq 0\}}$ is not regular.

Solution –

At first, we assume that **L** is regular and n is the number of states.

Let $w = a^n b^n$. Thus $|w| = 2n \geq n$.

By pumping lemma, let $w = xyz$, where $|xy| \leq n$.

Let $x = a^p$, $y = a^q$, and $z = a^r b^n$, where $p + q + r = n$, $p \neq 0$, $q \neq 0$, $r \neq 0$. Thus $|y| \neq 0$.

Let $k = 2$. Then $xy^2z = a^p a^{2q} a^r b^n$.

Number of as $= (p + 2q + r) = (p + q + r) + q = n + q$

Hence, $xy^2z = a^{n+q} b^n$. Since $q \neq 0$, xy^2z is not of the form $a^n b^n$.

Thus, xy^2z is not in L. Hence L is not regular.