

Partial Differential Equation

Homogeneous PDE :-

$$F(D, D') z = \phi(x, y) \text{ of degree } n$$

Case 1st:- $\phi(x, y)$ is function of $ax+by$.

Step 1: replace D by a and D' by b in $F(D, D')$ to get $F(a, b)$.

Step 2: Put $ax+by = u$ and integrate $\phi(u)$, n times w.r.t u .

$$\text{i.e. P.P.} = \frac{1}{F(a, b)} \int \int \int \dots \int \underbrace{\phi(u) du du du \dots du}_{n \text{ times}}$$

Step 3: replace ~~u~~ u by $ax+by$.

Note If $F(a, b) = 0$, then method fails.

Case of failure:- i.e. $F(D, D') z = \phi(ax+by)$; $F(a, b) = 0$

Step 1: Differentiate $F(D, D')$ partially w.r.t to D and multiply the expansion by x .

$$\begin{aligned} \text{i.e. P.P.} &= \frac{1}{F(D, D')} \phi(ax+by); F(a, b) = 0 \\ &= \cancel{x} \frac{1}{\cancel{x}} \frac{\partial}{\partial D} F(D, D') \phi(ax+by) \end{aligned}$$

We proceed this method as long as derivative $F(D, D')$ vanishes when D is replaced by a and D' by b .

Note If Highest derivative term in D' in $F(D, D')$ then in case of failure:

$$\begin{aligned} \text{P.P.} &= \frac{1}{F(D, D')} \phi(ax+by); F(a, b) = 0 \\ &= \cancel{y} \frac{1}{\cancel{y}} \frac{\partial}{\partial D'} F(D, D') \phi(ax+by) \end{aligned}$$

Case ii): If $\phi(x,y) = x^m y^n$ $m > n$

To obtain P.D., we expand $F(D, D')$ in an infinite series of ascending powers of D or D' or D/D' . Or $\frac{D'}{D}$

$$\text{Example 1. } \frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3$$

$$\text{The given P.D.E. } (D^3 - D'^3)z = x^3 y^3$$

$$\text{A.E. } m^3 - 1 = 0 \quad D \rightarrow m \quad D' \rightarrow 1$$

$$(m-1)(m^2 + m + 1) = 0$$

$$\Rightarrow m=1, \omega, \omega^2$$

where ω is cube root of unity

$$C.F. = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$$

$$P.I. = \frac{1}{D^3 - D'^3} \cdot (x^3 y^3)$$

$$= \frac{1}{D^3} \left\{ 1 - \frac{D'^3}{D^3} \right\} (x^3 y^3)$$

$$= \frac{1}{D^3} \left(1 - \frac{D'^3}{D^3} \right)^{-1} (x^3 y^3) \quad \because (1-x)^{-1} = 1 + x + x^2 + \dots$$

$$= \frac{1}{D^3} \left(1 + \frac{D'^3}{D^3} + \dots \infty \right) (x^3 y^3)$$

$$= \frac{1}{D^3} \left(x^3 y^3 + \frac{1}{D^3} x^3 D'^3 (y^3) + \dots \infty \right)$$

$$= \frac{1}{D^3} \left(x^3 y^3 + \frac{1}{D^3} x^3 \cdot 6 + 0 + 0 + \dots \right)$$

$$= y^3 \frac{1}{D^3} x^3 + 6 \cdot \frac{1}{D^6} x^3$$

$$= \frac{x^6 y^3}{6 \cdot 5 \cdot 4} + \frac{6 x^9}{4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9} = \frac{x^6 y^3}{120} + \frac{x^9}{10080} \text{ Ans.}$$

Complete solution = C.F + P.I.

$$Z = f_1(y+x) + f_2(y+\omega x) + f_3(y+\omega^2 x)$$

$$+ \frac{x^6 y^3}{120} + \frac{x^9}{10080}$$

$$\text{Q2} \quad \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = e^{x+2y} + x^3 + xy$$

$$\text{Soln A.E: } (D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3 + xy$$

$$\text{A.E. } m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$$

$$\text{C.F.} = f_1(y+x) + xf_2(y+x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2DD' + D'^2} \left\{ e^{x+2y} + x^3 + xy \right\} = P_1 + P_2 + P_3$$

$$P_1 = \frac{1}{D^2 - 2DD' + D'^2} e^{x+2y}, \text{ replace } D \text{ by } 1, D' \text{ by } 2$$

$$= \frac{1}{(D - D')^2} e^{x+2y} = \frac{1}{(1-2)^2} \iint e^u du du =$$

$$= \int e^u du = e^u = e^{x+2y}$$

$$P_2 = \frac{1}{(D - D')^2} x^3 = \text{Here } x^3 = (1x + 0y)^3 \text{ Let } u = 1x + 0y$$

$$= \frac{1}{(1-0)^2} \iint u^3 du du$$

$$= \frac{u^4}{4 \cdot 5} = \frac{x^4}{20}$$

$$P_3 = \frac{1}{(D - D')^2} xy = \frac{1}{D^2} \left\{ 1 - \frac{D'}{D} \right\}^{-2} xy$$

$$= \frac{1}{D^2} \left\{ 1 + 2 \frac{D'}{D} + \dots \right\} xy \text{ Leaving higher term.}$$

$$= \frac{1}{D^2} \left\{ xy + \frac{2}{D} D'(xy) + 0 + 0 + \dots \right\}$$

$$= \frac{1}{D^2} \left\{ xy + \frac{2}{D} x \right\} = \frac{1}{D^2} \left\{ xy + x^2 \right\}$$

$$= \frac{x^3}{2 \cdot 3} y + \frac{x^4}{3 \cdot 4} = \frac{x^3 y}{6} + \frac{x^4}{12}$$

Hence Soln $Z = \text{C.F.} + \text{P.I.}$

$$= f_1(y-x) + xf_2(y-x) + e^{x+2y} + \frac{x^5}{20} + \frac{x^4}{12}$$

$$+ \frac{x^3 y}{6}$$

Ans.

$$Q \quad \frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$$

$$\text{Soln PDE } (D^2 + 2DD' + D'^2)z = x^2 + xy + y^2$$

$$A.E \quad m^2 + 2m + 1 = 0$$

$$\Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$$

$$C.F. = 1 f_1(y-x) + x f_2(y-x)$$

$$P.I. = \frac{1}{F(D, D')} \phi(x, y) = \frac{1}{(D^2 + 2DD' + D'^2)} \cdot x^2 + xy + y^2$$

$$= P_1 + P_2 + P_3$$

$$\Rightarrow P_1 = \frac{1}{(D^2 + 2DD' + D'^2)^2} x^2 = \frac{1}{(D+D')^2} x^2 = \frac{1}{(D+D')^2} (1x+0y)^2$$

$$= \frac{1}{(1+0)^2} \iint u^2 du du; \quad \begin{matrix} \text{let } 1x+0y=u \\ \text{replace } D \rightarrow 1, D' \rightarrow 0 \end{matrix}$$

$$= \frac{u^4}{3 \cdot 4} = \frac{x^4}{12}$$

$$P_2 = \frac{1}{(D+D')^2} xy = \frac{1}{D^2 (1+\frac{D'}{D})^2} xy = \frac{1}{D^2} \left\{ 1 + \frac{D'}{D} \right\}^{-2} xy$$

$$= \frac{1}{D^2} \left\{ 1 - 2 \frac{D'}{D} + \dots \right\} xy \quad (\cancel{\text{higher terms}} \text{ leaving the higher terms})$$

$$= \frac{1}{D^2} \left\{ xy - \frac{2D'xy}{D} \right\}$$

$$= \frac{1}{D^2} \left\{ xy - \frac{2x}{D} \right\} = \frac{1}{D^2} \left\{ xy - \frac{2x^2}{2} \right\}$$

$$= \frac{1}{D} \left(\frac{x^2 y}{2} - \frac{x^3}{3} \right) = \frac{x^3 y}{6} - \frac{x^4}{12}$$

$$P_3 = \frac{1}{(D+D')^2} y^2 = \frac{1}{(D+0)^2} \iint u^2 du du$$

$$= \frac{1}{(0+1)^2} \left[\frac{u^4}{4 \cdot 3 \cdot 2} \right] = \frac{y^4}{6 \times 2} = \frac{y^4}{12}$$

$$\text{let } y = 0x + 1y = u$$

$$D \rightarrow 0 \quad D' \rightarrow 1$$

General Method to solve P.D.

$$\therefore P.D. = \frac{1}{F(D, D')} \phi(x, y); \quad F(D, D') \text{ is PDE of degree } n.$$

$$= \frac{1}{(D - m_1 D')(D - m_2 D') \dots (D - m_n D')} \phi(x, y) \checkmark$$

$$= \frac{1}{(D - m_1 D')} \cdot \frac{1}{(D - m_2 D')} \cdots \frac{1}{(D - m_n D')} \phi(x, y)$$

First we solve $\frac{1}{D - m_1 D'} \phi(x, y) = \int \phi(x, c - mx) dx$
 Here $y = c - mx$

In last again replace $c = y + mx$.

Similarly we solve P.D. for others factor $(D - m_2 D')$

$$\dots (D - m_n D').$$

or $r + \beta - 6t = y \cos x \quad p = \frac{\partial z}{\partial x}, q = \frac{\partial z}{\partial y}, r = \frac{\partial^2 z}{\partial x^2}$

Q.E.D: $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6 \frac{\partial^2 z}{\partial y^2} = y \cos x; \quad \beta = \frac{\partial^2 z}{\partial x \partial y}, t = \frac{\partial^2 z}{\partial y^2}$

$$\text{PDE: } (D^2 + DD' - 6D'^2)z = y \cos x;$$

$$\text{A.E. } m^2 + m - 6 = 0 \Rightarrow (m-2)(m+3) = 0$$

$$\Rightarrow m = 2, -3.$$

$$C.F. = f_1(y+2x) + f_2(y-3x) \quad y+mx = C.$$

$$P.D. = \frac{1}{D^2 + DD' - 6D'^2} y \cos x =$$

$$= \frac{1}{(D-2D')(D+3D')} y \cos x, \quad \text{Replace } y = c - mx, m = -3 \\ = \frac{1}{(D-2D')(D+3D')} y \cos x, \quad y = c + 3x$$

$$= \frac{1}{D-2D'} \int (c+3x) \cos x dx$$

$$= \frac{1}{D-2D'} \left[c \sin x + 3 \{ x \sin x + \cos x \} \right]$$

$$= \frac{1}{D-2D'} \left\{ (c+3x) \sin x + 3 \cos x \right\} = \frac{1}{D-2D'} \left\{ y \sin x + 3 \cos x \right\}$$

$c+3x \rightarrow y$

$$\begin{aligned}
 &= \frac{1}{D-2D'} \left\{ y \sin x + 3 \cos x \right\} \quad \because 3x+c=y \quad \underline{\underline{6}} \\
 &= \int (b-2x) \sin x dx + 3 \int \cos x dx \quad \because b-2x=y \\
 &= \int b \sin x dx + 2 \int x \sin x + 3 \int \cos x dx \\
 &= -b \cos x - 2 \left\{ -x \cos x + \sin x \right\} + 3 \sin x \\
 &= -(b-2x) \cos x - 2 \sin x + 3 \sin x \\
 &= -y \cos x + 1 \sin x \quad (b-2x=y)
 \end{aligned}$$

Hence complete solution

$$Z = C.F. + P.I.$$

$$= f_1(y+2x) + f_2(y-3x) - y \cos x + \sin x$$

$$2Q \quad (D^2 + 2DD' + D'^2)Z = 2 \cos y - x \sin y \quad Z = f(x,y)$$

~~PDE A.E.~~ $m^2 + 2m + 1 = 0$

$$m = -1, -1 \Rightarrow C.F. = f_1(y-x) + x f_2(y-x)$$

$$P.I. = \frac{1}{D^2 + 2DD' + D'^2} (2 \cos y - x \sin y) = P_1 - P_2$$

$$P_1 = \frac{1}{(D+D')^2} 2 \cos y = \frac{1}{(0+1)^2} 2 \cos y \quad \text{let } y = u = 0x + 1y$$

$$= \frac{u^3}{3} = \frac{y^3}{3} = -2 \cos u$$

$$= -2 \cos y \quad \dots$$

$$P_2 = \frac{1}{(D+D')^2} x \sin y = \frac{1}{(D+D')} \int x \sin(c+x) dx$$

$$= \frac{1}{D+D'} \left\{ -x \cos(c+x) + \sin(c+x) \right\} \quad \text{put } c+x=y$$

$$= \frac{1}{D+D'} \left\{ -x \cos y + \sin y \right\}$$

$$= \int \{-x \cos(b+x) + \sin(b+x)\} dx \quad y = b - mx$$

$$= - \left\{ x \sin(b+x) + \cos(b+x) \right\} - \left\{ \cos(b+x) \right\}$$

$$= -x \sin y - 2 \cos y \quad \text{Ans}$$

Hence complete soln

$$Z = C \cdot F_c + P \cdot I$$

$$= f_1(y-x) + x f_2(y-x) - 2 \cos y - \left\{ -x \sin y - 2 \cos y \right\}$$

$$= f_1(y-x) + x f_2(y-x) + x \sin y \quad \underline{\text{Ans}}$$

$$\rightarrow Q \quad (D^3 + 2D^2D' - DD'^2 - 2D'^3) Z = (y+2) e^x$$

$$\text{Soln. } A \cdot E. \quad m^3 + 2m^2 - m - 2 = 0$$

$$m^2(m+2) - 1(m+2) = 0$$

$$(m+2)(m^2-1) = 0$$

$$(m+2)(m-1)(m+1) = 0$$

$$\Rightarrow m = 1, -1, -2 \checkmark$$

$$C.F. = f_1(y+x) + f_2(y-x) + f_3(y-2x)$$

$$P.I. = \frac{1}{F \cdot (D, D')} \phi(x, y) =$$

$$= \frac{1}{(D+D')(D-D')(D+2D')} (y+2) e^x$$

$$= \frac{1}{(D+2D')(D-D')} \int (c+x+2) e^x dx$$

$$= \frac{1}{(D+2D')(D-D')} \left[(c+x+2) e^x - e^x \right] = \frac{1}{(D-2D')(D-D')} (c+x+1) e^x$$

$$= \frac{1}{(D+2D')(D-D')} (y+1) e^x = \frac{1}{D-2D'} \int [(b-x)+1] e^x dx$$

$$= \frac{1}{(D+2D')} \left\{ (b-x+\frac{1}{2}) e^x + e^x \right\} = \frac{1}{D+2D'} (b-x) e^x \quad \begin{cases} \text{Put} \\ b-x=y \end{cases}$$

$$= \frac{1}{D+2D'} \left\{ (y+2) e^x \right\} = \int [(d+2x)+2] e^x dx$$

$$= (d+2x+2) e^x - 2 e^x = (d+2x) e^x$$

$$= y e^x \quad \underline{\text{Ans}}$$

$$y = c - mx$$

$$y = c + x$$

$$y$$

$$(c+x+1) e^x$$

$$(b-x) e^x$$

$$(b-x) e^x$$

$$(b-x) e^x$$

$$(b-x) e^x$$

Non-Homogeneous P'linear PDE :-

If PDE: $F(D, D')Z = \phi(x, y) \quad (1)$ If $\phi(x, y) = 0$

If $F(D, D')$ is not homogeneous in D & D' . Then PDE (1) is called Non-homogeneous linear PDE.

• If $F(D, D') = (D - m_1 D' - a_1)(D - m_2 D' - a_2) \dots (D - m_n D' - a_n) = 0$

i. If all factors are distinct then

$$C.F. = e^{a_1 x} f_1(y + m_1 x) + e^{a_2 x} f_2(y + m_2 x) + \dots + e^{a_n x} f_n(y + m_n x)$$

2. If ~~all~~ factors are repeated e.g. $(D - m D' - a)^3 Z = 0$

$$\text{Then } C.F. = e^{ax} f_1(y + mx) + x e^{ax} f_2(y + mx) + x^2 e^{ax} f_3(y + mx)$$

Complete soln: $Z = C.F. + P.P. = C.F. \quad (P.P. = 0)$

• If $F(D, D')$ cannot be factorize into linear factors.

Here we use trial solution

$$\text{e.g. Let } (D - D'^2)Z = 0 \quad (2)$$

Let trial solution of (2)

$$Z = A e^{hx+ky} \quad \text{where } A, h, k \text{ are constants}$$

$$\text{from (2)} \quad DZ = \frac{\partial Z}{\partial x} = hAE^{hx+ky}, \quad D'Z = \frac{\partial Z}{\partial y} = kAE^{hx+ky}$$

$$D'^2 Z = \frac{\partial^2 Z}{\partial y^2} = k^2 A e^{hx+ky}$$

from (2); we get

$$Ahe^{hx+ky} - AK^2 e^{hx+ky} = 0$$

$$A(h - K^2) e^{hx+ky} = 0, \quad A \neq 0$$

$$\Rightarrow$$

$$h - K^2 = 0,$$

$$\Rightarrow$$

$$h = K^2$$

$$\text{Hence from (2)} \quad Z = A e^{K^2 x + ky}$$

Hence more general soln

$$Z = \sum A e^{K^2 x + ky}$$

arbitrary constants

where A & K are

and Σ denotes the no. of terms taken.

Ex 1: Solve the PDE $(D-D'-2)(D+D'+1)z=0$

Compare each factor with $(D-mD'-a)$, we get

$$m_1=1 \quad a_1=2 \quad m_2=-1 \quad a_2=-1$$

$$C.F. = e^{2x} f_1(y+x) + e^{-x} f_2(y-x)$$

$$P.I. = 0 \quad \therefore \phi(x,y) = 0$$

Then complete soln $Z = C.F. + P.I. = e^{2x} f_1(y+x) + e^{-x} f_2(y-x)$

Ex 2: Solve the PDE $DD'(D+5D'-1)(D-2D'-3)z=0$

Soln: Linear factors

$$D=0 \quad D'=0 \quad D+5D'-1=0 \quad D-2D'-3=0$$

$$\frac{\partial Z}{\partial x}=0 \quad \frac{\partial Z}{\partial y}=0 \quad m_1=5 \quad a_1=1 \quad m_2=+2 \quad a_2=3$$

Then part of C.F. are

$$\Rightarrow C.F. = f_1(y) \quad C.F. = f_2(x) \quad C.F. = f_3(y-5x) \quad C.F. = e^{3x} f_4(y+2x)$$

Complete soln $Z = f_1(y) + f_2(x) + e^x f_3(y-5x) + e^{3x} f_4(y+2x)$

$$P.I.: \text{Case i)} \phi(x,y) = e^{ax+by} \text{ then } P.I. = \frac{1}{F(D,D')} e^{ax+by}$$

Ex 3: Solve the $s+2s+t+2p+2q+z = e^{5x+3y}$. $P = \frac{\partial Z}{\partial x}$

$$\text{Given: } (D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = F(D, D')$$

$$\text{on factorization } \{(D+D')^2 + 2(D+D') + 1\}z = F(D, D')$$

$$\text{or } (D+D'+1)^2 z = F(D, D')$$

Here $F(D, D')$ having repeating factor with

$$m_1=-1, a_1=-1, m_2=-1, a_2=-1$$

$$\text{Then } C.F. = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x) \quad \left. \begin{array}{l} s+3t=11 \\ D \rightarrow 5 \\ D' \rightarrow 3 \end{array} \right.$$

$$P.I. = \frac{1}{F(D, D')} \phi(ax+by) = \frac{1}{(D+D'+1)^2} e^{5x+3y} \quad \left. \begin{array}{l} s+3t=11 \\ D \rightarrow 5 \\ D' \rightarrow 3 \end{array} \right.$$

$$= \frac{1}{(5+3+1)^2} e^{5x+3y}$$

$$= \frac{1}{81} e^{5x+3y} - \frac{1}{81} e^{5x+3y}$$

Hence complete soln

$$Z = e^{-x} f_1(y-x) + x e^{-x} f_2(y-x) + \frac{1}{81} e^{5x+3y} \quad \text{Ans}$$

Case 111d :- $\phi(x, y) = x^m y^n$, then P.I.

$$P.I. = \frac{1}{F(D, D')} x^m y^n = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1} x^m y^n$$

Expanding $[F(D, D')]^{-1}$ in powers of D/D' when ($m > n$) or $\{D/D'\}$ when $n > m\}$ or D or D' as per condition.

Ex1 solve the PDE $(D-D'^{-1})(D-D'^{-2})Z = e^{13x+2y} + 2x + 3y$

Soln On comparing each factor with $(D-mD'-a)$, we get

$$m_1 = +1 \quad a_1 = 1 \quad m_2 = +1 \quad a_2 = 2$$

$$C.F. = e^x f_1(y+x) + e^{2x} f_2(y+x) \checkmark$$

Particular Integral P.I. = $P_1 + P_2 + P_3$

$$P_1 = \frac{1}{(D-D'^{-1})(D-D'^{-2})}, \quad e^{13x+2y} = \frac{1}{(13-2-1)(13-2-2)} = \frac{1}{90} e^{13x+2y}$$

$$P_2 = \frac{1}{(-1)\{1-(D-D')\}(-2)\{1-(D-D')\}} \quad 2xe^x = \left\{1-(D-D')\right\}^{-1} \left\{1-\frac{(D-D')}{2}\right\}^{-1} x$$

$$= \left\{1+(D-D')\right\} \left\{1+\frac{D-D'}{2}\right\} x^2 \quad \text{expanding for only 2 terms}\\ \text{and leaving higher derivative term due to function } x \checkmark$$

$$= \left\{1+\frac{D-D'}{2}+(D-D')+\frac{(D-D')^2}{2}\right\} x^2$$

$$= \left\{1+\frac{3}{2}(D-D')+\frac{D^2+D'^2-2DD'}{2}\right\} x^2 = \left\{1+\frac{3}{2}(D-D')+\frac{D^2+D'^2}{2}-DD'\right\} x^2$$

$$= \left\{0 + \frac{3}{2}(0-0) + \frac{0}{2} + \frac{0}{2} - 0\right\} = \left(x + \frac{3}{2}\right) \checkmark$$

$$P_3 = \frac{1}{(D-D'^{-1})(D-D'^{-2})} 3y = \frac{1}{2} \left[1-(D-D')\right]^{-1} \left[1-\frac{(D-D')}{2}\right]^{-1} (3y)$$

$$= \frac{3}{2} \left\{1+\frac{3}{2}(D-D')+\frac{D^2+D'^2}{2}-DD'\right\} y = \frac{3}{2} \left\{y + \frac{3}{2}(0-1) + \frac{0}{2} + \frac{0}{2} - 0\right\}$$

$$= \frac{3}{2} \left\{y - \frac{3}{2}\right\}$$

$$\text{Complete soln: } Z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{90} e^{13x+2y} \\ + \left(x + \frac{3}{2}\right) + \frac{3}{2} \left(y - \frac{3}{2}\right)$$

Solve $(D+1)(D+D'-1) = e^{2x+3y} - \sin 2x + \sin 2x =$

Soln $F(D, D') = (D+1)(D+D'-1)$ Here $m_1=0$ $a_1=-1$ $m_2=-1$ $a_2=1$

$$C.F. = e^{-x} f_1(y) + e^x f_2(y-x)$$

$$P.P. = \frac{1}{F(D, D')} \phi(xy) = \beta_1 + \beta_2 + \beta_3$$

$$\beta_1 = \frac{1}{(D+1)(D+D'-1)} e^{2x+3y} = \frac{1}{(2+1)(2+3-1)} \times e^{2x+3y} = \frac{e^{2x+3y}}{12}$$

Complete soln $Z = C.F. + P.P. = e^{-x} f_1(y) + e^x f_2(y-x) + \frac{e^{2x+3y}}{12}$

Case And :- $F(D, D')z = \sin(ax+by)$ or $\cos(ax+by)$

$$P.P. = \frac{1}{F(D^2, DD', D'^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

$$D^2 \rightarrow -a^2, DD' = -ab, D'^2 = -b^2$$

$$P.P. = \frac{1}{F(-a^2, -ab, -b^2)} \sin(ax+by) \text{ or } \cos(ax+by)$$

If $F(D, D') = F(D^2, DD', D'^2, D, D')$ then P.P.

$$P.P. = \frac{1}{F(-a^2, -ab, -b^2, D, D')} \sin(ax+by) \text{ or } \cos(ax+by)$$

which can be evaluated further.

Ex: Solve $(D-D'-1)(D-D'-2)Z = \sin(2x+3y)$

Soln $F(D, D') = (D-D'-1)(D-D'-2)$ Here $m_1=1, m_2=1$

$$C.F. = e^x f_1(y+x) + e^{2x} f_2(y+2x) \quad a_1=1 \quad a_2=2$$

$$P.P. = \frac{1}{(D-D'-1)(D-D'-2)} \sin(2x+3y) = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2} \sin(2x+3y)$$

$$= \frac{1}{-4 + 12 - 9 - 3D + 3D' + 2} \sin(2x+3y) \quad D^2 \rightarrow -4, D'^2 \rightarrow -9, DD' \rightarrow -6$$

$$= \frac{1}{-3D + 3D' + 1} \sin(2x+3y) = -\frac{(3D-3D')+1}{\left\{ \frac{3D-3D'}{a} + 1 \right\} \left\{ \frac{3D-3D'}{b} + 1 \right\}} \sin(2x+3y)$$

$$\begin{aligned}
&= - \left\{ \frac{(3D-3D') + 1}{9D^2 + 9D'^2 - 18DD' - 1} \sin(2x+3y) \right\} \\
&= - \left\{ \frac{3D-3D'+1}{-36-81+108-1} \sin(2x+3y) \right\} \\
&= \frac{1}{10} (3D-3D'+1) \sin(2x+3y) \\
&= \frac{1}{10} \left\{ 6 \cos(2x+3y) - 9 \cos(2x+3y) \right. \\
&\quad \left. + \sin(2x+3y) \right\} \\
&= \frac{1}{10} \left\{ \sin(2x+3y) - 3 \cos(2x+3y) \right\}
\end{aligned}$$

Hence

$$Z = e^x f_1(y+x) + e^{2x} f_2(y+x) + \frac{1}{10} \left\{ \sin(2x+3y) - 3 \cos(2x+3y) \right\}$$

Q $D(D+D'-1)(D+3D'-2) \cancel{\equiv} x^2 - 4xy + y^2 \leftarrow$

Soln: factors: $D=0$ $D+D'-1=0$ $D+3D'-2=0$
 \Rightarrow part of C.F. $= f_1(y) = e^x f_2(y-x) = e^{+2x} f_3(y-3x)$
 $C.F. = f_1(y) + e^x f_2(y-x) + e^{+2x} f_3(y-3x) \leftarrow$

$$P.T. = \frac{1}{D(D+D'-1)(D+3D'-2)} x^2 - 4xy + y^2 =$$

$$P = \frac{1}{D(D+D'-1)(D+3D'-2)} \frac{x^2 - 4xy + y^2}{2D} = \frac{1}{2D} \left\{ \frac{1}{1-(D+D')}^{-1} \left\{ \frac{1}{1-(D+3D')}^{-1} \left(\frac{x^2 - 4xy + y^2}{2} \right) \right\} \right\}$$

$$\begin{aligned}
& \frac{1}{D(D+D'-1)(D+3D'-2)} (x^2 - 4xy + y^2) = \frac{1}{(-1)(-2)D} \left\{ 1 - (D+D') \right\}^{-1} \left\{ 1 - \left(\frac{D+3D'}{2} \right)^2 \right\} \\
&= \frac{1}{2D} \left\{ 1 + (D+D') + (D+D')^2 \right\} \left\{ 1 + \left(\frac{D+3D'}{2} \right) + \left(\frac{D+3D'}{2} \right)^2 \right\} \\
&= \frac{1}{2D} \left\{ 1 + \frac{D+3D'}{2} + \frac{D^2 + 9D'^2 + 6DD'}{4} + (D+D') + \frac{D^2 + 4DD' + 3D'^2}{2} \right. \\
&\quad \left. + (D+D') \left(D^2 + 9D'^2 + 6DD' \right) + (D^2 + 2DD' + D'^2) + \right. \\
&\quad \left. + \frac{(D^2 + D'^2 + 2DD')}{2} (D+3D') + \frac{(D^2 + 2DD' + D'^2)}{4} (D^2 + 9D'^2 + 6DD') \right. \\
&\quad \left. \times (x^2 - 4xy + y^2) \right\} \\
&= \frac{1}{2D} \left\{ 1 + \frac{3D+5D'}{2} + \frac{9D^2 + 19D'^2 + 22DD'}{4} \right\} (x^2 - 4xy + y^2) \\
&= \frac{1}{2D} \left\{ x^2 - 4xy + y^2 + \frac{3}{2}(2x-4y) + \frac{5}{2}(0-4x+2y) + \frac{9}{4}(2) \right. \\
&\quad \left. + \frac{19}{4}(0-0+2) + \frac{22}{4}(0-4+0) \right\} \\
&= \frac{1}{2D} \left\{ x^2 - 4xy + y^2 + \frac{3}{2}(2x-4y) + \frac{15}{2}(-4x+2y) + 22 \right\} \\
&= \frac{1}{2} \left\{ \frac{x^3}{3} - \frac{4x^2y}{2} + \frac{xy^2}{1} + \frac{3}{2}(x^2 - 4xy) + \frac{5}{2} \left(-\frac{4x^2}{2} + 2xy \right) - 22x \right\} \\
&= \frac{1}{2} \left\{ \frac{x^3}{3} - 2x^2y + xy^2 - \frac{7}{2}x^2 - 2xy - \frac{9}{2}x \right\} \checkmark
\end{aligned}$$

Case 17 :- when $\phi(x, y) = e^{ax+by}$.
 where y i) e^{ax+by} ii) $\sin(ax+by)$ or $\cos(ax+by)$
 iii) $x^m \cdot y^n$ iv) Constant (say 1)

Ex 1 $(D - 3D' - 2)^3 Z = 6e^{2x} \sin(3x+y)$

Soln Repeated factors $(D - 3D' - 2)^3$ which gives

$$m_1 = m_2 = m_3 = 3 \quad a_1 = a_2 = a_3 = 2$$

$$C.F. = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x) \checkmark$$

$$P.I. = \frac{1}{F(D, D')} \left[6e^{2x} \sin(3x+y) \right] = \frac{6 \cdot 1}{(D - 3D' - 2)^3} e^{2x+0y} \sin(3x+y)$$

$$= 6 \cdot \frac{e^{2x+0y}}{\{(D+2)-3(D+0)-2\}^3} \cdot \sin(3x+y)$$

$$= 6 \cdot e^{2x} \cdot \frac{1}{(D - 3D' - 1)^3} \sin(3x+y) \quad f(ax+by)$$

$$= 6x e^{2x} \cdot \frac{1}{3(D - 3D')^2} \sin(3x+y)$$

$$= 6x^2 e^{2x} \cdot \frac{1}{6(D - 3D')} \sin(3x+y)$$

$$= 6x^3 e^{2x} \cdot \frac{1}{3 \cdot 3x^0} \sin(3x+y) = x^3 e^{2x} \sin(3x+y) \checkmark \text{ Ans.}$$

$$Z = e^{2x} f_1(y+3x) + x e^{2x} f_2(y+3x) + x^2 e^{2x} f_3(y+3x) + x^3 e^{2x} \sin(3x+y) \text{ Ans.}$$

Ex $(D^2 - DD' + D' - 1)Z = \cos(x+2y) + e^y$

$$\therefore D^2 - DD' + D' - 1 = D(D-1) + (D^2-1) - (D-1) = (D-1)\{D+1-D'\}$$

$$\Rightarrow (D - D' + 1)(D - 1) = 0$$

$$\Rightarrow (D - D' + 1) = 0$$

$$m_1 = +1 \quad a_1 = -1$$

$$\text{part of C.F.} = f_1(y+x) x \cdot e^{-x}$$

$$C.F. = e^{-x} f_1(y+x) + f_2(y) e^x \checkmark$$

$$D \rightarrow D+a (=2)$$

$$D' \rightarrow D+b (=0)$$

$$D-3D' \text{ Homog. (1st case)}$$

$$\text{Now } D \rightarrow 3 \quad D' \rightarrow 1$$

$$\text{But } D-3D'=0$$

$$\Rightarrow \text{case failed.}$$

$$\text{Again case failed}$$

$$\text{Again case failed}$$

$$D-1=0 \Rightarrow D=1 \quad m_2=0 \quad a_2=1$$

$$\Rightarrow \text{part of C.F.} = f_2(y) e^x$$

$$P \cdot I = \frac{1}{D^2 - DD' + D' - 1} \cos(x+2y) + e^y = P_1 + P_2 \quad D^2 \rightarrow -a^2, D' \rightarrow -b^2, D \rightarrow -at$$

$$P_1 = \frac{1}{D^2 - DD' + D' - 1} \cos(x+2y) = \frac{1}{-1+2+D'-1} \cos(x+2y) = \frac{1}{D'} \cos(x+2y)$$

$$= +2 \sin(x+2y)$$

$$P_2 = \frac{1}{D^2 - DD' + D' - 1} e^y = \frac{1}{D^2 - DD' + D' - 1} e^y \times 1 \text{ when } D \rightarrow 0 + 0$$

$$= \cancel{e^y} \cancel{1} \quad \cancel{D^2 - DD' - D + D' - 2} = \frac{1}{0 - 0 + 1 - 1} \cancel{e^y} \text{ Care failed}$$

$$(D+0)^2 - (D+0)(D'+1) - (D'+1) - 1$$

$$= \cancel{e^y} \frac{1}{\cancel{D^2 - DD' - D + D' - 2}} = \cancel{e^y} \frac{1}{\cancel{D(D-D')}} \quad e^y = 0$$

$$e^y = 0 \quad \text{or } +1$$

$$P_2 = \frac{1}{D^2 - DD' + D' - 1} e^y = y \cdot \frac{1}{0 - D + 1 - 0} e^y = y \cdot \frac{e^y}{-0+1} = ye^y$$

Care failed

$$\text{Hence soln } z = e^{-x} f_1(y+x) + e^x f_2(y) + ye^y$$

Note $P_2 = \frac{1}{D^2 - DD' + D' - 1} e^y = x \cdot \frac{1}{2D - D'} e^y = x \cdot \frac{1}{2x0 - 1} e^y = -xe^y$

Care failed

$$\text{so } z = e^{-x} f_1(y+x) + e^x f_2(y) \neq xe^y \quad \text{Ans}$$

Ex: $9 - 4S + 4d + P - 2g = e^{x+y}$

Soln $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$

$$\{(D-2D')^2 + (D-2D')\}z = e^{x+y}$$

$$(D-2D'+1)(D-2D')z = e^{x+y}$$

$$m_1 = 2 \quad a_1 = -1 \quad m_2 = 2 \quad a_2 = 0$$

$$CF = e^{-x} f_1(y+2x) + f_2(y+2x) \checkmark$$

$$P \cdot I = \frac{1}{D^2 - 4DD' + 4D'^2 + D - 2D'} e^{x+y} \quad \text{Care failed}$$

$$= x \cdot \frac{1}{2D - 4D' + 1} e^{x+y} = x \cdot \frac{1}{2-4+1} e^{x+y}$$

$$= -x \cdot e^{x+y} \checkmark$$

$$z = e^{-x} f_1(y+2x) + f_2(y+2x) - x e^{x+y} \checkmark$$

Ex. $(D^2 - D')Z = xe^{ax+a^2y}$; find P.T. $\phi(x,y) = e^{ax+by}$

P.T. $= \frac{1}{(D-a)^2} xe^{ax+a^2y} = D \rightarrow D+a, D' = D'+a^2$

$$\begin{aligned}
 &= e^{ax+a^2y} \frac{x}{\frac{D^2 - D'}{(D+a)^2 - (D'+a^2)}} = e^{ax+a^2y} \cdot \frac{x}{D^2 + 2aD + a^2 - D' - a^2} \\
 &= e^{ax+a^2y} \cdot \frac{x}{D^2 + 2aD - D'} \\
 &= e^{ax+a^2y} \cdot \frac{x}{+2aD(\frac{1}{1} + (\frac{D^2 - D'}{2aD}))} = \frac{e^{ax+a^2y}}{2a} \cdot \frac{x}{1 + \frac{D^2 - D'}{2aD}} \\
 &= -\frac{e^{ax+a^2y}}{2a} \times \frac{x}{D} \left\{ 1 - \frac{D^2 - D'}{2aD} \right\} \text{ leaving Higher Derivative term} \\
 &= \frac{e^{ax+a^2y}}{2a} \times \frac{x}{D} \left\{ 1 - \frac{D}{2a} + \frac{D'}{2aD} \right\} x \\
 &= -\frac{e^{ax+a^2y}}{2a} \times \frac{x}{D} \left\{ x - \frac{1}{2a} + 0 \right\} \\
 &= +\frac{e^{ax+a^2y}}{2a} \times \left\{ \frac{x^2}{2} - \frac{x}{2a} \right\} \\
 &= e^{ax+a^2y} \left\{ \frac{x^2}{4a} - \frac{x}{4a^2} \right\} = e^{ax+a^2y} \times \frac{(ax^2 - x)}{4a^2}
 \end{aligned}$$

$$(D^2 - D'^2 - 3D + 3D')Z = xy + e^{x+2y} + \beta \sin(2x+3y) \quad \checkmark$$

$$\text{C.F. } (D^2 - D'^2 - 3D + 3D')Z = 0$$

$$(D - D')(D + D') - 3(D - D') = 0 \Rightarrow (D - D')(D + D' - 3) = 0$$

$$m_1 = 1 \quad \alpha_1 = 0 \quad m_2 = -1 \quad \alpha_2 = 3$$

$$\text{C.F. } = f_1(y+x) + e^{3x} f_2(y-x)$$

$$\text{P.I.} = P_1 + P_2 + P_3$$

$$P_1 = \frac{1}{(D - D')(D + D' - 3)} xy = \frac{1}{D} \left\{ 1 - \frac{D'}{D} \right\}^{-1} \left\{ 1 - \frac{D + D'}{3} \right\}^{-1} xy$$

$$= -\frac{1}{3D} \left\{ 1 + \frac{D'}{D} \right\} \left\{ 1 + \frac{D + D'}{3} \right\} xy = \cancel{\frac{1}{3D} \left\{ 1 + \frac{D + D'}{3} \right\} \left\{ 1 + \frac{D + D'}{D} \right\} xy}$$

$$= -\frac{1}{3D} \left\{ 1 + \frac{D + D'}{3} + \frac{D^2 + D' + 2DD'}{9} \right\} xy$$

$$= -\frac{1}{3D} \left\{ 1 + \frac{D + D'}{3} + \frac{2DD' + D' + \frac{1}{3}D^2 + \frac{D'^2}{3D} + \frac{D'(D+D')^2}{3}}{9} \right\} xy$$

$$= xy$$

$$= -\frac{1}{3D} \left\{ 1 + \frac{D + D'}{3} + \frac{2DD' + D' + \frac{1}{3}D^2}{9} \right\} xy$$

$$= -\frac{1}{3D} \left\{ 1 + \frac{D}{3} + \frac{2D'}{3} + \frac{2DD' + D'}{9} \right\} xy$$

$$= -\frac{1}{3D} \left\{ xy + \frac{1}{3}y + \frac{2}{3}x + \frac{2}{9} + \frac{1}{9}x \right\}$$

$$= -\frac{1}{3D} \left\{ xy + \frac{y}{3} + \frac{2x}{3} + \frac{2}{9} + \frac{x^2}{2} \right\}$$

$$= \frac{1}{3} \left\{ \frac{x^2}{2}y + \frac{xy}{3} + \frac{1}{3}x^2 + \frac{2}{9}x + \frac{x^3}{6} \right\}$$

$$P_2 = \frac{1}{(D - D')(D + D' - 3)} e^{x+2y} \quad \text{as if failed}$$

$$= x \cdot \frac{1}{2D - 0 - 3 + 0} e^{x+2y}$$

$$= x \cdot \frac{1}{2x - 3} e^{x+2y} = -x e^{x+2y} \quad \checkmark$$

P3 For practice.

$$2s+t-3q = 5 \cos(3x-2y) + e^{3x-2y}$$

$$2D^2 + (2DD' + D'^2 - 3D')z = 5 \cos(3x-2y); D = \frac{\partial}{\partial x}, D' = \frac{\partial}{\partial y}$$

$$D'(2D + D' - 3)z = 5 \cos(3x-2y)$$

$$D'(D + \frac{D'}{2} - \frac{3}{2})z = \frac{5}{2} \cos(3x-2y) + \frac{e^{3x-2y}}{2}$$

$$D' = 0$$

$$D + \frac{D'}{2} - 3 = 0$$

part of complex funct.

$$= f_1(x)$$

$$m = -\frac{1}{2}, a = \frac{3}{2}$$

$$= f_2(y - \frac{1}{2}x) + \frac{e^{3/2x}}{2}$$

$$\text{C.F.} = f_1(x) + f_2(y - \frac{1}{2}x) \times e^{\frac{3}{2}x}$$

$$\text{P.T.} = \frac{1}{2DD' + D'^2 - 3D'} \left\{ 5 \cos(3x-2y) + e^{3x-2y} \right\}$$

$$= \frac{1}{2(6) + (-4) - 3D'} \left\{ 5 \cos(3x-2y) + \frac{1}{2(-6) + 4 + 6} e^{3x-2y} \right\}$$

$$= 5 \frac{1}{8 - 3D'} \cos(3x-2y) + \frac{1}{(-2)} e^{3x-2y}$$

$$DD' = -6$$

$$= 5 \cdot \left\{ \frac{8 + 3D'}{(8 - 3D')(8 + 3D')} \cos(3x-2y) \right\} - \frac{1}{2} e^{3x-2y}$$

$$= 5 \cdot \left\{ \frac{8 + 3D'}{64 - 9D'^2} \cos(3x-2y) \right\} - \frac{1}{2} e^{3x-2y}$$

$$= 5 \left\{ \frac{8 + 3D'}{64 - 9(-4)} \cos(3x-2y) \right\} - \frac{1}{2} e^{3x-2y}$$

$$= \frac{5}{100} \left\{ 8 \cos(3x-2y) + 3(-2) \sin(3x-2y) \right\}$$

$$+ -\frac{1}{2} e^{3x-2y}$$

$$= \frac{1}{20} \left\{ 8 \cos(3x-2y) + 6 \sin(3x-2y) \right\}$$

$$- \frac{1}{2} e^{3x-2y}.$$

$$\text{Ex } (D^2 - D')z = \cos(x-3y) + e^{3y}$$

This is non-homog. Linear PDE; factorization is not possible. Then to find C.F., we consider

$$Z = Ae^{hx+ky}$$

$$D^2 z = Ah^2 e^{hx+ky} \quad D' = KAe^{hx+ky}$$

$$\text{from given PDE; } Ah^2 e^{hx+ky} - AKe^{hx+ky} = 0$$

$$A(h^2 - K)e^{hx+ky} = 0; \quad A \neq 0$$

$$\Rightarrow A \neq 0, \Rightarrow h^2 - K = 0 \Rightarrow K = h^2.$$

$$\text{Hence } C.F. = Ae^{hx+h^2y}$$

$$\text{General soln C.F.} = \sum Ae^{hx+h^2y}; \quad h^2 - K = 0$$

$$\text{P.I.} = \frac{1}{D^2 - D'} \left\{ \cos(x-3y) + e^{3y} \right\}_{0x+3y}$$

$$\begin{aligned} D^2 &\rightarrow -a^2 \\ &= \frac{1}{-1 - D'} \cos(x-3y) + \frac{1}{0-3} e^{3y} \\ &= \frac{-1 (1-D')}{(1+D')(1-D')} \cos(x-3y) - \frac{1}{3} e^{3y} \\ &= + \frac{(D'-1)}{1 - D'^2} \cos(x-3y) - \frac{1}{3} e^{3y} \end{aligned}$$

$$= \frac{+38 \sin(x-3y)}{1-(-9)} - \frac{1}{3} e^{3y} - \frac{\cos(x-3y)}{10}$$

$$= \frac{3}{10} 8 \sin(x-3y) - \frac{1}{3} e^{3y} - \frac{\cos(x-3y)}{10}$$

$$Z = \sum Ae^{hx+h^2y} + \frac{3}{10} 8 \sin(x-3y) - \frac{\cos(x-3y)}{10}$$

$$- \frac{1}{3} e^{3y} \underline{\text{Ans}}$$

$$\text{where } h - k^2 = 0$$

$$D(D-2D')(D+D')Z = e^{x+2y}(x^2+y^2)$$

C.F. $D=0$ $D-2D'=0$ $D+D'=0$

part of C.F.: $f_1(y)$, $f_2(y+2x)$, $f_3(y-x)$

$\left. \begin{array}{l} \text{Non Homog} \\ \xrightarrow{\text{Solve}} \text{Homog.} \\ \text{Reverse not} \\ \text{possible} \end{array} \right\}$

C.F. = $F_1(y) + F_2(y+2x) + F_3(y-x)$

P.I. = $\frac{1}{D(D-2D')(D+D')} e^{x+2y}(x^2+y^2)$

$D \rightarrow D+1, \quad D' \rightarrow D+2$

= $\frac{1}{(D+1)(D+1-2(D'+2))(D+1+D'+2)} e^{x+2y}(x^2+y^2)$

= $e^{x+2y} \frac{1}{(D+1)(D-2D'-3)(D+D'+3)} (x^2+y^2)$

= $e^{x+2y} \cdot \left\{ 1+D \right\}^{-1} \times \frac{1}{(-3)} \left(1 - \frac{D-2D'}{3} \right)^{-1} \times \frac{1}{3} \left(1 + \frac{D+D'}{3} \right)^{-1} (x^2+y^2)$