Applying the condition $y(0) = 2$, we get $\quad 2 = c_1$

Applying the condition $y\left(\frac{\pi}{2}\right) = -2$, we get $\quad -2 = c_2$

$\therefore$ Hence from (1), the particular solution is

$$y = 2(\cos x - \sin x)$$

TEST YOUR KNOWLEDGE

Solve the differential equations :

1. $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = 0$

2. $\frac{d^2y}{dx^2} + (a+b)\frac{dy}{dx} + aby = 0$

3. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$

4. $\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 9x = 0$

5. $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - y = 0$

6. $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$

7. $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$

8. $\frac{d^4y}{dx^4} + 8\frac{d^2y}{dx^2} + 16y = 0$

9. $\frac{d^5y}{dx^5} - \frac{d^3y}{dx^3} = 0$ *(U.P.T.U. 2009)*

10. $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0$

11. $(D^2 + 1)^2 (D - 1)\, y = 0$

12. $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + y = 0$

13. $(D^6 - 1)\, y = 0$

14. $(D^6 + 1)\, y = 0$

15. $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, given that when $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$

16. $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 8y = 0$ under the conditions $y(0) = 0$, $y'(0) = 0$ and $y''(0) = 2$

[G.B.T.U.(AG) SUM

Answers

1. $y = c_1 e^{3x} + c_2 e^{4x}$

2. $y = c_1 e^{-ax} + c_2 e^{-bx}$

3. $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

4. $x = (c_1 + c_2 t)e^{-3t}$

5. $y = (c_1 + c_2 x + c_3 x^2)e^x$

6. $y = (c_1 + c_2 x)\, e^x + c_3\, e^{-x}$

7. $y = e^{-x}(c_1 + c_2 x + c_3 x^2) + c_4 e^{4x}$

8. $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$

9. $y = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x} + c_5 e^x$

10. $y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$

11. $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x + c_5 e^x$

12. $y = e^{2x}(c_1 \cosh\sqrt{3}\,x + c_2 \sinh\sqrt{3}\,x)$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$

2. $\dfrac{d^2y}{dx^2} - 4y = (1 + e^x)^2$

3. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$

4. $(2D + 1)^2 y = 4e^{-x/2}$

5. $(D^2 - 2kD + k^2)\, y = e^{kx}$

6. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}$

7. $(D + 2)(D - 1)^3 y = e^x$

8. $\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 240y = 272\, e^{-x}$

9. $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\, y = e^{2x}$

10. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$

11. $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$

12. $y'' + 4y' + 13y = 18e^{-2x};\ y(0) = 0,\ y'(0) = 9.$

Answers

1. $y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$

2. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$

3. $y = e^{-2x}(c_1 \cos x + c_2 \sin x) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$

4. $y = \left(c_1 + c_2 x + \dfrac{x^2}{2}\right)e^{-x/2}$

5. $y = (c_1 + c_2 x)\, e^{kx} + \dfrac{x^2}{2}e^{kx}$

6. $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + e^{-x} \cdot \dfrac{x^3}{6}$

7. $y = (c_1 + c_2 x + c_3 x^2)\, e^x + c_4 e^{-2x} + \dfrac{x^3 e^x}{18}$

8. $y = c_1 e^{-15x} + c_2 e^{-16x} + \dfrac{136}{105}e^{-x}$

9. $y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + \dfrac{e^{2x}}{(2 + p)^2 + q}$

10. $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}\left[c_3 \cos\dfrac{\sqrt{7}}{2}x + c_4 \sin\dfrac{\sqrt{7}}{2}x\right] + \dfrac{1}{8}xe^x$

11. $y = c_1 e^{-x} + e^{x/2}\left(c_2 \cos\dfrac{\sqrt{3}}{2}x + c_3 \sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{9}e^{2x} + \dfrac{1}{3}xe^{-x}$

12. $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2).$

1.29.2. Case II. When Q = sin (ax + b) or cos (ax + b)

$$D \sin(ax + b) = a\cos(ax + b)$$

$$D^2 \sin(ax + b) = (-a^2)\sin(ax + b)$$

$$D^3 \sin(ax + b) = -a^3 \cos(ax + b)$$

Applying the condition ...

$$3 = c_1 + 6 \Rightarrow c_1 = -3$$

From (1),
$$\frac{dy}{dx} = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x) - e^{-x}(c_1 \cos 3x + c_2 \sin 3x) - 18 \sin 3x - 3 \cos 3x \qquad \ldots(2)$$

Applying the condition $\frac{dy}{dx} = 0$ when $x = 0$ in (2),

$$0 = 3c_2 - c_1 - 3 \qquad (\because \; c_1 = -3)$$

$$\Rightarrow \quad 0 = 3c_2$$

$$\Rightarrow \quad c_2 = 0$$

Substituting the values of c_1 and c_2 in equation (1), we get

$$y = (6 - 3e^{-x}) \cos 3x - \sin 3x$$

when $x = \frac{\pi}{2}$,
$$y = -\sin \frac{3\pi}{2} = 1$$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\frac{d^3y}{dx^3} + a^2 \frac{dy}{dx} = \sin ax$
2. $(D^2 - 4D + 3)y = \sin 3x \cos 2x$
3. $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$ [*U.P.T.U. (B. Pharm.) 2009, 2010*]
4. (*i*) $(D^2 + 9)\, y = \cos 2x + \sin 2x$ (*ii*) $(D^2 + 5D - 6)\, y = \sin 3x + \cos 2x$ [*G.B.T.U. 2010; G.B.T.U. (C.O.) 2011*]
5. $\frac{d^2y}{dx^2} + 2k \frac{dy}{dx} + k^2 y = a \cos px$
6. $(D^2 - 8D + 9)\, y = 40 \sin 5x$
7. $(D^2 - 4D + 4)y = e^{-4x} + 5 \cos 3x$
8. $(D^4 + 2D^3 - 3D^2)y = 3e^{2x} + 4 \sin x$
9. $(D^2 - 4D - 5)y = e^{2x} + 3 \cos (4x + 3)$
10. $(D^2 + 5D - 6)y = \sin 4x \sin x$ (*U.P.T.U. 2008*)
11. $(D^2 + 4)y = \cos x \cos 3x$ [*U.P.T.U. (SUM) 2009*]
12. $(D^4 + 2D^2n^2 + n^4)y = \cos mx \; ; \; m \neq n.$

Answers

1. $y = c_1 + c_2 \cos ax + c_3 \sin ax - \frac{x}{2a^2} \sin ax$
2. $y = c_1 e^x + c_2 e^{3x} + \frac{1}{884}(10 \cos 5x - 11 \sin 5x) + \frac{1}{20}(\sin x + 2 \cos x)$
3. $y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5} e^x - \frac{x}{4} \cos 2x$
4. (*i*) $y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}(\cos 2x + \sin 2x)$

   (*ii*) $y = c_1 e^x + c_2 e^{-6x} - \frac{1}{30}(\cos 3x + \sin 3x) + \frac{1}{20}(\sin 2x - \cos 2x)$

$$\text{P.I.} = \frac{}{D^2 - 1}(1)$$

$$= -(1 - D^2)^{-1}(1) = -(1 + D^2 + \ldots)(1) = -1$$

$\therefore$ General solution is

$$y = c_1 e^x + c_2 e^{-x} - 1 \quad \ldots(1)$$

when $x = 0, y = 0$

$\therefore$ From (1), $0 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 1 \quad \ldots(2)$

Also, y tends to a finite limit as $x \to -\infty$

This condition will be satisfied only when $c_2 = 0$

$\therefore$ From (2), $c_1 = 1$

Hence from (1), Particular solution is $y = e^x - 1$.

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^2y}{dx^2} - 4y = x^2$

2. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 3y = \cos x + x^2$

3. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 2y = x + \sin x$

4. $\dfrac{d^3y}{dx^3} + 2\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = e^{2x} + x^2 + x$

5. $(D^5 - D)y = 12e^x + 8 \sin x - 2x$

6. $\dfrac{d^2y}{dx^2} + y = e^{2x} + \cosh 2x + x^3$

7. $(D^3 + 8)y = x^4 + 2x + 1$

8. $(D^2 + 2D + 1)\, y = 2x + x^2$

9. $(D^2 + D - 6)y = x$

10. $(D^3 + 3D^2 + 2D)\, y = x^2$

11. $(D^6 - D^4)\, y = x^2$

12. $(D^2 - 1)\, y = 2x^4 - 3x + 1$ [U.P.T.U. (B.Pharm.) SUM 2009]

13. $(D^2 - 4D + 4)\, y = x^2 + e^x + \cos 2x$

14. $\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4y = x^2 + e^x$. [U.P.T.U. (B.Pharm.) SUM 2010]

Answers

1. $y = c_1 e^{2x} + c_2 e^{-2x} - \dfrac{1}{4}\left(x^2 + \dfrac{1}{2}\right)$

2. $y = e^x (c_1 \cos\sqrt{2}\, x + c_2 \sin \sqrt{2}\, x) + \dfrac{1}{4}(\cos x - \sin x) + \dfrac{1}{27}(9x^2 + 12x + 2)$

3. $y = c_1 e^x + c_2 e^{-2x} - \dfrac{1}{10}(\cos x + 3 \sin x) - \dfrac{1}{4}(2x + 1)$

4. $y = c_1 + (c_2 + c_3 x)\, e^{-x} + \dfrac{1}{3}x^3 - \dfrac{3}{2}x^2 + 4x + \dfrac{1}{18}e^{2x}$

5. $y = c_1 + (c_2 + 3x)\, e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + x^2 + 2x \sin x$

6. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{5}e^{2x} + \dfrac{1}{5}\cosh 2x + x^3 - 6x$

TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $(D - a)^2 y = e^{ax} f''(x)$
2. $(D^2 - 2D)y = e^x \sin x$
3. $(D^2 - 4D + 4)y = e^x \cos x$ [G.B.T.U. (C.O.) 2010]
4. $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^{2x}\cos x$ [M.T.U. (B. Pharm.) 2011]
5. $(D^2 - 2D + 5)y = e^{2x}\sin x$ [M.T.U. (AG) 2011]
6. $\dfrac{d^2y}{dx^2} + y = e^{-x} + \cos x + x^3 + e^x \sin x$
7. (*i*) $(D^2 - 3D + 2)\, y = xe^x + \sin 2x$ (U.P.T.U. 2008) (*ii*) $(D^2 - 1)\, y = xe^x + \cos^2 x$ [U.P.T.U. (SUM) 2007]
8. $(D - 1)^2 (D^2 + 1)^2 y = \sin^2 \dfrac{x}{2} + e^x + x$
9. $(D^2 - 6D + 13)y = 8e^{3x}\sin 4x + 2^x$
10. $(D^2 + 4)y = e^x \sin^2 x$
11. $(D^3 + 2D^2 + D)y = x^2 e^{2x} + \sin^2 x$
12. $(D^2 - 4D + 3)y = e^x \cos 2x + \cos 3x$
13. $(D^2 + 4D + 8)y = 12e^{-2x}\sin x \sin 3x$
14. $\dfrac{d^2y}{dx^2} + 2y = x^2 e^{3x} + e^x \cos 2x$
15. $(D - 1)^2 y = e^x \sec^2 x \tan x$
16. $(D^3 - 7D - 6)y = (x + 1)\, e^{2x}$
17. $(D^2 - 1)\, y = x^2 \cos x$ [U.P.T.U. (SUM) 2009]
18. $(D^2 - 1)\, y = x \sin x + x^2 e^x$
19. $(D^2 - 2D + 1)\, y = x \sin x$ (U.K.T.U. 2012)
20. $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = x^2 e^{-x}\cos x$ (G.B.T.U. 2012)

Answers

1. $y = e^{ax}\, [c_1 + c_2 x + f(x)]$
2. $y = c_1 + c_2 e^{2x} - \dfrac{1}{2} e^x \sin x$
3. $y = (c_1 + c_2 x)\, e^{2x} - \dfrac{e^x}{2} \sin x$
4. $y = e^x (c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x) + \dfrac{1}{13} e^{2x}\, (2 \sin x + 3 \cos x)$
5. $y = e^x\, (c_1 \cos 2x + c_2 \sin 2x) + \dfrac{e^{2x}}{10}\, (2 \sin x - \cos x)$
6. $y = c_1 \cos x + c_2 \sin x + \dfrac{1}{2} e^{-x} + \dfrac{1}{2} x \sin x + x^3 - 6x - \dfrac{1}{5} e^x\, (2 \cos x - \sin x)$
7. (*i*) $y = c_1 e^x + c_2 e^{2x} - e^x \left(\dfrac{x^2}{2} + x\right) + \dfrac{1}{20}\, (3 \cos 2x - \sin 2x)$

   (*ii*) $y = c_1 \ldots (x^2 - x) - \dfrac{1}{2} - \dfrac{1}{10} \cos 2x$

$\ldots \cos x + (c_5 + c_6 x) \sin x + \dfrac{1}{2} - \dfrac{x^2}{32} \sin x + \dfrac{x^2}{8} e^x + x + 2$