TEST YOUR KNOWLEDGE

Solve the following differential equations :

1. $\dfrac{d^3y}{dx^3} + y = 3 + 5e^x$
2. $\dfrac{d^2y}{dx^2} - 4y = (1 + e^x)^2$
3. $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 5y = -2\cosh x$
4. $(2D + 1)^2 y = 4e^{-x/2}$
5. $(D^2 - 2kD + k^2)\, y = e^{kx}$
6. $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}$
7. $(D + 2)(D - 1)^3 y = e^x$
8. $\dfrac{d^2y}{dx^2} + 31\dfrac{dy}{dx} + 240y = 272\,e^{-x}$
9. $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\, y = e^{2x}$
10. $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$
11. $\dfrac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$
12. $y'' + 4y' + 13y = 18e^{-2x}$; $y(0) = 0$, $y'(0) = 9$.

Answers

1. $y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\dfrac{\sqrt{3}}{2}x + c_3\sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{2}e^x$
2. $y = c_1e^{2x} + c_2e^{-2x} - \dfrac{1}{4} - \dfrac{2}{3}e^x + \dfrac{1}{4}xe^{2x}$
3. $y = e^{-2x}(c_1\cos x + c_2\sin x) - \dfrac{1}{10}e^x - \dfrac{1}{2}e^{-x}$
4. $y = \left(c_1 + c_2x + \dfrac{x^2}{2}\right)e^{-x/2}$
5. $y = (c_1 + c_2x)\,e^{kx} + \dfrac{x^2}{2}e^{kx}$
6. $y = (c_1 + c_2x + c_3x^2)\,e^{-x} + e^{-x}\cdot\dfrac{x^3}{6}$
7. $y = (c_1 + c_2x + c_3x^2)\,e^x + c_4\,e^{-2x} + \dfrac{x^3e^x}{18}$
8. $y = c_1e^{-15x} + c_2e^{-16x} + \dfrac{136}{105}e^{-x}$
9. $y = e^{-px}(c_1\cos qx + c_2\sin qx) + \dfrac{e^{2x}}{(2+p)^2 + q^2}$
10. $y = c_1\,e^x + c_2e^{-x} + e^{-x/2}\left[c_3\cos\dfrac{\sqrt{7}}{2}x + c_4\sin\dfrac{\sqrt{7}}{2}x\right] + \dfrac{1}{8}xe^x$
11. $y = c_1\,e^{-x} + e^{x/2}\left(c_2\cos\dfrac{\sqrt{3}}{2}x + c_3\sin\dfrac{\sqrt{3}}{2}x\right) + 3 + \dfrac{5}{9}e^{2x} + \dfrac{1}{3}xe^{-x}$
12. $y = e^{-2x}(-2\cos 3x + 3\sin 3x + 2)$.

1.29.2. Case II. When $Q = \sin(ax + b)$ or $\cos(ax + b)$

$$D\sin(ax + b) = a\cos(ax + b)$$

$$D^2\sin(ax + b) = (-a^2)\sin(ax + b)$$

$$D^3\sin(ax + b) = -a^3\cos(ax + b)$$

$$D^4\sin(ax + b) = a^4\sin(ax + b)$$

or $\qquad (D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$

$\qquad$ ....

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In general, $\qquad (D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$

$\therefore \qquad f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$

Operating on both sides by $\dfrac{1}{f(D^2)}$,

$$\frac{1}{f(D^2)}\{f(D^2)\sin(ax+b)\} = \frac{1}{f(D^2)}\{f(-a^2)\sin(ax+b)\}$$

or
$$\sin(ax+b) = f(-a^2)\frac{1}{f(D^2)}\sin(ax+b).$$

Dividing both sides by $f(-a^2)$,

$$\boldsymbol{\frac{1}{f(D^2)}\sin(ax+b) = \frac{1}{f(-a^2)}\sin(ax+b), \text{ provided } f(-a^2) \neq 0}$$

Similarly, $\quad \boldsymbol{\dfrac{1}{f(D^2)}\cos(ax+b) = \dfrac{1}{f(-a^2)}\cos(ax+b), \text{ provided } f(-a^2) \neq 0}$

Steps : When $\quad Q = \sin(ax+b)$ or $\cos(ax+b)$,

1. Replace $\qquad D^2$ by $-a^2$,

D^4 by a^4,

D^6 by $-a^6$,

D^8 by a^8 $\quad$ and so on.

2. By doing so, following possibilities arise :

(*a*) If denominator reduces to a constant, it will be final step in finding P.I.

(*b*) If denominator reduces into D only, we are then only to integrate the given function Q once.

(*c*) If denominator reduces to a factor of the form $\alpha D + \beta$ then we operate by its conjugate $\alpha D - \beta$ on both numerator and denominator from left hand side such as

$$\frac{\alpha D - \beta}{\alpha D - \beta}\cdot\left[\frac{1}{(\alpha D + \beta)}\sin(ax+b)\right]$$

By doing so, denominator will become $\alpha^2 D^2 - \beta^2$ which in turn reduces to a constant by replacing D^2 by $-a^2$.

Now, we operate $\sin(ax+b)$ by $(\alpha D - \beta)$ and consequently, find the required particular integral.

Case of failure : If $f(-a^2) = 0$, the above method fails. Then we proceed as follows :

$$\boldsymbol{\frac{1}{f(D^2)}\cos(ax+b) = x\,.\,\frac{1}{f'(D^2)}\cos(ax+b), \text{ provided } f'(-a^2) \neq 0}$$

$$\frac{1}{f(D^2)} \sin(ax+b) = x \cdot \frac{1}{f'(D^2)} \sin(ax+b), \text{ provided } f'(-a^2) \neq 0$$

If $f'(-a^2) = 0$, then

$$\frac{1}{f(D^2)} \sin(ax+b) = x^2 \cdot \frac{1}{f''(D^2)} \sin(ax+b), \text{ provided } f''(-a^2) \neq 0$$

$$\frac{1}{f(D^2)} \cos(ax+b) = x^2 \cdot \frac{1}{f''(D^2)} \cos(ax+b), \text{ provided } f''(-a^2) \neq 0$$

and so on.

Steps :

1. When $f(-a^2) = 0$, we differentiate the denominator w.r.t. D and multiply the expression by x simultaneously in the same step.
2. When $f'(-a^2) = 0$ (*i.e.*, step 1 fails) we again differentiate the reduced denominator in D w.r.t. D and again multiply the remaining expression by x simultaneously.
3. If there is another case of failure, above process is to be repeated again and again until we reach a constant in the denominator or any other possibility(ies) which we have discussed before in the same article.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following differential equation:*

$(D^2 + 4)y = \sin 3x + \cos 2x.$ [U.P.T.U. (SUM) 2008]

Sol. Auxiliary equation is

$$m^2 + 4 = 0$$

$$\Rightarrow \quad m = \pm 2i$$

$$\therefore \quad \text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}(\sin 3x) + \frac{1}{D^2+4}(\cos 2x)$$

$$= \frac{1}{-(3)^2+4} \sin 3x + x \cdot \frac{1}{2D}(\cos 2x)$$

$$= -\frac{1}{5}\sin 3x + \frac{x}{2}\left(\frac{\sin 2x}{2}\right) = -\frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{5}\sin 3x + \frac{x}{4}\sin 2x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 2. *Find the P.I. of $(D^3 + 1)y = \sin(2x + 1)$.*

Sol. $$\text{P.I.} = \frac{1}{D^3+1}\sin(2x+1) = \frac{1}{D(-2^2)+1}\sin(2x+1)$$ [Putting $D^2 = -2^2$]

$$= \frac{1}{1-4D}\sin(2x+1)$$

Operating N^r and D^r by $(1 + 4D)$

$$= \frac{1+4D}{(1+4D)(1-4D)}\sin(2x+1) = \frac{1+4D}{1-16D^2}\sin(2x+1)$$

$$= \frac{1+4D}{1-16(-2^2)}\sin(2x+1)$$ [Putting $D^2 = -2^2$]

$$= \frac{1}{65}[\sin(2x+1) + 4D\sin(2x+1)]$$

$$= \frac{1}{65}[\sin(2x+1) + 8\cos(2x+1)]$$ $\left[\because D \equiv \frac{d}{dx}\right]$

Example 3. *Solve the following differential equations:*

(*i*) $\frac{d^2y}{dx^2} + a^2y = \sin ax$ (U.P.T.U. 2008)

(*ii*) $(D^2 + 4)y = \cos^2 x$. [M.T.U. (B. Pharm.) 2011]

Sol. (*i*) The auxiliary equation is

$$m^2 + a^2 = 0 \Rightarrow m = \pm ai$$

$\therefore$ $$\text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2}(\sin ax) = x \cdot \frac{1}{2D}\sin ax$$

$$= \frac{x}{2}\left[\frac{-\cos ax}{a}\right] = -\frac{x}{2a}\cos ax$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1\cos ax + c_2 \sin ax - \frac{x}{2a}\cos ax$$

here c_1 and c_2 are arbitrary constants of integration.

(*ii*) The auxiliary equation is

$$m^2 + 4 = 0 \Rightarrow m = \pm 2i$$

$\therefore$ $$\text{C.F.} = c_1\cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\cos^2 x = \frac{1}{2}\left[\frac{1}{D^2+4}(1+\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}(e^{0x}) + \frac{1}{D^2+4}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + x \cdot \frac{1}{2D}(\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{x}{4}\sin 2x\right] = \frac{1}{8}(1 + x\sin 2x)$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(1 + x\sin 2x)$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve :* $\dfrac{d^4 y}{dx^4} - m^4 y = \cos mx.$

Sol. Auxiliary equation is

$$M^4 - m^4 = 0$$

$$(M^2 - m^2)(M^2 + m^2) = 0$$

$$M = \pm m, \pm mi$$

$$\therefore \quad \text{C.F.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$$

$$\text{P.I.} = \frac{1}{D^4 - m^4}(\cos mx) = x \cdot \frac{1}{4D^3}\cos mx$$

$$= \frac{x}{4} \cdot \frac{1}{D^2}\left(\frac{\sin mx}{m}\right) = -\frac{x}{4m^2}\left(\frac{\sin mx}{m}\right) = -\frac{x}{4m^3}\sin mx$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx - \frac{x}{4m^3}\sin mx$$

where c_1, c_2, c_3 and c_4 are arbitrary constants of integration.

Example 5. *Solve :* $\dfrac{d^3 y}{dx^3} - 3\dfrac{d^2 y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x + \cos x.$ (U.P.T.U. 2006)

Sol. Auxiliary equation is

$$m^3 - 3m^2 + 4m - 2 = 0$$

$$\Rightarrow \quad (m^2 - 2m + 2)(m - 1) = 0 \quad \Rightarrow \quad m = 1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{(D^3 - 3D^2 + 4D - 2)} e^x + \frac{1}{(D^3 - 3D^2 + 4D - 2)} \cos x$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4}(e^x) + \frac{1}{(-D + 3 + 4D - 2)}(\cos x)$$

$$= x \cdot \frac{1}{(7-6)} e^x + \frac{1}{3D+1} (\cos x) = x\, e^x + \frac{3D-1}{9D^2-1} (\cos x)$$

$$= xe^x - \frac{1}{10} (-3 \sin x - \cos x)$$

$\therefore$ Complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + xe^x + \frac{1}{10} (3 \sin x + \cos x)$$

where c_1, c_2, c_3 are arbitrary constants of integration.

Example 6. *Solve:* $(D^2 - 4D + 1)y = \cos x \cos 2x + \sin^2 x.$

Sol. Auxiliary equation is

$$m^2 - 4m + 1 = 0$$

$$\Rightarrow \qquad m = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}$$

$$\therefore \quad \text{C.F.} = e^{2x} (c_1 \cosh \sqrt{3}\, x + c_2 \sinh \sqrt{3}\, x)$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 1} (\cos x \cos 2x) + \frac{1}{D^2 - 4D + 1} (\sin^2 x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1} (\cos 3x) + \frac{1}{D^2 - 4D + 1} (\cos x)\right] + \frac{1}{D^2 - 4D + 1}\left(\frac{1 - \cos 2x}{2}\right)$$

$$= \frac{1}{2} (P_1 + P_2) + P_3 \qquad \ldots(1)$$

where,

$$P_1 = \frac{1}{D^2 - 4D + 1} (\cos 3x)$$

$$= \frac{1}{-9 - 4D + 1} (\cos 3x) = -\frac{1}{4(D+2)} \cos 3x$$

$$= -\frac{1}{4} \frac{D-2}{(D^2-4)} \cos 3x = -\frac{1}{4} \frac{(D-2)}{(-9-4)} \cos 3x = \frac{1}{52} (-3 \sin 3x - 2 \cos 3x)$$

$$P_2 = \frac{1}{D^2 - 4D + 1} (\cos x) = \frac{1}{-1 - 4D + 1} \cos x = -\frac{1}{4} \sin x$$

$$P_3 = \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1} (1) - \frac{1}{D^2 - 4D + 1} (\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 4D + 1} (e^{0x}) - \frac{1}{-4 - 4D + 1} (\cos 2x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{(0)^2 - 4(0) + 1} (e^{0x}) + \frac{1}{4D + 3} (\cos 2x)\right]$$