3. $y = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2} \sin ax \log \sin ax - \frac{x}{a} \cos ax$

4. $y = e^{-x}\left(c_1 \cos x + c_2 \sin x + \frac{\sin x \tan x}{2}\right)$

5. $y = e^x (c_1 \cos x + c_2 \sin x) - e^x \cos x \log (\sec x + \tan x)$.

1.30. HOMOGENEOUS LINEAR DIFFERENTIAL EQUATIONS (EULER-CAUCHY EQUATIONS)

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q \qquad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Cauchy's homogeneous linear equation.

Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^z \quad \text{or} \quad z = \log x$$

so that

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{dy}{dz} \cdot \frac{1}{x} \quad \text{or} \quad x \frac{dy}{dx} = \frac{dy}{dz} = \mathrm{D}y, \text{ where } \mathrm{D} \equiv \frac{d}{dz}$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2} \qquad \left(\because \frac{dz}{dx} = \frac{1}{x}\right)$$

or

$$x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = \mathrm{D}^2 y - \mathrm{D}y = \mathrm{D}(\mathrm{D}-1)y$$

Similarly, $x^3 \frac{d^3 y}{dx^3} = \mathrm{D}(\mathrm{D}-1)(\mathrm{D}-2)y$ and so on.

Substituting these values is equation (1), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

1.30.1. Steps for Solution

1. Put $x = e^z$ so that $z = \log x$ and Let $\mathrm{D} \equiv \frac{d}{dz}$

2. Replace $x \frac{d}{dx}$ by D,

   $x^2 \frac{d^2}{dx^2}$ by D(D – 1)

   $x^3 \frac{d^3}{dx^3}$ by D(D – 1) (D – 2) and so on.

3. By doing so, this type of equation reduces to linear differential equation with constant coefficients which is then solved as before.

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1.31. LEGENDRE'S LINEAR DIFFERENTIAL EQUATION

An equation of the form

$$(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = Q \qquad \ldots(1)$$

where a_i's are constants and Q is a function of x, is called Legendre's linear differential equation.

Such equations can be reduced to linear differential equations with constant co-efficients by the substitution

$$a + bx = e^z \text{ i.e. } z = \log(a+bx) \text{ so that } \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{b}{a+bx}\frac{dy}{dz}$$

or

$$(a+bx)\frac{dy}{dx} = b\frac{dy}{dz} = b\,\text{D}y, \text{ where D} \equiv \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{b}{a+bx}\frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{dy}{dx}$$

$$= -\frac{b^2}{(a+bx)^2}\frac{dy}{dz} + \frac{b}{a+bx}\frac{d^2y}{dz^2}\cdot\frac{b}{a+bx} = \frac{b^2}{(a+bx)^2}\left(\frac{d^2y}{dz^2} - \frac{dy}{dz}\right)$$

or

$$(a+bx)^2\frac{d^2y}{dx^2} = b^2(\text{D}^2y - \text{D}y) = b^2\,\text{D}(\text{D}-1)y$$

Similarly, $(a+bx)^3\dfrac{d^3y}{dx^3} = b^3\,\text{D}(\text{D}-1)(\text{D}-2)y.$

Substituting these values in equation (*i*), we get a linear differential equation with constant coefficients, which can be solved by the methods already discussed.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve :* $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$. [U.P.T.U. (C.O.) 2009]

Sol. Put $x = e^z$ so that $z = \log x$ and let $\text{D} \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[\text{D}(\text{D}-1)(\text{D}-2) + 2\text{D}(\text{D}-1) + 2]y = 10(e^z + e^{-z})$$

or

$$(\text{D}^3 - \text{D}^2 + 2)y = 10(e^z + e^{-z})$$

which is a linear equation with constant coefficients.

Its Auxiliary equation is

$$m^3 - m^2 + 2 = 0 \quad \text{or} \quad (m+1)(m^2 - 2m + 2) = 0$$

$$\therefore \quad m = -1, \frac{2 \pm \sqrt{4-8}}{2} = -1, 1 \pm i$$

$$\therefore \quad \text{C.F.} = c_1e^{-z} + e^z(c_1\cos z + c_3\sin z) = \frac{c_1}{x} + x\,[c_2\cos(\log x) + c_3\sin(\log x)]$$

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$$\text{P.I.} = 10\, \frac{1}{D^3 - D^2 + 2}\,(e^z + e^{-z}) = 10\left(\frac{1}{D^3 - D^2 + 2}\, e^z + \frac{1}{D^3 - D^2 + 2}\, e^{-z}\right)$$

$$= 10\left(\frac{1}{1^3 - 1^2 + 2}\, e^z + z\,.\,\frac{1}{3D^2 - 2D}\, e^{-z}\right) = 10\left(\frac{1}{2}\, e^z + z\,.\,\frac{1}{3(-1)^2 - 2(-1)}\, e^{-z}\right)$$

$$= 5e^z + 2ze^{-z} = 5x + \frac{2}{x}\log x$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = \frac{c_1}{x} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x}\log x$$

where c_1, c_2 and c_3 are arbitrary constants of integration.

Example 2. *Solve :* $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} + y = x + \log x.$

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \dfrac{d}{dz}$ then the given differential equation reduces to

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]\, y = e^z + z$$

$$\Rightarrow \qquad (D^3 + 1)y = e^z + z$$

Auxiliary equation is

$$m^3 + 1 = 0$$

$$\Rightarrow \qquad (m+1)(m^2 - m + 1) = 0 \qquad \Rightarrow \qquad m = -1,\ \frac{1 \pm \sqrt{3}i}{2}$$

$$\therefore \qquad \text{C.F.} = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2} z + c_3 \sin\frac{\sqrt{3}}{2} z\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}\,(e^z + z) = \frac{1}{D^3 + 1}\,(e^z) + \frac{1}{1 + D^3}\,(z)$$

$$= \frac{e^z}{2} + (1 + D^3)^{-1}\,(z) = \frac{e^z}{2} + (1 - D^3)\,(z) \qquad \text{| Leaving higher terms}$$

$$= \frac{e^z}{2} + z$$

$\therefore$ The complete solution is

$$y = c_1 e^{-z} + e^{z/2}\left(c_2 \cos\frac{\sqrt{3}}{2} z + c_3 \sin\frac{\sqrt{3}}{2} z\right) + \frac{e^z}{2} + z$$

$$\therefore \qquad y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\frac{\sqrt{3}}{2}(\log x) + c_3 \sin\frac{\sqrt{3}}{2}(\log x)\right] + \frac{x}{2} + \log x$$

where c_1, c_2 and c_3 are the arbitrary constants of integration.

Example 3. *Solve.* $x^2 \dfrac{d^2y}{dx^2} + x\,\dfrac{dy}{dx} - \lambda^2 y = 0$. (U.P.T.U. 2007)

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$\Rightarrow \quad [D(D-1) + D - \lambda^2]y = 0$$

$$(D^2 - \lambda^2)y = 0$$

Auxiliary equation is

$$\Rightarrow \quad m^2 - \lambda^2 = 0$$

$$\therefore \quad m = \pm \lambda$$

$$\text{C.F.} = c_1 e^{\lambda z} + c_2 e^{-\lambda z}$$

$$\text{P.I.} = 0$$

Hence, the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{\lambda z} + c_2 e^{-\lambda z} = c_1 x^{\lambda} + c_2 x^{-\lambda}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 4. *Solve:* $x^2 \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = e^x$.

Sol. Put $x = e^z$ so that $z = \log x$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$\{D(D-1) + 4D + 2\}y = e^{e^z}$$

$$(D^2 + 3D + 2)y = e^{e^z}$$

Auxiliary equation is

$$m^2 + 3m + 2 = 0$$

$$\Rightarrow \quad (m+1)(m+2) = 0 \quad \Rightarrow \quad m = -1, -2$$

$$\therefore \quad \text{C.F.} = c_1 e^{-z} + c_2\, e^{-2z}$$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2}\,(e^{e^z}) = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1}\,(e^{e^z}) - \frac{1}{D+2}\, e^{e^z} = e^{-z} \int e^z \,.\, e^{e^z}\, dz - e^{-2z} \int e^{2z}\, e^{e^z}\, dz$$

$$= e^{-z}\, e^{e^z} - e^{-2z}\,(e^z - 1)\, e^{e^z} = e^{-2z}\, e^{e^z}$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 e^{-z} + c_2 e^{-2z} + e^{-2z}\, e^{e^z} = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{x^2}\, e^x$$

where c_1 and c_2 are arbitrary constants of integration.

Example 5. *Solve:* $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1.$

Sol. Given equation is a Legendre's linear differential equation.

Put $3x + 2 = e^z$ *i.e.,* $z = \log(3x+2)$ so that $(3x+2)\frac{dy}{dx} = 3Dy.$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 D(D-1)y, \quad \text{where } D \equiv \frac{d}{dz}.$$

Substituting these values in the given equation, it reduces to

$$[3^2 D(D-1) + 3.3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

or
$$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3}$$

or
$$(D^2-4)y = \frac{1}{27}(e^{2z}-1)$$

which is a linear equation with constant co-efficients.

Its Auxiliary equation is $\quad m^2 - 4 = 0 \quad \therefore \quad m = \pm 2$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-2z} = c_1(3x+2)^2 + c_2(3x+2)^{-2}$$

$$\text{P.I.} = \frac{1}{27}\cdot\frac{1}{D^2-4}(e^{2z}-1) = \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right]$$

$$= \frac{1}{27}\left[z.\frac{1}{2D}e^{2z} - \frac{1}{0-4}e^{0z}\right] = \frac{1}{27}\left[\frac{z}{2}\int e^{2z}\,dz + \frac{1}{4}\right]$$

$$= \frac{1}{27}\left[\frac{z}{4}e^{2z} + \frac{1}{4}\right] = \frac{1}{108}(ze^{2z}+1) = \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$$

Hence the complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}[(3x+2)^2\log(3x+2)+1]$$

where c_1 and c_2 are arbitrary constants of integration.

Example 6. *By reducing to homogeneous, solve the differential equation*

$$(1+x)^2\frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 4\cos\{\log(1+x)\}.$$

Sol. Put $1 + x = e^z$ so that $z = \log(1+x)$ and let $D \equiv \frac{d}{dz}$ then the given differentia[l] equation reduces to

$$\{D(D-1) + D + 1\}y = 4\cos z$$

$$\Rightarrow \qquad (D^2+1)y = 4\cos z$$

Auxiliary equation is

$$m^2 + 1 = 0 \quad \Rightarrow \quad m = \pm i$$

$$\therefore \qquad \text{C.F.} = c_1\cos z + c_2\sin z$$

$$\text{P.I.} = \frac{1}{D^2+1}(4\cos z) = 4z\,.\,\frac{1}{2D}\cos z = 2z\sin z$$

Hence the complete solution is

$$y = c_1\cos z + c_2\sin z + 2z\sin z$$

$$= c_1\cos\{\log(1+x)\} + c_2\sin\{\log(1+x)\} + 2\log(1+x)\sin\{\log(1+x)\}$$

where c_1 and c_2 are arbitrary constants of integration.

Example 7. *Solve the differential equation:* (G.B.T.U. 2011)

$$(3x+2)^2 \frac{d^2y}{dx^2} - (3x+2)\frac{dy}{dx} - 12y = 6x.$$

Sol. Put $3x + 2 = e^z$ so that $z = \log(3x+2)$ and let $D \equiv \frac{d}{dz}$ then the given differential equation reduces to

$$[3^2 D(D-1) - 3D - 12]y = 6\left(\frac{e^z - 2}{3}\right) \quad ...(1)$$

$$(9D^2 - 12D - 12)y = 2e^z - 4$$

Auxiliary equation is

$$9m^2 - 12m - 12 = 0$$

$$\Rightarrow \quad (9m+6)(m-2) = 0$$

$$\Rightarrow \quad m = 2, -\frac{2}{3}$$

$$\therefore \quad \text{C.F.} = c_1 e^{2z} + c_2 e^{-\frac{2}{3}z}$$

$$\text{P.I.} = \frac{1}{9D^2 - 12D - 12} 2e^z - \frac{1}{9D^2 - 12D - 12} 4e^{0z} = -\frac{2}{15}e^z + \frac{1}{3}$$

Hence complete solution is

$$y = c_1 e^{2z} + c_2 e^{-\frac{2}{3}z} - \frac{2}{15}e^z + \frac{1}{3}$$

$$= c_1 (3x+2)^2 + c_2 (3x+2)^{-2/3} - \frac{2}{15}(3x+2) + \frac{1}{3}$$

ere c_1 and c_2 are arbitrary constants of integration.

Example 8. *Solve:* $(x+1)^2 \frac{d^2y}{dx^2} + (x+1)\frac{dy}{dx} = (2x+3)(2x+4).$

[M.T.U. (SUM) 2011; G.B.T.U. (C.O.) 2011]

Sol. Put $x + 1 = e^z$ so that $z = \log(x+1)$ and let $D \equiv \frac{d}{dz}$ then the given differential ation reduces to

$$[D(D-1) + D]y = (2e^z + 1)(2e^z + 2)$$

$$D^2 y = 4e^{2z} + 6e^z + 2$$

Auxiliary equation is

$$m^2 = 0 \quad \Rightarrow \quad m = 0, 0$$

$$\therefore \quad \text{C.F.} = c_1 + c_2 z$$

$$\text{P.I.} = \frac{1}{D^2}(4e^{2z} + 6e^z + 2) = e^{2z} + 6e^z + z^2$$

Hence complete solution is

$$y = \text{C.F.} + \text{P.I.} = c_1 + c_2 z + e^{2z} + 6e^z + z^2$$

$$= c_1 + c_2 \log(x+1) + (x+1)^2 + 6(x+1) + [\log(x+1)]^2$$

c_1 and c_2 are arbitrary constants of integration.

TEST YOUR KNOWLEDGE

Solve:

1. $\dfrac{d^3y}{dx^3} - \dfrac{4}{x}\dfrac{d^2y}{dx^2} + \dfrac{5}{x^2}\dfrac{dy}{dx} - \dfrac{2y}{x^3} = 1$

2. $x^2\dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

3. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2$

4. $x^2\dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4$

5. (*i*) $x^4\dfrac{d^3y}{dx^3} + 2x^3\dfrac{d^2y}{dx^2} - x^2\dfrac{dy}{dx} + xy = 1$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x^2$

6. (*i*) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2\log x$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - 4x\dfrac{dy}{dx} + 6y = x$

7. (*i*) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2\log x$ (*ii*) $x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x\log x$

8. (*i*) $x^2y'' + xy' - y = x^3e^x$. (*ii*) $\left(\dfrac{d}{dx} + \dfrac{1}{x}\right)^2 y = x^{-4}$. (M.T.U. 2012)

9. $x^2\dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 12y = x^3\log x$

10. $(x^2D^2 - xD + 4)y = \cos(\log x) + x\sin(\log x)$

11. (*i*) $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = \sin(\log x)$ (*ii*) $x^2\dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x\log x$

12. (*i*) $x^2\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^{-1}$ (*ii*) $x^3y''' + xy' - y = 3x^4$

13. (*i*) $x^3\dfrac{d^3y}{dx^3} + 2x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} - 3y = x + x^2$ (*ii*) $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x^m$

14. (*i*) $x^4\dfrac{d^4y}{dx^4} + 6x^3\dfrac{d^3y}{dx^3} + 9x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1+\log x)^2$

    (*ii*) $[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m\log x$

15. $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\cdot\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

16. $(x+a)^2\dfrac{d^2y}{dx^2} - 4(x+a)\dfrac{dy}{dx} + 6y = x$

17. $(1+2x)^2\dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2$

18. $(2x+3)^2\dfrac{d^2y}{dx^2} - 2(2x+3)\dfrac{dy}{dx} - 12y = 6x$

19. $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$

20. $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \dfrac{(\log x)\sin(\log x) + 1}{x}$.