Conditional Probability

Let $A, B \in S$ (sample space). Then

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad P(B) > 0$$

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \quad P(A) > 0$$

Ques (1) Two dice are rolled what is probability of getting sum 8 if it is given that first die shows a '3'.

Solution: Here $n(S) = 36$

Let A = getting sum of 8

$= \{(2,6), (6,2), (3,5), (5,3), (4,4)\}$

$n(A) = 5 \Rightarrow P(A) = \frac{5}{36}$

B = First die shows a '3'

$B = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$

$n(B) = 6 \Rightarrow P(B) = \frac{6}{36} = \frac{1}{6}$

$A \cap B = \{(3,5)\} \Rightarrow n(A \cap B) = 1$

$P(A \cap B) = \frac{1}{36}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{6}{36} = \frac{1}{6} \text{ Ans}$$

Ques (2) A family has two children. Given that one children is a boy. Find the probability that other child is also a boy.

Solution: Here $S = \{BB, BG, GB, GG\} \Rightarrow n(S) = 4$

Let B = one children is a boy

$= \{BB, BG, GB\} \Rightarrow n(B) = 3$

$P(B) = \frac{3}{4}$

Let A = both children are boys

$A = \{BB\} \Rightarrow n(A) = 1$

$P(A) = \frac{1}{4}$. Further $A \cap B = \{BB\}$ $\therefore n(A \cap B) = 1$

$P(A \cap B) = \frac{1}{4}$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3} \text{ Ans}$$

~~Independent~~

Multiplication Theorem of Probability: $A, B \subseteq S$ then

$$P(A \cap B) = P(A|B)\, P(B) \quad -(1)$$

$$P(A \cap B) = P(B|A)\, P(A) \quad -(2)$$

Independent Events: Two events are said to be independent if occurence or non-occurence of one event does not affect the occurence or non-occurence of another events.

(*) If two events are not independent then they are called dependent events.

Note: For independent events we have $P(A|B) = P(A)$

$P(B|A) = P(B)$

So, Now from eqn (1) and (2) we have

$$\boxed{P(A \cap B) = P(A) \cdot P(B)}$$

Ques (1) what is the probability of drawing a king twice in succession from a deck of 52 cards if

(I) with replacement (II) without replacement.

Solution: Let A = a king in first draw

B = a king in second draw

(I) with replacement (A and B are independent here)

$P(A) = \frac{4}{52} = \frac{1}{13}$, $P(B) = \frac{4}{52} = \frac{1}{13}$

$$P(A \cap B) = P(A)\, P(B) = \frac{1}{13} \times \frac{1}{13} = \frac{1}{169}$$

(II) Without replacement (A and B are dependent events now)

$$P(A \cap B) = \frac{4C_1}{52C_1} \times \frac{3C_1}{51C_1} = \frac{4}{52} \times \frac{3}{51} = \frac{12}{52 \times 51} = \frac{1}{221}$$

Binomial Distribution (B.D.)

$$P(X=r) = nC_r \, p^r q^{n-r} \quad r = 0, 1, 2, \dots n \quad -(1)$$

$\Downarrow$

Probability of rth successes in n trials.
($n \rightarrow$ finite number)

In Equation (1) n = total number of trials
p = probability of success in each trial
q = probability of failure in each trial

Note: If p is the probability of ~~fail~~ success for an event and q is the probability of failure for same event then

$$\boxed{p+q=1}$$

In Equation (1) n and p are known as parameter of B.D..

Ques (1) A coin is tossed 5 times. What is the probability of getting at least 3 heads.

Solution: Here, we have $n=5$, $p=\frac{1}{2}$, $q=\frac{1}{2}$

Let X denotes the number of heads in 5 tosses of a coin.

$P(X \geq 3) = P$(at least three heads)

$= P(X=3) + P(X=4) + P(X=5)$

$= 5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2 + 5C_4 \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right) + 5C_5 \left(\frac{1}{2}\right)^5$

$= 5C_3 \left(\frac{1}{2}\right)^5 + 5C_4 \left(\frac{1}{2}\right)^5 + 5C_5 \left(\frac{1}{2}\right)^5$

$= \left(\frac{1}{2}\right)^5 (5C_3 + 5C_4 + 5C_5) = \left(\frac{1}{2}\right)^5 (10+5+1) = \frac{16}{32}$

$= 1/2$ Ans (1)

Ques (2) The probability that a student entering the College will graduate is 0.4. Determine the probability that out of 5 students (I) None (II) at least one (III) all will graduate.

Solution: $n=5$, $p=0.4$, $q=1-0.4=0.6$

Let X denotes the no of students being graduated.

(I) $P(X=0) = 5C_0 (0.4)^0 (0.6)^5$

None will be graduate $= 5C_0 (0.6)^5 = 1 \times (0.6)^5 = 0.07776$ Ans

(II) $P(X \geq 1) = P$(At least one graduated)

$= 1 - P(X=0)$

$= 1 - 0.07776$ (from first part)

$= 0.92224$

(III) $P(X=5) = 5C_5 (0.4)^5 (0.6)^0 = (0.4)^5 = 0.01024$ Ans
(All will graduate)

(Constants of Binomial Distribution)

$P(X=r) = nC_r p^r q^{n-r}$ $r=0,1,2,\ldots n$

Mean of B.D. $= np$

Variance of B.D. $= npq$

Ques (1) In a B.D. the mean and variance are 12 and 4 respectively. Find the parameters n and p.

Solution: Here, we have Mean $=12 \Rightarrow np=12$ — (1)

Variance $=4 \Rightarrow npq=4$ — (2)

Dividing eqn (2) by Equation (1) we have

$\frac{npq}{np} = \frac{4}{12} \Rightarrow q = \frac{1}{3}$

$p = 1-q = 1-\frac{1}{3} = \frac{2}{3}$

From Eq^n (1) we have $n \times \frac{2}{3} = 12$

$$n = \frac{36}{2} = 18$$

$n = 18, \quad p = \frac{2}{3}$ **Ans**

Ques (2) Find the B.D. when the sum of its mean and variance is 4.8 for 5 trials.

Solution: Here, we have $n = 5$

$$\text{Mean} + \text{variance} = 4.8$$

$$np + npq = 4.8$$

$$np(1+q) = 4.8$$

$$5p(1+1-p) = 4.8 \quad (\because n = 5 \;\&\; q = 1-p)$$

$$5p(2-p) = 4.8 \Rightarrow p(2-p) = \frac{4.8}{5}$$

$$2p - p^2 = 0.96$$

$$p^2 - 2p + 0.96 = 0$$

$$p^2 - (1.2 + 0.8)p + 0.96 = 0$$

$$p^2 - 1.2p - 0.8p + 0.96 = 0$$

$$p(p - 1.2) - 0.8(p - 1.2) = 0$$

$$(p - 1.2)(p - 0.8) = 0$$

$$p = 1.2 \text{ or } p = 0.8$$

$p = 0.8$ ($\because$ the prob. can't Exceed to 1)

$$q = 1 - p = 1 - 0.8 = 0.2$$

Now, we have $n = 5, \; p = 0.8 \; q = 0.2$

$$P(X = r) = {}^5C_r \, (0.8)^r \, (0.2)^{5-r}$$ **Ans**

$$r = 0, 1, 2, 3, 4, 5$$

Ques (3) Obtain the mean and Standard deviation of a B.D. for which $P(X=3) = 16\,P(X=7)$ and $n = 10$.

Solution: Here, we have $n = 10$,

$$P(X=3) = 16\,P(X=7)$$

$${}^{10}C_3\, p^3 q^7 = 16 \cdot {}^{10}C_7\, p^7 q^3$$

We know that ${}^nC_r = {}^nC_{n-r} \Rightarrow {}^{10}C_3 = {}^{10}C_7$

$$p^3 q^7 = 16\, p^7 q^3$$

$$p^3 q^7 - 16 p^7 q^3 = 0 \Rightarrow p^3 q^3 (q^4 - 16 p^4) = 0$$

$$q^4 - 16 p^4 = 0 \quad (\because p^3 q^3 \neq 0)$$

$$q^4 = 16 p^4 \Rightarrow q = (16 p^4)^{\frac{1}{4}}$$

$$q = (2^4 p^4)^{\frac{1}{4}}$$

$$q = 2p$$

$$1 - p = 2p \Rightarrow 3p = 1$$

$$\Rightarrow p = \frac{1}{3}$$

$$q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}$$

$n = 10,\ p = \frac{1}{3},\ q = \frac{2}{3}$

Mean $= np = 10 \times \frac{1}{3} = \frac{10}{3}$ Ans

Variance $= npq = 10 \times \frac{1}{3} \times \frac{2}{3} = \frac{20}{9}$

S.D. $= \sqrt{\text{Variance}} = \sqrt{\frac{20}{9}} = \frac{\sqrt{20}}{3}$ Ans

Ques (3) If the mean and variance of a B.D. are respectively 9 and 6 find the distribution.

Ans: $P(X = r) = {}^{27}C_r \left(\frac{1}{3}\right)^r \left(\frac{2}{3}\right)^{27-r}$ $r = 0, 1, 2 \ldots 27$

(Do by your self)