Coefficient of $x^2 = 0$

$$\Rightarrow \quad 4.3.a_4 - 2.1.\ a_2 - 4a_2 + 2a_2 = 0$$

$$\Rightarrow \quad 12a_4 - 4a_2 = 0 \quad \Rightarrow \quad a_4 = \frac{a_2}{3} = -\frac{a_0}{3} \quad \Rightarrow \quad \boxed{a_4 = -\frac{a_0}{3}}$$

Coefficient of $x^3 = 0$

$$\Rightarrow \quad 5.4.a_5 - 3.2.\ a_3 - 6a_3 + 2a_3 = 0$$

$$\Rightarrow \quad 20a_5 - 10a_3 = 0 \quad \Rightarrow \quad \boxed{a_5 = 0}$$

Coefficient of $x^4 = 0$

$$\Rightarrow \quad 6.5.a_6 - 4.3.\ a_4 - 8a_4 + 2a_4 = 0$$

$$\Rightarrow \quad 30a_6 - 18a_4 = 0 \quad \Rightarrow a_6 = \frac{3}{5}a_4 \quad \Rightarrow \quad \boxed{a_6 = -\frac{a_0}{5}}$$

Also, $a_7 = 0,\ a_9 = 0$ and so on.

Substituting these values in assumed solution (1), we get

$$y = a_0 + a_1 x - a_0 x^2 - \frac{a_0}{3}x^4 - \frac{a_0}{5}x^6 - \ldots\ldots$$

$$\Rightarrow \quad y = a_0\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \ldots\right) + a_1 x$$

where a_0 and a_1 are constants.

Example 5. *Solve in series the Legendre's differential equation*

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + p(p+1)y = 0. \qquad \text{[G.B.T.U. (C.O.) 2010]}$$

Sol. Here, $\quad P(x) = \dfrac{-2x}{1-x^2},\ Q(x) = \dfrac{p(p+1)}{1-x^2}$

Since both $P(x)$ and $Q(x)$ are analytic at $x = 0$ $\therefore$ $x = 0$ is an *ordinary point* of the given differential equation.

Let the solution be $\quad y = a_0 + a_1 x + a_2 x^2 + \ldots\ldots + a_n x^n + \ldots\ldots = \sum_{n=0}^{\infty} a_n x^n \qquad \ldots(1)$

$$\therefore \quad \frac{dy}{dx} = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad \ldots(2)$$

$$\frac{d^2y}{dx^2} = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \qquad \ldots(3)$$

Substituting the above values in the given equation, we get

$$(1 - x^2)\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - 2x\sum_{n=0}^{\infty} n a_n x^{n-1} + p(p+1)\sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n[n(n-1) + 2n - p(p+1)]x^n = 0$$

$$\Rightarrow \quad \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n (n-p)(n+p+1)x^n = 0$$

This is an identity in x.

Coefficient of $x^n = 0$

$$(n+2)(n+1)\,a_{n+2} - (n-p)(n+p+1)\,a_n = 0$$

$\Rightarrow$

$$a_{n+2} = \frac{(n-p)(n+p+1)}{(n+2)(n+1)}\,a_n$$

$\therefore$

Putting $n = 0, 2, 4, \ldots\ldots$ etc., we get

$$a_2 = \frac{-p(p+1)}{2.1}\,a_0$$

$$a_4 = \frac{(2-p)(3+p)}{4.3}\,a_2 = \frac{(p-2)(p)(p+1)(p+3)}{4!}\,a_0 \text{ etc.}$$

Again, putting $n = 1, 3, 5, \ldots\ldots$ etc., we get

$$a_3 = \frac{(1-p)(p+2)}{3.2}\,a_1 = -\frac{(p-1)(p+2)}{3!}\,a_1$$

$$a_5 = \frac{(3-p)(p+4)}{5.4}\,a_3 = \frac{(p-3)(p-1)(p+2)(p+4)}{5!}\,a_1 \text{ etc.}$$

Substituting these values in eqn. (1), we get

$$y = a_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{(p-2)p(p+1)(p+3)}{4!}x^4 - \ldots\ldots\right]$$

$$+ a_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-3)(p-1)(p+2)(p+4)}{5!}x^5 + \ldots\ldots\right]$$

Note. Above method is an *aliter* to the method of solution in series discussed before and preferred when, we get the recurrence relation in between a_n and a_{n+2}.

Example 6. *Solve the differential equation*

$$y'' + (x-1)^2 y' - 4(x-1)\,y = 0$$

in series about the ordinary point $x = 1$.

Sol. Put $x = t + 1$ (or $x - 1 = t$)

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt} \qquad \left(\because \frac{dt}{dx} = 1\right)$$

$\Rightarrow$

$$\frac{d}{dx} \equiv \frac{d}{dt}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ The given equation becomes,

$$\frac{d^2y}{dt^2} + t^2y' - 4ty = 0 \qquad \ldots(1)$$

Now, $t = 0$ is an *ordinary point*. | given

Assume the solution to be ...(2)

$$y = a_0 + a_1t + a_2t^2 + a_3t^3 + \ldots + a_nt^n + \ldots$$

then

$$y' = a_1 + 2a_2t + 3a_3t^2 + \ldots + n\,a_n\,t^{n-1} + \ldots$$

and

$$y'' = 2a_2 + 3.2.\,a_3t + \ldots + n\,(n-1)\,a_n\,t^{n-2} + \ldots$$

Substituting these values in eqn. (1), we get

$$[2a_2 + 3.2.\, a_3 t + 4.3.\, a_4\, t^2 + \ldots + n(n-1)\, a_n\, t^{n-2} + \ldots] + t^2\, [a_1 + 2a_2 t + 3a_3\, t^2 + 4a_4 t^3 + \ldots + n\, a_n t^{n-1} + \ldots] - 4t\, [a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots + a_n t^n + \ldots] = 0$$

Coefficient of $t^0 = 0$

$\Rightarrow \quad 2a_2 = 0 \quad \Rightarrow \quad \boxed{a_2 = 0}$

Coefficient of $t = 0$

$\Rightarrow \quad 3.2.\, a_3 - 4a_0 = 0 \quad \Rightarrow \quad \boxed{a_3 = \dfrac{2a_0}{3}}$

Coefficient of $t^2 = 0$

$\Rightarrow \quad 4.3.\, a_4 + a_1 - 4a_1 = 0$

$\Rightarrow \quad 12a_4 = 3a_1 \quad \Rightarrow \quad \boxed{a_4 = \dfrac{a_1}{4}}$

Coefficient of $t^3 = 0$

$\Rightarrow \quad 5.4.\, a_5 + 2a_2 - 4a_2 = 0 \quad \Rightarrow \quad \boxed{a_5 = 0}$

Coefficient of $t^4 = 0$

$\Rightarrow \quad 6.5.\, a_6 + 3a_3 - 4a_3 = 0$

$$a_6 = \frac{a_3}{6.5} = \frac{2a_0}{6.5.3} \quad \Rightarrow \quad \boxed{a_6 = \frac{a_0}{45}}$$

Now, coefficient of $t^n = 0$

$\Rightarrow \quad (n+2)\,(n+1)\, a_{n+2} + (n-1)\, a_{n-1} - 4a_{n-1} = 0$

$$\Rightarrow \quad a_{n+2} = -\frac{(n-5)}{(n+2)(n+1)}\, a_{n-1}$$

Putting $n = 5, 6, 7, 8, \ldots$, we get

$$a_7 = 0$$

$$a_8 = \frac{-1}{8.7}\, a_5 = 0$$

$$a_9 = \frac{-2}{9.8}\, a_6 = \frac{-2}{9.8}\,\frac{a_0}{45} = -\frac{a_0}{1620}$$

and so on.

Substituting these values in (2), we get

$$y = a_0 + a_1 t + \frac{2}{3} a_0\, t^3 + \frac{a_1}{4} t^4 + \frac{a_0}{45} t^6 - \frac{a_0}{1620}\, t^9 + \ldots$$

$$= a_0 \left(1 + \frac{2}{3} t^3 + \frac{1}{45} t^6 - \frac{1}{1620} t^9 + \ldots\right) + a_1 \left(t + \frac{t^4}{4}\right)$$

$$\Rightarrow \quad y = a_0 \left[1 + \frac{2}{3}(x-1)^3 + \frac{1}{45}(x-1)^6 - \frac{1}{1620}(x-1)^9 + \ldots\right] + a_1 \left[(x-1) + \frac{(x-1)^4}{4}\right]$$

where a_0 and a_1 are constants.

TEST YOUR KNOWLEDGE

Solve the following equations in series : [Dashes denote differentiation w.r.t. x]

1. $\dfrac{d^2y}{dx^2} - y = 0$

2. $y'' + x^2y = 0$

3. (i) $y'' + xy' + y = 0$ (ii) $y'' - xy' + y = 0$

4. (i) $y'' - xy' + x^2y = 0$ (ii) $y'' + xy' + x^2y = 0$

5. $(1 - x^2)\, y'' + 2xy' + y = 0$

6. $(2 + x^2)\, y'' + xy' + (1 + x)\, y = 0$

7. $(x^2 + 1)\, y'' + xy' - xy = 0$ [U.P.T.U.(C.O.) 2008]

8. (i) $(x^2 - 1)\, y'' + 4xy' + 2y = 0$ (ii) $(x^2 - 1)\, y'' + xy' - y = 0$

9. (i) $y'' + xy' + (x^2 + 2)\, y = 0$ (ii) $(x^2 - 1)y'' + 3xy' + xy = 0$; $y(0) = 4$, $y'(0) = 6$.

10. (i) $y'' - xy' + 2y = 0$ near $x = 1$ (ii) $y'' + (x - 3)\, y' + y = 0$ near $x = 2$.

Answers

1. $y = a_0\left(1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \dfrac{x^6}{6!} + \ldots\right) + a_1\left(x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \ldots\right) = a_0 \cosh x + a_1 \sinh x$

2. $y = a_0\left(1 - \dfrac{x^4}{3.4} + \dfrac{x^8}{3.4.7.8} - \ldots\right) + a_1\left(x - \dfrac{x^5}{4.5} + \dfrac{x^9}{4.5.8.9} - \ldots\right)$

3. (i) $y = a_0\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{2.4} - \dfrac{x^6}{2.4.6} + \ldots\right) + a_1\left(x - \dfrac{x^3}{3} + \dfrac{x^5}{3.5} - \dfrac{x^7}{3.5.7} + \ldots\right)$

   (ii) $y = a_0\left(1 - \dfrac{x^2}{2!} - \dfrac{x^4}{4!} - \dfrac{3}{6!}x^6 - \dfrac{3.5}{8!}x^8 + \ldots\right) + a_1 x$

4. (i) $y = a_0\left(1 - \dfrac{x^4}{12} - \dfrac{x^6}{90} - \ldots\right) + a_1\left(x + \dfrac{x^3}{6} - \dfrac{x^5}{40} - \dfrac{x^7}{144} + \ldots\right)$

   (ii) $y = a_0\left(1 - \dfrac{x^4}{12} + \dfrac{x^6}{90} - \ldots\right) + a_1\left(x - \dfrac{x^3}{6} - \dfrac{x^5}{40} - \ldots\right)$

5. $y = a_0\left(1 - \dfrac{x^2}{2} + \dfrac{x^4}{8} + \ldots\right) + a_1\left(x - \dfrac{x^3}{2} + \dfrac{x^5}{40} + \ldots\right)$

6. $y = a_0\left(1 - \dfrac{x^2}{4} - \dfrac{x^3}{12} + \dfrac{5x^4}{96} + \ldots\right) + a_1\left(x - \dfrac{x^3}{6} - \dfrac{x^4}{24} + \ldots\right)$

7. $y = a_0\left(1 + \dfrac{x^3}{6} - \dfrac{3x^5}{40} + \ldots\right) + a_1\left(x - \dfrac{x^3}{6} + \dfrac{x^4}{12} + \dfrac{3x^5}{40} - \ldots\right)$

8. (i) $y = a_0\,(1 + x^2 + x^4 + \ldots) + a_1\,(x + x^3 + x^5 + \ldots)$ (ii) $y = a_0\left(1 + \dfrac{x^2}{2} + \dfrac{x^4}{4} + \ldots\right) + a_1 x$

9. (i) $y = c_0\left(1 - x^2 + \dfrac{x^4}{4} + \ldots\right) + c_1\left(x - \dfrac{x^3}{2} + \dfrac{3}{40}x^5 - \ldots\right)$

   (ii) $y = 4 + 6x + \dfrac{11}{3}x^3 + \dfrac{1}{2}x^4 + \dfrac{11}{4}x^5 + \ldots$

10. (i) $y = a_0\left[1-(x-1)^2 - \frac{1}{3}(x-1)^3 - \dots\right] + a_1\left[(x-1) + \frac{1}{2}(x-1)^2 - \dots\right]$

(ii) $y = a_0\left[1-\frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{12}(x-2)^4 + \dots\right]$

$$+ a_1\left[(x-2) + \frac{1}{2}(x-2)^2 - \frac{1}{6}(x-2)^3 - \frac{1}{6}(x-2)^4 + \dots\right]$$

2.5. FROBENIUS METHOD : SERIES SOLUTION WHEN $X = 0$ IS A REGULAR SINGULAR POINT OF THE DIFFERENTIAL EQUATION

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

Steps for solution :

1. Assume $y = a_0 x^m + a_1 x^{m+1} + a_2 x^{m+2} + \dots$ ...(1)
2. Substitute from (1) for y, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ in given equation.
3. Equate to zero the coefficient of *lowest power* of x. This gives a quadratic equation in m which is known as the *Indicial equation.*
4. Equate to zero, the coefficients of other powers of x to find $a_1, a_2, a_3, \dots$ interms of a_0.
5. Substitute the values of $a_1, a_2, a_3, \dots$ in (1) to get the series solution of the given equation having a_0 as arbitrary constant. Obviously, this is not the complete solution of given equation since the complete solution must have two independent arbitrary constants.

   The method of complete solution depends on the nature of roots of the indicial equation.

2.5.1. Case I. When Roots are Distinct and do not Differ by an Integer

e.g., $$m_1 = \frac{1}{2},\ m_2 = 1$$

Let m_1 and m_2 be the roots then complete solution is

$$\boxed{y = c_1(y)_{m_1} + c_2(y)_{m_2}}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Solve in series the differential equation*

$$2x(1-x)\frac{d^2y}{dx^2} + (5-7x)\frac{dy}{dx} - 3y = 0.$$

Sol. Comparing the given equation with

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0, \text{ we get}$$

$$P(x) = \frac{5-7x}{2x(1-x)}, \ Q(x) = \frac{-3}{2x(1-x)}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point*.

Now,
$$x\,P(x) = \frac{5-7x}{2(1-x)}$$
$$x^2\,Q(x) = \frac{-3x}{2(1-x)}$$

At $x = 0$, both $x\,P(x)$ and $x^2\,Q(x)$ are analytic, hence $x = 0$ is a *regular singular point*.

Let us assume

$$y = a_0\,x^m + a_1 x^{m+1} + a_2\,x^{m+2} + a_3 x^{m+3} + \ldots \qquad \ldots(1)$$

Then, $$y' = m a_0 x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots$$

and $$y' = m(m-1)\,a_0\,x^{m-2} + (m+1)\,m\,a_1\,x^{m-1} + (m+2)\,(m+1)\,a_2\,x^m + (m+3)\,(m+2)\,a_3\,x^{m+1} + \ldots$$

Substituting these values in given equation, we get

$$2x\,(1-x)\,[m(m-1)\,a_0\,x^{m-2} + (m+1)\,m a_1\,x^{m-1} + (m+2)\,(m+1)\,a_2\,x^m + (m+3)\,(m+2)\,a_3\,x^{m+1} + \ldots]$$
$$+ (5-7x)\,[m\,a_0\,x^{m-1} + (m+1)\,a_1\,x^m + (m+2)\,a_2\,x^{m+1} + (m+3)\,a_3\,x^{m+2} + \ldots]$$
$$-3\,[a_0\,x^m + a_1\,x^{m+1} + a_2\,x^{m+2} + a_3\,x^{m+3} + \ldots] = 0$$

Now, coefficient of lowest power of $x = 0$

$\Rightarrow$ Coefficient of $x^{m-1} = 0$

$\Rightarrow$ $2m\,(m-1)\,a_0 + 5m\,a_0 = 0$

$\Rightarrow$ $(2m^2 + 3m)\,a_0 = 0$

$\Rightarrow$ $2m^2 + 3m = 0 \qquad (\because\ a_0 \neq 0)$

This is called indicial equation

$$m(2m+3) = 0$$

$\Rightarrow$ $\boxed{m = 0, -3/2}$

Roots are distinct and do not differ by an integer.

Now, Coefficient of $x^m = 0$

$\Rightarrow$ $2(m+1)\,m\,a_1 - 2m\,(m-1)\,a_0 + 5(m+1)\,a_1 - 7ma_0 - 3a_0 = 0$

$\Rightarrow$ $(m+1)\,(2m+5)\,a_1 = (2m^2 - 2m + 7m + 3)\,a_0$

$$a_1 = \frac{(m+1)(2m+3)}{(m+1)(2m+5)}\,a_0$$

$\Rightarrow$ $\boxed{a_1 = \frac{2m+3}{2m+5}\,a_0}$

Coefficient of $x^{m+1} = 0$

$\Rightarrow$ $2(m+2)\,(m+1)\,a_2 - 2(m+1)\,m\,a_1 + 5\,(m+2)\,a_2 - 7(m+1)\,a_1 - 3a_1 = 0$

$\Rightarrow$ $(m+2)\,(2m+7)\,a_2 = (2m^2 + 2m + 7m + 7 + 3)\,a_1$

$= (2m^2 + 9m + 10)\,a_1 = (2m+5)\,(m+2)\,a_1$

$\Rightarrow$ $$a_2 = \frac{2m+5}{2m+7}\,a_1 = \frac{2m+5}{2m+7}\cdot\frac{2m+3}{2m+5}\,a_0$$

$$\Rightarrow \quad \boxed{a_2 = \frac{2m+3}{2m+7}\, a_0}$$

Similarly, $$a_3 = \frac{2m+7}{2m+9}\, a_2 = \frac{2m+7}{2m+9} \cdot \frac{2m+3}{2m+7}\, a_0$$

$$\Rightarrow \quad \boxed{a_3 = \frac{2m+3}{2m+9}\, a_0}$$

and so on.

Hence, from (1),

$$y = x^m \left[a_0 + \frac{2m+3}{2m+5}\, a_0\, x + \frac{2m+3}{2m+7}\, a_0\, x^2 + \frac{2m+3}{2m+9}\, a_0\, x^3 + \dots \right]$$

$$\Rightarrow \quad y = a_0\, x^m \left[1 + \left(\frac{2m+3}{2m+5} \right) x + \left(\frac{2m+3}{2m+7} \right) x^2 + \left(\frac{2m+3}{2m+9} \right) x^3 + \dots \right] \quad \dots(2)$$

Now, $$y_1 = (y)_{m=0}$$

$$\boxed{y_1 = a_0 \left[1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right]} \quad \dots(3)$$

Also, $$y_2 = (y)_{m=-3/2} = a_0\, x^{-3/2}\, (1 + 0 \,.\, x + 0 \,.\, x^2 + 0 \,.\, x^3 + \dots)$$

$$\boxed{y_2 = a_0\, x^{-3/2}} \quad \dots(4)$$

Hence the complete solution is given by

$$y = c_1\, y_1 + c_2\, y_2 = c_1\, a_0 \left(1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right) + c_2 a_0\, x^{-3/2}$$

$$\Rightarrow \quad y = A \left(1 + \frac{3}{5} x + \frac{3}{7} x^2 + \frac{3}{9} x^3 + \dots \right) + Bx^{-3/2}$$

where A and B are constants.

Example 2. *Solve in series the differential equation*

$$2x^2 \frac{d^2 y}{dx^2} + (2x^2 - x) \frac{dy}{dx} + y = 0.$$

Sol. Comparing the given equation with $\frac{d^2 y}{dx^2} + P(x) \frac{dy}{dx} + Q(x)\, y = 0$, we get

$$P(x) = \frac{2x^2 - x}{2x^2} = 1 - \frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{1}{2x^2}$$

At $x = 0$, Both $P(x)$ and $Q(x)$ are not analytic, hence $x = 0$ is a *singular point.*

Now, $$x\, P(x) = x - \frac{1}{2} \quad \text{and} \quad x^2\, Q(x) = \frac{1}{2}$$

Since both $x\, P(x)$ and $x^2\, Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a *regular singular point.*

Let us assume

$$y = a_0 x^m + a_1\, x^{m+1} + a_2\, x^{m+2} + a_3\, x^{m+3} + \dots \quad \dots(1)$$