Fuzzy Sets and Fuzzy Logic

Theory and Applications

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1. Introduction

- Uncertainty
  - When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false. It may be true only to some degree, the degree to which x is actually a member of A.
  - For example: the weather today
    - Sunny: If we define any cloud cover of 25% or less is sunny.
    - This means that a cloud cover of 26% is not sunny?
    - "Vagueness" should be introduced.

- The crisp set v.s. the fuzzy set
  - The crisp set is defined in such a way as to partition the individuals in some given universe of discourse into two groups: members and nonmembers.
    - However, many classification concepts do not exhibit this characteristic.
    - For example, the set of tall people, expensive cars, or sunny days.
  - A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.
    - For example: a fuzzy set representing our concept of sunny might assign a degree of membership of 1 to a cloud cover of 0%, 0.8 to a cloud cover of 20%, 0.4 to a cloud cover of 30%, and 0 to a cloud cover of 75%.

2. Fuzzy sets: basic types

- A membership function:
  - A characteristic function: the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set. μ_A
  - Larger values denote higher degrees of set membership.
- A set defined by membership functions is a fuzzy set.
- The most commonly used range of values of membership functions is the unit interval [0,1].
- The universal set X is always a crisp set.
- Notation:
  - The membership function of a fuzzy set A is denoted by :

    $$\mu_A : X \rightarrow [0,1]$$

  - Alternatively, the function can be denoted by A and has the form

    $$A : X \rightarrow [0,1]$$

  - We use the second notation.

2. Fuzzy sets: basic types

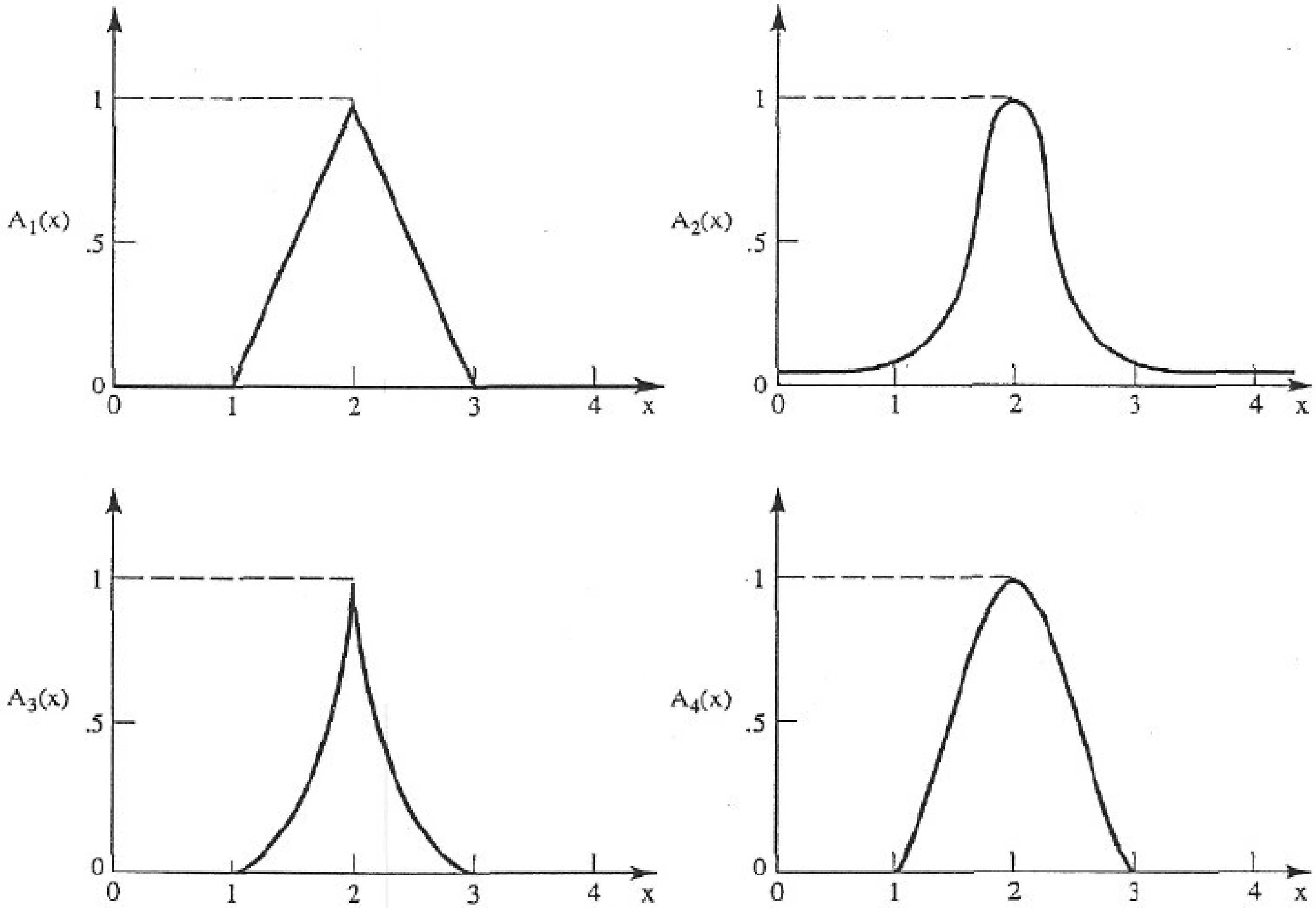


Figure 1.2 Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.

2. Fuzzy sets: basic types

- An example:
  - Define the seven levels of education:

0 – no education
1 – elementary school
2 – high school
3 – two-year college degree
4 – bachelor's degree
5 – master's degree
6 – doctoral degree

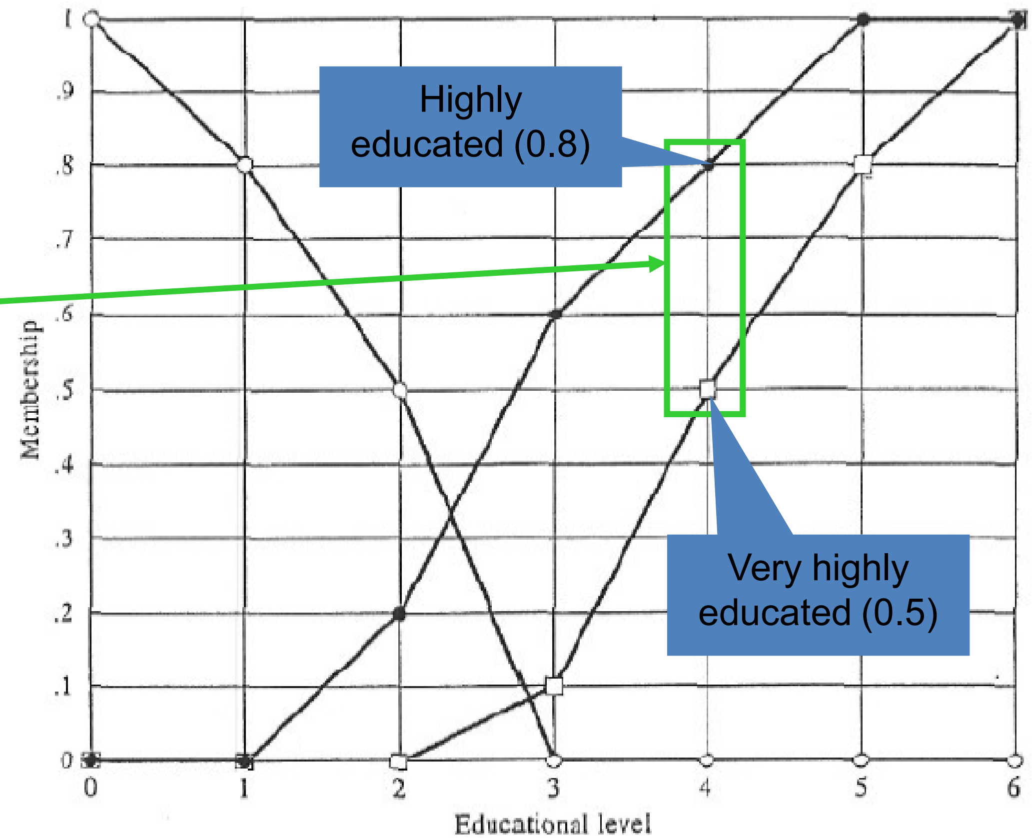


Figure 1.3 Examples of fuzzy sets expressing the concepts of people that are little educated (○), highly educated (●), and very highly educated (□).

2. Fuzzy sets: basic types

- Several fuzzy sets representing linguistic concepts such as low, medium, high, and so one are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.
- For example:

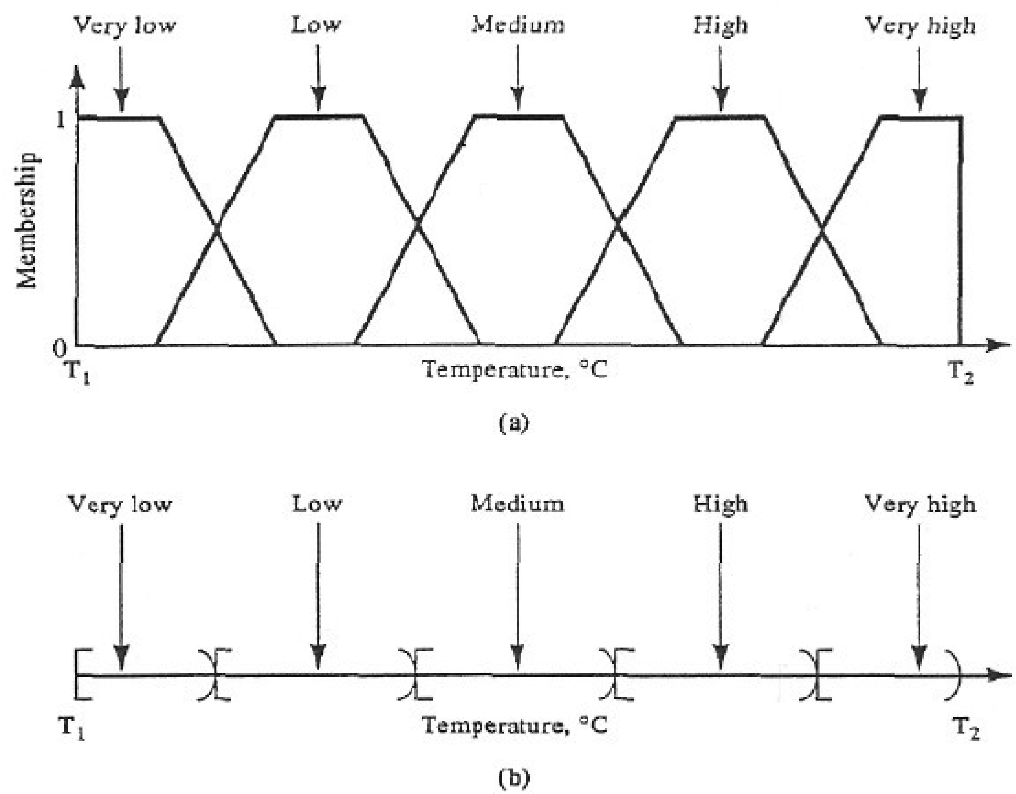


Figure 1.4 Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

2. Fuzzy sets: basic types

- Given a universal set X, a fuzzy set is defined by a function of the form

$$A : X \to [0,1]$$

This kind of fuzzy sets are called ordinary fuzzy sets.

- Interval-valued fuzzy sets:
  - The membership functions of ordinary fuzzy sets are often overly precise.
  - We may be able to identify appropriate membership functions only approximately.
  - Interval-valued fuzzy sets: a fuzzy set whose membership functions does not assign to each element of the universal set one real number, but a closed interval of real numbers between the identified lower and upper bounds.

$$A : X \to \varepsilon([0,1]), \quad \mathcal{E}([0, 1]) \subset \mathcal{P}([0, 1]).$$

Power set

2. Fuzzy sets: basic types

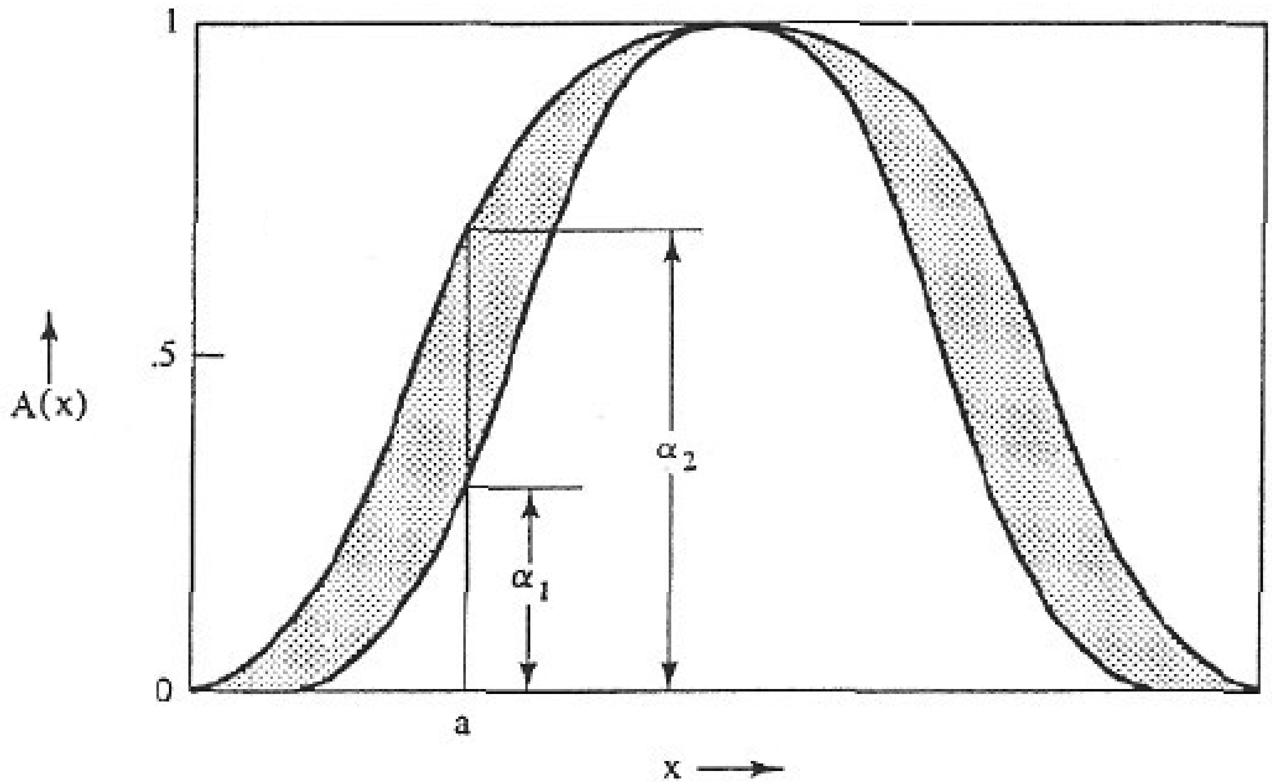


Figure 1.5 An example of an interval-valued fuzzy set $\boxed{(A(a) = [\alpha_1, \alpha_2])}$.

3 Fuzzy sets: basic concepts

- The standard complement of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation

  $$\bar{A}(x) = 1 - A(x)$$

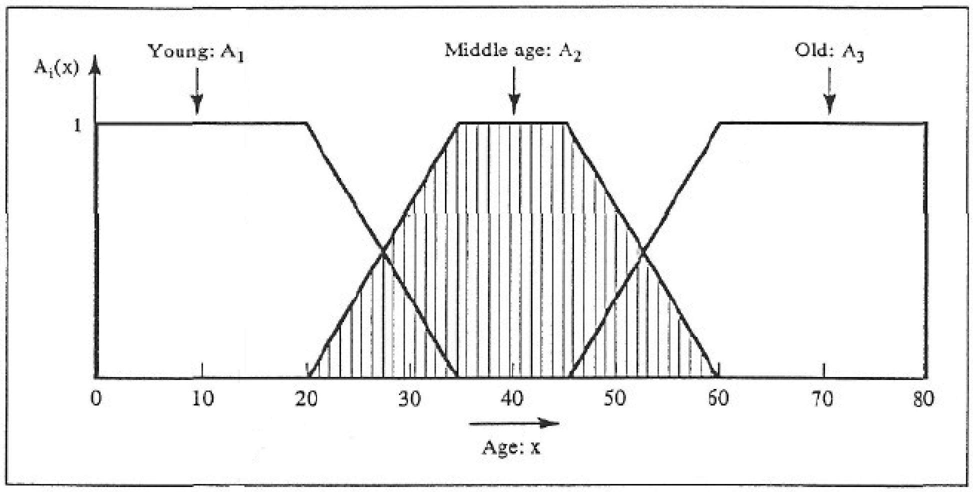
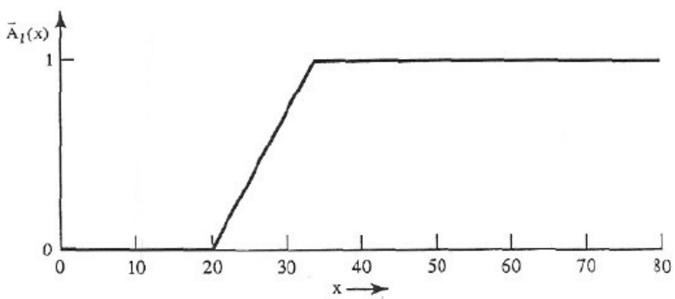
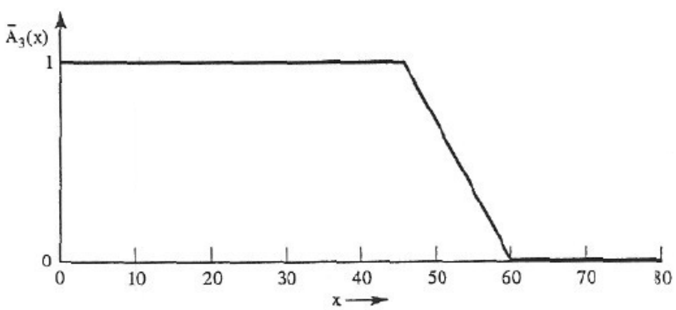


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.





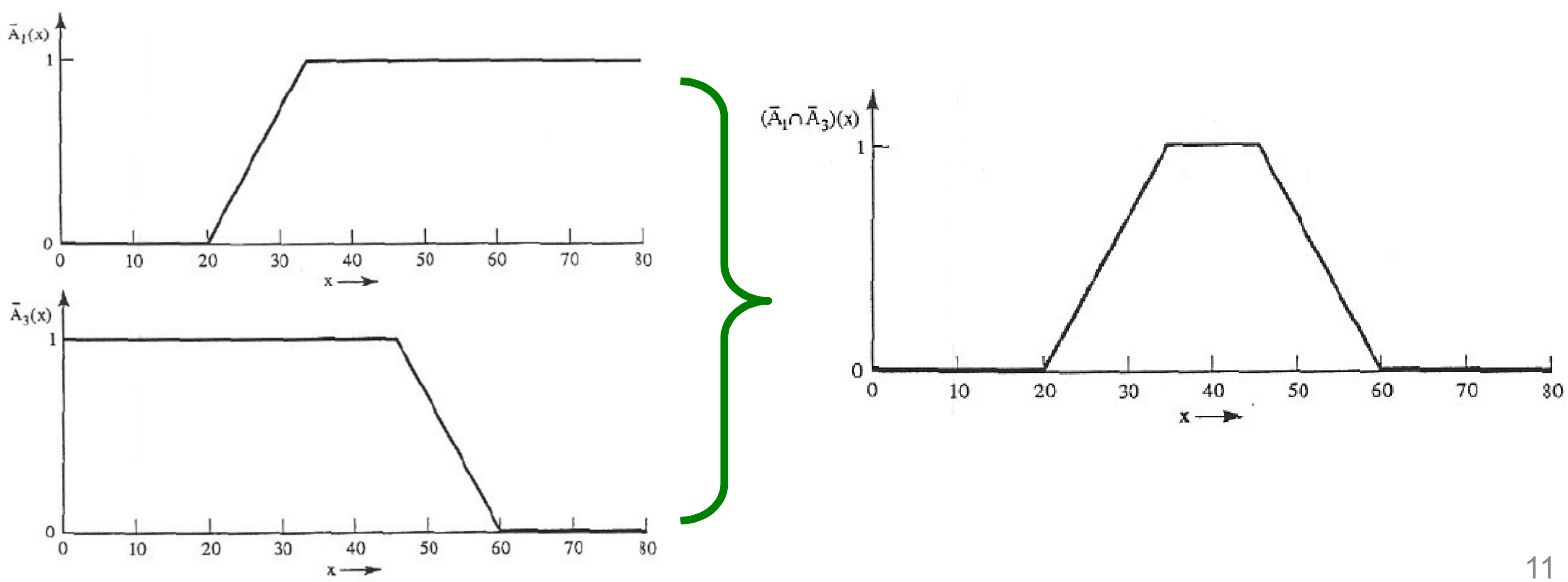
3. Fuzzy sets: basic concepts

- Given two fuzzy sets, A and B, their standard intersection and union are defined for all $x \in X$ by the equations

$$(A \cap B)(x) = \min[A(x), B(x)],$$

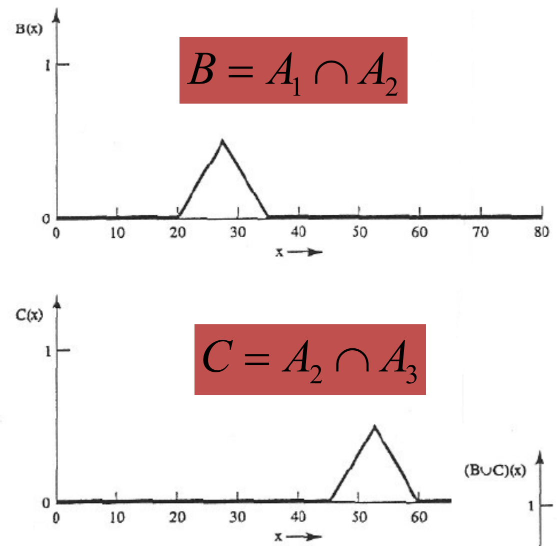
$$(A \cup B)(x) = \max[A(x), B(x)],$$

where min and max denote the minimum operator and the maximum operator, respectively.

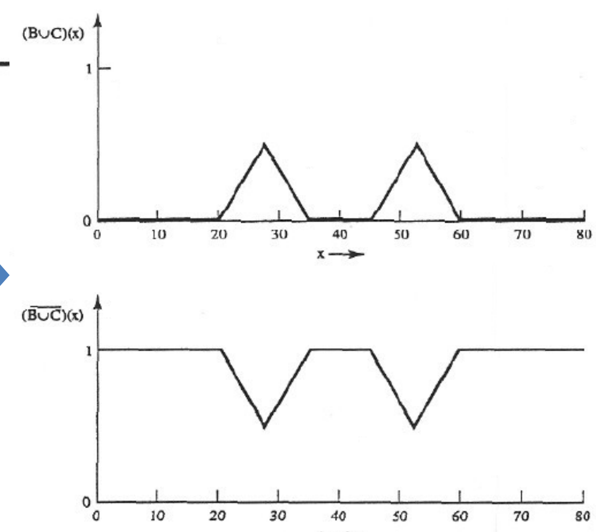


3. Fuzzy sets: basic concepts

- Another example:
  - A_1, A_2, A_3 are normal.
  - B and C are subnormal.
  - B and C are convex.
  - $B \cup C$ and $\overline{B \cup C}$ are not convex.



Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.



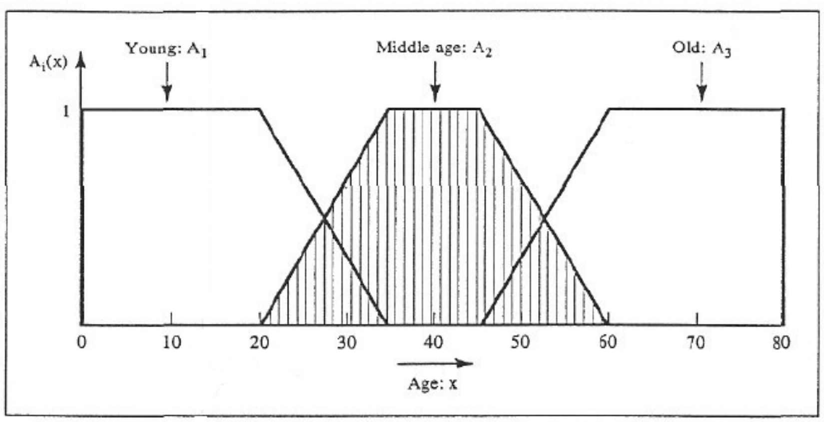


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

Figure 1.13 Illustration of standard operation on fuzzy sets $B = A_1 \cap A_2$ and $C = A_2 \cap A_3$ (A_1, A_2, A_3 are given in Fig. 1.7).

3. Fuzzy sets: basic concepts

- Discussions:
  - Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.
  - The fuzzy intersection and fuzzy union will satisfies all the properties of the Boolean lattice listed in Table 1.1 except the law of contradiction and the law of excluded middle.

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

3. Fuzzy sets: basic concepts

- The law of contradiction

$$A \cap \overline{A} = \phi$$

- To verify that the law of contradiction is violated for fuzzy sets, we need only to show that

$$\min[A(x), 1 - A(x)] = 0$$

is violated for at least one $x \in X$.

  - This is easy since the equation is obviously violated for any value $A(x) \in (0,1)$, and is satisfied only for $A(x) \in \{0,1\}$.

3. Fuzzy sets: basic concepts

- To verify the law of absorption,

  $A \cup (A \cap B) = A$

  - This requires showing that $\max[A(x), \min[A(x), B(x)]] = A(x)$ is satisfied for all $x \in X$
  - Consider two cases:

    (1) $A(x) \le B(x)$

    $\Rightarrow \max[A(x), \min[A(x), B(x)]] = \max[A(x), A(x)] = A(x)$

    (2) $A(x) > B(x)$

    $\Rightarrow \max[A(x), \min[A(x), B(x)]] = \max[A(x), B(x)] = A(x)$

$\Rightarrow \max[A(x), \min[A(x), B(x)]] = A(x)$

3. Fuzzy sets: basic concepts

- Given two fuzzy set $A, B \in \mathcal{F}(X)$,

  we say that A is a subset of B and write $A \subseteq B$ iff

  $$A(x) \leq B(x)$$

  for all $x \in X$

  - $A \subseteq B$ iff $A \cap B = A$ and $A \cup B = B$ for any $A, B \in \mathcal{F}(X)$.

Conception of Fuzzy Logic

- Many decision-making and problem-solving tasks are too complex to be defined precisely
- however, people succeed by using imprecise knowledge
- Fuzzy logic resembles human reasoning in its use of approximate information and uncertainty to generate decisions.

Natural Language

- Consider:
  - Joe is tall -- what is tall?
  - Joe is very tall -- what does this differ from tall?

- Natural language (like most other activities in life and indeed the universe) is not easily translated into the absolute terms of 0 and 1.

"false" "true"

Fuzzy Logic

- An approach to uncertainty that combines real values [0…1] and logic operations

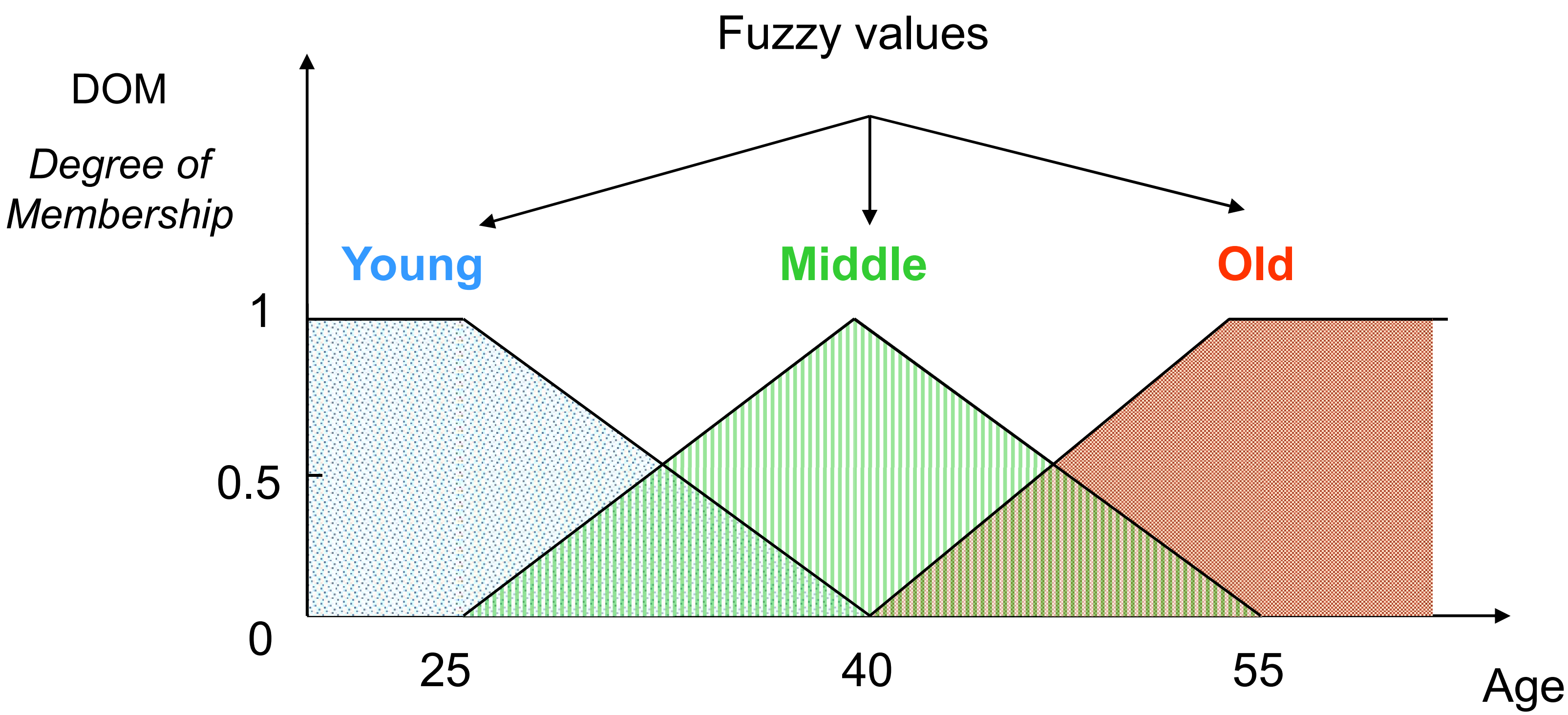
- Fuzzy logic is based on the ideas of fuzzy set theory and fuzzy set membership often found in natural (e.g., spoken) language.

Example: “Young”

- Example:
  - Ann is 28, 0.8 in set “Young”
  - Bob is 35, 0.1 in set “Young”
  - Charlie is 23, 1.0 in set “Young”

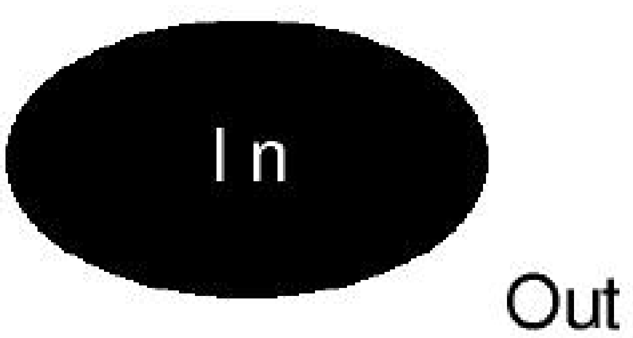
- Unlike statistics and probabilities, the *degree* is not describing *probabilities* that the item is in the set, but instead describes *to what extent* the item is the set.

Membership function of fuzzy logic

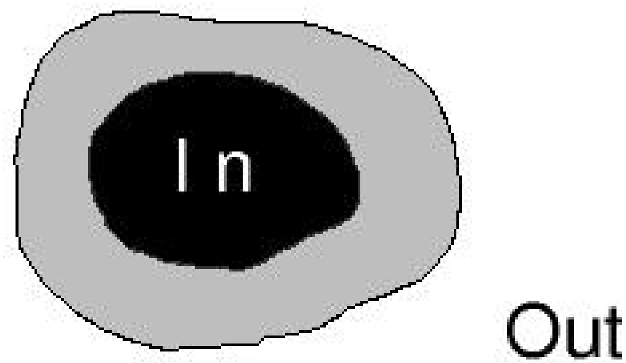


Fuzzy values have associated degrees of membership in the set.

Crisp set vs. Fuzzy set

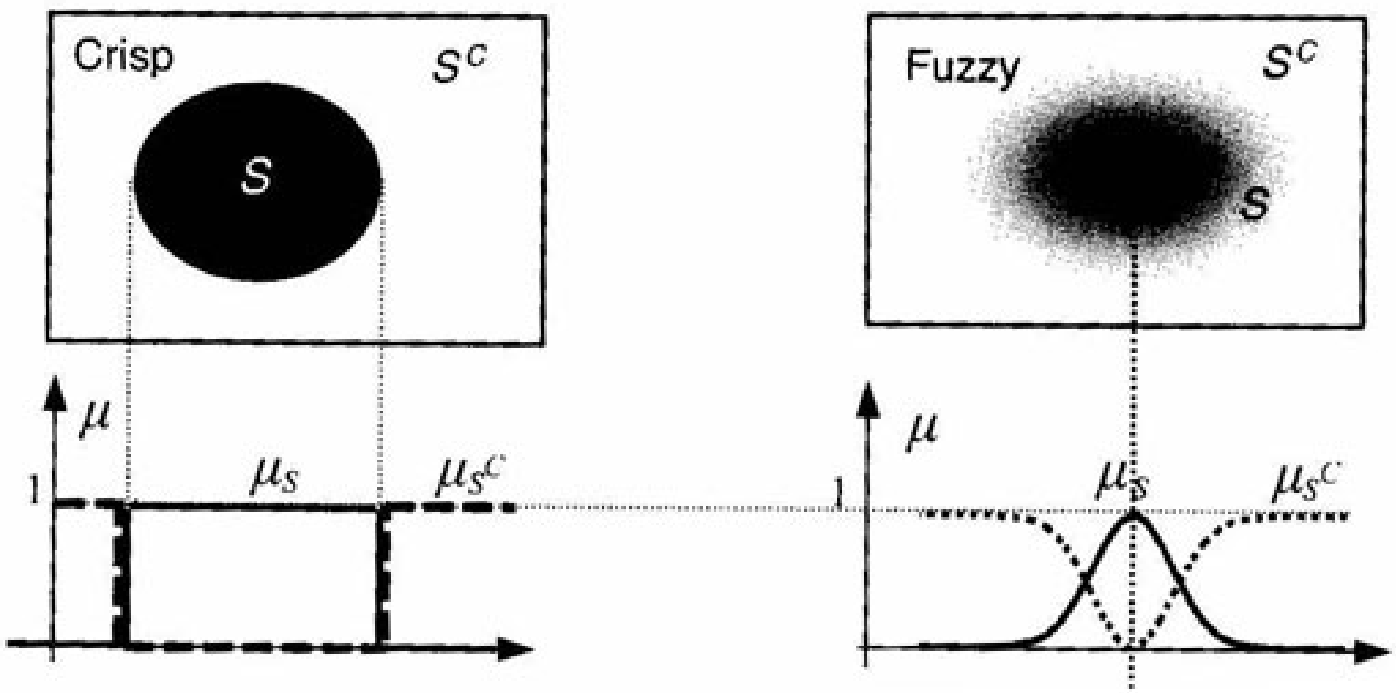


A traditional crisp set



A fuzzy set

Crisp set vs. Fuzzy set



Benefits of fuzzy logic

- You want the value to switch gradually as *Young* becomes *Middle* and *Middle* becomes *Old*. This is the idea of fuzzy logic.

Fuzzy Set Operations

- Fuzzy union ($\cup$): the union of two fuzzy sets is the maximum (MAX) of each element from two sets.
- E.g.
  - A = {1.0, 0.20, 0.75}
  - B = {0.2, 0.45, 0.50}
  - $A \cup B$ = {MAX(1.0, 0.2), MAX(0.20, 0.45), MAX(0.75, 0.50)}
    = {1.0, 0.45, 0.75}

- Fuzzy intersection ($\cap$): the intersection of two fuzzy sets is just the MIN of each element from the two sets.
- E.g.
  - $A \cap B$ = {MIN(1.0, 0.2), MIN(0.20, 0.45), MIN(0.75, 0.50)} = {0.2, 0.20, 0.50}

Fuzzy Set Operations

- The *complement* of a fuzzy variable with DOM *x* is (1-x).
- Complement ($\_^c$): The *complement* of a fuzzy set is composed of all elements' *complement.*
- Example.
  - $A^c = \{1 – 1.0, 1 – 0.2, 1 – 0.75\} = \{0.0, 0.8, 0.25\}$

Crisp Relations

- Ordered pairs showing connection between two sets:

  (a,b): a is related to b

  (2,3) are related with the relation “<“

- Relations are set themselves

  < = {(1,2), (2, 3), (2, 4), ….}

- Relations can be expressed as matrices

<	1	2
1	×	☺
2	×	×

…

Fuzzy Relations

- Triples showing connection between two sets:

  (a,b,#): a is related to b with degree #

- Fuzzy relations are set themselves

- Fuzzy relations can be expressed as matrices

…

Fuzzy Relations Matrices

- Example: Color-Ripeness relation for tomatoes

$R_1(x, y)$	unripe	semi ripe	ripe
green	1	0.5	0
yellow	0.3	1	0.4
Red	0	0.2	1

Where is Fuzzy Logic used?

- Fuzzy logic is used directly in very few applications.

- Most applications of fuzzy logic use it as the underlying logic system for decision support systems.

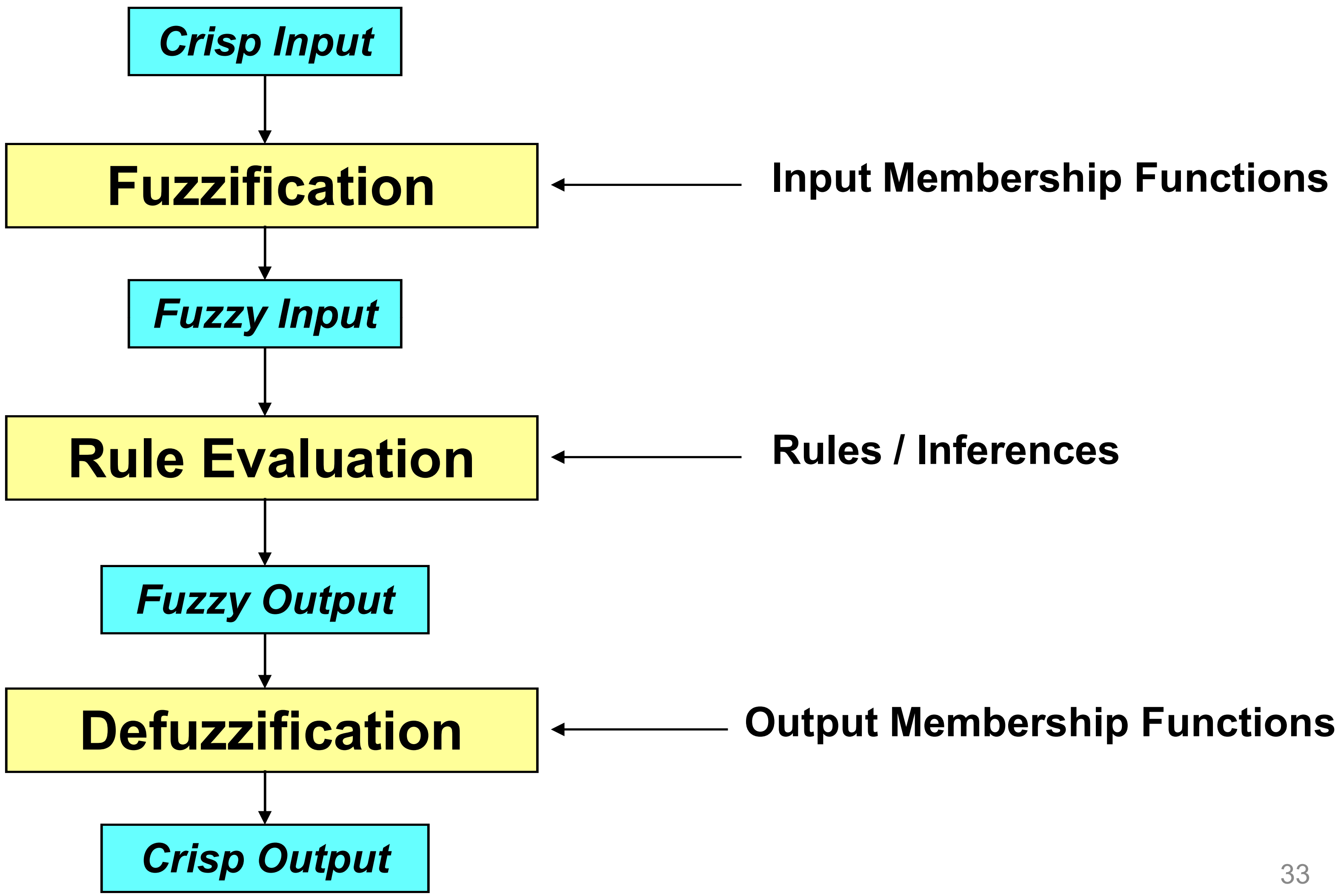
Fuzzy Expert System

- Fuzzy expert system is a collection of membership functions and rules that are used to reason about data.

- Usually, the rules in a fuzzy expert system are have the following form:

  “if x is low and y is high then z is medium”

Operation of Fuzzy System



Building Fuzzy Systems

- Fuzzification
- Inference
- Composition
- Defuzzification

Fuzzification

- Establishes the fact base of the fuzzy system. It identifies the input and output of the system, defines appropriate IF THEN rules, and uses raw data to derive a membership function.
- Consider an air conditioning system that determine the best circulation level by sampling temperature and moisture levels. The inputs are the current temperature and moisture level. The fuzzy system outputs the best air circulation level: “none”, “low”, or “high”. The following fuzzy rules are used:

  1. If the room is hot, circulate the air a lot.
  2. If the room is cool, do not circulate the air.
  3. If the room is cool and moist, circulate the air slightly.

  - A knowledge engineer determines membership functions that map temperatures to fuzzy values and map moisture measurements to fuzzy values.

Inference

- Evaluates all rules and determines their truth values. If an input does not precisely correspond to an IF THEN rule, partial matching of the input data is used to interpolate an answer.
- Continuing the example, suppose that the system has measured temperature and moisture levels and mapped them to the fuzzy values of .7 and .1 respectively. The system now infers the truth of each fuzzy rule. To do this a simple method called MAX-MIN is used. This method sets the fuzzy value of the THEN clause to the fuzzy value of the IF clause. Thus, the method infers fuzzy values of 0.7, 0.1, and 0.1 for rules 1, 2, and 3 respectively.

Composition

- Combines all fuzzy conclusions obtained by inference into a single conclusion. Since different fuzzy rules might have different conclusions, consider all rules.
- Continuing the example, each inference suggests a different action
  - rule 1 suggests a "high" circulation level
  - rule 2 suggests turning off air circulation
  - rule 3 suggests a "low" circulation level.
- A simple MAX-MIN method of selection is used where the maximum fuzzy value of the inferences is used as the final conclusion. So, composition selects a fuzzy value of 0.7 since this was the highest fuzzy value associated with the inference conclusions.

Defuzzification

- Convert the fuzzy value obtained from composition into a “crisp” value. This process is often complex since the fuzzy set might not translate directly into a crisp value.Defuzzification is necessary, since controllers of physical systems require discrete signals.
- Continuing the example, composition outputs a fuzzy value of 0.7. This imprecise value is not directly useful since the air circulation levels are “none”, “low”, and “high”. The defuzzification process converts the fuzzy output of 0.7 into one of the air circulation levels. In this case it is clear that a fuzzy output of 0.7 indicates that the circulation should be set to “high”.

Defuzzification

- There are many defuzzification methods. Two of the more common techniques are the centroid and maximum methods.
- In the centroid method, the crisp value of the output variable is computed by finding the variable value of the center of gravity of the membership function for the fuzzy value.
- In the maximum method, one of the variable values at which the fuzzy subset has its maximum truth value is chosen as the crisp value for the output variable.

Application of Fuzzy Logic

- Automatic Control System
- Domestic application and embedded system
- Used in washing machine ,air conditioner
- Fuzzy inference diagnosis of diabetes and prostate cancer
- Pattern recognition and classification

Limitation of fuzzy Logic

- Determining exact fuzzy rules and membership function is a hard task
- Verification and validation of a fuzzy knowledge-based system requires extensive testing with hardware
- Stability is an important concern for fuzzy control

Thank You