The Hessian matrix: Eigenvalues, concavity, and curvature

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1 Introduction

Students of courses in multivariable calculus are often taught the so-called "D-test" for optimizing functions of two variables:

Theorem 1.1. *Suppose $z = f(x, y)$ has continuous second partial derivatives. Let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$, and suppose (x_0, y_0) is a critical point of f.*

a. If $D(x_0, y_0) > 0$ and

  a. if $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum value of f;
  b. if $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum value of f.

b. If $D(x_0, y_0) < 0$ then (x_0, y_0) is a saddle point of f.

c. In all other cases, no conclusion can be drawn without further information.

This theorem is an application of a theorem upon which we can improve, namely:

Theorem 1.2. *Suppose $z = f(x, y)$ has continuous second partial derivatives at (x_0, y_0), and let $D(x, y) = f_{xx}f_{yy} - f_{xy}^2$.*

a. If $D(x_0, y_0) > 0$ and

  a. if $f_{xx}(x_0, y_0) > 0$, then f is concave up at (x_0, y_0);
  b. if $f_{xx}(x_0, y_0) < 0$, then f is concave down at (x_0, y_0).

b. If $D(x_0, y_0) < 0$ then the concavity of f is inconsistent at (x_0, y_0).

c. In all other cases, no conclusion can be drawn without further information.

Note that Theorem 1.2 says nothing about critical points. It's valid anywhere f has continuous second partial derivatives.

Theorems 1.1 and 1.2 are fine, as far as they go, but they don't go far enough for my tastes. In this document, you will learn about the relationship between curvature, the concavity of a surface, and the eigenvalues of the Hessian matrix of f. We will begin with a look at the local quadratic approximation, to see how the Hessian matrix can be involved.

2 The Hessian matrix and the local quadratic approximation

Recall that the Hessian matrix of $z = f(x, y)$ is defined to be

$$H_f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix},$$

at any point at which all the second partial derivatives of f exist.

Example 2.1. If $f(x, y) = 3x^2 - 5xy^3$, then $H_f(x, y) = \begin{bmatrix} 6 & -15y^2 \\ -15y^2 & -30xy \end{bmatrix}$. Note that the Hessian matrix is a function of x and y. Also note that $f_{xy} = f_{yx}$ in this example. This is because f is a polynomial, so its mixed second partial derivatives are continuous, so they are equal.[1]

All of the examples in this document will enjoy the property that $f_{xy} = f_{yx}$, an assumption that is very often reasonable. Therefore, we will assume the Hessian matrix of f reduces to

$$H_f(x, y) = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}.$$

The quantity D of the "D-test" mentioned in the introduction is actually the determinant of the Hessian matrix:

$$\det\left(H_f(x, y)\right) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{xy} = f_{xx}f_{yy} - f_{xy}^2 = D.$$

From now on, we'll call D the *Hessian determinant of f*. What does the Hessian determinant of f have to do with optimizing f?

Recall that the local quadratic approximation to $z = f(x, y)$ at (x_0, y_0) is

$$f(x, y) \approx f(x_0, y_0) + \vec{\nabla} f(x_0, y_0) \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} x - x_0 & y - y_0 \end{bmatrix} H_f(x_0, y_0) \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix},$$

for (x, y) sufficiently close to (x_0, y_0). If you've read the document *Increments, differentials, and local approximations* (click here), then you know that our excuse for using the Hessian matrix in this approximation is that second derivatives give information about concavity. We wanted to use information about concavity to improve on the local *linear* approximation, which only uses information about "slope."

Problem is, f has four second partial derivatives—four measures of concavity.[2] That's too many to keep track of, so it would be nice to have some way to combine them. One way is to calculate the Hessian determinant, which is the "D" of the "D-test." Another way is to calculate the so-called "eigenvalues" of the Hessian matrix, which are the subject of the next section. Until then, let the following exercise and theorem amuse and amaze you.

[1] If the mixed second partial derivatives are not continuous at some point, then they may or may not be equal there.
[2] Yes, two of them are the same in this document, so there are only three different ones. I'm going for drama here, OK?

Exercise 2.1. The points $(0,0)$, $(\pi/2, \pi/2)$, and $(\pi/2, -\pi/2)$ are critical points of $f(x, y) = \sin x \sin y$.

a. Calculate the local quadratic approximation to f at each of the three given critical points.

b. Graph the surface $z = f(x, y)$ together with the local quadratic approximations at the three given critical points.

c. What does the concavity of the local quadratic approximation at a given point have to do with the concavity of the surface at that point?

The results of the foregoing exercise are not coïncidental:

Theorem 2.2. *Suppose $f(x, y)$ has continuous third partial derivatives.*

a. If the local quadratic approximation to f is concave up at (x_0, y_0), so is the surface $z = f(x, y)$.

b. If the local quadratic approximation to f is concave down at (x_0, y_0), so is the surface.

c. If the concavity of the local quadratic approximation to f at (x_0, y_0) is inconsistent, then so is the concavity of the surface.

Apparently, the Hessian *matrix* somehow "knows" whether the surface is concave up or down. However, the Hessian *determinant* mixes up the information inherent in the Hessian matrix in such a way as to not be able to tell up from down: recall that if $D(x_0, y_0) > 0$, then additional information is needed, to be able to tell whether the surface is concave up or down. (We typically use the sign of $f_{xx}(x_0, y_0)$, but the sign of $f_{yy}(x_0, y_0)$ will serve just as well.)

The eigenvalues of H_f will tell us whether the surface is concave up, concave down, or a little of both, at any given point. To find out how, read on.

3 The eigenvalues of the Hessian matrix

Introducing eigenvalues to students who have never heard of them is a bit problematic. There's no good way for me to convince you that they're good for anything—that is, until you've seen some of their uses. Eigenvalues give information about a matrix; the Hessian matrix contains geometric information about the surface $z = f(x, y)$. We're going to use the eigenvalues of the Hessian matrix to get geometric information about the surface.

Here's the definition:

Definition 3.1. Let A be a square (that is, $n \times n$) matrix, and suppose there is a scalar λ and a vector $\vec{x}$ for which

$$A\vec{x} = \lambda\vec{x}.$$

Then

a. the ordered pair $(\lambda, \vec{x})$ is an *eigenpair of* A,

b. λ is an *eigenvalue of* A, and

c. $\vec{x}$ is an *eigenvector of* A *associated with* λ.

If you find the definition to be unenlightening, then think of it this way: If $(\lambda, \vec{x})$ is an eigenpair of A, then multiplying the *matrix* A by the vector $\vec{x}$ simplifies down to multiplying the *scalar* λ by $\vec{x}$. Sounds like a savings in computation, yes?

Example 3.1. The pairs $\left(5, \begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ and $\left(-1, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$ are eigenpairs of $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$. To see why, compare

$$\begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3(2)+4(1) \\ 2(2)+1(1) \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} \qquad \text{with} \qquad 5 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

$$\text{and} \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 3(1)+4(-1) \\ 2(1)+1(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{with} \qquad -1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Exercise 3.2. Again let $A = \begin{bmatrix} 3 & 4 \\ 2 & 1 \end{bmatrix}$.

a. Use the definition to determine whether $\begin{bmatrix} -\pi \\ \pi \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are eigenvectors of A associated with $\lambda = -1$.

b. Is either of the given vectors an eigenvector of A associated with $\lambda = 5$?

c. What is the point of this exercise?

If you're curious, there is a standard catechism for calculating eigenvalues by hand. You can learn it in a linear algebra course. For the purposes of this document, I will assume you can calculate eigenvalues by using a computer algebra system (CAS).[3] The only other thing I want you to know about eigenvalues at this point is that *every* 2×2 *matrix has exactly 2 eigenvalues.* Of course, the two eigenvalues might be the same number. In this case, we say that the eigenvalue has multiplicity 2. Your CAS knows about this, and has some way of telling you if an eigenvalue has multiplicity 2. So please give your CAS output the attention it deserves.

Exercise 3.3. Recall from Exercise 2.1 that $(0, 0)$, $(\pi/2, \pi/2)$, and $(\pi/2, -\pi/2)$ are critical points of $f(x, y) = \sin x \sin y$.

a. Calculate the Hessian matrix $H_f(x, y)$.

b. Fill in Table 1. (This will require you to evaluate H_f and find its eigenvalues at each of the three given critical points. You can figure out the concavity—whether up, down, or inconsistent—from the work you did for Exercise 2.1.)

Table 1: Eigenvalues of the Hessian matrix of $f(x, y) = \sin x \sin y$ at selected critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity at (x_0, y_0)
$(0, 0)$			
$(\pi/2, \pi/2)$			
$(\pi/2, -\pi/2)$			

c. Examine the table and the graph you made in Exercise 2.1. How can the eigenvalues help you classify the concavity of the surface at each critical point?

Exercise 3.4. Now let $f(x, y) = x^3 - x + y^3 - y$.

a. Graph the surface $z = f(x, y)$. (I call it "the Devil's easy chair". Can you see why?)

[3] You should be able to find instructions for this on my faculty website. If not, just do an Internet search or ask someone who knows what eigenvalues are.

b. Verify that the complete list of critical points of f is $(-1/\sqrt{3}, -1/\sqrt{3})$, $(-1/\sqrt{3}, 1/\sqrt{3})$, $(1/\sqrt{3}, -1/\sqrt{3})$, and $(1/\sqrt{3}, 1/\sqrt{3})$.

c. Calculate the Hessian matrix $H_f(x, y)$.

d. Fill in Table 2, except for the "concavity" column. (This will require you to evaluate H_f and find its eigenvalues at each of the three given critical points.)

Table 2: Eigenvalues of the Hessian matrix of $f(x, y) = x^3 - x + y^3 - y$ at selected critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity at (x_0, y_0)
$(-1/\sqrt{3}, -1/\sqrt{3})$			
$(-1/\sqrt{3}, 1/\sqrt{3})$			
$(1/\sqrt{3}, -1/\sqrt{3})$			
$(1/\sqrt{3}, 1/\sqrt{3})$			

e. Examine the graph of the Devil's easy chair and fill in the "concavity" column.

f. Examine the table carefully and tell how eigenvalues of Hessian matrices can help you classify the concavity of the surface at each critical point.

The foregoing exercises should give you reason to believe the following theorem.

Theorem 3.5. *Suppose the function $z = f(x, y)$ has continuous second partial derivatives. Let λ_1 and λ_2 be the eigenvalues of $H_f(x_0, y_0)$.*

a. If $\lambda_1 < 0$ and $\lambda_2 < 0$ then the surface is concave down at (x_0, y_0).

b. If $\lambda_1 < 0$ and $\lambda_2 > 0$ or if $\lambda_1 > 0$ and $\lambda_2 < 0$ then the concavity of the surface is inconsistent at (x_0, y_0).

c. If $\lambda_1 > 0$ and $\lambda_2 > 0$ then the surface is concave up at (x_0, y_0).

d. If either $\lambda_1 = 0$ or $\lambda_2 = 0$ (or both), then no conclusion can be drawn without further information.

Note that the above theorem says nothing about complex eigenvalues. As stated, it simply doesn't apply when eigenvalues are complex.[4] Also note that the theorem says nothing about critical points. But if you apply it to critical points, you get an "eigenvalue test for extreme values."

Theorem 3.6. *Suppose the function $z = f(x, y)$ has continuous second partial derivatives. Let (x_0, y_0) be a critical point of f, and let λ_1 and λ_2 be the eigenvalues of $H_f(x_0, y_0)$.*

a. If $\lambda_1 < 0$ and $\lambda_2 < 0$ then f has a local maximum value at (x_0, y_0).

b. If $\lambda_1 < 0$ and $\lambda_2 > 0$ or if $\lambda_1 > 0$ and $\lambda_2 < 0$ then f has a saddle point at (x_0, y_0).

c. If $\lambda_1 > 0$ and $\lambda_2 > 0$ then f has a local minimum value at (x_0, y_0).

d. If either $\lambda_1 = 0$ or $\lambda_2 = 0$ (or both), then no conclusion can be drawn without further informaiton.

Example 3.2. Let's use the eigenvalues of the Hessian matrix to classify the critical points of $f(x, y) = x^4 + y^3 - 5x^2y - 2xy^2 + x^2$. First, we need some derivatives:

$$\vec{\nabla} f(x, y) = \begin{bmatrix} 4x^3 - 10xy - 2y^2 + 2x \\ 3y^2 - 5x^2 - 4xy \end{bmatrix} \qquad \text{and} \qquad H_f(x, y) = \begin{bmatrix} 12x^2 - 10y + 2 & -10x - 4y \\ -10x - 4y & 6y - 4x \end{bmatrix}$$

[4] If your Hessian matrix has complex eigenvalues, ask your instructor how to interpret them. It's not hard; it's just beyond the scope of the typical first course in multivariable calculus.

Ugh. Solving $\vec{\nabla} f = \vec{0}$ could be a nightmare. Fortunately, any CAS worth the name can solve this system of equations. Mine says[5] that the critical points are approximately $(0.0669, 0.1417)$, $(-0.3969, 0.3121)$, $(-1.2597, 0.9905)$, $(7.4786, 15.8520)$, and $(0, 0)$.

Table 3 gives the results my CAS and I got for eigenvalues of the Hessian matrix at the critical points and for the classification.

Conclusion: f has a local minimum value of about -1024.0845 at the points $(-1.2597, 0.9905)$ and $(7.4786, 15.8520)$ (approximately). It also has saddle points at $(0.0669, 0.1417)$ and $(-0.3969, 0.3121)$ (approximately). The critical point $(0, 0)$ cannot be classified using the methods of this document.

Table 3: Classification of critical points, using the eigenvalues of the Hessian matrix, for Example 3.2.

(x_0, y_0)	Eigenvalues of $H_f(x_0, y_0)$	Concavity	Classification	Value of f
$(0, 0)$	0, 2	Can't tell	Can't tell	0
$(0.0669, 0.1417)$	-0.6260,1.8454	Inconsistent	Saddle point	N/A
$(-0.3969, 0.3121)$	-0.9203,5.1502	Inconsistent	Saddle point	N/A
$(-1.2597, 0.9905)$	2.4241,19.6947	Up	Local minimum	-1024.0845
$(7.4786, 15.8520)$	26.1057,553.7338	Up	Local minimum	-1024.0845

Your turn:

Exercise 3.7. Find the critical points of $f(x, y) = x^2 + y^3 - x^2 y + xy^2$ and classify them all by using the eigenvalues of the appropriate Hessian matrices.

Exercise 3.8. Find the critical points of $f(x, y) = x^3 + y^4 - 3x^2 y$ and classify them all by using the eigenvalues of the appropriate Hessian matrices.

4 Concavity and curvature

Recall the following definition from class:

Definition 4.1. Suppose $z = f(x, y)$ has continuous second partial derivatives at (x_0, y_0).

a. If the concavity of the surface $z = f(x, y)$ is consistent at (x_0, y_0), we say the *curvature of the surface is positive at* (x_0, y_0).

b. If the concavity of the surface is inconsistent at (x_0, y_0), we say the *curvature of the surface is negative at* (x_0, y_0).

c. If the surface is flat at (x_0, y_0) (in the sense that $D(x_0, y_0) = 0$), then we say *curvature of the surface is 0 at* (x_0, y_0).

Note that curvature is positive precisely when the Hessian determinant is positive, and that curvature is negative precisely when the Hessian determinant is negative. Also, curvature is 0 exactly when the Hessian determinant is 0. This is our excuse for using the adjectives "positive," "negative," and "0" to describe curvature.

Exercise 4.2. Classify the curvature of the surface $z = \sin x \sin y$ at each of the three critical points given in Exercise 2.1. (Hint: You've already done almost all of the work for this exercise.)

[5] I actually checked this "by hand," which means I did it step by step in the CAS. The five points listed above really are the only critical points of f.

Exercise 4.3. Classify the curvature of the Devil's easy chair at each of its four critical points.

Here's a theorem from Linear Algebra that connects the sign of the curvature of a surface with the eigenvalues of the Hessian matrix.

Theorem 4.4. *The product of the eigenvalues of a 2×2 matrix is the determinant of that matrix. In symbols, if A is 2×2 and if its eigenvalues are λ_1 and λ_2, then*

$$\det(A) = \lambda_1 \lambda_2.$$

Example 4.1. Let's apply this theorem to the Devil's easy chair. This means calculating the determinant of H_f (at each critical point) in two ways: by the "$f_{xx}f_{yy} - f_{xy}^2$" method, and by multiplying the eigenvalues of H_f together.

- The eigenvalues of $H_f(-1/\sqrt{3}, -1/\sqrt{3})$ are $\lambda_1 = -2\sqrt{3}$ and $\lambda_2 = -2\sqrt{3}$. Note that
$$\det\left(H_f(-1/\sqrt{3}, -1/\sqrt{3})\right) = 12 = (-2\sqrt{3})(-2\sqrt{3}) = \lambda_1\lambda_2.$$
- The eigenvalues of $H_f(-1/\sqrt{3}, 1/\sqrt{3})$ are $\lambda_1 = -2\sqrt{3}$ and $\lambda_2 = 2\sqrt{3}$. Observe that
$$\det\left(H_f(-1/\sqrt{3}, 1/\sqrt{3})\right) = -12 = (-2\sqrt{3})(2\sqrt{3}) = \lambda_1\lambda_2.$$
- The eigenvalues of $H_f(1/\sqrt{3}, -1/\sqrt{3})$ are $\lambda_1 = 2\sqrt{3}$ and $\lambda_2 = -2\sqrt{3}$. Notice that
$$\det\left(H_f(1/\sqrt{3}, -1/\sqrt{3})\right) = -12 = (2\sqrt{3})(-2\sqrt{3}) = \lambda_1\lambda_2.$$
- The eigenvalues of $H_f(1/\sqrt{3}, 1/\sqrt{3})$ are $\lambda_1 = 2\sqrt{3}$ and $\lambda_2 = 2\sqrt{3}$. As you expect,
$$\det\left(H_f(1/\sqrt{3}, 1/\sqrt{3})\right) = 12 = (2\sqrt{3})(2\sqrt{3}) = \lambda_1\lambda_2.$$

Exercise 4.5. Apply Theorem 4.4 to the three critical points of $f(x, y) = \sin x \sin y$ given in Exercise 2.1.

Exercise 4.6. I said that Theorem 4.4 connects the sign of the curvature with the eigenvalues of the Hessian matrix. What is this connection?

5 Conclusion

In summary, the eigenvalues of the Hessian matrix give you precisely the information you need for classifying critical points (unless, of course one of them is 0). The D-test, on the other hand, multiplies the eigenvalues together, hiding some of the information the eigenvalues contain. Table 4 shows how it all fits together.

If you're curious whether the ideas in this document still apply when there are three or more independent variables, the short answer is "Yes." The long answer is surprisingly long and complicated, and won't make sense to you until you know what a "minor" of an entry in a matrix is. Until then, stick to functions of two variables!

Exercise 5.1. Find all 13 critical points of $f(x, y) = x^3y^3 - 3xy^3 - 3x^3y + 9xy$, and use the eigenvalues of the Hessian matrix to classify them all. Also decide whether curvature is positive, negative, or 0 at each critical point. (Hint: A nice graph of the surface might help.) **Warning: Failure to use a CAS effectively will likely drive you crazy**.

Table 4: Relationship between the eigenvalues of the Hessian matrix, the Hessian determinant, and the concavity and curvature of a given surface. The point (x_0, y_0) need not be a critical point.

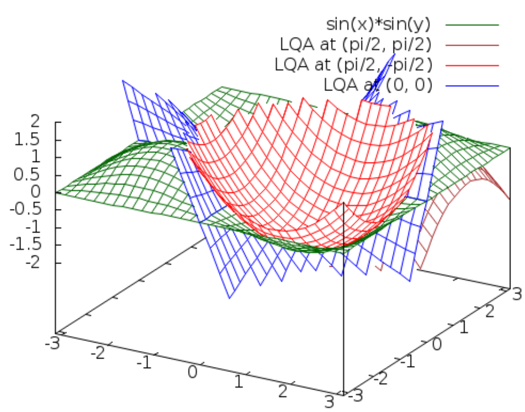
Eigenvalues of $H_f(x_0, y_0)$	D	Concavity	Curvature	Notes
$\lambda_1 > 0,\ \lambda_2 > 0$	$D > 0$	Consistent	Positive	The surface is concave up (and if (x_0, y_0) is a critical point, then f has a local minimum value there), but D can't tell you this. You need more information.
$\lambda_1 > 0,\ \lambda_2 < 0$	$D < 0$	Inconsistent	Negative	If (x_0, y_0) is a critical point, then f has a saddle point there.
$\lambda_1 < 0,\ \lambda_2 > 0$	$D < 0$	Inconsistent	Negative	If (x_0, y_0) is a critical point, then f has a saddle point there.
$\lambda_1 < 0,\ \lambda_2 < 0$	$D > 0$	Consistent	Positive	The surface is concave down (and if (x_0, y_0) is a critical point, then f has a local maximum value there), but D can't tell you this. You need more information.

6 Answers to the exercises

Exercise 2.1. The points $(0, 0)$, $(\pi/2, \pi/2)$, and $(\pi/2, -\pi/2)$ are critical points of $f(x, y) = \sin x \sin y$.

a. The required local quadratic approximations are

- $z \approx xy$ at $(0, 0)$,
- $z \approx 1 - \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 - \frac{1}{2}\left(y + \frac{\pi}{2}\right)^2$, at $(\pi/2, \pi/2)$, and
- $z \approx -1 + \frac{1}{2}\left(x - \frac{\pi}{2}\right)^2 + \frac{1}{2}\left(y + \frac{\pi}{2}\right)^2$, at $(\pi/2, -\pi/2)$.



b.

c. At any critical point, the surface and the local quadratic approximation either 1) are both concave up, 2) are both concave down, or 3) both have a saddle point.

Exercise 3.2.

a. $\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}-\pi\\ \pi\end{bmatrix}=\begin{bmatrix}\pi\\ -\pi\end{bmatrix}=-1\begin{bmatrix}-\pi\\ \pi\end{bmatrix}$; yes.

$\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}11\\ 4\end{bmatrix}\neq-1\begin{bmatrix}1\\ 2\end{bmatrix}$; no.

b. $\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}-\pi\\ \pi\end{bmatrix}=\begin{bmatrix}\pi\\ -\pi\end{bmatrix}\neq 5\begin{bmatrix}-\pi\\ \pi\end{bmatrix}$; no.

$\begin{bmatrix}3 & 4\\ 2 & 1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}11\\ 4\end{bmatrix}\neq 5\begin{bmatrix}1\\ 2\end{bmatrix}$; no.

c. A given vector doesn't have to be an eigenvector of A, and if it *is*, it doesn't have to be associated with both eigenvalues.[6]

Exercise 3.3.

a. $H_f(x,y)=\begin{bmatrix}-\sin x\sin y & \cos x\cos y\\ \cos x\cos y & -\sin x\sin y\end{bmatrix}$.

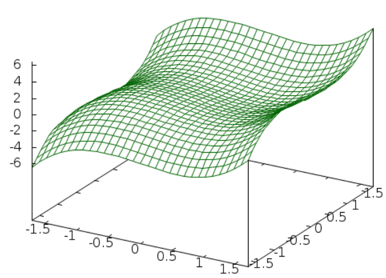
b. See table 5.

Table 5: Eigenvalues of the Hessian matrix of $f(x,y)=\sin x\sin y$ at selected critical points, with concavity.

Critical point (x_0,y_0)	$H_f(x_0,y_0)$	Eigenvalues of $H_f(x_0,y_0)$	Concavity at (x_0,y_0)
$(0,0)$	$\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$	$-1,\ -1$	Down
$(\pi/2,\pi/2)$	$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$	$1,\ 1$	Up
$(\pi/2,-\pi/2)$	$\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$	$-1,\ 1$	Inconsistent

c. Evidently, if both eigenvalues are positive, the concavity is "up." If both are negative, then "down." If there's a positive eigenvalue and a negative eigenvalue, then the concavity is "inconsistent."

Exercise 3.4.

a. 

(I call it "the Devil's easy chair" because you can sit in it if you like, but if you slip out, it's a loooooong way down!)

[6]If you take an introductory linear algebra course, you will find out that if $\lambda_1\neq\lambda_2$, then an eigenvector of A associated with λ_1 *cannot* be an eigenvector of A associated with λ_2

b. Stationary points: $\vec{\nabla} f = \begin{bmatrix} 3x^2 - 1 \\ 3y^2 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $x^2 = 1/3$, so that $x = \pm 1/\sqrt{3}$, and similarly for y. Knowing that $x = \pm 1/\sqrt{3}$ tells us nothing about y (we say that x and y are *uncoupled*). Therefore, we have to match each value of x that we've found with each value of y that we've found, to get all the critical points. I abbreviate the resulting list of four critical points this way: $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$.

   Singular points: $\vec{\nabla} f$ exists everywhere in the plane because f is a polynomial. So, there are no singular points. Therefore, the complete list of critical points of f is $(\pm 1/\sqrt{3}, \pm 1/\sqrt{3})$.

c. $H_f(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & 6y \end{bmatrix}$.

d. See Table 6.

e. See the rightmost column of Table 6.

f. Evidently, if both eigenvalues are positive, the concavity is "up." If both negative, then "down." If one of each, then "inconsistent."

Table 6: Eigenvalues of the Hessian matrix of $f(x, y) = x^3 - x + y^3 - y$ at all critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity at (x_0, y_0)
$(-1/\sqrt{3}, -1/\sqrt{3})$	$\begin{bmatrix} -6/\sqrt{3} & 0 \\ 0 & -6/\sqrt{3} \end{bmatrix}$	$-2\sqrt{3},\ -2\sqrt{3}$	Down
$(-1/\sqrt{3}, 1/\sqrt{3})$	$\begin{bmatrix} -6/\sqrt{3} & 0 \\ 0 & 6/\sqrt{3} \end{bmatrix}$	$-2\sqrt{3},\ 2\sqrt{3}$	Inconsistent
$(1/\sqrt{3}, -1/\sqrt{3})$	$\begin{bmatrix} 6/\sqrt{3} & 0 \\ 0 & -6/\sqrt{3} \end{bmatrix}$	$2\sqrt{3},\ -2\sqrt{3}$	Inconsistent
$(1/\sqrt{3}, 1/\sqrt{3})$	$\begin{bmatrix} 6/\sqrt{3} & 0 \\ 0 & 6/\sqrt{3} \end{bmatrix}$	$2\sqrt{3},\ 2\sqrt{3}$	Up

Exercise 3.7. See Table 7.

Table 7: Eigenvalues of the Hessian matrix of $f(x, y) = x^2 + y^3 - x^2 y + xy^2$ at all critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity at (x_0, y_0)
$(-2/3, 2/3)$	$\begin{bmatrix} 2/3 & 8/3 \\ 8/3 & 8/3 \end{bmatrix}$	$\frac{-5+\sqrt{73}}{3} < 0,\ \frac{5+\sqrt{73}}{3} > 0$	Inconsistent
$(0, 0)$	$\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$	$0,\ 2$	Unknown (so far)
$(18/5, 6/5)$	$\begin{bmatrix} -2/5 & -24/5 \\ -24/5 & 72/5 \end{bmatrix}$	$\frac{35+\sqrt{1945}}{5} > 0,\ \frac{-35+\sqrt{1945}}{5} < 0$	Inconsistent

(If you're interested, I calculated the directional second derivatives at (0,0) in all possible directions, and found them all to be positive except the y-direction. $f_{yy}(0, 0) = 0$, so the surface is flat in that direction. So: At the origin, the surface is concave up in every direction but one, in which direction it's flat. Sounds like "concave up" to me.)

Exercise 3.8. See Table 8

Table 8: Eigenvalues of the Hessian matrix of $f(x,y) = x^3 + y^4 - 3x^2y$ at selected critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity at (x_0, y_0)
$(0,0)$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$	0,0	Unknown (so far)
$(6,3)$	$\begin{bmatrix} 6/\sqrt{3} & 0 \\ 0 & 6/\sqrt{3} \end{bmatrix}$	$2\sqrt{3}$, $2\sqrt{3}$	Up

(If you're interested, I calculated the directional second derivatives at (0,0) in all possible directions, and got 0 for every one of them. So the surface is pretty flat at the origin, but it may still be concave up (like $y = x^4$ is), concave down (line $y = -x^4$ is), or have inconsistent concavity there (like $y = x^5$ does). More work is needed to find out, and I'm out of time. Sorry!)

Exercise 4.2. The curvature of the surface $z = \sin x \sin y$ is:

- Negative at $(\pi/2, -\pi/2)$
- Positive at $(\pi/2, \pi/2)$
- 0 at $(0,0)$

Exercise 4.3. The curvature of the Devil's easy chair is:

- Positive at $(-1/\sqrt{3}, -1/\sqrt{3})$,
- Negative at $(-1/\sqrt{3}, 1/\sqrt{3})$,
- Negative at $(1/\sqrt{3}, -1/\sqrt{3})$,
- Positive at $(1/\sqrt{3}, 1/\sqrt{3})$,

Exercise 4.5. At $(\pi/2, \pi/2)$, the determinant of the Hessian matrix is 1, which is the product of the eigenvalues: $-1 \times -1 = 1$. Likewise at $(\pi/2, -\pi/2)$, except there the eigenvalues are 1 and 1.

At $(0,0)$, the determinant of the Hessian matrix is -1, which again is the product of the eigenvalues: $-1 \times 1 = -1$.

Exercise 4.6. If the two eigenvalues have the same sign, curvature is positive. If they have opposite signs, curvature is negative. (If either is 0, then we can't say anything about curvature, without more information.)

Exercise 5.1. Because f is a polynomial function, it has no singular points. Stationary points: $\vec{\nabla} f = \begin{bmatrix} 3\,x^2\,y^3 - 3\,y^3 - 9\,x^2\,y + 9\,y \\ 3\,x^3\,y^2 - 9\,x\,y^2 - 3\,x^3 + 9\,x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has the following 13 solutions:

$$(0,0), \quad (0, \pm\sqrt{3}), \quad (\pm\sqrt{3}, 0), \quad (\pm\sqrt{3}, \pm\sqrt{3}), \quad \text{and} \quad (\pm 1, \pm 1)$$

The Hessian matrix is

$$H_f(x,y) = \begin{bmatrix} 6\,x\,y^3 - 18\,x\,y & 9\,x^2\,y^2 - 9\,y^2 - 9\,x^2 + 9 \\ 9\,x^2\,y^2 - 9\,y^2 - 9\,x^2 + 9 & 6\,x^3\,y - 18\,x\,y \end{bmatrix}.$$

Table 9 gives the eigenvalues of the Hessian matrix at each critical point, along with conclusions about concavity and curvature. Note well that if curvature is negative, there can only be one classification for the critical point: a saddle point; but if the curvature is positive, the critical point may be a maximum point or a minimum point.

Table 9: Eigenvalues of the Hessian matrix of $f(x, y) = x^3y^3 - 3xy^3 - 3x^3y + 9xy$ at all critical points, with concavity.

Critical point (x_0, y_0)	$H_f(x_0, y_0)$	Eigenvalues of $H_f(x_0, y_0)$	Concavity	Curvature
$(0, 0)$	$\begin{bmatrix} 0 & 9 \\ 9 & 0 \end{bmatrix}$	-9, 9	Inconsistent	Negative
$(0, -\sqrt{3})$	$\begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}$	-18, 18	Inconsistent	Negative
$(0, \sqrt{3})$	$\begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}$	-18, 18	Inconsistent	Negative
$(-\sqrt{3}, 0)$	$\begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}$	-18, 18	Inconsistent	Negative
$(\sqrt{3}, 0)$	$\begin{bmatrix} 0 & -18 \\ -18 & 0 \end{bmatrix}$	-18, 18	Inconsistent	Negative
$(-\sqrt{3}, -\sqrt{3})$	$\begin{bmatrix} 0 & 36 \\ 36 & 0 \end{bmatrix}$	-36, 36	Inconsistent	Negative
$(-\sqrt{3}, \sqrt{3})$	$\begin{bmatrix} 0 & 36 \\ 36 & 0 \end{bmatrix}$	-36, 36	Inconsistent	Negative
$(\sqrt{3}, -\sqrt{3})$	$\begin{bmatrix} 0 & 36 \\ 36 & 0 \end{bmatrix}$	-36, 36	Inconsistent	Negative
$(\sqrt{3}, \sqrt{3})$	$\begin{bmatrix} 0 & 36 \\ 36 & 0 \end{bmatrix}$	-36, 36	Inconsistent	Negative
$(-1, -1)$	$\begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix}$	-12, -12	Down	Positive
$(-1, 1)$	$\begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$	12, 12	Up	Positive
$(1, -1)$	$\begin{bmatrix} 12 & 0 \\ 0 & 12 \end{bmatrix}$	12, 12	Up	Positive
$(1, 1)$	$\begin{bmatrix} -12 & 0 \\ 0 & -12 \end{bmatrix}$	-12, -12	Down	Positive