A PROJECT REPORT
ON

STRUCTURAL VIBRATION ANALYSIS OF SYSTEMS

BY

K.BADARI VISHAL 2018A4PS0807G

BIRLA INSTITUTE OF TECHNOLOGY & SCIENCE, PILANI
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Under

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Abstract: This report discusses the background and the basic ideas of Vibration with considerations like Degrees of Freedom, starting with single degree of freedom to multiple degrees of freedom, followed by analysis based on applying various other factors such as damping, exciting forces on the system, etc. The report also discusses further on solving the Discrete & Continuous systems followed by two classical approximate methods namely, Rayleigh's Method and Rayleigh – Ritz method.

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Introduction

BASIC IDEAS BEHIND VIBRATION

Vibration is the mechanical oscillation of a particle, member, or a body from its position of equilibrium. It is the study that relates the motion of physical bodies to the forces acting on them. The basic concepts in the mechanics of vibration are space, time, and mass (or forces). When a body is disturbed from its position, then by the elastic property of the material of the body, it tries to come back to its initial position. Such equilibriums are called stable Equilibrium.

Applications of vibration's study to natural phenomena and engineering
Vibrations occur in almost all engineering devices and can be used as a tool in pile-driving, vibratory testing of materials, and electronic units to filter out unwanted frequencies. It is also used in complex earthquake simulations.

However vibrations can cause huge structural failures and resulting in heavy economic and human life. This primarily happens when external loads frequency match the natural frequency thereby increasing the amplitude beyond acceptable limits. Common examples of such failure are **"The 1940 Tacoma Narrows Bridge"** which collapsed due to the phenomenon of resonance.

Causes of Vibration
The main causes of vibration are as follows:
- Unequal distribution of forces in a moving or rotating machinery
- External forces like wind, tides, blasts, or earthquakes
- Friction between two bodies
- Change of magnetic or electric fields
- Movement of vehicles, etc.

Requirements for Vibration

The main requirements for the vibration are as follows:

- There should be a restoring force.
- The mean position of the body should be in equilibrium.
- There must be inertia (i.e., we must have mass).

Discrete and Continuous Systems

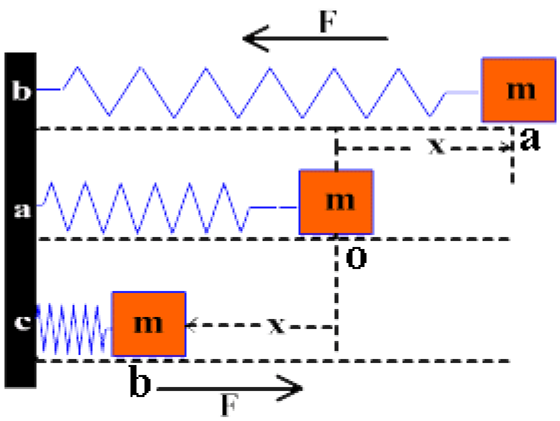
Dynamic system models may be divided into two classes, discrete and continuous (or distributed). The systems do depend on system parameters such as mass, damping, and stiffness (these will be defined later). Discrete systems are described mathematically by the variables that depend only on time. On the other hand, continuous systems are described by variables that depend on time and space. As such, the equations of motion of discrete systems are described by ordinary differential equations (ODEs), whereas the equations of motion for continuous systems are governed by partial differential equations (PDEs). Because ODE contains only one independent variable, i.e. time, and PDE contains more than one independent variable, such as time and space coordinates.

To describe a system, we need to know the variables or coordinates that describe the system and this follows a term known as degrees of freedom (DOF) and the DOF is defined as the minimum number of independent variables required to fully describe the motion of a system. If the time dependence is eliminated from the equation of discrete system, then it will be governed by a set of simultaneous algebraic equations and the continuous system will be governed by boundary value problems.

Most of the mechanical, structural, and aerospace systems can be described by using a finite number of DOFs. Continuous systems have infinite number of DOFs. Initially we try and formulate various vibration problems that occur in nature and look at the solutions later.

One Degree of Freedom Systems

We first formulate the basic governing equation of one degree of freedom system.



Note here the governing equation is very simple to understand,
In springs the spring force is proportional to the displacement i.e.

$$\mathbf{F = -kx}$$

(Note the negative sign emphasizes the fact that force vector is opposite to displacement vector.)

Thus,

$$\mathbf{-kx = mx''}$$

x'' is the second derivative with respect to time.
Thus if we write

$$\mathbf{k/m = \omega\text{^}2}$$

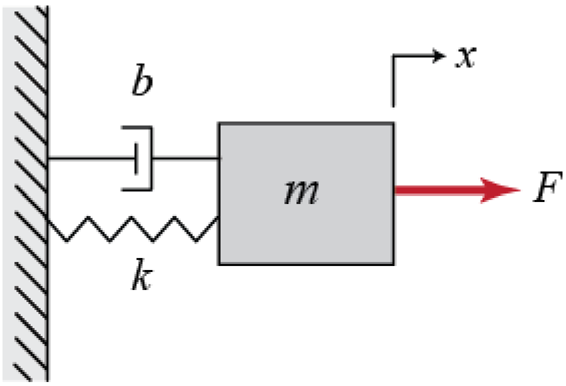
We get

$$\mathbf{x'' + \omega\text{^}2x = 0}$$

This is the governing equation for degree one freedom systems.

Simple Pendulum
Metal Thin Strip with a Mass at One End
Torsion of a Rod Having a Pulley at One End
All the above have the same governing equation.

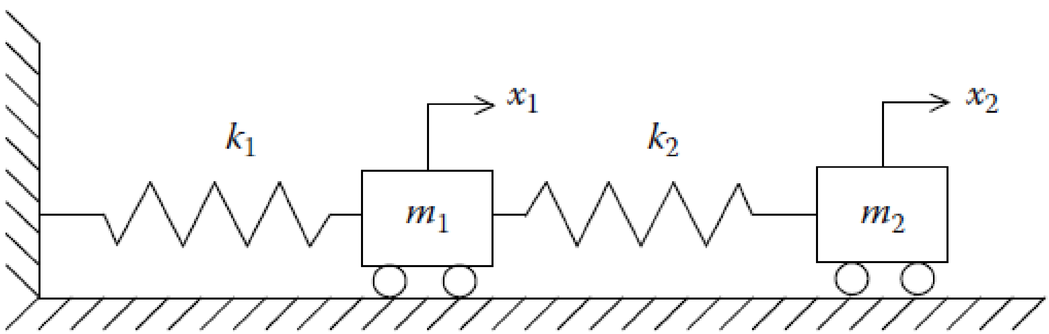
Degree one systems with damping



The above image shows a damped oscillator with damping force proportional to velocity.

$$mx'' + bx' + kx = F$$

Equation of Motion for Two Degree of Freedom System



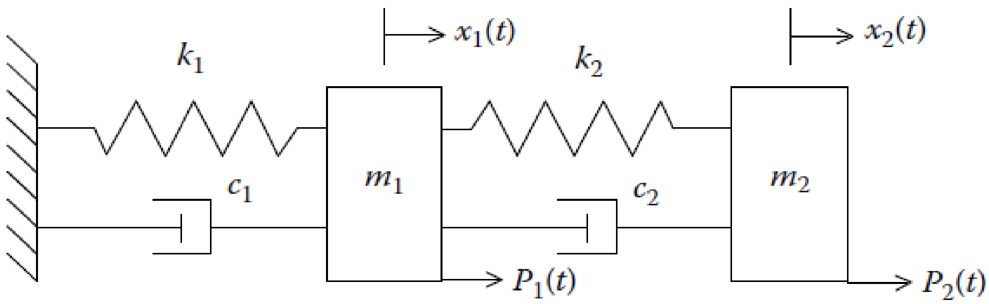
$$m_1\ddot{x}_1 + k_1x_1 = k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0$$

Above equations can be written in matrix form as -

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Two degree of freedom with damping



$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P\}$$

where

$$\{P\} = \begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix}, \text{ the force vector}$$

$$\{x\} = \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \text{ the displacement vector}$$

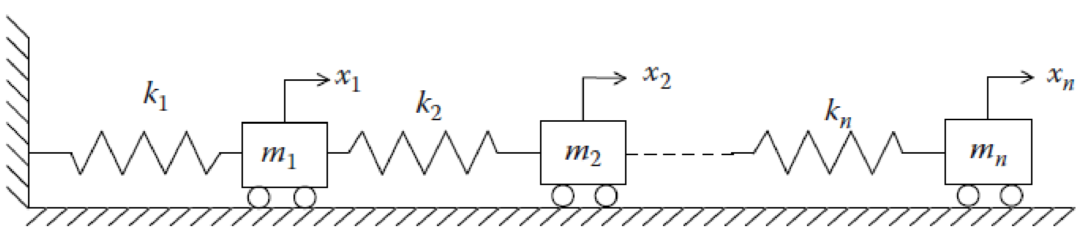
$$[M] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}, \text{ the mass matrix}$$

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix}, \text{ the damping matrix}$$

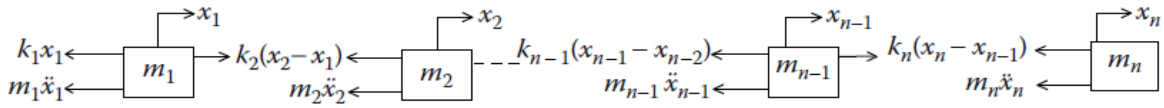
$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \text{ the stiffness matrix}$$

Multi degree of freedom

Just like we formulated the governing equation for two degrees of freedom we can generalize these to *n* degrees of freedom.



Equations for each mass are given as –



$$k_1x_1 - k_2(x_2 - x_1) + m_1\ddot{x}_1 = 0$$

$$k_2(x_2 - x_1) - k_3(x_3 - x_2) + m_2\ddot{x}_2 = 0$$

$$k_3(x_3 - x_2) - k_4(x_4 - x_3) + m_3\ddot{x}_3 = 0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$k_{n-1}(x_{n-1} - x_{n-2}) - k_n(x_n - x_{n-1}) + m_{n-1}\ddot{x}_{n-1} = 0$$

$$k_n(x_n - x_{n-1}) + m_n\ddot{x}_n = 0$$

When written in matrix form, this is equivalent to -

$$[M]\{\ddot{x}\} + [K]\{x\} = \{0\}$$

where

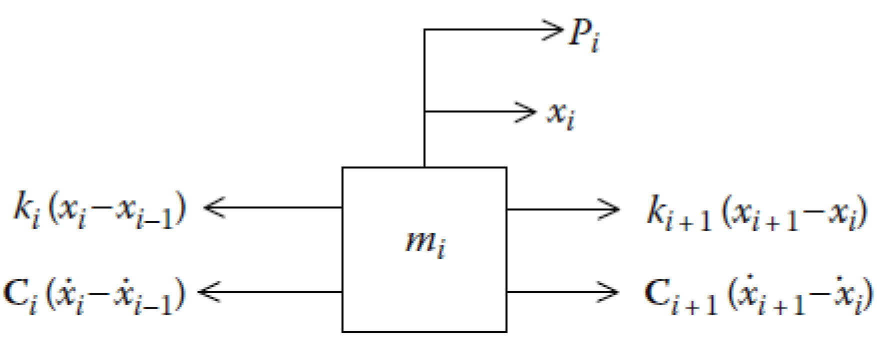
$$[M] = \begin{bmatrix} m_1 & 0 & \dots & \dots & 0 \\ 0 & m_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & m_{n-1} & 0 \\ 0 & \dots & \dots & 0 & m_n \end{bmatrix}$$

followed by K, x’’, x’ and x which are given as -

$$[K] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \dots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \dots & 0 \\ \dots & \dots & & & \dots \\ 0 & \dots & -k_{n-1} & k_{n-1} + k_n & -k_n \\ 0 & & \dots & -k_n & k_n \end{bmatrix}$$

$$\{\ddot{x}\} = \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \vdots \\ \ddot{x}_{n-1} \\ \ddot{x}_n \end{Bmatrix}, \{x\} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{Bmatrix} \quad \text{and} \quad \{0\} = \begin{Bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{Bmatrix}$$

N degrees of freedom with damping and external force



As in the case of two degrees of freedom we get a matrix equation to govern the motion of these masses which is as follows –

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{P\}$$

$$[C] = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \dots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \dots & 0 \\ \dots & \dots & & & \dots \\ 0 & \dots & -c_{n-1} & c_{n-1} + c_n & -c_n \\ 0 & & \dots & -c_n & c_n \end{bmatrix}$$

$$\{P\} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_{n-1} \\ P_n \end{Bmatrix}$$

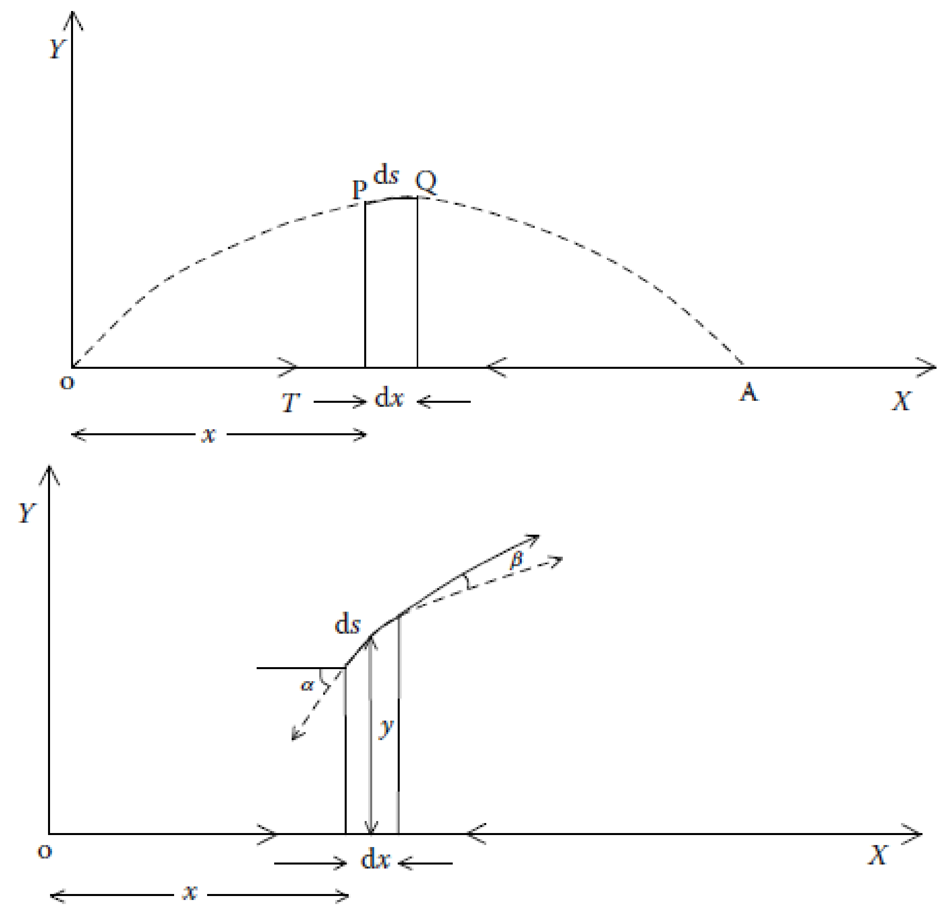
$$\{\dot{x}\} = \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{Bmatrix}$$

Continuous systems

Till now, we described the discrete systems, which are defined by finite number of DOFs and the corresponding differential equation is ordinary. But in the structural or mechanical systems such as strings, beams, rods, membranes, plates, and shells, elasticity and mass are considered to be distributed and hence these are called distributed or continuous systems. The continuous systems are designated by infinite number of DOFs.

Displacement of continuous systems is described by a continuous function of position and time and consequently will be governed by partial differential equations (PDEs).

Transverse vibration of a string



Consider a uniform elastic string stretched tightly between two fixed points O and A under tension T. Taking O as origin, OA as the axis of X and a line OY perpendicular to OX as the axis of Y, let y(x, t) denote the transverse displacement of any point of string at distance x from O at time t.

To study the motion, the following assumptions are made:

1. Entire motion takes place in the XY-plane, i.e., each particle of the string moves in a direction perpendicular to X-axis.
2. String is perfectly flexible and offers no resistance to bending.
3. Tension in the string is large enough so that the weight of the string can be neglected.
4. Transverse displacement y and the slope $\partial y/\partial x$ are small so that their squares and higher powers can be considered negligible.

Now, let m be the mass per unit length of the string and consider a differential element PQ $(= ds)$ at a distance x from O. Then $ds = \sqrt{1 + (\partial y/\partial x)^2} \approx dx$.

$$(m\ dx)\frac{\partial^2 y}{\partial t^2} = T_2 \sin\beta - T_1 \sin\alpha \text{ (along the vertical direction)}$$

$$0 = T_2 \cos\beta - T_1 \cos\alpha \text{ (along the horizontal direction)}$$

$$\begin{aligned}\frac{m dx}{T}\frac{\partial^2 y}{\partial t^2} &= \tan\beta - \tan\alpha \\ &= \left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x\end{aligned}$$

Expanding the first term in the above equation by Taylor's series, the equation of motion for vibration of string may be written as

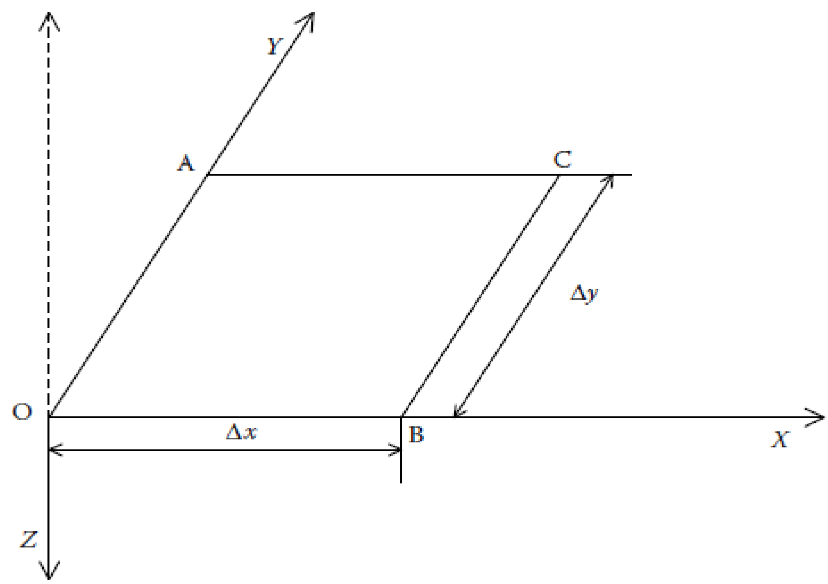
$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \tag{1.36}$$

The above equation is the governing equation for many vibration scenarios example the longitudinal vibration of a rod.

For a transverse vibration, however, the governing equation is quite different. The following is the governing equation for the same –

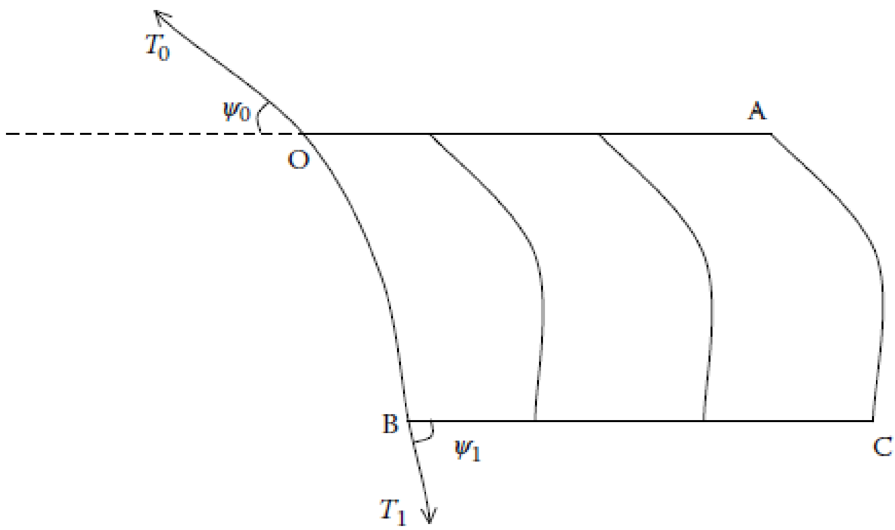
$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} = 0$$

Now we look at the topic that is of our primary interest i.e. vibration of flat membranes.



A few assumptions are made before constructing the equation which are given as follows –

1. Displacement z of any point of the membrane is perpendicular to the plane of the lemna during vibration.
2. Tension on the membrane is uniform, and therefore, we let the tension per unit length along the boundary OB, BC, CA, and AO of the membrane be P.
3. Let m be the mass of the membrane per unit area so that the mass of the elementary portion of the membrane is $m\Delta x\Delta y$.
4. As we are assuming that the displacements are perpendicular to the XY-plane during vibrations, we also assume that there is no sideway motion.



$$T_0 \cos\psi_0 = T_1 \cos\psi_1 = P\Delta y$$

Note here T0 and T1 are tensions acting along edges OA and BC. Assuming small displacement we take *cos* terms equal to 1.
Thus,

$$T_0 = T_1 = P\Delta y$$

Now considering vertical tension –

$$V_t \approx T_1 \sin\psi_1 - T_0 \sin\psi_0$$

We assume that -

$$\sin\ \psi_1 = \tan\ \psi_1 \ \text{ and } \ \sin\ \psi_0 = \tan\ \psi_0.$$

Thus,

$$V_t \approx P\Delta y \left\{ \left(\frac{\partial z}{\partial x}\right)_{\psi_1} - \left(\frac{\partial z}{\partial x}\right)_{\psi_0} \right\}$$

We now expand the above using Taylor expansion up to second order accuracy and obtain -

$$V_t \approx P\Delta x \Delta y \frac{\partial^2 z}{\partial x^2}$$

Similarly, considering the vertical component of tension for the faces OB and AC, the expression may be written as

$$\overline{V}_t \approx P\Delta x\Delta y \frac{\partial^2 z}{\partial y^2} \tag{1.52}$$

Thus adding the contributions from the two -

$$m\Delta x\Delta y \frac{\partial^2 z}{\partial t^2} = P\Delta x\Delta y \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

In a standard form it is written as -

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

For circular membranes a similar argument can be made in polar coordinates to get -

$$\frac{\partial^2 z}{\partial r^2} + \frac{1}{r}\frac{\partial z}{\partial r} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} = \frac{m}{P}\frac{\partial^2 z}{\partial t^2}$$

Initial and Boundary Conditions

We should now understand that for most part, discrete and continuous systems represent different mathematical models of the same physical system. It has already been mentioned that motion of discrete system is governed by only time variable, whereas continuous systems are governed by variables that depend not only on time but also on the space coordinates.

Essential or geometric boundary condition: These boundary conditions are demanded by the geometry of the body. This is a specified condition placed on displacements or slopes on the boundary of a physical body. Essential boundary conditions are also known as Dirichlet boundary conditions.

Natural or force (or dynamic) boundary condition: These boundary conditions are demanded by the condition of shearing force and bending moment balance.

Accordingly, this is a condition on bending moment and shear. Natural boundary conditions are also known as Neumann boundary conditions.

Energy method

Let us consider a mass m, which is attached to a massless spring with stiffness k. Let the position of equilibrium be O and x be the extension for the motion. Then, the potential energy V of the system may be written as -

$$V = \int_0^x kx \, \mathrm{d}x = \frac{1}{2}kx^2$$

and the kinetic energy T of the system becomes

$$T = \frac{1}{2}m\dot{x}^2$$

Since energy in this system is conserved, i.e.

KE + PE = constant

Thus differentiating the above with respect to time gives -

$$m\ddot{x} + kx = 0$$

which is the same as above.

Multiple degree of freedom

Essential equations of KE and PE are given below.

$$V = \frac{1}{2}\left[k_1x_1^2 + k_2(x_2 - x_1)^2 + \cdots + k_n(x_n - x_{n-1})^2 + k_{n+1}x_n^2\right]$$

$$T = \frac{1}{2}\left[m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + \cdots + m_n\dot{x}_n^2\right]$$

Vibration of string

If S and m denote the tensile force and mass per unit length of the string, then the potential energy V and the kinetic energy T for vibration of string are, respectively, given as

$$V = \frac{S}{2}\int_0^l \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x \tag{1.77}$$

$$T = \frac{m}{2}\int_0^l (\dot{y})^2 \mathrm{d}x \tag{1.78}$$

Vibration of membrane

$$V = \frac{S}{2}\int\int\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]\mathrm{d}x\,\mathrm{d}y$$

$$T = \frac{m}{2}\int\int (\dot{u})^2 \mathrm{d}x\,\mathrm{d}y$$

where -

S is the tensile force

m is the mass per unit area

Now we proceed on to how to solve these equations either analytically or numerically.

Analysis of some vibration problems

So far we have discussed the basic one degree of freedom condition with forces. Given the governing equation -

$$m\ddot{x} + c\ddot{x} + kx = P(t)$$

where P is the exciting force and x is the displacement.

Now, we will try to see the various cases that are associated with this general equation and also study the solutions of the differential equations thus formed.

CASES

- Free vibration without damping
- Free vibration with damping without the force
- Harmonic distributing force with damping
- Undamped system with sinusoidal force

The first case is the most basic with substituting the general solution of the differential equation as $A\sin(\omega t) + B\cos(\omega t)$ and solving it.

In the second case, we substitute an exponential result as a general solution to convert the second order differential equation to a\its characteristic quadratic polynomial and its solution comes out as -

$$q = \frac{-c \pm \sqrt{(c^2 - 4mk)}}{2m}$$

The square root term can be positive, negative or zero. If positive, it gets a profile of exponentials. If negative, we get a sinusoidal profile. It is convenient to express damping in terms of critical damping = $2m\omega_n$

So the damping ratio becomes $c/(2m\omega_n)$. Now it can be known that $\zeta>1$(overdamped), $\zeta=1$(critically damped), $\zeta<1$(underdamped).

In the third case, the force is harmonic, as in it fluctuates in a sinusoidal fashion.

$$P = P_0 \cos(\omega t),\ \omega/2\pi_{|\text{frequency}}\ P_{0\,|\text{constant}}$$

So we have the solution as the summation of a steady state solution and a transient solution.

$$\text{P.I.} = \frac{P_0 \cos(\omega t - \phi)}{\left[(k - m\omega^2)^2 + c^2\omega^2\right]^{1/2}}$$

$$\tan\phi = \frac{c\omega}{(k - m\omega^2)}$$

Hence, the solution has a cosine term and a sine term as the steady state solution and the particular integral (P.I.). Applying the below given conditions-

$$\mathbf{x(0) = x_0 \text{ and } dx/dt_{|x=0} = \dot{x}_0}$$

we can get our solution.

Multiple degree of freedom system

We had taken an example of two degrees of freedom systems. Now we can generalize it with multiple degrees of freedom.

Reduction to an Eigenvalue Problem for General System

Equations can be written in a matrix form as -

$$\mathbf{[M]\{x''\} + [K]\{x\} = \{0\}}$$

Where,
[M] : mass or inertia matrix
[K] : stiffness matrix
{x} : displacement

Generally M and K are symmetric matrices and we can define a dynamic matrix $[W] = [M]^{-1}[K]$

$$\{x''\} + [W]\{x\} = \{0\}$$

Now searching for harmonic solutions, we get a form of e^{iwt}
With a frequency -

$$-\omega^2\{A\} + [w]\{A\} = \{0\}$$

$([W] - \lambda[I])\{A\} = \{0\}$ is the eigenvalue representation of this.
We, therefore, have

$$\det([W] - \lambda[I]) = 0$$

which is the characteristic equation of [W].

Each eigenvalue will contribute to a different natural frequency of oscillation.

$$[W]\{A\}_i = \lambda_i\{A\}_i$$

Orthogonality

$$[W]\{A\}_i = \lambda_i\{A\}_i$$
$$[K]\{A\}_i = \lambda_i[M]\{A\}_i$$

Similarly for another random mode -

$$[K]\{A\}_j = \lambda_j[M]\{A\}_j$$

Now multiplying with their transpose and also taking the transpose of the whole, we get our orthogonal conditions -

$$0 = (\lambda_i - \lambda_j)\{A\}_j^T[M]\{A\}_i$$

We now know that the eigenvalues are distinct so,

$$\{A\}_j^T[M]\{A\}_i = 0$$

Substituting in the K expression with transpose, we simplify it as -

$$\mathbf{\{A\}_j^T[K]\,\{A\}_i = 0}$$

The above two equations represent weighted orthogonality relationships satisfied by the eigenvectors of the inverse dynamical matrix [W]. Simple type of orthogonality may occur for which [W] must be symmetric.

Modal Matrix

It is a matrix made of the modal columns placed side by side. Properties include that this diagonalizes the dynamical matrix. Next, we wish to find new properties of it and generate a new matrix P and define it as -

$$\mathbf{P = A^TMA}$$

And get the fact that P is a diagonal matrix and is called the general mass matrix and **$S = A^TKA$** is defined as a spectral matrix

Relation between P, S and λ: **$[S] = [P][\lambda]$**

SOLUTION OF DYNAMICAL PROBLEM

$$[A]^T[M][A]\{\ddot{y}\} + [A]^T[K][A]\{y\} = \{0\}$$

is the general equation. Further, using the definition of P and S -

$$[P]\{\ddot{y}\} + [S]\{y\} = \{0\}$$

Finally we get the final results as -

$$\{\ddot{y}\} + [\lambda]\{y\} = \{0\}$$

Each normal coordinate performs a SHM oscillator, we have the sinusoidal profile.

So we use the following form -

$$\{y\} = [C]\{a\} + [D]\{b\}$$

$$[C] = \begin{bmatrix} \cos(\omega_1 t) & 0 & 0 & \dots \\ 0 & \cos(\omega_2 t) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

$$[D] = \begin{bmatrix} \sin(\omega_1 t) & 0 & 0 & \dots \\ 0 & \sin(\omega_2 t) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

And we get the general solution from this as

$$\{x\} = [A][C][A]^{-1}\{x_0\} + [A][D][\omega]^{-1}[A]^{-1}\{v_0\}$$

Classical solution for forced vibrations without damping

Matrix equation of motion is

$$\mathbf{[M]\{\ddot{x}\} + [K]\{x\} = \{Q(t)\}}$$

, simplify this and get –

$$\mathbf{[P]\{\ddot{y}\} + [S]\{y\} = [A]^T[Q]}$$

And generalize $\mathbf{[A]^T\{Q\} = F}$ as a generalized force matrix.

Hence, the final set of linear equations is represented by

$$\ddot{y}_k + \omega_k^2 y_k = (1/p_{kk})F_k, \quad k = 1, 2, \dots, n$$

Modal damping in forced vibration

Equations with damping and force (preferably excitation force) is given by

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{F\}$$

Where general case of solving this is to neglect the damping and solve the homogeneous equation and determine the normal coordinates. But here we write it in terms of the mass and spectral matrix like -

$$[P]\{\ddot{y}\} + [A]^T[C][A]\{\dot{y}\} + [S]\{y\} = [A]^T\{F\}$$

A^TCA is not a diagonal matrix here but P and S are diagonal matrices. So if C ∝ [M] or [K], it becomes diagonal and we say that the system is uncoupled.

The uncoupled equation set is as follows:

$$P_i\ddot{y}_i + \overline{C}_i\dot{y}_i + S_i y_i = \overline{F}_i, \quad i = 1, 2, 3$$

<u>Normal mode summation</u>

It is helpful to cut down the size of computation. Essentially, the displacement of the structure under forced excitation is approximated by the sum of a limited number of normal modes of the system multiplied by normal coordinates.

Response computation

$$y_i(t) = \frac{\dot{y}_i(0)}{\omega_i}\sin\omega_i t + y_i(0)\cos\omega_i t + \frac{1}{\omega_i}\int_0^t \overline{F}_i(\tau)\sin\omega_i(t-\tau)\mathrm{d}\tau, \quad i = 1, 2$$

is the solution for each DOF. After getting y, get x as {x} = [A]{y} and from the initial conditions ,we get -

$$\{y\}_{t=0} = [A]^{-1}\{x_0\}$$

$$\{\dot{y}\}_{t=0} = [A]^{-1}\{\dot{x}_0\}$$

And obtain the final response of each DOF.

Continuous Systems

As stated earlier, Continuous systems are described by variables that depend on both Space & Time. The equations of motion for continuous systems, as such, are governed by partial differential equations (PDEs).

Further discussion involves the analysis of certain Continuous Systems, namely:

1. Vibration of a Taut String
2. Transverse vibration of an Elastic Beam
3. Vibration of Membrane
   (i) Rectangular Membrane
   (ii) Circular Membrane

1. Vibration of a Taut String

The governing equation is given as follows -

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

The approach for obtaining solution(s) involves the separation of variables.

y(x,t) = X(x)f(t)

Substituting the same in the governing equation, we arrive at a situation where both sides of the equation can be equal to a common value which is a constant.

$$\frac{1}{f}\frac{\mathrm{d}^2 f}{\mathrm{d}t^2} = c^2\frac{1}{\mathrm{X}}\frac{\mathrm{d}^2\mathrm{X}}{\mathrm{d}x^2} = a, \quad a > 0$$

Solving the separated variables individually further, we arrive at -

$$f(t) = A\cos\omega t + B\sin\omega t$$

$$\mathrm{X}(x) = \mathrm{C}\cos\frac{\omega}{c}x + D\sin\frac{\omega}{c}x$$

Further steps involve clubbing these equations and finding out necessary coefficients based on Boundary conditions (like y(0,t) = 0, y(l,t) = 0, X(0) = X(l) = 0).
Finally, the general solution is given by -

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\left(A'_n \cos\frac{n\pi c}{l}t + B'_n \sin\frac{n\pi c}{l}t\right)$$

For finding the coefficients, we utilize the Initial conditions given by -

$$y(x,0) = u(x); \quad \frac{\partial y}{\partial t}(x,0) = v(x), \quad 0 \le x \le l$$

Applying the Fourier series representation to the above variables, we obtain the coefficients -

$$A'_n = \frac{2}{l}\int_0^l u(x)\sin\frac{n\pi x}{x}\,\mathrm{d}x$$

$$B'_n = \frac{2}{n\pi c}\int_0^l v(x)\sin\frac{n\pi x}{l}\,\mathrm{d}x$$

2. Transverse vibration of an Elastic beam
The governing equation for a transverse vibration of an Elastic beam is given by -

$$\frac{\partial^4 y}{\partial x^4} + \frac{1}{a^2}\frac{\partial^2 y}{\partial t^2} = 0$$

We further apply the concept of separation of variables.

y = X(x)T(t)

Substituting the expressions for y in the governing equation, we arrive at a similar situation as in the case before where we can equate both the sides to a common value.

In this case, it is given by -

$$-\frac{a^2}{X}\frac{\mathrm{d}^4X}{\mathrm{d}x^4} = \frac{1}{T}\frac{\mathrm{d}^2T}{\mathrm{d}t^2} = -\omega^2$$

After solving for X and T individually, we arrive at -

$$T = \overline{E}\sin\omega t + \overline{F}\cos\omega t$$

$$X = \overline{A}\cos kx + \overline{B}\sin kx + \overline{C}\cosh\, kx + \overline{D}\sinh\, kx$$

So, on clubbing the above two expressions, y is given by -

$$y = \sum_{n=1}^{\infty}(\overline{A}_n\cos k_nx + \overline{B}_n\sin k_nx + \overline{C}_n\cosh\, k_nx + \overline{D}_n\sinh\, k_nx)$$
$$\times\, (\overline{E}_n\sin\omega_nt + \overline{F}_n\cos\omega_nt)$$

Boundary conditions for this beam are further described as follows:
(i) Fixed/Clamped end

$$y = 0, \quad \frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

(ii) Simply supported end

$$y = 0, \quad \frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$$

(iii) Free end

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0, \quad \frac{\mathrm{d}^3y}{\mathrm{d}x^2} = 0$$

For the sake of convenience in solving, we may write X as -

$$X = A_1(\sin kx - \sinh\, kx) + A_2(\sin kx + \sinh\, kx) + A_3(\cos kx - \cosh\, kx)$$
$$+ A_4(\cos kx + \cosh\, kx)$$

X is the obtained from applying the initial and boundary conditions.

Ultimately, the final general solution for y is obtained based on solving for X from respective Boundary and Initial conditions. In the case of a simply supported end, the general solution is as follows -

$$y = XT = \sum_{n=1}^{\infty} A_n \sin k_n x (E_n \sin \omega_n t + F_n \cos \omega_n t)$$

3. Vibration of Membrane

(i) Rectangular membrane

The governing equation is given by -

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

Defining z further based on separation of variables approach -

$$z(x,\ y,\ t) = X(x)Y(y)\sin(\omega t + \varepsilon)$$

Substituting the above in our governing equation, we arrive at -

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} + \frac{\omega^2}{c^2} = 0$$

Solving separately for X and Y, we obtain individual expressions for X and Y -

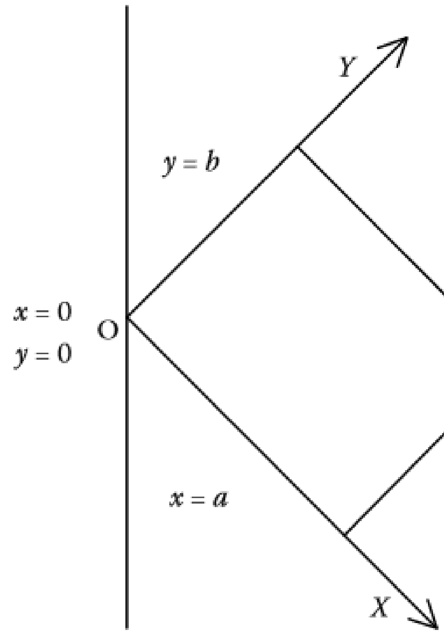
$$X = A \cos k_1 x + B \sin k_1 x$$

$$Y = C \cos k_2 y + D \sin k_2 y$$

Where k_1 and k_2 are given by -

$$k_1^2 + k_2^2 = \frac{\omega^2}{c^2}$$

The membrane and hence the respective conditions for z are defined as follows:



$$z = 0, \quad x = 0, \quad x = a$$

$$z = 0, \quad y = 0, \quad y = b$$

Based on above introduced conditions and minor adjustments in expressing the terms, we arrive at our general equation given by -

$$z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi}{a} x \sin \frac{n\pi}{b} y [B_{mn} \sin \omega_{mn} t + C_{mn} \cos \omega_{mn} t]$$

Where the frequency parameter is given by -

$$\omega_{mn}^2 = c^2 \left[\left(\frac{m\pi}{a} \right)^2 + \left(\frac{n\pi}{b} \right)^2 \right], \quad m = 1, 2, \ldots \quad \text{and} \quad n = 1, 2, \ldots$$

Further findings like B_{mn} and C_{mn} are to be obtained based on the Initial Conditions of the system.

(ii) Circular Membrane

The governing equation in the case of a Circular membrane is given by -

$$\frac{\partial^2 z}{\partial t^2} = c^2 \left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \right)$$

Defining z as a product of two variables (basically applying the concept of separation of variables) -

z = *u(x,y) sin*(ω□ + □)

Substituting the same in our governing equation and transforming it into polar form by substituting y = r sinθ and x = r cosθ we arrive at -

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\omega^2}{c^2}u=0$$

Having obtained the equation in polar form, we apply the separation of variables approach again, this time for u.

u = *R(r) V*(θ**)**

After substituting the above expression into the polar form of the equation, we may further write the resulting equation in the form of two differential equations -

$$\frac{1}{R}\left(r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2}+r\frac{\mathrm{d}R}{\mathrm{d}r}+\frac{\omega^2}{c^2}r^2R\right)=n^2$$

$$\frac{1}{V}\frac{\mathrm{d}^2V}{\mathrm{d}\theta^2}=-n^2$$

Solving for R (which happens to be a differential equation known as “Bessel’s equation”) and V individually, we get -

$$R=DJ_n\left(\frac{\omega r}{c}\right)$$

$$V=A\cos n\theta+B\sin n\theta$$

Therefore, the solution of membrane vibration in polar coordinates comes as -

$$z=RV\sin(\omega t+\varepsilon)=DJ_n\left(\frac{\omega r}{c}\right)\{A\cos n\theta+B\sin n\theta\}\sin(\omega t+\varepsilon)$$

Approximate Methods

Not always can exact solutions (as analyzed in before sections) be obtained as most of them happen to be in the form of infinite series of principal modes. Hence there arises a need for "Approximate methods". Two classical methods employed for further analysis are -

(i) Rayleigh's Method

(ii) Rayleigh - Ritz Method

(i) Rayleigh's Method

This classical approximate method is applicable to all continuous systems. This method takes into account the principle that maximum kinetic and potential energies of the system must be equal since no energy is lost and no energy is fed into the system over one cycle of vibration. This gives rise to a quotient known as Rayleigh quotient.

As an example, consider finding frequency expression for a string using this method. Let V and K denote the Potential and Kinetic energies of the vibration of the string respectively, then -

$$V = \frac{S}{2}\int_0^l \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 \mathrm{d}x \quad \text{And} \quad T = \frac{m}{2}\int_0^l (\dot{y})^2 \mathrm{d}x$$

Where S and m denote tensile force and mass per unit length of the fixed-fixed uniform string.

Abiding by the principle that $V_{max} = T_{max}$ where V_{max} and T_{max} are given by -

$$V_{\max} = \frac{S}{2}\int_0^l \left(\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x \quad \text{And} \quad T_{\max} = \frac{\omega^2 m}{2}\int_0^l (Y(x))^2 \mathrm{d}x$$

We arrive at the Rayleigh quotient, computed as -

$$\omega^2 = \frac{\frac{S}{2}\int_0^l \left(\frac{\mathrm{d}Y(x)}{\mathrm{d}x}\right)^2 \mathrm{d}x}{\frac{m}{2}\int_0^l (Y(x))^2 \mathrm{d}x}$$

The natural frequency of the system may be further computed after having known the deflection function Y(x).

Consider the deflection function Y(x) of a fixed-fixed uniform string, then -

$$\mathbf{Y(x)} = \boldsymbol{x\ (l - x)}$$

Substituting the above expression in the frequency equation, we obtain the natural frequency of the vibration of the string as -

$$\omega = 3.162\sqrt{\frac{S}{ml^2}}$$

(ii) Rayleigh - Ritz Method

This method basically is an extension of Rayleigh's Method. In this method, we will consider a linear combination of the several assumed functions satisfying certain boundary conditions in order to obtain a closer approximation to the exact values of the natural modes of vibration.

Consider that for approximating the deflection function Y(x) for vibration of a string, we have selected *n* assumed functions. Therefore, the deflection expression is given by -

$$Y(x) = \sum_{i=1}^{n} c_i y_i(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Where y_1, y_2, … so on are known functions that satisfy the boundary conditions of the problem and c_1, c_2, … so on are constants that are to be determined.

Substituting the deflection expression into the Rayleigh quotient expression and incorporating the condition for stationarity of the natural frequencies (where partial derivative of $\omega$with respect to each coefficient is made equal to zero), we have with ourselves *n* algebraic equations in *n* unknowns $c_1, c_2, ..., c_n$, which can be solved for *n* natural frequencies and mode shapes.

As an example, consider having two deflection terms -

$\boldsymbol{y_1 = x\ (l - x)}$ and $\boldsymbol{y_2 = x^2\ (l - x)^2}$

Substitute the above expressions for Y(x) and then for the Rayleigh quotient. Incorporate the condition of stationarity of the natural frequencies -

$$\frac{\partial \omega^2}{\partial c_1} = 0, \quad \frac{\partial \omega^2}{\partial c_2} = 0$$

Solving for c_1 and c_2 further, we get two simultaneous equations for two unknowns (c_1 and c_2) -

$$c_1\left(\frac{Sl^3}{3} - \frac{m\omega^2 l^5}{30}\right) + c_2\left(\frac{Sl^5}{15} - \frac{m\omega^2 l^7}{140}\right) = 0$$
$$c_1\left(\frac{Sl^5}{15} - \frac{m\omega^2 l^7}{140}\right) + c_2\left(\frac{2Sl^7}{105} - \frac{m\omega^2 l^9}{630}\right) = 0$$

When written in matrix notations, the above simultaneous equations turn into -

$$\begin{bmatrix} \left(\frac{Sl^3}{3} - \frac{m\omega^2 l^5}{30}\right) & \left(\frac{Sl^5}{15} - \frac{m\omega^2 l^7}{140}\right) \\ \left(\frac{Sl^5}{15} - \frac{m\omega^2 l^7}{140}\right) & \left(\frac{2Sl^7}{105} - \frac{m\omega^2 l^9}{630}\right) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solving for constants c_1 and c_2, we equate the determinant of the coefficient matrix to zero. Computed results are as follows -

$$\omega_1 = 3.142\sqrt{\frac{S}{ml^2}}$$

$$\omega_2 = 10.120\sqrt{\frac{S}{ml^2}}$$

Conclusion

The background knowledge of Vibration and the methods of analysis for Vibration problems were discussed. Based on the type of the system (i.e. whether Discrete or Continuous), the analysis for obtaining solutions from the governing equations was performed. From the analysis of the Vibration problems discussed, exact solutions were obtained.

Classical approximate methods such as Rayleigh's Method and Rayleigh-Ritz method which are to be utilized in the cases where obtaining exact solutions is intricate and difficult are discussed.

References

1. Chakraverty, S. *"Vibration of Plates"*, Taylor & Francis Group.