# Solutions to Exercises from ‘Algebra: Chapter 0’

Abstract. Solutions to exercises from the book 'Algebra: Chapter 0' by Paolo Aluffi.

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# Preface

This document contains my attempt at writing (hopefully correct!) solutions to exercises from Aluffi's book, while engaging in some self-study of modern abstract algebra with the ultimate aim of teaching myself some modern algebraic geometry.

CHAPTER 1

# Preliminaries: Set theory and categories

## 1. Naive set theory

EXERCISE 1.1. Let $U = \{x \mid x \notin x\}$. Then, $U \notin U \iff U \in U$, a contradiction. This is Russell's paradox. Either we assume the *set of all sets* doesn't exist, or we need to give up the axiom of *unrestricted comprehension* in set theory.

EXERCISE 1.2. Suppose $\sim$ is an equivalence relation on a set $S$. For every element $a \in S$, define the *equivalence class* of $a$ (with respect to $\sim$) by

$$[a]_\sim := \{b \in S \mid b \sim a\}.$$

Then, we note that due to *reflexivity*, the equivalence class $[a]_\sim$ of every element $a \in S$ contains $a$, and hence, is nonempty. Also, $[a]_\sim \subset S$, and therefore, $\bigcup_{a \in S} [a]_\sim = S$. Finally, we show the equivalence classes are mutually disjoint. Indeed, for any two elements $a, b \in S$, if $[a]_\sim$ and $[b]_\sim$ are disjoint, then there is nothing to prove. So, suppose $[a]_\sim \cap [b]_\sim$ is nonempty. Then, there exists some $c \in S$ that belongs to such an intersection. Thus, $c \sim a$ and $c \sim b$. By symmetry, $a \sim c$, and thus, by transitivity, $a \sim b$, which by symmetry again, implies $b \sim a$. Therefore, for all $x \in [a]_\sim$, we have $x \sim a$, and since $a \sim b$, by transitivity, $x \sim b$, which implies $x \in [b]_\sim$, from which we conclude $[a]_\sim \subset [b]_\sim$. We can similarly show $[b]_\sim \subset [a]_\sim$. Hence, $[a]_\sim = [b]_\sim$. This establishes equivalence classes are mutually disjoint. Hence, the set $\mathscr{P}_\sim$ of equivalence classes of $S$ is indeed a partition of $S$.

EXERCISE 1.3. Suppose $\mathscr{P}$ is a partition on a set $S$. Define a relation $\sim$ on $S$ as follows: For any two elements $a, b \in S$, $a \sim b$ iff $a$ and $b$ belong to the same set in the partition. Then, it is easy to check $\sim$ is indeed an equivalence relation on $S$. $\mathscr{P}$ is, therefore, the corresponding partition of the aforesaid equivalence relation, and we are done.

EXERCISE 1.4. Note the set of equivalence relations on a set $S$ are in a one-to-one correspondence with the set of partitions of $S$. Thus, the number of different equivalence relations that may be defined on $S = \{1, 2, 3\}$ equals the number of partitions of $S$, and this number equals 5, since the partitions of $S$ are

$$\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{2,3\},\{1\}\}, \{\{1,2,3\}\}.$$

The above partitions are also written as $1|2|3, 12|3, 13|2, 23|1, 123$.

EXERCISE 1.5. An example of a relation $R$ (defined on a set $S$) that is reflexive and symmetric but not transitive is the following:

$$R = \{(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2)\}, \text{ where } S = \{1,2,3\}.$$

EXERCISE 1.6. Define a relation $\sim$ on the set $\mathbb{R}$ of real numbers by setting

$$a \sim b \iff b - a \in \mathbb{Z}.$$

We claim $\sim$ is an equivalence relation. To that end, note, for all $a \in \mathbb{R}$, we have $a \sim a$, since $a - a = 0 \in \mathbb{Z}$. Therefore, $\sim$ is reflexive. Also, if $a \sim b$, then $b - a \in \mathbb{Z}$, which implies $a - b \in \mathbb{Z}$, and thus, $b \sim a$. Therefore, $\sim$ is symmetric. Finally, suppose $a \sim b$ and $b \sim c$. Then, $b-a, c-b \in \mathbb{Z}$, and thus, $c-a = (c-b)+(b-a) \in \mathbb{Z}$. Thus, $\sim$ is transitive. Therefore, $\sim$ is an equivalence relation on $\mathbb{R}$.

(Description of $\sim$) Note all reals that have the same decimal expansion belong to the same equivalence class under $\sim$. Thus, $[0]_\sim = \mathbb{Z}$, and for any $0 < \alpha < 1, [\alpha]_\sim = \{n+\alpha \mid n \in \mathbb{Z}\}$. This takes care of all the reals, since each real can always be written as $n + \alpha$, for some $n \in \mathbb{Z}$ and $0 < \alpha < 1$. Therefore, a 'compelling' description for $\mathbb{R}/\sim$ is the unit interval $[0, 1]$, such that the endpoints, 0 and 1, are 'glued' together. In other words, it is a 'loop' or a 1-sphere.

Define a relation $\approx$ on the plane $\mathbb{R} \times \mathbb{R}$ as follows:

$$(a_1, a_2) \approx (b_1, b_2) \iff b_1 - a_1 \in \mathbb{Z} \text{ and } b_2 - a_2 \in \mathbb{Z}.$$

Then, just as above, it is easy to show $\approx$ defines an equivalence relation on $\mathbb{R} \times \mathbb{R}$. We note $[(0, 0)]_\sim = \{(m, n) \mid m, n \in \mathbb{Z}\}$, and for any $0 < \alpha, \beta < 1, [(\alpha, \beta)]_\sim = \{(m + \alpha, n + \beta) \mid m, n \in \mathbb{Z}\}$. Thus, a 'compelling' description of $\mathbb{R} \times \mathbb{R}/\approx$ is the unit square $[0, 1] \times [0, 1]$ with the four corners joined together, so that it forms a 2-sphere.

## 2. Functions between sets

EXERCISE 2.1. We claim the number of bijections from a set $S$ with $n$ elements to itself is $n!$. To begin with, any element in $S$ can be mapped to any of the $n$ possible elements in $S$. Then, the next element in $S$ can be mapped to any of the remaining $n - 1$ elements in $S$, and so on, with the last element in $S$ being mapped to the last remaining element in $S$. Thus, the number of bijections equals $n \cdot (n - 1) \cdot \ldots \cdot 1 = n!$, which proves our claim.