Solutions to Exercises from 'Algebra: Chapter 0'

ABSTRACT. Solutions to exercises from the book 'Algebra: Chapter 0' by Paolo Aluffi.

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Preface

This document contains my attempt at writing (hopefully correct!) solutions to exercises from Aluffi's book, while engaging in some self-study of modern abstract algebra with the ultimate aim of teaching myself some modern algebraic geometry.

CHAPTER 1

Preliminaries: Set theory and categories

1. Naive set theory

Exercise 1.1. Let $U = \{x \mid x \notin x\}$. Then, $U \notin U \iff U \in U$, a contradiction. This is Russell's paradox. Either we assume the *set of all sets* doesn't exist, or we need to give up the axiom of *unrestricted comprehension* in set theory.

Exercise 1.2. Suppose $\sim$ is an equivalence relation on a set S. For every element $a \in S$, define the *equivalence class* of a (with respect to $\sim$) by

$$[a]_\sim := \{b \in S \mid b \sim a\}.$$

Then, we note that due to *reflexivity*, the equivalence class $[a]_\sim$ of every element $a \in S$ contains a, and hence, is nonempty. Also, $[a]_\sim \subset S$, and therefore, $\bigcup_{a \in S} [a]_\sim = S$. Finally, we show the equivalence classes are mutually disjoint. Indeed, for any two elements $a, b \in S$, if $[a]_\sim$ and $[b]_\sim$ are disjoint, then there is nothing to prove. So, suppose $[a]_\sim \cap [b]_\sim$ is nonempty. Then, there exists some $c \in S$ that belongs to such an intersection. Thus, $c \sim a$ and $c \sim b$. By symmetry, $a \sim c$, and thus, by transitivity, $a \sim b$, which by symmetry again, implies $b \sim a$. Therefore, for all $x \in [a]_\sim$, we have $x \sim a$, and since $a \sim b$, by transitivity, $x \sim b$, which implies $x \in [b]_\sim$, from which we conclude $[a]_\sim \subset [b]_\sim$. We can similarly show $[b]_\sim \subset [a]_\sim$. Hence, $[a]_\sim = [b]_\sim$. This establishes equivalence classes are mutually disjoint. Hence, the set $\mathscr{P}_\sim$ of equivalence classes of S is indeed a partition of S.

Exercise 1.3. Suppose $\mathscr{P}$ is a partition on a set S. Define a relation $\sim$ on S as follows: For any two elements $a, b \in S$, $a \sim b$ iff a and b belong to the same set in the partition. Then, it is easy to check $\sim$ is indeed an equivalence relation on S. $\mathscr{P}$ is, therefore, the corresponding partition of the aforesaid equivalence relation, and we are done.

Exercise 1.4. Note the set of equivalence relations on a set S are in a one-to-one correspondence with the set of partitions of S. Thus, the number of different equivalence relations that may be defined on $S = \{1, 2, 3\}$ equals the number of partitions of S, and this number equals 5, since the partitions of S are

$$\{\{1\}, \{2\}, \{3\}\}, \{\{1, 2\}, \{3\}\}, \{\{1, 3\}, \{2\}\}, \{\{2, 3\}, \{1\}\}, \{\{1, 2, 3\}\}.$$

The above partitions are also written as $1|2|3, 12|3, 13|2, 23|1, 123$.

Exercise 1.5. An example of a relation R (defined on a set S) that is reflexive and symmetric but not transitive is the following:

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (2, 3), (3, 2)\}, \text{ where } S = \{1, 2, 3\}.$$