Introduction to Categories and Categorical Logic

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ (f is right cancellative).

PROPOSITION 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

PROOF. We first show (1).
($\impliedby$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$\begin{array}{ll} & f(x) = f(x') \\ \implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\ \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\ \implies & f \circ \bar{x} = f \circ \bar{x'} \\ \implies & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\ \implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\ \implies & x = x' \end{array}$$

This shows that f is injective.

($\implies$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$\begin{array}{ll} & (f \circ g)(a) = (f \circ h)(a) \\ \implies & f(g(a)) = f(h(a)) \\ \implies & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\ \implies & g = h \end{array}$$

This establishes that f is monic. And, we are done. □

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

PROOF. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. $\square$

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \to H$ is a monoid homomorphism. Prove that h is a group homomorphism.

PROOF. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. $\square$

Exercise 6

Check that **Mon**, $\mathbf{Vect}_k$, **Pos**, and **Top** are indeed categories.

PROOF. (**Mon**) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \to (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \to (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \to (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m))(g(f(m'))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that **Mon** is indeed a category.

($\mathbf{Vect}_k$) The objects are vector spaces over a field k, and morphisms are linear maps between vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha(g \circ f)(x)$. This establishes $g \circ f : U \to W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that $\mathbf{Vect}_k$ is also a category.

(**Pos**) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \to Q$ and $g : Q \to R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \to R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(**Top**) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \to (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$,which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \to (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. $\square$