

# Introduction to Categories and Categorical Logic

Vishal Lama



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## CHAPTER 1

# Introduction to Categories and Categorical Logic

### 1. Introduction

We say that a function  $f : X \rightarrow Y$  is:

*injective*    if  $\forall x, x' \in X. f(x) = f(x') \implies x = x'$ ,  
*surjective*   if  $\forall y \in Y. \exists x \in X. f(x) = y$ ,

*monic*        if  $\forall g, h. f \circ g = f \circ h \implies g = h$     ( $f$  is left cancellative),  
*epic*         if  $\forall g, h. g \circ f = h \circ f \implies g = h$     ( $f$  is right cancellative).

PROPOSITION 1. Let  $f : X \rightarrow Y$ . Then,

- (1)  $f$  is injective  $\iff f$  is monic.
- (2)  $f$  is surjective  $\iff f$  is epic.

PROOF. We first show (1).

( $\Leftarrow$ ) Suppose  $f$  is monic. Fix a one-element set  $\mathbf{1} = \{\bullet\}$ . Then, note that elements  $x \in X$  are in 1-1 correspondence with functions  $\bar{x} : \mathbf{1} \rightarrow X$ , defined by  $\bar{x}(\bullet) := x$ . Then, for all  $x, x' \in X$ , we have

$$\begin{aligned}
 & f(x) = f(x') \\
 \implies & f(\bar{x}(\bullet)) = f(\bar{x}'(\bullet)) \\
 \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x}')(\bullet) \\
 \implies & f \circ \bar{x} = f \circ \bar{x}' \\
 \implies & \bar{x} = \bar{x}' \quad (\text{since } f \text{ is monic}) \\
 \implies & \bar{x}(\bullet) = \bar{x}'(\bullet) \\
 \implies & x = x'
 \end{aligned}$$

This shows that  $f$  is injective.

( $\Rightarrow$ ) Suppose  $f$  is injective. Let  $f \circ g = f \circ h$  for all  $g, h : A \rightarrow X$ . Then, for all  $a \in A$ ,

$$\begin{aligned}
 & (f \circ g)(a) = (f \circ h)(a) \\
 \implies & f(g(a)) = f(h(a)) \\
 \implies & g(a) = h(a) \quad (\text{since } f \text{ is injective}) \\
 \implies & g = h
 \end{aligned}$$

This establishes that  $f$  is monic. And, we are done. □

### Exercise 2

Show that  $f : X \rightarrow Y$  is surjective iff it is epic.

SOLUTION. ( $\implies$ ) Suppose  $f : X \rightarrow Y$  is epic. And, assume, for the sake of contradiction,  $f$  is *not* surjective. Then, there exists some  $y_0 \in Y$ , such that, for all  $x \in X$ ,  $f(x) \neq y_0$ . Define mappings  $g, h : Y \rightarrow Y \cup \{Y\}$  by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that  $g \neq h$ .

Then, for all  $x \in X$ ,  $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$ . This implies  $g \circ f = h \circ f$ , which implies  $g = h$ , since  $f$  is epic. The last conclusion contradicts the fact that  $g \neq h$ . Thus, we conclude  $f$  is surjective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is surjective. Then, for any  $y \in Y$ , there exists an  $x \in X$ , such that  $f(x) = y$ . Now, assume, for all  $g, h : Y \rightarrow Z$ ,  $g \circ f = h \circ f$ . Then, for all  $y \in Y$ ,  $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$ , which implies  $g = h$ , showing that  $f$  is epic.

And, this completes our proof.  $\square$

### Exercise 5

Suppose  $G$  and  $H$  are groups (and hence monoids), and that  $h : G \rightarrow H$  is a monoid homomorphism. Prove that  $h$  is a group homomorphism.

SOLUTION. We need only show that  $h$  preserves inverses. To that end, suppose  $g^{-1}$  is the inverse of  $g \in G$ . Then,  $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$ . This establishes  $h$  preserves inverses, and we are done.  $\square$

### Exercise 6

Check that **Mon**, **Vect<sub>k</sub>**, **Pos**, and **Top** are indeed categories.

SOLUTION. (**Mon**) The objects are monoids  $(M, \cdot, 1_M)$ , and morphisms are monoid homomorphisms. Given monoid homomorphisms,  $f : (M, \cdot, 1_M) \rightarrow (N, \cdot, 1_N)$  and  $g : (N, \cdot, 1_N) \rightarrow (P, \cdot, 1_P)$ , the function  $g \circ f : (M, \cdot, 1_M) \rightarrow (P, \cdot, 1_P)$  is also a monoid homomorphism, because for all  $m, m' \in M$ , we have  $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m)))(g(f(m')))) = ((g \circ f)(m))((g \circ f)(m'))$ . Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms  $f, g$  and  $h$  with the appropriate domains and codomains,  $h \circ (g \circ f) = (h \circ g) \circ f$ . This establishes that **Mon** is indeed a category.

(**Vect<sub>k</sub>**) The objects are vector spaces over a field  $k$ , and morphisms are linear maps between vector spaces. Suppose  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are linear maps. Then, for all  $x, y \in U$ , we have  $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$ . Also, for all  $\alpha \in k$ , we have  $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha (g \circ f)(x)$ . This establishes  $g \circ f : U \rightarrow W$  is a linear map as well. The identity map  $1_U$  for any vector space  $U$  is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that **Vect<sub>k</sub>** is also a category.

**(Pos)** The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose  $h : P \rightarrow Q$  and  $g : Q \rightarrow R$  are monotone functions. Then, for all  $x, y \in P$ ,  $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$ , which shows  $g \circ h : P \rightarrow R$  is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

**(Top)** The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps  $f : (X, T_X) \rightarrow (Y, T_Y)$  and  $g : (Y, T_Y) \rightarrow (Z, T_Z)$ , we can show that  $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$  is also a continuous map. First, note that for any  $T \subset Z$ ,  $x \in (g \circ f)^{-1}(T)$  iff  $(g \circ f)(x) \in T$  iff  $g(f(x)) \in T$  iff  $f(x) \in g^{-1}(T)$  iff  $x \in f^{-1}(g^{-1}(T))$ . Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set  $T \in T_Z$ , we have  $g^{-1}(T) \in T_Y$ , which implies  $f^{-1}(g^{-1}(T)) \in T_X$ , which implies  $(g \circ f)^{-1}(T) \in T_X$  (by using the result above.) Hence,  $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$  is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category.  $\square$

### Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon** have?)

**SOLUTION.** (Monoid as a one-object category) Given a monoid  $(M, \cdot, 1)$ , we can construct its corresponding category as follows. We write **BM** for the corresponding category with a single object  $\bullet$ , where  $\mathbf{Hom}_{\mathbf{BM}}(\bullet, \bullet) := M$ . We note then that the composition map in **BM** is reflected in the binary operation  $\cdot : M \times M \rightarrow M$ , where  $\mathbf{id}_\bullet := 1$ . Then, the associative and identity laws for the category **BM** follow directly from the associative and identity laws, respectively, satisfied by the monoid  $(M, \cdot, 1)$ . This shows any monoid can be seen or interpreted as a one-object category.  $\square$

### Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

**SOLUTION.** Let  $(P, \leq)$  be a preorder. Then, we define the corresponding category **C** as follows. The objects of **C** are the elements of the set  $P$ , and for all  $x, y \in P$ , we define a morphism  $x \rightarrow y$  iff  $x \leq y$ . Then, for every object  $x \in \mathbf{C}$ , the identity morphism  $1_x : x \rightarrow x$  corresponds exactly to the reflexive property  $x \leq x$  for all  $x \in P$ . Note that each homset in **C** has at most one element. Also, for every  $x \rightarrow y$  and  $y \rightarrow z$  in **C,  $x \rightarrow z$  follows from the fact that  $x \leq y$  and  $y \leq z$  and the transitivity of the  $\leq$  relation on  $P$ . This defines a composition map for morphisms in **C**. In addition, for all morphisms  $x \rightarrow y$ ,  $y \rightarrow z$ , and  $z \rightarrow w$ , their associativity follows immediately from the transitivity of  $\leq$ . Lastly, the unit laws**

also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element.  $\square$

### Exercise 10

Show that the inverse, if it exists, is unique.

SOLUTION. Suppose  $i : A \rightarrow B$  is an isomorphism, with inverse  $j : B \rightarrow A$ , in a category  $\mathbf{C}$ . Suppose  $j' : B \rightarrow A$  is also an inverse of  $i$ . Then,  $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$ , and we are done.  $\square$

### Exercise 11

Show that  $\cong$  is an equivalence relation on the objects of a category.

SOLUTION. Let  $\mathbf{C}$  be some category.

(*Reflexivity*) For any object  $X \in \mathbf{C}$ ,  $X \cong X$  follows from the fact that the identity morphism  $1_X : X \rightarrow X$  is an isomorphism.

(*Symmetry*) If  $X \cong Y$ , then there exists an isomorphism  $i : X \rightarrow Y$ . But, the inverse,  $i^{-1} : Y \rightarrow X$ , of  $i$  is also an isomorphism. Hence,  $Y \cong X$ .

(*Transitivity*) Suppose  $X \cong Y$  and  $Y \cong Z$ . Then, there exist isomorphisms  $i : X \rightarrow Y$  and  $j : Y \rightarrow Z$ . Then, we claim that  $j \circ i : X \rightarrow Z$  is also an isomorphism. Indeed, it's trivial to show that its inverse is the morphism  $i^{-1} \circ j^{-1} : Z \rightarrow X$ . This implies  $X \cong Z$ .

We thus conclude that  $\cong$  is an equivalence relation on the objects of a category.  $\square$

### Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

SOLUTION. (**Set**) We claim the following:

- (1)  $f : X \rightarrow Y$  is injective iff  $f$  has a left inverse.
- (2)  $f : X \rightarrow Y$  is surjective iff  $f$  has a right inverse.

We first show (1).

( $\implies$ ) Suppose  $f : X \rightarrow Y$  has a left inverse,  $g : Y \rightarrow X$ , say. Then,  $g \circ f = 1_X$ . Assume for any  $x, x' \in X$ ,  $f(x) = f(x')$ . Then,  $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$ , which implies  $f$  is injective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is injective. If  $X$  is empty, then  $f$  is an empty function corresponding to each  $Y$ . In this case,  $1_X$  is also an empty function, and we thus have  $g \circ f = 1_X$  for any  $g : Y \rightarrow X$ . That is,  $f$  has a left inverse. On the other hand, if  $X$  is nonempty, choose some  $x_0 \in X$ . Define  $g : Y \rightarrow X$  by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all  $x \in X$ ,  $(g \circ f)(x) = g(f(x)) = x = 1_X(x)$ , which implies  $g \circ f = 1_X$ , thus showing that  $g$  is a left inverse of  $f$ .

We now show (2).



( $\implies$ ) Suppose  $f : X \rightarrow Y$  has a right inverse,  $g : Y \rightarrow X$ , say. Then,  $f \circ g = 1_Y$ . Therefore, for all  $y \in Y$ ,  $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$ , where  $x = g(y)$ . This shows  $f$  is surjective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is surjective. Now, consider an indexed family of nonempty sets  $\{f^{-1}(y)\}_{y \in Y}$ . Then, using the axiom of choice, we conclude there exists a function  $g : Y \rightarrow X$ , such that  $g(y) \in f^{-1}(y)$  for all  $y \in Y$ . Then, for all  $y \in Y$ ,  $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$ , which implies  $f \circ g = 1_Y$ , thus proving  $f$  has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category **C**, if  $f : X \rightarrow Y$  has both a left inverse,  $g : Y \rightarrow X$ , say, and a right inverse,  $h : Y \rightarrow X$ , say, then  $g = h$ . Indeed,  $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$ , and we are done.

(Grp)

(Top)

(Pos)

□

**Opposite Categories and Duality.** Given a category **C**, the opposite category **C<sup>op</sup>** is given by taking the same objects as **C**, and

$$\mathbf{C}^{\text{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from **C**.

If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in **C<sup>op</sup>**, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in **C**. Therefore, composition  $g \circ f$  in **C<sup>op</sup>** is defined as  $f \circ g$  in **C**. This leads to the **principle of duality**: a statement  $S$  is true about a category **C** iff its dual (*i.e.* the one obtained from  $S$  by reversing all the arrows) is true about **C<sup>op</sup>**. For example, a morphism  $f$  is monic in **C<sup>op</sup>** iff it is epic in **C**. We say monic and epic are *dual notions*.

### Exercise 14

If  $P$  is a preorder, for example  $(\mathbb{R}, \leq)$ , describe **P<sup>op</sup>** explicitly.

SOLUTION. An arrow  $a \leq_{P^{\text{op}}} b$  in **P<sup>op</sup>** is precisely the arrow  $b \leq_P a$  in  $P$ . When  $P = (\mathbb{R}, \leq)$ , **P<sup>op</sup>** describes the “greater than or equal” preorder relation on  $\mathbb{R}$ . □

**Subcategories.** Let **C** be a category. Suppose we are given the collections

$$\begin{aligned} \mathbf{Ob}(\mathbf{D}) &\subseteq \mathbf{Ob}(\mathbf{C}), \\ \forall A, B \in \mathbf{Ob}(\mathbf{D}). \mathbf{D}(A, B) &\subseteq \mathbf{C}(A, B). \end{aligned}$$

We say **D** is a **subcategory** of **C** if it is itself a category. In particular, **D** is:

- A **full** subcategory of **C** if for any  $A, B \in \mathbf{Ob}(\mathbf{D})$ ,  $\mathbf{D}(A, B) = \mathbf{C}(A, B)$ .
- A **lluf** subcategory of **C** if  $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$ .

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

### Exercise 16

How many categories  $\mathbf{C}$  with  $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$  are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids.  $\square$

#### Exercises.

- (1) Consider the following properties of an arrow  $f$  in a category  $\mathbf{C}$ .
  - $f$  is *split monic* if for some  $g$ ,  $g \circ f$  is an identity arrow.
  - $f$  is *split epic* if for some  $g$ ,  $f \circ g$  is an identity arrow.
  - a. Prove that if  $f$  and  $g$  are arrows such that  $g \circ f$  is monic, then  $f$  is monic.
  - b. Prove that if  $f$  is split epic then it is epic.
  - c. Prove that if  $f$  and  $g \circ f$  are iso then  $g$  is iso.
  - d. Prove that if  $f$  is monic and split epic then it is iso.
  - e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map

$$i : (\mathbb{N}, +, 0) \rightarrow (\mathbb{Z}, +, 0)$$

of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if  $\{X_i\}_{i \in I}$  is a family of nonempty sets, we can form a set  $X = \{x_i \mid i \in I\}$ , where  $x_i \in X_i$  for all  $i \in I$ .

- f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
  - g. Is it always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.
- (2) Give a description of partial orders as categories of a special kind.

SOLUTION.

- (1)
  - a. Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow C$  such that  $g \circ f$  is monic. Assume, for all  $i, j : Z \rightarrow A$ ,  $f \circ i = f \circ j$ . Then,  $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$ , which implies  $i = j$ , since  $g \circ f$  is monic. This implies  $f$  is monic, and we are done.
  - b. Suppose  $f : A \rightarrow B$  is split epic. Then, there exists a  $g : B \rightarrow A$  such that  $f \circ g = 1_B$ . Assume, for all  $i, j : B \rightarrow C$ ,  $i \circ f = j \circ f$ . Then,  $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$ , which shows  $f$  is epic.
  - c. Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow C$  such that  $f$  and  $g \circ f$  are iso. We claim that the inverse of  $g$  is  $f \circ (g \circ f)^{-1} : C \rightarrow B$ . Indeed,  $g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = 1_C$ , and  $(f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} = f \circ f^{-1} = 1_B$ , which establishes  $g$  is also an iso.
  - d. Suppose  $f : A \rightarrow B$  is monic and split epic. The latter implies  $f$  has a right inverse,  $g : B \rightarrow A$ , say, where  $f \circ g = 1_B$ . Note that  $g \circ f : A \rightarrow A$  and  $1_A : A \rightarrow A$ . Now,  $f \circ (g \circ f) = (f \circ g) \circ f =$

$1_B \circ f = f = f \circ 1_A$ , which implies  $g \circ f = 1_A$ , since  $f$  is monic (left cancellative). Thus,  $g$  is also a left inverse of  $f$ , and hence,  $f$  is iso.

- e. It is easy to prove the inclusion map  $\mathbb{N} \hookrightarrow \mathbb{Z}$  is really a monoid homomorphism. Indeed,  $i(0) = 0$ , and, for all  $n_1, n_2 \in \mathbb{N}$ ,  $i(n_1 + n_2) = n_1 + n_2 = i(n_1) + i(n_2)$ .

Next, we show that  $i$  is monic. Assume, for all monoid homomorphisms  $g, h : X \rightarrow \mathbb{N}$ ,  $i \circ g = i \circ h$ . Then, for all  $x \in X$ ,  $(i \circ g)(x) = (i \circ h)(x)$ , which implies  $i(g(x)) = i(h(x))$ , which implies  $g(x) = h(x)$ , which implies  $g = h$ . This shows the inclusion map is monic.

We now show the inclusion map is epic. First, assume, for all monoid homomorphisms  $g, h : (\mathbb{Z}, +, 0) \rightarrow (X, \star, 1_X)$ ,  $g \circ i = h \circ i$ . Then, for all  $n \in \mathbb{N}$ ,  $(g \circ i)(n) = (h \circ i)(n)$ , which implies  $g(i(n)) = h(i(n))$ , which implies  $g(n) = h(n)$ . We now claim that for all  $n \geq 1$ ,  $g(-n) = h(-n)$ . To that end, we use induction on  $n$ . Note that  $g(-1) = g(-1) \star 1_X = g(-1) \star h(0) = g(-1) \star h(1 + (-1)) = g(-1) \star h(1) \star h(-1) = g(-1) \star g(1) \star h(-1) = g(-1 + 1) \star h(-1) = g(0) \star h(-1) = 1_X \star h(-1) = h(-1)$ . Now, assume the proposition holds for some  $n \geq 1$ . Then,  $g(-(n+1)) = g(-n + (-1)) = g(-n) \star g(-1) = h(-n) \star h(-1) = h(-n + (-1)) = h(-(n+1))$ . Hence, by induction,  $g(-n) = h(-n)$  for all  $n \geq 1$ . Combining the results from above, we thus conclude  $g(z) = h(z)$  for all  $z \in \mathbb{Z}$ . In other words,  $g = h$ , which implies  $i$  is epic.

Clearly, the inclusion map  $\mathbb{N} \hookrightarrow \mathbb{Z}$  is not iso.

- f. Suppose  $f : X \rightarrow Y$  is epic in **Set**. Then, from an earlier result about **Set**, we conclude  $f$  is surjective. Now, consider the family of nonempty sets  $\{f^{-1}(b)\}_{b \in B}$ . Each of the sets in the family is nonempty, because  $f$  is surjective. Therefore, using the Axiom of Choice, we can choose some element from each nonempty set in the family to construct a function  $g : Y \rightarrow X$ , given by  $g(y) := x$  if  $x \in f^{-1}(y)$ . In addition, for all  $y \in Y$ ,  $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$ , which implies  $f \circ g = 1_Y$ . This shows  $f$  has a right inverse, thus proving  $f$  is split epic.

- g. It isn't always the case that an arrow which is epic is split epic. For example, in the category **Mon**, the inclusion map  $\mathbb{N} \hookrightarrow \mathbb{Z}$  is epic (as shown in (e) above.) Now, if we assume that it is also split epic, then there exists a monoid homomorphism  $g : \mathbb{Z} \rightarrow \mathbb{N}$ , such that  $i \circ g = 1_{\mathbb{Z}}$ . This implies  $(i \circ g)(-1) = 1_{\mathbb{Z}}(-1)$ , which implies  $i(g(-1)) = -1$ , which implies  $g(-1) = -1$ , which implies  $-1 \in \mathbb{N}$ , which is absurd. We thus conclude the aforesaid inclusion map is *not* split epic, even though it is epic. And this proves our original claim.

- (2) Suppose  $(P, \leq)$  is a poset. Then, its corresponding category **C** is defined as follows. The objects of **C** are the elements of  $P$ , and for all  $x, y \in P$ ,  $x \rightarrow y$  iff  $x \leq y$ . The reflexivity of  $\leq$  corresponds to the identity arrows, and transitivity to arrow composition. Note that there is at most one arrow for every pair of objects in the category. Anti-symmetry of  $\leq$  corresponds to the fact that the only isomorphisms in **C** are the identity arrows.

□

## 2. Some Basic Constructions

**Initial and Terminal Objects.** An object  $I$  in a category  $\mathbf{C}$  is *initial* if, for every object  $A$ , there exists a unique arrow  $I \rightarrow A$ , which we write  $\iota_A : I \rightarrow A$ .

An object  $T$  in a category  $\mathbf{C}$  is *terminal* if, for every object  $A$ , there exists a unique arrow  $A \rightarrow T$ , which we write  $\tau_A : A \rightarrow T$ .

Note that initial and terminal objects are dual notions:  $T$  is terminal in  $\mathbf{C}$  iff it is initial in  $\mathbf{C}^{\text{op}}$ . We sometimes write  $\mathbf{1}$  for the terminal object and  $\mathbf{0}$  for the initial object.

### Exercise 18

Verify the following claims. In each case, identify the canonical arrows.

- (1) In **Set**, the empty set is an initial object while any one-element set  $\{\bullet\}$  is terminal.
- (2) In **Pos**, the poset  $(\emptyset, \emptyset)$  is an initial object while  $(\{\bullet\}, \{(\bullet, \bullet)\})$  is terminal.
- (3) In **Top**, the space  $(\emptyset, \{\emptyset\})$  is an initial object while  $(\{\bullet\}, \{\emptyset, \{\bullet\}\})$  is terminal.
- (4) In **Vect** $_k$ , the one-element space  $\{0\}$  is both initial and terminal.
- (5) In a poset, seen as a category, an initial object is a least element, while a terminal object is a greatest element.

SOLUTION.

- (1) In **Set**, for any set (object)  $A$ , the function  $(\emptyset, A, \emptyset)$  is the unique function (arrow) from  $\emptyset$  to  $A$ . Therefore, the empty set is (the) initial object in **Set**. And, for every set  $A$ , the function  $A \rightarrow \{\bullet\}$  that maps every element of  $A$  to  $\bullet$  is the unique function from  $A$  to  $\{\bullet\}$ . This establishes that any one-element set is terminal in **Set**.
- (2) For any poset  $(P, \leq)$ , there exists a unique (empty) monotone function  $(\emptyset, \emptyset) \xrightarrow{(\emptyset, P, \emptyset)} (P, \leq)$ . Hence, the poset  $(\emptyset, \emptyset)$  is an initial object in **Pos**. And, for any poset  $(P, \leq)$ , there exists a unique monotone function  $(P, \leq) \rightarrow (\{\bullet\}, \{(\bullet, \bullet)\})$ , defined by  $x \mapsto \bullet$  for all  $x \in P$ . Hence,  $(\{\bullet\}, \{(\bullet, \bullet)\})$  is terminal in **Pos**.
- (3) For any topological space  $(X, T_X)$ , the unique empty function

$$(\emptyset, \{\emptyset\}) \xrightarrow{(\emptyset, X, \emptyset)} (X, T_X)$$

is continuous, since for every open set  $T \in T_X$ , its preimage under the aforesaid function is the empty set, which is open. Hence,  $(\emptyset, \{\emptyset\})$  is initial in **Top**.

And, for any topological space  $(X, T_X)$ , the unique function  $(X, T_X) \rightarrow (\{\bullet\}, \{\emptyset, \{\bullet\}\})$ , defined by  $x \mapsto \bullet$  for all  $x \in X$ , is continuous, since the preimage of  $\emptyset$  under the aforesaid function is  $\emptyset$ , which is open, and the preimage of  $\{\bullet\}$  is  $X$ , which is also open. Hence,  $(\{\bullet\}, \{\emptyset, \{\bullet\}\})$  is terminal in **Top**.

- (4) Assuming the ground field is  $k$ , for any vector space  $V$ , the unique linear map  $\{0\} \rightarrow V$ , defined by  $0 \mapsto 0_V$  is a unique arrow from  $\{0\}$  to  $V$  in **Vect** $_k$ . Also, the unique linear map  $V \rightarrow \{0\}$ , defined by  $v \mapsto 0$  for all  $v \in V$ , is a unique arrow from  $V$  to  $\{0\}$  in **Vect** $_k$ . This shows that  $\{0\}$  is both initial and terminal in **Vect** $_k$ .

- (5) In a poset  $(P, \leq)$ , seen as a category, if  $\perp$  is an initial object, then there exists a unique arrow  $\perp \rightarrow p$  for all  $p \in P$ . This implies  $\perp \leq p$  for all  $p \in P$ , when seen as a set. Hence, an initial object in the category corresponding to  $(P, \leq)$  is a least element in  $P$ . Arguing similarly, we conclude that a terminal object in the category corresponding to  $(P, \leq)$  is a greatest element in  $P$ .

□

### Exercise 19

Identify the initial and terminal objects in **Rel**.

SOLUTION. In **Rel**, the empty set  $\emptyset$  is both the initial object and the terminal object. Indeed, for any set  $A$ , the empty relation  $\emptyset (\subseteq \emptyset \times A)$  is a unique relation from  $\emptyset$  to  $A$ , and the empty relation  $\emptyset (\subseteq A \times \emptyset)$  is also a unique relation from  $A$  to  $\emptyset$ . □

### Exercise 20

Suppose a monoid, viewed as a category, has either an initial or a terminal object. What must the monoid be?

SOLUTION. The category corresponding to a monoid  $(M, \cdot, 1_M)$  contains just a single object. If this object is initial, then all morphisms must be the identity morphism on this initial object, which implies  $M = \{1_M\}$ . The argument is similar if the aforesaid object is terminal, which would again imply  $M = \{1_M\}$ . Thus, in either case, the monoid must be the trivial monoid. □

A fundamental fact about initial and terminal objects is that they are *unique up to (unique) isomorphism*. This is characteristic of all such “universal” definitions. Hence, if initial objects exist in a category, we can speak of *the* initial object. Similarly for terminal objects.

### Exercise 22

Let **C** be a category with an initial object **0**. For any object  $A$ , show the following:

- (1) If  $A \cong \mathbf{0}$ , then  $A$  is an initial object.
- (2) If there exists a monomorphism  $f : A \rightarrow \mathbf{0}$ , then  $f$  is an iso, and hence  $A$  is initial.

SOLUTION.

- (1) Suppose  $A \cong \mathbf{0}$ . Then, there exists an isomorphism  $f : A \xrightarrow{\sim} \mathbf{0}$ . For any object  $B$ , there exists a unique morphism  $\iota_B : \mathbf{0} \rightarrow B$ , and hence,  $\iota_B \circ f : A \rightarrow B$ . This proves the existence of a morphism  $A \rightarrow B$  for any object  $B$  in the category. We now show that such a morphism is indeed unique. Let  $g, h : A \rightarrow B$  be a pair of morphisms for any object  $B$  in the category **C**. Then, we have

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{g} B$$

and

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{h} B.$$

Since  $\mathbf{0}$  is initial, we must have  $g \circ f^{-1} = h \circ f^{-1}$ . Therefore,  $g = g \circ 1_A = g \circ (f^{-1} \circ f) = (g \circ f^{-1}) \circ f = (h \circ f^{-1}) \circ f = h \circ (f^{-1} \circ f) = h \circ 1_A = h$ . This proves uniqueness, and we are done.

- (2) Suppose  $A \xrightarrow{f} \mathbf{0}$  is a monomorphism. We claim the unique arrow  $\mathbf{0} \xrightarrow{\iota_A} A$  is the inverse of  $f$ . To that end, we show  $\iota_A$  is both a left and a right inverse of  $f$ . Indeed,  $f \circ \iota_A : \mathbf{0} \rightarrow \mathbf{0}$ , and since  $\mathbf{0}$  is initial, we must have  $f \circ \iota_A = 1_{\mathbf{0}}$ , which implies  $\iota_A$  is a right inverse of  $f$ . Now, note  $\iota_A \circ f : A \rightarrow A$  and  $1_A : A \rightarrow A$ . Also,  $f \circ (\iota_A \circ f) = (f \circ \iota_A) \circ f = 1_{\mathbf{0}} \circ f = f = f \circ 1_A$ , and since  $f$  is left cancellative, we have  $\iota_A \circ f = 1_A$ , which shows  $\iota_A$  is a left inverse of  $f$ . Thus,  $f$  has both a left inverse and a right inverse, implying it is iso, and hence, using the result obtained in (1) above, we conclude  $A$  is an initial object in  $\mathbf{C}$ .

□

**Products and Coproducts.** We can express a general notion of product that is meaningful in any category, such that, if a product exists, it is characterized uniquely up to unique isomorphism. Given a particular mathematical context (*i.e.* a category), we can then verify if a product exists in that category. The concrete construction appropriate to the context will enter only into the proof of *existence*; all of the useful *properties* of a product follow from the general definition.

### Exercise 24

Verify  $\mathbf{Pair}(A, B)$  is a category, where  $A$  and  $B$  are arbitrary objects in some category.

**SOLUTION.** Let  $A$  and  $B$  be some arbitrary objects in some category  $\mathbf{C}$ . Now, given morphisms  $f : (P, p_1, p_2) \rightarrow (Q, q_1, q_2)$  and  $g : (Q, q_1, q_2) \rightarrow (R, r_1, r_2)$  in  $\mathbf{Pair}(A, B)$ , it is easy to check that  $g \circ f : P \rightarrow R$  in  $\mathbf{C}$ . Also, we have

$$q_1 \circ f = p_1, q_2 \circ f = p_2$$

and

$$r_1 \circ g = q_1, r_2 \circ g = q_2$$

So,  $r_1 \circ (g \circ f) = (r_1 \circ g) \circ f = q_1 \circ f = p_1$ , and,  $r_2 \circ (g \circ f) = (r_2 \circ g) \circ f = q_2 \circ f = p_2$ , which implies  $g \circ f : (P, p_1, p_2) \rightarrow (R, r_1, r_2)$  in  $\mathbf{Pair}(A, B)$ . Associativity of morphisms in  $\mathbf{Pair}(A, B)$  follows directly from the associativity of morphisms in  $\mathbf{C}$ . Finally, for all  $(P, p_1, p_2)$  in  $\mathbf{Pair}(A, B)$ , the identity morphism  $1_P : P \rightarrow P$  is the identity morphism for  $(P, p_1, p_2)$ , since  $p_1 \circ 1_P = p_1$  and  $p_2 \circ 1_P = p_2$ . And, this proves that  $\mathbf{Pair}(A, B)$  is indeed a category. □

We say  $(A \times B, \pi_1, \pi_2)$  is a **product** of  $A$  and  $B$  if it is *terminal* in  $\mathbf{Pair}(A, B)$ . Products are specified by triples  $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2}$ , where  $\pi_i$ 's are called *projections*. For economy (and if projections are obvious), we say  $A \times B$  is the product of  $A$  and  $B$ . We say a category  $\mathbf{C}$  has **(binary) products** if each pair of objects  $A, B$  has a product in  $\mathbf{C}$ . Since, products are terminal objects, they are unique up to (unique) isomorphism.

Unpacking the uniqueness condition from  $\mathbf{Pair}(A, B)$  back to  $\mathbf{C}$ , we obtain the following more concise definition of products that we use in practice.

**(Equivalent definition of product)** Let  $A, B$  be objects in a category  $\mathbf{C}$ . A product of  $A$  and  $B$  is an object  $A \times B$  together with a pair of arrows  $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$  such that for every triple  $A \xleftarrow{f} C \xrightarrow{g} B$ , there exists a *unique* morphism

$$\langle f, g \rangle : C \rightarrow A \times B$$

such that the corresponding diagram commutes. That is,

$$\begin{aligned}\pi_1 \circ \langle f, g \rangle &= f \\ \pi_2 \circ \langle f, g \rangle &= g\end{aligned}$$

We call  $\langle f, g \rangle$  the *pairing* of  $f$  and  $g$ .

### Exercise 26

Verify the following claims.

- (1) In **Set**, products are the usual cartesian products.
- (2) In **Pos**, products are cartesian products with the pointwise order.
- (3) In **Top**, products are cartesian products with the product topology.
- (4) In  $\mathbf{Vect}_k$ , products are direct sums.
- (5) In a poset, seen as a category, products are *greatest lower bounds*.

SOLUTION.

- (1) Let  $A, B$  be arbitrary sets in **Set**. We claim  $A \times B$  equipped with the canonical projection functions is the cross product of  $A$  and  $B$ . Indeed, given any  $A \xleftarrow{f} C \xrightarrow{g} B$ , we show  $\langle f, g \rangle : C \rightarrow A \times B$ , defined by

$$c \mapsto (f(c), g(c)),$$

is the unique function that makes the following diagram commute:

$$\begin{array}{ccccc} & & C & & \\ & f \swarrow & \vdots & \searrow g & \\ A & \xleftarrow{\pi_1} & A \times B & \xrightarrow{\pi_2} & B \end{array}$$

(*Existence*) It is easy to check that  $\langle f, g \rangle$  is indeed a function from  $C$  to  $A \times B$ .

(*Commutativity*) For all  $c \in C$ ,  $(\pi_1 \circ \langle f, g \rangle)(c) = f(c)$  and  $(\pi_2 \circ \langle f, g \rangle)(c) = g(c)$ , which imply the above diagram commutes.

(*Uniqueness*) Suppose  $h : C \rightarrow A \times B$  such that  $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ . Then, for all  $c \in C$ ,  $(\pi_1 \circ h)(c) = f(c)$  and  $(\pi_2 \circ h)(c) = g(c)$ , which imply  $\pi_1(h(c)) = f(c)$  and  $\pi_2(h(c)) = g(c)$ , which imply  $h(c) = (f(c), g(c)) = \langle f, g \rangle(c)$ , thus proving  $h = \langle f, g \rangle$ , and thereby, showing the uniqueness of  $\langle f, g \rangle$ .

Hence,  $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$  is the cross product of  $A$  and  $B$ .

- (2) Let  $(P, \leq)$  and  $(Q, \leq)$  be posets. Let  $(P \times Q, \leq)$  be the cartesian product of  $P$  and  $Q$  with the pointwise order. That is, for all  $a, c \in P$  and  $b, d \in Q$ ,  $(a, b) \leq (c, d)$  iff  $a \leq c$  and  $b \leq d$ . We claim  $(P \times Q, \leq)$  equipped with the canonical projection functions (which are monotone) is the cross product

of  $(P, \leq)$  and  $(Q, \leq)$ . Given any  $(P, \leq) \xleftarrow{f} (R, \leq) \xrightarrow{g} (Q, \leq)$ , where  $f, g$  are monotone functions, the function  $\langle f, g \rangle : (R, \leq) \rightarrow (P \times Q, \leq)$ , defined by

$$r \mapsto (f(r), g(r))$$

is the unique monotone function that makes the following diagram commute:

$$\begin{array}{ccccc} & & (R, \leq) & & \\ & \swarrow f & \vdots \langle f, g \rangle & \searrow g & \\ (P, \leq) & \xleftarrow{\pi_1} & (P \times Q, \leq) & \xrightarrow{\pi_2} & (Q, \leq) \end{array}$$

(*Existence*) It is easy to check that  $\langle f, g \rangle$  is indeed a set function from  $R$  to  $P \times Q$ . And, for all  $r_1, r_2 \in R$ , if  $r_1 \leq r_2$ , then  $f(r_1) \leq f(r_2)$  and  $g(r_1) \leq g(r_2)$  (since  $f, g$  are monotone), which implies  $(f(r_1), g(r_1)) \leq (f(r_2), g(r_2))$ , which implies  $\langle f, g \rangle(r_1) \leq \langle f, g \rangle(r_2)$ , which implies  $\langle f, g \rangle$  is monotone.

(*Commutativity*) For all  $r \in R$ , we have

$$\begin{aligned} (\pi_1 \circ \langle f, g \rangle)(r) &= f(r), \\ (\pi_2 \circ \langle f, g \rangle)(r) &= g(r). \end{aligned}$$

The above implies that the above diagram does commute.

(*Uniqueness*) Suppose  $h : (R, \leq) \rightarrow (P \times Q, \leq)$  is a monotone function such that  $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ . Then, for all  $r \in R$ ,  $\pi_1 \circ h(r) = f(r)$  and  $\pi_2 \circ h(r) = g(r)$ , which imply  $\pi_1(h(r)) = f(r)$  and  $\pi_2(h(r)) = g(r)$ , which imply  $h(r) = (f(r), g(r))$ , which implies  $h(r) = \langle f, g \rangle(r)$ , which implies  $h = \langle f, g \rangle$ , thus showing that  $\langle f, g \rangle$  with the commutativity property is indeed unique.

Hence, we conclude the cartesian product  $(P \times Q, \leq)$  with the pointwise order is the product of any posets  $(P, \leq)$  and  $(Q, \leq)$ .

(3)

(4)

(5) In a poset  $(P, \leq)$ , seen as a category, the product  $a \times b$  of two elements  $a, b \in P$  is an element in  $P$  satisfying  $a \times b \leq a$  and  $a \times b \leq b$ , such that for all elements  $c \in P$ , if  $c \leq a$  and  $c \leq b$ , then  $c \leq a \times b$ . This is precisely the definition of the *greatest lower bound* of any two elements  $a, b \in P$ , seen as a set. Therefore, products are greatest lower bounds in posets.

□

The following proposition shows that the uniqueness of the pairing arrow can be specified purely equationally by the equation:

$$\forall h : C \rightarrow A \times B. h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$$

PROPOSITION 27. For any triple  $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ , the following statements are equivalent:

(I) For any triple  $A \xleftarrow{f} C \xrightarrow{g} B$ , there exists a unique morphism  $\langle f, g \rangle : C \rightarrow A \times B$  such that  $\pi_1 \circ \langle f, g \rangle = f$  and  $\pi_2 \circ \langle f, g \rangle = g$ .



- (II) For any triple  $A \xleftarrow{f} C \xrightarrow{g} B$ , there exists a morphism  $\langle f, g \rangle : C \rightarrow A \times B$  such that  $\pi_1 \circ \langle f, g \rangle = f$  and  $\pi_2 \circ \langle f, g \rangle = g$ , and moreover, for any  $h : C \rightarrow A \times B$ ,  $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$ .

PROOF. ((I)  $\implies$  (II)) Suppose (I) holds. Assume  $A \xleftarrow{f} C \xrightarrow{g} B$ . Then, by (I), there exists a (unique) morphism  $\langle f, g \rangle : C \rightarrow A \times B$  such that  $\pi_1 \circ \langle f, g \rangle = f$  and  $\pi_2 \circ \langle f, g \rangle = g$ . Now, let  $h : C \rightarrow A \times B$ . Note  $A \xleftarrow{\pi_1 \circ h} C \xrightarrow{\pi_2 \circ h} B$ . Thus, by (I), there exists a unique morphism  $\langle \pi_1 \circ h, \pi_2 \circ h \rangle : C \rightarrow A \times B$  such that

$$\begin{aligned}\pi_1 \circ \langle \pi_1 \circ h, \pi_2 \circ h \rangle &= \pi_1 \circ h, \\ \pi_2 \circ \langle \pi_1 \circ h, \pi_2 \circ h \rangle &= \pi_2 \circ h.\end{aligned}$$

The above implies  $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$ . This proves (II).

((II)  $\implies$  (I)) Suppose (II) holds. Assume  $A \xleftarrow{f} C \xrightarrow{g} B$ . Then, by (II), there exists a morphism  $\langle f, g \rangle : C \rightarrow A \times B$  such that  $\pi_1 \circ \langle f, g \rangle = f$  and  $\pi_2 \circ \langle f, g \rangle = g$ . We claim such a morphism is unique. So, suppose  $h : C \rightarrow A \times B$  such that  $\pi_1 \circ h = f$  and  $\pi_2 \circ h = g$ . Then, by (II), we have  $h = \langle \pi_1 \circ h, \pi_2 \circ h \rangle$ , which implies  $h = \langle f, g \rangle$ . This proves (I), and our proof is complete.  $\square$

**Cartesian product of morphisms.** Given  $f_1 : A_1 \rightarrow B_1$  and  $f_2 : A_2 \rightarrow B_2$ , we define the *cartesian product of morphisms*  $f_1$  and  $f_2$  by

$$f_1 \times f_2 := \langle f_1 \circ \pi_1, f_2 \circ \pi_2 \rangle : A_1 \times A_2 \rightarrow B_1 \times B_2.$$

The following proposition provides some useful properties of products.

PROPOSITION 28. For any  $f : A \rightarrow B$ ,  $g : A \rightarrow C$ ,  $h : A' \rightarrow A$ , and any  $p : B \rightarrow B'$ ,  $q : C \rightarrow C'$ , the following hold:

- (1)  $\langle f, g \rangle \circ h = \langle f \circ h, g \circ h \rangle$
- (2)  $(p \times q) \circ \langle f, g \rangle = \langle p \circ f, q \circ g \rangle$ .

PROOF.

- (1) Note  $\langle f, h \rangle \circ h : A' \rightarrow B \times C$ . Therefore, by (II) of Proposition 27,  $\langle f, h \rangle \circ h = \langle \pi_1 \circ (\langle f, g \rangle \circ h), \pi_2 \circ (\langle f, g \rangle \circ h) \rangle = \langle f \circ h, g \circ h \rangle$ .
- (2)  $(p \times q) \circ \langle f, g \rangle = \langle p \circ \pi_1, q \circ \pi_2 \rangle \circ \langle f, g \rangle = \langle p \circ \pi_1 \circ \langle f, g \rangle, q \circ \pi_2 \circ \langle f, g \rangle \rangle = \langle p \circ f, q \circ g \rangle$ .

$\square$

**General Products.** The notion of products can be generalized to arbitrary arities as follows. In a category  $\mathbf{C}$ , a product for a family of objects  $\{A_i\}_{i \in I}$  is an object  $P$  and morphisms

$$p_i : P \rightarrow A_i \quad (i \in I)$$

such that, for all objects  $B$  and arrows

$$f_i : B \rightarrow A_i \quad (i \in I)$$

there is a *unique* arrow  $g : B \rightarrow P$  such that, for all  $i \in I$ , the following diagram commutes

$$\begin{array}{ccc} B & \overset{g}{\dashrightarrow} & P \\ & \searrow f_i \quad \swarrow p_i & \\ & A_i & \end{array}$$

Again, if such a product exists, it is unique up to (unique) isomorphism. We write  $P = \prod_{i \in I} A_i$  for the product object, and  $g = \langle f_i \mid i \in I \rangle$  for the unique morphism in the definition.

### Exercise 29

What is the product of the empty family?

SOLUTION. The product of the empty family is an object  $T$ , such that for every object with an arrow to the empty family, there is a unique arrow from that object to  $T$  making the corresponding diagram commute. Since there are no diagrams, this means there is a unique arrow from every object to  $T$ , and this is precisely the definition of a terminal object. Hence, the product of an empty family is a terminal object.  $\square$

### Exercise 30

Show that if a category has binary and nullary products, then it has all finite products.

SOLUTION.  $\square$