Introduction to Categories and Categorical Logic

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ (f is right cancellative).

Proposition 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

Proof. We first show (1).
($\impliedby$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$\begin{array}{ll} & f(x) = f(x') \\ \implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\ \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\ \implies & f \circ \bar{x} = f \circ \bar{x'} \\ \implies & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\ \implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\ \implies & x = x' \end{array}$$

This shows that f is injective.

($\implies$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$\begin{array}{ll} & (f \circ g)(a) = (f \circ h)(a) \\ \implies & f(g(a)) = f(h(a)) \\ \implies & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\ \implies & g = h \end{array}$$

This establishes that f is monic. And, we are done. $\square$

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

Proof. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. $\square$