

# Introduction to Categories and Categorical Logic

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## CHAPTER 1

# Introduction to Categories and Categorical Logic

### 1. Introduction

We say that a function  $f : X \rightarrow Y$  is:

*injective*    if  $\forall x, x' \in X. f(x) = f(x') \implies x = x'$ ,  
*surjective*   if  $\forall y \in Y. \exists x \in X. f(x) = y$ ,

*monic*        if  $\forall g, h. f \circ g = f \circ h \implies g = h$     ( $f$  is left cancellative),  
*epic*         if  $\forall g, h. g \circ f = h \circ f \implies g = h$     ( $f$  is right cancellative).

PROPOSITION 1. *Let  $f : X \rightarrow Y$ . Then,*

- (1)  *$f$  is injective  $\iff f$  is monic.*
- (2)  *$f$  is surjective  $\iff f$  is epic.*

PROOF. We first show (1).

( $\Leftarrow$ ) Suppose  $f$  is monic. Fix a one-element set  $\mathbf{1} = \{\bullet\}$ . Then, note that elements  $x \in X$  are in 1-1 correspondence with functions  $\bar{x} : \mathbf{1} \rightarrow X$ , defined by  $\bar{x}(\bullet) := x$ . Then, for all  $x, x' \in X$ , we have

$$\begin{aligned} & f(x) = f(x') \\ \implies & f(\bar{x}(\bullet)) = f(\bar{x}'(\bullet)) \\ \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x}')(\bullet) \\ \implies & f \circ \bar{x} = f \circ \bar{x}' \\ \implies & \bar{x} = \bar{x}' \quad (\text{since } f \text{ is monic}) \\ \implies & \bar{x}(\bullet) = \bar{x}'(\bullet) \\ \implies & x = x' \end{aligned}$$

This shows that  $f$  is injective.

( $\Rightarrow$ ) Suppose  $f$  is injective. Let  $f \circ g = f \circ h$  for all  $g, h : A \rightarrow X$ . Then, for all  $a \in A$ ,

$$\begin{aligned} & (f \circ g)(a) = (f \circ h)(a) \\ \implies & f(g(a)) = f(h(a)) \\ \implies & g(a) = h(a) \quad (\text{since } f \text{ is injective}) \\ \implies & g = h \end{aligned}$$

This establishes that  $f$  is monic. And, we are done. □

### Exercise 2

Show that  $f : X \rightarrow Y$  is surjective iff it is epic.

PROOF. ( $\implies$ ) Suppose  $f : X \rightarrow Y$  is epic. And, assume, for the sake of contradiction,  $f$  is *not* surjective. Then, there exists some  $y_0 \in Y$ , such that, for all  $x \in X$ ,  $f(x) \neq y_0$ . Define mappings  $g, h : Y \rightarrow Y \cup \{Y\}$  by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that  $g \neq h$ .

Then, for all  $x \in X$ ,  $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$ . This implies  $g \circ f = h \circ f$ , which implies  $g = h$ , since  $f$  is epic. The last conclusion contradicts the fact that  $g \neq h$ . Thus, we conclude  $f$  is surjective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is surjective. Then, for any  $y \in Y$ , there exists an  $x \in X$ , such that  $f(x) = y$ . Now, assume, for all  $g, h : Y \rightarrow Z$ ,  $g \circ f = h \circ f$ . Then, for all  $y \in Y$ ,  $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$ , which implies  $g = h$ , showing that  $f$  is epic.

And, this completes our proof.  $\square$

### Exercise 5

Suppose  $G$  and  $H$  are groups (and hence monoids), and that  $h : G \rightarrow H$  is a monoid homomorphism. Prove that  $h$  is a group homomorphism.

PROOF. We need only show that  $h$  preserves inverses. To that end, suppose  $g^{-1}$  is the inverse of  $g \in G$ . Also, let  $h(g) = h'$ , and  $h(g^{-1}) = h''$ . Then,  $h'h'' = h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H$ . And, also  $h''h' = h(g^{-1})h(g) = h(g^{-1}g) = h(1_G) = 1_H$ . This shows that  $h'$  and  $h''$  are inverses of each other, thereby implying  $h$  preserves inverses, which is what we set out to prove.  $\square$