Introduction to Categories and Categorical Logic

Vishal Lama

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ $\quad$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ $\quad$ (f is right cancellative).

PROPOSITION 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

PROOF. We first show (1).
($\Longleftarrow$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$\begin{array}{ll} & f(x) = f(x') \\ \implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\ \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\ \implies & f \circ \bar{x} = f \circ \bar{x'} \\ \implies & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\ \implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\ \implies & x = x' \end{array}$$

This shows that f is injective.

($\Longrightarrow$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$\begin{array}{ll} & (f \circ g)(a) = (f \circ h)(a) \\ \implies & f(g(a)) = f(h(a)) \\ \implies & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\ \implies & g = h \end{array}$$

This establishes that f is monic. And, we are done. $\square$

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

SOLUTION. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. □

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \to H$ is a monoid homomorphism. Prove that h is a group homomorphism.

SOLUTION. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. □

Exercise 6

Check that $\mathbf{Mon}, \mathbf{Vect}_k, \mathbf{Pos}$, and $\mathbf{Top}$ are indeed categories.

SOLUTION. ($\mathbf{Mon}$) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \to (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \to (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \to (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m))(g(f(m'))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that $\mathbf{Mon}$ is indeed a category.

($\mathbf{Vect}_k$) The objects are vector spaces over a field k, and morphisms are linear maps between vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha(g \circ f)(x)$. This establishes $g \circ f : U \to W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that $\mathbf{Vect}_k$ is also a category.

(**Pos**) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \to Q$ and $g : Q \to R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \to R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(**Top**) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \to (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$,which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \to (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. □

Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon**) have?)

SOLUTION. (Monoid as a one-object category) Given a monoid $(M, \cdot, 1)$, we can construct its corresponding category as follows. We write $\mathbf{B}M$ for the corresponding category with a single object $\bullet$, where $\mathbf{Hom}_{\mathbf{B}M}(\bullet, \bullet) := M$. We note then that the composition map in $\mathbf{B}M$ is reflected in the binary operation $\_ \cdot \_ : M \times M \to M$, where $\mathbf{id}_\bullet := 1$. Then, the associative and identity laws for the category $\mathbf{B}M$ follow directly from the associative and identity laws, respectively, satisfied by the monoid $(M, \cdot, 1)$. This shows any monoid can be seen or interpreted as a one-object category. □

Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

SOLUTION. Let $(P, \leq)$ be a preorder. Then, we define the corresponding category $\mathbf{C}$ as follows. The objects of $\mathbf{C}$ are the elements of the set P, and for all $x, y \in P$, we define a morphism $x \to y$ iff $x \leq y$. Then, for every object $x \in \mathbf{C}$, the identity morphism $1_x : x \to x$ corresponds exactly to the reflexive property $x \leq x$ for all $x \in P$. Note that each homset in $\mathbf{C}$ has at most one element. Also, for every $x \to y$ and $y \to z$ in $\mathbf{C}$, $x \to z$ follows from the fact that $x \leq y$ and $y \leq z$ and the transitivity of the $\leq$ relation on P. This defines a composition map for morphisms in $\mathbf{C}$. In addition, for all morphisms $x \to y$, $y \to z$, and $z \to w$, their associativity follows immediately from the transitivity of $\leq$. Lastly, the unit laws also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element. □

Exercise 10

Show that the inverse, if it exists, is unique.

SOLUTION. Suppose $i : A \to B$ is an isomorphism, with inverse $j : B \to A$, in a category **C**. Suppose $j' : B \to A$ is also an inverse of i. Then, $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$, and we are done. □

Exercise 11

Show that $\cong$ is an equivalence relation on the objects of a category.

SOLUTION. Let **C** be some category.
(*Reflexivity*) For any object $X \in \mathbf{C}$, $X \cong X$ follows from the fact that the identity morphism $1_X : X \to X$ is an isomorphism.
(*Symmetry*) If $X \cong Y$, then there exists an isomorphism $i : X \to Y$. But, the inverse, $i^{-1} : Y \to X$, of i is also an isomorphism. Hence, $Y \cong X$.
(*Transitivity*) Suppose $X \cong Y$ and $Y \cong Z$. Then, there exist isomorphisms $i : X \to Y$ and $j : Y \to Z$. Then, we claim that $j \circ i : X \to Z$ is also an isomorphism. Indeed, its trivial to show that its inverse is the morphism $i^{-1} \circ j^{-1} : Z \to X$. This implies $X \cong Z$.
We thus conclude that $\cong$ is an equivalence relation on the objects of a category. □

Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

SOLUTION. (**Set**) We claim the following:

(1) $f : X \to Y$ is injective iff f has a left inverse.
(2) $f : X \to Y$ is surjective iff f has a right inverse.

We first show (1).

($\implies$) Suppose $f : X \to Y$ has a left inverse, $g : Y \to X$, say. Then, $g \circ f = 1_X$. Assume for any $x, x' \in X, f(x) = f(x')$. Then, $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$, which implies f is injective.

($\impliedby$) Suppose $f : X \to Y$ is injective. If X is empty, then f is an empty function corresponding to each Y. In this case, 1_X is also an empty function, and we thus have $g \circ f = 1_X$ for any $g : Y \to X$. That is, f has a left inverse. On the other hand, if X is nonempty, choose some $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all $x \in X, (g \circ f)(x) = g(f(x)) = x = 1_X(x)$, which implies $g \circ f = 1_X$, thus showing that g is a left inverse of f.

We now show (2).

($\implies$) Suppose $f : X \to Y$ has a right inverse, $g : Y \to X$, say. Then, $f \circ g = 1_Y$. Therefore, for all $y \in Y$, $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$, where $x = g(y)$. This shows f is surjective.

($\impliedby$) Suppose $f : X \to Y$ is surjective. Now, consider an indexed family of nonempty sets $\{f^{-1}(y)\}_{y \in Y}$. Then, using the axiom of choice, we conclude there exists a function $g : Y \to X$, such that $g(y) \in f^{-1}(y)$ for all $y \in Y$. Then, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$, thus proving f has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category $\mathbf{C}$, if $f : X \to Y$ has both a left inverse, $g : Y \to X$, say, and a right inverse, $h : Y \to X$, say, then $g = h$. Indeed, $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$, and we are done.

(**Grp**)
(**Top**)
(**Pos**) □

Opposite Categories and Duality. Given a category $\mathbf{C}$, the opposite category $\mathbf{C^{op}}$ is given by taking the same objects as $\mathbf{C}$, and

$$\mathbf{C^{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from $\mathbf{C}$.
If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathbf{C^{op}}$, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in $\mathbf{C}$. Therefore, composition $g \circ f$ is $\mathbf{C^{op}}$ is defined as $f \circ g$ in $\mathbf{C}$. This leads to the ***principle of duality***: a statement S is true about a category $\mathbf{C}$ iff its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about $\mathbf{C^{op}}$. For example, a morphism f is monic in $\mathbf{C^{op}}$ iff it is epic in $\mathbf{C}$. We say monic and epic are *dual notions.*

Exercise 14

If P is a preorder, for example $(\mathbb{R}, \leq)$, describe $P^{\mathbf{op}}$ explicitly.

Solution. An arrow $a \leq_{P^{\mathbf{op}}} b$ in $P^{\mathbf{op}}$ is precisely the arrow $b \leq_P a$ in P. When $P = (\mathbb{R}, \leq)$, $P^{\mathbf{op}}$ describes the "greater than or equal" preorder relation on $\mathbb{R}$. □

Subcategories. Let $\mathbf{C}$ be a category. Suppose we are given the collections

$$\mathbf{Ob}(\mathbf{D}) \subseteq \mathbf{Ob}(\mathbf{C}),$$
$$\forall A, B \in \mathbf{Ob}(\mathbf{D}).\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B).$$

We say $\mathbf{D}$ is a ***subcategory*** of $\mathbf{C}$ if it is itself a category. In particular, $\mathbf{D}$ is:

- A ***full*** subcategory of $\mathbf{C}$ if for any $A, B \in \mathbf{Ob}(\mathbf{D})$, $\mathbf{D}(A, B) = \mathbf{C}(A, B)$.
- A ***lluf*** subcategory of $\mathbf{C}$ if $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$.

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

Exercise 16

How many categories $\mathbf{C}$ with $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$ are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids. $\square$

Exercises.

(1) Consider the following properties of an arrow f in a category $\mathbf{C}$.
- f is *split monic* if for some g, $g \circ f$ is an identity arrow.
- f is *split epic* if for some g, $f \circ g$ is an identity arrow.

a. Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.
b. Prove that if f is split epic then it is epic.
c. Prove that if f and $g \circ f$ are iso then g is iso.
d. Prove that if f is monic and split epic then it is iso.
e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map
$$i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$$
of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if $\{X_i\}_{i \in I}$ is a family of nonempty sets, we can form a set $X = \{x_i \mid i \in I\}$, where $x_i \in X_i$ for all $i \in I$.

f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
g. Is is always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.

(2) Give a description of partial orders as categories of a special kind.

SOLUTION.

(1)

a. Suppose $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is monic. Assume, for all $i, j : Z \to A$, $f \circ i = f \circ j$. Then, $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$, which implies $i = j$, since $g \circ f$ is monic. This implies f is monic, and we are done.
b. Suppose $f : A \to B$ is split epic. Then, there exists a $g : B \to A$ such that $f \circ g = 1_B$. Assume, for all $i, j : B \to C$, $i \circ f = j \circ f$. Then, $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$, which shows f is epic.
c. Suppose $f : A \to B$ and $g : B \to C$ such that f and $g \circ f$ are iso. We claim that the inverse of g is $f \circ (g \circ f)^{-1} : C \to B$. Indeed, $g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = 1_C$, and $(f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} = f \circ f^{-1} = 1_B$, which establishes g is also an iso.
d. Suppose $f : A \to B$ is monic and split epic. The latter implies f has a right inverse, $g : B \to A$, say, where $f \circ g = 1_B$. Note that $g \circ f : A \to A$ and $1_A : A \to A$. Now, $f \circ (g \circ f) = (f \circ g) \circ f =$

$1_B \circ f = f = f \circ 1_A$, which implies $g \circ f = 1_A$, since f is monic (left cancellative). Thus, g is also a left inverse of f, and hence, f is iso.

e. It is easy to prove the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is really a monoid homomorphism. Indeed, $i(0) = 0$, and, for all $n_1, n_2 \in \mathbb{N}$, $i(n_1 + n_2) = n_1 + n_2 = i(n_1) + i(n_2)$.
Next, we show that i is monic. Assume, for all monoid homomorphisms $g, h : X \to \mathbb{N}$, $i \circ g = i \circ h$. Then, for all $x \in X$, $(i \circ g)(x) = (i \circ h)(x)$, which implies $i(g(x)) = i(h(x))$, which implies $g(x) = h(x)$, which implies $g = h$. This shows the inclusion map is monic.
We now show the inclusion map is epic. First, assume, for all monoid homomorphisms $g, h : (\mathbb{Z}, +, 0) \to (X, \star, 1_X)$, $g \circ i = h \circ i$. Then, for all $n \in \mathbb{N}$, $(g \circ i)(n) = (h \circ i)(n)$, which implies $g(i(n)) = h(i(n))$, which implies $g(n) = h(n)$. We now claim that for all $n \geq 1$, $g(-n) = h(-n)$. To that end, we use induction on n. Note that $g(-1) = g(-1) \star 1_X = g(-1) \star h(0) = g(-1) \star h(1 + (-1)) = g(-1) \star h(1) \star h(-1) = g(-1) \star g(1) * h(-1) = g(-1+1) \star h(-1) = g(0) \star h(-1) = 1_X \star h(-1) = h(-1)$. Now, assume the proposition holds for some $n \geq 1$. Then, $g(-(n+1)) = g(-n + (-1)) = g(-n) \star g(-1) = h(-n) \star h(-1) = h(-n + (-1)) = h(-(n+1))$. Hence, by induction, $g(-n) = h(-n)$ for all $n \geq 1$. Combining the results from above, we thus conclude $g(z) = h(z)$ for all $z \in \mathbb{Z}$. In other words, $g = h$, which implies i is epic.
Clearly, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is not iso.

f. Suppose $f : X \to Y$ is epic in **Set**. Then, from an earlier result about **Set**, we conclude f is surjective. Now, consider the family of nonempty sets $\{f^{-1}(b)\}_{b \in B}$. Each of the sets in the family is nonempty, because f is surjective. Therefore, using the Axiom of Choice, we can choose some element from each nonempty set in the family to construct a function $g : Y \to X$, given by $g(y) := x$ if $x \in f^{-1}(b)$. In addition, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$. This shows f has a right inverse, thus proving f is split epic.

g. It isn't always the case that an arrow which is epic is split epic. For example, in the category **Mon**, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is epic (as shown in (e) above.) Now, if we assume that it is also split epic, then there exists a monoid homomorphism $g : \mathbb{Z} \to \mathbb{N}$, such that $i \circ g = 1_{\mathbb{Z}}$. This implies $(i \circ g)(-1) = 1_{\mathbb{Z}}(-1)$, which implies $i(g(-1)) = -1$, which implies $g(-1) = -1$, which implies $-1 \in \mathbb{N}$, which is absurd. We thus conclude the aforesaid inclusion map is *not* split epic, even though it is epic. And this proves our original claim.

(2) Suppose $(P, \leq)$ is a poset. Then, its corresponding category **C** is defined as follows. The objects of **C** are the elements of P, and for all $x, y \in P$, $x \to y$ iff $x \leq y$. The reflexivity of $\leq$ corresponds to the identity arrows, and transitivity to arrow composition. Note that there is at most one arrow for every pair of objects in the category. Anti-symmetry of $\leq$ corresponds to the fact that the only isomorphisms in **C** are the identity arrows.

$\square$

2. Some Basic Constructions

Initial and Terminal Objects. An object I in a category $\mathbf{C}$ is ***initial*** if, for every object A, *there exists a unique arrow* $I \to A$, which we write $\iota_A : I \to A$.

An object T in a category $\mathbf{C}$ is ***terminal*** if, for every object A, *there exists a unique arrow* $A \to T$, which we write $\tau_A : A \to T$.

Note that initial and terminal objects are dual notions: T is terminal in $\mathbf{C}$ iff it is initial in $\mathbf{C}^{\mathbf{op}}$. We sometimes write $\mathbf{1}$ for the terminal object and $\mathbf{0}$ for the initial object.

Exercise 18

Verify the following claims. In each case, identify the canonical arrows.

(1) In **Set**, the empty set is an initial object while any one-element set $\{\bullet\}$ is terminal.
(2) In **Pos**, the poset $(\varnothing, \varnothing)$ is an initial object while $(\{\bullet\}, \{(\bullet, \bullet)\})$ is terminal.
(3) In **Top**, the space $(\varnothing, \{\varnothing\})$ is an initial object while $(\{\bullet\}, \{\varnothing, \{\bullet\}\})$ is terminal.
(4) In $\mathbf{Vect}_k$, the one-element space $\{0\}$ is both initial and terminal.
(5) In a poset, seen as a category, an initial object is a least element, while a terminal object is a greatest element.

Solution.

(1) In **Set**, for any set (object) A, the function $(\varnothing, A, \varnothing)$ is the unique function (arrow) from $\varnothing$ to A. Therefore, the empty set is (the) initial object in **Set**. And, for every set A, the function $A \to \{\bullet\}$ that maps every element of A to $\bullet$ is the unique function from A to $\{\bullet\}$. This establishes that any one-element set is terminal in **Set**.
(2) For any poset $(P, \leq)$, there exists a unique (empty) monotone function $(\varnothing, \varnothing) \xrightarrow{(\varnothing, P, \varnothing)} (P, \leq)$. Hence, the poset $(\varnothing, \varnothing)$ is an initial object in **Pos**. And, for any poset $(P, \leq)$, there exists a unique monotone function $(P, \leq) \to (\{\bullet\}, \{(\bullet, \bullet)\})$, defined by $x \mapsto \bullet$ for all $x \in P$. Hence, $(\{\bullet\}, \{(\bullet, \bullet)\})$ is terminal in **Pos**.
(3) For any topological space (X, T_X), the unique empty function
$$(\varnothing, \{\varnothing\}) \xrightarrow{(\varnothing, X, \varnothing)} (X, T_X)$$
is continuous, since for every open set $T \in T_X$, its preimage under the aforesaid function is the empty set, which is open. Hence, $(\varnothing, \{\varnothing\})$ is initial in **Top**.
And, for any topological space (X, T_X), the unique function $(X, T_X) \to (\{\bullet\}, \{\varnothing, \{\bullet\}\})$, defined by $x \mapsto \bullet$ for all $x \in X$, is continuous, since the preimage of $\varnothing$ under the aforesaid function is $\varnothing$, which is open, and the preimage of $\{\bullet\}$ is X, which is also open. Hence, $(\{\bullet\}, \{\varnothing, \{\bullet\}\})$ is terminal in **Top**.
(4) Assuming the ground field is k, for any vector space V, the unique linear map $\{0\} \to V$, defined by $0 \mapsto 0_V$ is a unique arrow from $\{0\}$ to V in $\mathbf{Vect}_k$. Also, the unique linear map $V \to \{0\}$, defined by $v \mapsto 0$ for all $v \in V$, is a unique arrow from V to $\{0\}$ in $\mathbf{Vect}_k$. This shows that $\{0\}$ is both initial and terminal in $\mathbf{Vect}_k$.

(5) In a poset $(P, \leq)$, seen as a category, if $\perp$ is an initial object, then there exists a unique arrow $\perp \to p$ for all $p \in P$. This implies $\perp \leq p$ for all $p \in P$, when seen as a set. Hence, an initial object in the category corresponding to $(P, \leq)$ is a least element in P. Arguing similarly, we conclude that a terminal object in the category corresponding to $(P, \leq)$ is a greatest element in P.

□

Exercise 19

Identify the initial and terminal objects in **Rel**.

Solution. In **Rel**, the empty set $\varnothing$ is both the initial object and the terminal object. Indeed, for any set A, the empty relation $\varnothing$ ($\subseteq \varnothing \times A$) is a unique relation from $\varnothing$ to A, and the empty relation $\varnothing$ ($\subseteq A \times \varnothing$) is also a unique relation from A to $\varnothing$. □

Exercise 20

Suppose a monoid, viewed as a category, has either an initial or a terminal object. What must the monoid be?

Solution. The category corresponding to a monoid $(M, \cdot, 1_M)$ contains just a single object. If this object is initial, then all morphisms must be the identity morphism on this initial object, which implies $M = \{1_M\}$. The argument is similar if the aforesaid object is terminal, which would again imply $M = \{1_M\}$. Thus, in either case, the monoid must be the trivial monoid. □

A fundamental fact about initial and terminal objects is that they are *unique up to (unique) isomorphism*. This is characteristic of all such “universal” definitions. Hence, if initial objects exist in a category, we can speak of *the* initial object. Similarly for terminal objects.

Exercise 22

Let $\mathbf{C}$ be a category with an initial object $\mathbf{0}$. For any object A, show the following:

(1) If $A \cong \mathbf{0}$, then A is an initial object.
(2) If there exists a monomorphism $f : A \to \mathbf{0}$, then f is an iso, and hence A is initial.

Solution.

(1) Suppose $A \cong \mathbf{0}$. Then, there exists an isomorphism $f : A \xrightarrow{\sim} \mathbf{0}$. For any object B, there exists a unique morphism $\iota_B : \mathbf{0} \to B$, and hence, $\iota_B \circ f : A \to B$. This proves the existence of a morphism $A \to B$ for any object B in the category. We now show that such a morphism is indeed unique. Let $g, h : A \to B$ be a pair of morphisms for any object B in the category $\mathbf{C}$. Then, we have

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{g} B$$

and

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{h} B.$$

Since $\mathbf{0}$ is initial, we must have $g \circ f^{-1} = h \circ f^{-1}$. Therefore, $g = g \circ 1_A = g \circ (f^{-1} \circ f) = (g \circ f^{-1}) \circ f = (h \circ f^{-1}) \circ f = h \circ (f^{-1} \circ f) = h \circ 1_A = h$. This proves uniqueness, and we are done.

(2) Suppose $A \xrightarrow{f} \mathbf{0}$ is a monomorphism. We claim the unique arrow $\mathbf{0} \xrightarrow{\iota_A} A$ is the inverse of f. To that end, we show ι_A is both a left and a right inverse of f. Indeed, $f \circ \iota_A : \mathbf{0} \to \mathbf{0}$, and since $\mathbf{0}$ is initial, we must have $f \circ \iota_A = 1_{\mathbf{0}}$, which implies ι_A is a right inverse of f. Now, note $\iota_A \circ f : A \to A$ and $1_A : A \to A$. Also, $f \circ (\iota_A \circ f) = (f \circ \iota_A) \circ f = 1_{\mathbf{0}} \circ f = f = f \circ 1_A$, and since f is left cancellative, we have $\iota_A \circ f = 1_A$, which shows ι_A is a left inverse of f. Thus, f has both a left inverse and a right inverse, implying it is iso, and hence, using the result obtained in (1) above, we conclude A is an initial object in $\mathbf{C}$.

□

Products and Coproducts. We can express a general notion of product that is meaningful in any category, such that, if a product exists, it is characterized uniquely up to unique isomorphism. Given a particular mathematical context (*i.e.* a category), we can then verify if a product exists in that category. The concrete construction appropriate to the context will enter only into the proof of *existence*; all of the useful *properties* of a product follow from the general definition.

Exercise 24

Verify $\mathbf{Pair}(A, B)$ is a category, where A and B are arbitrary objects in some category.

SOLUTION. Let A and B be some arbitrary objects in some category $\mathbf{C}$. Now, given morphisms $f : (P, p_1, p_2) \to (Q, q_1, q_2)$ and $g : (Q, q_1, q_2) \to (R, r_1, r_2)$ in $\mathbf{Pair}(A, B)$, it is easy to check that $g \circ f : P \to R$ in $\mathbf{C}$. Also, we have

$$q_1 \circ f = p_1,\ q_2 \circ f = p_2$$

and

$$r_1 \circ g = q_1,\ r_2 \circ g = q_2$$

So, $r_1 \circ (g \circ f) = (r_1 \circ g) \circ f = q_1 \circ f = p_1$, and, $r_2 \circ (g \circ f) = (r_2 \circ g) \circ f = q_2 \circ f = p_2$, which implies $g \circ f : (P, p_1, p_2) \to (R, r_1, r_2)$ in $\mathbf{Pair}(A, B)$. Associativity of morphisms in $\mathbf{Pair}(A, B)$ follows directly from the associativity of morphisms in $\mathbf{C}$. Finally, for all (P, p_1, p_2) in $\mathbf{Pair}(A, B)$, the identity morphism $1_P : P \to P$ is the identity morphism for (P, p_1, p_2), since $p_1 \circ 1_P = p_1$ and $p_2 \circ 1_P = p_2$. And, this proves that $\mathbf{Pair}(A, B)$ is indeed a category. □

We say $(A \times B, \pi_1, \pi_2)$ is a ***product*** of A and B if it is *terminal* in $\mathbf{Pair}(A, B)$. Products are specified by triples $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2}$, where pi_i's are called *projections.* For economy (and if projections are obvious), we say $A \times B$ is the product of A and B. We say a category $\mathbf{C}$ has ***(binary) products*** if each pair of objects A, B has a product in $\mathbf{C}$. Since, products are terminal objects, they are unique up to (unique) isomorphism.

Unpacking the uniqueness condition from $\mathbf{Pair}(A, B)$ back to $\mathbf{C}$, we obtain the following more concise definition of products that we use in practice.

(**Equivalent definition of product**) Let A, B be objects in a category $\mathbf{C}$. A product of A and B is an object $A \times B$ together with a pair of arrows $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ such that for every triple $A \xleftarrow{f} C \xrightarrow{g} B$, there exists a *unique* morphism

$$\langle f, g\rangle : C \to A \times B$$

such that the corresponding diagram commutes. That is,

$$\pi_1 \circ \langle f, g\rangle = f$$
$$\pi_2 \circ \langle f, g\rangle = g$$

We call $\langle f, g\rangle$ the *pairing* of f and g.

Exercise 26

Verify the following claims.

(1) In **Set**, products are the usual cartesian products.
(2) In **Pos**, products are cartesian products with the pointwise order.
(3) In **Top**, products are cartesian products with the product topology.
(4) In $\mathbf{Vect}_k$, products are direct sums.
(5) In a poset, seen as a category, products are *greatest lower bounds*.

SOLUTION.

(1) Let A, B be arbitrary sets in **Set**. We claim $A \times B$ equipped with the canonical projection functions is the cross product of A and B. Indeed, given any $A \xleftarrow{f} C \xrightarrow{g} B$, we show $\langle f, g\rangle : C \to A \times B$, defined by

$$c \mapsto (f(c), g(c)),$$

is the unique function that makes the following diagram commute:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

(*Existence*) It is easy to check that $\langle f, g\rangle$ is indeed a function from C to $A \times B$.

(*Commutativity*) For all $c \in C$, $(\pi_1 \circ \langle f, g\rangle)(c) = f(c)$ and $(\pi_2 \circ \langle f, g\rangle)(c) = g(c)$, which imply the above diagram commutes.

(*Uniqueness*) Suppose $h : C \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, for all $c \in C$, $(\pi_1 \circ h)(c) = f(c)$ and $(\pi_2 \circ h)(c) = g(c)$, which imply $\pi_1(h(c)) = f(c)$ and $\pi_2(h(c)) = g(c)$, which imply $h(c) = (f(c), g(c)) = \langle f, g\rangle(c)$, thus proving $h = \langle f, g\rangle$, and thereby, showing the uniqueness of $\langle f, g\rangle$.

Hence, $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ is the cross product of A and B.

(2) Let $(P, \leq)$ and $(Q, \leq)$ be posets. Let $(P \times Q, \leq)$ be the cartesian product of P and Q with the pointwise order. That is, for all $a, c \in P$ and $b, d \in Q$, $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. We claim $(P \times Q, \leq)$ equipped with the canonical projection functions (which are monotone) is the cross product

of $(P, \leq)$ and $(Q, \leq)$. Given any $(P, \leq) \xleftarrow{f} (R, \leq) \xrightarrow{g} (Q, \leq)$, where f, g are monotone functions, the function $\langle f, g\rangle : (R, \leq) \to (P \times Q, \leq)$, defined by

$$r \mapsto (f(r), g(r))$$

is the unique monotone function that makes the following diagram commute:

$$\begin{tikzcd} & (R, \leq) \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\ (P, \leq) & (P \times Q, \leq) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & (Q, \leq) \end{tikzcd}$$

(*Existence*) It is easy to check that $\langle f, g\rangle$ is indeed a set function from R to $P \times Q$. And, for all $r_1, r_2 \in R$, if $r_1 \leq r_2$, then $f(r_1) \leq f(r_2)$ and $g(r_1) \leq g(r_2)$ (since f, g are monotone), which implies $(f(r_1), g(r_1)) \leq (f(r_2), g(r_2))$, which implies $\langle f, g\rangle(r_1) \leq \langle f, g\rangle(r_2)$, which implies $\langle f, g\rangle$ is monotone.

(*Commutativity*) For all $r \in R$, we have

$$(\pi_1 \circ \langle f, g\rangle)(r) = f(r),$$
$$(\pi_2 \circ \langle f, g\rangle)(r) = g(r).$$

The above implies that the above diagram does commute.

(*Uniqueness*) Suppose $h : (R, \leq) \to (P \times Q, \leq)$ is a monotone function such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, for all $r \in R$, $\pi_1 \circ h(r) = f(r)$ and $\pi_2 \circ h(r) = g(r)$, which imply $\pi_1(h(r)) = f(r)$ and $\pi_2(h(r)) = g(r)$, which imply $h(r) = (f(r), g(r))$, which implies $h(r) = \langle f, g\rangle(r)$, which implies $h = \langle f, g\rangle$, thus showing that $\langle f, g\rangle$ with the commutativity property is indeed unique.

Hence, we conclude the cartesian product $(P \times Q, \leq)$ with the pointwise order is the product of any posets $(P, \leq)$ and $(Q, \leq)$.

(3)
(4)
(5) In a poset $(P, \leq)$, seen as a category, the product $a \times b$ of two elements $a, b \in P$ is an element in P satisfying $a \times b \leq a$ and $a \times b \leq b$, such that for all elements $c \in P$, if $c \leq a$ and $c \leq b$, then $c \leq a \times b$. This is precisely the definition of the *greatest lower bound* of any two elements $a, b \in P$, seen as a set. Therefore, products are greatest lower bounds in posets.

□

The following proposition shows that the uniqueness of the pairing arrow can be specified purely equationally by the equation:

$$\forall h : C \to A \times B.\, h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$$

Proposition 27. *For any triple $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$, the following statements are equivalent:*

(I) *For any triple $A \xleftarrow{f} C \xrightarrow{g} B$, there exists a unique morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$.*

(II) *For any triple* $A \xleftarrow{f} C \xrightarrow{g} B$*, there exists a morphism* $\langle f, g\rangle : C \to A \times B$ *such that* $\pi_1 \circ \langle f, g\rangle = f$ *and* $\pi_2 \circ \langle f, g\rangle = g$*, and moreover, for any* $h : C \to A \times B$*,* $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$*.*

Proof. ((I) $\Longrightarrow$ (II)) Suppose (I) holds. Assume $A \xleftarrow{f} C \xrightarrow{g} B$. Then, by (I), there exists a (unique) morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. Now, let $h : C \to A \times B$. Note $A \xleftarrow{\pi_1 \circ h} C \xrightarrow{\pi_2 \circ h} B$. Thus, by (I), there exists a unique morphism $\langle \pi_1 \circ h, \pi_2 \circ h\rangle : C \to A \times B$ such that

$$\pi_1 \circ \langle \pi_1 \circ h, \pi_2 \circ h\rangle = \pi_1 \circ h,$$
$$\pi_2 \circ \langle \pi_1 \circ h, \pi_2 \circ h\rangle = \pi_2 \circ h.$$

The above implies $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$. This proves (II).

((II) $\Longrightarrow$ (I)) Suppose (II) holds. Assume $A \xleftarrow{f} C \xrightarrow{g} B$. Then, by (II), there exists a morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. We claim such a morphism is unique. So, suppose $h : C \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, by (II), we have $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$, which implies $h = \langle f, g\rangle$. This proves (I), and our proof is complete. $\square$

Cartesian product of morphisms. Given $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, we define the *cartesian product of morphisms* f_1 and f_2 by

$$f_1 \times f_2 := \langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle : A_1 \times A_2 \to B_1 \times B_2.$$

The following proposition provides some useful properties of products.

Proposition 28. *For any* $f : A \to B$*,* $g : A \to C$*,* $h : A' \to A$*, and any* $p : B \to B'$*,* $q : C \to C'$*, the following hold:*

(1) $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$
(2) $(p \times q) \circ \langle f, g\rangle = \langle p \circ f, q \circ g\rangle$.

Proof.

(1) Note $\langle f, h\rangle \circ h : A' \to B \times C$. Therefore, by (II) of Proposition 27, $\langle f, h\rangle \circ h = \langle \pi_1 \circ (\langle f, g\rangle \circ h), \pi_2 \circ (\langle f, g\rangle \circ h)\rangle = \langle f \circ h, f \circ g\rangle$.
(2) $(p \times q) \circ \langle f, g\rangle = \langle p \circ \pi_1, q \circ \pi_2\rangle \circ \langle f, g\rangle = \langle p \circ \pi_1 \circ \langle f, g\rangle, q \circ \pi_2 \circ \langle f, g\rangle = \langle p \circ f, q \circ g\rangle$.

$\square$

General Products. The notion of products can be generalized to arbitrary arities as follows. In a category $\mathbf{C}$, a product for a family of objects $\{A_i\}_{i \in I}$ is an object P and morphisms

$$p_i : P \to A_i \ (i \in I)$$

such that, for all objects B and arrows

$$f_i : B \to A_i \ (i \in I)$$

there is a *unique* arrow $g : B \to P$ such that, for all $i \in I$, the following diagram commutes

$$\begin{tikzcd} B \arrow[rr, dashed, "g"] \arrow[dr, "f_i"'] & & P \arrow[dl, "p_i"] \\ & A_i & \end{tikzcd}$$

Again, if such a product exists, it is unique up to (unique) isomorphism. We write $P = \prod_{i \in I} A_i$ for the product object, and $g = \langle f_i \mid i \in I \rangle$ for the unique morphism in the definition.

Exercise 29

What is the product of the empty family?

SOLUTION. The product of the empty family is an object T, such that for every object with an arrow to the empty family, there is a unique arrow from that object to T making the corresponding diagram commute. Since there are no diagrams, this means there is a unique arrow from every object to T, and this is precisely the definition of a terminal object. Hence, the product of an empty family is a terminal object. $\square$