Introduction to Categories and Categorical Logic

Vishal Lama

Contents

CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ (f is right cancellative).

Proposition 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

Proof. We first show (1).
($\Longleftarrow$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$
\begin{array}{ll}
 & f(x) = f(x') \\
\implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\
\implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\
\implies & f \circ \bar{x} = f \circ \bar{x'} \\
\implies & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\
\implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\
\implies & x = x'
\end{array}
$$

This shows that f is injective.

($\Longrightarrow$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$
\begin{array}{ll}
 & (f \circ g)(a) = (f \circ h)(a) \\
\implies & f(g(a)) = f(h(a)) \\
\implies & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\
\implies & g = h
\end{array}
$$

This establishes that f is monic. And, we are done. $\square$

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.