

# Introduction to Categories and Categorical Logic

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## Contents

Chapter 1. Introduction to Categories and Categorical Logic	5
1. Introduction	5



## CHAPTER 1

# Introduction to Categories and Categorical Logic

### 1. Introduction

We say that a function  $f : X \rightarrow Y$  is:

*injective*    if  $\forall x, x' \in X. f(x) = f(x') \implies x = x'$ ,  
*surjective*   if  $\forall y \in Y. \exists x \in X. f(x) = y$ ,

*monic*        if  $\forall g, h. f \circ g = f \circ h \implies g = h$     ( $f$  is left cancellative),  
*epic*         if  $\forall g, h. g \circ f = h \circ f \implies g = h$     ( $f$  is right cancellative).

PROPOSITION 1. *Let  $f : X \rightarrow Y$ . Then,*

- (1)  *$f$  is injective  $\iff f$  is monic.*
- (2)  *$f$  is surjective  $\iff f$  is epic.*

PROOF. We first show (1).

( $\Leftarrow$ ) Suppose  $f$  is monic. Fix a one-element set  $\mathbf{1} = \{\bullet\}$ . Then, note that elements  $x \in X$  are in 1-1 correspondence with functions  $\bar{x} : \mathbf{1} \rightarrow X$ , defined by  $\bar{x}(\bullet) := x$ . Then, for all  $x, x' \in X$ , we have

$$\begin{aligned}
 & f(x) = f(x') \\
 \implies & f(\bar{x}(\bullet)) = f(\bar{x}'(\bullet)) \\
 \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x}')(\bullet) \\
 \implies & f \circ \bar{x} = f \circ \bar{x}' \\
 \implies & \bar{x} = \bar{x}' \quad (\text{since } f \text{ is monic}) \\
 \implies & \bar{x}(\bullet) = \bar{x}'(\bullet) \\
 \implies & x = x'
 \end{aligned}$$

This shows that  $f$  is injective.

( $\Rightarrow$ ) Suppose  $f$  is injective. Let  $f \circ g = f \circ h$  for all  $g, h : A \rightarrow X$ . Then, for all  $a \in A$ ,

$$\begin{aligned}
 & (f \circ g)(a) = (f \circ h)(a) \\
 \implies & f(g(a)) = f(h(a)) \\
 \implies & g(a) = h(a) \quad (\text{since } f \text{ is injective}) \\
 \implies & g = h
 \end{aligned}$$

This establishes that  $f$  is monic. And, we are done. □

### Exercise 2

Show that  $f : X \rightarrow Y$  is surjective iff it is epic.

SOLUTION. ( $\implies$ ) Suppose  $f : X \rightarrow Y$  is epic. And, assume, for the sake of contradiction,  $f$  is *not* surjective. Then, there exists some  $y_0 \in Y$ , such that, for all  $x \in X$ ,  $f(x) \neq y_0$ . Define mappings  $g, h : Y \rightarrow Y \cup \{Y\}$  by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that  $g \neq h$ .

Then, for all  $x \in X$ ,  $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$ . This implies  $g \circ f = h \circ f$ , which implies  $g = h$ , since  $f$  is epic. The last conclusion contradicts the fact that  $g \neq h$ . Thus, we conclude  $f$  is surjective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is surjective. Then, for any  $y \in Y$ , there exists an  $x \in X$ , such that  $f(x) = y$ . Now, assume, for all  $g, h : Y \rightarrow Z$ ,  $g \circ f = h \circ f$ . Then, for all  $y \in Y$ ,  $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$ , which implies  $g = h$ , showing that  $f$  is epic.

And, this completes our proof.  $\square$

### Exercise 5

Suppose  $G$  and  $H$  are groups (and hence monoids), and that  $h : G \rightarrow H$  is a monoid homomorphism. Prove that  $h$  is a group homomorphism.

SOLUTION. We need only show that  $h$  preserves inverses. To that end, suppose  $g^{-1}$  is the inverse of  $g \in G$ . Then,  $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$ . This establishes  $h$  preserves inverses, and we are done.  $\square$

### Exercise 6

Check that **Mon**, **Vect<sub>k</sub>**, **Pos**, and **Top** are indeed categories.

SOLUTION. (**Mon**) The objects are monoids  $(M, \cdot, 1_M)$ , and morphisms are monoid homomorphisms. Given monoid homomorphisms,  $f : (M, \cdot, 1_M) \rightarrow (N, \cdot, 1_N)$  and  $g : (N, \cdot, 1_N) \rightarrow (P, \cdot, 1_P)$ , the function  $g \circ f : (M, \cdot, 1_M) \rightarrow (P, \cdot, 1_P)$  is also a monoid homomorphism, because for all  $m, m' \in M$ , we have  $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m)))(g(f(m')))) = ((g \circ f)(m))((g \circ f)(m'))$ . Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms  $f, g$  and  $h$  with the appropriate domains and codomains,  $h \circ (g \circ f) = (h \circ g) \circ f$ . This establishes that **Mon** is indeed a category.

(**Vect<sub>k</sub>**) The objects are vector spaces over a field  $k$ , and morphisms are linear maps between vector spaces. Suppose  $f : U \rightarrow V$  and  $g : V \rightarrow W$  are linear maps. Then, for all  $x, y \in U$ , we have  $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$ . Also, for all  $\alpha \in k$ , we have  $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha (g \circ f)(x)$ . This establishes  $g \circ f : U \rightarrow W$  is a linear map as well. The identity map  $1_U$  for any vector space  $U$  is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that **Vect<sub>k</sub>** is also a category.

**(Pos)** The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose  $h : P \rightarrow Q$  and  $g : Q \rightarrow R$  are monotone functions. Then, for all  $x, y \in P$ ,  $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$ , which shows  $g \circ h : P \rightarrow R$  is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

**(Top)** The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps  $f : (X, T_X) \rightarrow (Y, T_Y)$  and  $g : (Y, T_Y) \rightarrow (Z, T_Z)$ , we can show that  $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$  is also a continuous map. First, note that for any  $T \subset Z$ ,  $x \in (g \circ f)^{-1}(T)$  iff  $(g \circ f)(x) \in T$  iff  $g(f(x)) \in T$  iff  $f(x) \in g^{-1}(T)$  iff  $x \in f^{-1}(g^{-1}(T))$ . Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set  $T \in T_Z$ , we have  $g^{-1}(T) \in T_Y$ , which implies  $f^{-1}(g^{-1}(T)) \in T_X$ , which implies  $(g \circ f)^{-1}(T) \in T_X$  (by using the result above.) Hence,  $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$  is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category.  $\square$

### Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon** have?)

**SOLUTION.** (Monoid as a one-object category) Given a monoid  $(M, \cdot, 1)$ , we can construct its corresponding category as follows. We write **BM** for the corresponding category with a single object  $\bullet$ , where  $\text{Hom}_{\mathbf{BM}}(\bullet, \bullet) := M$ . We note then that the composition map in **BM** is reflected in the binary operation  $\cdot : M \times M \rightarrow M$ , where  $\text{id}_\bullet := 1$ . Then, the associative and identity laws for the category **BM** follow directly from the associative and identity laws, respectively, satisfied by the monoid  $(M, \cdot, 1)$ . This shows any monoid can be seen or interpreted as a one-object category.  $\square$

### Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

**SOLUTION.** Let  $(P, \leq)$  be a preorder. Then, we define the corresponding category **C** as follows. The objects of **C** are the elements of the set  $P$ , and for all  $x, y \in P$ , we define a morphism  $x \rightarrow y$  iff  $x \leq y$ . Then, for every object  $x \in \mathbf{C}$ , the identity morphism  $1_x : x \rightarrow x$  corresponds exactly to the reflexive property  $x \leq x$  for all  $x \in P$ . Note that each homset in **C** has at most one element. Also, for every  $x \rightarrow y$  and  $y \rightarrow z$  in **C**,  $x \rightarrow z$  follows from the fact that  $x \leq y$  and  $y \leq z$  and the transitivity of the  $\leq$  relation on  $P$ . This defines a composition map for morphisms in **C**. In addition, for all morphisms  $x \rightarrow y$ ,  $y \rightarrow z$ , and  $z \rightarrow w$ , their associativity follows immediately from the transitivity of  $\leq$ . Lastly, the unit laws

also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element.  $\square$

### Exercise 10

Show that the inverse, if it exists, is unique.

SOLUTION. Suppose  $i : A \rightarrow B$  is an isomorphism, with inverse  $j : B \rightarrow A$ , in a category  $\mathbf{C}$ . Suppose  $j' : B \rightarrow A$  is also an inverse of  $i$ . Then,  $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$ , and we are done.  $\square$

### Exercise 11

Show that  $\cong$  is an equivalence relation on the objects of a category.

SOLUTION. Let  $\mathbf{C}$  be some category.

(*Reflexivity*) For any object  $X \in \mathbf{C}$ ,  $X \cong X$  follows from the fact that the identity morphism  $1_X : X \rightarrow X$  is an isomorphism.

(*Symmetry*) If  $X \cong Y$ , then there exists an isomorphism  $i : X \rightarrow Y$ . But, the inverse,  $i^{-1} : Y \rightarrow X$ , of  $i$  is also an isomorphism. Hence,  $Y \cong X$ .

(*Transitivity*) Suppose  $X \cong Y$  and  $Y \cong Z$ . Then, there exist isomorphisms  $i : X \rightarrow Y$  and  $j : Y \rightarrow Z$ . Then, we claim that  $j \circ i : X \rightarrow Z$  is also an isomorphism. Indeed, it's trivial to show that its inverse is the morphism  $i^{-1} \circ j^{-1} : Z \rightarrow X$ . This implies  $X \cong Z$ .

We thus conclude that  $\cong$  is an equivalence relation on the objects of a category.  $\square$

### Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

SOLUTION. (**Set**) We claim the following:

- (1)  $f : X \rightarrow Y$  is injective iff  $f$  has a left inverse.
- (2)  $f : X \rightarrow Y$  is surjective iff  $f$  has a right inverse.

We first show (1).

( $\implies$ ) Suppose  $f : X \rightarrow Y$  has a left inverse,  $g : Y \rightarrow X$ , say. Then,  $g \circ f = 1_X$ . Assume for any  $x, x' \in X$ ,  $f(x) = f(x')$ . Then,  $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$ , which implies  $f$  is injective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is injective. If  $X$  is empty, then  $f$  is an empty function corresponding to each  $Y$ . In this case,  $1_X$  is also an empty function, and we thus have  $g \circ f = 1_X$  for any  $g : Y \rightarrow X$ . That is,  $f$  has a left inverse. On the other hand, if  $X$  is nonempty, choose some  $x_0 \in X$ . Define  $g : Y \rightarrow X$  by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all  $x \in X$ ,  $(g \circ f)(x) = g(f(x)) = x = 1_X(x)$ , which implies  $g \circ f = 1_X$ , thus showing that  $g$  is a left inverse of  $f$ .

We now show (2).



( $\implies$ ) Suppose  $f : X \rightarrow Y$  has a right inverse,  $g : Y \rightarrow X$ , say. Then,  $f \circ g = 1_Y$ . Therefore, for all  $y \in Y$ ,  $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$ , where  $x = g(y)$ . This shows  $f$  is surjective.

( $\impliedby$ ) Suppose  $f : X \rightarrow Y$  is surjective. Now, consider an indexed family of nonempty sets  $\{f^{-1}(y)\}_{y \in Y}$ . Then, using the axiom of choice, we conclude there exists a function  $g : Y \rightarrow X$ , such that  $g(y) \in f^{-1}(y)$  for all  $y \in Y$ . Then, for all  $y \in Y$ ,  $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$ , which implies  $f \circ g = 1_Y$ , thus proving  $f$  has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category **C**, if  $f : X \rightarrow Y$  has both a left inverse,  $g : Y \rightarrow X$ , say, and a right inverse,  $h : Y \rightarrow X$ , say, then  $g = h$ . Indeed,  $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$ , and we are done.

(Grp)  
(Top)  
(Pos)

□

**Opposite Categories and Duality.** Given a category **C**, the opposite category **C<sup>op</sup>** is given by taking the same objects as **C**, and

$$\mathbf{C}^{\text{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from **C**.

If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in **C<sup>op</sup>**, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in **C**. Therefore, composition  $g \circ f$  in **C<sup>op</sup>** is defined as  $f \circ g$  in **C**. This leads to the **principle of duality**: a statement  $S$  is true about a category **C** iff its dual (*i.e.* the one obtained from  $S$  by reversing all the arrows) is true about **C<sup>op</sup>**. For example, a morphism  $f$  is monic in **C<sup>op</sup>** iff it is epic in **C**. We say monic and epic are *dual notions*.

### Exercise 14

If  $P$  is a preorder, for example  $(\mathbb{R}, \leq)$ , describe **P<sup>op</sup>** explicitly.

SOLUTION. An arrow  $a \leq_{P^{\text{op}}} b$  in **P<sup>op</sup>** is precisely the arrow  $b \leq_P a$  in  $P$ . When  $P = (\mathbb{R}, \leq)$ , **P<sup>op</sup>** describes the “greater than or equal” preorder relation on  $\mathbb{R}$ . □

**Subcategories.** Let **C** be a category. Suppose we are given the collections

$$\begin{aligned} \mathbf{Ob}(\mathbf{D}) &\subseteq \mathbf{Ob}(\mathbf{C}), \\ \forall A, B \in \mathbf{Ob}(\mathbf{D}). \mathbf{D}(A, B) &\subseteq \mathbf{C}(A, B). \end{aligned}$$

We say **D** is a **subcategory** of **C** if it is itself a category. In particular, **D** is:

- A **full** subcategory of **C** if for any  $A, B \in \mathbf{Ob}(\mathbf{D})$ ,  $\mathbf{D}(A, B) = \mathbf{C}(A, B)$ .
- A **lluf** subcategory of **C** if  $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$ .

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

### Exercise 16

How many categories  $\mathbf{C}$  with  $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$  are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids.  $\square$

#### Exercises.

- (1) Consider the following properties of an arrow  $f$  in a category  $\mathbf{C}$ .
  - $f$  is *split monic* if for some  $g$ ,  $g \circ f$  is an identity arrow.
  - $f$  is *split epic* if for some  $g$ ,  $f \circ g$  is an identity arrow.
  - a. Prove that if  $f$  and  $g$  are arrows such that  $g \circ f$  is monic, then  $f$  is monic.
  - b. Prove that if  $f$  is split epic then it is epic.
  - c. Prove that if  $f$  and  $g \circ f$  are iso then  $g$  is iso.
  - d. Prove that if  $f$  is monic and split epic then it is iso.
  - e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map

$$i : (\mathbb{N}, +, 0) \rightarrow (\mathbb{Z}, +, 0)$$

of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if  $\{X_i\}_{i \in I}$  is a family of nonempty sets, we can form a set  $X = \{x_i \mid i \in I\}$ , where  $x_i \in X_i$  for all  $i \in I$ .

- f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
  - g. Is it always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.
- (2) Give a description of partial orders as categories of a special kind.

SOLUTION.

- (1)
  - a. Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow C$  such that  $g \circ f$  is monic. Assume, for all  $i, j : Z \rightarrow A$ ,  $f \circ i = f \circ j$ . Then,  $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$ , which implies  $i = j$ , since  $g \circ f$  is monic. This implies  $f$  is monic, and we are done.
  - b. Suppose  $f : A \rightarrow B$  is split epic. Then, there exists a  $g : B \rightarrow A$  such that  $f \circ g = 1_B$ . Assume, for all  $i, j : B \rightarrow C$ ,  $i \circ f = j \circ f$ . Then,  $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$ , which shows  $f$  is epic.
  - c. Suppose  $f : A \rightarrow B$  and  $g : B \rightarrow C$  such that  $f$  and  $g \circ f$  are iso. We claim that the inverse of  $g$  is  $f \circ (g \circ f)^{-1} : C \rightarrow B$ . Indeed,  $g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = 1_C$ , and  $(f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} = f \circ f^{-1} = 1_B$ , which establishes  $g$  is also an iso.
  - d. Suppose  $f : A \rightarrow B$  is monic and split epic. The latter implies  $f$  has a right inverse,  $g : B \rightarrow A$ , say, where  $f \circ g = 1_B$ . Note that  $g \circ f : A \rightarrow A$  and  $1_A : A \rightarrow A$ . Now,  $f \circ (g \circ f) = (f \circ g) \circ f =$

- $1_B \circ f = f = f \circ 1_A$ , which implies  $g \circ f = 1_A$ , since  $f$  is monic (left cancellative). Thus,  $g$  is also a left inverse of  $f$ , and hence,  $f$  is iso.
- e. It is easy to prove the inclusion map  $\mathbb{N} \hookrightarrow \mathbb{Z}$  is really a monoid homomorphism. Indeed,  $i(0) = 0$ , and, for all  $n_1, n_2 \in \mathbb{N}$ ,  $i(n_1 + n_2) = n_1 + n_2 = i(n_1) + i(n_2)$ .

Next, we show that  $i$  is monic. Assume, for all monoid homomorphisms  $g, h : X \rightarrow \mathbb{N}$ ,  $i \circ g = i \circ h$ . Then, for all  $x \in X$ ,  $(i \circ g)(x) = (i \circ h)(x)$ , which implies  $i(g(x)) = i(h(x))$ , which implies  $g(x) = h(x)$ , which implies  $g = h$ . This shows the inclusion map is monic.

We now show the inclusion map is epic. First, assume, for all monoid homomorphisms  $g, h : (\mathbb{Z}, +, 0) \rightarrow (X, \star, 1_X)$ ,  $g \circ i = h \circ i$ . Then, for all  $n \in \mathbb{N}$ ,  $(g \circ i)(n) = (h \circ i)(n)$ , which implies  $g(i(n)) = h(i(n))$ , which implies  $g(n) = h(n)$ . We now claim that for all  $n \geq 1$ ,  $g(-n) = h(-n)$ . To that end, we use induction on  $n$ . Note that  $g(-1) = g(-1) \star 1_X = g(-1) \star h(0) = g(-1) \star h(1 + (-1)) = g(-1) \star h(1) \star h(-1) = g(-1) \star g(1) \star h(-1) = g(-1 + 1) \star h(-1) = g(0) \star h(-1) = 1_X \star h(-1) = h(-1)$ . Now, assume the proposition holds for some  $n \geq 1$ . Then,  $g(-(n+1)) = g(-n + (-1)) = g(-n) \star g(-1) = h(-n) \star h(-1) = h(-n + (-1)) = h(-(n+1))$ . Hence, by induction,  $g(-n) = h(-n)$  for all  $n \geq 1$ . Combining the results from above, we thus conclude  $g(z) = h(z)$  for all  $z \in \mathbb{Z}$ . In other words,  $g = h$ , which implies  $i$  is epic.

Clearly, the inclusion map  $\mathbb{N} \hookrightarrow \mathbb{Z}$  is not iso.

- f. Suppose  $f : X \rightarrow Y$  is epic in **Set**. Then, from an earlier result about **Set**, we conclude  $f$  is surjective. Now, consider the family of nonempty sets  $\{f^{-1}(b)\}_{b \in B}$ . Each of the sets in the family is nonempty, because  $f$  is surjective. Therefore, using the Axiom of Choice, we can choose some element from each nonempty set in the family to construct a function  $g : Y \rightarrow X$ , given by  $g(y) := x$  if  $x \in f^{-1}(y)$ . In addition, for all  $y \in Y$ ,  $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$ , which implies  $f \circ g = 1_Y$ . This shows  $f$  has a right inverse, thus proving  $f$  is split epic.

□