Introduction to Categories and Categorical Logic

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ $\quad$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ $\quad$ (f is right cancellative).

Proposition 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

Proof. We first show (1).
($\Longleftarrow$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$
\begin{array}{ll}
 & f(x) = f(x') \\
\Longrightarrow & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\
\Longrightarrow & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\
\Longrightarrow & f \circ \bar{x} = f \circ \bar{x'} \\
\Longrightarrow & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\
\Longrightarrow & \bar{x}(\bullet) = \bar{x'}(\bullet) \\
\Longrightarrow & x = x'
\end{array}
$$

This shows that f is injective.

($\Longrightarrow$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$
\begin{array}{ll}
 & (f \circ g)(a) = (f \circ h)(a) \\
\Longrightarrow & f(g(a)) = f(h(a)) \\
\Longrightarrow & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\
\Longrightarrow & g = h
\end{array}
$$

This establishes that f is monic. And, we are done. $\square$

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

SOLUTION. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. □

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \to H$ is a monoid homomorphism. Prove that h is a group homomorphism.

SOLUTION. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. □

Exercise 6

Check that $\mathbf{Mon}, \mathbf{Vect}_k, \mathbf{Pos}$, and $\mathbf{Top}$ are indeed categories.

SOLUTION. ($\mathbf{Mon}$) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \to (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \to (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \to (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m))(g(f(m'))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that $\mathbf{Mon}$ is indeed a category.

($\mathbf{Vect}_k$) The objects are vector spaces over a field k, and morphisms are linear maps between vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha(g \circ f)(x)$. This establishes $g \circ f : U \to W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that $\mathbf{Vect}_k$ is also a category.

(**Pos**) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \to Q$ and $g : Q \to R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \to R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(**Top**) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \to (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$,which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \to (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. □

Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon**) have?)

SOLUTION. (Monoid as a one-object category) Given a monoid $(M, \cdot, 1)$, we can construct its corresponding category as follows. We write $\mathbf{B}M$ for the corresponding category with a single object $\bullet$, where $\mathbf{Hom}_{\mathbf{B}M}(\bullet, \bullet) := M$. We note then that the composition map in $\mathbf{B}M$ is reflected in the binary operation $\_ \cdot \_ : M \times M \to M$, where $\mathbf{id}_\bullet := 1$. Then, the associative and identity laws for the category $\mathbf{B}M$ follow directly from the associative and identity laws, respectively, satisfied by the monoid $(M, \cdot, 1)$. This shows any monoid can be seen or interpreted as a one-object category. □

Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

SOLUTION. Let $(P, \leq)$ be a preorder. Then, we define the corresponding category $\mathbf{C}$ as follows. The objects of $\mathbf{C}$ are the elements of the set P, and for all $x, y \in P$, we define a morphism $x \to y$ iff $x \leq y$. Then, for every object $x \in \mathbf{C}$, the identity morphism $1_x : x \to x$ corresponds exactly to the reflexive property $x \leq x$ for all $x \in P$. Note that each homset in $\mathbf{C}$ has at most one element. Also, for every $x \to y$ and $y \to z$ in $\mathbf{C}$, $x \to z$ follows from the fact that $x \leq y$ and $y \leq z$ and the transitivity of the $\leq$ relation on P. This defines a composition map for morphisms in $\mathbf{C}$. In addition, for all morphisms $x \to y$, $y \to z$, and $z \to w$, their associativity follows immediately from the transitivity of $\leq$. Lastly, the unit laws also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element. □

Exercise 10

Show that the inverse, if it exists, is unique.

SOLUTION. Suppose $i : A \to B$ is an isomorphism, with inverse $j : B \to A$, in a category $\mathbf{C}$. Suppose $j' : B \to A$ is also an inverse of i. Then, $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$, and we are done. □

Exercise 11

Show that $\cong$ is an equivalence relation on the objects of a category.

SOLUTION. Let $\mathbf{C}$ be some category.
(*Reflexivity*) For any object $X \in \mathbf{C}$, $X \cong X$ follows from the fact that the identity morphism $1_X : X \to X$ is an isomorphism.
(*Symmetry*) If $X \cong Y$, then there exists an isomorphism $i : X \to Y$. But, the inverse, $i^{-1} : Y \to X$, of i is also an isomorphism. Hence, $Y \cong X$.
(*Transitivity*) Suppose $X \cong Y$ and $Y \cong Z$. Then, there exist isomorphisms $i : X \to Y$ and $j : Y \to Z$. Then, we claim that $j \circ i : X \to Z$ is also an isomorphism. Indeed, its trivial to show that its inverse is the morphism $i^{-1} \circ j^{-1} : Z \to X$. This implies $X \cong Z$.
We thus conclude that $\cong$ is an equivalence relation on the objects of a category. □

Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

SOLUTION. (**Set**) We claim the following:

(1) $f : X \to Y$ is injective iff f has a left inverse.
(2) $f : X \to Y$ is surjective iff f has a right inverse.

We first show (1).

($\implies$) Suppose $f : X \to Y$ has a left inverse, $g : Y \to X$, say. Then, $g \circ f = 1_X$. Assume for any $x, x' \in X, f(x) = f(x')$. Then, $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$, which implies f is injective.

($\impliedby$) Suppose $f : X \to Y$ is injective. If X is empty, then f is an empty function corresponding to each Y. In this case, 1_X is also an empty function, and we thus have $g \circ f = 1_X$ for any $g : Y \to X$. That is, f has a left inverse. On the other hand, if X is nonempty, choose some $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all $x \in X, (g \circ f)(x) = g(f(x)) = x = 1_X(x)$, which implies $g \circ f = 1_X$, thus showing that g is a left inverse of f.

We now show (2).

($\Longrightarrow$) Suppose $f : X \to Y$ has a right inverse, $g : Y \to X$, say. Then, $f \circ g = 1_Y$. Therefore, for all $y \in Y$, $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$, where $x = g(y)$. This shows f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Now, consider an indexed family of nonempty sets $\{f^{-1}(y)\}_{y \in Y}$. Then, using the axiom of choice, we conclude there exists a function $g : Y \to X$, such that $g(y) \in f^{-1}(y)$ for all $y \in Y$. Then, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$, thus proving f has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category **C**, if $f : X \to Y$ has both a left inverse, $g : Y \to X$, say, and a right inverse, $h : Y \to X$, say, then $g = h$. Indeed, $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$, and we are done.

(**Grp**)
(**Top**)
(**Pos**) □

Opposite Categories and Duality. Given a category **C**, the opposite category $\mathbf{C^{op}}$ is given by taking the same objects as **C**, and

$$\mathbf{C^{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from **C**.
If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathbf{C^{op}}$, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in **C**. Therefore, composition $g \circ f$ is $\mathbf{C^{op}}$ is defined as $f \circ g$ in **C**. This leads to the ***principle of duality***: a statement S is true about a category **C** iff its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about $\mathbf{C^{op}}$. For example, a morphism f is monic in $\mathbf{C^{op}}$ iff it is epic in **C**. We say monic and epic are *dual notions.*

Exercise 14

If P is a preorder, for example $(\mathbb{R}, \leq)$, describe $P^{\mathbf{op}}$ explicitly.

SOLUTION. An arrow $a \leq_{P^{\mathbf{op}}} b$ in $P^{\mathbf{op}}$ is precisely the arrow $b \leq_P a$ in P. When $P = (\mathbb{R}, \leq)$, $P^{\mathbf{op}}$ describes the "greater than or equal" preorder relation on $\mathbb{R}$. □

Subcategories. Let **C** be a category. Suppose we are given the collections

$$\mathbf{Ob}(\mathbf{D}) \subseteq \mathbf{Ob}(\mathbf{C}),$$
$$\forall A, B \in \mathbf{Ob}(\mathbf{D}). \mathbf{D}(A, B) \subseteq \mathbf{C}(A, B).$$

We say **D** is a ***subcategory*** of **C** if it is itself a category. In particular, **D** is:

- A ***full*** subcategory of **C** if for any $A, B \in \mathbf{Ob}(\mathbf{D})$, $\mathbf{D}(A, B) = \mathbf{C}(A, B)$.
- A ***lluf*** subcategory of **C** if $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$.

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

Exercise 16

How many categories $\mathbf{C}$ with $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$ are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids. □

Exercises.

(1) Consider the following properties of an arrow f in a category $\mathbf{C}$.
- f is *split monic* if for some g, $g \circ f$ is an identity arrow.
- f is *split epic* if for some g, $f \circ g$ is an identity arrow.

a. Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.
b. Prove that if f is split epic then it is epic.
c. Prove that if f and $g \circ f$ are iso then g is iso.
d. Prove that if f is monic and split epic then it is iso.
e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map
$$i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$$
of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if $\{X_i\}_{i \in I}$ is a family of nonempty sets, we can form a set $X = \{x_i \mid i \in I\}$, where $x_i \in X_i$ for all $i \in I$.

f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
g. Is is always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.

(2) Give a description of partial orders as categories of a special kind.

SOLUTION.

(1)

a. Suppose $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is monic. Assume, for all $i, j : Z \to A$, $f \circ i = f \circ j$. Then, $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$, which implies $i = j$, since $g \circ f$ is monic. This implies f is monic, and we are done.

b. Suppose $f : A \to B$ is split epic. Then, there exists a $g : B \to A$ such that $f \circ g = 1_B$. Assume, for all $i, j : B \to C$, $i \circ f = j \circ f$. Then, $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$, which shows f is epic.

c. Suppose $f : A \to B$ and $g : B \to C$ such that f and $g \circ f$ are iso. We claim that the inverse of g is $f \circ (g \circ f)^{-1} : C \to B$. Indeed, $g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = 1_C$, and $(f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} = f \circ f^{-1} = 1_B$, which establishes g is also an iso.

d. Suppose $f : A \to B$ is monic and split epic. The latter implies f has a right inverse, $g : B \to A$, say, where $f \circ g = 1_B$. Note that $g \circ f : A \to A$ and $1_A : A \to A$. Now, $f \circ (g \circ f) = (f \circ g) \circ f =$

$1_B \circ f = f = f \circ 1_A$, which implies $g \circ f = 1_A$, since f is monic (left cancellative). Thus, g is also a left inverse of f, and hence, f is iso.

e. It is easy to prove the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is really a monoid homomorphism. Indeed, $i(0) = 0$, and, for all $n_1, n_2 \in \mathbb{N}$, $i(n_1 + n_2) = n_1 + n_2 = i(n_1) + i(n_2)$.
Next, we show that i is monic. Assume, for all monoid homomorphisms $g, h : X \to \mathbb{N}$, $i \circ g = i \circ h$. Then, for all $x \in X$, $(i \circ g)(x) = (i \circ h)(x)$, which implies $i(g(x)) = i(h(x))$, which implies $g(x) = h(x)$, which implies $g = h$. This shows the inclusion map is monic.
We now show the inclusion map is epic. First, assume, for all monoid homomorphisms $g, h : (\mathbb{Z}, +, 0) \to (X, \star, 1_X)$, $g \circ i = h \circ i$. Then, for all $n \in \mathbb{N}$, $(g \circ i)(n) = (h \circ i)(n)$, which implies $g(i(n)) = h(i(n))$, which implies $g(n) = h(n)$. We now claim that for all $n \geq 1$, $g(-n) = h(-n)$. To that end, we use induction on n. Note that $g(-1) = g(-1) \star 1_X = g(-1) \star h(0) = g(-1) \star h(1 + (-1)) = g(-1) \star h(1) \star h(-1) = g(-1) \star g(1) * h(-1) = g(-1 + 1) \star h(-1) = g(0) \star h(-1) = 1_X \star h(-1) = h(-1)$. Now, assume the proposition holds for some $n \geq 1$. Then, $g(-(n+1)) = g(-n + (-1)) = g(-n) \star g(-1) = h(-n) \star h(-1) = h(-n + (-1)) = h(-(n+1))$. Hence, by induction, $g(-n) = h(-n)$ for all $n \geq 1$. Combining the results from above, we thus conclude $g(z) = h(z)$ for all $z \in \mathbb{Z}$. In other words, $g = h$, which implies i is epic.
Clearly, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is not iso.

f. Suppose $f : X \to Y$ is epic in **Set**. Then, from an earlier result about **Set**, we conclude f is surjective. Now, consider the family of nonempty sets $\{f^{-1}(b)\}_{b \in B}$. Each of the sets in the family is nonempty, because f is surjective. Therefore, using the Axiom of Choice, we can choose some element from each nonempty set in the family to construct a function $g : Y \to X$, given by $g(y) := x$ if $x \in f^{-1}(b)$. In addition, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$. This shows f has a right inverse, thus proving f is split epic.

g. It isn't always the case that an arrow which is epic is split epic. For example, in the category **Mon**, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is epic (as shown in (e) above.) Now, if we assume that it is also split epic, then there exists a monoid homomorphism $g : \mathbb{Z} \to \mathbb{N}$, such that $i \circ g = 1_{\mathbb{Z}}$. This implies $(i \circ g)(-1) = 1_{\mathbb{Z}}(-1)$, which implies $i(g(-1)) = -1$, which implies $g(-1) = -1$, which implies $-1 \in \mathbb{N}$, which is absurd. We thus conclude the aforesaid inclusion map is *not* split epic, even though it is epic. And this proves our original claim.

$\square$