Introduction to Categories and Categorical Logic

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

injective	if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective	if $\forall y \in Y. \exists x \in X. f(x) = y$,
monic	if $\forall g, h. f \circ g = f \circ h \implies g = h$ (f is left cancellative),
epic	if $\forall g, h. g \circ f = h \circ f \implies g = h$ (f is right cancellative).

Proposition 1. *Let $f : X \to Y$. Then,*

1. *f is injective $\iff$ f is monic.*
2. *f is surjective $\iff$ f is epic.*

Proof. We first show (1).
($\Longleftarrow$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$
\begin{array}{ll}
 & f(x) = f(x') \\
\implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\
\implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\
\implies & f \circ \bar{x} = f \circ \bar{x'} \\
\implies & \bar{x} = \bar{x'} \quad \text{(since } f \text{ is monic)} \\
\implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\
\implies & x = x'
\end{array}
$$

This shows that f is injective.

($\Longrightarrow$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$
\begin{array}{ll}
 & (f \circ g)(a) = (f \circ h)(a) \\
\implies & f(g(a)) = f(h(a)) \\
\implies & g(a) = h(a) \quad \text{(since } f \text{ is injective)} \\
\implies & g = h
\end{array}
$$

This establishes that f is monic. And, we are done. □

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

SOLUTION. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. □

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \to H$ is a monoid homomorphism. Prove that h is a group homomorphism.

SOLUTION. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. □

Exercise 6

Check that $\mathbf{Mon}, \mathbf{Vect}_k, \mathbf{Pos}$, and $\mathbf{Top}$ are indeed categories.

SOLUTION. ($\mathbf{Mon}$) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \to (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \to (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \to (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m))(g(f(m'))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that $\mathbf{Mon}$ is indeed a category.

($\mathbf{Vect}_k$) The objects are vector spaces over a field k, and morphisms are linear maps between vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha(g \circ f)(x)$. This establishes $g \circ f : U \to W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that $\mathbf{Vect}_k$ is also a category.

(**Pos**) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \to Q$ and $g : Q \to R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \to R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(**Top**) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \to (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$,which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \to (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. □

Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon**) have?)

Solution. (Monoid as a one-object category) Given a monoid $(M, \cdot, 1)$, we can construct its corresponding category as follows. We write $\mathbf{B}M$ for the corresponding category with a single object $\bullet$, where $\mathbf{Hom}_{\mathbf{B}M}(\bullet, \bullet) := M$. We note then that the composition map in $\mathbf{B}M$ is reflected in the binary operation $\_ \cdot \_ : M \times M \to M$, where $\mathbf{id}_\bullet := 1$. Then, the associative and identity laws for the category $\mathbf{B}M$ follow directly from the associative and identity laws, respectively, satisfied by the monoid $(M, \cdot, 1)$. This shows any monoid can be seen or interpreted as a one-object category. □

Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

Solution. Let $(P, \leq)$ be a preorder. Then, we define the corresponding category $\mathbf{C}$ as follows. The objects of $\mathbf{C}$ are the elements of the set P, and for all $x, y \in P$, we define a morphism $x \to y$ iff $x \leq y$. Then, for every object $x \in \mathbf{C}$, the identity morphism $1_x : x \to x$ corresponds exactly to the reflexive property $x \leq x$ for all $x \in P$. Note that each homset in $\mathbf{C}$ has at most one element. Also, for every $x \to y$ and $y \to z$ in $\mathbf{C}$, $x \to z$ follows from the fact that $x \leq y$ and $y \leq z$ and the transitivity of the $\leq$ relation on P. This defines a composition map for morphisms in $\mathbf{C}$. In addition, for all morphisms $x \to y$, $y \to z$, and $z \to w$, their associativity follows immediately from the transitivity of $\leq$. Lastly, the unit laws also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element. □

Exercise 10

Show that the inverse, if it exists, is unique.

Solution. Suppose $i : A \to B$ is an isomorphism, with inverse $j : B \to A$, in a category $\mathbf{C}$. Suppose $j' : B \to A$ is also an inverse of i. Then, $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$, and we are done. □

Exercise 11

Show that $\cong$ is an equivalence relation on the objects of a category.

Solution. Let $\mathbf{C}$ be some category.
(*Reflexivity*) For any object $X \in \mathbf{C}$, $X \cong X$ follows from the fact that the identity morphism $1_X : X \to X$ is an isomorphism.
(*Symmetry*) If $X \cong Y$, then there exists an isomorphism $i : X \to Y$. But, the inverse, $i^{-1} : Y \to X$, of i is also an isomorphism. Hence, $Y \cong X$.
(*Transitivity*) Suppose $X \cong Y$ and $Y \cong Z$. Then, there exist isomorphisms $i : X \to Y$ and $j : Y \to Z$. Then, we claim that $j \circ i : X \to Z$ is also an isomorphism. Indeed, its trivial to show that its inverse is the morphism $i^{-1} \circ j^{-1} : Z \to X$. This implies $X \cong Z$.
We thus conclude that $\cong$ is an equivalence relation on the objects of a category. □

Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

Solution. (**Set**) We claim the following:

(1) $f : X \to Y$ is injective iff f has a left inverse.
(2) $f : X \to Y$ is surjective iff f has a right inverse.

We first show (1).

($\implies$) Suppose $f : X \to Y$ has a left inverse, $g : Y \to X$, say. Then, $g \circ f = 1_X$. Assume for any $x, x' \in X, f(x) = f(x')$. Then, $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$, which implies f is injective.

($\impliedby$) Suppose $f : X \to Y$ is injective. If X is empty, then f is an empty function corresponding to each Y. In this case, 1_X is also an empty function, and we thus have $g \circ f = 1_X$ for any $g : Y \to X$. That is, f has a left inverse. On the other hand, if X is nonempty, choose some $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all $x \in X, (g \circ f)(x) = g(f(x)) = x = 1_X(x)$, which implies $g \circ f = 1_X$, thus showing that g is a left inverse of f.

We now show (2).

($\Longrightarrow$) Suppose $f : X \to Y$ has a right inverse, $g : Y \to X$, say. Then, $f \circ g = 1_Y$. Therefore, for all $y \in Y$, $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$, where $x = g(y)$. This shows f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Now, consider an indexed family of nonempty sets $\{f^{-1}(y)\}_{y \in Y}$. Then, using the axiom of choice, we conclude there exists a function $g : Y \to X$, such that $g(y) \in f^{-1}(y)$ for all $y \in Y$. Then, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$, thus proving f has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category **C**, if $f : X \to Y$ has both a left inverse, $g : Y \to X$, say, and a right inverse, $h : Y \to X$, say, then $g = h$. Indeed, $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$, and we are done.

(**Grp**)
(**Top**)
(**Pos**) □

Opposite Categories and Duality. Given a category **C**, the opposite category $\mathbf{C^{op}}$ is given by taking the same objects as **C**, and

$$\mathbf{C^{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from **C**.
If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathbf{C^{op}}$, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in **C**. Therefore, composition $g \circ f$ is $\mathbf{C^{op}}$ is defined as $f \circ g$ in **C**. This leads to the ***principle of duality***: a statement S is true about a category **C** iff its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about $\mathbf{C^{op}}$. For example, a morphism f is monic in $\mathbf{C^{op}}$ iff it is epic in **C**. We say monic and epic are *dual notions*.

Exercise 14

If P is a preorder, for example $(\mathbb{R}, \leq)$, describe $P^{\mathbf{op}}$ explicitly.

Solution. An arrow $a \leq_{P^{\mathbf{op}}} b$ in $P^{\mathbf{op}}$ is precisely the arrow $b \leq_P a$ in P. When $P = (\mathbb{R}, \leq)$, $P^{\mathbf{op}}$ describes the "greater than or equal" preorder relation on $\mathbb{R}$. □

Subcategories. Let **C** be a category. Suppose we are given the collections

$$\mathbf{Ob}(\mathbf{D}) \subseteq \mathbf{Ob}(\mathbf{C}),$$
$$\forall A, B \in \mathbf{Ob}(\mathbf{D}). \mathbf{D}(A, B) \subseteq \mathbf{C}(A, B).$$

We say **D** is a ***subcategory*** of **C** if it is itself a category. In particular, **D** is:

- A ***full*** subcategory of **C** if for any $A, B \in \mathbf{Ob}(\mathbf{D})$, $\mathbf{D}(A, B) = \mathbf{C}(A, B)$.
- A ***lluf*** subcategory of **C** if $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$.

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.