

Introduction to Categories and Categorical Logic

Vishal Lama

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CHAPTER 1

Introduction to Categories and Categorical Logic

1. Introduction

We say that a function $f : X \rightarrow Y$ is:

injective if $\forall x, x' \in X. f(x) = f(x') \implies x = x'$,
surjective if $\forall y \in Y. \exists x \in X. f(x) = y$,

monic if $\forall g, h. f \circ g = f \circ h \implies g = h$ (f is left cancellative),
epic if $\forall g, h. g \circ f = h \circ f \implies g = h$ (f is right cancellative).

PROPOSITION 1. *Let $f : X \rightarrow Y$. Then,*

- (1) *f is injective $\iff f$ is monic.*
- (2) *f is surjective $\iff f$ is epic.*

PROOF. We first show (1).

(\Leftarrow) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \rightarrow X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$\begin{aligned} & f(x) = f(x') \\ \implies & f(\bar{x}(\bullet)) = f(\bar{x}'(\bullet)) \\ \implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x}')(\bullet) \\ \implies & f \circ \bar{x} = f \circ \bar{x}' \\ \implies & \bar{x} = \bar{x}' \quad (\text{since } f \text{ is monic}) \\ \implies & \bar{x}(\bullet) = \bar{x}'(\bullet) \\ \implies & x = x' \end{aligned}$$

This shows that f is injective.

(\Rightarrow) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \rightarrow X$. Then, for all $a \in A$,

$$\begin{aligned} & (f \circ g)(a) = (f \circ h)(a) \\ \implies & f(g(a)) = f(h(a)) \\ \implies & g(a) = h(a) \quad (\text{since } f \text{ is injective}) \\ \implies & g = h \end{aligned}$$

This establishes that f is monic. And, we are done. □

Exercise 2

Show that $f : X \rightarrow Y$ is surjective iff it is epic.

SOLUTION. (\implies) Suppose $f : X \rightarrow Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \rightarrow Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.

Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g \neq h$. Thus, we conclude f is surjective.

(\impliedby) Suppose $f : X \rightarrow Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \rightarrow Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.

And, this completes our proof. \square

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \rightarrow H$ is a monoid homomorphism. Prove that h is a group homomorphism.

SOLUTION. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. \square

Exercise 6

Check that **Mon**, **Vect_k**, **Pos**, and **Top** are indeed categories.

SOLUTION. (**Mon**) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \rightarrow (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \rightarrow (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \rightarrow (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m)))(g(f(m')))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that **Mon** is indeed a category.

(**Vect_k**) The objects are vector spaces over a field k , and morphisms are linear maps between vector spaces. Suppose $f : U \rightarrow V$ and $g : V \rightarrow W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha (g \circ f)(x)$. This establishes $g \circ f : U \rightarrow W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that **Vect_k** is also a category.

(Pos) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \rightarrow Q$ and $g : Q \rightarrow R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \rightarrow R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(Top) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \rightarrow (Y, T_Y)$ and $g : (Y, T_Y) \rightarrow (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$, which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \rightarrow (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. \square

Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon** have?)

SOLUTION. (Monoid as a one-object category) Given a monoid $(M, \cdot, 1)$, we can construct its corresponding category as follows. We write **BM** for the corresponding category with a single object \bullet , where $\mathbf{Hom}_{\mathbf{BM}}(\bullet, \bullet) := M$. We note then that the composition map in **BM** is reflected in the binary operation $\cdot : M \times M \rightarrow M$, where $\mathbf{id}_\bullet := 1$. Then, the associative and identity laws for the category **BM** follow directly from the associative and identity laws, respectively, satisfied by the monoid $(M, \cdot, 1)$. This shows any monoid can be seen or interpreted as a one-object category. \square

Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

SOLUTION. Let (P, \leq) be a preorder. Then, we define the corresponding category **C** as follows. The objects of **C** are the elements of the set P , and for all $x, y \in P$, we define a morphism $x \rightarrow y$ iff $x \leq y$. Then, for every object $x \in \mathbf{C}$, the identity morphism $1_x : x \rightarrow x$ corresponds exactly to the reflexive property $x \leq x$ for all $x \in P$. Note that each homset in **C** has at most one element. Also, for every $x \rightarrow y$ and $y \rightarrow z$ in **C, $x \rightarrow z$ follows from the fact that $x \leq y$ and $y \leq z$ and the transitivity of the \leq relation on P . This defines a composition map for morphisms in **C**. In addition, for all morphisms $x \rightarrow y$, $y \rightarrow z$, and $z \rightarrow w$, their associativity follows immediately from the transitivity of \leq . Lastly, the unit laws**

also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element. \square

Exercise 10

Show that the inverse, if it exists, is unique.

SOLUTION. Suppose $i : A \rightarrow B$ is an isomorphism, with inverse $j : B \rightarrow A$, in a category \mathbf{C} . Suppose $j' : B \rightarrow A$ is also an inverse of i . Then, $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$, and we are done. \square

Exercise 11

Show that \cong is an equivalence relation on the objects of a category.

SOLUTION. Let \mathbf{C} be some category.

(*Reflexivity*) For any object $X \in \mathbf{C}$, $X \cong X$ follows from the fact that the identity morphism $1_X : X \rightarrow X$ is an isomorphism.

(*Symmetry*) If $X \cong Y$, then there exists an isomorphism $i : X \rightarrow Y$. But, the inverse, $i^{-1} : Y \rightarrow X$, of i is also an isomorphism. Hence, $Y \cong X$.

(*Transitivity*) Suppose $X \cong Y$ and $Y \cong Z$. Then, there exist isomorphisms $i : X \rightarrow Y$ and $j : Y \rightarrow Z$. Then, we claim that $j \circ i : X \rightarrow Z$ is also an isomorphism. Indeed, it's trivial to show that its inverse is the morphism $i^{-1} \circ j^{-1} : Z \rightarrow X$. This implies $X \cong Z$.

We thus conclude that \cong is an equivalence relation on the objects of a category. \square

Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

SOLUTION. (**Set**) We claim the following:

- (1) $f : X \rightarrow Y$ is injective iff f has a left inverse.
- (2) $f : X \rightarrow Y$ is surjective iff f has a right inverse.

We first show (1).

(\implies) Suppose $f : X \rightarrow Y$ has a left inverse, $g : Y \rightarrow X$, say. Then, $g \circ f = 1_X$. Assume for any $x, x' \in X$, $f(x) = f(x')$. Then, $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$, which implies f is injective.

(\impliedby) Suppose $f : X \rightarrow Y$ is injective. If X is empty, then f is an empty function corresponding to each Y . In this case, 1_X is also an empty function, and we thus have $g \circ f = 1_X$ for any $g : Y \rightarrow X$. That is, f has a left inverse. On the other hand, if X is nonempty, choose some $x_0 \in X$. Define $g : Y \rightarrow X$ by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = x = 1_X(x)$, which implies $g \circ f = 1_X$, thus showing that g is a left inverse of f .

We now show (2).

(\implies) Suppose $f : X \rightarrow Y$ has a right inverse, $g : Y \rightarrow X$, say. Then, $f \circ g = 1_Y$. Therefore, for all $y \in Y$, $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$, where $x = g(y)$. This shows f is surjective.

(\impliedby) Suppose $f : X \rightarrow Y$ is surjective. Now, consider an indexed family of nonempty sets $\{f^{-1}(y)\}_{y \in Y}$. Then, using the axiom of choice, we conclude there exists a function $g : Y \rightarrow X$, such that $g(y) \in f^{-1}(y)$ for all $y \in Y$. Then, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$, thus proving f has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category **C**, if $f : X \rightarrow Y$ has both a left inverse, $g : Y \rightarrow X$, say, and a right inverse, $h : Y \rightarrow X$, say, then $g = h$. Indeed, $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$, and we are done.

(Grp)
(Top)
(Pos)

□

Opposite Categories and Duality. Given a category **C**, the opposite category **C^{op}** is given by taking the same objects as **C**, and

$$\mathbf{C}^{\text{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from **C**.

If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in **C^{op}**, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in **C**. Therefore, composition $g \circ f$ in **C^{op}** is defined as $f \circ g$ in **C**. This leads to the **principle of duality**: a statement S is true about a category **C** iff its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about **C^{op}**. For example, a morphism f is monic in **C^{op}** iff it is epic in **C**. We say monic and epic are *dual notions*.

Exercise 14

If P is a preorder, for example (\mathbb{R}, \leq) , describe **P^{op}** explicitly.

SOLUTION. An arrow $a \leq_{P^{\text{op}}} b$ in **P^{op}** is precisely the arrow $b \leq_P a$ in P . When $P = (\mathbb{R}, \leq)$, **P^{op}** describes the “greater than or equal” preorder relation on \mathbb{R} . □

Subcategories. Let **C** be a category. Suppose we are given the collections

$$\begin{aligned} \mathbf{Ob}(\mathbf{D}) &\subseteq \mathbf{Ob}(\mathbf{C}), \\ \forall A, B \in \mathbf{Ob}(\mathbf{D}). \mathbf{D}(A, B) &\subseteq \mathbf{C}(A, B). \end{aligned}$$

We say **D** is a **subcategory** of **C** if it is itself a category. In particular, **D** is:

- A **full** subcategory of **C** if for any $A, B \in \mathbf{Ob}(\mathbf{D})$, $\mathbf{D}(A, B) = \mathbf{C}(A, B)$.
- A **lluf** subcategory of **C** if $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$.

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

Exercise 16

How many categories \mathbf{C} with $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$ are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids. \square

Exercises.

- (1) Consider the following properties of an arrow f in a category \mathbf{C} .
 - f is *split monic* if for some g , $g \circ f$ is an identity arrow.
 - f is *split epic* if for some g , $f \circ g$ is an identity arrow.
 - a. Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.
 - b. Prove that if f is split epic then it is epic.
 - c. Prove that if f and $g \circ f$ are iso then g is iso.
 - d. Prove that if f is monic and split epic then it is iso.
 - e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map

$$i : (\mathbb{N}, +, 0) \rightarrow (\mathbb{Z}, +, 0)$$

of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if $\{X_i\}_{i \in I}$ is a family of nonempty sets, we can form a set $X = \{x_i \mid i \in I\}$, where $x_i \in X_i$ for all $i \in I$.

- f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
 - g. Is it always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.
- (2) Give a description of partial orders as categories of a special kind.

SOLUTION.

- (1)
 - a. Suppose $f : A \rightarrow B$ and $g : B \rightarrow C$ such that $g \circ f$ is monic. Assume, for all $i, j : Z \rightarrow A$, $f \circ i = f \circ j$. Then, $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$, which implies $i = j$, since $g \circ f$ is monic. This implies f is monic, and we are done.
 - b. Suppose $f : A \rightarrow B$ is split epic. Then, there exists a $g : B \rightarrow A$ such that $f \circ g = 1_B$. Assume, for all $i, j : B \rightarrow C$, $i \circ f = j \circ f$. Then, $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$, which shows f is epic.

\square