Introduction to Categories and Categorical Logic

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CHAPTER 1

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1. Introduction

We say that a function $f : X \to Y$ is:

$$
\begin{array}{ll}
\textit{injective} & \text{if } \forall x, x' \in X. f(x) = f(x') \implies x = x', \\
\textit{surjective} & \text{if } \forall y \in Y. \exists x \in X. f(x) = y,
\end{array}
$$

$$
\begin{array}{lll}
\textit{monic} & \text{if } \forall g, h. f \circ g = f \circ h \implies g = h & (f \text{ is left cancellative}), \\
\textit{epic} & \text{if } \forall g, h. g \circ f = h \circ f \implies g = h & (f \text{ is right cancellative}).
\end{array}
$$

Proposition 1. *Let $f : X \to Y$. Then,*

(1) *f is injective $\iff$ f is monic.*
(2) *f is surjective $\iff$ f is epic.*

Proof. We first show (1).
($\Longleftarrow$) Suppose f is monic. Fix a one-element set $\mathbf{1} = \{\bullet\}$. Then, note that elements $x \in X$ are in 1-1 correspondence with functions $\bar{x} : \mathbf{1} \to X$, defined by $\bar{x}(\bullet) := x$. Then, for all $x, x' \in X$, we have

$$
\begin{array}{ll}
 & f(x) = f(x') \\
\implies & f(\bar{x}(\bullet)) = f(\bar{x'}(\bullet)) \\
\implies & (f \circ \bar{x})(\bullet) = (f \circ \bar{x'})(\bullet) \\
\implies & f \circ \bar{x} = f \circ \bar{x'} \\
\implies & \bar{x} = \bar{x'} \quad (\text{since } f \text{ is monic}) \\
\implies & \bar{x}(\bullet) = \bar{x'}(\bullet) \\
\implies & x = x'
\end{array}
$$

This shows that f is injective.

($\Longrightarrow$) Suppose f is injective. Let $f \circ g = f \circ h$ for all $g, h : A \to X$. Then, for all $a \in A$,

$$
\begin{array}{ll}
 & (f \circ g)(a) = (f \circ h)(a) \\
\implies & f(g(a)) = f(h(a)) \\
\implies & g(a) = h(a) \quad (\text{since } f \text{ is injective}) \\
\implies & g = h
\end{array}
$$

This establishes that f is monic. And, we are done. $\square$

Exercise 2

Show that $f : X \to Y$ is surjective iff it is epic.

SOLUTION. ($\Longrightarrow$) Suppose $f : X \to Y$ is epic. And, assume, for the sake of contradiction, f is *not* surjective. Then, there exists some $y_0 \in Y$, such that, for all $x \in X$, $f(x) \neq y_0$. Define mappings $g, h : Y \to Y \cup \{Y\}$ by:

$$g(y) := y$$

$$h(y) := \begin{cases} y & \text{if } y \neq y_0 \\ Y & \text{if } y = y_0 \end{cases}$$

Note that $g \neq h$.
Then, for all $x \in X$, $(g \circ f)(x) = g(f(x)) = h(f(x)) = (h \circ f)(x)$. This implies $g \circ f = h \circ f$, which implies $g = h$, since f is epic. The last conclusion contradicts the fact that $g = h$. Thus, we conclude f is surjective.

($\Longleftarrow$) Suppose $f : X \to Y$ is surjective. Then, for any $y \in Y$, there exists an $x \in X$, such that $f(x) = y$. Now, assume, for all $g, h : Y \to Z$, $g \circ f = h \circ f$. Then, for all $y \in Y$, $g(y) = g(f(x)) = (g \circ f)(x) = (h \circ f)(x) = h(f(x)) = h(y)$, which implies $g = h$, showing that f is epic.
And, this completes our proof. □

Exercise 5

Suppose G and H are groups (and hence monoids), and that $h : G \to H$ is a monoid homomorphism. Prove that h is a group homomorphism.

SOLUTION. We need only show that h preserves inverses. To that end, suppose g^{-1} is the inverse of $g \in G$. Then, $h(g)h(g^{-1}) = h(gg^{-1}) = h(1_G) = 1_H = h(1_G) = h(g^{-1}g) = h(g^{-1})h(g)$. This establishes h preserves inverses, and we are done. □

Exercise 6

Check that $\mathbf{Mon}, \mathbf{Vect}_k, \mathbf{Pos}$, and $\mathbf{Top}$ are indeed categories.

SOLUTION. ($\mathbf{Mon}$) The objects are monoids $(M, \cdot, 1_M)$, and morphisms are monoid homomorphisms. Given monoid homomorphisms, $f : (M, \cdot, 1_M) \to (N, \cdot, 1_N)$ and $g : (N, \cdot, 1_N) \to (P, \cdot, 1_P)$, the function $g \circ f : (M, \cdot, 1_M) \to (P, \cdot, 1_P)$ is also a monoid homomorphism, because for all $m, m' \in M$, we have $(g \circ f)(mm') = g(f(mm')) = g(f(m)f(m')) = (g(f(m))(g(f(m'))) = ((g \circ f)(m))((g \circ f)(m'))$. Also, for each monoid, the identity morphism is the identity function. It is also easy to check that for all monoid homomorphisms f, g and h with the appropriate domains and codomains, $h \circ (g \circ f) = (h \circ g) \circ f$. This establishes that $\mathbf{Mon}$ is indeed a category.

($\mathbf{Vect}_k$) The objects are vector spaces over a field k, and morphisms are linear maps between vector spaces. Suppose $f : U \to V$ and $g : V \to W$ are linear maps. Then, for all $x, y \in U$, we have $(g \circ f)(x + y) = g(f(x + y)) = g(f(x) + f(y)) = g(f(x)) + g(f(y)) = (g \circ f)(x) + (g \circ f)(y)$. Also, for all $\alpha \in k$, we have $(g \circ f)(\alpha x) = g(f(\alpha x)) = g(\alpha f(x)) = \alpha g(f(x)) = \alpha(g \circ f)(x)$. This establishes $g \circ f : U \to W$ is a linear map as well. The identity map 1_U for any vector space U is the identity morphism. The associativity of linear maps and the identity axiom follow from the property of functions. This shows that $\mathbf{Vect}_k$ is also a category.

(**Pos**) The objects are partially ordered sets, and morphisms are monotone functions between these sets. Suppose $h : P \to Q$ and $g : Q \to R$ are monotone functions. Then, for all $x, y \in P$, $x \leq y \implies h(x) \leq h(y) \implies g(h(x)) \leq g(h(y)) \implies (g \circ h)(x) \leq (g \circ h)(y)$, which shows $g \circ h : P \to R$ is a monotone function. The identity map is the identity morphism, and the associativity and identity axioms are satisfied by the property of functions. This establishes **Pos** is a category.

(**Top**) The objects are topological spaces, and morphisms are continuous maps between these spaces. Given continuous maps $f : (X, T_X) \to (Y, T_Y)$ and $g : (Y, T_Y) \to (Z, T_Z)$, we can show that $g \circ f : (X, T_X) \to (Z, T_Z)$ is also a continuous map. First, note that for any $T \subset Z$, $x \in (g \circ f)^{-1}(T)$ iff $(g \circ f)(x) \in T$ iff $g(f(x)) \in T$ iff $f(x) \in g^{-1}(T)$ iff $x \in f^{-1}(g^{-1}(T))$. Thus,

$$\text{for all } T \subset Z, (g \circ f)^{-1}(T) = f^{-1}(g^{-1}(T)).$$

Therefore, for any open set $T \in T_Z$, we have $g^{-1}(T) \in T_Y$,which implies $f^{-1}(g^{-1}(T)) \in T_X$, which implies $(g \circ f)^{-1}(T) \in T_X$ (by using the result above.) Hence, $g \circ f : (X, T_X) \to (Z, T_Z)$ is a continuous map. The associativity and identity axioms follow from the associativity and identity laws for functions. This establishes **Top** is a category. $\square$

Exercise 7

Check carefully that monoids correspond exactly to one-object categories. Make sure you understand the difference between such a category and **Mon**. (For example: how many objects does **Mon**) have?)

Solution. (Monoid as a one-object category) Given a monoid $(M, \cdot, 1)$, we can construct its corresponding category as follows. We write $\mathbf{B}M$ for the corresponding category with a single object $\bullet$, where $\mathbf{Hom}_{\mathbf{B}M}(\bullet, \bullet) := M$. We note then that the composition map in $\mathbf{B}M$ is reflected in the binary operation $\_ \cdot \_ : M \times M \to M$, where $\mathbf{id}_\bullet := 1$. Then, the associative and identity laws for the category $\mathbf{B}M$ follow directly from the associative and identity laws, respectively, satisfied by the monoid $(M, \cdot, 1)$. This shows any monoid can be seen or interpreted as a one-object category. $\square$

Exercise 8

Check carefully that preorders correspond exactly to categories in which each homset has at most one element. Make sure you understand the difference between such a category and **Pos**. (For example: how big can homsets in **Pos** be?)

Solution. Let $(P, \leq)$ be a preorder. Then, we define the corresponding category $\mathbf{C}$ as follows. The objects of $\mathbf{C}$ are the elements of the set P, and for all $x, y \in P$, we define a morphism $x \to y$ iff $x \leq y$. Then, for every object $x \in \mathbf{C}$, the identity morphism $1_x : x \to x$ corresponds exactly to the reflexive property $x \leq x$ for all $x \in P$. Note that each homset in $\mathbf{C}$ has at most one element. Also, for every $x \to y$ and $y \to z$ in $\mathbf{C}$, $x \to z$ follows from the fact that $x \leq y$ and $y \leq z$ and the transitivity of the $\leq$ relation on P. This defines a composition map for morphisms in $\mathbf{C}$. In addition, for all morphisms $x \to y$, $y \to z$, and $z \to w$, their associativity follows immediately from the transitivity of $\leq$. Lastly, the unit laws also follow from the same transitivity relation. Therefore, we conclude that every preorder corresponds precisely to a category in which each homset has at most one element. $\square$

Exercise 10

Show that the inverse, if it exists, is unique.

Solution. Suppose $i : A \to B$ is an isomorphism, with inverse $j : B \to A$, in a category $\mathbf{C}$. Suppose $j' : B \to A$ is also an inverse of i. Then, $j = 1_A \circ j = (j' \circ i) \circ j = j' \circ (i \circ j) = j' \circ 1_B = j'$, and we are done. $\square$

Exercise 11

Show that $\cong$ is an equivalence relation on the objects of a category.

Solution. Let $\mathbf{C}$ be some category.
(*Reflexivity*) For any object $X \in \mathbf{C}$, $X \cong X$ follows from the fact that the identity morphism $1_X : X \to X$ is an isomorphism.
(*Symmetry*) If $X \cong Y$, then there exists an isomorphism $i : X \to Y$. But, the inverse, $i^{-1} : Y \to X$, of i is also an isomorphism. Hence, $Y \cong X$.
(*Transitivity*) Suppose $X \cong Y$ and $Y \cong Z$. Then, there exist isomorphisms $i : X \to Y$ and $j : Y \to Z$. Then, we claim that $j \circ i : X \to Z$ is also an isomorphism. Indeed, its trivial to show that its inverse is the morphism $i^{-1} \circ j^{-1} : Z \to X$. This implies $X \cong Z$.
We thus conclude that $\cong$ is an equivalence relation on the objects of a category. $\square$

Exercise 12

Verify the claims that isomorphisms in **Set** correspond exactly to bijections, in **Grp** to group isomorphisms, in **Top** to homeomorphisms, and in **Pos** to isomorphisms.

Solution. (**Set**) We claim the following:

(1) $f : X \to Y$ is injective iff f has a left inverse.
(2) $f : X \to Y$ is surjective iff f has a right inverse.

We first show (1).

($\implies$) Suppose $f : X \to Y$ has a left inverse, $g : Y \to X$, say. Then, $g \circ f = 1_X$. Assume for any $x, x' \in X, f(x) = f(x')$. Then, $x = 1_X(x) = (g \circ f)(x) = g(f(x)) = g(f(x')) = (g \circ f)(x') = 1_X(x') = x'$, which implies f is injective.

($\impliedby$) Suppose $f : X \to Y$ is injective. If X is empty, then f is an empty function corresponding to each Y. In this case, 1_X is also an empty function, and we thus have $g \circ f = 1_X$ for any $g : Y \to X$. That is, f has a left inverse. On the other hand, if X is nonempty, choose some $x_0 \in X$. Define $g : Y \to X$ by

$$g(y) := \begin{cases} x_0 & \text{if } y \in Y \setminus \mathbf{Im}(f) \\ f^{-1}(y) & \text{if } y \in \mathbf{Im}(f) \end{cases}$$

Then, for all $x \in X, (g \circ f)(x) = g(f(x)) = x = 1_X(x)$, which implies $g \circ f = 1_X$, thus showing that g is a left inverse of f.

We now show (2).

($\implies$) Suppose $f : X \to Y$ has a right inverse, $g : Y \to X$, say. Then, $f \circ g = 1_Y$. Therefore, for all $y \in Y$, $y = 1_Y(y) = (f \circ g)(y) = f(g(y)) = f(x)$, where $x = g(y)$. This shows f is surjective.

($\impliedby$) Suppose $f : X \to Y$ is surjective. Now, consider an indexed family of nonempty sets $\{f^{-1}(y)\}_{y \in Y}$. Then, using the axiom of choice, we conclude there exists a function $g : Y \to X$, such that $g(y) \in f^{-1}(y)$ for all $y \in Y$. Then, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$, thus proving f has a right inverse.

Since in **Set** a bijection is a function which is both injective and surjective, using (1) and (2), we immediately conclude that bijections in **Set** correspond exactly to isomorphisms, and we are done.

In addition, in any category $\mathbf{C}$, if $f : X \to Y$ has both a left inverse, $g : Y \to X$, say, and a right inverse, $h : Y \to X$, say, then $g = h$. Indeed, $g = g \circ 1_Y = g \circ (f \circ h) = (g \circ f) \circ h = 1_X \circ h = h$, and we are done.

(**Grp**)
(**Top**)
(**Pos**) □

Opposite Categories and Duality. Given a category $\mathbf{C}$, the opposite category $\mathbf{C^{op}}$ is given by taking the same objects as $\mathbf{C}$, and

$$\mathbf{C^{op}}(A, B) = \mathbf{C}(B, A).$$

Composition and identities are inherited from $\mathbf{C}$.
If we have

$$A \xrightarrow{f} B \xrightarrow{g} C$$

in $\mathbf{C^{op}}$, this means

$$A \xleftarrow{f} B \xleftarrow{g} C$$

in $\mathbf{C}$. Therefore, composition $g \circ f$ is $\mathbf{C^{op}}$ is defined as $f \circ g$ in $\mathbf{C}$. This leads to the ***principle of duality***: a statement S is true about a category $\mathbf{C}$ iff its dual (*i.e.* the one obtained from S by reversing all the arrows) is true about $\mathbf{C^{op}}$. For example, a morphism f is monic in $\mathbf{C^{op}}$ iff it is epic in $\mathbf{C}$. We say monic and epic are *dual notions.*

Exercise 14

If P is a preorder, for example $(\mathbb{R}, \leq)$, describe $P^{\mathbf{op}}$ explicitly.

Solution. An arrow $a \leq_{P^{\mathbf{op}}} b$ in $P^{\mathbf{op}}$ is precisely the arrow $b \leq_P a$ in P. When $P = (\mathbb{R}, \leq)$, $P^{\mathbf{op}}$ describes the "greater than or equal" preorder relation on $\mathbb{R}$. □

Subcategories. Let $\mathbf{C}$ be a category. Suppose we are given the collections

$$\mathbf{Ob}(\mathbf{D}) \subseteq \mathbf{Ob}(\mathbf{C}),$$
$$\forall A, B \in \mathbf{Ob}(\mathbf{D}).\mathbf{D}(A, B) \subseteq \mathbf{C}(A, B).$$

We say $\mathbf{D}$ is a ***subcategory*** of $\mathbf{C}$ if it is itself a category. In particular, $\mathbf{D}$ is:

- A ***full*** subcategory of $\mathbf{C}$ if for any $A, B \in \mathbf{Ob}(\mathbf{D})$, $\mathbf{D}(A, B) = \mathbf{C}(A, B)$.
- A ***lluf*** subcategory of $\mathbf{C}$ if $\mathbf{Ob}(\mathbf{D}) = \mathbf{Ob}(\mathbf{C})$.

For example, **Grp** is a full subcategory of **Mon**, and **Set** is a lluf subcategory of **Rel**.

Exercise 16

How many categories $\mathbf{C}$ with $\mathbf{Ob}(\mathbf{C}) = \{\bullet\}$ are there? (Hint: what do such categories correspond to?)

SOLUTION. Each such category corresponds to a monoid. So, there are as many such categories as there are monoids. □

Exercises.

(1) Consider the following properties of an arrow f in a category $\mathbf{C}$.
- f is *split monic* if for some g, $g \circ f$ is an identity arrow.
- f is *split epic* if for some g, $f \circ g$ is an identity arrow.

a. Prove that if f and g are arrows such that $g \circ f$ is monic, then f is monic.
b. Prove that if f is split epic then it is epic.
c. Prove that if f and $g \circ f$ are iso then g is iso.
d. Prove that if f is monic and split epic then it is iso.
e. In the category **Mon** of monoids and monoid homomorphisms, consider the inclusion map
$$i : (\mathbb{N}, +, 0) \to (\mathbb{Z}, +, 0)$$
of natural numbers into the integers. Show that this arrow is both monic and epic. Is it an iso?

The **Axiom of Choice** in Set Theory states that if $\{X_i\}_{i \in I}$ is a family of nonempty sets, we can form a set $X = \{x_i \mid i \in I\}$, where $x_i \in X_i$ for all $i \in I$.

f. Show that in **Set** an arrow which is epic is split epic. Explain why this needs the Axiom of Choice.
g. Is is always the case that an arrow which is epic is split epic? Either prove that it is, or give a counterexample.

(2) Give a description of partial orders as categories of a special kind.

SOLUTION.

(1)

a. Suppose $f : A \to B$ and $g : B \to C$ such that $g \circ f$ is monic. Assume, for all $i, j : Z \to A$, $f \circ i = f \circ j$. Then, $(g \circ f) \circ i = g \circ (f \circ i) = g \circ (f \circ j) = (g \circ f) \circ j$, which implies $i = j$, since $g \circ f$ is monic. This implies f is monic, and we are done.
b. Suppose $f : A \to B$ is split epic. Then, there exists a $g : B \to A$ such that $f \circ g = 1_B$. Assume, for all $i, j : B \to C$, $i \circ f = j \circ f$. Then, $i = i \circ 1_B = i \circ (f \circ g) = (i \circ f) \circ g = (j \circ f) \circ g = j \circ (f \circ g) = j \circ 1_B = j$, which shows f is epic.
c. Suppose $f : A \to B$ and $g : B \to C$ such that f and $g \circ f$ are iso. We claim that the inverse of g is $f \circ (g \circ f)^{-1} : C \to B$. Indeed, $g \circ (f \circ (g \circ f)^{-1}) = (g \circ f) \circ (g \circ f)^{-1} = 1_C$, and $(f \circ (g \circ f)^{-1}) \circ g = f \circ (g \circ f)^{-1} \circ (g \circ f) \circ f^{-1} = f \circ f^{-1} = 1_B$, which establishes g is also an iso.
d. Suppose $f : A \to B$ is monic and split epic. The latter implies f has a right inverse, $g : B \to A$, say, where $f \circ g = 1_B$. Note that $g \circ f : A \to A$ and $1_A : A \to A$. Now, $f \circ (g \circ f) = (f \circ g) \circ f =$

$1_B \circ f = f = f \circ 1_A$, which implies $g \circ f = 1_A$, since f is monic (left cancellative). Thus, g is also a left inverse of f, and hence, f is iso.

e. It is easy to prove the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is really a monoid homomorphism. Indeed, $i(0) = 0$, and, for all $n_1, n_2 \in \mathbb{N}$, $i(n_1 + n_2) = n_1 + n_2 = i(n_1) + i(n_2)$.
Next, we show that i is monic. Assume, for all monoid homomorphisms $g, h : X \to \mathbb{N}$, $i \circ g = i \circ h$. Then, for all $x \in X$, $(i \circ g)(x) = (i \circ h)(x)$, which implies $i(g(x)) = i(h(x))$, which implies $g(x) = h(x)$, which implies $g = h$. This shows the inclusion map is monic.
We now show the inclusion map is epic. First, assume, for all monoid homomorphisms $g, h : (\mathbb{Z}, +, 0) \to (X, \star, 1_X)$, $g \circ i = h \circ i$. Then, for all $n \in \mathbb{N}$, $(g \circ i)(n) = (h \circ i)(n)$, which implies $g(i(n)) = h(i(n))$, which implies $g(n) = h(n)$. We now claim that for all $n \geq 1$, $g(-n) = h(-n)$. To that end, we use induction on n. Note that $g(-1) = g(-1) \star 1_X = g(-1) \star h(0) = g(-1) \star h(1 + (-1)) = g(-1) \star h(1) \star h(-1) = g(-1) \star g(1) * h(-1) = g(-1+1) \star h(-1) = g(0) \star h(-1) = 1_X \star h(-1) = h(-1)$. Now, assume the proposition holds for some $n \geq 1$. Then, $g(-(n+1)) = g(-n + (-1)) = g(-n) \star g(-1) = h(-n) \star h(-1) = h(-n + (-1)) = h(-(n+1))$. Hence, by induction, $g(-n) = h(-n)$ for all $n \geq 1$. Combining the results from above, we thus conclude $g(z) = h(z)$ for all $z \in \mathbb{Z}$. In other words, $g = h$, which implies i is epic.
Clearly, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is not iso.

f. Suppose $f : X \to Y$ is epic in **Set**. Then, from an earlier result about **Set**, we conclude f is surjective. Now, consider the family of nonempty sets $\{f^{-1}(b)\}_{b \in B}$. Each of the sets in the family is nonempty, because f is surjective. Therefore, using the Axiom of Choice, we can choose some element from each nonempty set in the family to construct a function $g : Y \to X$, given by $g(y) := x$ if $x \in f^{-1}(b)$. In addition, for all $y \in Y$, $(f \circ g)(y) = f(g(y)) = y = 1_Y(y)$, which implies $f \circ g = 1_Y$. This shows f has a right inverse, thus proving f is split epic.

g. It isn't always the case that an arrow which is epic is split epic. For example, in the category **Mon**, the inclusion map $\mathbb{N} \hookrightarrow \mathbb{Z}$ is epic (as shown in (e) above.) Now, if we assume that it is also split epic, then there exists a monoid homomorphism $g : \mathbb{Z} \to \mathbb{N}$, such that $i \circ g = 1_{\mathbb{Z}}$. This implies $(i \circ g)(-1) = 1_{\mathbb{Z}}(-1)$, which implies $i(g(-1)) = -1$, which implies $g(-1) = -1$, which implies $-1 \in \mathbb{N}$, which is absurd. We thus conclude the aforesaid inclusion map is *not* split epic, even though it is epic. And this proves our original claim.

(2) Suppose $(P, \leq)$ is a poset. Then, its corresponding category $\mathbf{C}$ is defined as follows. The objects of $\mathbf{C}$ are the elements of P, and for all $x, y \in P$, $x \to y$ iff $x \leq y$. The reflexivity of $\leq$ corresponds to the identity arrows, and transitivity to arrow composition. Note that there is at most one arrow for every pair of objects in the category. Anti-symmetry of $\leq$ corresponds to the fact that the only isomorphisms in $\mathbf{C}$ are the identity arrows.

□

2. Some Basic Constructions

Initial and Terminal Objects. An object I in a category $\mathbf{C}$ is ***initial*** if, for every object A, *there exists a unique arrow* $I \to A$, which we write $\iota_A : I \to A$.

An object T in a category $\mathbf{C}$ is ***terminal*** if, for every object A, *there exists a unique arrow* $A \to T$, which we write $\tau_A : A \to T$.

Note that initial and terminal objects are dual notions: T is terminal in $\mathbf{C}$ iff it is initial in $\mathbf{C}^{\mathbf{op}}$. We sometimes write $\mathbf{1}$ for the terminal object and $\mathbf{0}$ for the initial object.

Exercise 18

Verify the following claims. In each case, identify the canonical arrows.

(1) In **Set**, the empty set is an initial object while any one-element set $\{\bullet\}$ is terminal.
(2) In **Pos**, the poset $(\varnothing, \varnothing)$ is an initial object while $(\{\bullet\}, \{(\bullet, \bullet)\})$ is terminal.
(3) In **Top**, the space $(\varnothing, \{\varnothing\})$ is an initial object while $(\{\bullet\}, \{\varnothing, \{\bullet\}\})$ is terminal.
(4) In $\mathbf{Vect}_k$, the one-element space $\{0\}$ is both initial and terminal.
(5) In a poset, seen as a category, an initial object is a least element, while a terminal object is a greatest element.

SOLUTION.

(1) In **Set**, for any set (object) A, the function $(\varnothing, A, \varnothing)$ is the unique function (arrow) from $\varnothing$ to A. Therefore, the empty set is (the) initial object in **Set**. And, for every set A, the function $A \to \{\bullet\}$ that maps every element of A to $\bullet$ is the unique function from A to $\{\bullet\}$. This establishes that any one-element set is terminal in **Set**.
(2) For any poset $(P, \leq)$, there exists a unique (empty) monotone function $(\varnothing, \varnothing) \xrightarrow{(\varnothing, P, \varnothing)} (P, \leq)$. Hence, the poset $(\varnothing, \varnothing)$ is an initial object in **Pos**. And, for any poset $(P, \leq)$, there exists a unique monotone function $(P, \leq) \to (\{\bullet\}, \{(\bullet, \bullet)\})$, defined by $x \mapsto \bullet$ for all $x \in P$. Hence, $(\{\bullet\}, \{(\bullet, \bullet)\})$ is terminal in **Pos**.
(3) For any topological space (X, T_X), the unique empty function
$$(\varnothing, \{\varnothing\}) \xrightarrow{(\varnothing, X, \varnothing)} (X, T_X)$$
is continuous, since for every open set $T \in T_X$, its preimage under the aforesaid function is the empty set, which is open. Hence, $(\varnothing, \{\varnothing\})$ is initial in **Top**.
And, for any topological space (X, T_X), the unique function $(X, T_X) \to (\{\bullet\}, \{\varnothing, \{\bullet\}\})$, defined by $x \mapsto \bullet$ for all $x \in X$, is continuous, since the preimage of $\varnothing$ under the aforesaid function is $\varnothing$, which is open, and the preimage of $\{\bullet\}$ is X, which is also open. Hence, $(\{\bullet\}, \{\varnothing, \{\bullet\}\})$ is terminal in **Top**.
(4) Assuming the ground field is k, for any vector space V, the unique linear map $\{0\} \to V$, defined by $0 \mapsto 0_V$ is a unique arrow from $\{0\}$ to V in $\mathbf{Vect}_k$. Also, the unique linear map $V \to \{0\}$, defined by $v \mapsto 0$ for all $v \in V$, is a unique arrow from V to $\{0\}$ in $\mathbf{Vect}_k$. This shows that $\{0\}$ is both initial and terminal in $\mathbf{Vect}_k$.

(5) In a poset $(P, \leq)$, seen as a category, if $\perp$ is an initial object, then there exists a unique arrow $\perp \to p$ for all $p \in P$. This implies $\perp \leq p$ for all $p \in P$, when seen as a set. Hence, an initial object in the category corresponding to $(P, \leq)$ is a least element in P. Arguing similarly, we conclude that a terminal object in the category corresponding to $(P, \leq)$ is a greatest element in P.

□

Exercise 19

Identify the initial and terminal objects in **Rel**.

Solution. In **Rel**, the empty set $\varnothing$ is both the initial object and the terminal object. Indeed, for any set A, the empty relation $\varnothing$ ($\subseteq \varnothing \times A$) is a unique relation from $\varnothing$ to A, and the empty relation $\varnothing$ ($\subseteq A \times \varnothing$) is also a unique relation from A to $\varnothing$. □

Exercise 20

Suppose a monoid, viewed as a category, has either an initial or a terminal object. What must the monoid be?

Solution. The category corresponding to a monoid $(M, \cdot, 1_M)$ contains just a single object. If this object is initial, then all morphisms must be the identity morphism on this initial object, which implies $M = \{1_M\}$. The argument is similar if the aforesaid object is terminal, which would again imply $M = \{1_M\}$. Thus, in either case, the monoid must be the trivial monoid. □

A fundamental fact about initial and terminal objects is that they are *unique up to (unique) isomorphism*. This is characteristic of all such "universal" definitions. Hence, if initial objects exist in a category, we can speak of *the* initial object. Similarly for terminal objects.

Exercise 22

Let $\mathbf{C}$ be a category with an initial object $\mathbf{0}$. For any object A, show the following:

(1) If $A \cong \mathbf{0}$, then A is an initial object.
(2) If there exists a monomorphism $f : A \to \mathbf{0}$, then f is an iso, and hence A is initial.

Solution.

(1) Suppose $A \cong \mathbf{0}$. Then, there exists an isomorphism $f : A \xrightarrow{\sim} \mathbf{0}$. For any object B, there exists a unique morphism $\iota_B : \mathbf{0} \to B$, and hence, $\iota_B \circ f : A \to B$. This proves the existence of a morphism $A \to B$ for any object B in the category. We now show that such a morphism is indeed unique. Let $g, h : A \to B$ be a pair of morphisms for any object B in the category $\mathbf{C}$. Then, we have

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{g} B$$

and

$$\mathbf{0} \xrightarrow{f^{-1}} A \xrightarrow{h} B.$$

Since $\mathbf{0}$ is initial, we must have $g \circ f^{-1} = h \circ f^{-1}$. Therefore, $g = g \circ 1_A = g \circ (f^{-1} \circ f) = (g \circ f^{-1}) \circ f = (h \circ f^{-1}) \circ f = h \circ (f^{-1} \circ f) = h \circ 1_A = h$. This proves uniqueness, and we are done.

(2) Suppose $A \xrightarrow{f} \mathbf{0}$ is a monomorphism. We claim the unique arrow $\mathbf{0} \xrightarrow{\iota_A} A$ is the inverse of f. To that end, we show ι_A is both a left and a right inverse of f. Indeed, $f \circ \iota_A : \mathbf{0} \to \mathbf{0}$, and since $\mathbf{0}$ is initial, we must have $f \circ \iota_A = 1_{\mathbf{0}}$, which implies ι_A is a right inverse of f. Now, note $\iota_A \circ f : A \to A$ and $1_A : A \to A$. Also, $f \circ (\iota_A \circ f) = (f \circ \iota_A) \circ f = 1_{\mathbf{0}} \circ f = f = f \circ 1_A$, and since f is left cancellative, we have $\iota_A \circ f = 1_A$, which shows ι_A is a left inverse of f. Thus, f has both a left inverse and a right inverse, implying it is iso, and hence, using the result obtained in (1) above, we conclude A is an initial object in $\mathbf{C}$.

□

Products and Coproducts. We can express a general notion of product that is meaningful in any category, such that, if a product exists, it is characterized uniquely up to unique isomorphism. Given a particular mathematical context (*i.e.* a category), we can then verify if a product exists in that category. The concrete construction appropriate to the context will enter only into the proof of *existence*; all of the useful *properties* of a product follow from the general definition.

Exercise 24

Verify $\mathbf{Pair}(A, B)$ is a category, where A and B are arbitrary objects in some category.

Solution. Let A and B be some arbitrary objects in some category $\mathbf{C}$. Now, given morphisms $f : (P, p_1, p_2) \to (Q, q_1, q_2)$ and $g : (Q, q_1, q_2) \to (R, r_1, r_2)$ in $\mathbf{Pair}(A, B)$, it is easy to check that $g \circ f : P \to R$ in $\mathbf{C}$. Also, we have

$$q_1 \circ f = p_1,\ q_2 \circ f = p_2$$

and

$$r_1 \circ g = q_1,\ r_2 \circ g = q_2$$

So, $r_1 \circ (g \circ f) = (r_1 \circ g) \circ f = q_1 \circ f = p_1$, and, $r_2 \circ (g \circ f) = (r_2 \circ g) \circ f = q_2 \circ f = p_2$, which implies $g \circ f : (P, p_1, p_2) \to (R, r_1, r_2)$ in $\mathbf{Pair}(A, B)$. Associativity of morphisms in $\mathbf{Pair}(A, B)$ follows directly from the associativity of morphisms in $\mathbf{C}$. Finally, for all (P, p_1, p_2) in $\mathbf{Pair}(A, B)$, the identity morphism $1_P : P \to P$ is the identity morphism for (P, p_1, p_2), since $p_1 \circ 1_P = p_1$ and $p_2 \circ 1_P = p_2$. And, this proves that $\mathbf{Pair}(A, B)$ is indeed a category. □

We say $(A \times B, \pi_1, \pi_2)$ is a ***product*** of A and B if it is *terminal* in $\mathbf{Pair}(A, B)$. Products are specified by triples $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2}$, where pi_i's are called *projections.* For economy (and if projections are obvious), we say $A \times B$ is the product of A and B. We say a category $\mathbf{C}$ has ***(binary) products*** if each pair of objects A, B has a product in $\mathbf{C}$. Since, products are terminal objects, they are unique up to (unique) isomorphism.

Unpacking the uniqueness condition from $\mathbf{Pair}(A, B)$ back to $\mathbf{C}$, we obtain the following more concise definition of products that we use in practice.

(**Equivalent definition of product**) Let A, B be objects in a category $\mathbf{C}$. A product of A and B is an object $A \times B$ together with a pair of arrows $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ such that for every triple $A \xleftarrow{f} C \xrightarrow{g} B$, there exists a *unique* morphism

$$\langle f, g \rangle : C \to A \times B$$

such that the corresponding diagram commutes. That is,

$$\pi_1 \circ \langle f, g \rangle = f$$
$$\pi_2 \circ \langle f, g \rangle = g$$

We call $\langle f, g \rangle$ the *pairing* of f and g.

Exercise 26

Verify the following claims.

(1) In **Set**, products are the usual cartesian products.
(2) In **Pos**, products are cartesian products with the pointwise order.
(3) In **Top**, products are cartesian products with the product topology.
(4) In $\mathbf{Vect}_k$, products are direct sums.
(5) In a poset, seen as a category, products are *greatest lower bounds*.

SOLUTION.

(1) Let A, B be arbitrary sets in **Set**. We claim $A \times B$ equipped with the canonical projection functions is the cross product of A and B. Indeed, given any $A \xleftarrow{f} C \xrightarrow{g} B$, we show $\langle f, g \rangle : C \to A \times B$, defined by

$$c \mapsto (f(c), g(c)),$$

is the unique function that makes the following diagram commute:

$$\begin{tikzcd} & C \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g \rangle"] \arrow[dr, "g"] & \\ A & A \times B \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & B \end{tikzcd}$$

(*Existence*) It is easy to check that $\langle f, g \rangle$ is indeed a function from C to $A \times B$.
(*Commutativity*) For all $c \in C$, $(\pi_1 \circ \langle f, g \rangle)(c) = f(c)$ and $(\pi_2 \circ \langle f, g \rangle)(c) = g(c)$, which imply the above diagram commutes.
(*Uniqueness*) Suppose $h : C \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, for all $c \in C$, $(\pi_1 \circ h)(c) = f(c)$ and $(\pi_2 \circ h)(c) = g(c)$, which imply $\pi_1(h(c)) = f(c)$ and $\pi_2(h(c)) = g(c)$, which imply $h(c) = (f(c), g(c)) = \langle f, g \rangle(c)$, thus proving $h = \langle f, g \rangle$, and thereby, showing the uniqueness of $\langle f, g \rangle$.
Hence, $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ is the cross product of A and B.

(2) Let $(P, \leq)$ and $(Q, \leq)$ be posets. Let $(P \times Q, \leq)$ be the cartesian product of P and Q with the pointwise order. That is, for all $a, c \in P$ and $b, d \in Q$, $(a, b) \leq (c, d)$ iff $a \leq c$ and $b \leq d$. We claim $(P \times Q, \leq)$ equipped with the canonical projection functions (which are monotone) is the cross product

of $(P, \leq)$ and $(Q, \leq)$. Given any $(P, \leq) \xleftarrow{f} (R, \leq) \xrightarrow{g} (Q, \leq)$, where f, g are monotone functions, the function $\langle f, g\rangle : (R, \leq) \to (P \times Q, \leq)$, defined by

$$r \mapsto (f(r), g(r))$$

is the unique monotone function that makes the following diagram commute:

$$\begin{tikzcd}
 & (R, \leq) \arrow[dl, "f"'] \arrow[d, dashed, "\langle f,g\rangle"] \arrow[dr, "g"] & \\
(P, \leq) & (P \times Q, \leq) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & (Q, \leq)
\end{tikzcd}$$

(*Existence*) It is easy to check that $\langle f, g\rangle$ is indeed a set function from R to $P \times Q$. And, for all $r_1, r_2 \in R$, if $r_1 \leq r_2$, then $f(r_1) \leq f(r_2)$ and $g(r_1) \leq g(r_2)$ (since f, g are monotone), which implies $(f(r_1), g(r_1)) \leq (f(r_2), g(r_2))$, which implies $\langle f, g\rangle(r_1) \leq \langle f, g\rangle(r_2)$, which implies $\langle f, g\rangle$ is monotone.

(*Commutativity*) For all $r \in R$, we have

$$(\pi_1 \circ \langle f, g\rangle)(r) = f(r),$$
$$(\pi_2 \circ \langle f, g\rangle)(r) = g(r).$$

The above implies that the above diagram does commute.

(*Uniqueness*) Suppose $h : (R, \leq) \to (P \times Q, \leq)$ is a monotone function such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, for all $r \in R$, $\pi_1 \circ h(r) = f(r)$ and $\pi_2 \circ h(r) = g(r)$, which imply $\pi_1(h(r)) = f(r)$ and $\pi_2(h(r)) = g(r)$, which imply $h(r) = (f(r), g(r))$, which implies $h(r) = \langle f, g\rangle(r)$, which implies $h = \langle f, g\rangle$, thus showing that $\langle f, g\rangle$ with the commutativity property is indeed unique.

Hence, we conclude the cartesian product $(P \times Q, \leq)$ with the pointwise order is the product of any posets $(P, \leq)$ and $(Q, \leq)$.

(3)

(4)

(5) In a poset $(P, \leq)$, seen as a category, the product $a \times b$ of two elements $a, b \in P$ is an element in P satisfying $a \times b \leq a$ and $a \times b \leq b$, such that for all elements $c \in P$, if $c \leq a$ and $c \leq b$, then $c \leq a \times b$. This is precisely the definition of the *greatest lower bound* of any two elements $a, b \in P$, seen as a set. Therefore, products are greatest lower bounds in posets.

$\square$

The following proposition shows that the uniqueness of the pairing arrow can be specified purely equationally by the equation:

$$\forall h : C \to A \times B.\, h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$$

PROPOSITION 27. *For any triple* $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$, *the following statements are equivalent:*

(I) *For any triple* $A \xleftarrow{f} C \xrightarrow{g} B$, *there exists a unique morphism* $\langle f, g\rangle : C \to A \times B$ *such that* $\pi_1 \circ \langle f, g\rangle = f$ *and* $\pi_2 \circ \langle f, g\rangle = g$.

(II) *For any triple* $A \xleftarrow{f} C \xrightarrow{g} B$*, there exists a morphism* $\langle f, g\rangle : C \to A \times B$ *such that* $\pi_1 \circ \langle f, g\rangle = f$ *and* $\pi_2 \circ \langle f, g\rangle = g$*, and moreover, for any* $h : C \to A \times B$*,* $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$*.*

Proof. ((I) $\implies$ (II)) Suppose (I) holds. Assume $A \xleftarrow{f} C \xrightarrow{g} B$. Then, by (I), there exists a (unique) morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. Now, let $h : C \to A \times B$. Note $A \xleftarrow{\pi_1 \circ h} C \xrightarrow{\pi_2 \circ h} B$. Thus, by (I), there exists a unique morphism $\langle \pi_1 \circ h, \pi_2 \circ h\rangle : C \to A \times B$ such that

$$\pi_1 \circ \langle \pi_1 \circ h, \pi_2 \circ h\rangle = \pi_1 \circ h,$$
$$\pi_2 \circ \langle \pi_1 \circ h, \pi_2 \circ h\rangle = \pi_2 \circ h.$$

The above implies $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$. This proves (II).

((II) $\implies$ (I)) Suppose (II) holds. Assume $A \xleftarrow{f} C \xrightarrow{g} B$. Then, by (II), there exists a morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$. We claim such a morphism is unique. So, suppose $h : C \to A \times B$ such that $\pi_1 \circ h = f$ and $\pi_2 \circ h = g$. Then, by (II), we have $h = \langle \pi_1 \circ h, \pi_2 \circ h\rangle$, which implies $h = \langle f, g\rangle$. This proves (I), and our proof is complete. $\square$

Cartesian product of morphisms. Given $f_1 : A_1 \to B_1$ and $f_2 : A_2 \to B_2$, we define the *cartesian product of morphisms* f_1 and f_2 by

$$f_1 \times f_2 := \langle f_1 \circ \pi_1, f_2 \circ \pi_2\rangle : A_1 \times A_2 \to B_1 \times B_2.$$

The following proposition provides some useful properties of products.

Proposition 28. *For any* $f : A \to B$*,* $g : A \to C$*,* $h : A' \to A$*, and any* $p : B \to B'$*,* $q : C \to C'$*, the following hold:*

(1) $\langle f, g\rangle \circ h = \langle f \circ h, g \circ h\rangle$
(2) $(p \times q) \circ \langle f, g\rangle = \langle p \circ f, q \circ g\rangle$.

Proof.

(1) Note $\langle f, h\rangle \circ h : A' \to B \times C$. Therefore, by (II) of Proposition 27, $\langle f, h\rangle \circ h = \langle \pi_1 \circ (\langle f, g\rangle \circ h), \pi_2 \circ (\langle f, g\rangle \circ h)\rangle = \langle f \circ h, f \circ g\rangle$.
(2) $(p \times q) \circ \langle f, g\rangle = \langle p \circ \pi_1, q \circ \pi_2\rangle \circ \langle f, g\rangle = \langle p \circ \pi_1 \circ \langle f, g\rangle, q \circ \pi_2 \circ \langle f, g\rangle = \langle p \circ f, q \circ g\rangle$.

$\square$

General Products. The notion of products can be generalized to arbitrary arities as follows. In a category $\mathbf{C}$, a product for a family of objects $\{A_i\}_{i \in I}$ is an object P and morphisms

$$p_i : P \to A_i \ (i \in I)$$

such that, for all objects B and arrows

$$f_i : B \to A_i \ (i \in I)$$

there is a *unique* arrow $g : B \to P$ such that, for all $i \in I$, the following diagram commutes

$$\begin{tikzcd} B \arrow[rr, dashed, "g"] \arrow[dr, "f_i"'] & & P \arrow[dl, "p_i"] \\ & A_i & \end{tikzcd}$$

Again, if such a product exists, it is unique up to (unique) isomorphism. We write $P = \prod_{i \in I} A_i$ for the product object, and $g = \langle f_i \mid i \in I \rangle$ for the unique morphism in the definition.

Exercise 29

What is the product of the empty family?

Solution. The product of the empty family is an object T, such that for every object with arrows to (non-existent) members of the empty family, there is a unique arrow from that object to T making the corresponding diagram commute. Since there are no diagrams, this means there is a unique arrow from every object to T, and this is precisely the definition of a terminal object. Hence, the product of an empty family is a terminal object. $\square$

Exercise 30

Show that if a category has binary and nullary products, then it has all finite products.

Solution. Suppose $\mathbf{C}$ is a category with binary and nullary products. We claim, for all $n \in \mathbb{N}$, $P_n = \prod_{i=1}^{n} A_i$ with the corresponding projection functions $p_i : P \to A_i$, where A_i is an object in $\mathbf{C}$, exists. We use induction on n to prove our claim. (*Base case*) For $n = 0$, P is the nullary product, which exists by assumption. (*Inductive case*) Now, suppose a product P_n exists for some $n \geq 0$. Then, $P_{n+1} = \prod_{i=1}^{n+1} A_i = \prod_{i=1}^{n} A_i \times A_{n+1} = P_n \times A_{i+1}$, which is a binary product of objects, which exists due to the fact that $\mathbf{C}$ has binary products and that P_n exists (from our inductive hypothesis.) Hence, by induction, P_n exists for all $n \in \mathbb{N}$. $\square$

Coproducts. The dual notion to products are coproducts. Formally, coproducts in a category $\mathbf{C}$ are just products in $\mathbf{C}^{\mathbf{op}}$, interpreted back in $\mathbf{C}$.

Let A, B be objects in a category $\mathbf{C}$. A *coproduct* of A and B is an object $A + B$ together with a pair of arrows $A \xrightarrow{i_A} A + B \xleftarrow{i_B} B$, such that for every triple $A \xrightarrow{f} C \xleftarrow{g} B$, there exists a unique morphism

$$[f, g] : A + B \to C$$

such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "i_A"] \arrow[dr, "f"'] & A + B \arrow[d, dashed, "{[f,g]}"] & B \arrow[l, "i_B"'] \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

We call i_A and i_B *injections* and $[f, g]$ the *copairing* of f and g. As with pairings, the uniqueness of copairings can be specified by an equation:

$$\forall h : A + B \to C.\, h = [h \circ i_A, h \circ i_B]$$

Exercise 32

A coproduct in **Set** is given by *disjoint union* of sets, which can be defined concretely, *e.g.* by

$$X + Y := (\{1\} \times X) \bigcup (\{2\} \times Y)$$

We can define *injections*

$$X \xrightarrow{i_X} X + Y \xleftarrow{i_Y} Y$$
$$i_X(x) := (1, x), \quad i_Y(y) := (2, y).$$

Also, given functions $f : X \to Z$ and $g : Y \to Z$, we can define

$$[f, g] : X + Y \to Z$$
$$[f, g](1, x) := f(x), \quad [f, g](2, y) := g(y).$$

Check that the above construction does yield coproducts in **Set**.

Solution. For all $x \in X$, $([f, g] \circ i_X)(x) = [f, g](i_X(x)) = [f, g](1, x) = f(x)$, and, for all $y \in Y$, $([f, g] \circ i_Y)(y) = [f, g](i_Y(y)) = [f, g](2, y) = g(y)$. Therefore, $[f, g] \circ i_X = f$ and $[f, g] \circ i_Y = g$, proving that the corresponding diagram is indeed commutative. Let $h : X + Y \to Z$ be such that $h \circ i_X = f$ and $h \circ i_Y = g$. Then, for all $x \in X$ and $y \in Y$, $(h \circ i_X)(x) = f(x)$ and $(h \circ i_Y)(y) = g(y)$, which imply $h(1, x) = [f, g](1, x)$ and $h(2, g) = [f, g](2, y)$, which imply $h = [f, g]$, thus showing $[f, g]$ is indeed unique. This shows $X \xrightarrow{i_X} X + Y \xleftarrow{i_Y} Y$ as defined is a coproduct of X and Y, for any two objects X, Y in **C**. □

Exercise 33

Verify the following claims:

(1) In **Pos**, disjoint unions (with the inherited orders) are coproducts.
(2) In **Top**, topological disjoint unions are coproducts.
(3) In $\mathbf{Vect}_k$, direct sums are coproducts.
(4) In a poset, *least upper bounds* are coproducts.

Solution.

(1)
(2)
(3)
(4) In a poset $(P, \leq)$, for any two elements $p, q \in P$, the coproduct $p \times q$ is an element satisfying $p \leq p \times q$ and $q \leq p \times q$, such that for any element $r \in P$, if $p \leq r$ and $q \leq r$ then $p \times q \leq r$. Thus, $p \times q$ satisfies precisely the definition of the least upper bound of p and q. Hence, least upper bounds in a poset are coproducts.

□

Exercise 34

Dually to products, express coproducts as initial objects of a category $\mathbf{Copair}(A, B)$ of A, B-copairings.

Solution. Let A, B be objects in a category **C**. An A, B-copairing is a triple $A \xrightarrow{p_1} P \xleftarrow{p_2} B$, where P is an object in **C**. A morphism of A, B-copairings $f : (P, p_1, p_2) \to (Q, q_1, q2)$ is a morphism $f : P \to Q$ in **C** such that the following diagram commutes

$$\begin{tikzcd}
& Q & \\
A \arrow[ur, "q_1"] \arrow[r, "p_1"'] & P \arrow[u, "f"] & B \arrow[ul, "q_2"'] \arrow[l, "p_2"]
\end{tikzcd}$$

Then, it is easy to check that $\mathbf{Copair}(A, B)$ is a category of A, B-copairings.

We say $(A+B, i_A, i_B)$ is a **coproduct** of A and B if it is *initial* in $\mathbf{Copair}(A, B)$. □

Pullbacks and Equalisers. We consider two further constructions of interest: *pullbacks* and *equalisers*.

Pullbacks. Consider a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$. The ***pullback*** of f along g is a pair $A \xleftarrow{p} D \xrightarrow{q} B$ such that $f \circ p = g \circ q$, and, for any pair $A \xleftarrow{p'} D' \xrightarrow{q'} B$ such that $f \circ p' = g \circ q'$, there exists a unique $h : D' \to D$ such that the following diagram commutes.

$$\begin{tikzcd}
D' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "q'"] \arrow[ddr, bend right, "p'"'] & & \\
& D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\
& A \arrow[r, "f"'] & C
\end{tikzcd}$$

Examples of pullbacks:

- In $\mathbf{Set}$, the pullback of $A \xrightarrow{f} C \xleftarrow{g} B$ is defined as a *subset of the cartesian product*:
$$A \times_C B = \{(a, b) \in A \times B \mid f(a) = g(b)\}.$$
For example, consider a category $\mathbf{C}$ with
$$\mathrm{Ar}(\mathbf{C}) \xrightarrow{\mathrm{dom}} \mathrm{Ob}(\mathbf{C}) \xleftarrow{\mathrm{cod}} \mathrm{Ar}(\mathbf{C}).$$
Then, the pullback of **dom** along **cod** is the set of *composable morphisms, i.e.* pairs of morphisms (f, g) in $\mathbf{C}$ such that $f \circ g$ is well-defined.
- In $\mathbf{Set}$ again, subsets (*i.e.* inclusion maps) pull back to subsets:

$$\begin{tikzcd}
f^{-1}(U) \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Exercise 37

Let $\mathbf{C}$ be a category with a terminal object $\mathbf{1}$. Show that, for any $A, B \in \mathbf{Ob}(\mathbf{C})$, the pullback of $A \xrightarrow{\tau_A} \mathbf{1} \xleftarrow{\tau_B} B$ is the product of A and B, if it exists.

Solution. Suppose $\mathbf{C}$ is a category with a terminal object $\mathbf{1}$. Assume, for any $A, B \in \mathbf{Ob}(\mathbf{C})$, their product $A \xleftarrow{\pi_1} A \times B \xrightarrow{\pi_2} B$ exists. We show that this product is the pullback of $A \xrightarrow{\tau_A} \mathbf{1} \xleftarrow{\tau_B} B$. First, note $\tau_A \circ \pi_1 : A \times B \to \mathbf{1}$ and $\tau_B \circ \pi_w : A \times B \to \mathbf{1}$, but since $\mathbf{1}$ is terminal, we have $\tau_A \circ \pi_1 = \tau_B \circ \pi_2$. That is, the following diagram commutes.

$$\begin{tikzcd}
A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\tau_B"] \\
A \arrow[r, "\tau_A"'] & \mathbf{1}
\end{tikzcd}$$

Now, for any pair $A \xleftarrow{f} C \xrightarrow{g} B$, we again have $\tau_A \circ f = \tau_B \circ g$, since $\mathbf{1}$ is terminal. Also, there exists a unique morphism $\langle f, g\rangle : C \to A \times B$ such that $\pi_1 \circ \langle f, g\rangle = f$ and $\pi_2 \circ \langle f, g\rangle = g$, which implies the following diagram commutes.

$$\begin{tikzcd}
C \arrow[dr, dashed, "\langle f,g\rangle"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& A \times B \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & B \arrow[d, "\tau_B"] \\
& A \arrow[r, "\tau_A"'] & \mathbf{1}
\end{tikzcd}$$

And, this completes our proof. $\square$

Just as for products, pullbacks can equivalently be described as terminal objects in suitable categories. Given a pair of morphisms $A \xrightarrow{f} C \xleftarrow{g} B$, we define an (f, g)-*cone* to be a triple (D, p, q) such that the following diagram commutes.

$$\begin{tikzcd}
D \arrow[r, "q"] \arrow[d, "p"'] & B \arrow[d, "g"] \\
A \arrow[r, "f"'] & C
\end{tikzcd}$$

A morphism of (f, g)-cones $h : (D_1, p_1, q_1) \to (D_2, p_2, q_2)$ is a morphism $h : D_1 \to D_2$ such that the following diagram commutes.

$$\begin{tikzcd}
& D_1 \arrow[dl, "p_1"'] \arrow[d, "h"] \arrow[dr, "q_1"] & \\
A & D_2 \arrow[l, "p_2"] \arrow[r, "q_2"'] & B
\end{tikzcd}$$

We can thus form a category $\mathbf{Cone}(f, g)$. A pullback of f along g, if it exists, is exactly a terminal object in $\mathbf{Cone}(f, g)$. This also shows the uniqueness of pullbacks up to unique isomorphism.

Equalisers. Consider a pair of parallel arrows $\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \end{tikzcd}$. An ***equaliser*** of (f, g) is an arrow $e : E \to A$ such that $f \circ e = g \circ e$, and for any arrow $h : D \to A$ such that $f \circ h = g \circ h$, there is a unique arrow $\hat{h} : D \to E$ so that $h = e \circ \hat{h}$. That is, the following diagram commutes.

$$\begin{tikzcd} E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\ D \arrow[u, dashed, "\hat{h}"] \arrow[ur, "h"'] & & \end{tikzcd}$$

As for products, the uniqueness of the arrow from D to E can be expressed equationally:

$$\forall k : D \to E.\ \widehat{e \circ k} = k.$$

Exercise 39

Why is $\widehat{e \circ k}$ well-defined for any $k : D \to E$? Prove that the above equation is equivalent to the uniqueness requirement.

SOLUTION.
(Uniqueness requirement $\implies$ equation) Suppose the uniqueness requirement is satisfied. Assume $k : D \to E$. Then, $e \circ k : D \to A$. Therefore, there exists a unique $l : D \to E$ such that $e \circ l = e \circ k$. Now, note $l = k$ and $l = \widehat{e \circ k}$ (the latter from the satisfaction of the uniqueness requirement) both satisfy the previous equation, which implies $k = \widehat{e \circ k}$.
(Equation $\implies$ uniqueness requirement) Suppose $\forall k : D \to E, \widehat{e \circ k} = k$. Let $h : D \to A$ such that $f \circ h = g \circ h$, and assume $\hat{h} : D \to E$ such that $e \circ \hat{h} = h$. We show such an arrow $\hat{h}$ is unique. To that end, suppose $l : D \to E$ such that $e \circ l = h$. Then, $\hat{h} = \widehat{e \circ l} = l$, which is what we set out to proof, and we are done. □

Example of equaliser: In **Set**, the equaliser of f, g is given by the inclusion

$$\{x \in A \mid f(x) = g(x)\} \hookrightarrow A.$$

This allows *equationally defined subsets* to be defined as equalisers. For example, consider the pairs of maps $\mathbb{R}^2 \overset{f}{\underset{g}{\rightrightarrows}} \mathbb{R}$, where

$$f : (x, y) \mapsto x^2 + y^2, \qquad g : (x, y) \mapsto 1.$$

Then, the equaliser is the unit circle as a subset of $\mathbb{R}^2$.

Limits and Colimits. The notions introduced so far are all special cases of a general notion of *limits* in categories, as well as the dual notion of *colimits*. An

Limits	**Colimits**
Terminal Objects	Initial Objects
Products	Coproducts
Pullbacks	Pushouts
Equalisers	Coequalisers

TABLE 1. Examples of Limits and Colimits

import aspect of studying any kind of mathematical structure is to determine what limits and colimits the category of such structures has.

Exercises.

(1) Give an example of a category where some pair of objects lacks a product or coproduct.

(2) (*Pullback lemma*) Consider the following commutative diagram.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[d, "w"] \\ D \arrow[r, "h"'] & E \arrow[r, "i"'] & F \end{tikzcd}$$

Given that the right hand square $BCEF$ and the outer square $ACDF$ are pullbacks, prove that the left hand square $ABDE$ is a pullback.

(3) Consider $A \xrightarrow{f} C \xleftarrow{g} B$ with pullback $A \xleftarrow{p} D \xrightarrow{q} B$. For each $A \xleftarrow{p'} D' \xrightarrow{q'} B$ with $f \circ p' = g \circ q'$, let $\phi(p', q') : D' \to D$ be the arrow dictated by the pullback condition. Express uniqueness of $\phi(p', q')$ equationally.

Solution.

(1) In a discrete category with the naturals as objects, say, any pair of natural numbers (objects) lacks either a product or a coproduct.

(2) Suppose the following diagram is commutative.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[d, "w"] \\ D \arrow[r, "h"'] & E \arrow[r, "i"'] & F \end{tikzcd}$$

And, suppose the right hand square $BCEF$ and the outer square $ACDF$ are pullbacks. We claim the left hand square $ABDE$ is a pullback. To that end, let $D \xleftarrow{k} A' \xrightarrow{l} B$ such that $v \circ l = h \circ k \, (*)$. Note $D \xleftarrow{k} A' \xrightarrow{g \circ l} C$ and that $(i \circ h) \circ k = i \circ (v \circ l) = (w \circ g) \circ l = w \circ (g \circ l)$. And, since $ACDF$ is a pullback, there exists a unique morphism $p : A' \to A$ making the following diagram commute.

$$\begin{tikzcd} A' \arrow[drrr, bend left, "g \circ l"] \arrow[dr, dashed, "p"] \arrow[ddr, bend right, "k"'] & & & \\ & A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[d, "w"] \\ & D \arrow[r, "h"'] & E \arrow[r, "i"'] & F \end{tikzcd}$$

Now, we need only show $f \circ p = l$.

Note $E \xleftarrow{h \circ k} A' \xrightarrow{g \circ l} C$, and $i \circ (h \circ k) = (i \circ h) \circ k = (i \circ h) \circ (u \circ p) = (i \circ h \circ u) \circ p = (w \circ g \circ f) \circ p = w \circ (g \circ f \circ p) = w \circ (g \circ l)$. Since $BCEF$ is a pullback, there exists a unique morphism $q : A' \to B$ such that

$$g \circ q = g \circ l$$
$$v \circ q = h \circ k$$

Using $(*)$ and the commutativity of the above diagram, we see $q = l$ and $q = f \circ p$ both satisfy the above set of equations, which implies $f \circ p = l$.

Let $p' : A' \to A$ where $f \circ p' = l$ and $u \circ p' = k$. Then, this implies the following diagram commutes.

$$\begin{tikzcd}
A' \arrow[dr, "p'"] \arrow[drrr, bend left, "g\circ l"] \arrow[ddr, bend right, "k"'] & & & \\
& A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[r, "g"] \arrow[d, "v"] & C \arrow[d, "w"] \\
& D \arrow[r, "h"'] & E \arrow[r, "i"'] & F
\end{tikzcd}$$

Since $ACDF$ is a pullback, it implies $p' = p$, from which we conclude $ABDE$ is indeed a pullback.

(3) The uniqueness of $\phi(p', q')$ can be stated equationally as follows:

$$\forall h : D' \to D.\ h = \phi(p \circ h, p \circ h).$$

□

3. Functors

Part of the "categorical philosophy" is:

Don't just look at the objects; take the morphisms into account too.

Basics. A "morphism of categories" is a *functor*.

A ***functor*** $F : \mathbf{C} \to \mathbf{D}$ is given by:

- An object-map, assigning an object FA of $\mathbf{D}$ to every object A of $\mathbf{C}$.
- An arrow-map, assigning an arrow $Ff : FA \to FB$ of $\mathbf{D}$ to every arrow $f : A \to B$ in such a way that composition and identities are preserved:

$$F(g \circ f) = F(g) \circ F(f), \qquad F1_A = 1_{FA}.$$

We use the same symbol to denote object- and arrow-maps; in practice, this never causes confusion. Since functors preserve domains and codomains of arrows, for each pair of objects A, B of $\mathbf{C}$, there is a well-defined map

$$F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(FA, FB).$$

The conditions expressing preservation of composition and identities are called *functoriality.*

Examples of functors.

(1) Let $(P, \leq)$ and $(Q, \leq)$ be preorders (seen as categories.) A functor $F : (P, \leq) \to (Q, \leq)$ is specified by an object-map, $F : P \to Q$, say, and an appropriate arrow-map. The arrow-map corresponds to the condition:

$$\forall p_1, p_2 \in P.\ p_1 \leq p_2 \implies F(p_1) \leq F(p_2),$$

i.e. to monotonicity of F. Moreover, the functoriality conditions are trivial, since in the codomain $(Q, \leq)$, all homsets are singletons.
Hence, a functor between preorders is just a monotone map.

(2) Let $(M, \cdot, 1)$ and $(N, \cdot, 1)$ be monoids. A functor $F : (M, \cdot, 1) \to (N, \cdot, 1)$ is specified by a trivial object-map (since monoids are categories with a single object) and an arrow-map, $F : M \to N$, say. The functoriality conditions correspond to

$$\forall m_1, m_2 \in M.\ F(m_1 \cdot m_2) = F(m_1) \cdot F(m_2), \qquad F(1) = 1,$$

i.e. to F being a monoid homomorphism.

Hence, a functor between monoids is just a monoid homomorphism.

Other examples of functors.

- Inclusion of a subcategory, $\mathbf{C} \hookrightarrow \mathbf{D}$, is a functor (by taking the identity map for object- and arrow-map.)
- The *covariant* powerset functor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$:
$$X \mapsto \mathcal{P}X, \quad (f : X \to Y) \mapsto \mathcal{P}(f) := S \mapsto \{f(x) \mid x \in S\}.$$
- $U : \mathbf{Mon} \to \mathbf{Set}$ is the 'forgetful' or 'underlying' functor which sends a monoid to its set of elements, 'forgetting' the algebraic structure, and sends a homomorphism to the corresponding function between sets. There are similar forgetful functors for other categories of structured sets.
- (*Group theory examples*) The assignment of the commutator sub-group of a group extends to a functor from **Grp** to **Grp**; and the assignment of the quotient by this normal subgroup extends to a functor from **Grp** to **Ab** (the category of abelian groups and group homomorphisms.)
- (*Homology*) The basic idea of algebraic topology is that there are functorial assignments of algebraic objects (*i.e.* groups) to topological spaces, and variants of this idea ('(co)homology theories') are pervasive throughout modern pure mathematics.

Functors 'of several variables'

We can generalize the notion of a functor to a mapping from several domain categories to a codomain category. For this, we need the following definition.

(***Product category***) For categories $\mathbf{C}, \mathbf{D}$, define the ***product category*** $\mathbf{C} \times \mathbf{D}$ as follows. An object in $\mathbf{C} \times \mathbf{D}$ is a pair of objects from $\mathbf{C}$ and $\mathbf{D}$, and an arrow in $\mathbf{C} \times \mathbf{D}$ is a pair of arrows from $\mathbf{C}$ and $\mathbf{D}$. Identities and arrow composition are defined componentwise:

$$1_{(A,B)} := (1_A, 1_B), \qquad (f, g) \circ (f', g') := (f \circ f', g \circ g').$$

A functor 'of two variables', with domains $\mathbf{C}$ and $\mathbf{D}$, to $\mathbf{E}$ is simply a functor

$$F : \mathbf{C} \times \mathbf{D} \to \mathbf{E}.$$

For example, there are evident projection functors

$$\mathbf{C} \leftarrow \mathbf{C} \times \mathbf{D} \to \mathbf{D}.$$

Further Examples. Set-valued functors.

Many important constructions arise as functors $F : \mathbf{C} \to \mathbf{Set}$. Here are some examples:

- If G is a group, a functor $F : G \to \mathbf{Set}$ is *an action of G on a set.*
- If P is a poset representing time, a functor $F : P \to \mathbf{Set}$ is a notion of *set varying through time.* This is related to *Kripke semantics*, and to *forcing arguments* in set theory.

- Recall $\mathbb{2}_{\rightrightarrows}$ is the category $\bullet \rightrightarrows \bullet$. Then, functors $F : \mathbb{2}_{\rightrightarrows} \to \mathbf{Set}$ correspond to *directed graphs, i.e.* structures (V, E, s, t), where V is a set of vertices, E is a set of edges, and $s, t : E \to V$ specify the source and target vertices of each edge.

In the first example above, we note that for a group $(G, \cdot, 1)$, a functor $F : G \to \mathbf{Set}$ maps the unique object in the category corresponding to G to a set X, and each element in $g \in G$ to an endofunction on X, $g \bullet \_ : X \to X$. Then, functoriality amounts to the conditions

$$\forall g, h \in G.\ F(g \cdot h) = F(g) \circ F(h), \qquad F(1) = 1_X.$$

That is, for all $g, h \in G$ and all $x \in X$,

$$(g \cdot h) \bullet x = g \bullet (h \bullet x), \qquad 1 \bullet x = x.$$

We therefore see F defines a (*left*) *group action* of G on X.

Exercise 45

Verify that functors $F : \mathbb{2}_{\rightrightarrows} \to \mathbf{Set}$ correspond to directed graphs.

SOLUTION. Let $\mathbb{2}_{\rightrightarrows} :=$ $\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] & B \end{tikzcd}$ Then, let any functor

$$F : \mathbb{2}_{\rightrightarrows} \to \mathbf{Set}$$

map A to E, the set of edges of a directed graph, and, B to V, the set of vertices of the directed graph. Also, F maps f to the source function, and g to the target function, where both the source and target functions have domain E and codomain V. Then, it is easy to check F is indeed functorial, and thus we conclude that each such F corresponds to a directed graph, and we are done. □

Example: Lists

Data-type constructors are functors. As a basic example, we consider lists. There is a functor

$$\mathtt{List} : \mathbf{Set} \to \mathbf{Set}$$

which takes a set X to the set of all finite lists (sequences) of elements of X. `List` is functorial: its action on morphisms (*i.e.* functional programs) is given by *maplist*:

$$\frac{f : X \to Y}{\mathtt{List}(f) : \mathtt{List}(X) \to \mathtt{List}(Y)}$$

$$\mathtt{List}(f)[x_1, \ldots, x_n] := [f(x_1), \ldots, f(x_n)].$$

We can upgrade `List` to a functor $\mathtt{MList} : \mathbf{Set} \to \mathbf{Mon}$ by mapping each set X to the monoid $(\mathtt{List}(X), *, \epsilon)$ and $f : X \to Y$ to $\mathtt{List}(f)$, as above. The monoid operation $* : \mathtt{List}(X) \times \mathtt{List}(X) \to \mathtt{List}(X)$ is list concatenation, and ϵ is the empty list. We call $\mathtt{MList}(X)$ the ***free monoid*** over X.

Products as functors

If a category $\mathbf{C}$ has binary products, then there is automatically a functor

$$\_ \times \_ : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$$

which takes each pair (A, B) to the product $A \times B$, and each (f, g) to

$$f \times g := \langle f \circ \pi_1, g \circ \pi_2 \rangle.$$

Functoriality is demonstrated as follows

$$\begin{aligned}
(f \times g) \circ (f' \times g') &= (f \times g) \circ \langle f' \circ \pi_1, g' \circ g' \rangle \\
&= \langle f \circ f' \circ \pi_1, g \circ g' \circ \pi_2 \rangle \\
&= (f \circ f') \times (g \circ g'), \\
1_A \times 1_B &= \langle 1_A \circ \pi_1, 1_B \circ \pi_2 \rangle \\
&= \langle \pi_1 \circ 1_{A \times B}, \pi_2 \circ 1_{A \times B} \rangle \\
&= 1_{A \times B}.
\end{aligned}$$

The category of categories

There is a category **Cat** whose objects are categories, and whose arrows are functors. Identities in **Cat** are given by identity functors:

$$\mathrm{Id}_{\mathcal{C}} : \mathcal{C} \to \mathcal{C} := A \mapsto A,\ f \mapsto f.$$

Composition of functors is defined in the evident way. Note that if $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{E}$, then, for $f : A \to B$ in $\mathcal{C}$,

$$G \circ F(f) := G(F(f)) : G(F(A)) \to G(F(B)).$$

One usually makes some size restriction on the categories, so that **Cat** does not contain itself.

Note that product categories are products in **Cat**. For any pair of categories $\mathcal{C}, \mathcal{D}$, set

$$\mathcal{C} \xleftarrow{\pi_1} \mathcal{C} \times \mathcal{D} \xrightarrow{\pi_2} \mathcal{D}$$

where $\mathcal{C} \times \mathcal{D}$ the product category (defined previously) and π_i's the obvious projection functors. For any pair of functors $\mathcal{C} \xleftarrow{F} \mathcal{E} \xrightarrow{G} \mathcal{D}$, set

$$\langle F, G \rangle : \mathcal{E} \to \mathcal{C} \times \mathcal{D} := A \mapsto (FA, GA), \quad f \mapsto (Ff, Gf).$$

It is easy to check that $\langle F, G \rangle$ is indeed a functor. Moreover, satisfaction of the product diagram and uniqueness are shown exactly as in **Set**.

Contravariance. By definition, the arrow-map of a functor F is *covariant*: it preserves the direction of arrows. That is, if $f : A \to B$, then $Ff : FA \to FB$. A *contravariant* functor G does exactly the opposite: it reverses the direction of arrows. That is, if $f : A \to B$, then $Gf : FB \to FA$. The following is a concise way to express contravariance.

Let $\mathcal{C}, \mathcal{D}$ be categories. A ***contravariant*** functor G from $\mathcal{C}$ to $\mathcal{D}$ is a functor $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. (Equivalently, a functor $G : \mathcal{C} \to \mathcal{D}^{\mathrm{op}}$.)

More explicitly, a contravariant functor G is given by an assignment of:

- An object GA in $\mathcal{D}$ to every object A in $\mathcal{C}$.
- An arrow $Gf : GB \to GA$ in $\mathcal{D}$ to every arrow $f : A \to B$ in $\mathbf{C}$, such that

$$G(g \circ f) = G(f) \circ G(g), \qquad G(1_A) = 1_{GA}.$$

Note that functors of several variables can be covariant in some variables and contravariant in others, *e.g.*

$$F : \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \to \mathcal{E}.$$

Examples of Contravariant Functors

- The contravariant powerset functor, $\mathcal{P}^{\mathrm{op}} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$, is given by:
$$\mathcal{P}^{\mathrm{op}}(X) := \mathcal{P}(X),$$
$$\mathcal{P}(f : X \to Y) : \mathcal{P}(Y) \to \mathcal{P}(X) := T \mapsto \{x \in X \mid f(x) \in T\}.$$
- The dual space functor on vector spaces:
$$(\_)^* : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k := V \mapsto V^*$$

The above are both examples of the following idea: Send an object A into functions from A into some fixed object. For example, the powerset can be written as $\mathcal{P}(X) = 2^X$, where we think of a subset in terms of its characteristic function.

Hom-functors

We now consider some fundamental examples of **Set**-valued functors. Given a category $\mathcal{C}$ and an object A of $\mathcal{C}$, two functors to **Set** can be defined:

- The covariant Hom-functor at A,
$$\mathcal{C}(A, \_) : \mathcal{C} \to \mathbf{Set},$$
which is given by (recall that each $\mathcal{C}(A, B)$ is a set):
$$\mathcal{C}(A, \_)(B) := \mathcal{C}(A, B), \qquad \mathcal{C}(A, \_)(f : B \to C) := g \mapsto f \circ g.$$
We usually write $\mathcal{C}(A, \_)(f)$ as $\mathcal{C}(A, f)$. Functoriality reduces directly to the basic category axioms: associativity of composition and the unit laws for the identity.
- There is also a contravariant Hom-functor,
$$\mathcal{C}(A, \_) : \mathcal{C}^{\mathrm{op}} \to \mathbf{Set},$$
given by:
$$\mathcal{C}(A, \_)(B) := \mathcal{C}(B, A), \qquad \mathcal{C}(A, \_)(h : C \to B) := g \mapsto g \circ h.$$

Generalizing both of the above, we obtain a ***bivariant*** Hom-functor,
$$\mathcal{C}(\_, \_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}.$$

Exercise 47

Spell out the definition of $\mathcal{C}(\_, \_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$. Verify carefully that it is a functor.

SOLUTION. We define $\mathcal{C}(\_, \_) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathbf{Set}$ as follows:

For all (A, B) in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, $\mathcal{C}(\_, \_)(A, B) := \mathcal{C}(A, B)$, and
for all $f : A' \to A$ and $g : B \to B'$, $\mathcal{C}(\_, \_)(f, g) = h \mapsto g \circ h \circ f$, where $h \in \mathcal{C}(A, B)$.

Now, we show the above functor does satisfy the functoriality conditions. Indeed, for all $(A, B), (A', B'), (A'', B'')$ in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, if $f : A' \to A, f' : A'' \to A', g : B \to B', g' : B' \to B''$, then $\mathcal{C}(\_, \_)(f', g') \circ \mathcal{C}(\_, \_)(f, g) = (h' \mapsto g' \circ h' \circ f') \circ (h \mapsto g \circ h \circ f) = h \mapsto g' \circ (g \circ h \circ f) \circ f' = h \mapsto (g' \circ g) \circ h \circ (f \circ f') = \mathcal{C}(\_, \_)(f \circ f', g' \circ g)$. And, for any (A, B) in $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$, we have $\mathcal{C}(\_, \_)(1_{(A,B)}) = \mathcal{C}(\_, \_)(1_A, 1_B) = h \mapsto 1_B \circ h \circ 1_B = h \mapsto h = 1_{\mathcal{C}(A,B)}$. And this establishes that $\mathcal{C}(\_, \_)$ as defined above is indeed a functor. $\square$

Properties of Functors. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be:

- ***faithful*** if each map $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is injective;
- ***full*** if each map $F_{A,B} : \mathcal{C}(A, B) \to \mathcal{D}(FA, FB)$ is surjective;
- an ***embedding*** if F is full, faithful, and injective on objects;
- an ***equivalence*** if F is full, faithful, and *essentially surjective, i.e.* for every object B of $\mathcal{D}$, there is an object A of $\mathcal{C}$ such that $F(A) \cong B$;
- an ***isomorphism*** if there is a functor $G : \mathcal{D} \to \mathcal{C}$ such that

$$G \circ F = 1_{\mathcal{C}}, \qquad F \circ G = 1_{\mathcal{D}}.$$

We say categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic, $\mathcal{C} \cong \mathcal{D}$, if there is an isomorphism between them. This is just the usual notion of isomorphism applied to **Cat**.

Some examples of functors with various properties are as follows:

- The forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ is faithful, but not full. Note not all functions $f : M \to N$ yield an arrow $f : (M, \cdot, 1) \to (N, \cdot, 1)$. Similar properties hold for other forgetful functors.
- The free monoid functor $\mathtt{MList} : \mathbf{Set} \to \mathbf{Mon}$ is faithful, but not full.
- The product functor $- \times - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is generally neither faithful nor full.
- There is an equivalence between $\mathbf{FDVect}_k$, the category of finite dimensional vector spaces over the field k, and $\mathbf{Mat}_k$, the category of matrices with entries in k. These categories are very far from being isomorphic.

Preservation and Reflection

Let P be a property of arrows. We say a functor $F : \mathcal{C} \to \mathcal{D}$ *preserves* P if whenever f satisfies P, so does $F(f)$. We say F *reflects* P if whenever $F(f)$ satisfies P, so does f.

Exercise 49

Prove the following:

(1) All functors preserve isomorphisms, split monics, and split epics.
(2) Faithful functors reflect monics and epics.
(3) Full and faithful functors reflect isomorphisms.
(4) Equivalences preserve monics and epics.
(5) The forgetful functor $U : \mathbf{Mon} \to \mathbf{Set}$ preserves products.

SOLUTION.

(1) (Isomorphism) Suppose $f : A \to B$ in $\mathcal{C}$ is an isomorphism. Then, $F(f) \circ F(f^{-1}) = F(f \circ f^{-1}) = F(1_B) = 1_{FB}$. And, $F(f^{-1}) \circ F(f) = F(f^{-1} \circ f) = F(1_A) = 1_{FA}$. Thus, $F(f)$ is an isomorphism, and we are done.
(Split monic) Suppose $f : A \to B$ in $\mathcal{C}$ is split monic. Then, f has a left inverse $g : B \to A$ such that $g \circ f = 1_A$. Therefore, $F(g) \circ F(f) = F(g \circ f) = F(1_A) = 1_{FA}$, which implies $F(f)$ has a left inverse, from which we conclude $F(f)$ is split monic, and we are done.
(Split epic) Suppose $f : A \to B$ in $\mathcal{C}$ is split epic. Then, f has a right inverse $g : B \to A$ such that $f \circ g = 1_B$. Therefore, $F(f) \circ F(g) = F(f \circ g) = F(1_B) = 1_{FB}$, which implies $F(f)$ has a right inverse, from which we conclude $F(f)$ is split epic, and we are done.

(2) (Monic) Suppose $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor, and let $f : A \to B$ in $\mathcal{C}$ such that $F(f)$ is monic. Assume $g, h : C \to A$ such that $f \circ g = f \circ h$. Then, $F(f \circ g) = F(f \circ h)$, which implies $F(f) \circ F(g) = F(f) \circ F(h)$, which implies $F(g) = F(h)$, since $F(f)$ is left-cancellative, and since F is faithful, we have $g = h$, showing f is monic. Hence, we conclude faithful functors preserve monics.
(Epic) Suppose $F : \mathcal{C} \to \mathcal{D}$ is a faithful functor, and let $f : A \to B$ in $\mathcal{C}$ such that $F(f)$ is epic. Assume $g, h : B \to C$ such that $g \circ f = h \circ f$. Then, $F(g \circ f) = F(h \circ f)$, which implies $F(g) \circ F(f) = F(h) \circ F(f)$, which implies $F(g) = F(h)$, since $F(f)$ is right-cancellative, and since F is faithful, we have $g = h$, showing f is epic. Hence, we conclude faithful functors preserve epics.

(3) Suppose $F : \mathcal{C} \to \mathcal{D}$ is a full and faithful functor, and let $f : A \to B$ in $\mathcal{C}$ such that $F(f)$ is an isomorphism. Since F is full, there exists some $g : B \to A$ such that $F(g) = F(f)^{-1}$. Therefore, $F(f \circ g) = F(f) \circ F(g) = F(f) \circ F(f)^{-1} = 1_{FB} = F(1_B)$, which implies $f \circ g = 1_B$, since F is faithful. Similarly, it is easy to check $F(g \circ f) = F(1_A)$, which implies $g \circ f = 1_A$. Hence, f is an isomorphism, and we are done.

(4) (Monic) Suppose $F : \mathcal{C} \to \mathcal{D}$ is an equivalence. Then, F is full, faithful, and essentially surjective. Let $f : A \to B$ in $\mathcal{C}$ be monic. Let $g', h' : C' \to FA$ such that $F(f) \circ g' = F(f) \circ h'$. That is, the following diagram commutes:

$$\begin{tikzcd} C' \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] & FA \arrow[r, "F(f)"] & FB \end{tikzcd}$$

We claim $F(f)$ is monic, *i.e.* $F(f)$ is left-cancellative. To that end, since F is essentially surjective, there exists some C_0 in $\mathcal{C}$ such that $F(C_0) \cong C'$. In other words, there exists an isomorphism $k : F(C_0) \xrightarrow{\sim} C'$. In addition, since F is full, there exist $g, h : C_0 \to A$ in $\mathcal{C}$ such that $F(g) = g' \circ k$ and $F(h) = h' \circ k$. That is, the following diagram commutes:

$$\begin{tikzcd} F(C_0) \arrow[r, shift left, "F(g)"] \arrow[r, shift right, "F(h)"'] & FA \arrow[r, "F(f)"] & FB \end{tikzcd}$$

Therefore, $F(f) \circ F(g) = F(f) \circ F(h)$, which implies $F(f \circ g) = F(f \circ h)$, which implies $f \circ g = f \circ h$ (since F is faithful), which implies $g = h$, since f is monic. Thus, $g' = g' \circ 1_{C'} = g' \circ (k \circ k^{-1}) = (g' \circ k) \circ k^{-1} = F(g) \circ k^{-1} = F(h) \circ k^{-1} = (h' \circ k) \circ k^{-1} = h' \circ (k \circ k^{-1}) = h' \circ 1_{C'} = h'$, which shows $F(f)$ is indeed left-cancellative, and we are done.
(Epic) Suppose $F : \mathcal{C} \to \mathcal{D}$ is an equivalence. Then, F is full, faithful, and essentially surjective. Let $f : A \to B$ in $\mathcal{C}$ be epic. We claim $F(f)$ is epic, *i.e.* $F(f)$ is right-cancellative. To that end, let $g', h' : FB \to C'$ such that $g' \circ F(f) = h' \circ F(f)$, *i.e* the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "F(f)"] & FB \arrow[r, shift left, "g'"] \arrow[r, shift right, "h'"'] & C' \end{tikzcd}$$

Since F is essentially surjective, there exists some C_0 in $\mathcal{C}$ such that $C' \cong F(C_0)$. In other words, there exists an isomorphism $k : C' \to F(C_0)$. And, since F is full, there exist $g, h : B \to C_0$ such that $F(g) = k \circ g'$ and

$F(h) = k \circ h'$. That is, the following diagram commutes:

$$FA \xrightarrow{F(f)} FB \overset{F(g)}{\underset{F(h)}{\rightrightarrows}} F(C_0)$$

Therefore, $F(g) \circ F(f) = F(h) \circ F(f)$, which implies $F(g \circ f) = F(h \circ f)$, which implies $g \circ f = h \circ f$ (since F is faithful), which implies $g = h$, since f is epic. Thus, $g' = 1_{C'} \circ g' = (k^{-1} \circ k) \circ g' = k^{-1} \circ (k \circ g') = k^{-1} \circ F(g) = k^{-1} \circ F(h) = k^{-1} \circ (k \circ h') = (k^{-1} \circ k) \circ h' = 1_{C'} \circ h' = h'$, which shows $F(f)$ is indeed right-cancellative, and we are done.

(5)

$\square$

Exercise 50

Show the following:

(1) Functors do not, in general, reflect monics or epics.
(2) Faithful functors do not, in general, reflect isomorphisms.
(3) Full and faithful functors do not, in general, preserve monics or epics.

Solution. $\square$

Exercises.

(1) Consider the category $\mathbf{FDVect}_{\mathbb{R}}$ of finite dimensional vector spaces over $\mathbb{R}$, and $\mathbf{Mat}_{\mathbb{R}}$ of matrices over $\mathbb{R}$. Concretely, $\mathbf{Mat}_{\mathbb{R}}$ is defined as follows:

$$Ob(\mathbf{Mat}_{\mathbb{R}}) := \mathbb{N},$$
$$\mathbf{Mat}_{\mathbb{R}}(n, m) := \{M \mid M \text{ is an } n \times m \text{ matrix with entries in } \mathbb{R}\}.$$

Thus, objects are natural numbers, and arrows $n \to m$ are $n \times m$ real matrices. Composition is matrix multiplication, and the identity on n is the $n \times n$ identity matrix.
Now, let $F : \mathbf{Mat}_{\mathbb{R}} \to \mathbf{FDVect}_{\mathbb{R}}$ be the functor taking each n to the vector space $\mathbb{R}^n$ and each $M : n \to m$ to the linear function

$$FM : \mathbb{R}^n \to \mathbb{R}^m := (x_1, \ldots, x_n) \mapsto [x_1, \ldots, x_n]M$$

with the $1 \times m$ matrix $[x_1, \ldots, x_n]M$ considered as a vector in $\mathbb{R}^m$. Show that F is full, faithful, and essentially surjective, and hence, $\mathbf{FDVect}_{\mathbb{R}}$ and $\mathbf{Mat}_{\mathbb{R}}$ are equivalent categories. Are they isomorphic?

(2) Let $\mathcal{C}$ be a category with binary products such that, for each pair of objects A, B,

$$\mathcal{C}(A, B) \neq \varnothing \qquad (*)$$

Show that the product functor $F : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is faithful.
Would F still be faithful in the absence of condition $(*)$?

Solution.

(1)
(2)

$\square$

4. Natural Transformations

Morphisms between functors are *natural transformations*, just as functors are morphisms between categories.

Basics. Let us define natural transformations.

Let $F, G : \mathcal{C} \to \mathcal{D}$ be functors. A ***natural transformation***

$$t : F \to G$$

is a family of morphisms in $\mathcal{D}$ indexed by objects A of $\mathcal{C}$,

$$\{t_A : FA \to GA\}_{A \in Ob(\mathcal{C})}$$

such that for all $f : A \to B$, the following diagram commutes:

$$\begin{tikzcd} FA \arrow[r, "Ff"] \arrow[d, "t_A"'] & FB \arrow[d, "t_B"] \\ GA \arrow[r, "Gf"'] & GB \end{tikzcd}$$

This condition is known as *naturality.*

If each t_A is an isomorphism, we say t is a ***natural isomorphism***:

$$t : F \xrightarrow{\cong} G.$$

Examples of natural transformations:

- Let Id be the identity functor on **Set**, and $\times \circ \langle \mathrm{Id}, \mathrm{Id} \rangle$ be the functor taking each set X to $X \times X$ and each function f to $f \times f$. Then, there is a natural transformation $\Delta : \mathrm{Id} \to \times \circ \langle \mathrm{Id}, \mathrm{Id} \rangle$ given by:

  $$\Delta_X : X \to X \times X := x \mapsto (x, x).$$

  Naturality amounts to asserting that, for any function $f : X \to Y$, the following diagram commutes:

  $$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\Delta_X"'] & Y \arrow[d, "\Delta_Y"] \\ X \times X \arrow[r, "f \times f"'] & Y \times Y \end{tikzcd}$$

  We call Δ the *diagonal* transformation on **Set**. In fact, it is the *only* natural transformation between these functors.
- The diagonal transformation can be defined for any category $\mathcal{C}$ with binary products by setting, for each object A in $\mathcal{C}$,

  $$\Delta_A : A \to A \times A := \langle 1_A, 1_A \rangle.$$

  Projections also yield natural transformations. For example, the arrows

  $$\pi_{1(A,B)} : A \times B \to A$$

  specify a natural transformation $\pi_1 : - \times - \to \pi_1$. Note that $- \times -$, $\pi_1 : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ are the functors for product and first projection, respectively.
- Let $\mathcal{C}$ be a category with terminal object T, and let $K_T : \mathcal{C} \to \mathcal{C}$ be the functor mapping all objects to T and all arrows to 1_T. Then, the canonical arrows

  $$\tau_A : A \to T$$

  specify a natural transformation $\tau : \mathrm{Id} \to K_T$ (where Id) is the identity functor on $\mathcal{C}$.

- Recall the functor $\texttt{List} : \mathbf{Set} \to \mathbf{Set}$ that takes a set X to the set of finite lists with elements in X. We can define the following natural transformations,

$$\begin{aligned}\texttt{reverse} &: \texttt{List} \to \texttt{List},\\ \texttt{unit} &: \mathrm{Id} \to \texttt{List},\\ \texttt{flatten} &: \texttt{List} \circ \texttt{List} \to \texttt{List},\end{aligned}$$

  by setting, for each set X,

$$\begin{aligned}\texttt{reverse}_X &: \texttt{List}(X) \to \texttt{List}(X) := [x_1, \ldots, x_n] \mapsto [x_n, \ldots, x_1]\\ \texttt{unit}_X &: X \to \texttt{List}(X) := x \mapsto [x]\\ \texttt{flatten}_X &: \texttt{List}(\texttt{List}(X)) \to \texttt{List}(X)\\ &:= [[x_1^1, \ldots, x_{n_1}^1], \ldots, [x_1^k, \ldots, x_{n_k}^k] \mapsto [x_1^1, \ldots \ldots, x_{n_k}^k].\end{aligned}$$

- Consider the functor $P := \times \circ \langle U, U \rangle$ with $U : \mathbf{Mon} \to \mathbf{Set}$ being the forgetful functor. That is,

$$P : \mathbf{Mon} \to \mathbf{Set} := (M, \cdot, 1) \mapsto M \times M,\ f \mapsto f \times f.$$

  Then, the monoid operation yields a natural transformation $t : P \to U$ defined by:

$$t_{(M,\cdot,1)} : M \times M \to M := (m, m') \mapsto m \cdot m'.$$

  Naturality corresponds to asserting that, for any $f : (M, \cdot, 1) \to (N, \cdot, 1)$, the following diagram commutes:

$$\begin{tikzcd} M \times M \arrow[r, "f \times f"] \arrow[d, "t_M"'] & N \times N \arrow[d, "t_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

  That is, for any $m_1, m_2 \in M$, $f(m_1) \cdot f(m_2) = f(m_1 \cdot m_2)$.
- If V is a finite dimensional vector space, then V is isomorphic to both its first dual V^* and to its second dual V^{**}.
  However, while it is naturally isomorphic to its second dual, there is no natural isomorphism to the first dual. This was the original example that motivated Samuel Eilenberg and Saunders Mac Lane to define the concept of natural transformation; in this case, naturality captures *basis independence*.

Exercise 52

Verify naturality of diagonal transformations, projections, and terminals, for a category $\mathcal{C}$ with finite products.

Solution. Suppose $\mathcal{C}$ is a category with finite products.

(Naturality of diagonal transformation) We show that for all objects A, B in $\mathcal{C}$ and all arrows $f : A \to B$, the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ A \times A \arrow[r, "f \times f"'] & B \times B \end{tikzcd}$$

Note that $\Delta_A : A \to A \times A := \langle 1_A, 1_A \rangle$. Now, $\Delta_B \circ f = \langle 1_B, 1_B \rangle \circ f = \langle 1_B \circ f, 1_B \circ f \rangle = \langle f, f \rangle$. And, $(f \times f) \circ \Delta_A = (f \times f) \circ \langle 1_A, 1_A \rangle = \langle f \circ 1_A, f \circ 1_A \rangle = \langle f, f \rangle$, which shows $\Delta_B \circ f = (f \times f) \circ \Delta_A$, thus proving the above diagram commutes.

(Naturality of projection) We claim that for all $f : A \to B$ and $g : C \to D$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{tikzcd} A \times C \arrow[r, "f \times g"] \arrow[d, "\pi_1"'] & B \times D \arrow[d, "\pi_1"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

Indeed, $\pi_1 \circ (f \times g) = \pi_1 \circ \langle f \circ \pi_1, g \circ \pi_2 \rangle = f \circ \pi_1$, thus establishing our claim.

(Naturality of terminal projection) Let T be a terminal object in $\mathcal{C}$. The fact that the canonical arrows $\tau_A : A \to T$ specify a natural transformation is demonstrated by showing the following diagram commutes:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\tau_A"'] & B \arrow[d, "\tau_B"] \\ T \arrow[r, "1_T"'] & T \end{tikzcd}$$

Indeed, note $\tau_B \circ f : A \to T$ and $1_T \circ \tau_A : A \to T$, and since T is a terminal object, we must have $\tau_B \circ f = 1_T \circ \tau_A$, thereby proving the above diagram commutes. □

Exercise 53

Prove that the diagonal is the only natural transformation $\mathrm{Id} \to \times \circ \langle \mathrm{Id}, \mathrm{Id} \rangle$ on **Set**. Similarly, prove that the first projection is the only natural transformation $\times \to \pi_1$ on **Set**.

SOLUTION. □