MS&E 260

Introduction to Operations Management

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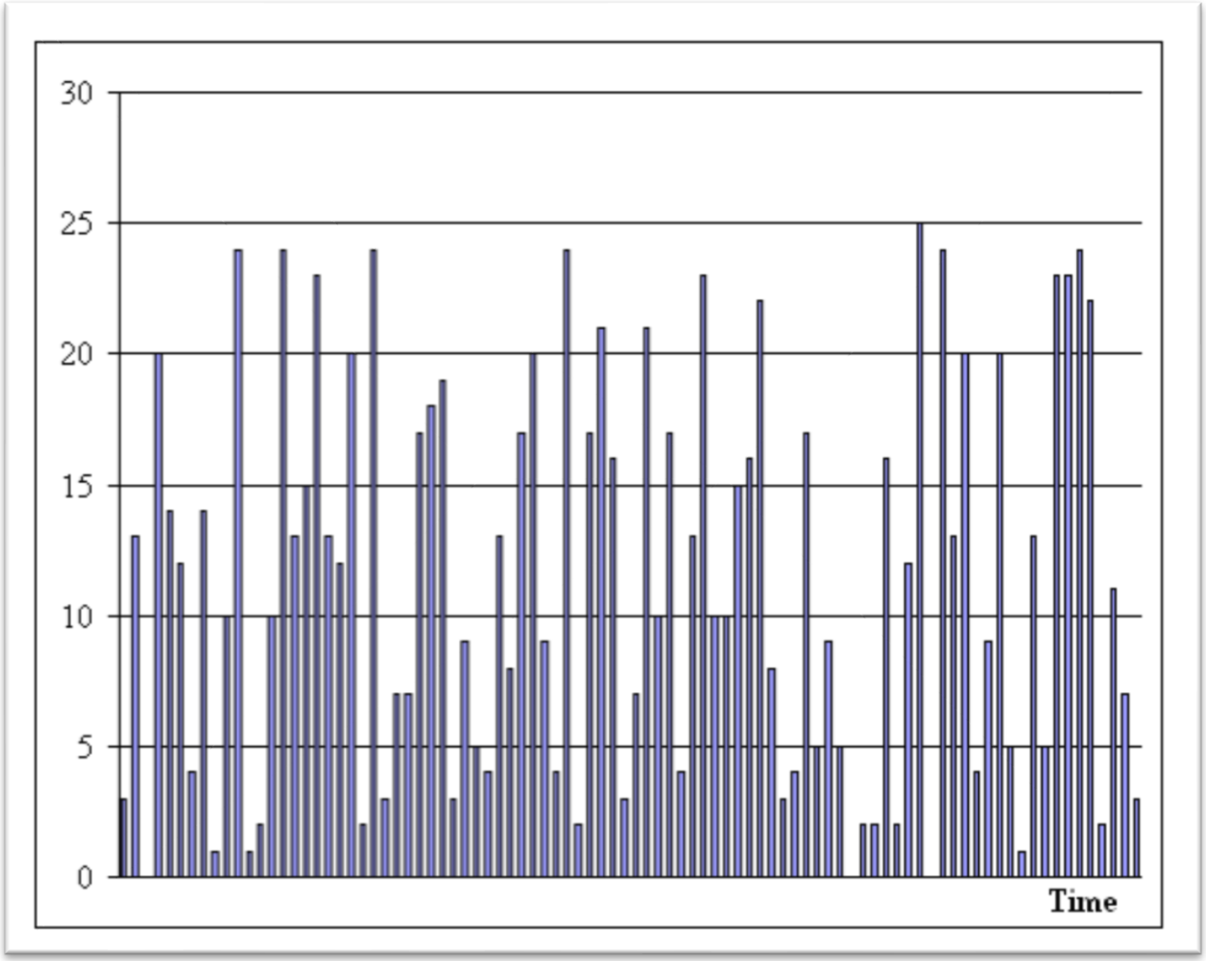
Inventory Management with Uncertain Demand

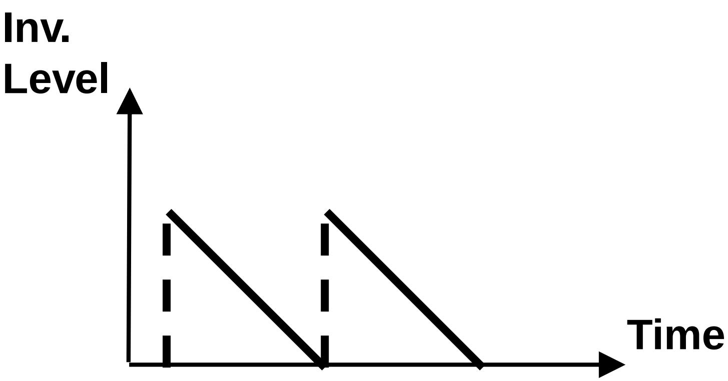
A Typical Demand Sequence



- In many real-world applications, the constant demand rate assumption does not hold. So, we need to consider stochastic inventory control models.

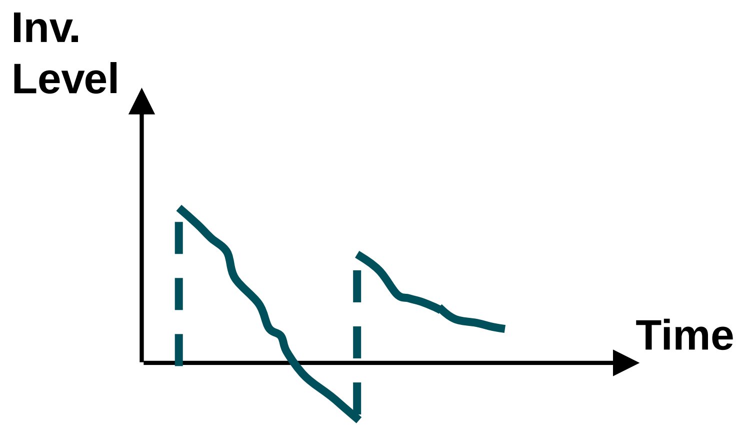
Deterministic vs. Stochastic Inventory

- EOQ models assumed deterministic demand and lead times



- **Objective**: To minimize the cost or to maximize the profit

- Now, we relax those assumptions



- **Objective**: To minimize the **expected** cost or to maximize the **expected** profit

Inventory Control Subject to Uncertain Demand

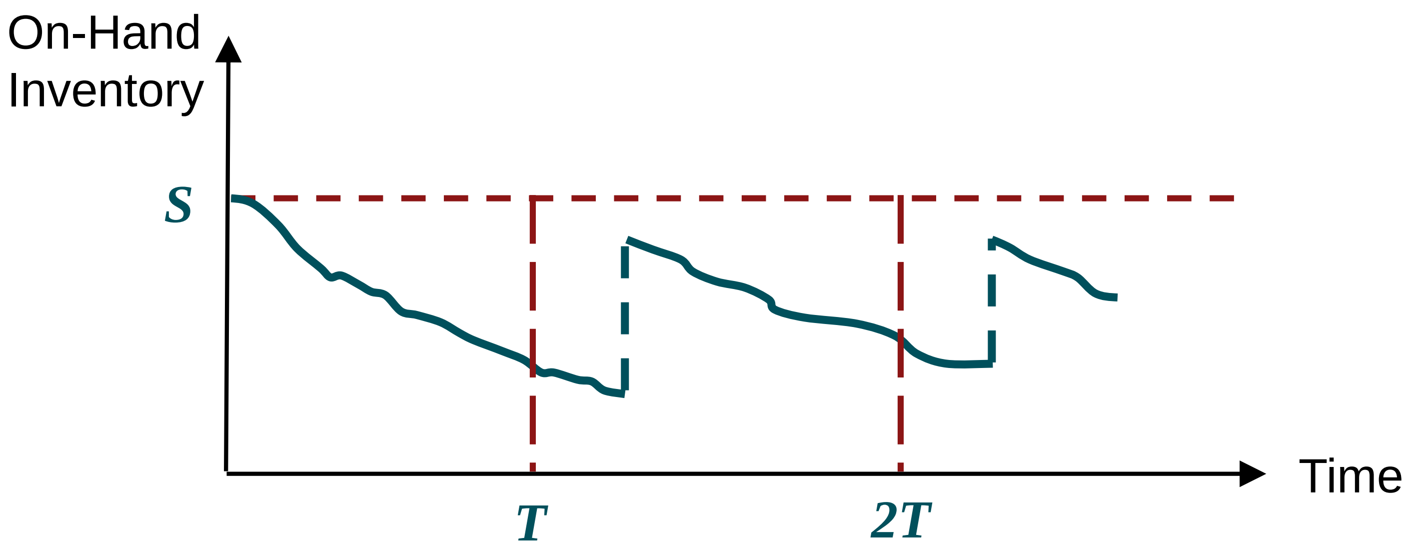
- Two types of inventory control models
  - **Fixed time period** - Periodic review
    - One period (Newsvendor model)
    - Multiple periods
  - **Fixed order quantity** - Continuous review
    - (Q,R) models

Types of Inventory Control Policies: Fixed Time Period

- Fixed time period policies
  - The time between orders is constant, but the quantity ordered each time varies with demand and the current level of inventory
  - Inventory is reviewed and replenished in given time intervals, such as a week or month (i.e., review period)
  - Depending on the current inventory level an order size is determined to (possibly) increase the inventory level up-to a prespecified level (i.e., order-up-to level)
  - **Periodic** review policy

Periodic Review

- (T, S) System
  - Every T periods order up to S units

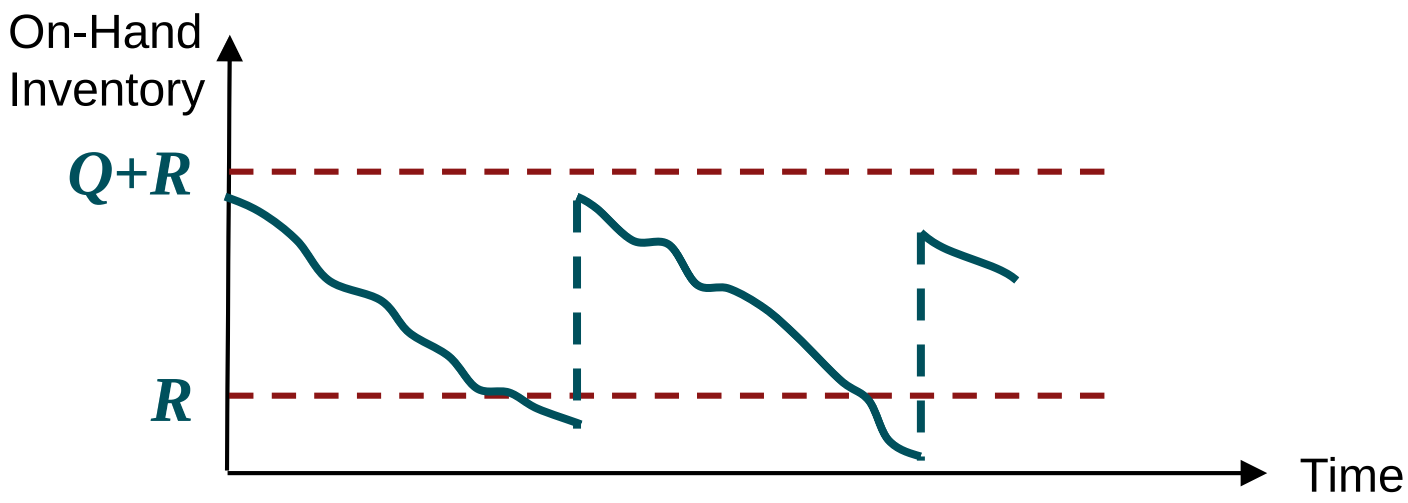


Types of Inventory Control Policies: Fixed Order Quantity

- Fixed order quantity policies
  - The order quantity is always the same but the time between the orders will vary depending on demand and the current inventory levels
  - Inventory levels are continuously monitored and an order is placed whenever the inventory level drops below a prespecified reorder point.
  - **Continuous** review policy

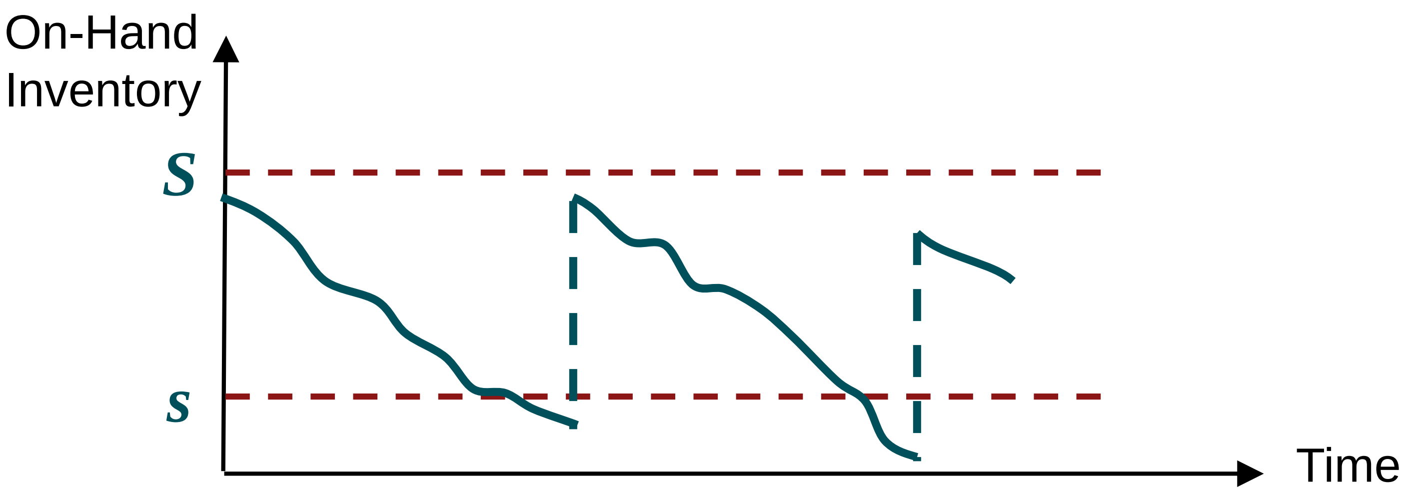
Continuous Review

- (Q, R) System
  - When the inventory level reaches R (order point), order exactly Q units (order quantity)



Continuous/Periodic Review Policy

- (s, S) System
  - (continuous review) When inventory level reaches s (order point), order up to S (order-up-to level)
  - Recall Periodic Review (T, S): Review your system every T periods. If inventory level is below s (order point), order up to S (order-up-to level); otherwise, do not order

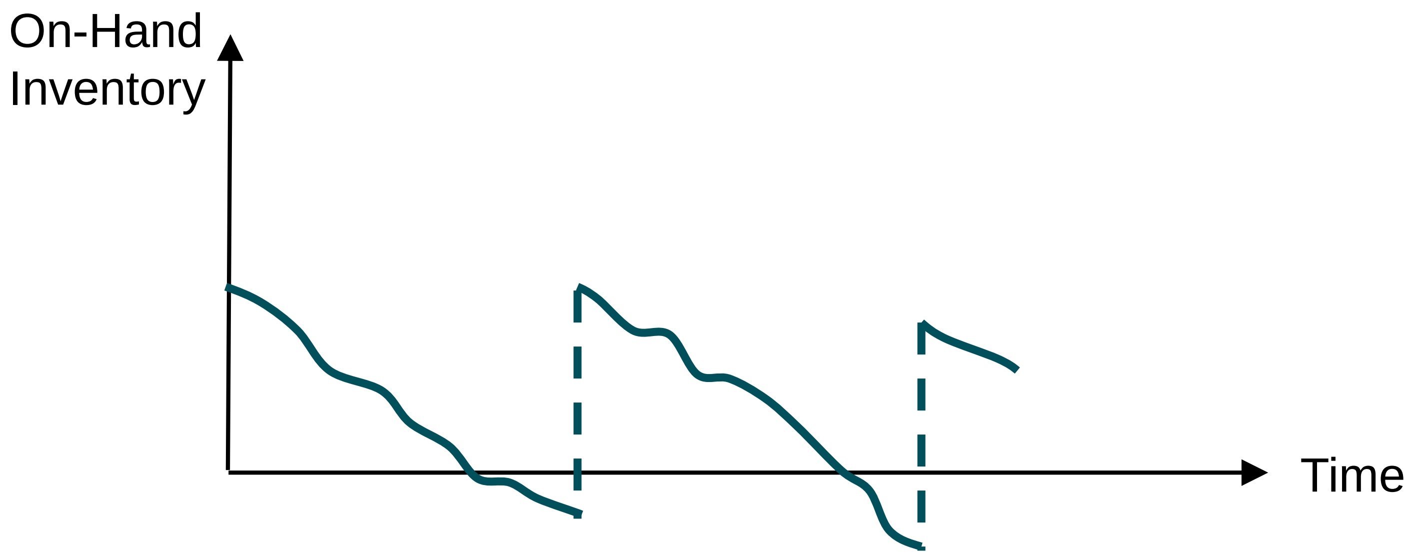


Continuous vs. Periodic Preferences

- A Continuous Review System is preferred:
  - When/Why?
  - Examples?

- A Periodic Review System is preferred:
  - When/Why?
  - Examples?

Inventory Systems with Stockouts



- Stockout costs: Goodwill, lost sales revenue, etc.
- Fill-rate constraint: Fraction of demand met from stock
- Stockout probability constraint: Fraction of order cycles in which stockout does not occur

Factors Influencing Safety Stock Policy

Low safety stock

- Zero stockout cost
- Zero lead time
- Constant lead time
- Deterministic demand

High safety stock

- High stockout cost
- Long lead time
- Variable lead time
- Highly variable demand

Inventory Control Subject to Uncertain Demand

- Inventory Systems with Uncertain (Stochastic) Demand
  - **Newsvendor Model (single period)**
  - Continuous Review (Q,R) Model (multiple periods)
    - Per-Unit Backorder Penalty
    - Service Level (Fill Rate)

Newsvendor Model

- Motivation:
  - At the start of each day, a newspaper vendor must decide on the number of papers to purchase and sell
  - Daily sales cannot be predicted exactly, and are represented by the random variable, D
- Relevant costs:
  - c_o = unit cost of overage (not enough demand)
    - Cost of having positive inventory left over at the end of period
  - c_u = unit cost of underage (too much demand)
    - Cost of unsatisfied demand
- Example: Fashion
  - Retail: Stanford Shopping Center
  - Overage: Gilroy Outlet Mall

Newsvendor Model

- Need to know:
  - Demand cdf: normal, uniform, discrete, etc.
  - Overage/underage costs

$$G(Q, D) = c_o \max(0, Q - D) + c_u \max(0, D - Q)$$

with expected cost function:

$$G(Q) = E[G(Q, D)]$$

Newsvendor Model: Finding the Optimal Order Quantity

$G(Q, D)$: total overage + underage cost (if Q is ordered and demand is D)

$$G(Q, D) := \begin{cases} c_u(D - Q) & if\ D \geq Q \\ c_o(Q - D) & if\ D \leq Q \end{cases}$$

$G(Q)$ = expected overage + underage cost if Q is ordered

$$G(Q) = E[G(Q, D)] = \int_0^\infty G(Q, x) f(x) dx$$

$$= \int_0^Q c_o(Q - x) f(x) dx + \int_Q^\infty c_u(x - Q) f(x) dx \Rightarrow$$

$$\boxed{F(Q) = \frac{c_u}{c_u + c_o}}$$

Critical Ratio!

Intuition

$$\boxed{c_o P(D \leq Q^*) = c_u P(D \geq Q^*)}$$

Expected overage costs:

$$c_o P(D \leq Q^*)$$

Expected underage costs:

$$c_u P(D \geq Q^*)$$

Q^*: order amount where expected overage costs = expected underage costs

Newsvendor Model Example: Hotel Room Reservation

- An organization needs to reserve rooms for a conference. Rooms can be reserved at a cost of $50 per room. Demand for rooms is normally distributed with mean 5,000 and standard deviation 2,000. If the number of rooms required exceeds the number of rooms reserved, extra rooms will have to be found at neighboring hotels at a cost of $80 per room. Inconvenience of staying at another hotel is estimated at $10.
- How many rooms should be reserved to minimize the expected cost?

Newsvendor Model Example: Hotel Room Reservation

- D = number of rooms actually required by demand
- Q = number of rooms reserved
- What are c_o and c_u?
- What is the critical ratio?
- What is Q^*?

- $c_o = booking\ price = \$50$
- $c_u = alternative\ price - reservation\ price + inconvenience\ value = \$80 - \$50 + \$10 = \$40$
- $critical\ ratio = F(Q) = \frac{c_u}{c_u + c_o} = \frac{40}{40+50} = \frac{4}{9}$
  - $\Rightarrow$ Probability of satisfying all demand during the time period if Q^* units are purchased at the start of the period is 0.44

Newsvendor Model Example: Hotel Room Reservation

- Recall: $D \sim N(5000{,}2000)$
- We have lookup tables for Standard Normal distribution $(\mu = 0, \sigma = 1)$
  - $\Rightarrow$ Convert to Standard Normal

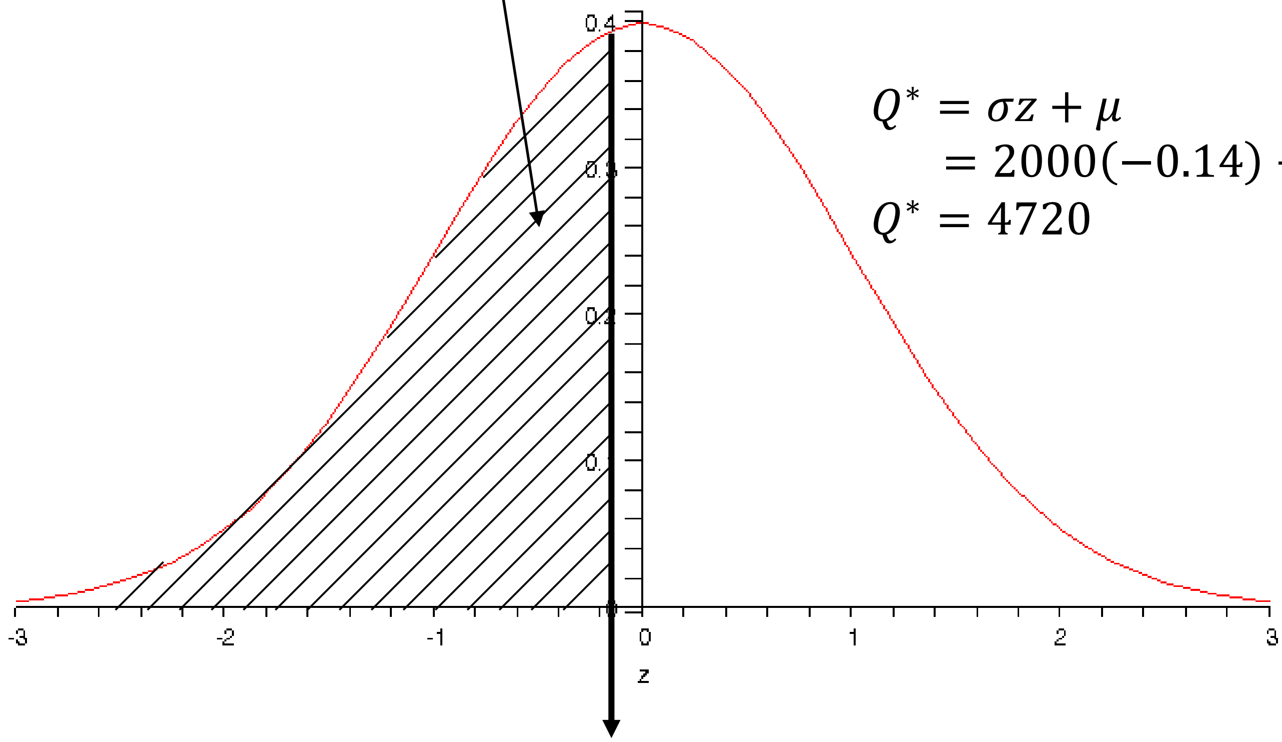
$$F(Q) = P(D \leq Q) = \frac{4}{9} = 0.4444$$

$$P(D \leq Q) = P(D - \mu \leq Q - \mu) = P\left(\frac{D - \mu}{\sigma} \leq \frac{Q - \mu}{\sigma}\right) = P(Z \leq z) = \Phi(z)$$

- Find z from the lookup table [alternatively, use NORM.INV(*) function in Excel]
- $Q = \sigma z + \mu$

Newsvendor Model Example: Hotel Room Reservation

$P(Z \leq z) = \Phi(z) = 0.4444$

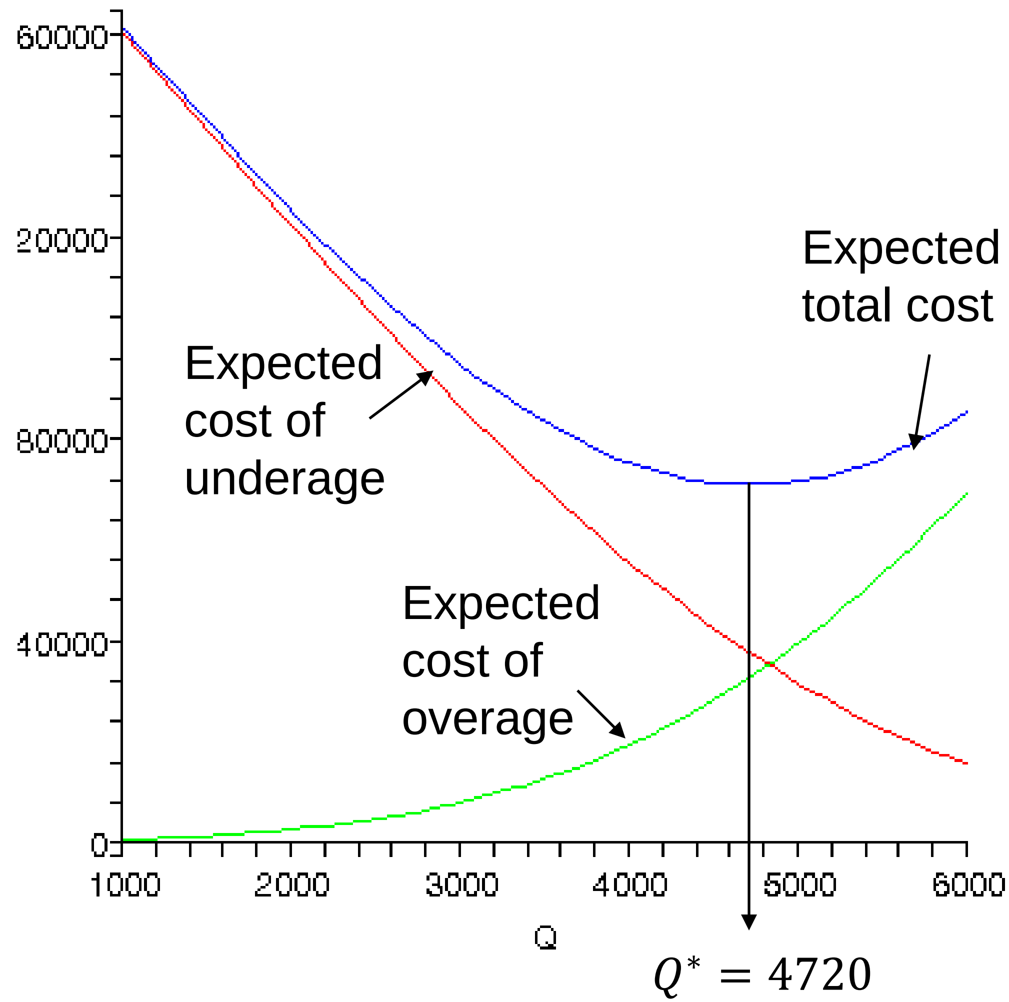


$$Q^* = \sigma z + \mu$$
$$= 2000(-0.14) + 5000$$
$$Q^* = 4720$$

z = - 0.14

Newsvendor Model Example: Hotel Room Reservation

Why is Q*<5000, i.e., less than the expected demand?



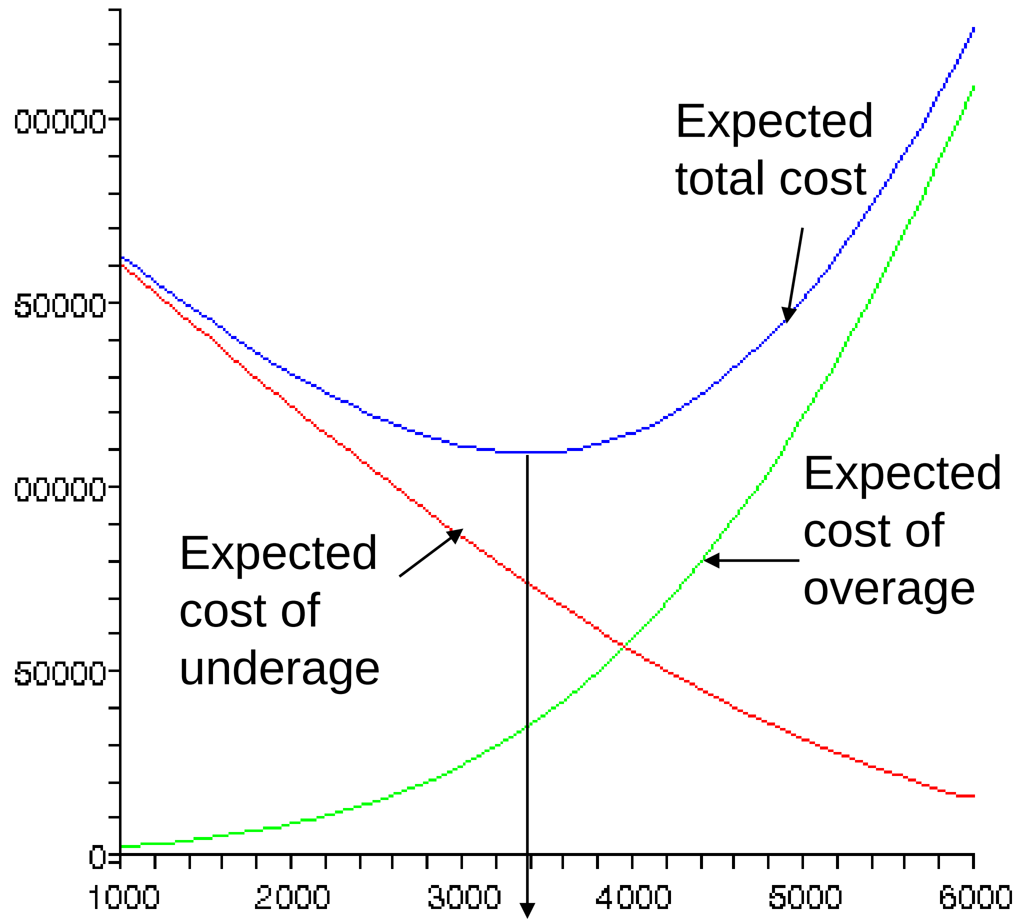
Newsvendor Model Example: Hotel Room Reservation

- What if $c_o = \$150$?
  - ($c_o = \$50$, $c_u = \$40$ in the original problem)

Newsvendor Model Example: Hotel Room Reservation

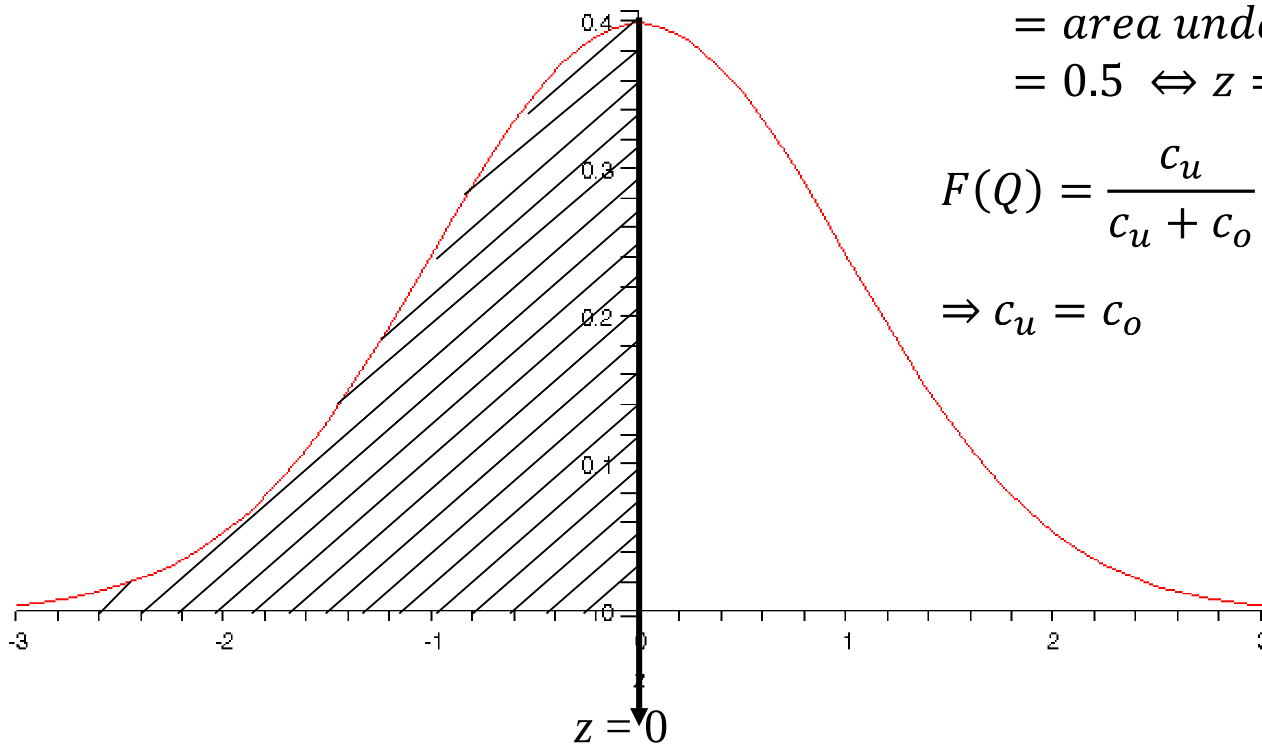
$$F(Q) = \frac{c_u}{c_u + c_o} = \frac{40}{40 + 150}$$
$$= 0.21 \Rightarrow z = -0.805$$

$$Q^* = \sigma z + \mu =$$
$$2000(-0.805) + 5000 =$$
$$3390$$



$Q^* = 3390$

When Do We Have $Q^* =$ Expected Demand?



$$P(Z \le z) = \Phi(z)$$
$$= area\ under\ the\ curve$$
$$= 0.5 \iff z = 0$$

$$F(Q) = \frac{c_u}{c_u + c_o} = 0.5$$

$$\Rightarrow c_u = c_o$$

Optimal Quantity vs. Expected Demand

$$c_u > c_o \Rightarrow F(Q) = \frac{c_u}{c_u + c_o} > 0.5 \Rightarrow z > 0$$

$$\Rightarrow Q^* > \mu$$

- If shortages are more costly, order more than expected demand

$$c_u < c_o \Rightarrow F(Q) = \frac{c_u}{c_u + c_o} < 0.5 \Rightarrow z < 0$$

$$\Rightarrow Q^* < \mu$$

- If excess inventory is more costly, order less than expected demand

$$c_u = c_o \Rightarrow F(Q) = \frac{c_u}{c_u + c_o} = 0.5 \Rightarrow z = 0$$

$$\Rightarrow Q^* = \mu$$

- If shortages and excess inventory cost the same, order expected demand

Note: Assuming the demand distribution is symmetric around its mean

The Impact of Standard Deviation

- What happens to the optimal quantity as the standard deviation increases? (Assuming the demand distribution is symmetric around its mean)

$$c_u > c_o \Rightarrow z > 0 \Rightarrow Q^* = \sigma z + \mu \text{ increases in } \sigma$$

- If shortages are more costly, Q^* increases due to increasing standard deviation term

$$c_u < c_o \Rightarrow z < 0 \Rightarrow Q^* = \sigma z + \mu \text{ decreases in } \sigma$$

- If excess inventory is more costly, Q^* decreases due to decreasing standard deviation term

$$c_u = c_o \Rightarrow z = 0 \Rightarrow Q^* = \mu \text{ does not change in } \sigma$$

- If shortages and excess inventory cost the same, order expected demand regardless of standard deviation (since $z = 0$)

Newsvendor Model Example: Fashion Bags

- The buyer for What-a-Markup Fashion Bags must decide on the quantity of a premium woman's handbag to procure in Italy for the following Christmas Season. The unit cost of the handbag to the store is **$28.50** and the handbag will sell for **$150**. A discount firm purchases any handbags not sold by the end of the season for **$20**. In addition, the store accountants estimate that there is cost of **$0.40 for each dollar tied up in inventory** at the end of the season (after all sales have been made).

Newsvendor Model Example: Fashion Bags

- Given Input:
  - Unit cost $c = \$28.50$
  - Selling price $p = \$150$
  - Salvage value $s = \$20$
  - Cost of inventory $= \$0.40$ for each dollar tied up in inventory at the end of the season
- Computed input:
  - Holding cost per bag $h = (0.40)(28.50) = \$11.40$
  - $c_u = p - c = \$150.00 - \$28.50 = \$121.50$
  - $c_o = c - s + h = \$28.50 - \$20.00 + (\$0.40 \times \$28.50) = \$19.90$
  - $F(Q) = \frac{c_u}{c_u + c_o} = \frac{121.50}{121.50 + 19.90} = 0.86$

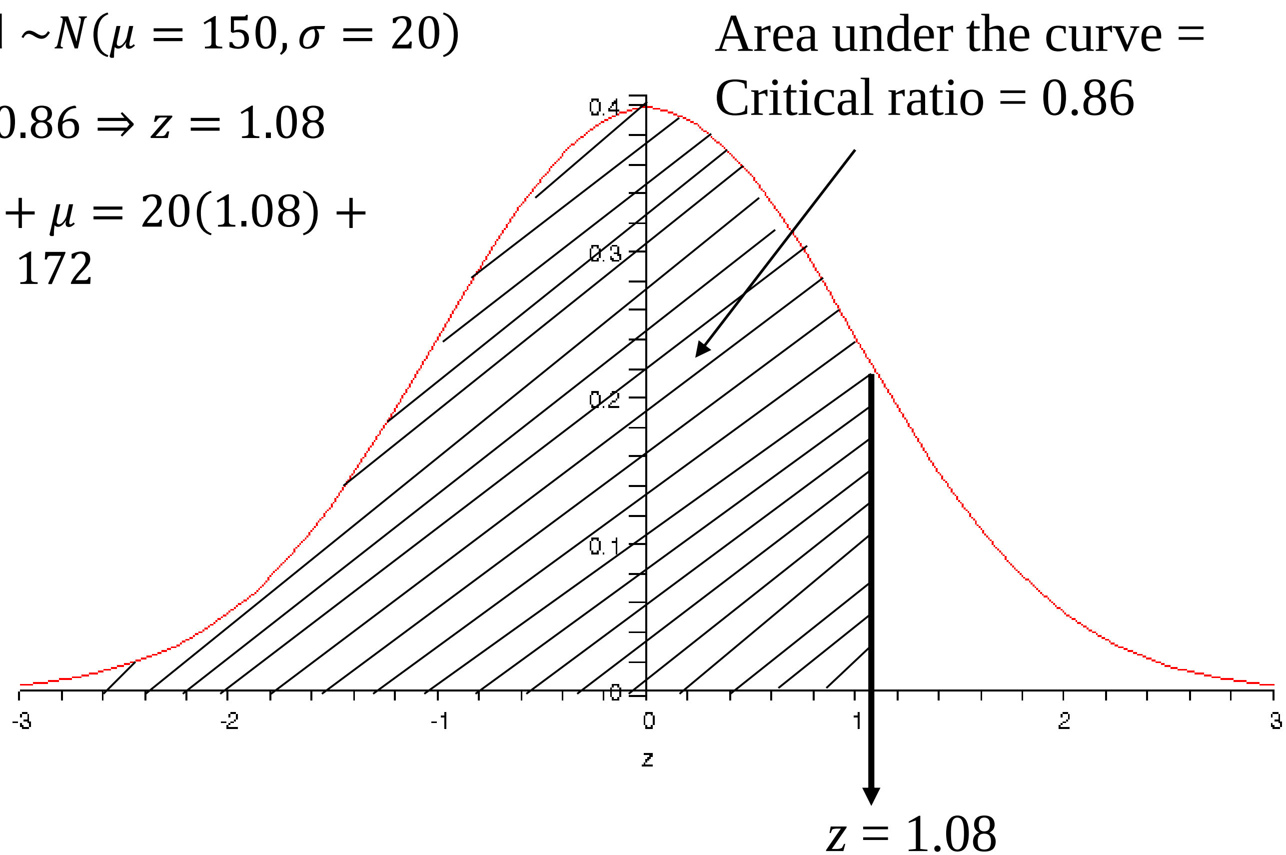
Newsvendor Model Example: Fashion Bags

(Assuming Normally Distributed Demand)

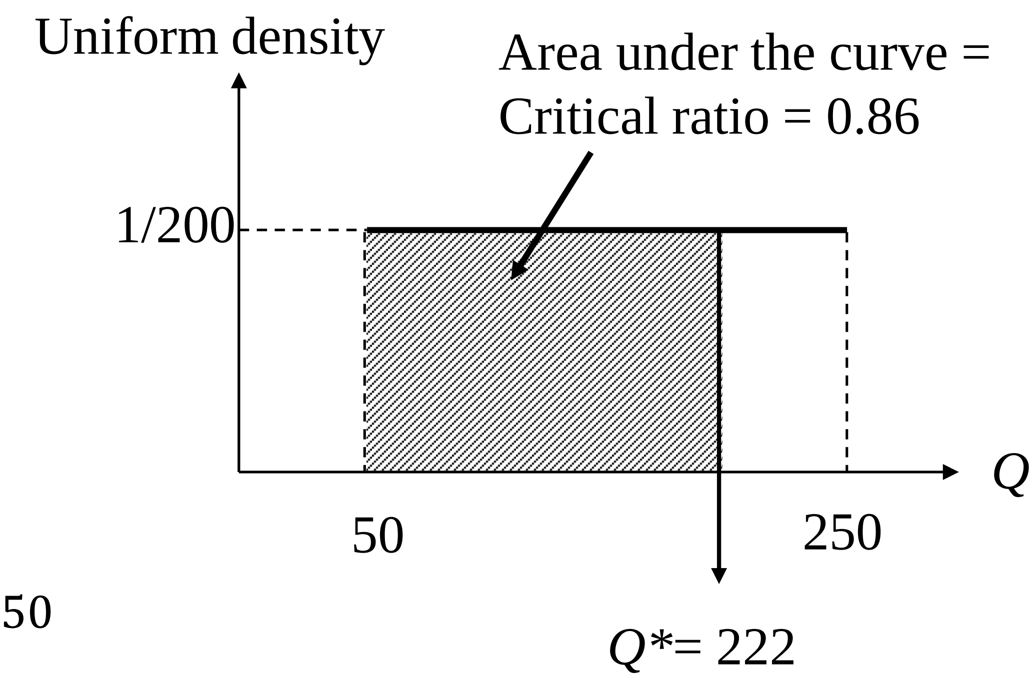
Demand $\sim N(\mu = 150, \sigma = 20)$

$F(Q) = 0.86 \Rightarrow z = 1.08$

$Q^* = \sigma z + \mu = 20(1.08) + 150 = 172$



Newsvendor Model Example: Fashion Bags
(Assuming Uniformly Distributed Demand)



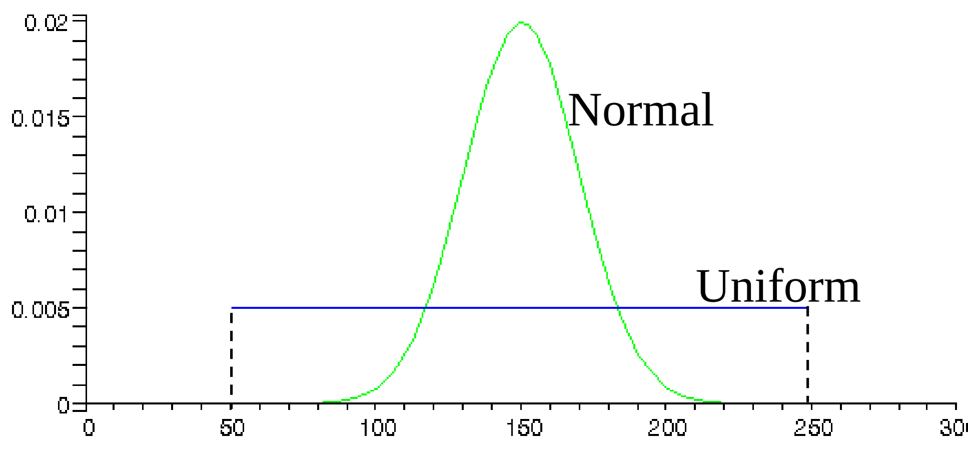
Demand $\sim U(50{,}250)$, $\mu = 150$

$$0.86 = (Q^* - 50)\frac{1}{200}$$

$$\Rightarrow Q^* = 222$$

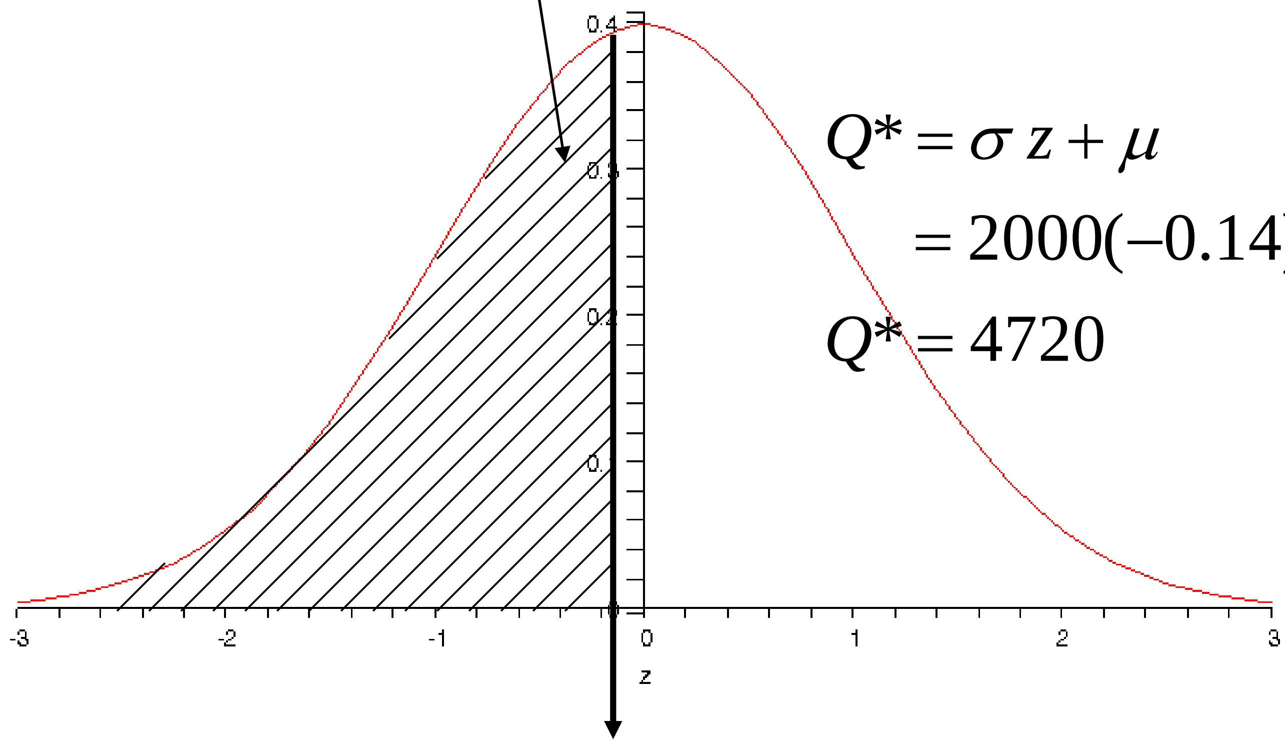
Newsvendor Model Example: Fashion Bags

- Even though both the Normal and the Uniform distributions have the same mean ($\mu = 150$), why did we get different quantities?
  - Normal distribution: $Q^* = 172$
  - Uniform distribution: $Q^* = 222$
- Because of the variance / the shape of the distribution
  - Normal: $\sigma = 20$
  - Uniform: $\sigma = 57.7$



Newsvendor Model Example: Hotel Room Reservation

$$P(Z \le z) = \Phi(z) = \text{Area under the curve}$$
$$= 0.444$$



$$Q^* = \sigma z + \mu$$
$$= 2000(-0.14) + 5000$$
$$Q^* = 4720$$

z = - 0.14

Newsvendor Example: Expected Fill Rate

- Expected Fill Rate = Expected Sales / Expected Demand

$$= Expected\ Sales/\mu$$
$$= (\mu - L(z)\sigma)/\mu$$
$$= 1 - L(z)\sigma/\mu$$
$$= 1 - (0.4728)\,(2000)/5000 = 81.0\%$$

- Note that

$$L(z) = \phi(z) - z\big(1 - \Phi(z)\big)$$

- **Just use '=NORMDIST(z,0,1,FALSE) – z*NORMSDIST(-z,0,1,TRUE)' in Excel to find L(z)!**

Loss Function

- The Loss Function $L(z)$ is the expected amount that demand D is greater than z
  - Analogous example: When rolling dice, Loss Function for rolling more than a seven
- When the demand is normally distributed, $L(z)$ is the standard loss function
  - i.e. $L(z)$ is the expected number of lost sales as a fraction of the standard deviation
  - **Therefore, $L(z)\sigma$ is the expected total lost sales**

Example: Rolling Dice, L(7)

Dice Roll	(a) Prob	(b) Amt X exceeds 7	(a) * (b)
2	0.0278	0	0
3	0.0556	0	0
4	0.0833	0	0
5	0.1111	0	0
6	0.1389	0	0
7	0.1667	0	0
8	0.1389	1	0.139
9	0.1111	2	0.222
10	0.0833	3	0.250
11	0.0556	4	0.222
12	0.0278	5	0.139
		L(7) = Total of last column	**0.972**

Summary of Newsvendor Model

- Single period
- Depending on the relationship between the cost of shortage or excess inventory, we may order more or less than expected demand
- Optimal order quantity
  - Increases as shortage cost increases
  - Decreases as holding cost increases
- Higher variability may cause an increase or a decrease in the optimal order quantity
- As σ increases, Q^* will deviate more from the mean