MS&E 260

Introduction to Operations Management

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Inventory Management with Deterministic Demand (EOQ Model)

Fundamentals of Inventory Control

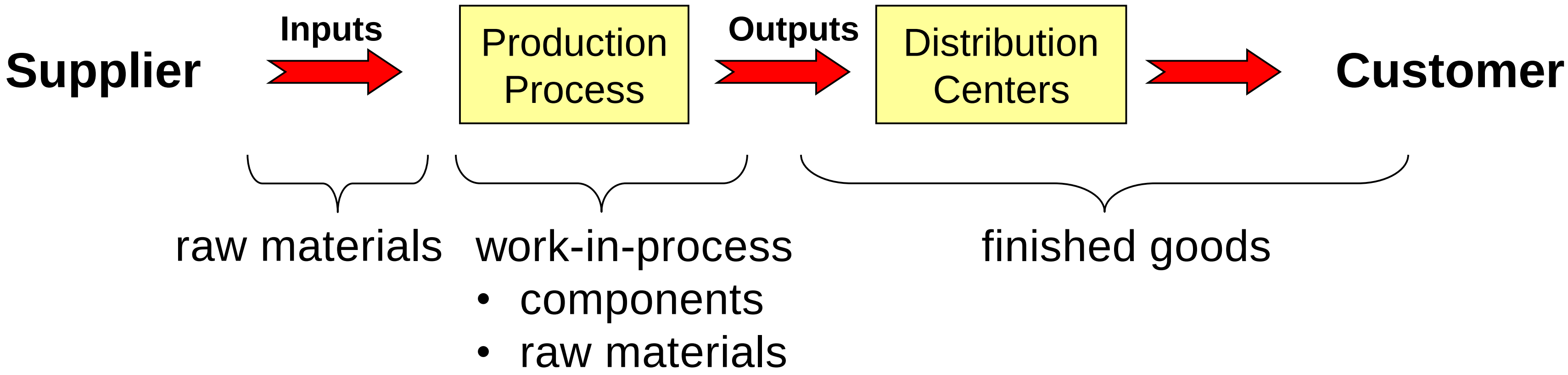
Why We Care About Inventory

- Macro level:
  - Total investment by firms in inventory in the US: 20% - 25% of GNP
  - US Inventory levels (07/2007): $1.57 Trillion
    - 30% held by retailers
    - 23% held by wholesalers
    - 29% held by manufacturers
- Firm level:
  - Sales growth: availability to consume when they need it
  - Cost reduction: cash flow, obsolescence

Product Variety and Inventory Management

- What if the number of models and unit sales increase for a company?
  - Must manage higher *product variety*
    - Can be achieved through inventory management
- Potential improvements in inventory management
  - Direct sales (e.g. Dell, no fixed retail position)
  - Postponement of the creation or delivery of final product until demand uncertainty decreases (e.g. HP)
  - Coordination (usually via contracts) to align incentives across supply chain often through information sharing (e.g. Seven-Eleven Japan)
  - Collaboration (e.g. Walmart) is less formal with agreements for cooperation between supply chain participants (often information sharing)

Where is Inventory Held in the Supply Chain?



Motivation for Holding Inventory

- Economies of scale: **cycle stock**
  - Spread a fixed cost over a large number of items (shipping, machine setup)
- Uncertainties: **safety stock**
  - Demand uncertainty: consumer preferences
  - Supply uncertainty: disruption in supply line
  - Lead time uncertainty: elapsed time from order placement to arrival
- Speculation: **anticipation stock**
  - Resources with increasing value: precious metals, crude oil, labor
- Smoothing: **anticipation stock**
  - Seasonal demand
- Lead times (supply chain): **pipeline stock**
  - Transportation and logistics
  - Long transit time between supplier to manufacturer to retail
  - Production schedule lead times

Cycle Stock

- Created by ordering in large quantities, so we order less frequently
- The longer the cycle, the bigger quantity Q and the bigger the inventory
- Helps with customer service, ordering costs, setups, transportation rates and material costs
- Average cycle inventory $= {}^{Q}/_{2}$

Safety Stock

- Created by placing orders before they are needed
- Helps with customer service and hidden costs of missing parts
- Protects against the three types of uncertainty
  - Demand
  - Lead Time
  - Supply

Anticipation Stock

- Created by smoothing output rates, overbuying before price increase or capacity shortage
- Used to absorb uneven rates in demand and supply

Pipeline (Transit) Stock

- Created by the time spent to move and produce inventory
- Can be in any of three stages
  - Inbound
  - Within Plant
  - Outbound
- Using Little's Law:
  - Average pipeline inventory $= d \times L$
  - where $d =$ demand and $L =$ lead time

Inventory Calculation and Application

- Suppose management has decided to establish 3 distribution centers (DCs) in different regions of the country in order to save on transportation costs. For one of the company's products the ***average weekly demand*** at each DC will be ***50 units***. The product is valued at $650 per unit. ***Average shipment sizes*** into each DC will be ***350 units per trip***. The ***average lead time*** will be ***2 weeks***. Each DC will carry ***1 week's supply as safety stock***, since the demand during lead time sometimes exceeds its average of 100 units (50 units/week x 2 weeks). Anticipation inventory should be negligible.
- For each distribution center:
  - Cycle inventory = ?
  - Safety inventory = ?
  - Pipeline inventory = ?
  - Total inventory at DC = ?

Inventory Trade-offs

Pressures for Smaller Inventories

- Inventory Holding Costs
- Opportunity Costs
- Storage and handling
- Taxes and Insurance
- Shrinkage

Pressures for Larger Inventories

- Customer Service
  - Stockouts
  - Backorders
- Ordering Costs
- Setup Costs
- Labor and Equipment utilization
- Transportation costs
- Quantity Discounts

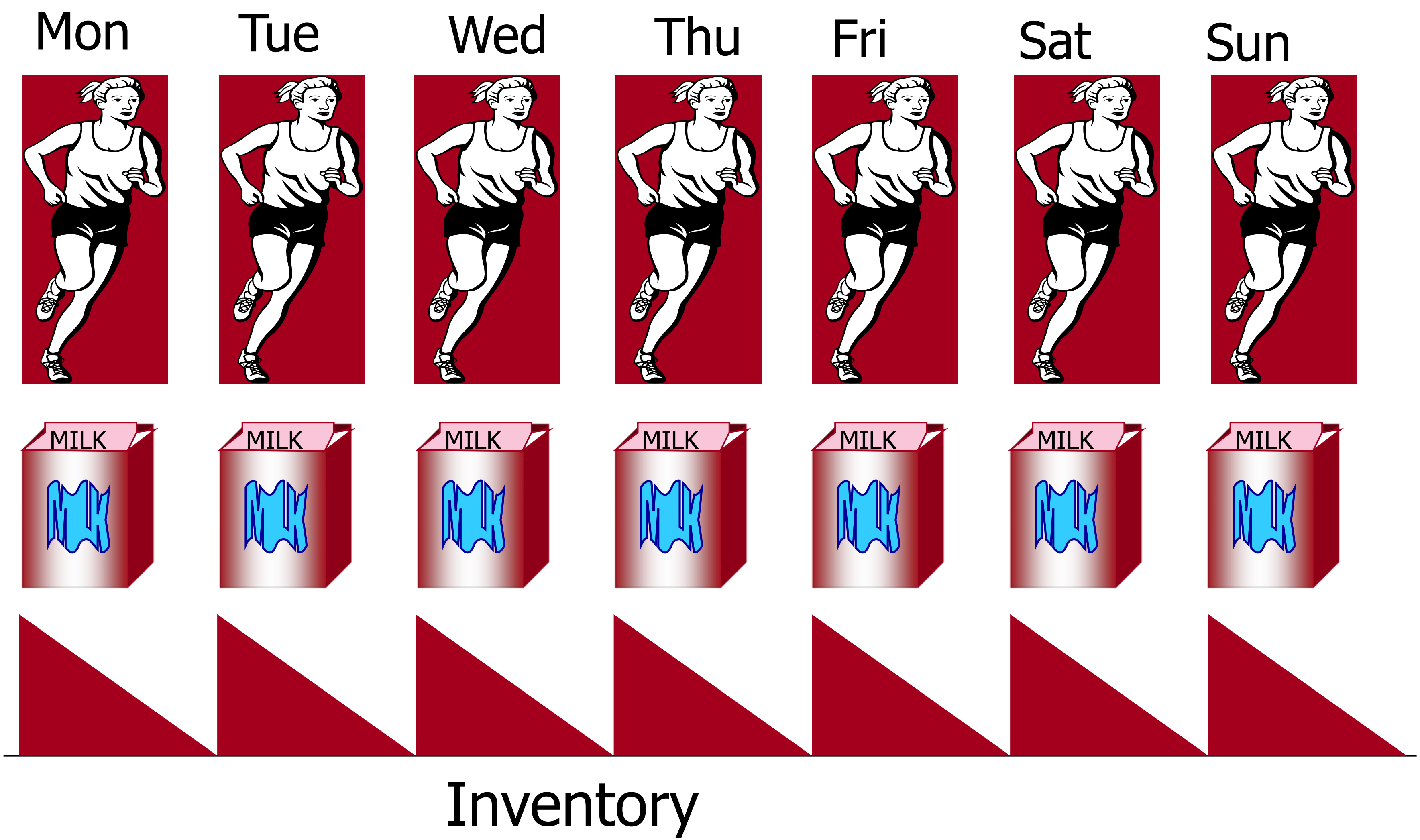
Outline of Inventory Management Lectures

- Basic trade-offs and models
  - Economic order quantity: cycle stock
  - Newsvendor model: safety stock
- Replenishment models
  - Review policies

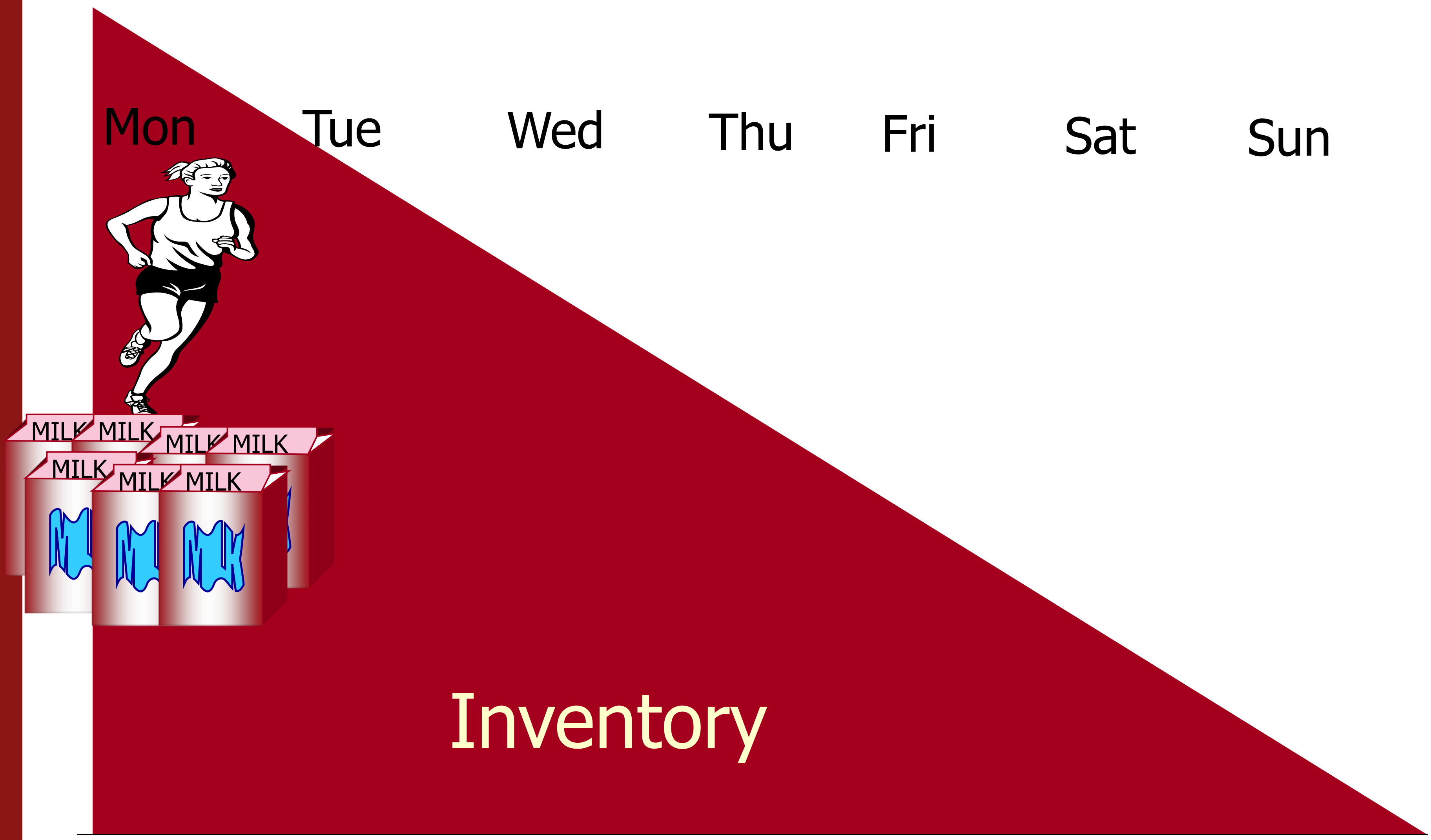
Inventory Management with Deterministic Demand (EOQ Model)

- Basic model
- Quantity discount models
- Finite production models

Running to the Store a Lot…



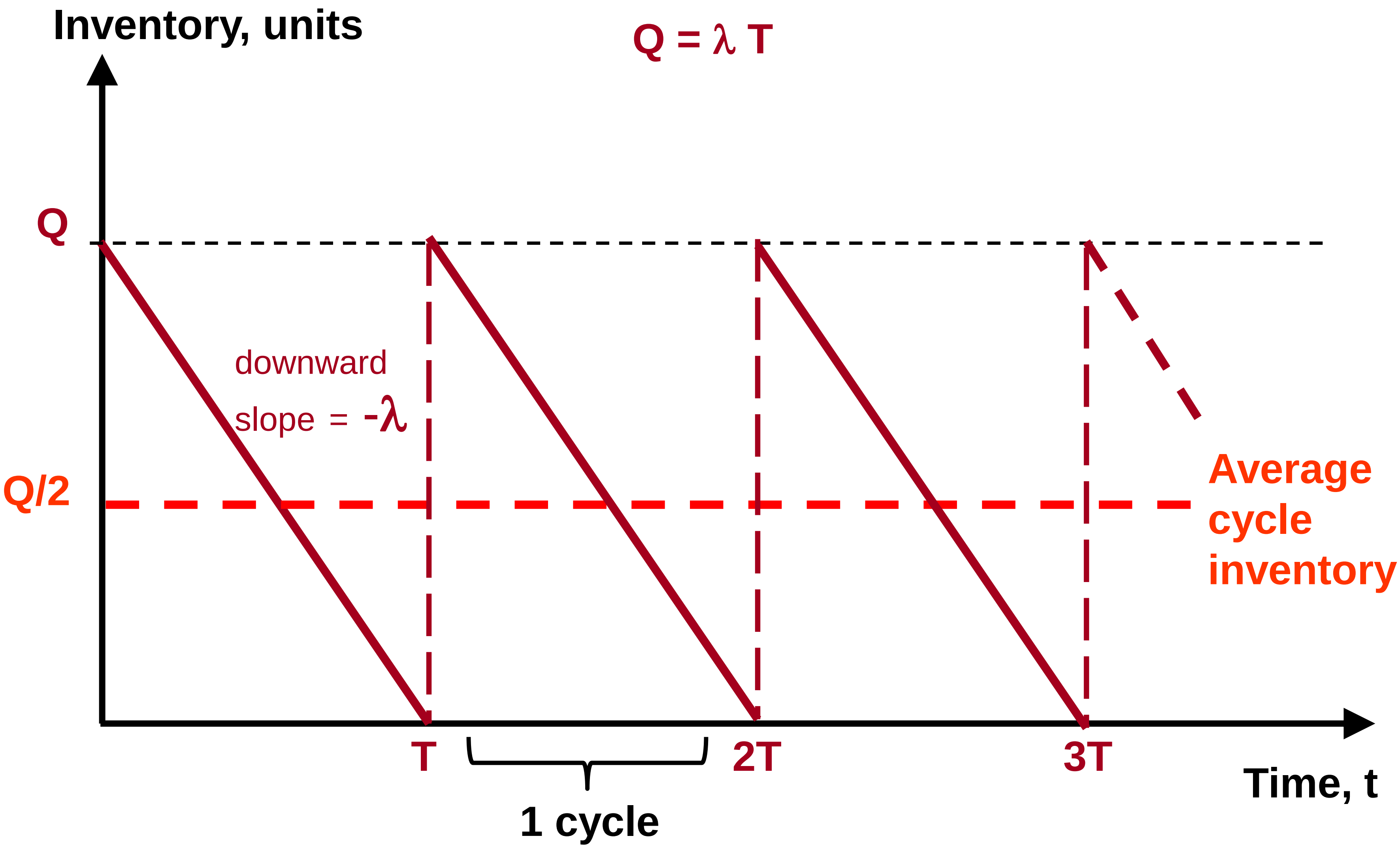
vs. Running to the Store a Little



Economic Order Quantity (EOQ)

- Assumptions:
  - Consider a single inventory item
  - Demand is fixed (deterministic) at λ units/time
  - Shortages are not allowed
  - Order quantity is fixed at Q per cycle
  - Orders are received instantaneously (no lead time)
- Cost structure:
  - Fixed and marginal order costs per cycle $(K + cQ)$
  - Holding cost at h per unit held per unit time
- Objective: Determine order quantity Q^* to minimize sum of ordering cost and inventory holding cost

EOQ: Graphical Concept



EOQ Derivation

- Cost function
  - Total Cost = Setup (Ordering) + Holding + Purchase

$$TC(Q) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)h + \lambda c$$

$Q =$ Order (production) quantity: the **decision to be made**

$\lambda =$ Demand rate

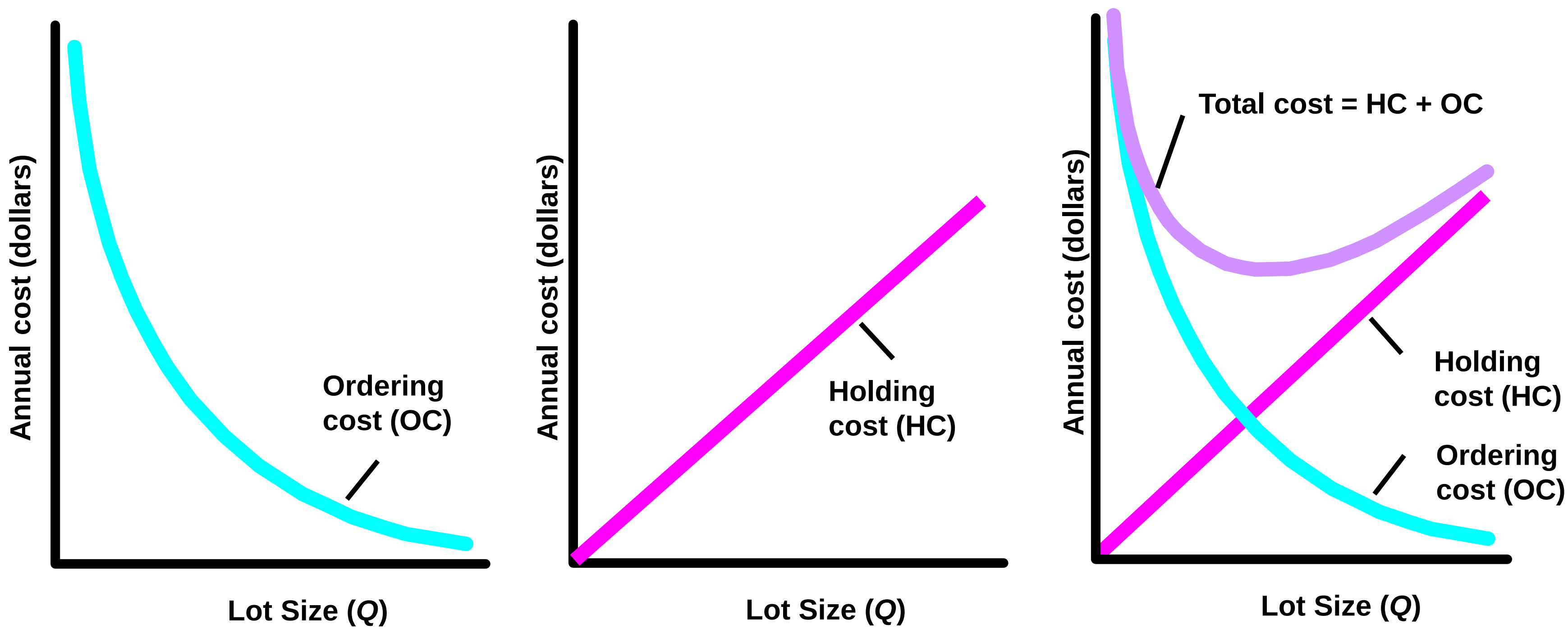
$K =$ Fixed ordering (or setup) cost

$c =$ Cost per unit in inventory

$i =$ Annual interest rate

$h =$ Holding cost per unit per year $= ic$

Graphs of Annual Holding, Ordering and Total Costs



$$TC(Q) = \underbrace{\left(\frac{\lambda}{Q}\right)K}_{\text{OC}} + \underbrace{\left(\frac{Q}{2}\right)h}_{\text{HC}} + \lambda c$$

EOQ Derivation

- Cost function
  - Total Cost = Setup (Ordering) + Holding + Purchase

$$TC(Q) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)h + \lambda c$$

  - Q^* minimizes the total cost function

$$\frac{d[TC(Q)]}{dQ} = \left(-\frac{\lambda}{Q^2}\right)K + \left(\frac{h}{2}\right) = 0$$

$$Q^* = \sqrt{\frac{2K\lambda}{h}}$$

EOQ Example

- Given:
  $\lambda = 1{,}000$ units per year $\quad c = \$400$
  $i = 25\%$ per year $\quad K = \$20$
- What is the total cost?
- How often should we order?

$$Q^* = \sqrt{\frac{2K\lambda}{h}} = \sqrt{\frac{2K\lambda}{ic}} = \sqrt{\frac{2(20)(1000)}{(0.25)(400)}} = 20$$

$$TC(Q^*) = \left(\frac{\lambda}{Q^*}\right)K + \left(\frac{Q^*}{2}\right)ic + \lambda c$$

$$= \left(\frac{1000}{20}\right)20 + \left(\frac{20}{2}\right)(0.25)(400) + (1000)(400) = \$402{,}000$$

- Number of orders per year $= \frac{1000}{20} = 50$

Effect of Changes

- What happens to cycle inventory if the demand rate increases?

- What happens to lot sizes if setup costs decrease?

- What happens if interest rates drop?

- How critical are errors in estimating λ, h, and K?

$$Q^* = \sqrt{\frac{2K\lambda}{h}}$$

Properties of the EOQ Solutions

- Q is increasing with K and λ
- Q is decreasing with h
- Q changes as the square root of these quantities
- Q is independent of the proportional order cost c
  - (except as it relates to the value of $h = ic$)
- Observe:

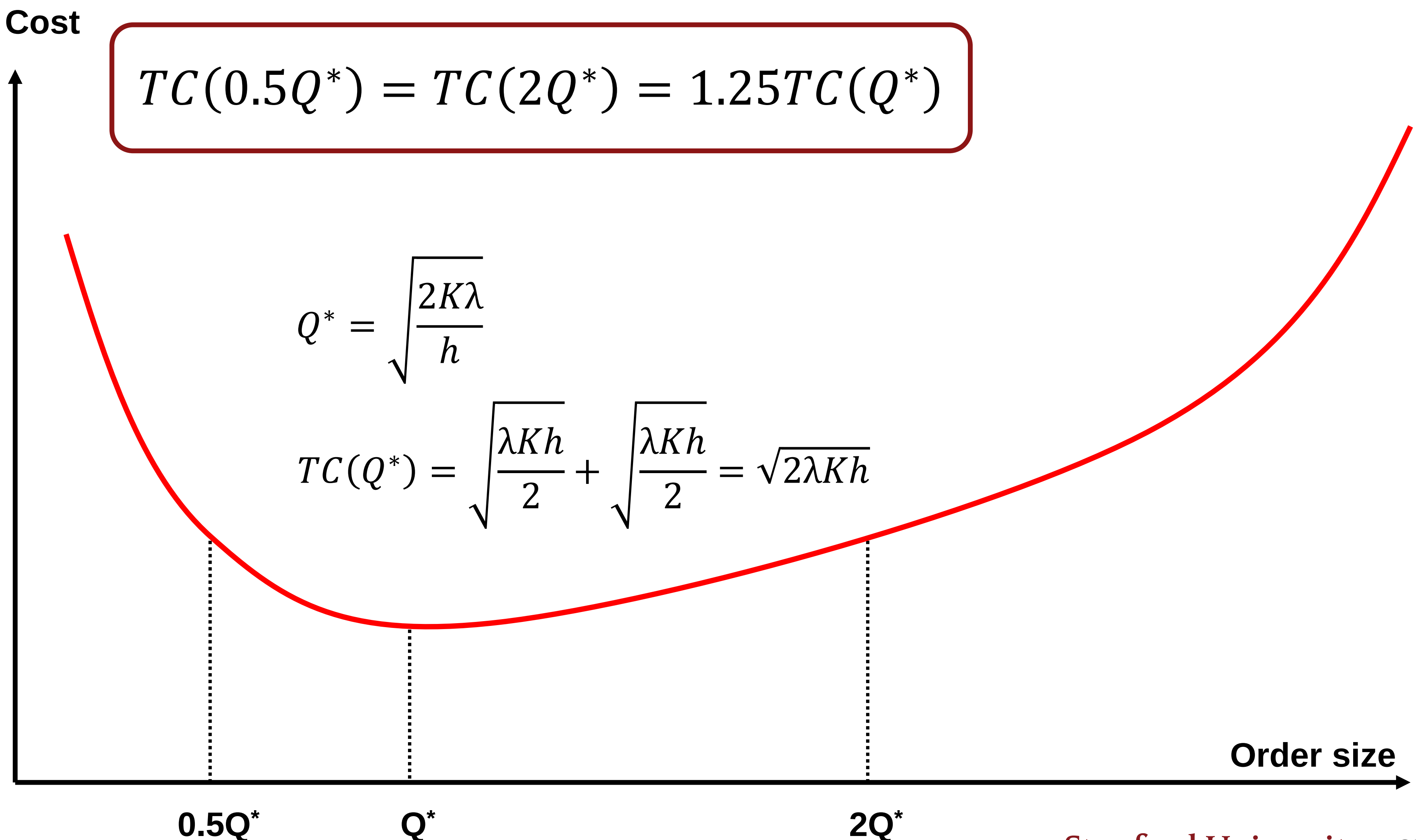
$$\frac{\lambda K}{Q^*} = \frac{hQ^*}{2}$$

EOQ Robustness (with respect to holding and setup costs)

$$TC(0.5Q^*) = TC(2Q^*) = 1.25TC(Q^*)$$

$$Q^* = \sqrt{\frac{2K\lambda}{h}}$$

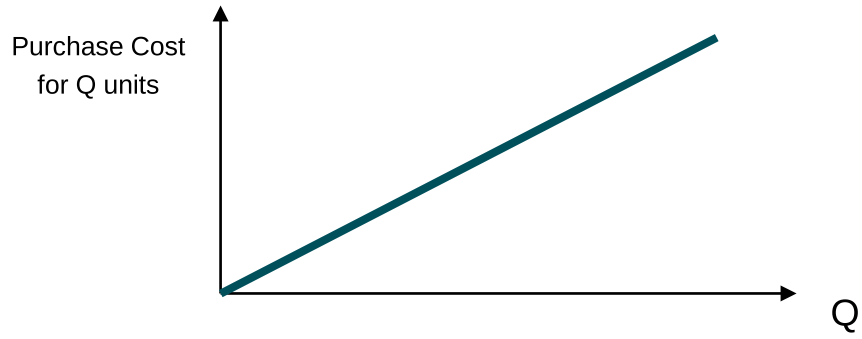
$$TC(Q^*) = \sqrt{\frac{\lambda K h}{2}} + \sqrt{\frac{\lambda K h}{2}} = \sqrt{2\lambda K h}$$



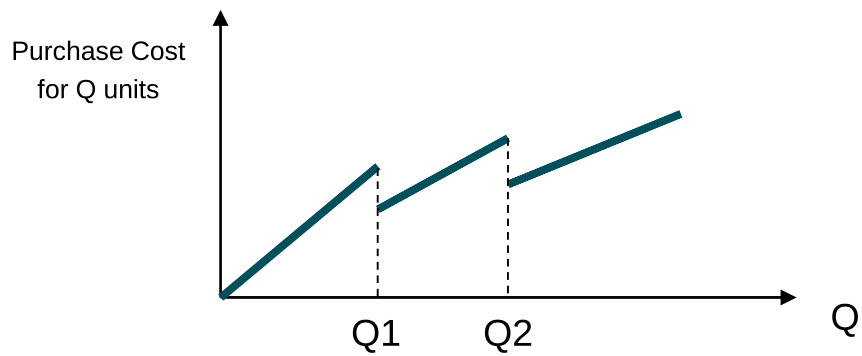
EOQ with Quantity Discounts

EOQ with Quantity Discounts

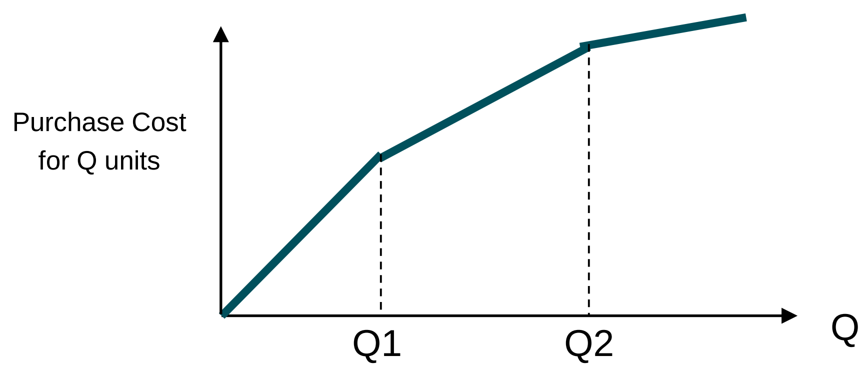
- No Discount



- All-Unit Discount
  - The discount is applied to ALL of the units in the order



- Incremental Discount
  - The discount is applied only to the number of units above the breakpoint



EOQ with All-Unit Discount

- Standard EOQ cost function
  - Total Cost = Setup (Ordering) + Holding + Purchase

    $= \left(\frac{\lambda}{Q}\right) K \quad + \left(\frac{Q}{2}\right) h \quad + \lambda c$

- New cost function
  - Replace c by the new unit cost c_0, or c_1, or …, or c_m
  - c_j = price per unit in j^{th} quantity range

$$TC_j = \left(\frac{\lambda}{Q}\right) K + \left(\frac{Q}{2}\right)(ic_j) + \lambda c_j$$

EOQ with All-Unit Discount

- To find Q^*:

(1) Let Q_j = EOQ obtained using cost c_j

(2) If Q_j is in quantity range for price c_j, set $Q_j^* = Q_j$

If Q_j is not in quantity range for price c_j, pick the boundary point of the quantity range which is closest to Q_j; set $Q_j^* = Q_j$

Use Q_j^* to calculate TC_j

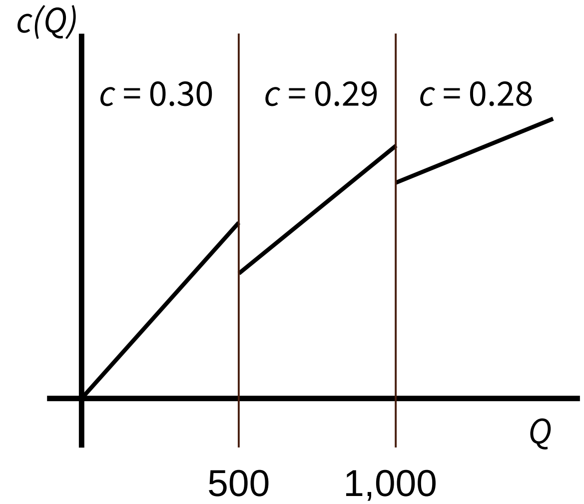
(3) Select minimum TC_j over $j = 0, \dots, n$; call this TC_j^*

Q^* = value of Q_j^* corresponding to TC_j^*

EOQ with All-Unit Discount

- Trash bag problem*: Determine the number of trash bags to order.
  - Cost to place order $K = \$8$
  - Annual demand $\lambda = 600$
  - Annual interest rate for holding cost $i = 0.2$

$$c = \begin{cases} 0.30 & for\ 0 \leq Q < 500 \\ 0.29 & for\ 500 \leq Q < 1{,}000 \\ 0.28 & for\ 1{,}000 \leq Q \end{cases}$$



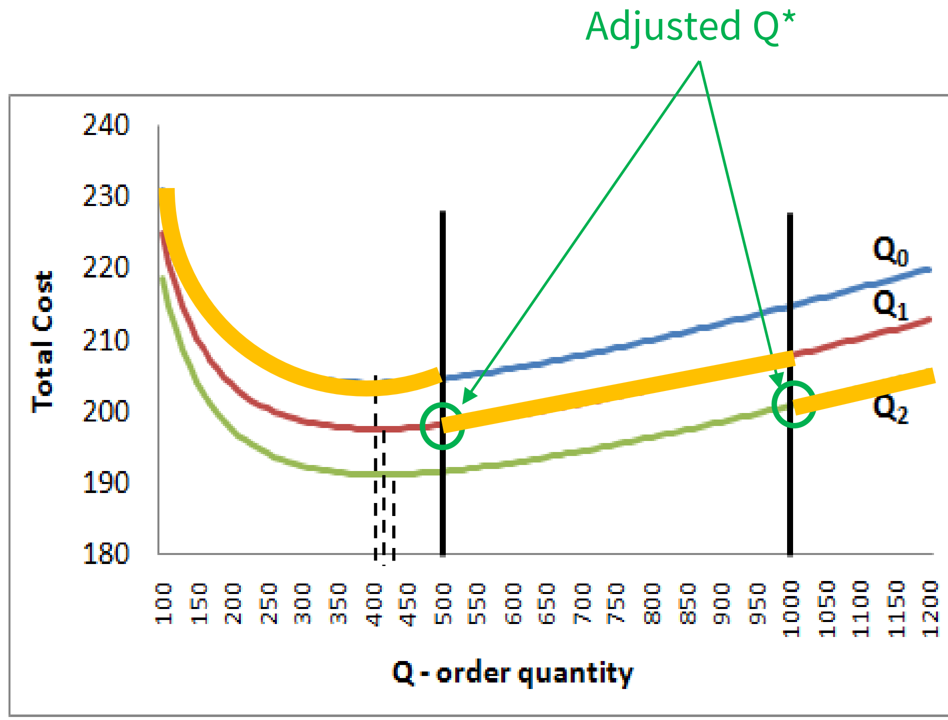
*See Nahmias, pg. 220

EOQ with All-Unit Discount

$$Q_0^* = \sqrt{\frac{2K\lambda}{ic_0}} = \sqrt{\frac{2(8)(600)}{(0.2)(0.30)}} = 400$$

$$Q_1^* = \sqrt{\frac{2K\lambda}{ic_1}} = \sqrt{\frac{2(8)(600)}{(0.2)(0.29)}} = \cancel{406}\ 500$$

$$Q_2^* = \sqrt{\frac{2K\lambda}{ic_2}} = \sqrt{\frac{2(8)(600)}{(0.2)(0.28)}} = \cancel{414}\ 1000$$



Total Cost = Setup (Ordering) + Holding + Purchase

$$= \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)h + \lambda c$$

EOQ with All-Unit Discount

- To determine the optimal Q^*, check the total cost for our ***adjusted*** Q^* values

$$TC(Q^*) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)ic + \lambda c$$

$$TC(Q^* = 400) = \left(\frac{600}{400}\right)8 + \left(\frac{400}{2}\right)(0.20)(0.30) + (600)(0.30) = \$204.00$$

$$TC(Q^* = 500) = \left(\frac{600}{500}\right)8 + \left(\frac{500}{2}\right)(0.20)(0.29) + (600)(0.29) = \$198.10$$

$$TC(Q^* = 1{,}000) = \left(\frac{600}{1{,}000}\right)8 + \left(\frac{1{,}000}{2}\right)(0.20)(0.28) + (600)(0.28) = \$200.80$$

- Answer: Order 500 units
  - Average annual cost = \$198.10, $c = 0.29$

EOQ with Incremental Discount

- Standard EOQ cost function
  - Total Cost = Setup (Ordering) + Holding + Purchase

    $$= \left(\frac{\lambda}{Q}\right) K \quad + \left(\frac{Q}{2}\right) h \quad + \lambda c$$

- New cost function
  - Replace c by the new unit cost $C_{AVG} = \frac{C(Q)}{Q}$, where:

Range of Q	Value of C
$0 \le Q < Q_1$	$C_{AVG} = C_0$
$Q_1 \le Q < Q_2$	$C_{AVG} = [Q_1 C_0 + (Q - Q_1) C_1]/Q$
$Q_1 \le Q < Q_3$	$C_{AVG} = [Q_1 C_0 + (Q_2 - Q_1) C_1 + (Q - Q_2) C_2]/Q$

EOQ with Incremental Discount

- To find Q^*:

(1) Determine an algebraic expression for $C(Q)$ and for $C(Q)/Q$ in each price interval

(2) For each price interval find the feasible value Q_j^* that minimizes TC_j

(3) Select minimum TC_j over $j = 0, \ldots, n$; call this TC_j^*

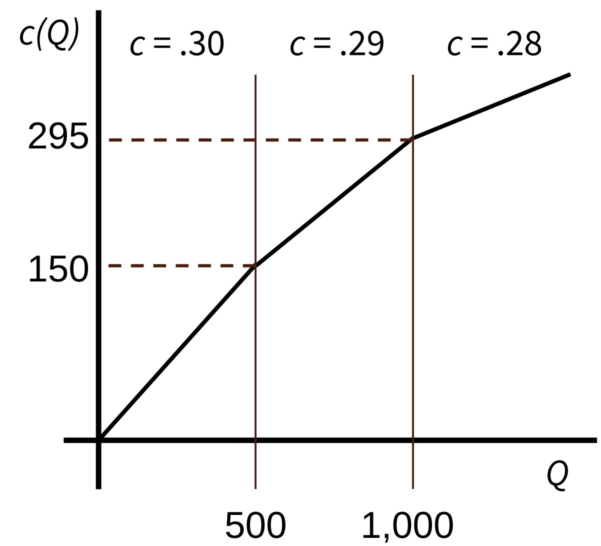
$Q^* =$ value of Q_j^* corresponding to TC_j^*

EOQ with Incremental Discount

- Trash bag example again but with incremental discounts*:
  - Cost to place order $K = \$8$
  - Annual demand $\lambda = 600$
  - Annual interest rate for holding cost $i = 0.2$

$$C(Q) = \begin{cases} 0.30Q & for\ 0 \le Q < 500 \\ 150 + 0.29(Q - 500) & for\ 500 \le Q < 1{,}000 \\ 295 + 0.28(Q - 1000) & for\ 1{,}000 \le Q \end{cases}$$

$$C(Q)/Q = \begin{cases} 0.30 & for\ 0 \le Q < 500 \\ 0.29 + 5/Q & for\ 500 \le Q < 1{,}000 \\ 0.28 + 15/Q & for\ 1{,}000 \le Q \end{cases}$$



*See Nahmias, pg. 223

EOQ with Incremental Discount

- First, write out the average annual cost function for the new $C(Q)/Q$ value. (Remember: the old c is replaced by the new unit cost $C_{AVG} = \frac{C(Q)}{Q}$):

$$TC(Q^*) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)ic + \lambda c$$

- For $C(Q)/Q = 0.30$:

$$TC(Q) = \left(\frac{600}{Q}\right)8 + \left(\frac{Q}{2}\right)(0.20)(0.30) + (600)(0.30)$$

- For $C(Q)/Q = 0.29 + 5/Q$:

$$TC(Q) = \left(\frac{600}{Q}\right)8 + \left(\frac{Q}{2}\right)(0.20)(\mathbf{0.29 + 5/Q}) + (600)(\mathbf{0.29 + 5/Q})$$

$$= \left(\frac{600}{Q}\right)(8 + \mathbf{5}) + \left(\frac{Q}{2}\right)(0.20)(\mathbf{0.29}) + \left(\frac{5}{2}\right)(0.20) + (600)(0.29)$$

- For $C(Q)/Q = 0.28 + 15/Q$:

$$TC(Q) = \left(\frac{600}{Q}\right)8 + \left(\frac{Q}{2}\right)(0.20)(\mathbf{0.28 + 15/Q}) + (600)(\mathbf{0.28 + 15/Q})$$

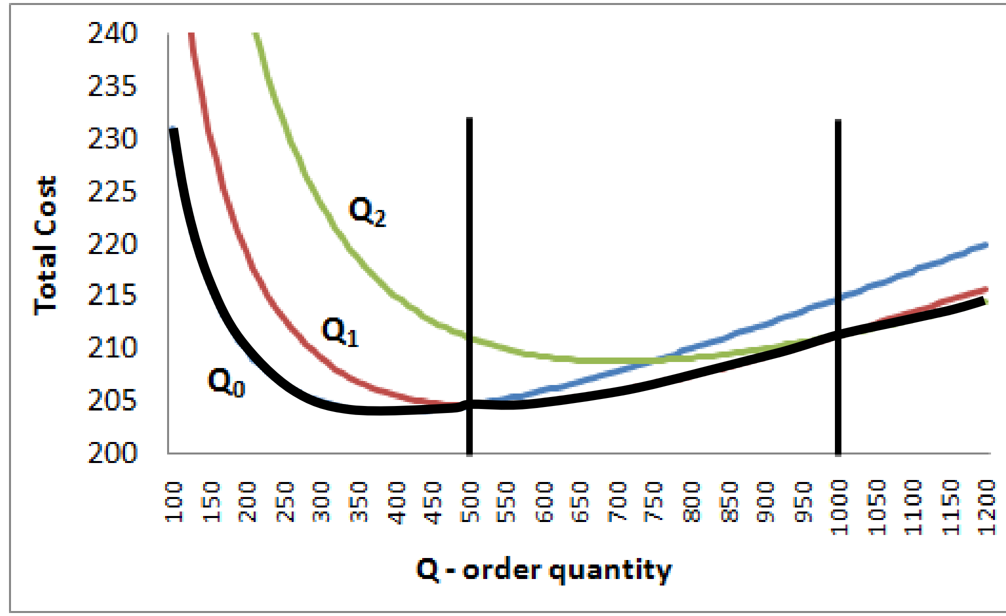
$$= \left(\frac{600}{Q}\right)(8 + \mathbf{15}) + \left(\frac{Q}{2}\right)(0.20)(\mathbf{0.28}) + \left(\frac{15}{2}\right)(0.20) + (600)(0.28)$$

EOQ with Incremental Discount

$$Q_0^* = \sqrt{\frac{2K\lambda}{ic_0}} = \sqrt{\frac{2(8)(600)}{(0.2)(0.30)}} = 400$$

$$Q_1^* = \sqrt{\frac{2(K+5)\lambda}{ic_1}} = \sqrt{\frac{2(13)(600)}{(0.2)(0.29)}}$$
$$= 519$$

$$\cancel{Q_2^* = \sqrt{\frac{2(K+15)\lambda}{ic_2}} = \sqrt{\frac{2(23)(600)}{(0.2)(0.28)}} = 702}$$



EOQ with Incremental Discount

- Finally, calculate the average annual cost

$$TC(Q) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)i(C(Q)/Q) + \lambda(C(Q)/Q)$$

- For $C(Q)/Q = 0.30$:

$$TC(Q = 400) = \left(\frac{600}{400}\right)8 + \left(\frac{400}{2}\right)(0.20)(0.30) + (600)(0.30) = \$204.00$$

- For $C(Q)/Q = 0.29 + 5/Q$:

$$TC(Q = 519)$$
$$= \left(\frac{600}{519}\right)8 + \left(\frac{519}{2}\right)(0.20)(0.29 + 5/519) + (600)(0.29 + 5/519)$$
$$= \$204.58$$

- Answer: Order 400 units
  - Average annual cost = $204.00, $c = 0.30$

Properties of the Optimal Solutions

- For all-unit discounts:
  - The optimal will occur at the bottom of one of the cost curves or at a breakpoint
- For incremental discounts:
  - The optimal will always occur at a realizable EOQ value
  - Compare costs at all realizable EOQs

EOQ Extension: Finite Production Rate

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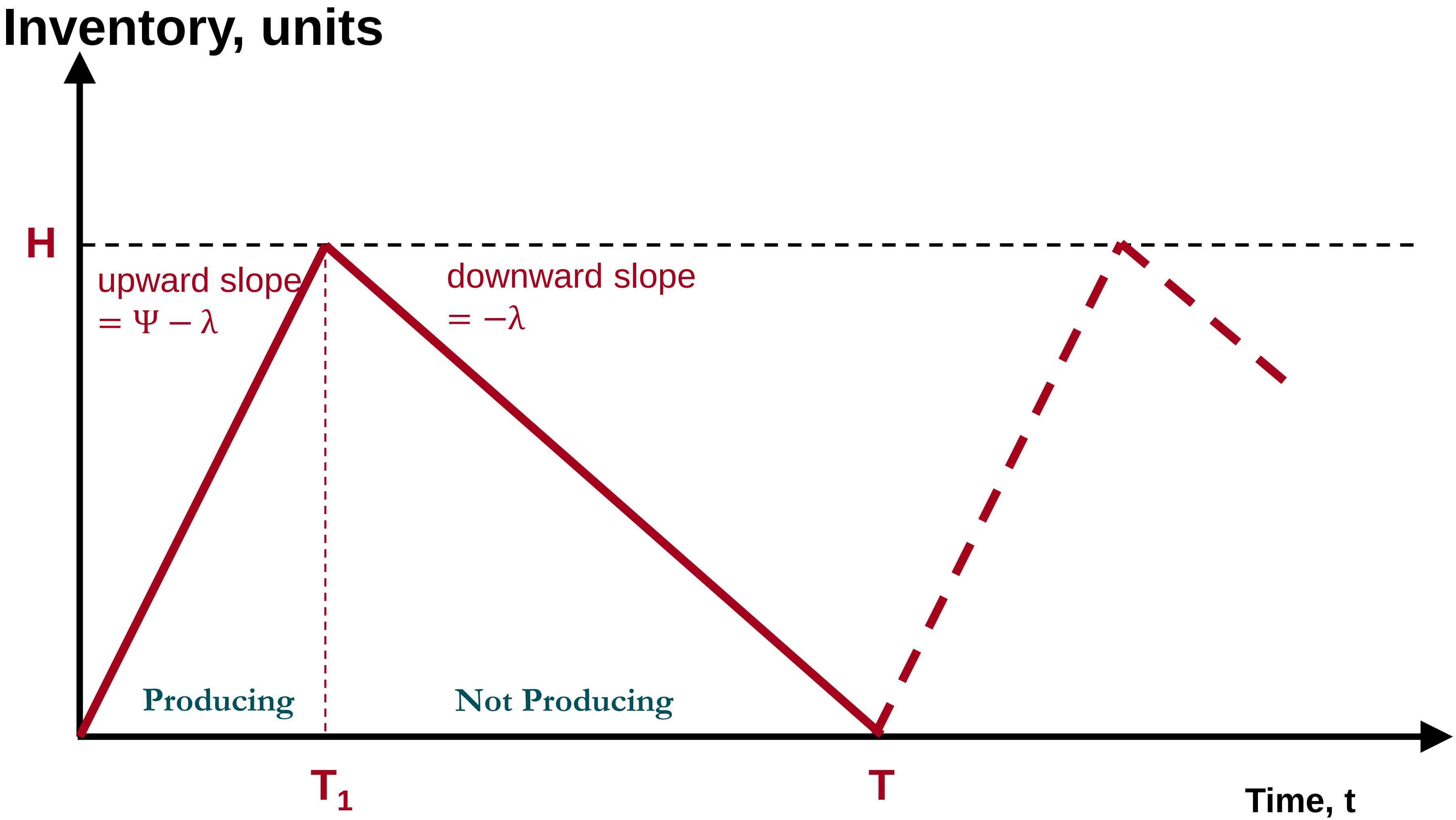
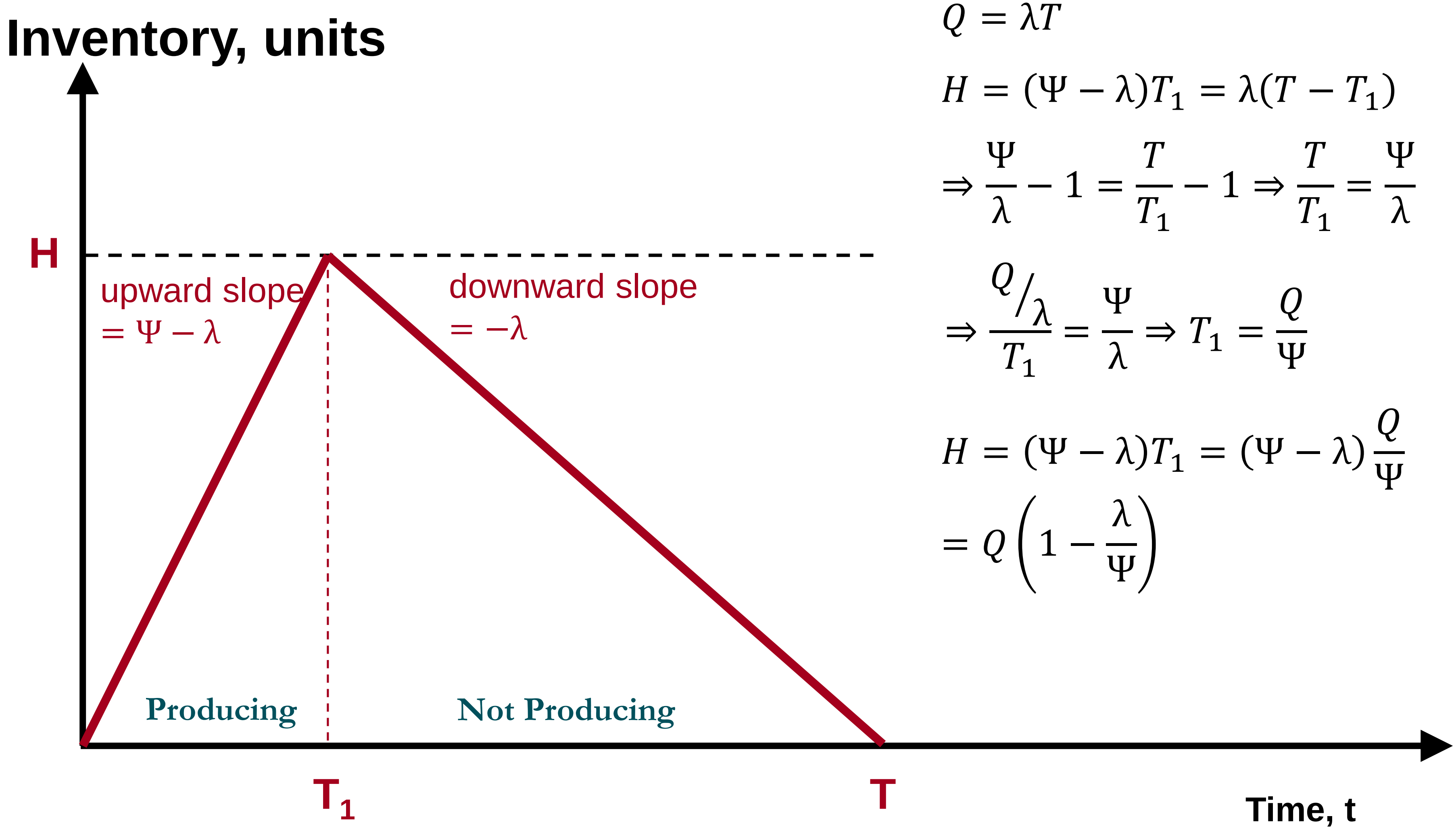
- Suppose replenishment is not instantaneous, but production rate Ψ is greater than demand rate λ

- Then the optimal production quantity to minimize average annual holding and set up costs has the same form as the EOQ

EOQ Extension: Finite Production Rate



EOQ Extension: Finite Production Rate



$$Q = \lambda T$$

$$H = (\Psi - \lambda) T_1 = \lambda (T - T_1)$$

$$\Rightarrow \frac{\Psi}{\lambda} - 1 = \frac{T}{T_1} - 1 \Rightarrow \frac{T}{T_1} = \frac{\Psi}{\lambda}$$

$$\Rightarrow \frac{Q/\lambda}{T_1} = \frac{\Psi}{\lambda} \Rightarrow T_1 = \frac{Q}{\Psi}$$

$$H = (\Psi - \lambda) T_1 = (\Psi - \lambda) \frac{Q}{\Psi}$$

$$= Q\left(1 - \frac{\lambda}{\Psi}\right)$$

Finite Production Rate Derivation

- Suppose replenishment is not instantaneous, but production rate Ψ is greater than demand rate λ

$$H = \text{maximum inventory level} = Q\left(1 - \frac{\lambda}{\Psi}\right)$$

$$\frac{H}{2} = \text{average inventory level} = \frac{Q}{2}\left(1 - \frac{\lambda}{\Psi}\right)$$

$$TC(Q) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)\left(1 - \frac{\lambda}{\Psi}\right)h + \lambda c$$

$$Q^* = \sqrt{\frac{2K\lambda}{h\left(1 - \frac{\lambda}{\Psi}\right)}}$$

EOQ Extension: Finite Production Rate

- Suppose that items are produced internally at a rate $\Psi > \lambda$
- Then the optimal production quantity to minimize average annual holding and set up costs has the same form as the EOQ, namely:

$$Q^* = \sqrt{\frac{2K\lambda}{h'}}$$

where

$$h' = h\left(1 - \frac{\lambda}{\Psi}\right)$$

EOQ Finite Production Rate Example

- Given:

  $\lambda = 1{,}000$ units per year
  $\Psi = 4{,}000$ units per year
  $K = \$20$
  $c = \$400$
  $i = 25\%$ per year

- How often should we order?

$$Q^* = \sqrt{\frac{2K\lambda}{h\left(1 - \frac{\lambda}{\Psi}\right)}} = \sqrt{\frac{2(20)(1000)}{(0.25)(400)(0.75)}} = 23$$

$$TC(Q) = \left(\frac{\lambda}{Q}\right)K + \left(\frac{Q}{2}\right)\left(1 - \frac{\lambda}{\Psi}\right)h + \lambda c$$

$$= \left(\frac{1000}{23}\right)20 + \left(\frac{23}{2}\right)(0.25)(400)(0.75) + (1000)(400) = \$401{,}732$$

- Number of orders per year $= \frac{1000}{23} = 43.5$

EOQ Summary

- Demand is assumed to be constant and deterministic
- The same quantity Q is ordered every time
- Costs considered are ordering cost, purchase cost, and holding cost
- Can use calculus to solve for optimal order quantity
- There are some extensions of the EOQ models: lead time, internal production, quantity discounts, fixed costs, per-unit costs (handling/warehouse), etc.