Cheat sheet:

**Little’s Law:**

Conservation of Flow (at equilibrium):

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|  | 𝐿 = Average number of flow units in process  λ = Process throughput  𝑊 = Average process cycle time |

**Cycle Stock**

Spread a fixed cost over a large number of items (shipping, machine setup)

Created by ordering in large quantities, so we order less frequently

Average cycle inventory = 𝑄/2

**Safety Stock**

Created by placing orders before they are needed

**Anticipation Stock**

Created by smoothing output rates, overbuying before price increase or capacity shortage

**Pipeline (Transit) Stock**

* Created by the time spent to move and produce inventory
* Average pipeline inventory = 𝑑 × 𝐿
* where 𝑑 = demand and 𝐿 = lead time

**Economic Order Quantity (EOQ)**

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| Total Cost = Setup (Ordering) + Holding + Purchase | λ = Demand units/time  𝑄 = Order quantity /cycle  𝑐 = Cost per unit in inventory  𝑖 = Annual interest rate  ℎ = Holding cost per unit per year = 𝑖 𝑐  *K = Fixed costs* |

**EOQ with All-Unit Discount**

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| - Replace 𝑐 by the new unit cost 𝑐0, or 𝑐1,or …, or 𝑐𝑚 | 𝑐𝑗 = price per unit in 𝑗th quantity range  To find 𝑄∗:  1. Let 𝑄𝑗 = EOQ obtained using cost 𝑐𝑗  2. 𝑄𝑗 is in quantity range for price 𝑐𝑗, set 𝑄𝑗∗ = 𝑄𝑗  - If 𝑄𝑗 is not in quantity range for price 𝑐𝑗, pick the boundary point of the quantity range which is closest to 𝑄𝑗; set 𝑄𝑗∗ = 𝑄𝑗  - Use 𝑄𝑗∗ to calculate 𝑇𝐶𝑗  3. Select minimum 𝑇𝐶𝑗 over 𝑗 = 0,…, 𝑛; call this 𝑇𝐶𝑗∗  𝑄∗ = value of 𝑄𝑗∗ corresponding to 𝑇𝐶𝑗∗ |

**EOQ with Incremental Discount**

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| - Replace 𝑐 by the new unit cost 𝑐AVG  Range of Q Value of C  0 ≤ 𝑄 < 𝑄1 𝐶𝐴𝑉𝐺 = 𝐶0  𝑄1 ≤ 𝑄 < 𝑄2 𝐶𝐴𝑉𝐺 = [𝑄1 𝐶0 + (𝑄 − 𝑄1) 𝐶1 ] / 𝑄  𝑄1 ≤ 𝑄 <𝑄3 𝐶𝐴𝑉𝐺 = [𝑄1 𝐶0 + (𝑄2 − 𝑄1) 𝐶1 + (𝑄 − 𝑄2) 𝐶2] / 𝑄  To find 𝑄∗:  1. Determine an algebraic expression for 𝐶(𝑄) and for 𝐶(𝑄)/𝑄 in each price interval  2. For each price interval find the feasible value 𝑄𝑗∗ that minimizes 𝑇𝐶𝑗  3. Select minimum 𝑇𝐶𝑗 over 𝑗 = 0,…,𝑛; call this 𝑇𝐶𝑗∗  𝑄∗ = value of 𝑄𝑗\* corresponding to 𝑇𝐶𝑗\* |

**EOQ: Finite Production Rate**

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|  | Ψ = Production rate units/time, should be greater than λ  λ = Demand units/time  𝑄 = Order quantity /cycle  𝑐 = Cost per unit in inventory  𝑖 = Annual interest rate  ℎ = Holding cost per unit per year = 𝑖 𝑐  *K = Fixed costs*  H = Maximum Level of inventory at any instant of time |

**Inventory Control Subject to Uncertain Demand**

Two types of inventory control models

1. **Fixed time period** : Periodic review
   1. One period (**Newsvendor model**)
   2. Multiple periods (**𝑇,𝑆 System)**

* Every 𝑇 periods order up to 𝑆 units
* Let 𝑇 = 𝐸𝑂𝑄 / 𝜆
* Then, the order-up-to level 𝑆 should cover the demand during 𝑇 + 𝜏 periods  
  𝑆 = (𝑚𝑒𝑎𝑛 𝑑𝑒𝑚𝑎𝑛𝑑 + 𝑠𝑎𝑓𝑒𝑡𝑦 𝑠𝑡𝑜𝑐𝑘) 𝑜𝑣𝑒𝑟 (𝑇 + 𝜏) 𝑝𝑒𝑟𝑖𝑜𝑑𝑠   
   = 𝜇𝑇+𝜏 + 𝑧𝜎𝑇+𝜏
* Determine 𝑧 as in a (𝑄, 𝑅) system

1. **Fixed order quantity** - Continuous review
   1. (𝑄, 𝑅) System  
      When the inventory level reaches 𝑅 (order point), order exactly 𝑄 units (order quantity)
2. **Continuous/Periodic Review Policy**
   1. (𝑠, 𝑆) System

* (continuous review) When inventory level reaches 𝑠 (order point), order up to 𝑆 (order-up-to level)
* Recall Periodic Review 𝑇, 𝑆 : Review your system every 𝑇 periods. If inventory level is below s (order point), order up to S (order-up-to level); otherwise, do not order
* Define two levels, 𝑠 < 𝑆, and let 𝑢 be the starting inventory at the beginning of a period. Then  
  If 𝑢 ≤ 𝑠, order 𝑆 – 𝑢  
  If 𝑢 > 𝑠, do not order
* In general, computing the optimal values of 𝑠 and 𝑆 is much more difficult than computing 𝑄 and 𝑅
* But, we can use a (𝑄, 𝑅) approximation:   
  𝑠 = 𝑅 and 𝑆 = 𝑅 + 𝑄

**Newsvendor Model**

Demand cdf: normal, uniform, discrete, etc.

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| Generally:  *cu = p - c*  *co = c – s + h*  *G (Q) = expected overage + underage cost*  F(Q) = Probability of satisfying all demand during the time period if 𝑄 units are purchased  Or probability that demand will not exceed Q  𝑐𝑜𝑃( 𝐷 ≤ Q\* ) = 𝑐𝑢𝑃(𝐷 ≥ 𝑄\* )  Stock out probability = 1 - F(Q\*) | *c = unit cost*  *p = selling price per unit*  *s = salvage value ( selling overage somewhere else at a discounted rate)*  *h = inventory cost per unit*  𝑐𝑜 = unit cost of overage (not enough demand), Cost of having positive inventory left over at the end of period  𝑐𝑢 = unit cost of underage (too much demand), Cost of unsatisfied demand  𝑄 is ordered, D is Demand  𝑄\*: order amount where  expected overage costs = expected underage costs  Expected underage costs: 𝑐𝑢𝑃(𝐷 ≥ 𝑄\* )  Expected overage costs : 𝑐𝑜𝑃( 𝐷 ≤ Q\* ) |
| **Normal Distribution demand**  𝑐𝑢 > 𝑐𝑜 ⇒ 𝑧 > 0 ⇒ 𝑄\* = 𝜎 𝑧 + 𝜇 increases in 𝜎  𝑐𝑢 < 𝑐𝑜 ⇒ 𝑧 < 0 ⇒ 𝑄\* = 𝜎 𝑧 + 𝜇 decreases in 𝜎  𝑐𝑢 = 𝑐𝑜 ⇒ 𝑧 = 0 ⇒ 𝑄\* = 𝜇 does not change in 𝜎  If shortages and excess inventory cost the same, order expected demand regardless of standard deviation (since 𝑧 = 0) | |
| 𝑄\* = 𝜎 𝑧 + 𝜇  Loss Function 𝐿 (𝑧) : is the expected amount that demand 𝐷 is greater than 𝑧.  When the demand is normally distributed, 𝐿 (𝑧) is the standard loss function  𝐿 (𝑧) is the expected number of lost sales as a fraction of the standard deviation  Therefore,  expected total lost sales = 𝑳(𝒛)𝝈  Expected sales = ( 𝜇 − 𝐿 (𝑧) 𝜎 )  *Expected Fill Rate = Expected Sales/Expected Demand*  = ( 𝜇 − 𝐿 (𝑧) 𝜎 ) / 𝜇 | μ = mean  σ = Standard deviation  z = from z table find z value of F (Q)  z = [alternatively, use NORM.INV(\*) function in Excel]  *Expected sales = mean demand – loss function of critical ratio multiplied by deviation.* |
| **Uniformly Distribution demand** | |
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**(Q, R) Model**

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| **Optimal Q and R**   1. First calculate Q using EOQ model 2. Calculate F(R) using that Q and  using find z from z table. 3. Using z value calculate R = 𝜎 z + 𝜇 4. Using z value calculate 𝑛 (𝑅) = 𝜎 𝐿(𝑧) 5. Using n(R) calculate Q 6. Unless otherwise specified, stop when either Q or R converges to one value | *R : stock amount when to place next order (Stock to meet demand during lead time)*  *𝐾 : Setup cost per order*  *ℎ : Holding cost per unit per unit time*  *𝑐 : Purchase price (cost) per unit*  *𝑝 : Penalty cost per unit of unsatisfied demand*  *λ : Expected demand per unit time*  *τ : Lead time*  *D : demand during the lead time*  *𝑓 (𝑥) : probability distribution*  *𝜇 : mean demand during the lead time*  *𝜎 : Standard deviation during the lead time*  𝑠 = Safety stock  = average inventory level before an order arrives  = 𝑟𝑒𝑜𝑟𝑑𝑒𝑟 𝑙𝑒𝑣𝑒𝑙 − 𝑒𝑥𝑝𝑒𝑐𝑡𝑒𝑑 𝑑𝑒𝑚𝑎𝑛𝑑 𝑑𝑢𝑟𝑖𝑛𝑔 𝑙𝑒𝑎𝑑 𝑡𝑖𝑚𝑒  = 𝑅 – 𝜇  𝑛(𝑅) = expected amount of shortage per cycle  = expected number of stockouts  = 𝜎 𝐿(𝑧)  = Standard Loss function (D ~ Normal)  F(R) = the proportion of order cycles in which no stock-outs occur  *= proportion of demand satisfied on time, which is the Type II service* |
| **Costs**  Expected Total Cost per unit time = holding cost + fixed cost + shortage cost  *Cost of uncertainty (COU) = Total cost in QR model - total cost in EOQ model* |
| *Mean and variance during Lead Time:*  *Eg: if* 𝜇 = 240 and *𝜎 = 600 yearly, and lead time is 2 months or 1/6 year :* | |
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**Service Levels**

* Type I service level (𝛼)  
  The proportion of cycles in which no stockouts occur  
  Example: 90% Type I service level ⇒ There are no stock outs in 9 out of 10 cycles (on average)

* Type II service level (fill rate, 𝛽)  
  𝛽 : Fraction of demand satisfied on time  
  1 – 𝛽 : Fraction of demand not met on time (stock out)

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| λ = expected arrival rate.  = 1/ E[A] (in arrivals per unit time)  𝜇 = 1/E[S]  = rate of service (in service per unit time)  capacity utilization = 𝜌𝑖 = 𝜆𝑖 / 𝜇𝑖  buildup rate = 𝜆𝑖 − 𝜇𝑖  total queue (in people hours) = area under the queue buildup diagram curve  Average queue = area under the queue buildup diagram curve / total time. (people)  Using little’s Law:  Avg wait (W) = avg queue (L) / avg. throughput (λ)  Utilization in case of 1 server = ρ = λ / 𝜇  Utilization in case of N server = ρ = λ / N 𝜇 | A = inter arrival distribution  E[A] = mean time between arrivals  S = distribution of service  E[S] = mean time of service. |

**M/M/1 Queue**

Memoryless arrival rate /Memoryless service rate / server is 1

Arrival rate 𝜆 follows a Poisson distribution (inter arrival follows exponential distribution)

Exponential service rate with 𝜇

𝑁 = 1 (single server)

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| Little’s Law ⇒ L = 𝜆 W  ⇒ Lq = 𝜆 Wq  𝜌 = 𝜆 / 𝜇  Idle time = no work time = 1 – utilization  = 1- 𝜌  = 1 - 𝜆 / N 𝜇  Probability of no line up = no work to do = idle time | 𝜆 = demand/input/arrival rate into system  𝜇 = service rate per server  𝑁 = number of servers  𝜌 = utilization rate = 𝜆 / 𝑁 𝜇  𝑊𝑞 = expected time customer spends in the queue in steady state, Wait time in queue  𝑊 = expected time customer spends in the system in steady state,  = Wait time plus service time  = Throughput time  𝐿𝑞 = expected number of customers in the queue in steady state, Number of customers waiting  𝐿 = expected number of customers in the system in steady state, Number of customers in the system |

**G/G/N Queueing Model**

General arrival distribution / General service distribution / N servers

average arrival rate 𝜆 = 1 / 𝐸[𝐴]

N servers average capacity utilization 𝜌 = 𝜆 / (𝑁 𝜇)

inter-arrival time distribution A = 𝐶𝐴 = 𝜎[𝐴] /𝐸[𝐴]

average individual service rate 𝜇 = 1 /𝐸[𝑆]

average system service rate = 𝑁 𝜇

service time distribution S = 𝐶𝑆 = 𝜎[𝑆] / 𝐸[𝑆]

𝐶𝐴 = coefficient of variation: inter-arrival times = 𝜎A / 𝜇A

𝐶𝑆 = coefficient of variation: service times = 𝜎S / 𝜇S

𝑊 = average time in the system

Exponential Distribution: 𝜇 = 𝜎 = 1 / 𝜆

**Service Pooling: Efficiency**

* Mitigate the impact of unpredictable variability!

Capacity Optimization

* Incorporate capacity costs and waiting cost into the model

**Model 1: Optimize Capacity in M/M/1**

* Given:
  + 𝑐 = service cost (per unit time per service rate)
  + ℎ = waiting cost (cost per unit time per customer in system)
  + 𝜆 = arrival rate (given)
  + 𝜇 = decision variable
* Objective: minimize long run average cost per unit time
* Assume 𝜌 = 𝜆 / 𝜇 < 1. Otherwise?
* Average cost per unit time: 𝐶(𝜇)= 𝑐 𝜇 + ℎ 𝐿(𝜇)

**Model 2: Optimize Arrival Rate in M/M/1**

* Given:
  + 𝑟 = reward per entering customer
  + ℎ = waiting cost (cost per unit time per customer in system)
  + 𝜆 = arrival rate (given)
  + 𝜇 = service rate (given)
* Objective: maximize the expected net benefit per unit of time
* Assume 𝜌 = 𝜆 / 𝜇 < 1
* Average cost per unit time: 𝐵 (𝜆) = 𝑟 𝜆 + ℎ 𝐿(𝜆)

*Note: x + = max (x, 0)*