Assignment 4

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Question

Explain Newton's method with equality constraints. Also explain Newton step with infeasible start.
Ans. -
Lets consider a problem with linear equality constraints.

$$\begin{aligned} \min_{x \in \mathcal{R}^N} \quad & f(x) \\ s.t. \quad & h(x) = 0 \end{aligned}$$

Here, assume f(x) is convex and twice differentiable.
Now, Let g(x) is gradient of f(x).
Let H(x) is Hessian of f(x) and
J(x) is Jacobian.
for non-constraint probelm, Newton' Method is:

$$x_+ = x + \Delta x_{nt}$$

here, Newton's step is:

$$\Delta x_{nt} = -H(x)^{-1} g(x)$$

To solve Δx_{nt}, We have to solve this linear equation:

$$\begin{bmatrix} H(x) & J(x)^T \\ J(x) & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \lambda \end{bmatrix} = \begin{bmatrix} -g(x) \\ -h(x) \end{bmatrix}$$

Assume that h(x) = Ax and solution is feasible for h(x) = 0, then linear system:

$$\begin{bmatrix} H(x^k) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

Let's suppose H(x) is strictly convex, and A is full rank matrix then linear system is solvable:

$$\begin{bmatrix} \Delta x \\ \lambda \end{bmatrix} = \begin{bmatrix} H(x^k) & A^T \\ A & 0 \end{bmatrix}^{-1} \begin{bmatrix} -g \\ 0 \end{bmatrix}$$

This is Newton's step with feasible start equation.

Algorithm 1 Newton's method with equality constraints

given starting point x $\in$ **dom** f and Ax = b, tolerance ϵ ¿ 0
repeat
Compute the Newton step and decrement $\Delta x_{nt}, \lambda(x)$.
Stopping criterion. quit if $\lambda^2/2 \leq \epsilon$
Line search. Choose step size t by backtracking line search.
Update. x := x + tΔx_{nt}

Newton Step with Infeasible start:

The Newton step of f(x) at an infeasible point x for the linear equality constrained problem is given by the solution of:

$$\begin{bmatrix} H(x) & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} \Delta x_{nt} \\ \lambda \end{bmatrix} = \begin{bmatrix} -g(x) \\ Ax - b \end{bmatrix}$$

Let x be the an infeasible point. Our goal is to find a step
Δ x s.t. x + Δx satisfies approximately the optimality condition. After linearization we get:

$$\begin{aligned} A(x+) &= b, \\ g(x) + H(x) + A^T\lambda &= 0, \end{aligned}$$

i.e., the definition of the Newton step

These are the primal and dual conditions of Newton's method. We have to update x and λ both in order to satisfy the optimal conditions i.e. we have to update till Ax = b.

Algorithm 2 Newton's method with equality constraints

given starting point x $\in$ **dom** f,λ tolerance ϵ ¿ 0, $\alpha \in (0, 1/2), \beta \in (0, 1)$
repeat
Compute primal and dual Newton steps $\Delta x_{nt}, \Delta\lambda_{nt}$
Backtracking line search on $\parallel r \parallel_2$
    t:=1
    while $\parallel r(x + t\Delta x_{nt}, \lambda + t\Delta\lambda_{nt} \parallel_2 > (1 - \alpha t) \parallel r(x, \lambda) \parallel_2$. $t := \beta t$
Update, $x := x + t\Delta x_{nt}, \lambda := \lambda + t\Delta\lambda_{nt}$
    until Ax=b and $\parallel r(x, \lambda) \parallel_2 \leq \epsilon$