Assignment 8

Vishal Kumar, MIT2019090

Question

$$\min_{x_1,x_2\in R^2} \frac{1}{2}x_1^2 - x_2^2 \;\; \textit{subject to } 1 - x_1^2 - x_2^2 \geq 0$$

Or, we can write it as:

$$\min_{x_1,x_2\in R^2} \frac{1}{2}x_1^2 - x_2^2 \;\; \textit{subject to } x_1^2 + x_2^2 - 1 \leq 0$$

Now, the Lagrangian is:

$$\mathcal{L}(x1, x2, \lambda) = \frac{1}{2}x_1^2 - x_2^2 + \lambda(x_1^2 + x_2^2 - 1)$$

a.) Ans.-

KKT conditions are:

$$i.)\quad \frac{\partial \mathcal{L}}{\partial x_1} = x_1(2\lambda + 1) = 0$$

$$ii.)\quad \frac{\partial \mathcal{L}}{\partial x_2} = x_2(2\lambda - 2) = 0$$

$$iii.)\quad \lambda(x_1^2 + x_2^2 - 1) = 0$$

$$iv.)\quad \lambda \geq 0$$

$$v.)\quad x_1^2 + x_2^2 - 1 \geq 0$$

From (i), either $x_1 = 0$ or $2\lambda + 1 = 0$ i.e. λ = -1/2 which does not satisfy (iv) condition. So, we can say that $x_1 = 0$.

From (ii), either $x_2 = 0$ or 2λ - 2 = 0 i.e. λ = 1 which satisfy KKT conditions.

From (iii), either $\lambda = 0$ or $x_1^2 + x_2^2 - 1 = 0$. Let's consider $\lambda = 1$, then $x_1 = 0$ and $x_1^2 + x_2^2 - 1 = 0$ i.e. x_2 = 1/-1
If $\lambda = 0$, then from (i) and (ii), $x_1 = 0$ and $x_2 = 0$

Now, the solution set:

For $\lambda = 0$ then $(x_1, x_2) = (0, 0)$
For $\lambda = 1$ then $(x_1, x_2) = (0, 1)$ or (0, -1)

b.) Ans.-

Slater's Equation

$$x_1^2 + x_2^2 - 1 < 0; \text{for any } x_1, x_2 \in R^2$$

In this problem, this condition holds true for $(x_1, x_2) = (0, 0)$
Now, we have to find out local minima with first order necessary conditions:

$$f(x) = \frac{1}{2}x_1^2 - x_2^2$$

$$\frac{\partial f}{\partial x_1} = x_1 = 0 \tag{1}$$

$$\frac{\partial f}{\partial x_2} = 2x_2 = 0 \tag{2}$$

From (1) and (2), we get local minima as (0,0) which is one of the KKT points.

c.) Ans.-

For x^* to be a global minimizer, we have to check if x^* satisfies these two conditions :

$$\mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x, \lambda^*); \text{ for all } x$$

$$\mathcal{L}(x^*, \lambda^*) \leq \mathcal{L}(x, \lambda^{)}; \text{ for all } \lambda \geq 0$$

$$\mathcal{L}(x^*, \lambda^*) = -1 \ ; \ for \ (x_1, x_2, \lambda) = (0, 1, 1) \ and \ (0, -1, 1)$$

For any x_1, x_2 that satisfy $x_1^2 + x_2^2 - 1 = 0$, we will get minimum value for $\mathcal{L}(x, \lambda^{)}, \mathcal{L}(x, \lambda^*) = -1$, which is equal to $\mathcal{L}(x^*, \lambda^*)$
Thus, we can say that the points $(0, 1), (0, -1)$ are global minimizers of original equation.

d.) Ans.-

The constraint is as follows :

$$x_1^2 + x_2^2 - 1 \leq 0$$

For point (0, 1):

$$x_1^2 + x_2^2 - 1 = 0 \leq 0$$

For point (0, -1):

$$x_1^2 + x_2^2 - 1 = 0 \leq 0$$

For point (0, 0):

$$x_1^2 + x_2^2 - 1 = -1 \leq 0$$

So, all the points in the solution satisfies the constraint.