Assignment 12

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Question

Discuss Semidefinite Programming (SDP) including Cone of PSD Matrices and SDP duality. Explain the barrier method for SDP.
Ans.-

Let $X \in S^n$. We can think of X as a matrix, or equivalently, as an array of n^2 components of the form $(x_{11}, ..., x_{nn})$. We can also just think of X as an object (a vector) in the space S^n . All three different equivalent ways of looking at X will be useful.

Now, the linear function of X can be written as, C(X) or $C \bullet X$, where

$$C \bullet X := \sum_{i=1}^{n} \sum_{j=1}^{n} C_{ij} X_{ij}$$

If X is a symmetric matrix, there is no loss of generality in assuming that the matrix C is also symmetric.

Now, let's define a semidefinite program. A semidefinite program (SDP) is an optimization problem of the form:

$$\begin{aligned} SDP: \quad minimise \quad & C \bullet X \\ s.t. \quad A_i \bullet X =& b_i, \quad i = 1, ., ., m, \\ X &\succeq 0 \end{aligned}$$

Semidefinite Programming Duality

The dual problem can be defined as:

$$\begin{aligned} SDD: \quad maximise \quad & \sum_{i=1}^{m} y_i b_i \\ s.t. \quad \sum_{i=1}^{m} y_i A_i + S =& C, \\ S &\succeq 0 \end{aligned}$$

And, the constraints of SDD state that the matrix S defined as:

$$S = C - \sum_{i=1}^{m} y_i A_i$$

must be semi-definite, that means:

$$C - \sum_{i=1}^{m} y_i A_i \succeq 0$$

Barrier Method for SDP:

For SDP, we need a barrier function whose values approach $+\infty$ as points X approach the boundary of the semi-definite cone S_n^+.
Let's consider the logarithmic barrier problem BSDP(θ) parameterized by the positive barrier parameter θ:

$$\begin{aligned} BSDP(\theta): \quad & C \bullet X - \theta ln(det(X) \\ & s.t. \quad A_i \bullet X = b_i, \quad i = 1, ., ., m, \\ & X \succ 0 \end{aligned}$$

Now, the objective function:

$$\Delta f_\theta(X) = C - \theta X^{-1}$$

Now, X = LL^T, because X is symmetric.

$$S = \theta X^{-1} = \theta L^{-T} L{-}1,$$

and,

$$\frac{1}{\theta} L^T S L = I$$

So, the KKT conditions of BSDP are:

- $A_i \bullet X = b_i$, i=1,.,.,m
- X $\succ$ 0, X = LL^T
- $\sum_{i=1}^{m} y_i A_i$ + S = C
- I - $\frac{1}{\theta} L^T S L = 0$