UNIT 2 Advanced Counting Principles

Structure **Page No.**

2.0 INTRODUCTION

In this unit, we continue our discussion of the previous unit on combinatorial techniques. We particularly focus on two principles of counting – the pigeonhole principle and the principle of inclusion-exclusion.

In Sec. 2.2 you will see how obvious the pigeonhole principle is. Its proof is very simple, and amazingly, it has several useful applications. We shall also include some of these in this section.

In Sec. 2.3, we focus on the principle (or formula) of inclusion-exclusion. As you will see, this principle tells us how many elements do not fit into any of n categories. We prove this result and also give a generalisation. Following this, in Sec. 2.4 we give several important applications of inclusion-exclusion.

We shall continue our discussion on combinatorial techniques in the next unit.

2.1 OBJECTIVES

After studying this unit, you should be able to:

- prove the pigeonhole principle, and state the generalised pigeonhole principle;
- identify situations in which these principles apply, and solve related problems;
- prove the principle of inclusion-exclusion;
- apply inclusion-exclusion for counting the number of surjective functions, derangements and for finding discrete probability.

2.2 PIGEONHOLE PRINCIPLE

Let us start with considering a situation where we have 10 boxes and 11 objects to be placed in them. Wouldn't you agree that regardless of the way the objects are placed in the two boxes at least one box will have more than one object in it? On the face of it, this seems obvious. This is actually an application of the pigeonhole principle, which we now state.

Theorem 1 (The Pigeonhole Principle): Let there be n boxes and (n+1) objects. Then, under any assignment of objects to the boxes, there will always be a box with more than one object in it.
This can be reworded as: if m pigeons occupy n pigeonholes, where m > n, then there is at least one pigeonhole with two or more pigeons in it.

Proof: Let us label the n pigeonholes 1, 2, …, n, and the m pigeons $p_1, p_2, …, p_m$. Now, beginning with p_1, we assign one each of these pigeons the holes numbered 1, …, n, respectively. Under this assignment, each hole has one pigeon, but there are still (m–n) pigeons left. So, in whichever way we place these pigeons, at least one hole will have more than one pigeon in it. This completes the proof!

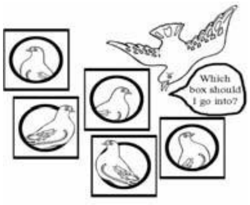

Fig. 1 The pigeonhole principle

This result appears very trivial, but has many applications. For example, using it you can show that:

- if 8 people are picked in any way from a group, at least 2 of them will have been born on the same weekday.
- in any group of 13 people, at least two are born in the same month.

Let us consider some examples of its application, in detail.

Example 1: Assuming that friendship is mutual, show that in any group of people we can always find two persons with the same number of friends in the group.

Solution: If there are n persons in the group, then let the number of friends the ith person has be f(i), i = 1,…, n. Clearly, f(i) can take values only between 0 and (n – 1).

If some f(i) is 0, it means that the ith person does not have any friends in the group. In this case, no person can be friends with all the other (n – 1) people. So, no f(i) can be (n – 1). So, only one of the values 0 or (n – 1) can be present among the f(i)'s. So, the n f(i)'s can take only (n – 1) distinct values. Therefore, by the pigeonhole principle, two f(i)'s must be equal. Then the corresponding i's have the same number of friends in the group.

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Example 2: Suppose 5 points are chosen at random within or on the boundary of an equilateral triangle of side 1 metre. Show that we can find two points at a distance of at most ½ metre.

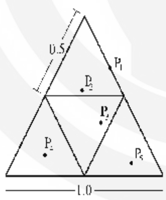

Fig. 2

Solution: Divide the triangle into four equilateral triangles of side ½ m by joining the midpoints of the sides by three line segments (see Fig. 2). These four triangles may now be considered as boxes and the five points as objects. By the pigeonhole principle, at least one of these smaller triangles will have two points in or on it. Clearly, the distance between these two points is at most ½ metre.

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Example 3: Given any ten different positive integers less than 107, show that there will be two disjoint subsets with the same sum.

Solution: The highest numbers we could be given would be 97, 98, …, 106, which add up to 1015. So, consider pigeonholes marked 0, 1, 2,…, 1015. The set of 10 positive integers have 2^{10} = 1024 subsets. Put a subset in the pigeonhole marked with the sum of the numbers in the set. The 1024 subsets have to be put in 1016 pigeonholes. So, some pigeonhole will have more than one subset with the same sum.

Now, note that two subsets that we get with the same sum, may not be disjoint. But, by dropping the common elements in them, we are left with disjoint subsets with the same sum.

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Here are some related exercises for you to do.

E1) If 10 points are chosen in an equilateral triangle of side 3 cms., show that we can find two points at a distance of at most 1 cm.

E2) On 11 occasions a pair of persons from a group of 5 was called for a function. Show that some pair of persons must have attended the function at least twice.

E3) Four persons were found in a queue, independently, on 25 occasions. Show that at least on two occasions they must have been in the queue in the same order.

As you know, **mathematics develops through a process of generalisation**. You know that the principle is valid for n+1 objects and n boxes. It is natural to ask: what if we have, say, 4n+1 objects and 4 boxes? Can we prove a similar principle? In fact, we can, as given below.

Theorem 2 (The Generalized Pigeonhole Principle): If nm + 1 objects are distributed among m boxes, then at least one box will contain more than n objects.

This can be reworded as: Let k and n be positive integers. If k balls are put into n boxes, then some box contains at least [k/n] + 1 balls, where [x] denotes the greatest integer less than x.

Proof: We prove this by contradiction (see Unit 2, Block 1). Suppose all the m boxes have at most n objects in them. Then the total number of objects is at most nm, a contradiction. Hence, the theorem.

Applying this result, we see, for example, that suppose 479 students are enrolled in the course Discrete Mathematics, consisting of 6 units. Then, at least $\left\lfloor \frac{479}{6} \right\rfloor + 1 = 80$ students are studying the same unit at a given point of time.

Let us consider a few more examples of the application of this principle.

Example 4: Show that in any group of 30 people, we can always find 5 people who were born on the same day of the week.

Solution: 30 people can be assigned to 7 days of the week. Then at least $\left\lceil \frac{30}{7} \right\rceil + 1 = 5$ of them must have been born on the same day.

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Example 5: 20 cards, numbered from 1 to 20, are placed face down on a table. 12 cards are selected one at a time and turned over. If two of the cards add up to 21, the player loses. Is it possible to win this game?

Solution: The pairs that can add up to 21 are (1, 20), (2, 19),…, (10,11). So, there are 10 such pairs. In turning 12 cards, at least one of these pairs will be included. Therefore, the player will lose.

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Example 6: Show that every sequence of $n^2 + 1$ distinct integers includes either an increasing subsequence of n + 1 numbers or a decreasing subsequence of n + 1 numbers.

Solution: Let the sequence be $a_1, a_2, \ldots, a_{n^2+1}$. Suppose there is no increasing subsequence of n + 1 numbers. For each of these a_ks, let s(k) be the length of the longest increasing subsequence beginning at a_k. Since all n^2+1 of the s(k)'s are between 1 and n, at least $\left\lceil \frac{n^2+1}{n} \right\rceil + 1 = n + 1$ of these numbers are the same. (The s(k)'s are the objects and the numbers from 1 to n are the boxes.)

Now, if i < j and s(i) = s(j), then $a_i > a_j$. Otherwise a_i followed by the longest increasing subsequence starting at a_j would be an increasing subsequence of length s(j) + 1 starting at a_i. This is a contradiction, since s(i) = s(j).

Therefore, all the n + 1 integers a_k, for which s(k) = m, must form a decreasing subsequence of length at least n + 1.

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Example 7: Take n integers, not necessarily distinct. Show that the sum of some of these numbers is a multiple of n.

Solution: Let S(m) be the sum of the first m of these numbers. If for some r and m, r < m, S(m) – S(r) is divisible by n, then $a_{r+1} + a_{r+2} + \ldots + a_m$ is a multiple of n. This also means that S(r) and S(m) leave the same remainder when divided by n. So, if we cannot find such pairs m and r, then it means that the n numbers S(1), S(2),…, S(n) leave different remainders when divided by n. But there are only n possible remainders, viz., 0, 1, 2,…, (n – 1). So, one of these numbers must leave a remainder of 0. This means that one of the S(i) s is divisible by n. This completes the proof.

In fact, in this example we have proved that one of the sums of consecutive terms is divisible by n.

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You may like to try some exercises now.

E4) If any set of 11 integers is chosen from 1, …,20, show that we can find among them two numbers such that one divides the other.

E5) If 100 balls are placed in 15 boxes, show that two of the boxes must have the same number of balls.

E6) If $a_1, a_2, \ldots, a_n$ is a permutation of 1, 2,…, n and n is odd, show that the product $(a_1 - 1)(a_2 - 2)\ldots(a_n - n)$ must be even.

There are several corollaries to Theorem 2. We shall present one of them here.

Theorem 3: If a finite set S is partitioned into s subsets, then at least one of the subsets has $\frac{|S|}{k}$ or more elements.

Proof: Let $A_1, \ldots, A_k$ be a partition of S. (This means that $A_i \cap A_j = \phi$ for $i \neq j$ and $S = A_1 \cup A_2 \cup \ldots \cup A_k$.) Then the average value of $|A_i|$ is $\frac{1}{k}[\,|A_1| + \ldots + |A_k|\,] = \frac{|S|}{k}$.

So the largest A_i has at least $\frac{|S|}{k}$ elements.

A consequence of this result is the following theorem.

Theorem 4: Consider a function f: S → T, where S and T are finite sets satisfying $|S| > r\,|T|$. Then at least one of the sets $f^{-1}(t)$, $t \in T$, has more than r elements. ($f^{-1}(t)$ denotes the inverse image of the set {t}, i.e., $f^{-1}(t) = \{x \in S : f(x) = t\}$.)

Proof: The family $\{f^{-1}(t): t \in T\}$ partitions S into k sets with $k \le |T|$. By Theorem 3, some set in this family, say $f^{-1}(t')$, has at least $\frac{|S|}{k}$ members. Since $\frac{|S|}{k} \ge \frac{|S|}{T} > r$ by our hypothesis, $f^{-1}(t')$ has more than r elements.

Corollary: If $f: S \to T$ and $|S| > |T|$, then **f is not injective.**

Proof: Putting r = 1 in Theorem 4, we see that at least one of the sets $f^{-1}(t)$ has more than one element.

We conclude this section with some more extensions of the pigeonhole principle.

Theorem 5: Suppose we put an infinity of objects in a finite number of boxes. Then at least one box must have an infinity of objects.

Proof: If every box contains only a finite number of objects, then the total number of objects must be finite. Hence the theorem.

Theorem 6 (A generalisation of Theorem 3): Let $A_1, A_2, \ldots, A_k$ be subsets of a finite set S such that each element of S is in at least t of the sets A_i. Then the average number of elements in the A_is is at least $t.\frac{|S|}{k}$. (Note that, in this statement, the sets A_i may overlap.)

We leave the proof to you to do, and give you some related exercises now.

E7) Every positive integer is given one of the seven colours in VIBGYOR. Show that at least one of the colours must have been used infinitely many times.

E8) Let A be a fixed 10-element subset of $\{1, 2, \ldots, 50\}$. Show that A possesses two different 5-element subsets, the sum of whose elements are equal.

E9) The positive integers are grouped into 100 sets. Show that at least one of the sets has an infinity of even numbers. Is it necessary that at least one set should have an infinity of even numbers and an infinity of odd numbers?

Let us now consider another very important counting principle.

2.3 INCLUSION-EXCLUSION PRINCIPLE

Let us begin with considering the following situation: In a sports club with 54 members, 34 play tennis, 22 play golf, and 10 play both. There are 11 members who play handball, of whom 6 play tennis also, 4 play golf also, and 2 play both tennis and golf. How many play none of the three sports?

To answer this, let S represent the set of all members of the club. Let T represent the set of tennis playing members, G represent the set of golf playing members, and H represent the set of handball playing members. Let us represent the number of elements in A by $|A|$. Consider the number $|S| - |T| - |G| - |H|$. Is this the answer to the problem? No, because those who are in T as well as G have been subtracted twice. To compensate for this double subtraction, we may now consider the number $|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T|$. Is this the answer? No, because those playing all the three games have been subtracted thrice and then added thrice. But those members have to be totally excluded also. Hence, we now consider the number

$|S|-|T|-|G|-|H|+|T\cap G|+|G\cap H|+|H\cap T|-|T\cap G\cap H|$. This is the correct answer. This reduces to 54 – 34 – 22 – 11 + 10 + 6 + 4 – 2 = 5.

To solve this problem we have made inclusions and exclusions alternately to arrive at the correct answer. This is a simple case of **the principle of inclusion and exclusion**. It is also known as the **sieve principle** because we subject the objects to sieves of a progressively finer mesh to arrive at a certain grading.

Let us state and prove this principle now.

A^c, or $\bar{A}$, denotes the complement of the set A

Theorem 7 (The inclusion-exclusion formula): Let $A_1, A_2, \ldots, A_n$ be n sets in a universal set U consisting of N elements. Let S_k denote the sum of the sizes of all the sets formed by intersecting k of the A_is at a time. Then the number of elements in none of the sets $A_1, A_2, \ldots, A_n$ is given by

$$\left|\bar{A}_1\cap \bar{A}_2\cap \ldots \cap \bar{A}_n\right| = N - S_1 + S_2 - S_3 + \ldots + (-1)^k S_k + \ldots + (-1)^n S_n.$$

RHS is short for 'right-hand side'.

Proof: The proof is on the same lines of the counting argument given in the 'sports club' example at the beginning of this section. If an element is in none of the A_is, then it should be counted only once, as part of 'N' in the RHS of the formula above. It is not counted in any of the S_ks since it is in none of the A_is.

Next, an element in exactly one A_i, say A_r, is counted once in N, and once in S_1, and in none of the other S_ks. So the net count is 1–1 = 0.

Finally, take an element x in exactly m of the A_is. This is counted once in N, m times in S_1, C(m, 2) times in S_2 (since x is in C(m, 2) intersections $A_i\cap A_j$),…, C(m, k) times in S_k for $k \le m$. x is not counted in any S_k for $k > m$. So the net count of x in the RHS of the formula is

$1 - C(m,1) + C(m,2) - \ldots + (-1)^k C(m, k) + \ldots + (-1)^m C(m, m) = 0$, by Identity 2 in Sec. 2.5.

So the only elements that have a net count of 1 in the RHS are those in $\bigcap_{i=1}^{n} \bar{A}_i$. The rest have a net count of 0. Hence the formula.

From this result, we immediately get the following one.

Corollary: Given the situation of Theorem 7,

$$\left|A_1 \cup A_2 \cup \ldots \cup A_n\right| = S_1 - S_2 + \ldots + (-1)^{k-1} S_k + \ldots + (-1)^{n-1} S_n.$$

Why don't you try and prove this result? (see E 10.)

What the inclusion-exclusion principle tells us is that to calculate the size of $A_1\cup A_2\cup\ldots\cup A_n$, calculate the size of all possible intersections of the sets A_1, $A_2,\ldots,A_n$. Add the results obtained by intersecting an odd number of the sets, and then subtract the results obtained by intersecting an even number of the sets. Therefore, this principle is ideally suited to situations in which

i) we just want the size of $A_1\cup A_2\cup\ldots\cup A_n$, not a listing of its elements, and

ii) multiple intersections are fairly easy to count.

Now let us consider some examples in which Theorem 7 is applied.

Example 8: How many ways are there to distribute r distinct objects into five (distinct) boxes with

i) at least one empty box?

ii) no empty box ($r \ge 5$)?

Solution: Let U be all possible distributions of r distinct objects into five boxes. Let A_i denote the set of possible distributions with the ith box being empty.

i) Then the required number of distributions with at least one empty box is $|A_1 \cup A_2 \cup \ldots \cup A_5|$. We have N = 5^r. Also, $|A_i| = (5-1)^r$, the number of distributions in which the objects are put into one of the remaining four boxes. Similarly, $|A_i \cap A_j| = (5-2)^r$, and so forth. Thus, by the corollary above, we have

$$|A_1 \cup \ldots \cup A_5| = S_1 - S_2 + S_3 - S_4 + S_5$$
$$= C(5,1)4^r - C(5,2)3^r + C(5,3)2^r - C(5,4)1^r + 0$$

ii) $\left|\bar{A}_1 \cap \bar{A}_2 \cap \ldots \cap \bar{A}_5\right| = 5^r - C(5,1)4^r + C(5,2)3^r - C(5,3)2^r + C(5,4)1^r$, by Theorem 7.

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Example 9: How many solutions are there to the equation x + y + z + w = 20, where x, y, z, w are positive integers such that $x \le 6$, $y \le 7$, $z \le 8$, $w \le 9$?

Solution: To use inclusion-exclusion, we let the objects be the solutions (in positive integers) of the given equation. A solution is in A_1 if x > 6, in A_2 if y > 7, in A_3 if z > 8, and in A_4 if w > 9. Then what we need is $\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right|$.

Now, to find the total number of **positive** solutions to the given equation, we rewrite it as $x_1+y_1+z_1+w_1 = 16$, where $x_1 = x+1$, $y_1 = y+1$, $z_1 = z+1$, $w_1 = w+1$. Any non-negative solution of this equation will be a positive solution of the given equation. So, the number of positive solutions is

N = C(16+4–1, 16) (see Example 11 of Unit 2)
= C(19, 3).

Similarly, $|A_1| = C(13,3), |A_2| = C(12,3), |A_3| = C(11,3)$,

$|A_4| = C(10,3)$, $|A_1 \cap A_2| = C(6,3), |A_2 \cap A_3| = C(5,3)$, and so on. **Note** that for a solution to be in 3 or more A_i s, the sum of the respective variables would exceed 20, which is not possible. By inclusion- exclusion, we obtain

$\left|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4\right|$ = C(19, 3) – C(13, 3) – C(12, 3) – C(11, 3) – C(10, 3) + C(6, 3) + C(5, 3) + C(4, 3) + C(4, 3) + C(3, 3) = 217.

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Now you may try the following exercises.

E10) Prove the corollary to Theorem 7.

E11) How many numbers from 0 to 999 are not divisible by either 5 or 7?

Let us now consider applications of the inclusion-exclusion principle to some specific problem types.

2.4 APPLICATIONS OF INCLUSION-EXCLUSION

In this section we shall consider three broad kinds of applications — for counting the number of surjective functions, finding probability and finding the number of derangements.

2.4.1 Application to Surjective Functions

Let us first recall that a function $f : S \to T$ is called **surjective** (or **onto**) if $f(S) = T$, that is, if for every $t \in T$ $\exists s \in S$ such that $f(s) = t$. Now let us prove a very useful result regarding the number of such functions.

Theorem 8: The number of functions from an m-element set **onto** a k-element set is

$\sum_{i=0}^{k} (-1)^i C(k,i)(k-i)^m$, where $1 \le k \le m$.

Proof: We will use the inclusion-exclusion principle to prove this. For this, we define the objects to be all the functions (not just the onto functions) from M, an m-element set, to K, a k-element set. For these objects, we will define A_i to be the set of all $f: M \to K$ for which the ith element of K is not in f(M). Then what we want is $\left|\bar{A}_1 \cap \ldots\ldots \cap \bar{A}_k\right|$.

Now, the total number of functions from M to K is k^m. Also, the number of mappings that exclude a specific set of i elements in K is $(k - i)^m$, and there are C(k, i) such sets. Therefore, $|A_i| = (k-1)^m, |A_i \cap A_j| = (k-2)^m$, and so on.

Now, applying Theorem 7, we get

$$\left|\bar{A}_1 \cap \ldots \cap \bar{A}_k\right| = k^m - C(k,1)(k-1)^m + C(k,2)(k-2)^m - \ldots + (-1)^{k-1}C(k,k-1)1^m$$

Hence the result.

For example, the number of functions from a five-element set onto a three-element set are $\sum_{i=0}^{k} (-1)^i C(k,i)(k-i)^m$ for m = 5 and k = 3, that is, $3^5 - 3.2^5 + 3.1^5 = 150$.

Why don't you try some exercises now?

E12) Eight people enter an elevator. At each of four floors it stops, and at least one person leaves the elevator. After four floors the elevator is empty. In how many ways can this happen?

E13) How many six-digit numbers contain exactly three different digits?

Now we look at another application.

2.4.2 Application to Probability

An important application of the principle of inclusion-exclusion is used in probability. We have the following theorem.

Theorem 9: Suppose $A_1, A_2, \ldots, A_n$ are n events in a probability space. Then

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{r=1}^{n} (-1)^{r+1} \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r})$$

Proof: Let us begin by observing that $A_1 \cup A_2 \cup \ldots \cup A_n$ means that at least one of the events $A_1, A_2, \ldots, A_n$ occurs. Now, let the ith property be that an outcome belongs to the event A_i. By De Morgan's law, $\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}$ is the complement of $A_1 \cup A_2 \cup \ldots \cup A_n$. But the principle of inclusion-exclusion gives

$$\left| \overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n \right| = N - \sum (-1)^r \sum_{1 \le i_1 < i_2 < \ldots < i_r \le n} \left| A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_r} \right|,$$ where N is the total number of outcomes.

Now, we divide throughout by N and note that

$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - P(\overline{A}_1 \cap \overline{A}_2 \cap \ldots \cap \overline{A}_n)$, to get the result.

Let us consider an example of the use of this result.

Example 12: Find the probability of a student in a college studying Japanese, given the following data:

All students have to study at least one language out of Hindi, Spanish and Japanese. 65 study Hindi, 45 study Spanish and 42 study Japanese. Further, 20 study Hindi and Spanish, 25 study Hindi and Japanese, 15 study Spanish and Japanese, and 8 study all 3 languages.

Solution: The total number of students is $|H \cup S \cup J|$, where H, S and J denote the number of students studying Hindi, Spanish and Japanese, respectively. By the inclusion-exclusion principle,

$$\begin{aligned} |H \cup S \cup J| &= |H| + |S| + |J| - |H \cap S| - |H \cap J| - |S \cap J| + |H \cap S \cap J| \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Therefore, the required probability is $\frac{|J|}{100} = 0.42$.

* * *

You could do the following exercises now.

E14) What is the probability that a 13-card hand has at least one card in each suit?

E15) What is the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5 nor 7?

Let us now come to the use of inclusion-exclusion for counting the number of a particular kind of permutation.

2.4.3 Application to Derangements

As you know, a permutation of a set is an arrangement of the elements of a set.

So, for example, a rearrangement $1 \to 1, 2 \to 2, 4 \to 3, 3 \to 4$ is a permutation of the 4-element set {1, 2, 3, 4}. In this permutation, the position of the elements 1 and 2 are **fixed**, but the positions of 3 and 4 have been interchanged. Now consider the permutation $1 \to 4, 2 \to 1, 3 \to 2, 4 \to 3$, of {1, 2, 3, 4}, in which the position of

every element has been changed. This is an example of a derangement, a term we shall now define.

Definition: A **derangement** of a set S is a permutation of the elements of S which does not fix any element of S, i.e., it is a rearrangement of the elements of S in which the position of every element is altered.

So, if we treat a permutation as a 1-to-1 function from S to S, then a derangement is a function $f:S \to S$ such that $f(s) \neq s \ \forall \ s \in S$.

We have the following theorem regarding the number of derangements.

Theorem 10: The number of derangements of an n-element set is $D_n = n! \sum_{i=0}^{n} \frac{(-1)^i}{i!}$.

Proof: Let A_i be the set of all permutations of the n-element set that fix i $\forall$ i = 1, …, n. Then

$$D_n = \left| \bigcap_{i=1}^{n} \bar{A}_i \right| = n! - \sum_i |A_i| + \sum_{i<j} |A_i \cap A_j| - \ldots + (-1)^n |A_1 \cap \ldots \cap A_n|$$

$$= n! - C(n, 1)(n-1)! + C(n, 2)(n-2)! - \ldots + (-1)^n C(n, n)0!$$

$$= n! \left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$$ which is the expression we wanted.

Remark: The expression $\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \ldots + (-1)^n \frac{1}{n!}\right)$ is the beginning of the expansion for e^{-1}. Even for moderately large values of n, D_n is very close to $n!e^{-1} = 0.36788\, n!$.

As an extension of Theorem 10, we have the following results.

Theorem 11: For a set of n objects, the number of permutations in which

(i) only r of these n objects are deranged is

$n! - C(r, 1)(n-1)! + C(r, 2)(n-2)! - \ldots + (-1)^r C(r, r)(n-r)!$;

(ii) exactly r elements are fixed is $C(n, r)\, D_{n-r}$.

We will not prove these formulae here, but shall consider some examples of their applications.

Example 12: Let n books be distributed to n children. The books are returned and distributed to the children again later on. In how many ways can the books be distributed so that no child will get the same book twice?

Solution: The required number is $(n!)^2 e^{-1}$, since corresponding to each first distribution, there are $(n!)e^{-1}$ ways of distributing again.

* * *

Example 13: Suppose 10 people have exactly the same briefcases, which they leave at a counter. The cases are handed back to the people randomly. What is the probability that no one gets the right case?

Solution: The number of possibilities favourable to the event is D_{10}. The total number of possibilities is 10!. Thus, the probability that none will get the right briefcase is $D_{10}/10! = 0.36788$.

* * *

Note that, since $D_n \approx n!e^{-1}$, the possibility in all such examples is essentially e^{-1}, which is independent of n.

You may now try the following exercises.

E16) Each of the n guests at a party puts on a coat when s/he leaves. None of them gets the correct coat. In how many ways can this happen? In how many ways can just one of the guests get the right coat?

E17) In how many ways can the integers 1, 2, 3,…,9 be permuted such that no odd integer will be in its natural position.

E18) Find the number of permutations in which exactly four of the nine integers 1, 2,…,9 are fixed.

With this we come to the end of this unit. In the next unit we shall continue our discussion on 'counting' from a slightly different perspective. Let us now summarise what we have covered in this unit.

2.5 SUMMARY

In this unit, you have studied the following points.

1. The pigeonhole principle, stated in several forms, its proof, and its applications.

2. The generalized pigeonhole principle, its proof, and applications.

3. The inclusion-exclusion principle, and its proof.

4. Finding the number of surjective functions, the discrete probability and the number of derangements, by using the inclusion-exclusion principle.

2.6 SOLUTIONS /ANSWERS

E1) By drawing lines parallel to the sides and through the points trisecting each side, we can divide the equilateral triangle into 9 equilateral triangles of side 1 cm. Thus, if 10 points are chosen, at least two of them must lie in one of the 9 triangles.

E2) 5 persons can be paired in C(5, 2) =10 ways. Hence, if pairs are invited 11 times, at least one pair must have been invited twice or more times, by the pigeonhole principle.

E3) Four persons can be arranged in a line in 4! = 24 ways. Hence, if we consider 25 occasions, at least on two occasions the same ordering in the queue must have been found, by the pigeonhole principle.

E4) Consider the following grouping of numbers:

{1, 2, 4, 8, 16}, {3, 9, 18}, {5, 15}, {6, 12}, {7, 14}, {10, 20}, {11}, {13}, {17}, {19}.

There are 10 groupings, exhausting all the 20 integers from 1 to 20. If 11 numbers are chosen it is impossible to select at most one from each group. So two numbers have to be selected from some group. Obviously one of them will divide the other.

E5) Suppose $x_1, x_2,\ldots, x_{15}$ are the number of balls in the 15 boxes, listed in increasing order, assuming that all these numbers are different. Then, clearly, $x_i \geq i - 1$ for $i = 1, 2,\ldots,15$. But then, $\sum_{i=1}^{15} x_i \geq 14.15/2 = 105$.

But the total number of balls is only 100, a contradiction. Thus, the x_is cannot all be different.

E6) In the sequence $a_1, a_2,\ldots,a_n$, there are $(n+1)/2$ odd numbers and $(n-1)/2$ even numbers because n is odd. Hence, it is impossible to pair all the a_is with numbers from 1, 2,…, n with opposite parity (evenness and oddness). Hence, in at least one pair (i, a_i), both the numbers will be of the same parity. This means that the factor (a_i-i) will be even, and hence the product will be even.

E7) Consider the seven colours as containers, and the integers getting the respective colour as their contents. Then we have a distribution of an infinite number of objects among 7 containers. Hence, by Theorem 5, at least one container must have an infinity of objects, that is, the colour of that container must have been used an infinite number of times.

E8) Let H be the family of 5-element subsets B of A. For each B in H, let f(B) be the sum of the numbers in B. Obviously, we must have

$f(B) \geq 1 + 2 + 3 + 4 + 5 = 15$, and $f(B) \leq 46 + 47 + 48 + 49 + 50 = 240$.

Hence, $f: H \to T$ where $T = \{15, 16,\ldots, 240\}$.

Since $|T| = 226$ and $|H| = C(10,5) = 252$, by Theorem 4, H contains different sets with the same image under f, that is different sets, the sums of whose elements are equal.

E9) The 100 collections can be considered as containers. There are an infinity of even numbers. When these even numbers are distributed into 100 containers, at least one container must have an infinity of them, by Theorem 5.

E10) The inclusion-exclusion formula can be rewritten as

$$\left|\bar{A}_1 \cap \ldots \cap \bar{A}_n\right| = N - (S_1 - S_2 + \ldots + (-1)^{n-1} S_n.$$

Also, we know that $\left|\bar{A}_1 \cap \ldots \cap \bar{A}_n\right| = N - |A_1 \cup \ldots \cup A_n|$.

Hence the result.

E11) Let the objects be the integers 0, 1,…, 999. Let A_1 be the set of numbers divisible by 5, and A_2 the set of numbers divisible by 7. Now, $N = 1000$, $|A_1| = 200$, $|A_2| = 143$ and $|A_1 \cap A_2| = 29$. So, by Theorem 7, the answer is $1000 - 200 - 143 + 29 = 686$.

E12) The answer to this problem is clearly the number of functions from an 8-element set (the set of people) onto a set of 4-elements (the set of floors). This number is

$$\sum_{i-0}^{4} C(4,i)(4-i)^8 = 4^8 - 4.3^8 + 6.2^8 - 4.1^8.$$

E13) We can choose three digits in $C(10,3) = 120$ ways.

The number of 6-digit numbers, using all the three digits, is the same as the number of functions from a 6-set onto a 3-set. This number is

$3^6 – 3.2^6 + 3.1^6 = 540$.

Hence, the answer is 120.540 = 64800. This will include numbers starting with 0 also.

E14) The total number of ways in which 13 cards can be chosen from a deck of 52 cards is C(52, 13).

If A_i is a choice of cards, none of which are from the ith suit, for i = 1,2,3,4, then $|A_i| = C(39,13), |A_i \cap A_j| = C(26,13)$, and $C(A_i \cap A_j \cap A_k) = C(13,13)$.

So, $\left|\cap \bar{A}_i\right| = C(52,13) - 4C(39,13) + C(4,2)C(26,13) - C(4,3)\, C(13, 13)$

Hence, the required probability is $\dfrac{\left|\cap \bar{A}_i\right|}{C(52,13)}$.

E15) If A,B,C,D are the integers divisible by 2, 3, 5, 7, respectively, then

$$\left|\bar{A} \cap \ldots \cap \bar{D}\right| = 10{,}000 - \left[\frac{10000}{2}\right] - \left[\frac{10000}{3}\right] - \left[\frac{10000}{5}\right] - \left[\frac{10000}{7}\right]$$

$$+ \left[\frac{10000}{6}\right] + \left[\frac{10000}{15}\right] + \left[\frac{10000}{35}\right] + \left[\frac{10000}{14}\right] + \left[\frac{10000}{21}\right] + \left[\frac{10000}{10}\right]$$

$$- \left[\frac{10000}{30}\right] - \left[\frac{10000}{42}\right] - \left[\frac{10000}{105}\right] - \left[\frac{10000}{70}\right] + \left[\frac{10000}{210}\right]$$

= 2285, where [x] denotes the greatest integer $\le$ x.

Hence, the required probability is $\dfrac{2285}{10000} = 0.23$.

E16) If A_r is the event that the rth person gets the right coat, then by Theorem 7,

$$\left|\cap \bar{A}_i\right| = n! - \sum_r |A_r| + \sum_{r,s} |A_r \cap A_s| - \ldots$$

$$= n! - n(n-1)! + C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots$$

$$= C(n,2)(n-2)! - C(n,3)(n-3)! + \ldots$$

$$= n! \left(\sum_{r=0}^{n} (-1)^r \frac{1}{r!}\right)$$

The number of ways in which only one person receives the correct coat is the sum of all possible intersections of (n–1) $\bar{A}_i$ s. This is

$$n!\; n(n-1)! \left(\sum_{r=0}^{n-1} (-1)^r \frac{1}{r!}\right) = n! \left(\sum_{r=0}^{n-1} (-1)^r \frac{1}{r!}\right).$$

E17) 1, 3, 5, 7, 9 are the odd integers.
By Theorem 11(i), the required number of ways is
9! – C(5, 1)8! + C(5, 2)7! – C(5,3)6! + C(5, 4)5! – C(5, 5)4!

E18) By Theorem 11(ii), the required number of permutations is
$C(9,4)D_{9-4} = C(9,4)D_5$.