UNIT 1 BASIC PROPERTIES OF GRAPHS

Structure **Page Nos.**

1.0 INTRODUCTION

In our everyday life, we come across various problems requiring us to look at structures of objects and some family of subsets of those objects. For example, we may need to put up an electric network where different electrical gadgets are the objects, and they are to be connected by electric wires. The lengths of these wires may not be important, but it is important to know how the wires are connected. This means that it is important to know which gadgets are connected to the endpoints of the wires.

Another example is that of the public transport system in a city. Various places are the objects here, the bus routes are the connections, and we need to know the places connected to the railway station, say. Yet another problem could be that of establishing communication links between different centres.

All these problems can be represented pictorially with a set of dots called vertices and a set of edges connecting various pairs of dots. Such representations are called graphs. The solutions to the given problems can be obtained by analysing their graphs. Ideas given by various mathematicians to solve such problems gave birth to a branch of mathematics called **graph theory.**

In this unit we shall begin with defining a graph and study some of its basic properties. In Sec.1.2 and Sec.1.3 we have defined various types of graphs. Throughout the sections, these graphs and their properties are illustrated with the help of examples.

Next, Sec.1.4 is devoted to the study of subgraphs.

In the following units of this block you would notice how these simple basic ideas help us to solve many tough problems of day-to-day life. We can have graphs with vertices representing points in space, people, animal species, sports teams etc., and edges might represent roads, telephone lines, communication channels etc.

Before getting into the subject matter, let us take a look at the broad objectives of this unit.

1.1 OBJECTIVES

After studying this unit, you should be able to

- identify different ways of representing a graph;
- identify complete graphs, paths, cycles;

- obtain the complement of a graph;
- write the degree sequence of a graph and obtain the number of edges of a graph using the degrees of vertices;
- identify graphs isomorphic to a given graph;
- obtain a subgraph of G induced by a subset of V(G);
- draw a regular graph on p vertices having degree r, where p and r are integers with r < p, such that at least one of them is even.

1.2 GRAPHS

You must have used the term 'graph' while studying the calculus of real-valued functions of a real variable. It is a set of the form $\{(x, f(x))$: x is in the domain of the function f$\}$. Such a set helps us to analyse the function f. The graphs that we will define presently are a little different. Before giving a formal definition of a graph let us look at some simple examples.

Example 1: Take two points x_1, x_2 in the plane and join them by any line. This line may be a straight line or an arc. There are many ways of joining these points. In Fig.1 we have shown three different ways.

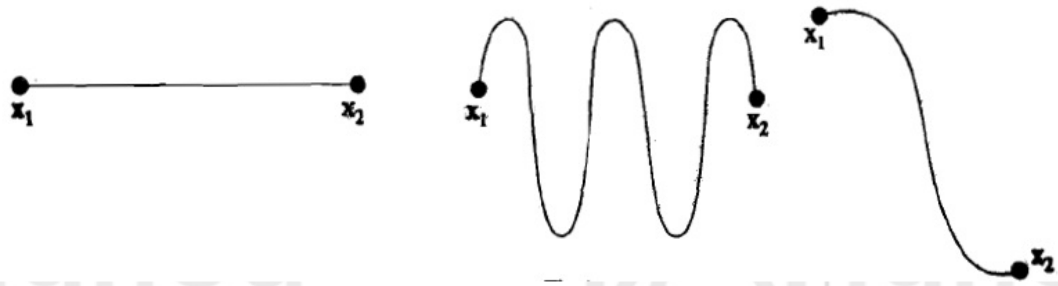

Fig.1

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Similarly in Examples 2 and 3 we have shown different ways of joining 4 points.

Example 2: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_i to x_{i+1} by a line for $1 \leq i \leq 3$. Then join x_4 to x_1. In Fig.2 we have given two different ways of doing this.

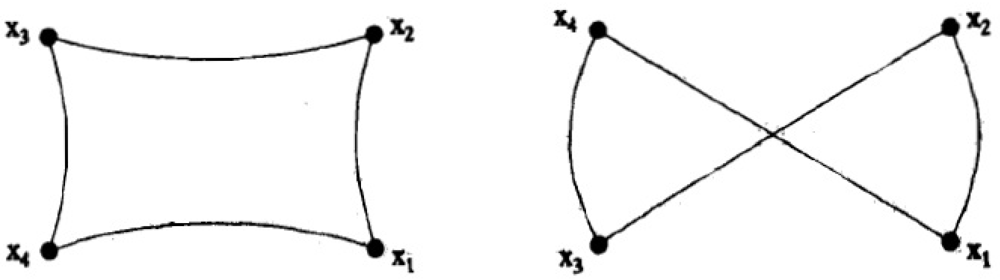

Fig.2

As far as our study in this block is concerned these drawings represent the same object.

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Example 3: Take four points x_1, x_2, x_3, x_4 in the plane. Join x_1 to the three other points by lines.

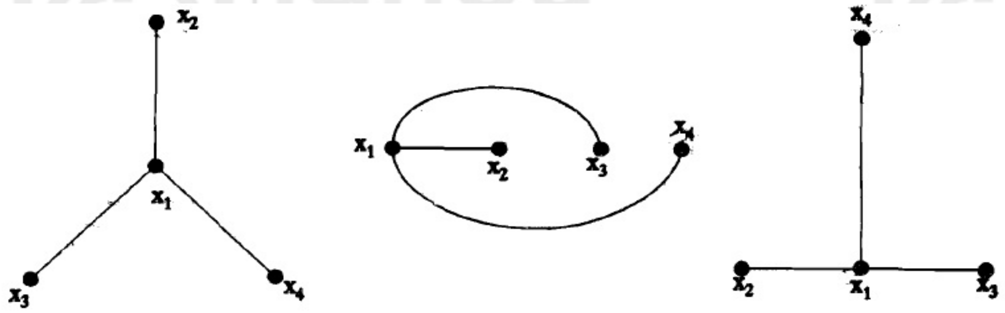

Fig.3

Again in Fig.3 above, the three drawings represent the same object.

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From the examples above, you may have observed the points are important as objects but their positions are not important. Similarly, it is important to know which pairs are joined to each other, but the lines or the curves joining them are not important.

In the diagrams above, each point is called a **vertex,** and the curve joining any pair is called an **edge**. So, for instance, in Example 1 we have two **vertices** (plural of 'vertex') x_1 and x_2, and one edge.

In this way, to each drawing corresponds two sets — one consisting of vertices, say V, and one of edges**,** say E.

Now, note that, to identify an edge, we need to know which vertices it joins. So, we denote an edge joining x_1 and x_2 by (x_1, x_2). So, any edge is given by a pair of points from V. Now, we are in a position to define the objects we wanted to.

Definition : An **undirected graph** G is a finite non-empty set V together with a set E consisting of pairs of points of V. The set V is called the **vertex set** of G, the set E is called the **edge set** of G. To show the relationship between V, E and G, we write G = (V(G), E(G)).

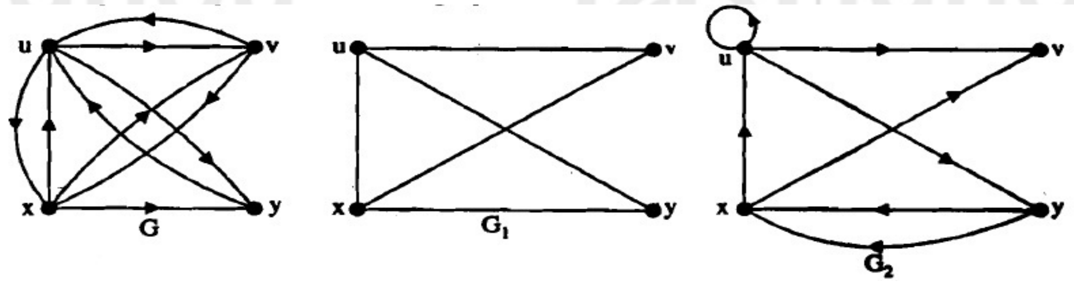
If $|V| = p$ and $|E| = q$, then G is a **(p,q)-graph**.

Now, suppose the edges have a direction, then the edge going from a vertex x_1 to a vertex x_2 is not the same as the one going from x_2 to x_1. So, $(x_1, x_2) \neq (x_2, x_1)$. In this case, the edges are represented by **ordered** pairs, that is, elements of $V \times V$, where V is the vertex set. This leads us to the following definition.

Definition: A **directed graph,** or **digraph,** G consists of a finite non-empty set V together with a subset E of the Cartesian product set $V \times V$. We call V the **vertex set** of G and E the **edge set** of G, and we write G = (V(G), E(G)).

In Fig.4 we have shown directed as well as undirected graphs. **Note that**, in an undirected graph, if (u,v) is an edge, so is (v, u). So, the relation E(G) is a symmetric binary relation on V(G). However, in a digraph E(G) need not be symmetric.

Sometimes, it may happen that in a graph there is a **loop**, i.e., an edge joins a vertex to itself as in Fig.4 (c), where (u, u) shows a loop. It may also happen that there are two or more edges joining the same vertices, as in Fig.4 (c), where there are two edges joining y to x. Such edges are called parallel or multiple edges, and a graph with any parallel edges is called a **multigraph**.



(a) **(b)** **(c)**

Fig.4

An undirected graph is either a simple graph or a multigraph.

Definition: An undirected graph without loops or parallel edges is called a **simple graph**.

Why don't you try an exercise now ?

E1) There are four basic blood types: A,B, AB and O. Type O can donate to any of the four types. A and B can donate to AB as well as to their own types, but type AB can only donate to AB. Draw a graph that presents this information.

So far, we have seen several kinds of graphs. However, in this block, **we shall only discuss simple graphs**, and **shall just refer to them as graphs**. Also, whenever there is no confusion, **we shall write V and E in place of V(G) and E(G)**.

Now, let us introduce some terminology.

- If e is an edge joining the vertices u and v of a graph G=(V,E), we will **denote it as uv**. In this case, u and v are called **adjacent vertices** (or **neighbours**), and are the **endpoints** of e. We also say that e is **incident** with u and v.

- If distinct edges e_1 and e_2 of G have at least one vertex in common, then e_1 and e_2 are called **adjacent edges**.

For an example of these concepts, consider the graph G_1 of Fig.4. G_1 = (V, E), where V = {u, v, x, y} and E = {uv,ux,uy,vx,xy}. So, G_1 is a (4, 5)-graph. The only non-adjacent vertices of G_1 are v and y. The edges uv and vx are adjacent, since both are incident with the vertex v. The edges uv and xy are non-adjacent. Two other ways of representing the graph G_1 are shown in Fig.5. Thus, **there is no unique way of drawing a graph**; the relative placing of the points and curves have no special significance.

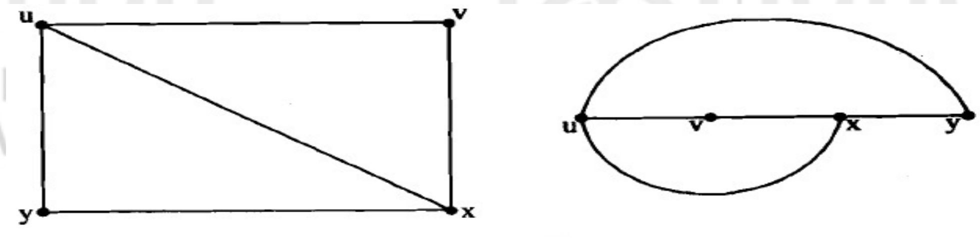


Fig.5

Remark : Since a diagram of a graph, as in Figures 1-5, completely describe the graph, **we often refer to the diagram of a graph G as G itself**.

It is interesting to know that the structure of molecules can also be represented by graphs (see Fig.6). Various atoms are represented by the vertices and the structural bonds are represented by the edges. For example, butane as well as isobutane are both the hydrocarbons $C_4 H_{10}$. But the manner in which the bonds are present between the carbon and hydrogen atoms makes the difference. In both the compounds each carbon atom is attached to four other atoms. Unlike isobutane, in butane there is no carbon atom which is attached to all the other carbon atoms.

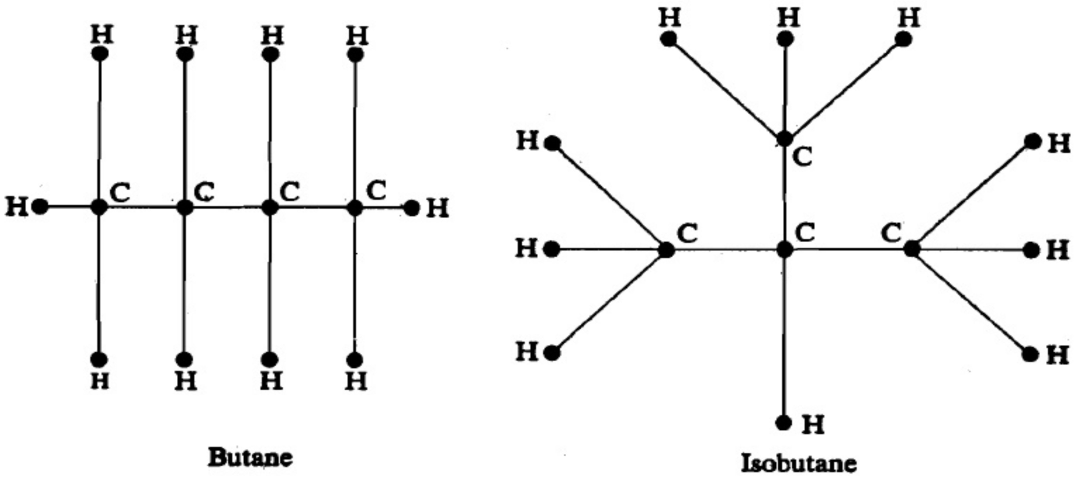


Fig.6

You may now try some exercises.

E2) Take three vertices x, y, z and draw all possible (3,2)-graphs on these vertices.

E3) Write down the vertex set V and edge set E of each graph in Examples 1, 2 and 3.

We now define some graphs which are used for data communication and parallel processing.

A) **A complete graph** is a graph in which any two vertices are adjacent, i.e., each vertex is joined to every other vertex by an edge. We denote the complete graph on n vertices by $\mathbf{K}_{n.}$

In Fig.7, we have shown K_n for various n. K_1 is just a single vertex; K_2 consists of two vertices and an edge; K_3 is often called a triangle. The last two figures in Fig.7 show two ways of representing K_4.

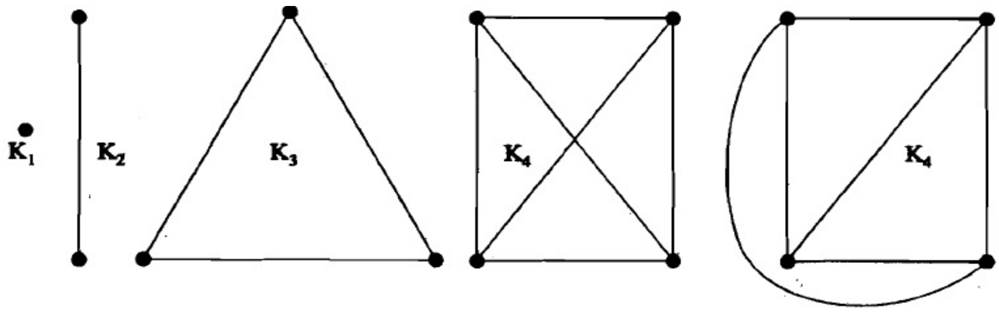


Fig.7

B) **Star topology**: Various computers, printers and plotters on a campus can be connected using a local area network. Some of these networks are based on graphs like the one given in Fig.8, called a **star topology**. In this graph n vertices are adjacent to one central vertex. This represents n devices connected to a central control device from which messages are sent to them.

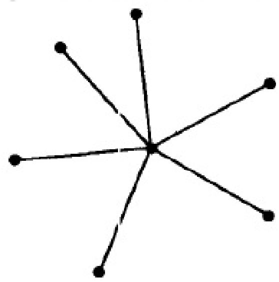


Fig.8 : Star topology

C) Cycles: A **cycle C_n** is a graph on n vertices $\{x_1, \ldots, x_n\}$ where $E(C_n) = \{x_i x_{i+1} : 1 \le i \le n-1\} \cup \{x_n x_1\}$. For instance, C_{16} is shown in Fig.9.

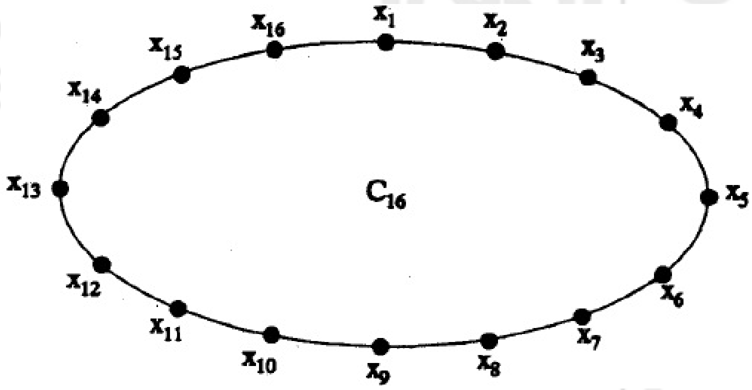


Fig.9 : C_{16}, a cycle on 16 vertices

You may now try the following exercise.

E4) How many edges do C_n and K_n have, $n \ge 3$?

We now take up some definitions related to certain algorithms that you have studied in MCS-031.

Definition: Let G = (V, E) be a (p, q)-graph. By its **complement** $\overline{G}$, we mean the graph with $V(\overline{G}) = V(G)$ and $E(\overline{G}) = \{xy : xy \notin E(G), x, y \in V(G)\}$.

Note that $\overline{G}$ is a (p, $\overline{q}$)-graph, where $\overline{q}$ = (number of pairs of elements of V) – q. Since in a set V with p elements, there can be $C(p, 2) = \frac{p(p-1)}{2}$ such pairs of elements, $\overline{q} = \frac{p(p-1)}{2} - q$.

Remark : The complement of $\overline{G}$ is G. Can you prove this ?

Let us consider some examples, Fig. 10 shows C_5 and its complement. Check whether $\overline{C_5}$ is also a cycle or not.

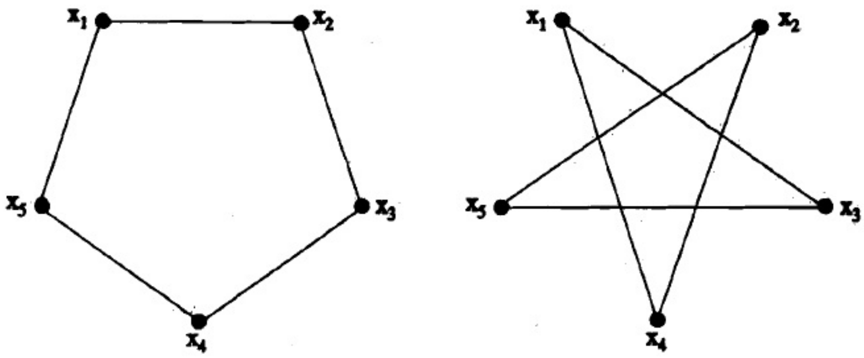
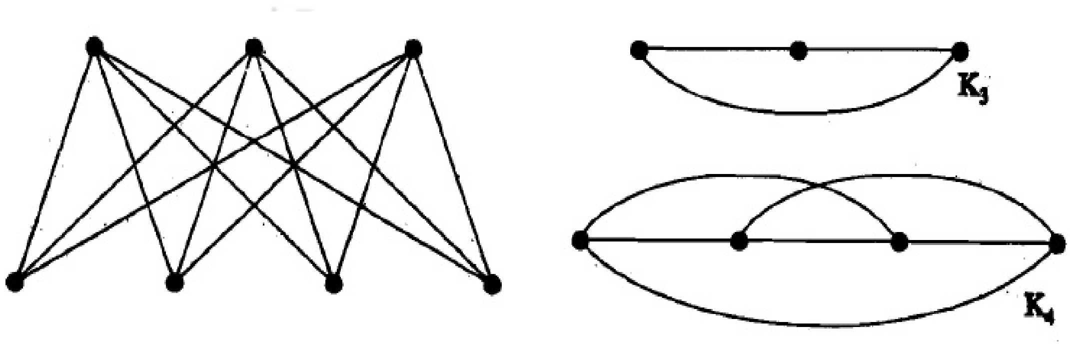


Fig.10

Two graphs G and H are **disjoint** if $V(G) \cap V(H) = \phi$

For another example, consider the graph shown in Fig.11 (a). Its complement breaks into two disjoint graphs. One is K_3 and the other is K_4 (see Fig.11 (b)).



(a) (b)

Fig.11

Consider the complements of all the graphs you have seen so far.

Notice that C_5 is a (5, 5) graph, and $\overline{C}_5$ has 5 edges. Also, in Fig.11, G is a (7, 12)-graph and $\overline{G}$ has 9 edges. Do you see any relation between the number of vertices of G and number of edges of $\overline{G}$? You may try the following exercises and look for the answer.

E5) Three graphs G_1, G_2 and G_3 are listed below

$G_1 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_5, u_1 u_6, u_2u_3, u_2u_5, u_3u_4, u_4u_5\})$

$G_2 = (\{u_1,u_2, u_3, u_4,u_5\}, \{u_1u_2,u_1u_3, u_1u_4,u_1u_5, u_2u_4, u_2u_5, u_3u_4, u_3u_5\})$

$G_3 = (\{u_1, u_2, u_3, u_4, u_5, u_6\}, \{u_1u_2, u_1u_4, u_1u_5, u_2u_3, u_3u_4, u_3 u_6, u_5u_6\})$

Find $\overline{G_1}, \overline{G_2}$ and $\overline{G_3}$.

E6) If G is a (p, q)-graph, then how many edges can $\overline{G}$ have ?

So far we haven't really looked at the ways in which graphs are related to the real-life situations given in Sec.1.0. Let us consider some graphs that will help us see this connection.

1.3 DEGREE, REGULARITY AND ISOMORPHISM

You may recall that in the beginning we defined two vertices of a graph G to be adjacent if they are joined by an edge. Such vertices are also called **neighbours**. The set of all neighbours of a fixed vertex x of G is called the **neighbourhood set of x**, denoted by $\mathbf{N_G(x)}$. Since our graphs are simple, there is a one-one correspondence between N_G (x) and the set of all edges of G incident with the vertex x. Related to this set, we get the following number.

Definition: By the **degree** of a vertex x in G, we mean the number of edges incident with x. We denote the degree of x by $\mathbf{d_G(x)}$.

By definition, $d_G(x) = |N_G(x)|$, where $|N_G(x)|$ denotes the number of elements of the set $N_G(x)$.

Since in a (p, q)-graph G the maximum number of edges incident with a vertex x can be (p –1), we have

$\mathbf{0 \le d_G(x) \le (p-1)}$ for every vertex x in G.

Whenever there is no possibility of confusion, we will simply write d(x) instead of $d_G(x)$.

Also a vertex x of a graph G is called an **even vertex** if d_G (x) is even; otherwise it is called an **odd vertex**. A vertex with degree 0 is called an **isolated vertex**.

Now let us look at the following example.

Example 4 : Consider the graph G shown in Fig. 12. First consider the vertex x_1. Clearly, three edges are incident with it, so that d (x_1) = 3. Likewise you may observe that d (x_2) = 4, d (x_3) = 5, d (x_4) = 6 and d (x_5) = 7.

We can write these observations as d $(x_i) = i + 2$ for $1 \le i \le 5$.

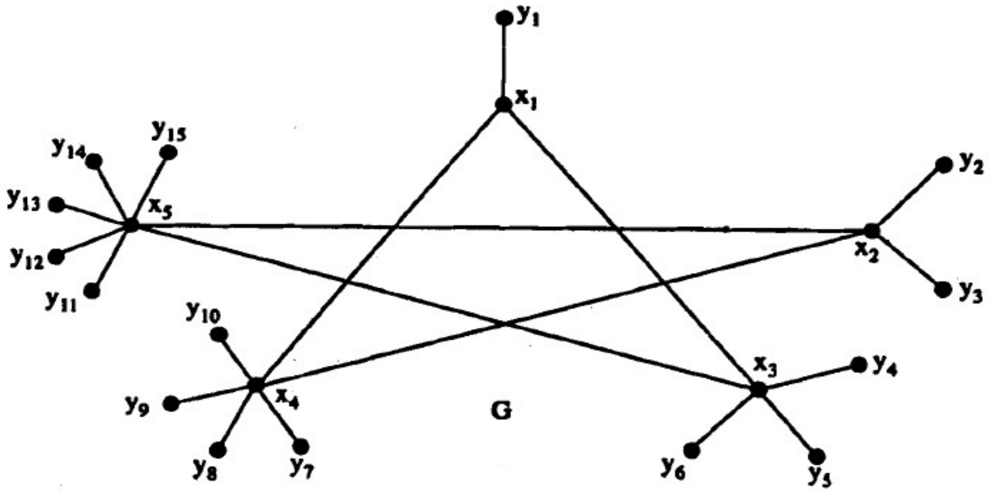


Fig .12

In the same way, we can write d (y_j) = 1 for $1 \le j \le 15$.

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You may now try the following exercises.

E7) Write down the degrees of all the vertices for the graphs given in Figures 5 to 9.

E8) If G is a (p, q)-graph and x is a vertex in G, show that the degree of x in $\overline{G}$ is $p - 1 - d_G(x)$.

Note that in Example 4

$$\begin{aligned} d(x_1) + d(x_2) + \ldots + d(x_5) + d(y_1) + \ldots + d(y_{15}) &= 40 \\ &= 2 \times 20 \\ &= 2 \times \text{ (number of edges in G)} \end{aligned}$$

Does a similar relationship hold for the graphs in Fig.5 to Fig.9 ? In fact, it should, because of the following theorem.

Theorem 1 (Handshaking Theorem) : If G is a (p, q)-graph with

$V(G) = \{v_1, \ldots, v_p\}$, and if $d_i = d_G(v_i)$, $1 \le i \le p$, then $2q = \sum_{i=1}^{p} d_i$.

That is, **the sum of the degrees of the vertices of G is twice the number of edges.**

Proof: Consider the set S = {(x, e): $x \in V(G)$, $e \in E(G)$, x is an endpoint of e}. Choose a vertex $v_i \in V$. This can be done in p ways. Now, since $d_i = d(v_i)$, there are precisely d_i edges incident with this vertex v_i. These edges give d_i elements of the set S. Adding over all the vertices of G, we get

$$|S| = \sum_{i=1}^{p} d_i . \qquad (1)$$

Now choose an edge e in E (G). This can be done in q ways. This edge has precisely two endpoints, and they give two elements of S. Summing over every edge $e \in E(G)$, we get

$$|S| = 2q \qquad (2)$$

This is because every edge is counted twice, once for each vertex it contains. Equating (1) and (2) we get the required result.

The next result immediately follows from Theorem 1.

Corollary 1 : The sum of the degrees of all the vertices of any graph is even.

You have already verified Theorem 1 for the graphs in Fig.5 to Fig.9. Now let us look at a useful application of Theorem 1.

So far, in the discussion, you must have noticed that for a simple (p, q)-graph G the edge set E(G) is a subset of the set of all subsets of size 2 of elements of V (G). This means $q \le \frac{p(p-1)}{2}$. But then you may wonder : is it always possible to go the other way round? That is, for any pair of positive integers (p, q) with $q \le \frac{p(p-1)}{2}$, is it always possible to find a (p, q)-graph?

Theorem 1 gives us a necessary condition on p and q under which a (p, q)-graph exists. It helps us to see that **there does not always exist a graph with vertices having given degrees**.

For instance, can we construct a graph on 12 vertices with 2 of them having degree1, three having degree 3, and the remaining seven having degree 10 ? This is not possible. Why? If such a graph existed, the sum of the degrees of all its vertices would be 1+1+3+3+3+10+10+10+10+10+10+10=81, which is not even.
So, the condition of Theorem 1 would not be satisfied.

Theorem 1 can also be used to obtain another result which we discuss now.
Corollary 2 : Any graph can only have an even number of odd vertices.

Proof : Let G be a (p, q)- graph and let $\{x_1,\ldots,x_t\}$ be the set of its odd vertices and $\{x_{t+1},\ldots,x_p\}$ be the set of its even vertices. Let $d_G(x_i) = 2c_i+1$, $1 \le i \le t$ and $d_G(x_i) = 2r_i$, $t+1 \le i \le p$.

Then Theorem 1 says that $2q = \sum_{1}^{p} d_G(x_i)$

$$\Rightarrow 2q = \sum_{1}^{t}(2c_i+1) + \sum_{t+1}^{p}(2r_i) = 2(c_1+c_2+\cdots+c_t)+t+2(r_{t+1}+\cdots+r_p),$$

which shows that t is even.

What this result says is that **if a graph has any odd vertices**, then their number has to be even. So, for instance, we can't have a graph with only 1 odd vertex. In some graphs, like K_{10}, all the ten vertices are odd vertices. On the other hand, in K_{11} all the vertices have degree 10, that is, none of the vertices are odd.

We now give the reason for the name given to Theorem 1.

Corollary 3 : At any party, the number of people who shake the hands of an odd number of people is even.

You can see this from Theorem 1 if you consider a handshake between two hands as an edge between two adjacent vertices.

Let us now consider some numbers that help us judge the type of graph we are dealing with.

Definitions: If G = (V, E) is a (p, q)-graph, then
$\delta(G) = \min\{d_G(x) : x \in V(G)\}$ is called the **minimum vertex degree of G,** and
$\Delta(G) = \max\{d_G(x) : x \in V(G)\}$ is called the **maximum vertex degree** of G.
Clearly, $\delta(G)$ and $\Delta(G)$ are non-negative integers.

We can, in fact, re-number the vertices of V (G) as $\{v_1, \ldots, v_p\}$ with $d_i = d(v_i)$, $1 \le i \le p$ such that $d_1 \ge d_2 \ge \ldots \ge d_p$, that is, place the vertices in decreasing order of their degrees. This is called the **degree sequence** of the graph G.

For instance, the degree sequence of the graph G in Fig.11 is 7, 6, 5, 4, 3, $\underbrace{1,1,\ldots,1}_{15\,\text{times}}$.

And now some exercises for you.

E9) Write down $\delta(G)$ and $\Delta(G)$ for all the graphs in Examples 1,2, 3.

E10) For each of the number sequences given below, give an example of a graph having this as a degree sequence, if possible. Otherwise explain why such a graph does not exist.

(i) (3,2,2,2,1) (ii) (3,2,2,2,1,1) (iii) (4,3,2,1,0)
(iv) (4,4,3,3,2,2) (v) (5,5,5,4,4,3,3)

E11) Let G be a (p, q)-graph, each of whose vertices has degree k or k+1. If G has m vertices of degree k and r vertices of degree k+1, then show that m = (k+1) p – 2q.

Let us now consider graphs that have a constant degree sequence, that is, each of their vertices has the same degree. For example, the degree sequence of C_5 and its complement is 2,2,2,2,2, that is, it is a constant 2. Such graphs have a special status in graph theory, and we name them as follows.

Definition: A (p, q)-graph G is said to be **regular, with degree of regularity r,** if $d_G(x) = r$ for every vertex $x \in V(G)$. In this case we also say that G is an **r-regular graph**. Of course, $0 \le r \le (p-1)$.

You have seen that K_3 is regular. What about K_n for n>3? In fact, it is (n–1)-regular.

As you will see in the next unit, 3-regular graphs, called **cubic graphs**, are important. A well known example of a cubic graph is the **Petersen graph**. This is named after the Danish mathematician J.P.C. Petersen. He worked in several areas of pure and applied mathematics. Two representations of the Petersen graph are shown in Fig.13. **Note** that it is a (10,15)-graph.

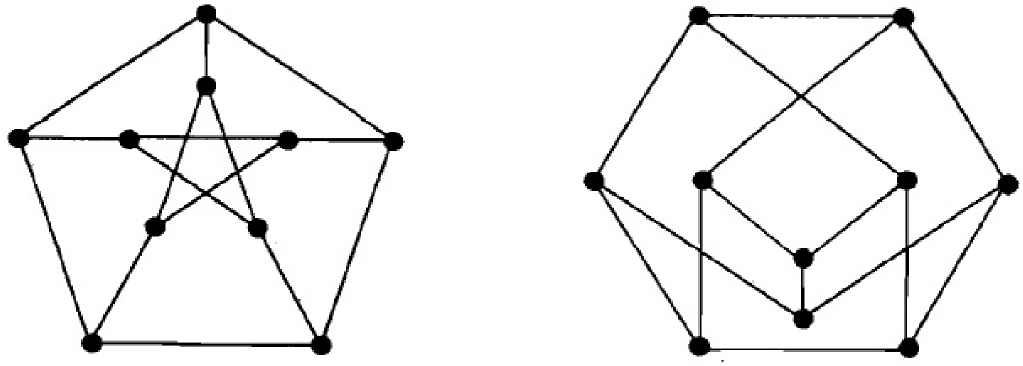


Fig.13

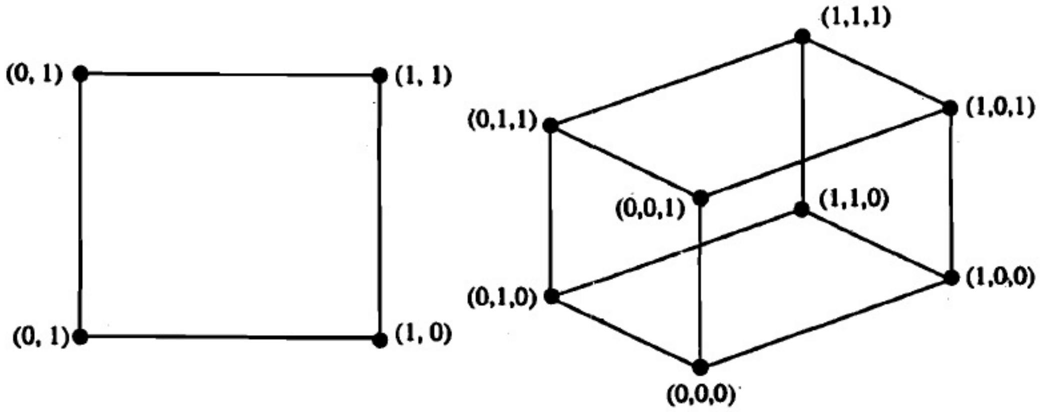
Fig. 14: **Julius Petersen (1839-1910)**

Let us now consider the following example of a regular graph.

Example 5 (Hypercube Q_n): Let the vertex set consist of all n-tuples with entries 0,1 only. The edge set is given by

$$E(Q_n) = \{ \mathbf{ab} : \mathbf{a} \text{ and } \mathbf{b} \text{ differ exactly at one coordinate}\}.$$

Here, by **a** we mean an n-tuple $(a_1, \ldots, a_n)$, where $a_i = 0$ or 1, for $1 \le i \le n$. In Fig.14, we show Q_2 and Q_3.



(a) (b)

Fig.15

Any vertex **a** is adjacent to precisely n other vertices. For example, (0,0, …,0) is adjacent to (1, 0, 0, …, 0), (0, 1, 0, 0, …0), … , (0, 0, …, 1). Hence, the hypercube Q_n is n-regular. You should check that Q_n has 2^n vertices and $n2^{n-1}$ edges.

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You have seen some examples of regular graphs, and you can think of some more. You also know that if G is an r-regular graph on p vertices, then by Theorem 1, 2q = pr. So, pr is even. Therefore, at least one of p or r is even. **Is the converse true** ? That is, given a pair of integers p, r, $0 \le r \le (p-1)$, where pr is even, can we always construct an r-regular graph on p vertices? This is true. The proof is by construction on the same lines as the examples we give below. We consider two cases — when r is even, and when r is odd.

Example 6 : We construct a 4-regular graph G with 12 vertices. Let $V(G)=\{x_1,\ldots,x_{12}\}$. Place the vertices in a circular manner. Join x_i to x_{i+1} by an edge for every i, $1 \le i \le 11$. Join x_{12} to x_1 also by an edge. Now all the vertices have acquired degree 2. Now, join each x_i to x_{i+2} for every i = 1,…10. Finally, join x_{11} to x_1 and x_{12} to x_2, as in Fig.16. You can see that the resulting graph is 4-regular.

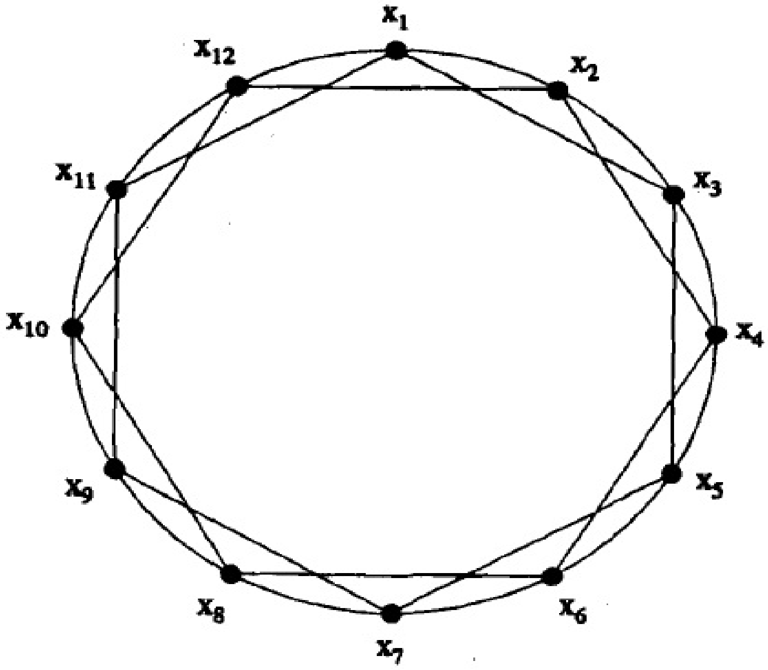

Fig.16

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Example 7 : We shall construct a 5-regular graph on 12 vertices now. We first construct a graph on 12 vertices and with regularity (r–1), i.e., 5 – 1 = 4 — in fact, the graph constructed in Example 6.
Now join x_i to x_{i+6} for every i, $1 \leq i \leq 6$.

We choose 6 because $\frac{p}{2} = 6$. Notice that the resulting graph is 5-regular (see Fig.17).

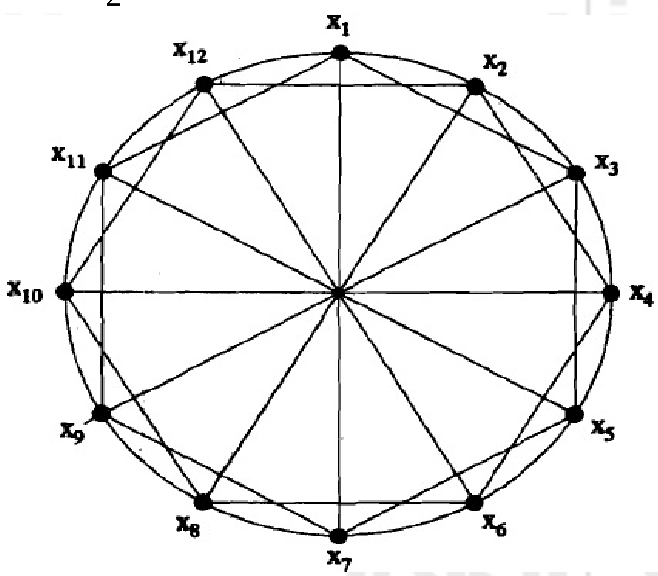

Fig. 17

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The constructions above can be generalized to obtain a regular graph on p vertices with degree of regularity r, where at least one of p and r is even. In the following exercise, we give you an opportunity to do this.

E12) Construct a 5-regular graph on 10 vertices.

Often, when we look at two graphs, they appear different when they may essentially be the same. For instance, the complement of C_5 is essentially C_5, though they don't

look the same. To see why we say this, let us look at the two graphs in Fig.10. Now, consider the function f: $V(C_5) \to V(\overline{C_5})$ defined by
$f(x_1) = x_1$, $f(x_2) = x_3$, $f(x_3) = x_5$, $f(x_4) = x_2$, $f(x_5) = x_4$.
With this definition, note that whenever $x_i x_j \in E(C_5), f(x_i)f(x_j) \in E(\overline{C_5})$. In other words, $x_i x_j$ is an edge of C_5 if and only if $f(x_i)f(x_j)$ is an edge of $\overline{C}_5$. In such a situation we say that f is an isomorphism between C_5 and $\overline{C}_5$.

This leads us to the following definition.

Definition : Let G = (V(G), E(G)), H = (V(H), E(H)) be two graphs. By an **isomorphism** f from the graph G to the graph H, we mean a map f: $V(G) \to V(H)$ such that

(i) f is one-one and onto; and
(ii) $x\,y \in E(G)$ if and only if $f(x)\,f(y) \in E(H)$.

In this case we say that G and H are **isomorphic.** Otherwise they are called **non-isomorphic**.

Note that two graphs G and H are isomorphic **if and only if** there is a one-one correspondence between V(G) and V(H) that preserves adjacencies and non-adjacencies. In this case we say that **the map f preserves the structure of G**.

Let us consider one more example.

Example 8 : Consider the two graphs G and H shown in Fig.18.

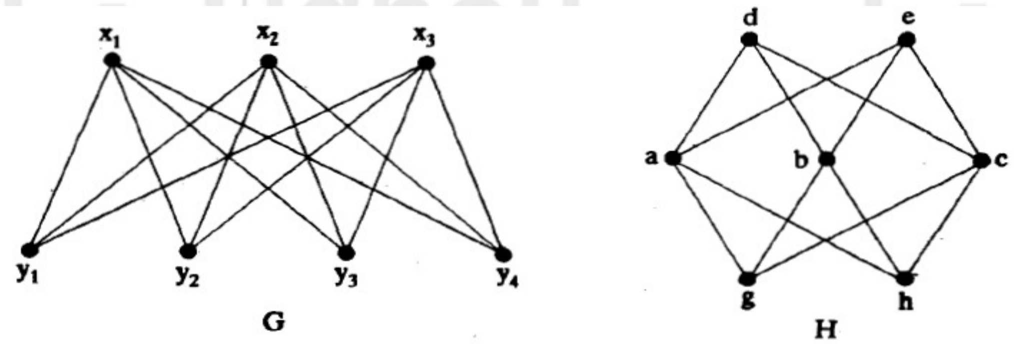


Fig.18

Define a map $f : V(G) \to V(H)$ as follows:
$f(x_1) = a$, $f(x_2) = b$, $f(x_3) = c$, $f(y_1) = d$, $f(y_2) = e$, $f(y_3) = g$, $f(y_4) = h$.
Observe that $uv \in E(G)$ **if and only if** $f(u)\,f(v) \in E(H)$.
The two graphs shown in Fig.18 are isomorphic under this correspondence.

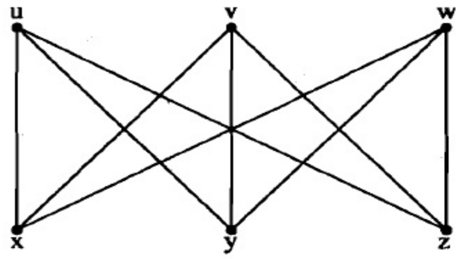
***

As you can see, if G and H are isomorphic, many properties of a vertex in V(G) are shared by its image in V(H). For example, you can check that $d_G(u) = d_H(f(u))$, for every $u \in V(G)$.

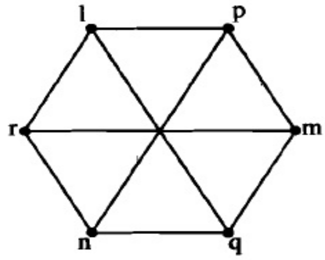
Here's an exercise about isomorphic graphs.

E13) Show that the graphs G and H given in Fig.19 are isomorphic.

G_1



G



H

Fig.19

G_2

Fig.20

As you have seen, in order to show that two graphs are isomorphic, it is enough to produce one isomorphism from one of them to the other. In some cases, as in Fig.20, it is clear that the 2 graphs are not isomorphic. However, if we are given two similar looking graphs, it is not easy to show that there **does not exist** any isomorphism between them. The six properties given below are of help in this matter. We shall state them, without proof.

Theorem 3 : Let f be an isomorphism from a graph G to a graph H. Then the following hold :

i) If G is a (p, q)-graph, then H is also a (p, q)-graph.

ii) The inverse map f^{-1} is an isomorphism from the graph H to the graph G.

iii) If g is an isomorphism from the graph H to a graph K, then the composite map $g \circ f$ is an isomorphism from the graph G to the graph K.

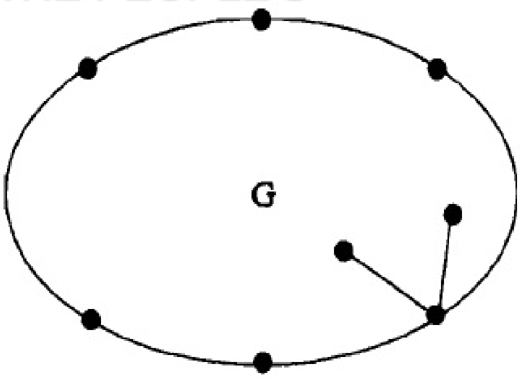
iv) f induces a bijective map $\tilde{f} : E(G) \rightarrow E(H)$, given by $\tilde{f}(xy) = f(x) f(y)$.

v) For every $x \in V(G)$, a vertex y belongs to $N_G(x)$ if and only if f (y) belongs to $N_H(f(x))$. (This means that $d_G(x) = d_H(f(x))$, for every $x \in V(G)$.) Thus, the degree sequence of the graph G is the same as the degree sequence of the graph H.

vi) If G has a set of vertices $\{x_1, \ldots, x_n\}$ such that $x_n x_1$ and $x_i x_{i+1}$ are in E (G) $\forall i$, $1 \le i \le (n-1)$, then the vertices $\{f(x_1), \ldots, f(x_n)\}$ in V (H) are such that $f(x_n) f(x_1)$ as well as $f(x_i) f(x_{i+1})$ are in E(H) $\forall i$, $1 \le i \le (n-1)$. Thus, for every positive integer $n \geq 3$, the number of copies of C_n in G is equal to the number copies of C_n in H.

Let us now consider the following examples where we have used these properties to show non-isomorphism of the two graphs.

Example 9 : Consider the two graphs shown in Fig.21. Both are (8, 8)-graphs and



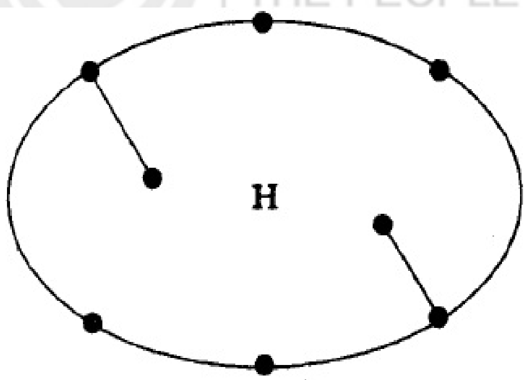


Fig.21

have a copy of C_6 inside them. Are they isomorphic ? The degree sequence of the graph G is 4, 2, 2, 2, 2, 2, 1, 1 and of the graph H is 3, 3, 2, 2, 2, 2, 1, 1. This contradicts (v) of Theorem 3. Therefore, G and H are not isomorphic.

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Example 10 : Consider the graphs G and H in Fig.22.

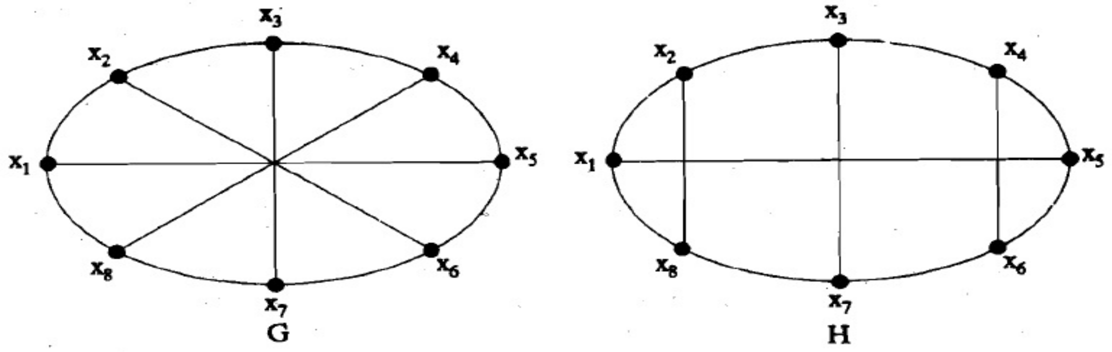

Fig.22

Both are (8, 12)-graphs and have a copy of C_8 inside them. Moreover, both have degree sequences 3, 3, 3, 3, 3, 3, 3, 3. They are still not isomorphic. This can be seen by observing that the graph G has no copy of a triangle inside it and the graph H has two triangles $\{x_1, x_2, x_8\}$ and $\{x_4, x_5, x_6\}$, which contradicts (vi) of Theorem 3.

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Example 11: Consider the graphs G and H shown in Fig. 23.

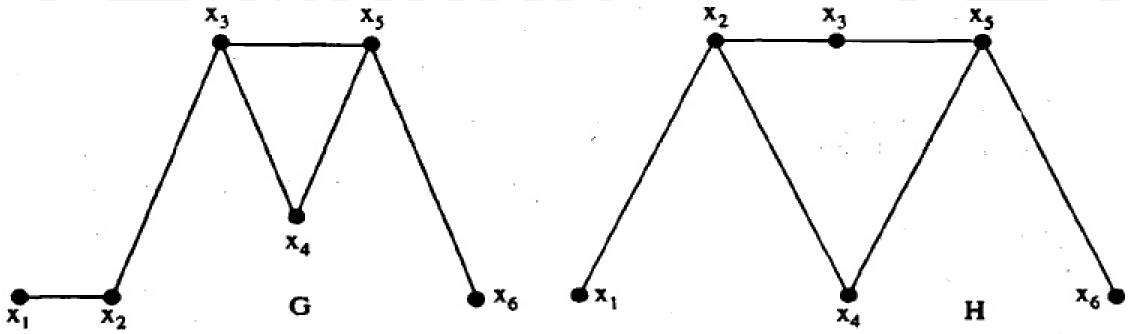

Fig.23

Both are (6, 6)-graphs having 3, 3, 2, 2, 1, 1 as their degree sequence. However, they are not isomorphic. In the graph G the two vertices x_3, x_5 having degree 3 are adjacent. Under any isomorphism (if it exists) they should be mapped to two adjacent vertices of degree 3. We observe that in the graph H the two vertices of degree 3 are not adjacent.

***

Notice that the two graphs shown in Fig. 6, corresponding to butane and isobutane, are not isomorphic. Unlike isobutane, no carbon atom is attached to all the other carbon atoms of butane.

And now the following exercises for you to try.

E14) Draw at least 3 non-isomorphic graphs on four vertices.

E15) A graph G is said to be **self complementary** if it is isomorphic to its complement $\overline{G}$. Show that for a self complementary (p, q)-graph G, either p or (p – 1) is divisible by 4.

It is often the case that a graph under study is contained within some larger graph also being investigated. When we talk of an electric circuit, it is often described in terms of various sub-circuits. Transport in a country is always divided into various sections, for example, the railway transport in India is divided into Central Railway, Western Railway, …etc. That is, whenever we study any system, it is important to study its subsystems. Likewise here in the next section we study subgraphs.

1.4 SUBGRAPHS

Let us start with considering the graph G = (V(G), E(G)) shown in Fig.24.

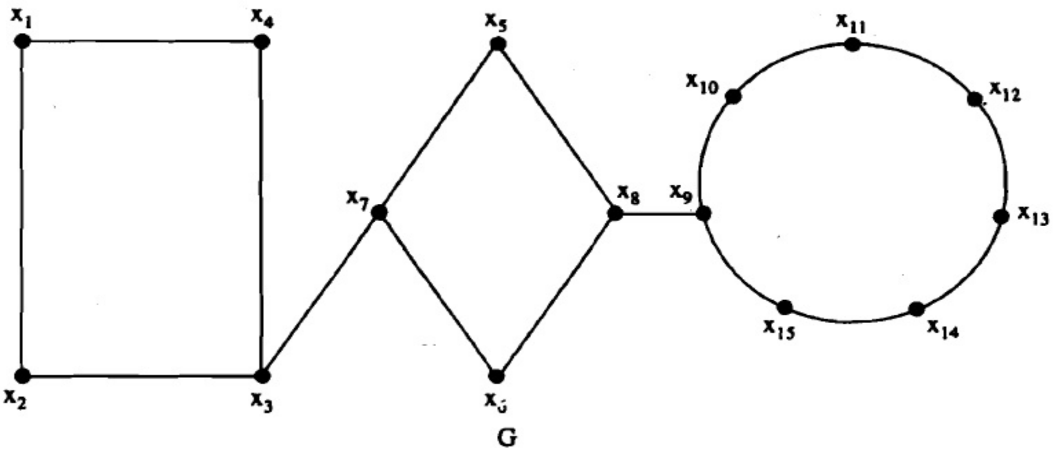

Fig.24

What if we just take a part of this graph G? Would this be a graph? Yes, it would. For example, consider the following.

Let V (G_1) = $\{x_1, x_2, x_3, x_4\}$, E (G_1) = $\{x_i\, x_{i+1}: 1 \le i \le 3\} \cup \{x_4 x_1\}$.
Note that G_1 is **isomorphic** to C_4.
If V (G_2) = $\{x_8, x_9\}$, E (G_2) = $\{x_8x_9\}$, then G_2 is **isomorphic** to K_2.
Also the graph G_3 is **isomorphic** to C_7, where
V (G_3) = $\{x_9, \ldots, x_{15}\}$, E (G_3) = $\{x_{15}\, x_9\} \cup \{x_i x_{i+1} : 9 \le i \le 14\}$

Note that all these graphs have one thing in common. Their vertex sets are subsets of V (G) and edge sets are subsets of E (G). In this sense, all these graphs are 'portions' of the graph G. Formally, we have the following definition.

Definition : Let G = (V (G), E (G)) be a graph. A **subgraph** H of the graph G is a graph, such that every vertex of H is a vertex of G, and every edge of H is an edge of G also, that is, V (H) $\subseteq$ V (G) and E (H) $\subseteq$ E (G).
Further, if H is a **subgraph** of a graph G, such that V (H) = V (G) and E (H) $\subseteq$ E (G), that is, H and G have exactly the same vertex set, then H is called a **spanning subgraph** of G.

So, G_1, G_2 and G_3 given above are subgraphs of G given in Fig.23. However, the graph H, with $V(H) = V(G_3), E(H) = E(G_3) \cup \{x_9 x_{12}\}$ is **not** a subgraph of the graph G, since the edge x_9x_{12} is not in E (G).

Note : You should write down the reasons why the following statements are true.

1) Every graph G is a subgraph of itself, i.e., **G is a subgraph of G**.
2) **For any v $\in$ V (G), {v} is a subgraph of G**.

Now for an example of a spanning subgraph.

Example 12: Consider G = K_4 on four vertices x_1, x_2, x_3, x_4 as shown in Fig.25. From the figure you can see G_1, G_2, G_3 are subgraphs of G, with $V(G_1)=V(G_2)=V(G_3)=V(G)$. So, G_1 , G_2 and G_3 are spanning subgraphs of the graph G.

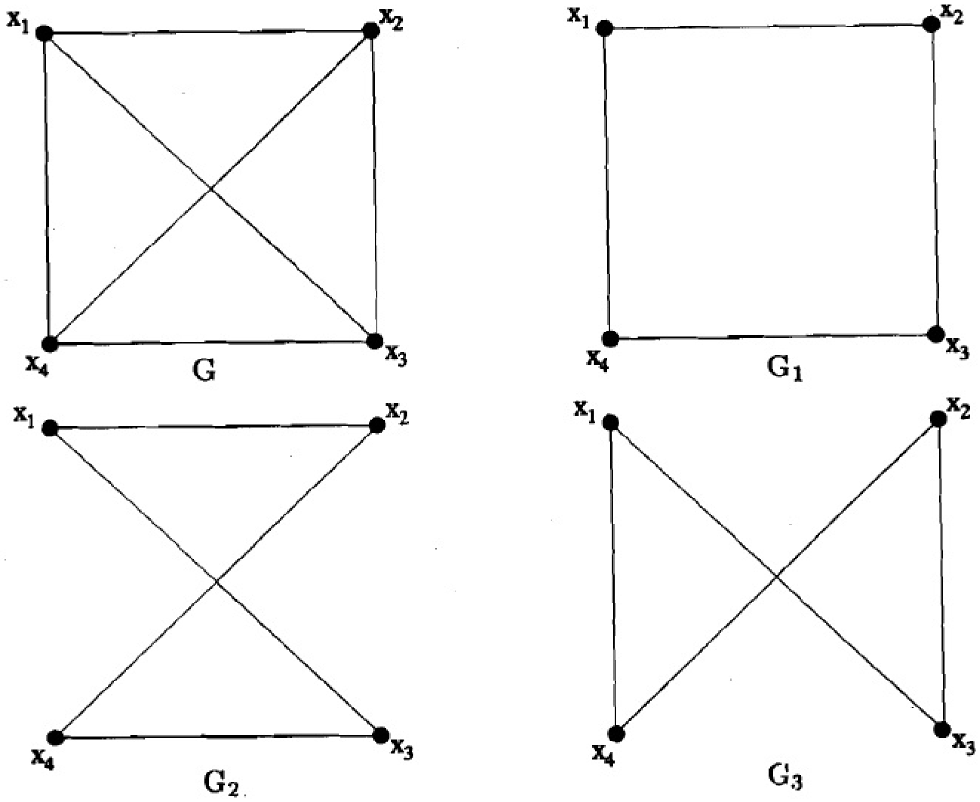

Fig.25

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Example 13: Consider the Petersen graph G, with the vertex set $\{x_i :1\le i\le 5\}\cup\{y_j :1\le 1\le 5\}$ shown in Fig.26. Consider the graph G_1, where $V(G_1)=\{y_j :1\le j\le 5\}, E(G_1)=\{y_1y_3, y_3y_5, y_5y_2, y_2y_4\}$.

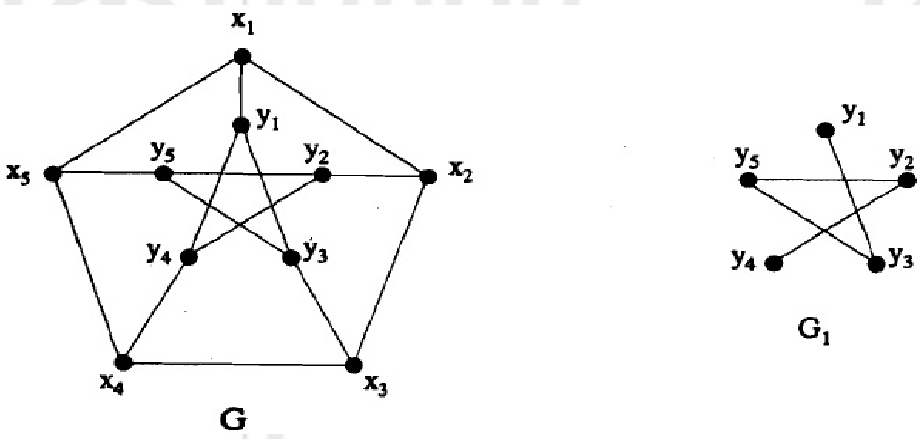

Fig.26 : The Petersen Graph

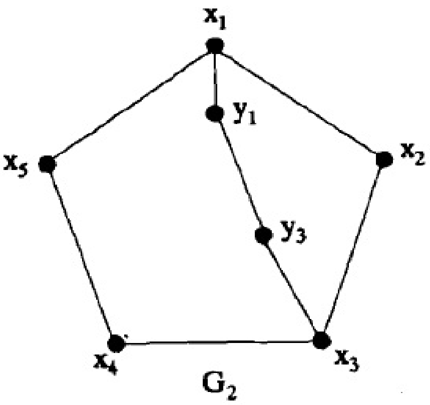

Fig.27

Here every edge of G_1 is an edge in G. On the other hand, $y_4\,y_1$ is an edge in G but not an edge in G_1. Thus, G_1 is a subgraph of G. So, is the graph G_2 shown in Fig.27. However, there is a difference in the two subgraphs. Though y_1 and y_4 lie in $V(G_1)$, y_1y_4 is in E(G) but not in $E(G_1)$. But, whenever two vertices of G_2 are joined by an edge in G, that edge belongs to E (G_2)

The property of the subgraph G_2 that we have just mentioned leads us to the following definition.

Definition: Let G be a graph and let $S \subseteq V(G)$. By **the subgraph of the graph G**, **induced by the set S**, we mean the subgraph H with V(H)=S and the edge set consisting of those edges of G which are joining the vertices in S. That is, $E(H) = \{x\,y : x \neq y, x \in S, y \in S, x\,y \in E(G)\}$. We denote H by **$<S>_G$**.

Note that two points of S are adjacent in $<S>_G$ if and only if they are adjacent in G.

For example, the subgraph G_2 in Example 13 is an induced subgraph of the graph G, induced by $\{x_1, x_2, x_3, x_4, x_5, y_1, y_3\}$, whereas the subgraph G_1 is not induced.

Note that for a vertex $v \in V(G)$, by **G – v we mean the subgraph $< V(G) - \{v\} >_G$**, which means a subgraph of G consisting of all points of G except v, and all edges of G except for the edges incident with v.
For a subset S of V (G), the subgraph **$< V(G) - S >_G$ is often written as G–S**.

We now illustrate various types of subgraphs, relating their minimum and maximum vertex degrees.

Example 14 : Consider the graph G shown in Fig.28 (a). Observe that Fig.28 (b)

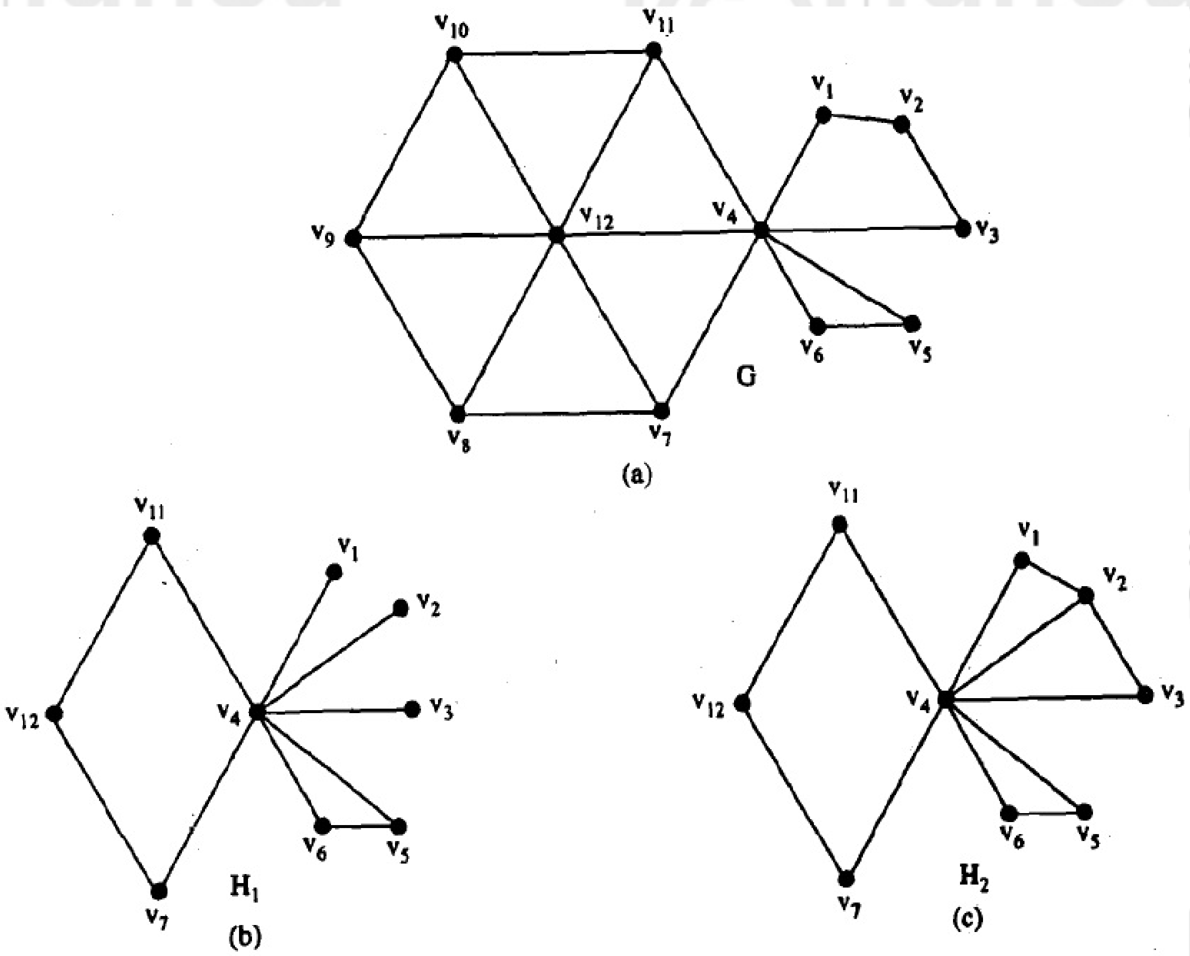

Fig.28

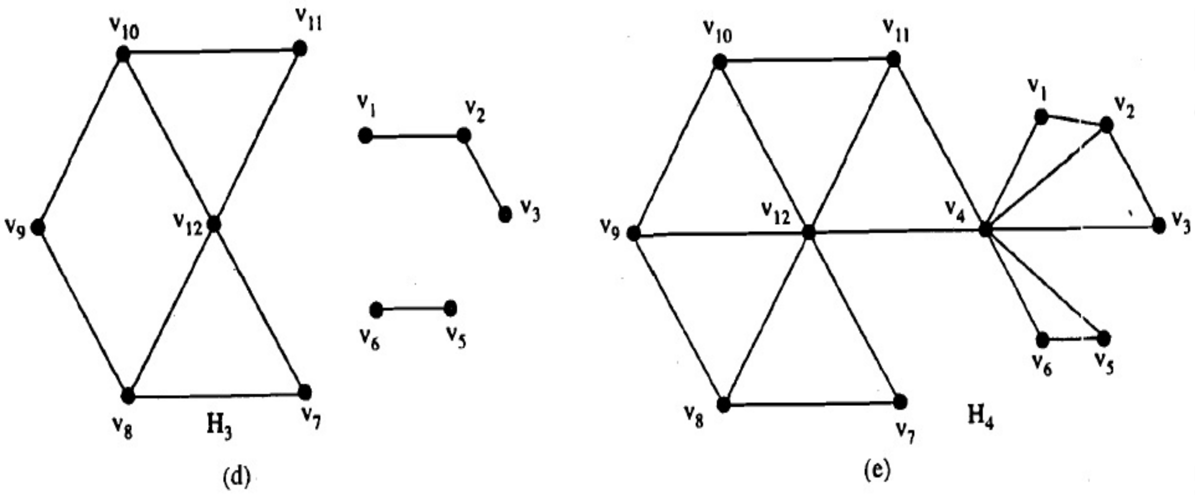

Fig.28 (contd.)

shows a subgraph H_1, Fig.28 (c) gives a vertex induced subgraph H_2 with $V(H_2) = V(H_1)$, Fig.28 (d) shows $H_3 = G - v_4$, and Fig.28 (e) gives the spanning subgraph H_4.

***

Now, a few exercises for you.

E16) Show that for a subgraph H of a graph G, $\Delta(H) \leq \Delta(G)$.

E17) Give an example of a subgraph H of a graph G with $\delta(G) < \delta(H)$ and $\Delta(H) < \Delta(G)$.

E18) Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k. How many vertices and edges does G – v have?

E19) Is every subgraph of a regular graph regular ? Give reasons for your answer.

We now end this unit by giving a summary of what we have covered here.

1.5 SUMMARY

1) A simple graph G consists of a finite non-empty set V of points together with a prescribed set E of 2 element subsets of V.

2) The complete graph K_n is a graph with n vertices such that every vertex is joined to every other vertex by an edge.

3) The path P_n is a graph on n vertices $\{x_1, x_2, \ldots, x_n\}$ in which any two consecutive edges are adjacent and where no edge and no vertex is repeated.

4) A cycle is a circuit in which the only repeated vertex is the first vertex, which is the same as the last vertex.

5) The complement of the (p, q)-graph G is a $(p, \overline{q})$-graph $\overline{G}$ where $\overline{q}$ = (number of pairs of elements of V) – q.

6) The number of edges incident with a vertex in a graph G gives the degree of the vertex. A graph having the same degree of all its vertices is regular.

7) In any graph the sum of the degrees of all its vertices is even.

8) There always exists an r-regular graph on p vertices, where p, r are non-negative integers and at least one of them is even.

9) For a graph G = (V (G), E (G)), a graph H = (V (H), E (H)) is a subgraph of G whenever V (H) $\subseteq$ V (G) and E (H) $\subseteq$ E (G).

10) A subgraph of a regular graph may or may not be regular.

11) A subgraph H of a graph G is a spanning subgraph of G if V(H) = V(G).

12) For any $S \subseteq V(G)$, the subgraph of G induced by S is H=$\langle S \rangle_G$, where V(H)=S and E(H)=$\{xy{:}x \neq y, x \in S, y \in S, xy \in E(G)\}$.

1.6 SOLUTIONS / ANSWERS

E1)

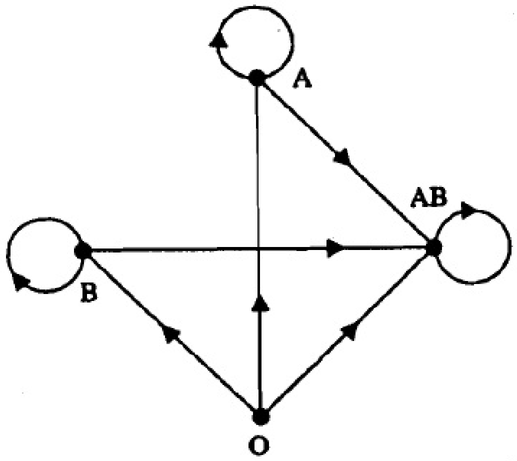

Fig.29

E2)

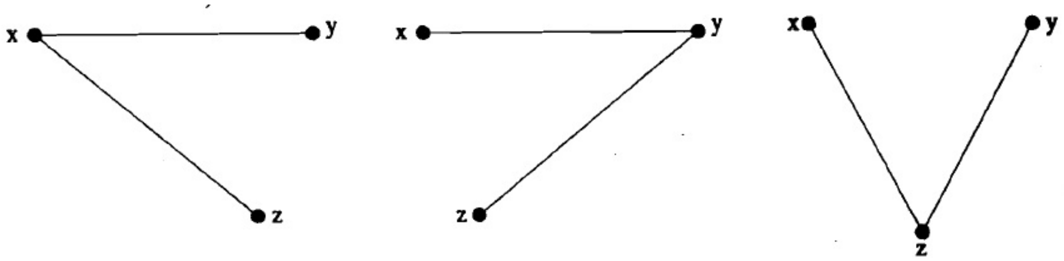

Fig.30

E3) Example 1, V = {x_1, x_2}, E = { x_1x_2}
Example 2, V = {x_1, x_2, x_3, x_4} E = {x_1x_2, x_2x_3, x_3x_4, x_4x_1}
Example 3, V = { x_1, x_2, x_3, x_4}, E = {x_1x_2, x_1x_3, x_1x_4}

E4) C_n has n edges and K_n has $\frac{n(n-1)}{2}$ edges.

E5) E ($\overline{G_1}$) = $\{u_1u_3, u_1u_4, u_2u_4, u_2u_6\ u_3u_5, u_3u_6, u_4u_6, u_5u_6\}$
E ($\overline{G_2}$) = $\{u_2u_3, u_4u_5\}$
E ($\overline{G_3}$) =$\{u_1u_3, u_1u_6, u_2u_4, u_2u_5, u_2u_6, u_3u_5, u_4u_5, u_4u_6\}$

E6) $\overline{G}$ can have $\frac{p(p-1)}{2} - q$ edges.

E7) For Fig.5, $d(x_i) = 4$, $1 \le i \le 5$, $d(y_i) = 2$, $1 \le i \le 7$.
For Fig.8, $d(x_i) = 2$, $1 \le i \le 5$.
Do the others similarly.

E8) $d_{\overline{G}}(x) = |N_{\overline{G}}(x)| = |\{y \in V(G) : xy \notin E(G)\}| = |V(G)| - 1 - |N_G(x)|$
$= p - 1 - d_G(x)$.

E9) (1) 1, 1 (2) 2, 2 (3) 1, 3

E10) ii) The graph has 3 vertices of odd degree, contradicting Corollary 1 of Theorem 1.
v) The sum of the degrees of all the vertices of a graph is odd, contradicting Corollary 1.
For the rest you can draw graphs whose degree sequence is the given one.

E11) km + (k+1)r = 2q (Using Theorem 1)
Also, m+r = p
Therefore, km+(k+1)(p–r) = 2q
E $\Rightarrow$ m = (k+1) p – 2q

E12) Here p = 10, r = 5. So $\frac{r-1}{2}$ is an integer. Take 10 vertices $\{x_1, x_2, \ldots, x_{10}\}$. Join x_i to x_{i+1} for $1 \le i \le 9$. Join x_{10} to x_1. Now all the vertices have acquired degree $\frac{r-1}{2} = 2$. Join x_i to x_{i+2} for i=1,..,8. Join x_9 to x_1 and x_{10} to x_2. We now have a 4-regular graph. Here $\frac{p}{2} = n = 5$.
Thus, to obtain a 5-regular graph join x_i to x_{i+5} for $1 \le i \le 5$ (see Fig.31).

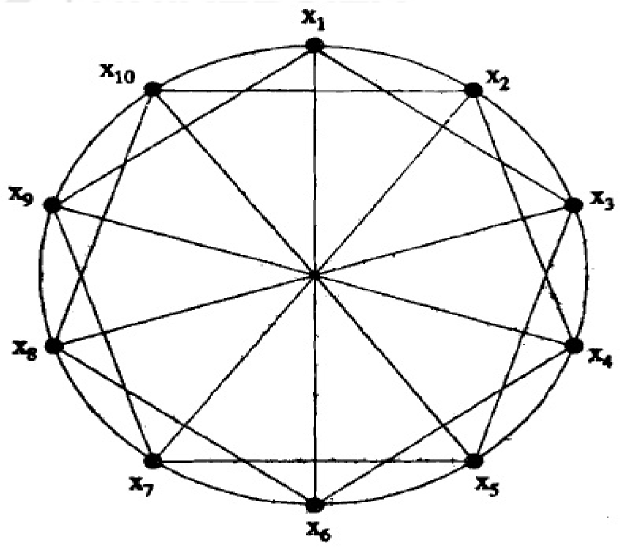

Fig.31

E13) Consider the function
$\phi : V(G) \to V(H) : \phi(u) = \ell, \phi(v) = m, \phi(w) = n, \phi(x) = p, \phi(y) = q, \phi(z) = r.$
You can check that ϕ is an isomorphism.

E14) If p = 4, then q $\le$ C(4, 2) = 6. So we want (4, q)-graphs, with $0 \le q \le 6$. We are giving several in Fig. 32.

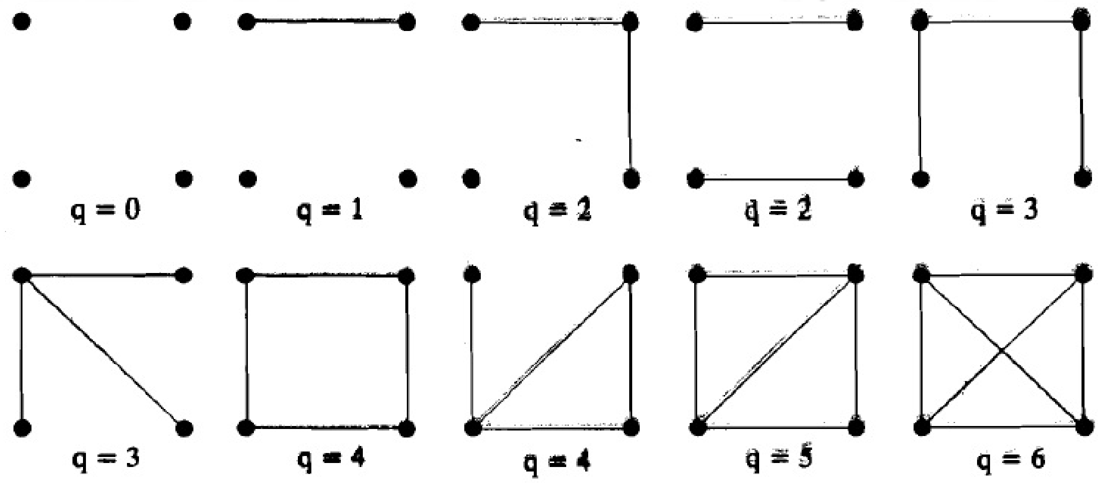

Fig.32

E15) Suppose G is a (p, q)-graph. Then
$E(G) \cup E(\overline{G})$ = {the set of all pairs of vertices in V (G)}. Thus,

$$q + \bar{q} = \frac{p(p-1)}{2}.$$

If the graph G is self complementary, then $q = \bar{q}$. Thus, $p(p-1) = 2q + 2\bar{q} = 4q$, that is 4 divides $p(p-1)$. Since only one of p or $(p-1)$ is even, this means either p or $(p-1)$ is divisible by 4.

E16) Let $x \in V(H)$ such that $d_H(x) = \Delta(H)$. Then, $N_H(x) \subseteq N_G(x)$. Thus,

$$\Delta(H) = |N_H(x)| \le |N_G(x)| \le \Delta(G).$$

E17) $\delta(G) = 1 < 2 = \delta(H)$
$\Delta(H) = 2 < 3 = \Delta(G)$

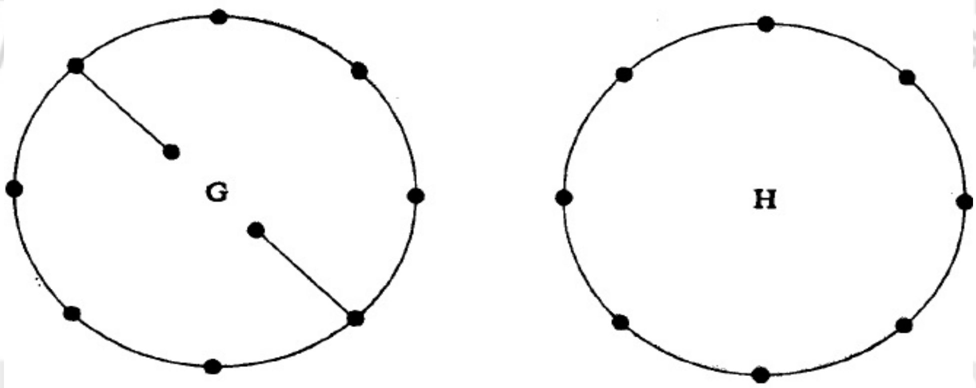

Fig.33

E18) G–v will have (n – 1) vertices and m – k edges

E19) No, for example, any cycle is regular. However, if you remove one of its edges, you get a subgraph which is not regular.