UNIT 3 GRAPH ALGORITHMS-1

Structure

3.0 INTRODUCTION

Graphs are most widely used mathematical structure. It is widely used in finding shortest path routes, shortest path between every pair of vertices, in computing maximum flow problem which has applications in a large range of problems related to airlines scheduling, maximum bipartite matching and image segmentation.
A graph can be used to model a social network which comprises millions of users or interest groups which can be represented as nodes. There are interdependencies among nodes through mutual interests and common friends. Many algorithms used in social networks are based on graph algorithms like Facebook's friends' suggestion algorithm, Google's page ranking algorithm, Linkdn's suggestion to join a group etc.

3.1 OBJECTIVES

After successful completion of this unit, the students will able to:

- Define different types of graphs
- Represent a graph through an adjacency matrix and an adjacency list and calculate complexities of each
- Write pseudo-codes for graph traversal techniques like breadth first search and depth first search and measure their time complexities
- Define directed acyclic graph, topological ordering and connected components
- Write pseudo codes for topological ordering and connected components

3.2 BASIC DEFINITION AND TERMINOLOGIES

Graph:

A graph G = (V,E) is a data structure comprising two set of objects, V={v_1,v_2,……} called vertices, and an another set E={e_1,e_2,……} called the edges.

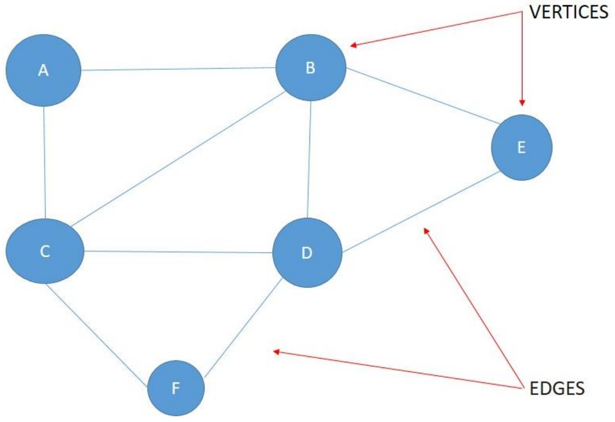


Figure 1: A graph

In the above graph set of vertices V = {A,B,C,D,E,F} and set of edges E = {A-B , A-C, B-E ,B-D,B-C,C-D,C-F,D-F, D-E,}.Vertices are unordered set of nodes V.

Edge = An edge is identified with an unordered pair of vertices (v_i,v_j), where v_i andv_j are the end vertices of the edge e_k

Graph Types:

Though a graph contains only vertices and edges, but there are many variations in them. Most of the variations are due to edges (directed / undirected, no of edges). In the following such graph types are listed with examples.

1. **Simple Graph:** Asimple graph (figure 2) is a graphin which each edge is connected with two different vertices.There is no self-loop and no parallel edges in a simple graph. A simple graph. A simple graph with multiple edges is called multigraph (figure 3).

Example:

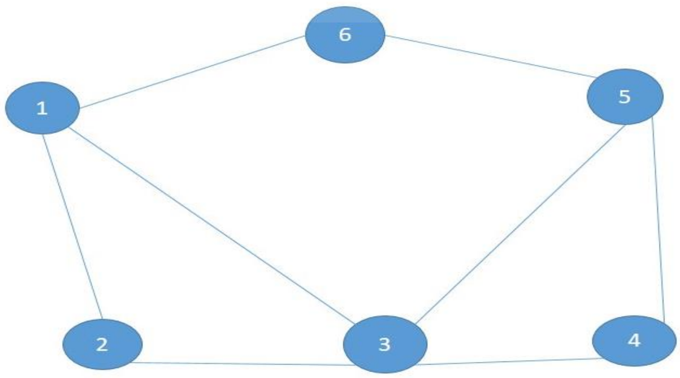


Figure-2 Simple Graph

The above graph is a simple graph, since no vertex has a self-loop and no two vertices have more than one edge connecting them.

Example: Multi graph

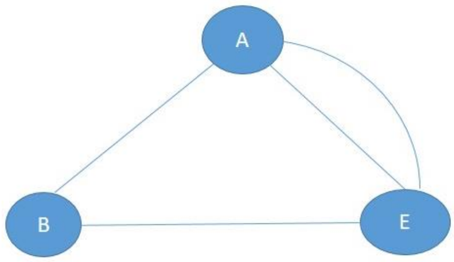


Figure-3 Multi Graph

2. **Undirected Graph:** A graph in which the edges do not have any direction and all the edges are in bi-direction (figure 4). In undirected graph if there is an edge from u to v then we can move from node u to node v and as well as from node v to node u.

Example:

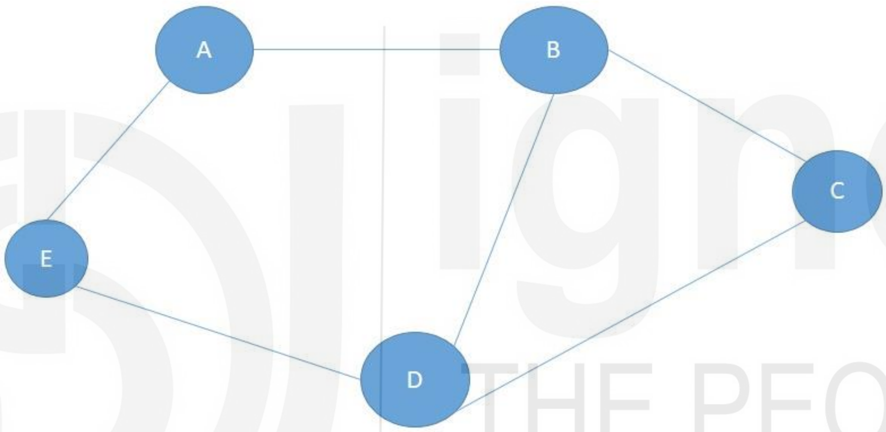


Figure-4: Undirected Graph

In the above undirected graph we can traverse through following paths:

From A: A – E, A – B
From B: B – A, B – D, B – C
From C: C – B, C –D
From D: D – E, D –B,D –C
From E: E –A, E –D

3. **Directed Graph:** A graph in which the edges have direction (figure 5). It is also called digraph. This is usually indicated with an arrow on the edge. In a directed graph if there is an edge from u to v then we can move from a node u to a node v only.With the help of directed graph we can represent asymmetrical relationships between nodes , roads network , hyperlinks connecting web pages etc.

Example:

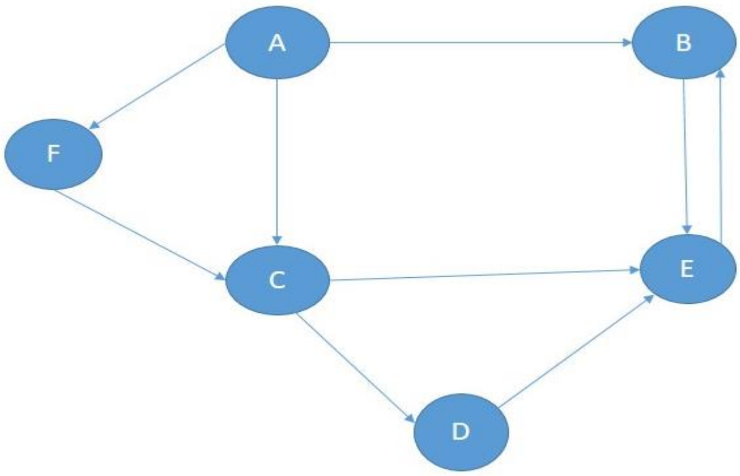


Figure -5 Directed Graph

In the above directed graph we can traverse through following paths:

From A: A – F, A – C, A –B
From B: B – E
From C: C – E, C – D
From D: D – E
From E: E – B
From F: F – C

4. **Subgraph:** A subgraph of G is a graph G' such that V(G') ⊆ V(G) and E(G') ⊆ E(G)
i.e., a graph whose vertices and edges are subsets ofanother graph. It is not necessary that a subgraph will have all the edges of graph or all the nodes.
Example:

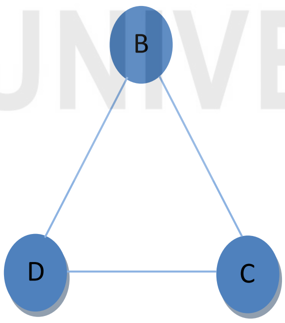


Figure-6: A sub graph of a figure-4

This is a subgraph of the graph which has the nodes A, B, C, D. In this there are C, B, D nodes.

5. **Connected Graph:** An undirected graph is said to be connected if for every pair of two different vertices v_i , v_j, there is a path between these two vertices. The graph G_1 is connected whereas G_2 is not connected

Example:

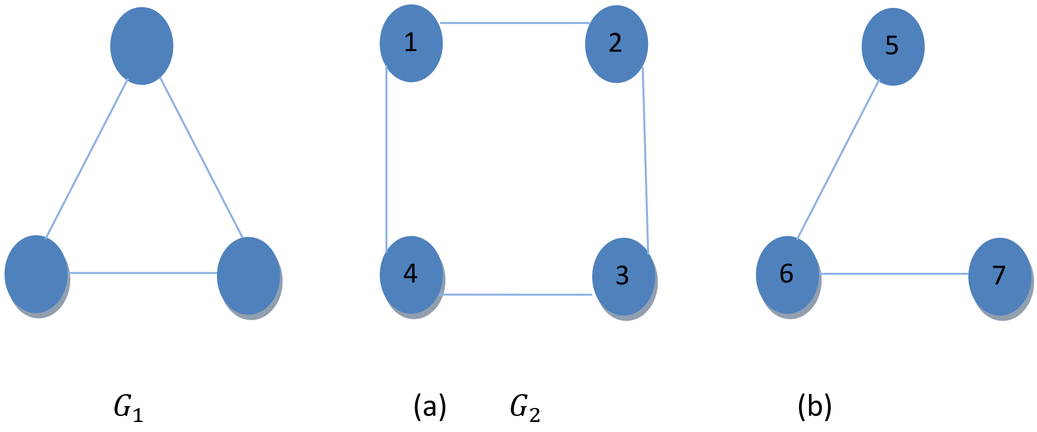


Figure 7: Connected graph

3.3 GRAPH REPRESENTATION

The purpose of graph representation is to convert it to a format that can be used by algorithms running on a computer. In order to have efficient algorithms, the logical representation of the graph plays a very critical role. When it comes to representations of graphs, two most standard and common computational representations are in practice such as Adjacency Matrix and Adjacency List. Apart from these other additional representations such as Linked lists, contiguous lists and combinations are also used in some cases. The selection of a particular representation depends on applications and functions one wants to perform on these graphs. In case the graph is sparse for which the number of edges also written as |E| is quite less than $|v|^2$,adjacency list is preferred but if the graph is dense where $|E|$ is close to $|v|^2$ an adjacency matrix is selected.

3.3.1 Adjacency Matrix:

The adjacency matrix of a graph with V vertices is V x V Boolean matrix with one row and one column for each of the graph's vertices
Adjacency matrix representation is typically used to represent both directed and undirected graphs, where the concentration of nodes is a graph is dense in nature. Usually in such graphs |E| is much close to $|V|^2$.
The Boolean matrix $A[i,j] = 1$if there is edges between $A_i \& A_j$
$A[i,j] = 0$, otherwise

Adjacency Matrix of the following graph(Figure 8)is given in Figure 9.

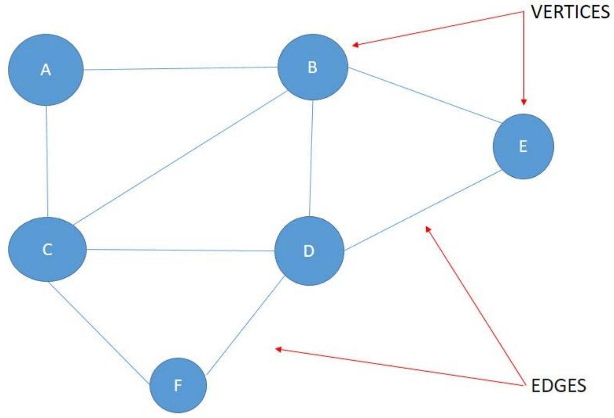


Figure 8: Example graph for matrix representation

In the given graph, if there is a link between the vertices we markas'1' in the adjacency matrix, else we markas '0'.

	A	B	C	D	E	F
A	0	1	1	1	0	0
B	1	0	1	1	1	0
C	1	1	0	1	0	1
D	0	1	1	0	1	1
E	0	1	0	1	0	0
F	0	0	1	1	0	0

Figure 9: Adjacency matrix representation of a graph given at figure 8

It is easy to determine if there is an edge between two vertices in a graph if it is represented throughan adjacency matrix.To find out how many edges are in a graph, it will require at least $O(V^2)$ time as there arev^2 entries of matrix have to be examined except the diagonal element which are zeroes.
Adjacency matrix for a directed graph may or may not be symmetric but in case of undirected graph, it is always symmetric
Space complexity to represent an adjacency matrix is $V^2\ bits$.

3.3.2 Adjacency List

Adjacency list representation is typically used to represent graphs, where the number edges |E| is much less than $|V|^2$.Adjacency list is represented as anarray of |V| linked lists.There is one linked list for every vertex node in agraphíEach Node in this linked list is a reference to the other vertices which share an edge with the current vertex. The figure 11, shows adjacency list of a given graph (Figure 10)

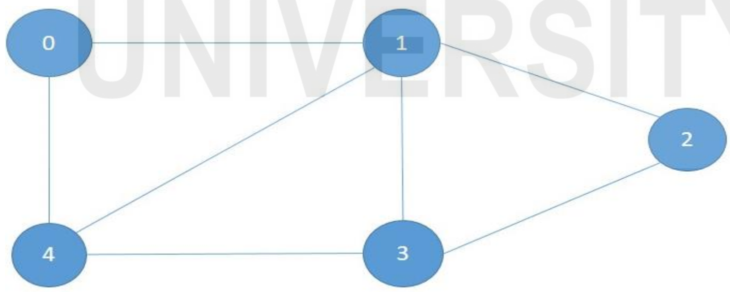


Figure 10: Example Graph for adjacency list

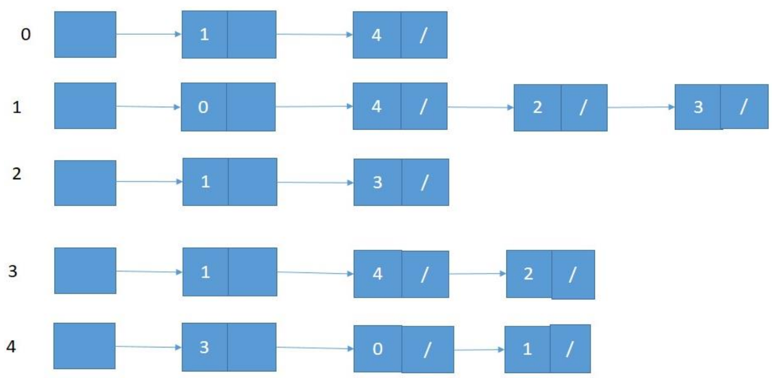


Figure 11: An adjacency list of the above graph

If there is undirected graph with V vertices and E edges, adjacency list representation requires |V| head nodes and 2|E| adjacency list nodes. In an undirected edge (i, j), i appear in j's adjacency list and j appears in i's adjacency list. In terms of space complexity it takes logV bits for the head nodes and log V and log E bits for list nodes where V and E are the number of vertices and edges.

3.4 GRAPH TRAVERSAL ALGORITHMS

Traversal algorithms are used to navigate across a given graph among all the nodes using all possible vertices. These algorithms will help us in finding the nodes, making paths that are shortest or feasible or prioritized in nature. Graph traversal can be carried out using breadth or depth as a criteria.In the data structure course, we have learnt to traverse a tree in preorder, inorder and postorder. One encounters a similar problem of traversing a graph: given a graph $G = (V, E)$ one wants to visit all the vertices in G from a given vertex. There are two key graphs traversal Depth First Search(DFS)&Breadth First Search (BFS) algorithms. In the next section we will examine both algorithms.

3.4.1 Depth First Search (DFS)

DFS starts with any arbitrary vertex in a graph and traverses to the deepest node as far as possible and then backtracks. The algorithm proceeds as follows:it starts with selecting any arbitrary vertex as start vertex then traverses the next node w adjacent to the starting vertex v. The process of traversing the unexplored nodes adjacent to the previous node $w$and continues till it finds that no adjacent node is left to be examined. If the last vertex is u which does not have any adjacent vertices, it backtracks to w and explores the unvisited adjacent nodes.

The traversal process terminates when all the vertices are traversed. Recursive implementationof DFS pseudo-codeis given below.

```
Function DFS( v)   // v is a starting vertex of a graph
Input G(V, E), visited [n], visited[n]    // visited[n] = 0
{
visited [v] = 1
for each vertex w adjacent to v do
{
If visited [w]  =  0 // if vertex w is not yet visited
                    DFS(w);               // Recursive call to DFS
}
} - // end of DFS (  )
```

Figure 12: pseudo-code of DFS

Let us apply the pseudo-code of DFS to thefollowing example graph

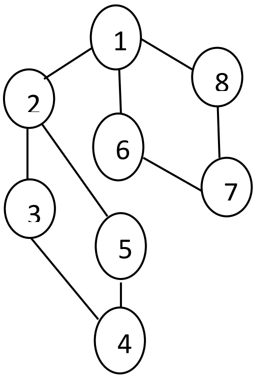


Figure 13: An example graph for DFS and BFS traversal

Output of DFS traversal of a graph at figure 12:1,2,3,4,5,6,7,8

What is the running time of DFS?

Suppose $G(V,E)$ is an undirected graph with V number of vertices and E edges and represented through adjacency list, then a sum of the length of all the adjacency list is $2|E|$.

DFS visits each node only once in the adjacency list. Therefore, the time to complete visiting all edges and the associated vertices is O(V+E). If a graph is represented through its adjacency matrix, the time to determine all vertices which are adjacent to the starting vertex v is V. Since at most V vertices are visited, the total running time is O(V^2)

3.4.2 Breadth First Search (BFS)

BFS is a graph searching algorithm which start from anyarbitrary vertex as astarting vertex and visits all its adjacent vertices at the first leveland then moves at the second level to visit all the unvisited vertices which are adjacent to vertices at the first level vertices and so on. Dijkstra's single source shortest path algorithm and Prim's minimum cost spanning tree problem uses ideas similar to BFS. BFS algorithm starts with any arbitrary vertex 1 and then visits all its adjacency vertices 2,6,8(i.e., nodes at the first level) first instead of discovering the deepest node as in DFS. At the next level (i.e., the second level) it will visit the adjacent vertices of 2,6 and 8 respectively. The adjacent vertices of 2 are 3 and 5, the adjacent vertex of 6 and 8 is 7 respectively.Finally, it will visit 4 which is a common adjacent vertex of 3 and 5 vertices.

The final output of BFS traversal of a graph is: 1, 2,6,8,3,5,7,4

The following algorithm describes the implementation details of BFS. Queue is a main data structure used to implement the algorithm

Pseudo code:

```
function BFS (v)// v is a starting  node of a graph
1.    let Q be queue data structure.
2.    Q.insert(v)//Insert s in queue until all its neighbour vertices are marked.
3.    Visited[v] =1 // Visited[ ] will mark a vertex '1' if it is visited, otherwise it is 0
```

```
print Q
while( Qisnot empty)
//Remove that vertex from queue,whoseneighbour will be visited now
  Q.remove()

//processing all the neighbours of v
for all neighbours w of v in a Graph G
if w isnot visited
Q.insert(w)//Stores w in Q to further visit its neighborand  markas visited.
Visited[ w] = 1
    Print Q.
```

Figure 14: BFS Pseudo-code

Complexities:

Time complexity:O (V + E),where $O(V)$ is a total time taken to complete queue operations (insertion and deletion of vertices. Insertion and deletion of a single vertex takes O(1) unit of time . Since there are V number of vertices in the graph, it will take O(V) time. The time taken to traverse each adjacency list only once is O(E). Therefore, the total time is O (V + E).

Check Your progress -1

Q1. Which is the most suitable graph representation scheme for a sparse graph?

Q2. What is a simple graph?

Q3.How does BFS graph traversal scheme work?

Q4. What is the running time of DFS?

3.5 DIRECTED ACYCLIC GRAPH AND TOPOLOGICAL ORDERING

A directed graph without a cycle is called a directed acyclic graph (or a (DAG (for short) which is a frequently used graph structure to represent precedence relation or dependence is a network. The following is a example of a directed acyclic task graph

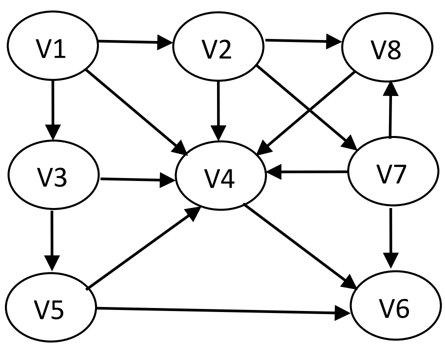


Figure 15: Directed Acyclic Graph

In this graph except vertex V1, all other vertices are dependent upon other vertices. Anymajor task can be broken down into several subtasks. The successful completion of the task is possible only when all the subtasks are completed successfully. Dependencies among subtasks are represented through a directed graph. In such representation, subtasks are represented through vertices and an edge between two vertices define a precedence relation. After showing dependency, the next task is to ordering is to ordering of these subtasks for execution. This is also called topological sorting or topological ordering

Topological sort or ordering of a $DAG\ G = (V, E)$ is a liner ordering of its vertices $V_1, V_2 \ldots . V_n$ so that if a graph G contains an edge $from\ v_i\ to\ v_j$ then v_i comes before v_j in the ordering. In other words, all edges between vertices representing tasks show forward direction in ordering or sorting. It the graph G contains a cycle then no topological order is possible.
Therefore,if a graph G has a topological sort, then G is a DAG.

Pseudo-code to compute topological sorting

```
Function topological_sort(G)
Input  G = (V, E) //G is a DAG.
{
search for a node V with zero in-degree (no incoming edges) and order it first in topological
sorting
remove V from G
topological_sort(G-{V})  // recursively compute topological sorting
append the ordering
}
```

Figure 16: Pseudo-code for topological sorting

Topological sorting time complexity

Finding out a vertex with no incoming edge, deleting it from a graph and appending it in the linear ordering would take O(n) time. Since there are n number of vertices is G, there will be n times loop, the total time will be $O(n^2)$. But if the graph is sparse where the $|E| << n^2\ and$ the graph is represented through an adjacency list it is possible to have $O(m + n)$ time complexity, where m is |E|.

Illustration of an example

The following figures show the application of the algorithm to the example givenin the figure 17

Step 1:V_1 - does not have incoming edges so it is deleted first.

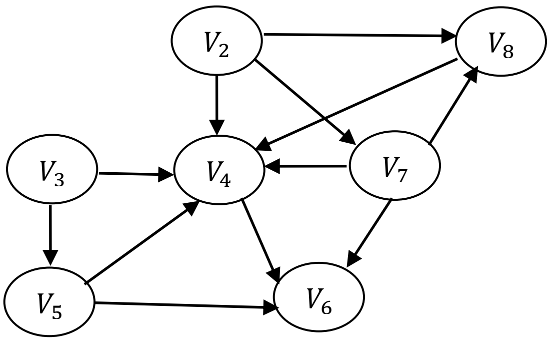


Figure 17 (a) : Deletion of V_1

Step 2 : In the graph there are the two nodes V_2&V_3 with no incoming edges. One can pick up either of the two vertices.Let us remove V_2 and append it after V_1, i.e., V_1,V_2

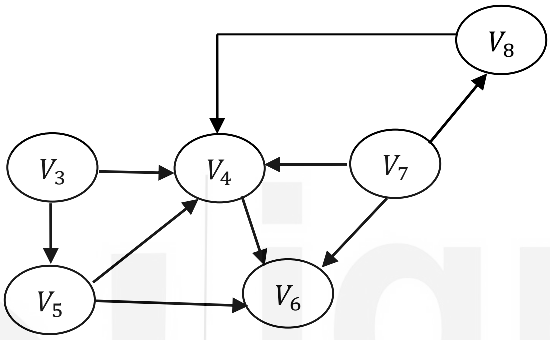


Figure 17(b): Deletion of V_2

Step 3: - $V_3$is the only node with no incoming edge. ThereforeV_3will be removed and appended after V_2,$i.e.$, $V_1V_2V_3$

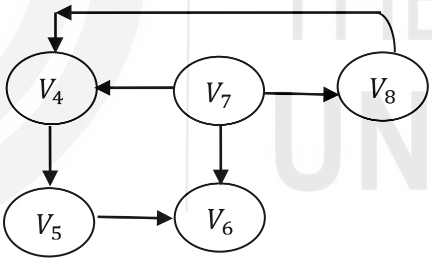


Figure 17(c): Deletion of V_3

*Step*4 :V_5 is the only node with no incoming edge.V_5 will be removed and appended to the list ,i.e.,

$V_1V_2V_3V_5$

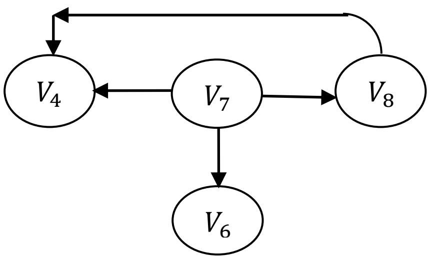


Figure 17(d): Deletion of V_5

Step 5 :V_7 is the only node with no incoming edge. Therefore V_7 will be removed and appended, i.e., $V_1 V_2 V_3 V_5 V_7$

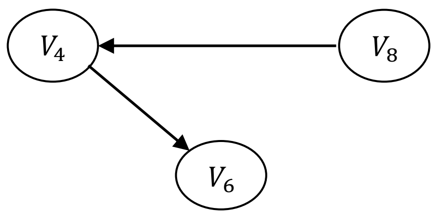


Figure 17(e): Deletion of V_7

Step 6: - V_8 will be removed and appended, i.e.,$V_1 V_2 V_3 V_5 V_7\, V_8$

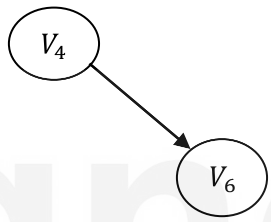
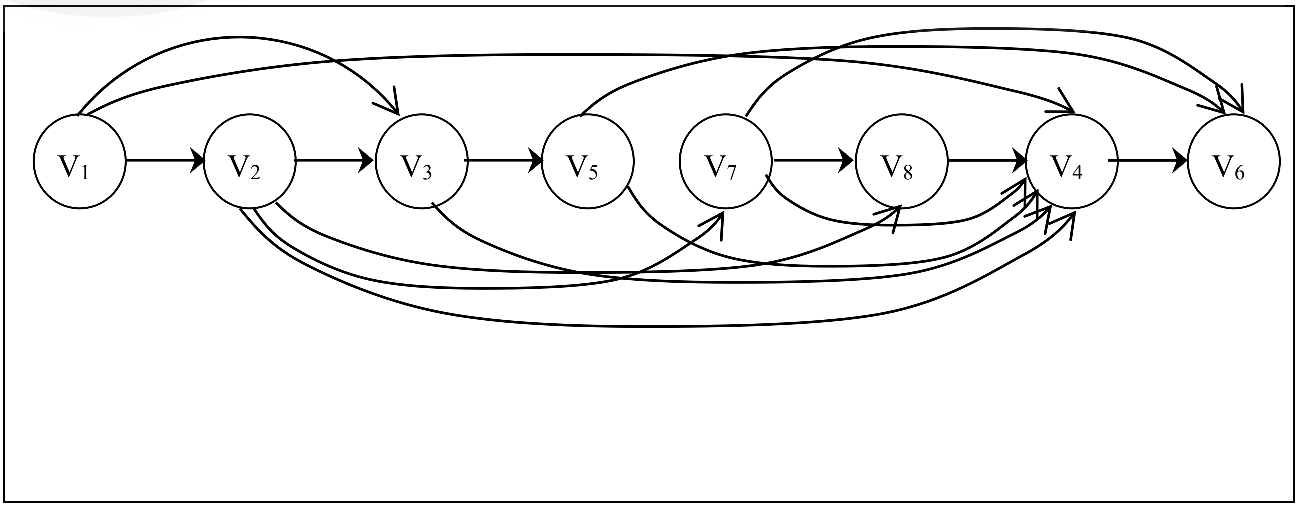


Figure 17(f) : Deletion of V_8

Step 7: V_4 will be removed and appended as: $V_1, V_2, V_3, V_5, V_7\,, V_8, V_4$

Step 8: FinallyV_6 will be appended:$V_1, V_2, V_3, V_5, V_7\,, V_8, V_4,V_6$

The linear ordering of vertices is shown in the following figure 18:



3.6 STRONGLY CONNECTED COMPONENTS (SCC):

In this section we define two terms: Strongly Connected Graph andStrongly Connected Components of a directed graphand then we apply an algorithm to find out whether a given directed graph is strongly connected component or not. A directed graph $G = (V, E)$ is strongly connected if for every two vertices $V_i$and $V_j$there is a pair of edges fromV_i to V_j and V_j to V_i

Strongly connected components is a maximal set of vertices M $\subseteq$ V such that every pair of vertices in M aremutually reachable. The following figure is an example of strongly connected components of a graph *G*.

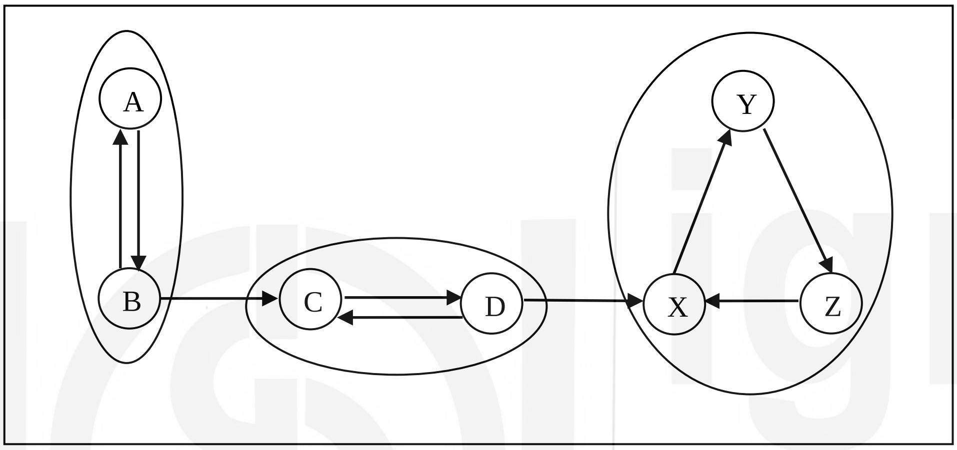


Figure 19: Example of SCC

In the above graph *G* there are 3 subsets of vertices, i.e., AB and CD and XYZ which are mutually reachable.

Pseudocode of Strongly Connected Components

- Perform DFS (Depth First Search) on the directed graph *G* and the number the vertices in the order they are visited
- Transpose of the original *G* (i.e. G^T)
- Perform DFS (Depth First Search) on G^T = (V, E^T) by starting the traversal at the highest numbered vertex
- Print the final result.

Figure 20: Pseudocode ofStrongly connected components

The prospered pseudocode for finding strongly connected components of a *G*= (V, E) uses the graph traversal scheme DFS first and then performs transpose of a graph G, i.e.,G^T(V, E^T) with all the edges of *G* reversed i.e., E^T consists of all the edges in reverse direction of *G*. To compute transpose of a directed graph , we have to make a traversal of adjacency list representation of a graph. If there is an edge from a vertex u to v in the main graph, the vertex v will be in the adjacency list of the vertex u. We reverse the edge from(u to v)to (v to u) in the transpose graph. This process will be continued for all the vertices in the adjacency list.

It is to be observed that a graph G and its transpose G^T have exactly the same strongly connected components.

Time Complexity:- It a graph is represented through adjacency lists, time to create $G^T$is O(V + E). The total time complexity is also O(V+E).

Check your progress -2

Q1. Define topological ordering.

Q2 Define strongly connected components of a graph**.**

3.7 SUMMARY

A graph is a very frequently used data structure for many basic and significant algorithms. There are two standard approaches to represent a graph: Adjacency matrix and adjacency lists which can be used to represent both directed as well as undirected graph. When the graph is a sparse, adjacency list is preferred because it is a more compact way to represent it. Many graph algorithms are based on the assumption that the graph is represented through an adjacency list. BFS and DFS are two simplest searching algorithms. Prim's minimum cost spanning tree problem and Dijkstra's single source shortest path algorithm uses ideas similar to BFS The unit also describes two applications of DFS: (i) directed acyclic graph and topological sorting (ii) decomposing a graph into strongly connected components.

3.8 SOLUTION TO CHECK YOUR PROGRESS

Check Your progress -1

Q1 . Which is the most suitable graph representation scheme for a sparse graph?

Ans. Adjacency list provides a compact way to represent a sparse graph

Q2What is a simple graph?

Ans. Asimple graph is a graphin which each edge is connected with two different vertices and no two edges are connected with same vertices. In this there is no self-loop and no parallel edges in a graph. A A simple graph with multiple edges is called multigraph

Q3 How does BFS graph traversal scheme work?

Ans.BFS is a graph searching algorithm which start from any arbitrary vertex as a starting vertex and visits all its adjacent vertices first and then moves at the second level to visit all the unvisited vertices which are adjacent to vertices at the first level vertices and so on

Q4 What is the running time of DFS?

Ans. Suppose $G(V,E)$ is an undirected graph with V number of vertices and E edges and represented through adjacency list, then a sum of the length of all the adjacency list is $2|E|$.
DFS visits each node only once in the adjacency list. Therefore the time to complete visiting all edges and the associated vertices is O(V+E). If a graph is represented through its adjacency matrix, the time to determine all vertices which are adjacent to the starting vertex v is V. Since at most V vertices are visited, the total running time is O(V^2)

Check your progress -2

Q1.Define topological ordering.

Topological sort or ordering of a $DAG\ G = (V,E)$ is a liner ordering of its vertices $V_1, V_2 \ldots . V_n$ so that if a graph G contains an edge $from\ \ v_i\ to\ v_j$ then v_i comes before $v_j$in the ordering. In other words, all edges between vertices representing tasks show forward direction in ordering or sorting. It the graph G contains a cycle then no topological order is possible. Therefore,if a graph G has a topological sort, then G is a DAG.

Q2 Define strongly connected components of a graph.

Strongly connected components of a graph is a maximal set of vertices M $\subseteq$ V such that every pair of vertices in M aremutually reachable.

3.9 FURTHER READINGS

1. Thomas H. Cormen, et el, Introduction to Algorithms, 3rd edition, prentice Hall of India, 2012
2. John Kleinberg and Eva Tardos, Algorithm Design, Pearson, 2012