UNIT 4 GRAPH COLOURINGS

Structure **Page Nos.**

4.0 INTRODUCTION

You must have seen political maps of India with different states coloured differently to distinguish between them. Have you ever wondered what the minimum number of colours required to colour the map is for any two states with a common boundary to have two different colours? This problem of finding the minimum number of colours needed to colour a given map is called the map colouring problem.

We can formulate the problem in terms of graph theory. We can construct a graph in such a way that each state of India corresponds to a vertex of India and two states are adjacent if and only if the corresponding vertices are adjacent. So, we have to colour the vertices of the graph in such a way that any pair of adjacent vertices have different colours. In the map colouring problem, we ask for the minimum number of colours needed to carry out such a colouring.

Note that the construction mentioned above leads to a special class of graphs called planar graphs. If we are interested in the map colouring problem alone, it is enough to restrict ourselves to such graphs. However, the general vertex colouring problem, which asks for the minimum number of colours needed to colour the vertices of a given graph, not necessarily planar, is interesting in itself. So, we start our unit by discussing this problem in Sec. 4.2.

Analogous to the colouring of vertices, is the colouring of edges. In Sec. 4.3, we have a brief discussion on edge colourings. This includes the definition of edge colouring, some examples of edge colouring and statements of some of the well known results in this field.

In Sec. 4.4, as a preparation for our study of the map colouring problem, we study planar graphs. In this section, we will prove some basic results about planar graphs. We will also prove a characterization of planar graphs due to Kuratowski.

In Sec. 4.5, we study the map colouring problem. We give a brief history of the four colour theorem, which says that any map can be coloured with four colours. However, the proof of this theorem is beyond the scope of this course.

4.1 OBJECTIVES

After studying this unit, you should be able to

- compute the vertex chromatic number of some simple graphs;

- compute the edge chromatic number for some simple graphs;
- in simple cases verify whether a given graph is planar or not using Kuratowski's theorem;
- explain the map colouring problem and relate it to the study of planar graphs.

4.2 VERTEX COLOURING

In this section, we start our study of colourings by considering the graph in Fig. 1. We have given a colouring of K_3 using three colours, namely, red (r), green (g) and blue (b).

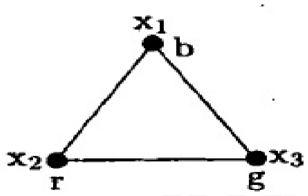

Fig.1

Why have we used three colours? It is because we want the adjacent vertices to have different colours. In K_3, any two vertices are adjacent. So we need to colour each of the vertices with different colours. Keep this example in mind when you read the definition given below.

'χ' is the Greek letter 'chi'.

Definition: A **vertex colouring** of a graph G is an assignment of colours to vertices of G in such a way that no two adjacent vertices have the same colour. A graph is called **k-vertex colourable** if it has a vertex colouring of k colours. The **minimum** number of colours required to colour a graph G is called the **vertex chromatic number of G,** usually denoted as χ**(G)**.

We will say that a graph is **k-chromatic** (or **k-colourable**) if it has chromatic number k.

In Fig. 1, we were able to use the names of the colours, red, green and blue, because we needed only three colours. Suppose we need, say, 20 colours, can we still use the names to refer to the colours? We may not remember the names of so many colours, and could call them Colour 1, Colour 2, etc. This will do just as well, because the names of the colours are not important as long as you can distinguish between the different colours. In the diagrams, we will denote the colours as $\boxed{1}$, $\boxed{2}$,….

Let us now look at some examples.

Example 1: Colour the graphs in Fig. 2 with the minimum possible number of colours. Also, find the chromatic numbers of the graphs.

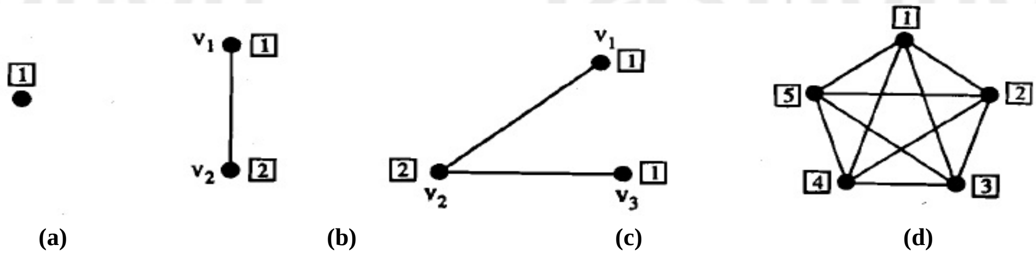

Fig. 2 : Some examples of colouring

Solution: In Fig. 2(a), K_1 has just one vertex. Let us colour this with $\boxed{1}$. So, one colour is clearly the minimum required for colouring this graph. Hence, its chromatic number is 1.

In Fig. 2(b), K_2 has two adjacent vertices. We assign $\boxed{1}$ to the vertex v_1 and $\boxed{2}$ to the vertex v_2. Thus, we have a 2-colouring. Is there a 1-colouring? No! The two vertices are adjacent and so we need at least two colours. In other words, the chromatic number $\chi(K_2) = 2$.

In Fig. 2(c), we have three vertices and we can colour them with three different colours. But, can we also have a two-colouring? Notice that, v_1 and v_3 are not adjacent. So, we can colour them with the same colour, say, $\boxed{1}$. v_2 is adjacent to both v_1 and v_3. So, we cannot assign $\boxed{1}$ to this. Let us assign $\boxed{2}$ to v_2. So, we have a 2-colouring. As we cannot have a 1-colouring, this graph has chromatic number 2.

In Fig. 2(d), we have K_5. In this any two vertices are adjacent, so we need as many colours as there are vertices, that is, we need five colours. So K_5 has chromatic number 5.

$\chi(K_n) = n \forall n \geq 1$ because any pair of vertices are adjacent in K_n.

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Remark: In the example above, we saw that the chromatic number of K_1 is 1. More generally, **a graph consists of isolated vertices, if and only if its chromatic number is 1.**

Example 2: Find the chromatic number of a bipartite graph with a non-empty edge set.

Solution: From Unit 2, you may recall that a graph G is bipartite if the vertex set of G can be partitioned into two **non-empty** disjoint subsets A and B
Such that any two vertices in a given set are non-adjacent. We get a 2-colouring of G by assigning $\boxed{1}$ to all the vertices in A and $\boxed{2}$ to all the vertices in B. (This is illustrated in a particular case in Fig. 3.) Further, note that, since A and B are non-empty and since the edge set of G is non-empty, at least one vertex in A is adjacent to a vertex in B and these two vertices must have different colours. So, we cannot manage with less than two colours. So, $\chi(G) = 2$ if G is a bipartite graph with a non-empty edge set.

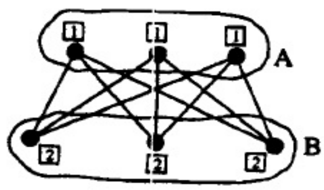

Fig.3

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Remark: We saw, in Example 2, that the chromatic number of a bipartite graph with non-empty edge set is 2. The converse is also true. Given a graph G and a 2-colouring of G, we can partition the edge set of G into two non-empty sets A and B defined as follows:

$$A = \{ v \in V(G) \mid v \text{ is assigned the colour } \boxed{1} \}$$
$$B = \{ v \in V(G) \mid v \text{ is assigned the colour } \boxed{2} \}$$

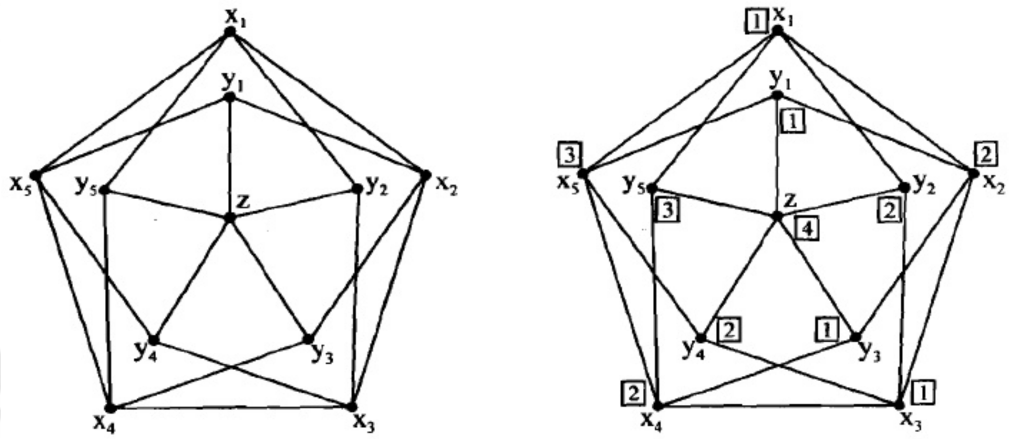
By the definition of colouring, no two vertices in A are adjacent, and similarly for B. Since A and B are disjoint, G is bipartite, by definition.

Here are some exercises to test your understanding of the examples above.

E1) What is the chromatic number of
- i) a tree with at least two vertices?
- ii) an even cycle C_{2n}, $n \geq 2$?
- iii) an odd cycle C_{2n+1}, $n \geq 1$?

Now, if a graph is k-colourable, are all its subgraphs k-colourable? Let us see. Let G be a k-colourable graph and H be its subgraph. We assign to each vertex of H the same colour that we assigned to it, considered as a vertex of G. If two vertices are non-adjacent in G, they are non-adjacent in H, and therefore this gives a colouring of H. In other words, $\chi(H) \leq k = \chi(G)$ for every subgraph H of G. We can also recast this statement in the following form. **If a graph G has a subgraph H with chromatic number k, the chromatic number of G must be at least k.** This fact helps us in finding the chromatic number of a graph sometimes. We illustrate this in the next example.

Example 3: Find the chromatic number of the Grötzsch graph (see Fig 4(a).)



Without colouring
(a)

With 4-colouring
(b)

Fig. 4 : The Grötzsch graph

Solution: Fig.4(b) gives a 4-colouring of this graph. Can this graph have a 3-colouring? Let us see. Since the outer 5-cycle is an odd cycle, it needs three colours. So, we need at least three colours. Let us suppose the colours of $x_1, \dots, x_5$ are as shown in Fig.4(b). Since y_1 is adjacent to x_2 and x_5, we have to give it a colour different from $\boxed{2}$ and $\boxed{3}$. So, we assign $\boxed{1}$ to it. Similarly, the colours of y_4 and y_5 must be $\boxed{2}$ and $\boxed{3}$, respectively. Since the vertex z is adjacent to vertices to which the colours $\boxed{1}$, $\boxed{2}$ and $\boxed{3}$ have been allotted, we have to use a fourth colour for this vertex. So, this graph is not 3-colourable. Therefore, this has chromatic number 4.

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In the examples and exercises above, we saw that if a graph G has a subgraph H with chromatic number $\chi(H) = n$, then $\chi(G) \geq n$. In particular, if a graph G has a subgraph H which is isomorphic to K_n (such a subgraph H is known as a **clique of size n**), the chromatic number of G is at least n.

However, **the converse is not true**, i.e., if a graph has chromatic number $\geq n$, it need not have a clique of size n. The Petersen graph provides a counter-example for this. Its chromatic number is 3 (see E2). Convince yourself (you need not prove it) that it does not contain a clique of size 3, i.e., a subgraph isomorphic to K_3.

More generally, in 1955, Mycielski proved that, for any integer k, there exists a k-chromatic graph without triangles. The proof of this result is beyond the scope of this course. However, it is not difficult to prove the much weaker result that if the chromatic number of a connected graph is greater than 2, it contains an odd cycle. We leave this as an exercise for you (see E3), along with some more exercises, to test your understanding of the material we have covered so far.

E2) Find a 3-colouring of the graph in Fig.5, and its chromatic number.

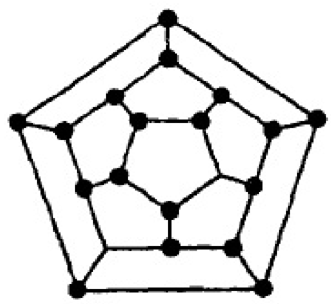


Fig.5

E3) Show that the chromatic number of the Petersen graph, given in Fig. 6, is 3. Also check that it does not contain K_3.

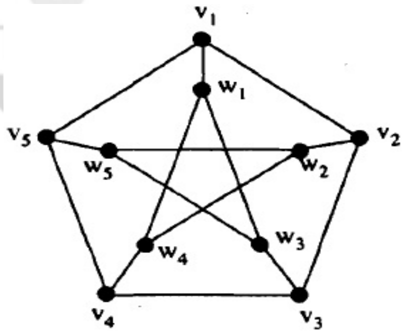


Fig.6 : The Petersen graph

E4) Show that if $\chi(G) \geq 3$ for a graph G, it contains an odd cycle

E5) Find the chromatic number of the following graph.

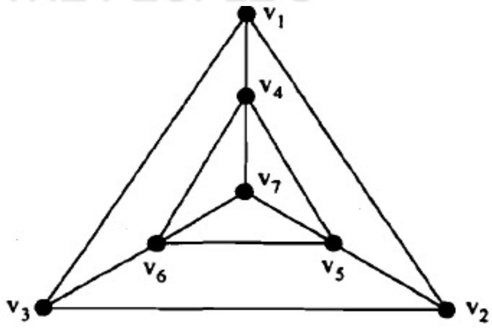


Fig. 7

E6) Construct a graph with chromatic number 5.

Recall that, we have shown that any 2-colourable graph is bipartite. How was this done? We had put all the vertices having the same colour in a single set. There were two colours and so we got two subsets. They were disjoint because no vertex can be assigned two colours.

The colour classes can be defined for any colouring of a graph G, not just for a χ (G)-colouring.

We are going to extend these ideas to n-colourable graphs. We do this through the concept of colour classes, which we now define.

Definition: For a k-colouring of a graph G, consider the set
$C_i = \{ x \in V(G) \mid x \text{ is assigned the colour } \boxed{i} \}$, for $1 \leq i \leq k$.
Clearly, $C_i \cap C_j = \phi$, for every $i \neq j$, and $V(G) = C_1 \cup \ldots .. \cup C_k$.
In particular, if $\chi(G) = n$, each of the n colours is assigned to at least one vertex. (Why?) So none of these subsets is empty. Therefore, we get a partition of the vertex set V(G) into n mutually disjoint non-empty subsets. The subsets $C_1, \ldots, C_n$ are called **the colour classes** of G given by the n-colouring.

For example, the colour classes of a 2-colourable graph give a bipartition of the vertex set of the graph, making it bipartite.

Let us now look at some examples of colour classes.

Example 4: Find the colour classes in the two different colourings of the graph given in Fig.8.

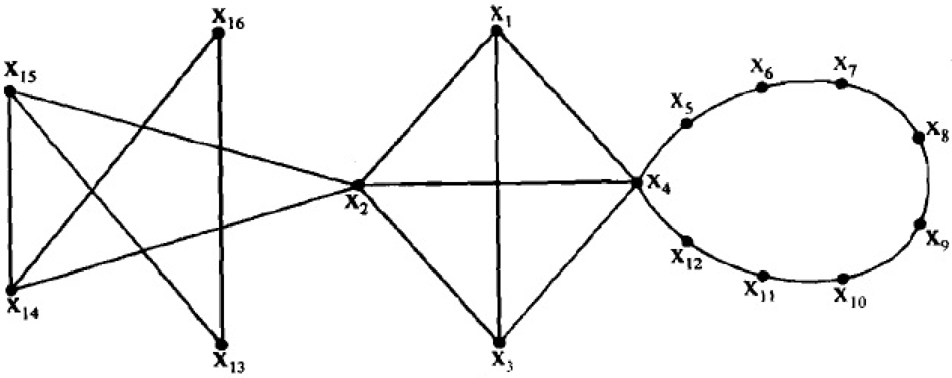
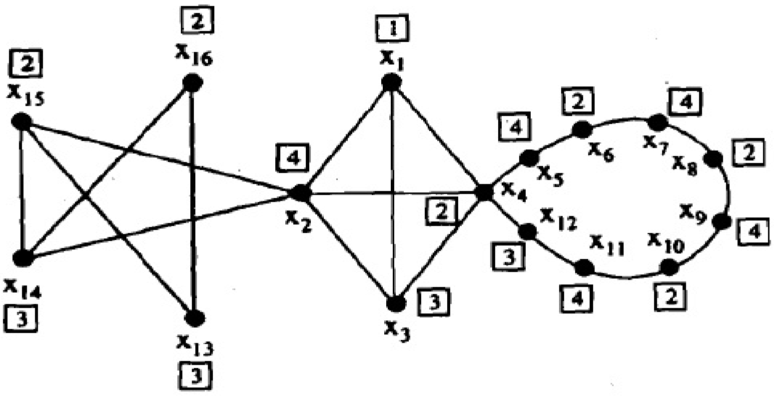
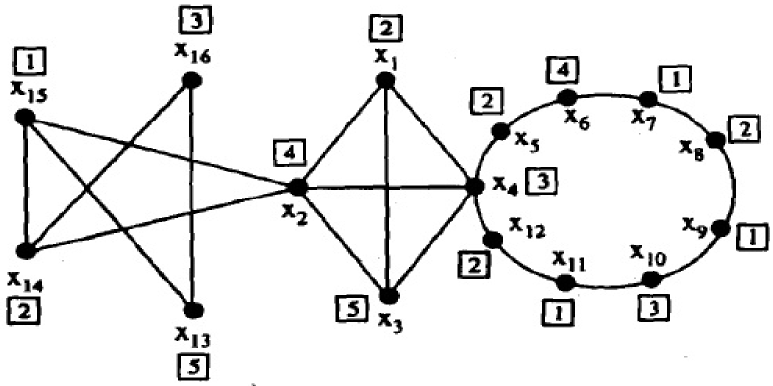


Fig. 8

Solution : The colour classes given by the colouring in Fig. 9(a) are $C_1 = \{x_1\}$, $C_2 = \{x_4, x_6, x_8, x_{10}, x_{15}, x_{16}\}$, $C_3 = \{x_3, x_{12}, x_{13}, x_{14}\}$ and $C_4 = \{x_2, x_5, x_7, x_9, x_{11}\}$. Check that $C_1 = \{x_7, x_9, x_{11}, x_{15}\}$, $C_2 = \{x_1, x_5, x_8, x_{12}, x_{14}\}$, $C_3 = \{x_4, x_{10}, x_{16}\}$, $C_4 = \{x_2, x_6\}$, and $C_5 = \{x_3, x_{13}\}$ are the colour classes corresponding to the colouring in Fig. 9(b).



(a)



(b)
Fig.9

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Try some exercises to test your understanding of the example above.

E7) Check whether 'xRy iff x and y lie in the same colour class' defines an equivalence relation on the vertices of a graph G.

E8) Colour the following graph in two different ways, and give the colour classes in each case.

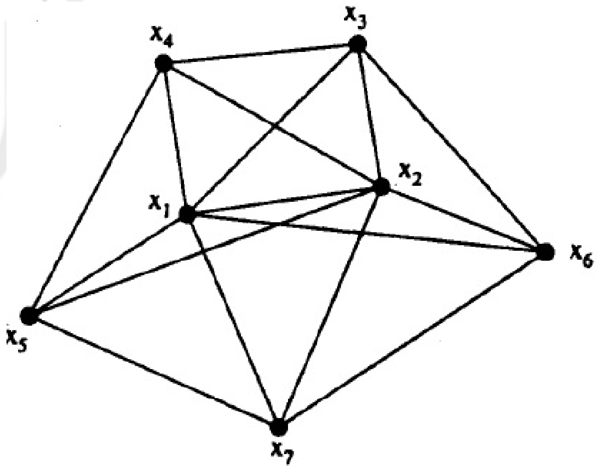

Fig. 10

We have seen that any colouring of a graph gives rise to colour classes.
You know that, if x, y are two vertices in a colour class C_i, then $xy \notin E(G)$. So, **each colour class consists of mutually non-adjacent vertices.** We now give a name to those subsets of the vertex set of a graph with this property.

Definition: A subset S of the vertex set V(G) of a graph G, is said to be an **independent set** if any two vertices in S are non-adjacent. An independent set is called **maximal** if it is not contained in any other independent set. The number of vertices in a largest independent set of G, is called the **independence number** of the graph G, and is denoted by α**(G)**.

So, for example, each colour class is an independent set, and, in fact, a maximal independent set. However, all independent sets are not dependent on a particular colouring.

Example 5 : Find three different maximal independent sets in the graph given in Fig. 11.

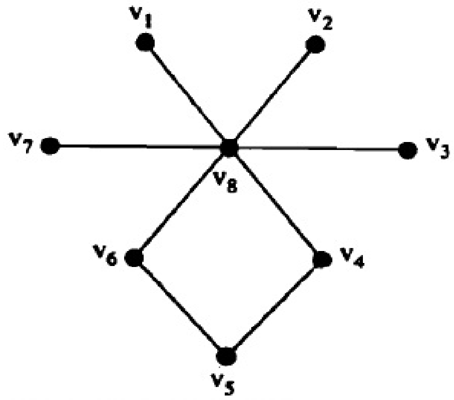

Fig. 11

Solution : The graph has the following maximal independent sets :
$\{v_8, v_5\}$, $\{v_5, v_1, v_2, v_3, v_7\}$, $\{v_1, v_2, v_3, v_4, v_6, v_7\}$
Let us check this.

Firstly, $\{v_8, v_5\}$ is a maximal independent set because all the other vertices are adjacent to one of these two vertices. So, if any more vertices are added, the resulting set will no longer be an independent set.

In the same way, you can check that the other two sets are also maximal independent sets.

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Now test your understanding of independent sets by trying the following exercises.

E9) Find an independent set of cardinality 4 in the graph given below:

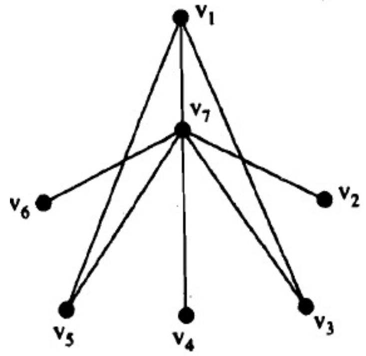

Fig. 12

E10) Find $\alpha(G)$ for the graphs given in Fig.8 and Fig. 10.

If we can colour the vertices of a graph, can we colour the edges of a graph? Is it interesting, or useful, to do so? In the next chapter, we will answer these questions.

4.3 EDGE COLOURING

In this section, we consider the problem of colouring the edges of a graph in such a way that no two edges with a common vertex receive the same colour. We will not prove any of the important results in this subject although we will state some of them. The purpose of this section is to give a brief introduction to edge colouring. We begin by defining edge colouring.

Definition: A **k-edge colouring** of a graph G is an assignment of k colours to the edges of G in such a way that no two edges incident with the same vertex have the same colour. A graph is **k-edge colourable** if it has a k-edge colouring. The **minimum** number of colours required to colour the edges of a graph is called the **edge chromatic number of G,** usually denoted by $\chi'(G)$.

Let us now look at some examples of edge colouring. The easiest case is the edge colouring of those graphs which have edge chromatic number 1.

Example 6 : Find all the graphs that have edge chromatic number 1.

Solution: Suppose a graph G has edge chromatic number 1. Since the edge chromatic number is one, the graph is 1-edge colourable, and no two edges share an end vertex, that is, the graph must be the union of some isolated vertices and some components consisting of two vertices and an edge (see Fig.13).
Conversely, a graph which is the union of isolated vertices and components having two vertices each has edge chromatic number 1.

Fig.13

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Example 7 : Colour the edges of the graphs K_3, K_4, K_5.

Solution : The colouring of K_3, K_4, K_5 is given in Fig. 14. Here no two adjacent edges have received the same colour. In all the cases, we have used the least possible colours.

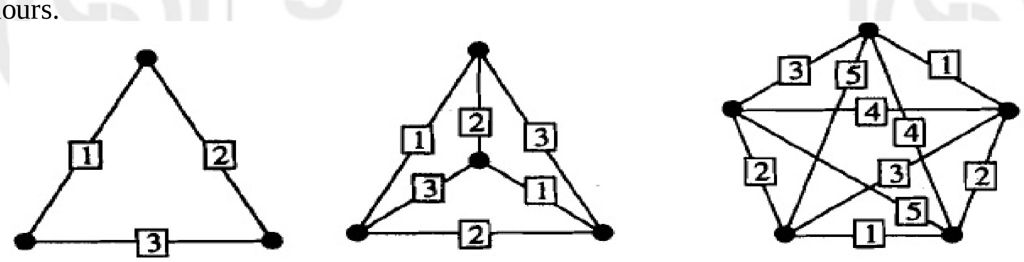

Fig.14

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Example 8 : Give an edge colouring of the Petersen graph.

Solution: Fig. 15 gives a 4-edge colouring of the Petersen graph.

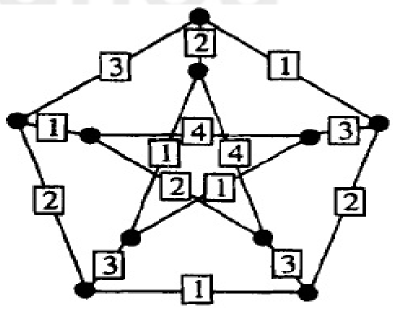

Fig. 15

Again no two adjacent edges have received the same colour. You can quickly check that three colours will not be enough. So $\chi'(G) = 4$.

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Example 9: Give edge colourings of all the trees on 5 vertices.

Solution : In Fig.16, we have presented all types of possibilities, with colourings.

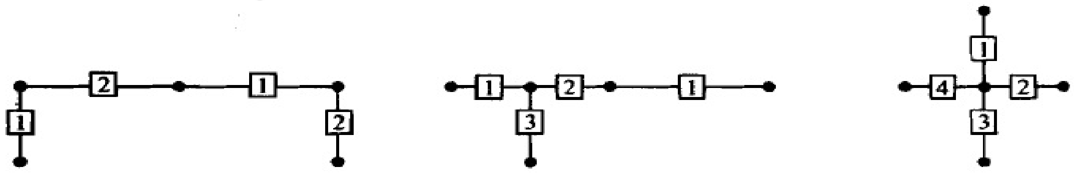

Fig. 16

Again we have used the least possible number of colours.

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Example 10: Find the edge-chromatic number of C_n.

Solution: As in the case of vertex colouring, if n is even, the edge chromatic number is 2. We can colour the edges alternately with the two colours. If n is odd, the edge chromatic number is 3. We have illustrated this in the case of C_4 and C_5 in Fig.17 .

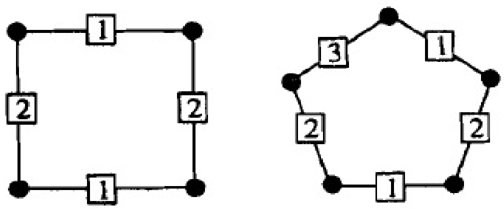

Fig. 17

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If G is a graph and $v \in V(G)$ such that $d_G(v) = \Delta(G)$, then all the edges incident on v must receive different colours. Hence, any edge colouring of G will need at least $\Delta(G)$ colours, that is, $\Delta(G) \leq \chi'(G)$.

Regarding an upper bound for $\chi'(G)$, in 1964 Vizing proved the following result.
Theorem 1: For any graph G, $\chi'(G) \leq \Delta(G) + 1$.

So, for any graph G, $\mathbf{\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1}$.

Therefore, there are only two possibilities for the edge chromatic number of a graph G, either $\Delta(G)$ or $\Delta(G) + 1$. We now present some of the results known in this direction, without proof.

1) The **edge chromatic number of K_n** is n, if n is odd ($\neq 1$) and $n - 1$ if n is even. Recall that, K_n is $(n-1)$-regular. So $\Delta(K_n) = n - 1$.

2) For a **bipartite graph** G, $\chi'(G) = \Delta(G)$ (proved by König in 1916).

Here are some related exercises now.

E11) What is the edge chromatic number of $K_{m,n}$?

E12) Consider the tree T given in Fig.18. Give an explicit $\Delta(T)$-edge colouring of T.

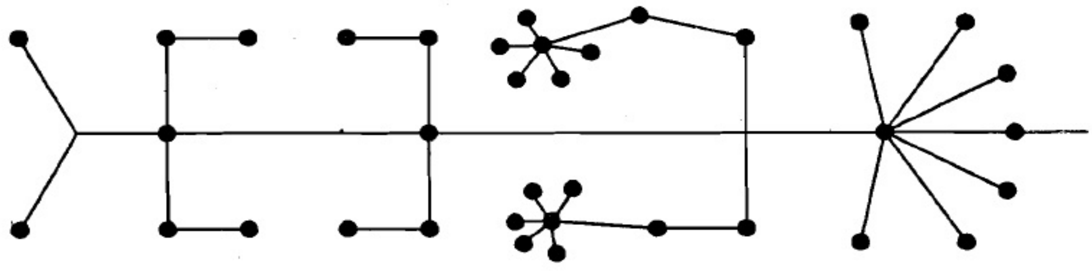


Fig.18

In the introduction, we mentioned that the map colouring problem can be reduced to finding the minimum number of colours needed to colour a special class of graphs called planar graphs. In the next section, we define planar graphs and prove some basic results that will be useful in the study of the map colouring problem.

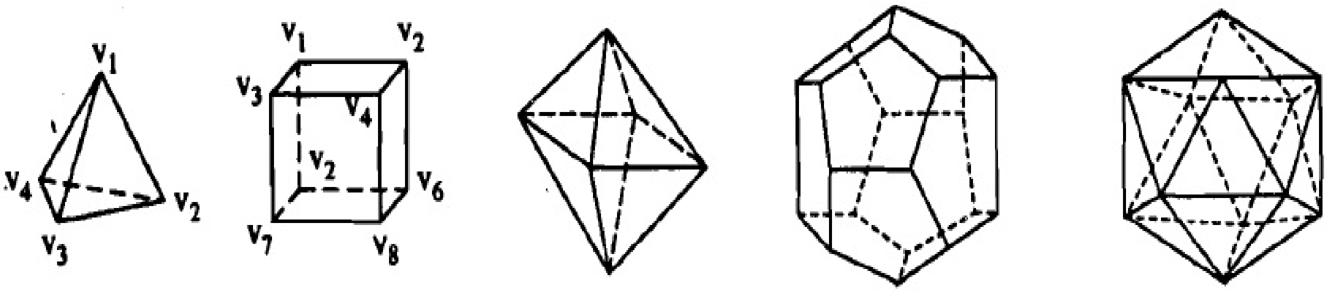
4.4 PLANAR GRAPHS

In transistor radios and television sets, you must have seen printed circuit boards. These boards have slots for various components and these slots are connected to each other. The connection between these slots must be made in such a way that no two connections cross each other. Given an electronic circuit, is it always possible to design a printed circuit board corresponding to it?

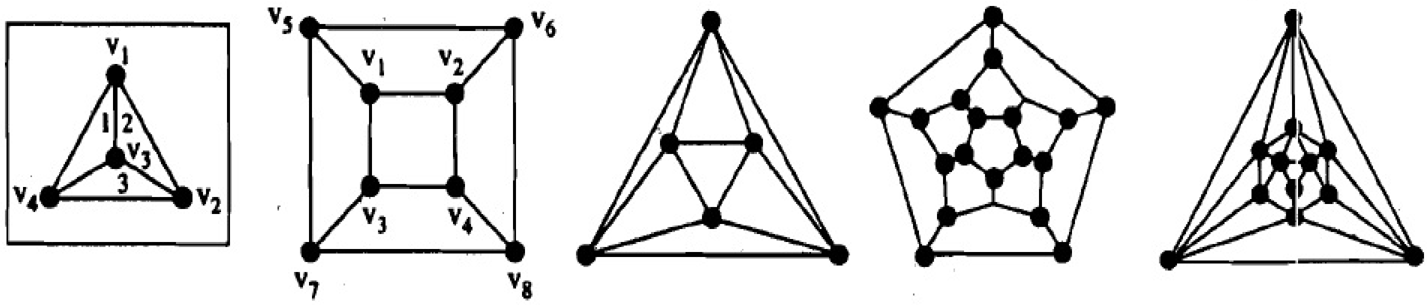
This can be formulated as a problem in graph theory. We replace the electronic components by vertices and the connections between them by edges. If the resulting graph can be drawn in such a way that no two of the edges cross each other except at the vertices, then we can design a printed circuit board for the given circuit. Let's start by seeing what such graphs are called.

Definition: A graph G is called **planar** if it can be drawn on a plane in such a way that no two edges cross each other at any point except possibly at a common end vertex. Such a drawing is called a **plane drawing.**

To see some examples of planar graphs, consider Fig.19. In this figure, we have given the five regular solids called platonic solids. In the second row, we have given the corresponding planar graphs. In each of these graphs, the vertices correspond to the vertices of the associated solid and the edges correspond to the edges of the solid.



tetrahedron **cube** **octahedron** **dodecahedron** **icosahedron**



tetrahedron **cube** **octahedron** **dodecahedron** **icosahedron**

Fig. 19 : Regular solids and the corresponding planar graphs

Next, we introduce the concept of a region. Look at the tetrahedron in Fig. 19 (a). It has four faces. The planar graph corresponding to it is given in Fig. 19(A). It divides the plane into four faces, or regions, which we have numbered from 1 to 4 in the figure. Similarly, the graph of the cube, given in Fig. 19(B) divides the plane into six regions.

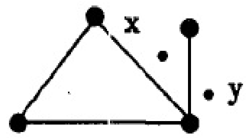


Fig.20 : x and y are just points in the plane, they are not vertices.

In all the cases above, it is very clear what the different regions are. But, look at the graph in Fig. 20. Into how many regions does it divide the plane? Two or three? Do the points x and y lie in the same region or in different regions? To avoid such confusion we need to define the concept of a region carefully. Here is the definition.

Definition: Given a plane drawing of a planar graph G, by a **region** (or **face)** of G, we mean a maximal portion of the plane for which any two points a, b in it can be joined by a curve which lies completely in that portion of the plane.

If R is a region of a planar graph G, by the **boundary of R** we mean all those points x in the plane corresponding to the vertices and edges of G having the property that x can be joined to any point in that region by a simple curve all of whose points, except x, are in that region.

There is always one unbounded region of G, and it is called the **exterior region** of G. Any other region is called an **interior region**.

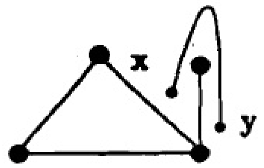


Fig.21

Let us go back to Fig. 20 again. Armed with this definition, we can answer the question we raised. As you can see in Fig. 21, the points x and y can be joined by a curve that does not cross any of the edges. So, there are only two regions, the region inside the triangle and the region outside it. Both the points lie in the exterior region of the triangle.

Let us now look at an example to understand these concepts better.

Example 11: Find the number of regions in the graphs given in Fig.22.

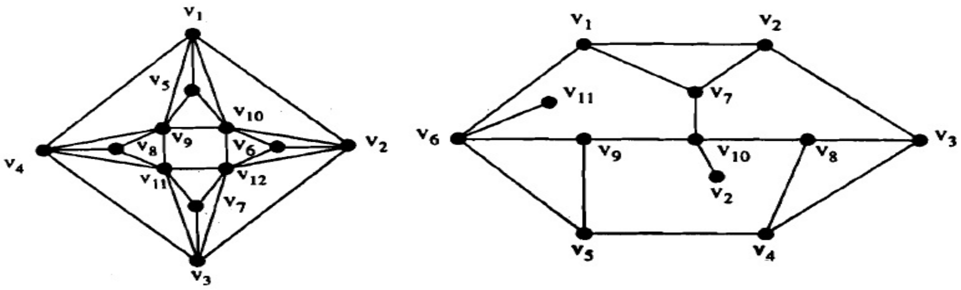


Fig. 22

Solution: The graph in Fig. 22 (a) has 8 regions, 7 interior and 1 exterior region. In the graph in Fig. 22(b), there are 3 regions.

***

Now, try the following exercises.

E13) Find the number of regions in a tree, a cycle and K_4.

E14) Is a subgraph of a planar graph planar ? Why ?

Now, given a planar (p,q)-graph, is there any relationship between the number of regions, r, and p,q ? Let us calculate the quantity $p - q + r$ for all the planar graphs in Fig. 13 and for the graph in Fig. 16(b), to see if we can find an answer to this.

Graph	P	q	r	p − q + r
Fig. 16(b)	6	7	3	2
Tetrahedron	4	6	4	2
Cube	8	12	6	2
Octahedron	6	12	8	2
Dodecahedron	20	30	12	2
Icosahedron	12	24	18	2

As you can see, $p - q + r$ is 2 for all these planar graphs. In fact, the following theorem, proved by Euler in 1736, shows that this is true for all such graphs.

Theorem 2 (Euler's formula) : If G is a connected planar (p,q)-graph, then the number r of the regions of G is given by $r = q - p + 2$.

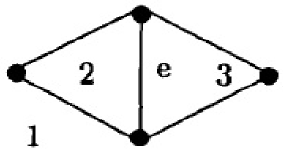
Proof: We apply induction on q, the number of edges of G, to show that $p - q + r = 2$.

If q = 0, then G just consists of 1 isolated vertex since it is connected. Hence, r = 1 and the formula holds.

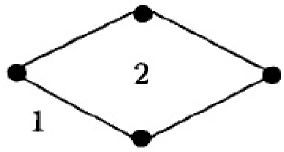
Now, assume that the formula holds for any plane drawing of a (p,t)-graph for every $t \le (q - 1)$, and suppose G is a (p, q)-graph.

If G is a tree, then p = q + 1 and r = 1, so that the formula holds.

If G is not a tree, then it contains a cycle. Let e be an edge of G that lies on a cycle of G, and consider the subgraph G – e of G. When we remove this edge e on the cycle, we are joining two regions to make one region out of them (e.g., if we remove e in Fig.23(a), the earlier regions 1 and 2 join to become one region in Fig.23(b).) So, G − e has p vertices, (q − 1) edges and (r − 1) regions.



(a)



(b)

Fig. 23

Now, by the induction assumption, Euler's formula holds for G − e. So, the number of regions = number of edges – number of vertices + 2.

i.e., $r - 1 = (q - 1) - p + 2.$

$\Rightarrow r = q - p + 2.$

Hence, by induction, the result is true for any connected planar graph.

Remark : Since $r = p - q + 2,$ where p and q are fixed once we fix a graph, the number of regions in a plane drawing of a planar graph is independent of the plane drawing.

The result we have just proved does not immediately help us to tell whether a graph is planar or not. However, we derive a necessary condition for planarity from it that is very useful. It tells us intuitively, that a planar graph can't have 'too many' edges. Recall that a graph on p vertices can have up to $\frac{p(p-1)}{2}$ edges. In the case of planar graphs, there is a much better bound. We give this bound in the next theorem.

Theorem 3: If G is a planar (p, q)-graph, with p ≥ 3, then q ≤ 3p − 6. Further, if G is also bipartite, we have q ≤ 2p − 4.

Proof : For p = 3, the result is clearly true. So, we assume $p \geq 4.$ Let G have r regions. For each region R of G, the number of edges lying on its boundary is at least 3. So, if S is the sum of the number of edges of each region, then $S \geq 3r.$

Also, every edge of G is counted once or twice while obtaining S. So $S \leq 2q.$ Thus, $3r \leq 2q.$ Using Euler's formula, we obtain $3(q - p + 2) \leq 2q,$ which gives us $q \leq 3p - 6.$

Now, if G is also bipartite, it will not contain any 3-cycle. Therefore, a region will be bounded by at least 4 edges, so that $S \geq 4r.$ Then, as argued above, we get $q \leq 2p - 4.$

Let us see how this result helps us to check whether a graph is planar or not.

Example 12: Show that K_5 is not planar.

Solution: Suppose K_5 is planar. Then the number of edges and vertices in K_5 satisfy the relation q ≤ 3p − 6 given in Theorem 3. K_5 has 5 vertices and 10 edges, so 10 ≤ 3 × 5 − 6, i.e. 10 ≤ 9, a contradiction. Thus, K_5 is non-planar.

Try the next exercise to check your understanding of Theorem 3.

E15) Show that $K_{3,3}$ is non-planar. Hence, show that given 3 houses, each having 3 outlets for electricity, gas and water, respectively, it is not possible to connect each of these utilities to each of the houses without the lines or mains crossing.

We have already seen that K_5 and $K_{3,3}$ are not planar. To prove this we used either of two necessary conditions. However, these conditions are not sufficient. For example, in the Grötzsch graph (see Fig.4), p = 11, q = 20 and $20 \le 33 - 6 = 27$. So, the condition in Theorem 3 is satisfied. But, as we shall show later, the Grötzsch graph is not planar.

So, the question is whether there is a necessary and sufficient condition for a graph to be planar. In 1930, K. Kuratowski, a Polish mathematician, proved a necessary and sufficient condition for a graph to be planar. We will state this theorem and illustrate its application through an example. To understand the statement, let us first consider Fig. 24 below.

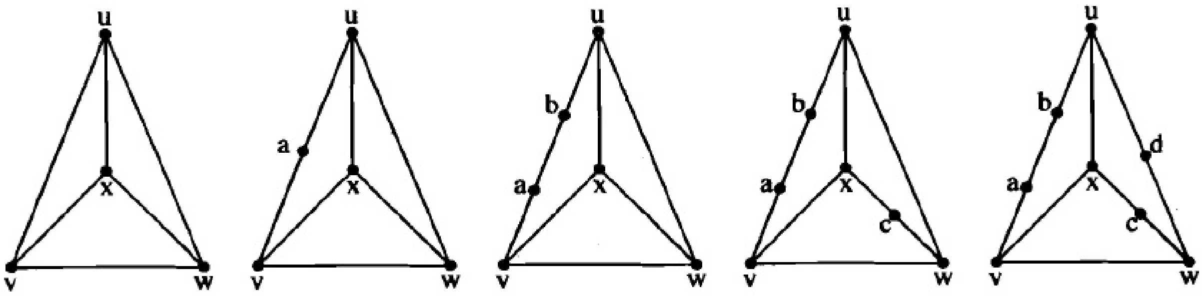


Fig. 24 : Subdivision of a graph

In this figure, we have started with K_4 and inserted vertices of degree 2 in some of the existing edges. For example, in Fig. 24(b), we have inserted a vertex a on the edge uv. In effect, this replaces the edge uv with two new edges va and au. We have made similar changes in the graphs in Fig. 24(b), Fig. 24(c), Fig. 24(d) and Fig. 24(e). In this way we have got subdivisions of the graph in Fig. 24(a), as you shall now see.

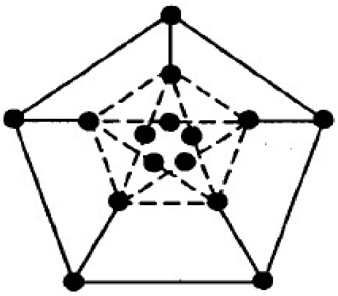


Fig.25

Definition: A graph G′ is a **subdivision** of a graph G if it can be obtained by adding one or more vertices of degree 2 on the existing edges of G. In other words, we 'subdivide' some of the existing edges.

Note : If a graph is planar, all its subdivisions are planar. If a graph G is non-planar, any subdivision of G is also non-planar. So, if a graph contains a non-planar subgraph or a subgraph which is a subdivision of a non-planar graph, it is non-planar. For example, the graph in Fig. 25 is non-planar since it contains as a subgraph a subdivision of K_5 (shown by dotted lines), which is a non-planar graph.

In proving the non-planarity of the graph in Fig. 25, is it just a coincidence that we found that it had a subdivision of K_5 as a subgraph? Here's an exercise about this.

E16) Check whether the graph given in Fig.26
i) is planar;
ii) has a subdivision of K_5.

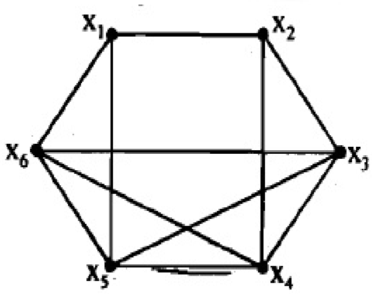


Fig.26

From E16 you see that non-planar graphs do not necessarily contain a subdivision of K_5. However, Kuratowski's theorem (stated below) says that a non-planar graph has to contain a subgraph which is a subdivision of K_5 or $K_{3,3}$. So, we need to restrict our search for non-planar subgraphs (or their subdivisions) to only these two graphs.

Theorem 4 (Kuratowski) : A graph G is non-planar if and only if it contains a subdivision of K_5 or $K_{3,3}$ as a subgraph.

We will not present the proof of this statement here. But, we shall look at an example to see how this theorem can be used to prove non-planarity.

Example 13: Show that the Grötzsch graph (see Fig. 4) is non-planar.

Solution: From Kuratowski's theorem we know that we have to look for a subgraph which is a subdivision of K_5 or K_4. But, in this case, which of these two should we look for? Note that a subdivision of a graph does not affect the degree of any of the vertices of a graph; it only introduces new vertices of degree 2.

So, if our graph contains a subdivision of K_5, it will contain at least 5 vertices of degree 4. If it contains a subdivision of $K_{3,3}$ it will have at least six vertices of degree 3.

Let us first check if our graph contains a subdivision of $K_{3,3}$. Since it contains only five vertices of degree 3, namely, y_1, y_2, y_3, y_4 and y_5, it cannot contain a subdivision of $K_{3,3}$.

So, let us check if it contains a subdivision of K_5. K_5 contains 5 vertices of degree 4. In the Grötzsch graph also there are 5 vertices of degree 4, namely x_1, x_2, x_3, x_4 and x_5. Let us remove the middle vertex, labelled as z. We get the graph given in Fig. 27(a).

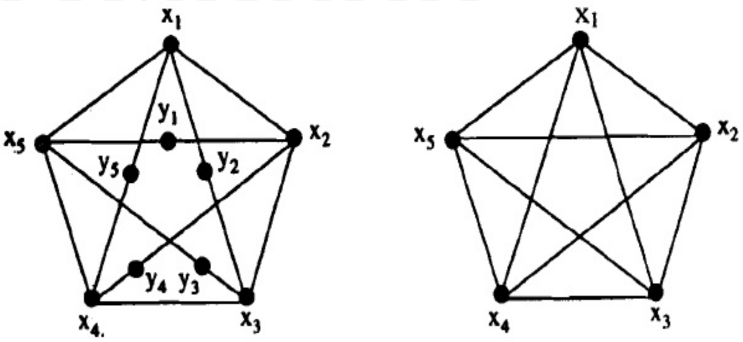


Fig. 27 : Non-planarity of the Grötzsch graph.

As you can see, it can be obtained from K_5 in Fig. 27(b) by adding degree two vertices to x_1x_3, x_1x_4, x_2x_4, x_2x_5, x_3x_5 and x_1x_5. So, it is non-planar.

***

Now, an exercise for you to try!

E17) Show that the Petersen graph (in Fig.5) is non-planar.
(**Hint** : Consider the graph obtained by removing the two horizontal edges.)

In the next section we will discuss the map colouring problem. We will show that this can be reduced to a colouring of planar maps.

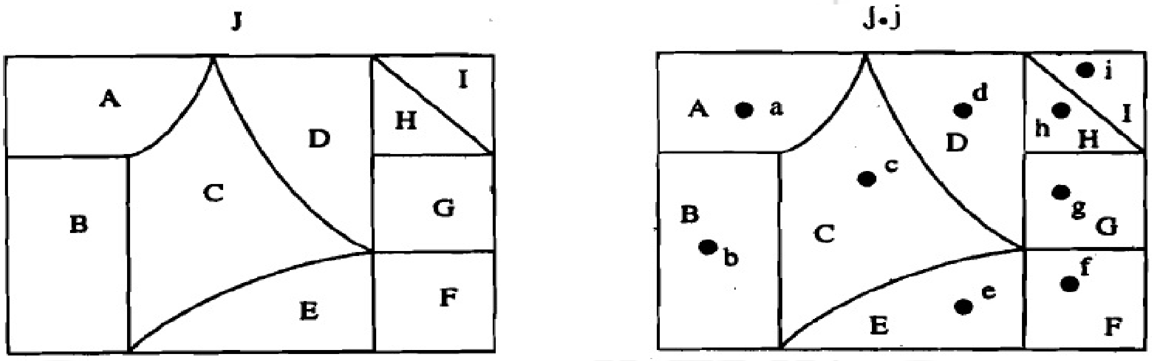
4.5 MAP COLOURING PROBLEM

The four colour problem asks whether any map of a part of the world can be coloured with 4 colours. We begin this section with a brief discussion of the history of the four colour problem. We then show how to construct a planar graph corresponding to a given map in such a way that colouring the graph is equivalent to colouring the map. So, if we can prove that any planar map can be coloured with four colours, we would have proved that any map can be coloured with four colours. In 1976, the American mathematicians, Kenneth Appel and Wolfgang Haken, proved that four colours are enough to colour planar graphs. They used nearly 1200 hours of computer time on some of the fastest computers available at that time to prove this by doing a case-by-case analysis. This gives an idea about the complexity of the proof, which we will not be giving in this course.

Now, for some background about this problem. In 1852, Francis Guthrie communicated the four colour problem to De Morgan through his brother Fredrick Guthrie, who was a student at the University College, London at that time. It appeared in print for the first time when Cayley published a paper on this problem in the Royal Geographical Society in 1879. In this paper, he outlines where the difficulties lie in this problem. In the same year, A.B. Kempe published a proof of the theorem in the American Journal of Mathematics. However, in 1890, P.J. Heawood pointed out a mistake in Kempe's proof. He also showed that the proof can be modified to show that five colours are enough to colour any map. Since then many mathematicians, G.D. Birkhoff, Veblen, Ore, Franklin among others, contributed to the solution of the problem. Appel and Haken finally solved the problem in 1976.

We now show how to construct a planar graph corresponding to a given map in such a way that colouring the vertices of the graph is equivalent to colouring the map.

Consider the map given in Fig. 28(a) below. There are 10 regions in the map, A, B, C, D, E, F, G, H, I and J, including the exterior region. In this map we add a vertex corresponding to each region of the map (see Fig.28(b)). Note that we have added a vertex corresponding to the exterior region, namely, J.



(a) (b)

Fig. 28

We join two vertices if the corresponding regions have an edge in common. For example, we have connected a and c because they have a common boundary (see Fig. 29 below). We have not connected the vertices a and e because they do not have a common boundary. We do not connect two vertices if the corresponding regions share only a point and not a boundary. For example, we have not connected c and g by an edge for this reason.

As you can see, we get a planar graph, and colouring this graph is equivalent to colouring the map. (We assume that the exterior region of the map is coloured with a single colour.) So, the four colour problem can be stated as follows:

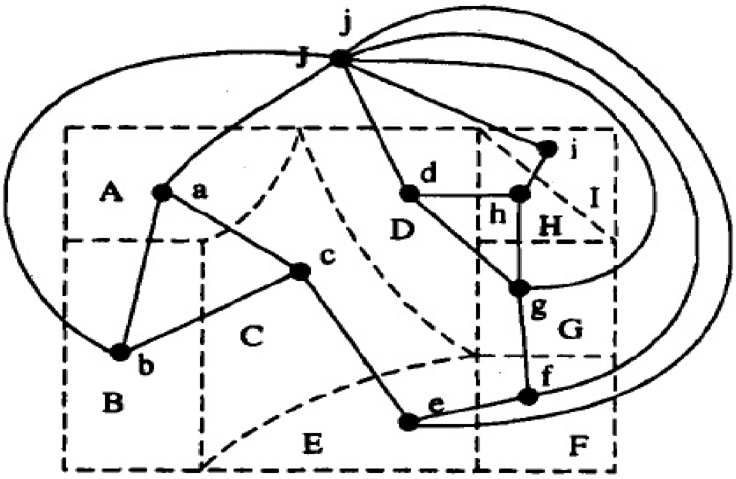


Fig.29

Is it possible to colour any planar graph with four colours?

The following theorem answers this question.

Theorem 5 (Appel-Haken) : Any planar graph can be coloured with four colours.

As we mentioned in the introduction we will not be proving this theorem.

Now, the question is whether we can get a better result. Can we colour a map with three colours always? No! For example, K_4 is planar, being the graph corresponding to a tetrahedron, and it cannot be coloured with three colours. So, we cannot improve the result in Theorem 5.

Why don't you try an exercise now ?

E18) Show that $\chi(K_5) = 5$. Does this contradict Theorem 5 ? Give reasons for your answer.

E19) Check whether there exists a non-planar graph with vertex chromatic number i) 1, ii) 2, iii) 3.

On doing E18, you will have realised that the result in Theorem 5 would not hold true for non-planar graphs.

We have now reached the end of this unit. Let us briefly summarise what we have learnt so far.

4.6 SUMMARY

In this unit we discussed the following concepts along with several examples and exercises.

i) **Vertex colouring of a graph:** A vertex colouring of a graph is an assignment of colours to its vertices in such a way that no two adjacent vertices receive the same colouring.

ii) **Vertex chromatic number of graph:** The chromatic number of a graph is the minimum number of colours required to colour the graph.

iii) **A colour class of a colouring:** For each colour of a colouring, the set of all vertices that are coloured with that colour is the colour class of that colour.

iv) **Independent set:** A subset of the vertex set is independent if any two vertices in the set are non-adjacent.

v) **Edge colouring of a graph:** An edge colouring of a graph is an assignment of colours to its edges in such a way that no two edges with a common vertex are given the same colour.

vi) **Edge chromatic number of a graph:** The edge chromatic number of a graph is the minimum number of colours needed to colour the edges of graph.

vii) **Planar graph:** A graph is planar if there is a plane drawing in which no two edges cross each other, except at vertices.

viii) **Subdivision of a graph:** A graph G_2 is a subdivision of another graph G_1 if it can be obtained from G_1 by inserting vertices of degree two in the existing edges.

In the process we also studied the following matters.

1) $\chi(K_n) = n;$ and $\chi(G) = 2$ **iff** G is a bipartite graph with $E(G) \neq \phi$.

2) If G has a subgraph isomorphic to K_n, then $\chi(G) \geq n$. But the converse is not true. However, if $\chi(G) \geq 3$, then G contains an odd cycle.

3) For a graph G, the colour classes corresponding to each of the $\chi(G)$ colours give maximal independent sets of V(G).

4) Vizing's bound for the edge chromatic number of a graph, namely, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$.

5) Euler's formula for planar graphs, which states that Number of vertices − Number of edges + Number of regions = 2, for any planar graph.

6) If G is a planar (p, q)-graph, with $p \geq 3$, then $q \leq 3p - 6$. Further, if G is also bipartite, we have $q \leq 2p - 4$.

7) Kuratowski's characterization of planar graphs, which says that a graph is planar if and only if it does not contain a subdivision of $K_{3,3}$ or K_5.

8) The four colour theorem (without proof), which says that any planar graph can be coloured with four colours.

4.7 SOLUTIONS/ANSWERS

E1) i) Trees do not contain cycles as subgraphs, and therefore, in particular, they are bipartitie. Since a tree is connected and we have assumed it has at least two vertices, it has chromatic number 2.

ii) Even cycles do not contain odd cycles as subgraphs. So, they are bipartite. Therefore, they have chromatic number 2.

iii) The chromatic number of an odd cycle is 3. Since it is not bipartite, its chromatic number is at least 3. We get a 3-colouring of C_{2n+1} as follows:Let $\{v_1, v_2, \ldots, v_{2n+1}\}$ be the vertex set of C_{2n+1}.

We assign $\boxed{1}$ to all the vertices in the set $\{v_i \in V(C_{2n+1}) \mid i \text{ odd}, 1 \le i \le 2n\}$, and $\boxed{2}$ to all the vertices in the set $\{v_i \mid i \text{ even}, 2 \le i \le 2n\}$.
Now, v_{2n+1} is adjacent to both v_1 and v_{2n}. So, we cannot assign $\boxed{1}$ or $\boxed{2}$ to this vertex. Therefore, we assign the colour $\boxed{3}$ to v_{2n+1}.

E2) A three-colouring of the Petersen graph is given below.

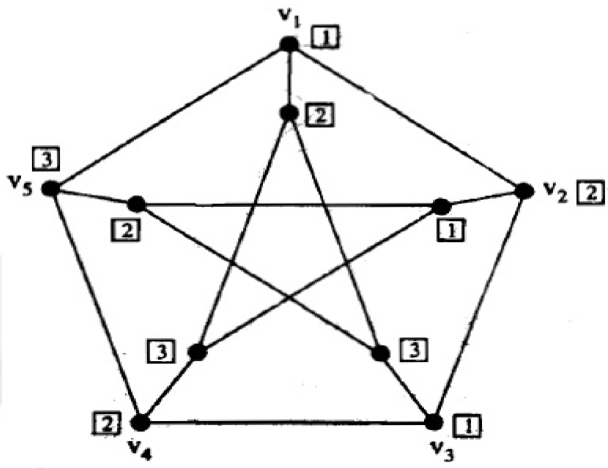


Fig. 30

Further, the Petersen graph contains a 5-cycle which has chromatic number three. So, the Petersen graph has chromatic number three.

E3) Since it has chromatic number greater than 2, it cannot be bipartite. So, it must contain an odd cycle.

E4) A 3-colouring of the graph is given in Fig.31. Also, it has cycles of length 5 as subgraphs, and we have already seen that cycles of odd length have chromatic number 3. Therefore, the chromatic number of this graph is 3.

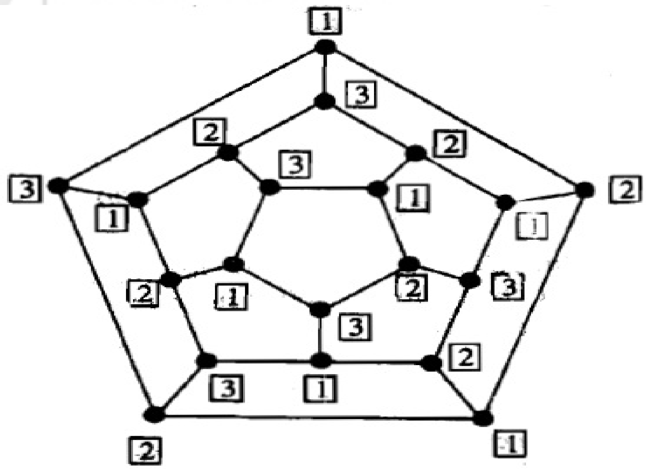


Fig. 31

E5) In the graph in Fig. 7, the graph induced by v_4, v_5, v_6, v_7 is K_4. So, it has a clique of size 4 and therefore we need at least 4 colours. We get a 4-colouring by assigning $\boxed{1}$ to v_1, $\boxed{2}$ to v_2, $\boxed{3}$ to v_3, $\boxed{2}$ to v_4 $\boxed{3}$ to v_5, $\boxed{1}$ to v_6 and $\boxed{4}$ to v_7. So, the chromatic number is 4.

E6) The figure given in Fig. 32 is 5-chromatic. It contains a clique of size 5, namely, the subgraph induced by the vertices w_1, w_2, w_3, w_4, w_5. So, we need at least 5 colours. We first give a 5-colouring to the subgraph isomorphic to

K_5 by assigning $\boxed{i}$ to w_i, $1 \le i \le 5$. Next, we assign $\boxed{2}$ to v_1, $\boxed{3}$ to v_2, $\boxed{1}$ to v_3, $\boxed{2}$ to v_4, and $\boxed{1}$ to v_5.

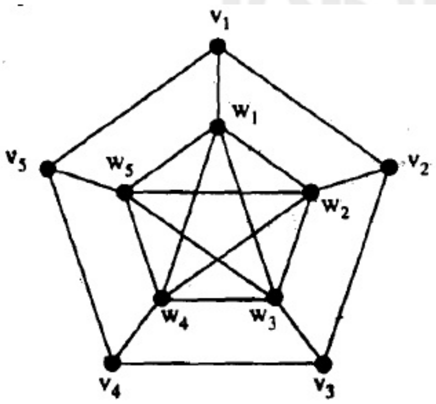
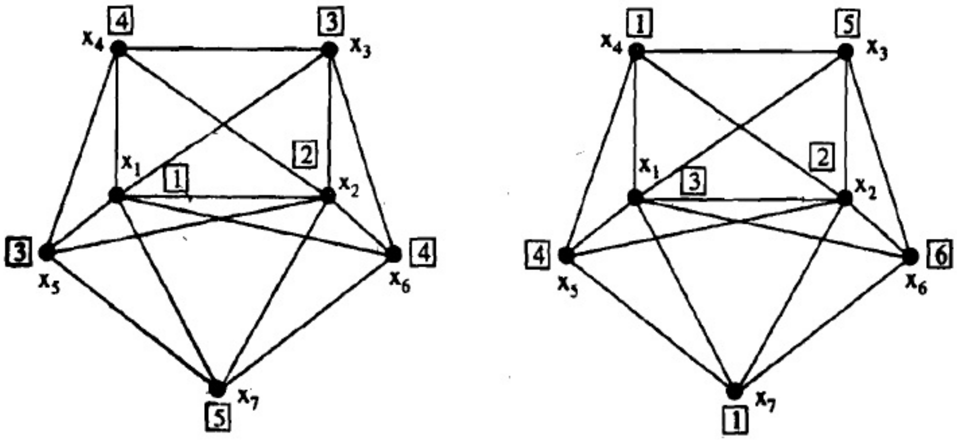


Fig. 32

E7) Firstly, every vertex lies in some colour class.
Next, $(x,x) \in R \forall x \in V(G)$.
Then, $(x,y) \in R \Leftrightarrow (y,x) \in R, x, y \in V(G)$.
Finally, you can see that $(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R, x, y, z \in V(G)$.

E8) Two different colourings are given in Fig. 33.



(a) **(b)**

Fig. 33

The colour classes for the colouring in Fig. 33(a) are $\{x_1\}$, $\{x_2\}$, $\{x_7\}$, $\{x_3, x_5\}$, $\{x_4, x_6\}$. The colour classes for the colouring in Fig. 33 (b) are $\{x_1\}$, $\{x_2\}$, $\{x_3\}$, $\{x_4, x_7\}$ and $\{x_5, x_6\}$.

E9) $\{v_1, v_2, v_4, v_6\}$

E10) For the graph in Fig.8, you can see that $\{x_{13}, x_{14}, x_3, x_5, x_7, x_9, x_{11}\}$ is an independent set and any other set has ≤ 7 elements. Thus α (G) = 7.

For the graph in Fig.10, note that for every vertex x_i, there are precisely two vertices in G not adjacent to x_i. But those two are adjacent. Hence, α (G) = 2.

E11) Since $K_{m,n}$ is a bipartite graph, by König's result,
$\chi'(K_{m,n}) = \Delta(K_{m,n}) = \min(m,n)$.

E12) The required Δ (T)- colouring is given in Fig. 34. You must remember that it is not unique.

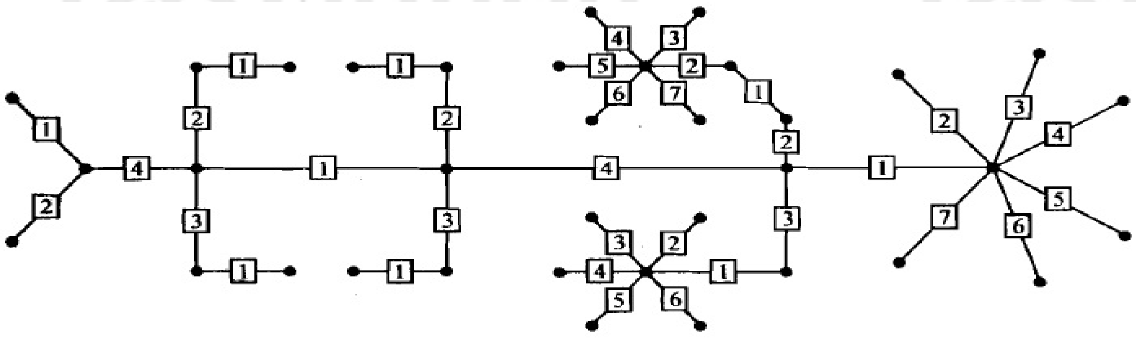


Fig.34

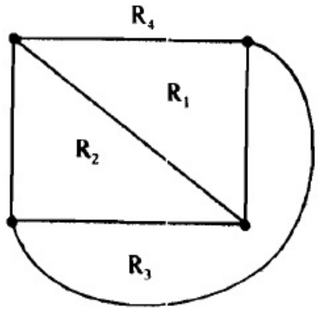


Fig.35

E13) For a tree, the number of regions is 1; for a cycle it is 2; for K_4 it is 4 (see Fig.35).

E14) Yes. This is because, if G can be drawn without its edges crossing each other, this would also hold for any subgraph of G, as its edge set is a subset of the edge set of G.

E15) Since $K_{3,3}$ is bipartite, we can apply Theorem 5. Here p = 6 and q = 9. But, $2p - 4 = 10 > 9 = q$. So, $K_{3,3}$ is not planar.
The situation given is modelled by $K_{3,3}$. Since $K_{3,3}$ is non-planar, some of its edges will cross each other. Therefore, the corresponding lines or mains of the utility will cross.

E16) If you consider the vertex sets $\{x_1, x_3, x_4\}$ and $\{x_2, x_5, x_6\}$, you will see that $K_{3,3}$ is a subgraph of this graph. Thus, the graph is non-planar. You can also check that it is not a subdivision of K_5.

E17) The graph obtained by deleting the two horizontal edges is shown in Fig. 36(a). We have redrawn Fig. 36(a) in Fig. 36(b) so that you can clearly see that it is a subdivision of $K_{3,3}$.

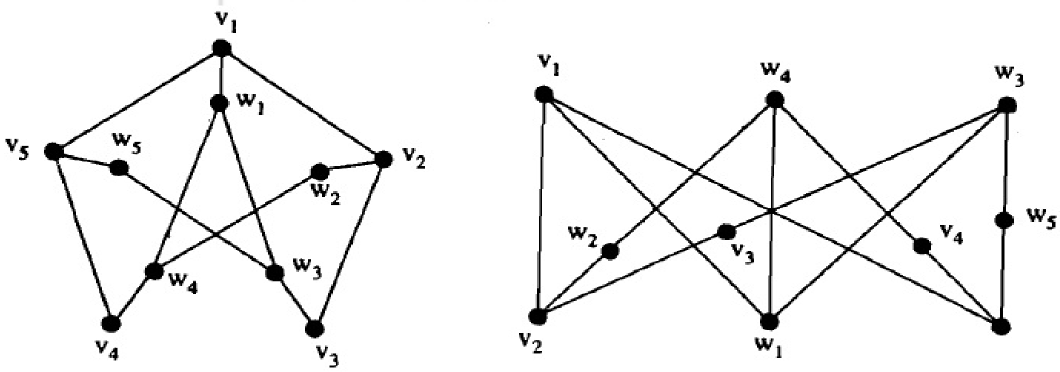


Fig. 36

E18) Since K_5 is non-planar, there is no contradiction. Use the arguments given in Sec.4.2 to show $\chi(K_5) = 5$.

E19) $\chi(G) = 1$ iff G consists of isolated vertices. In this case, there is no question of non-planarity.
$\chi(K_{3,3}) = 2$.
$\chi(G) = 3$, where G is the Petersen graph.