Exploring the Equation of State for Neutron Stars: A Review

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This review traces the historical and theoretical development of Equations of State (EOS) for neutron stars. Beginning with pioneering contributions, we revisit Fritz Zwicky's early hypotheses and the discovery of pulsars. We then delve into the theoretical intricacies of neutron star internal structure, emphasizing neutron degeneracy pressure and quantum effects within the core. The Brueckner-Bethe-Goldstone (BBG) theory takes center stage as a key tool in formulating EOS for neutron star cores. Alternative theoretical approaches are also explored. Moving from theory to practicality, we present numerical solutions for three EOS models, shedding light on the mathematical foundations governing these extreme environments. Concluding, we discuss the application of empirical data, such as mass and radius measurements, to calibrate and constrain the EOS. This integration bridges the theoretical and observational realms, enhancing our understanding of neutron stars and their EOS.

I. HISTORY

Neutron Stars did not get their present name until after their prediction by German astronomer Walter Baade and Swiss astronomer Fritz Zwicky in December of 1933. Prior to this, Lev Landau, played an important role in their initial description. Landau had written a paper called "On the theory of stars" around February 1931 which predated the discovery of neutron itself. The neutron was discovered around May of 1932 by James Chadwick.

FIG. 1. The illustration of Baade and Zwicky's observation of the supernova and their prediction of neutron star published in the *The Los Angeles Times* in January 1934 [8].

The actual paper written by Landau, published to the *Physikalische Zeitschrift der Sowjetunion* journal, was not submitted for publication until January of 1932. Many suggest that Chadwick's discovery of the neutron coincides with Landau's ideas on dense stars [8]. In his paper, Landau's first derived the upper limit to the mass of white dwarf, independent of Chandrasekhar, at which the gravitational potential is in equilibrium with the degeneracy pressure of relativistic electrons. Landau obtained an upper limit of $M \approx 1.5 M_{\odot}$. In the second part of his paper, he discussed the implications of star with mass greater than M and their fate. Moreover, he implied the existence of packed nuclei that violated the laws of quantum mechanics as understood during that period of time [8]. Two years later, Baade and Zwicky observed an enormous amount of energy released from a supernova. In the American Physical Society meeting at Stanford, they presented their predication that the large energy release was due to the transition of an ordinary star into a neutron star. The discovery of neutron star soon became popular and an illustration of the discovery was published in the *The Los Angeles Times* as shown in Figure 1.

Before the second World War, there were several theoretical progress made towards the description nuclear matter within neutron stars [6]. The initial attempts of describing nuclear matter were made in collaboration by J.Robert Oppenheimer and George Volkoff and independently by Richard Tolman around the same time. In the Oppenheimer-Volkoff model, they assumed a non-interacting degenerate fermions under hydrostatic equilibrium using Einstein's General Relativity as a framework. Their model was way too simple which resulted in a upper limit of $M = 0.7 M_{odot}$ which was less than the Chandrasekhar's limit for white dwarf. The conclusion reached by Oppenheimer and Volkoff was incorrect but it implied that the interaction between the fermions were important.

The first potential observational discovery was made during the 1960's with the development of X-ray detectors. In 1964, Bowyer et al. measured an X-ray source from the Crab Nebula. Initially, they assumed it be a neutron star, however their data indicated a radius of 10^{13} km which was far greater than the predicted radius of neutron star. Hence, their hypothesis from this observation was disregarded. However, now we know that a neutron star (pulsar) does indeed reside at the center of the Crab Nebula. The actual discovered of the neutron star goes Jocelyn Bell who was graduate student of Antony Hewish at the Cavendish Laboratory. In August 1967, Bell observed several "weak radio signals" which were due to the pulsar PSR B1919+21. Soon after, many others discovered multiple pulsating sources. Hewish received the 1974 Nobel Prize for the discovery of pulsar. Although, it was Bell who actually made the observations and the discovery of the pulsar [6]. Along with observations and further complex theoretical framework, the development of the Equations of States (EOS) of Neutrons commenced.

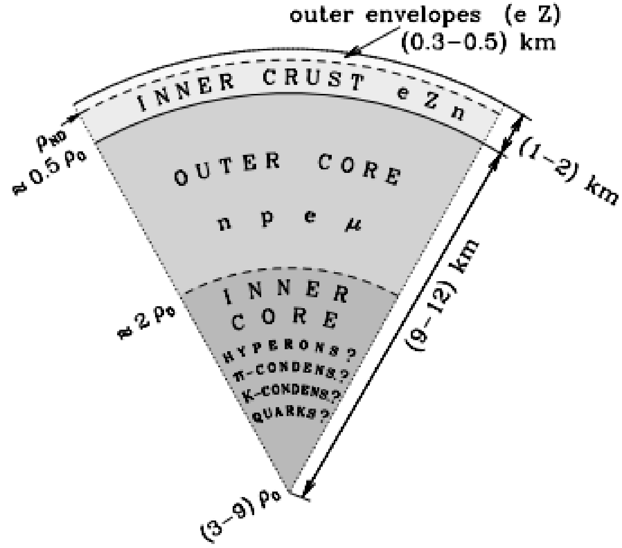

FIG. 2. The internal structure of a neutron star. ρ_o is known the nuclear matter density [4].

II. INTERNAL STRUCTURE OF NEUTRON STARS

Neutron stars are very dense stars with mass about $1.4M_{\odot}$ compressed into a radius of 10 km. The birth of a neutron star begins with the death of supergiant star with a mass around $8-25M_{\odot}$. As the star ages by burning hydrogen to helium, helium to carbon, it eventually reaching a stage with an iron core. The star is unable to generate any more fusion energy because combining iron nuclei requires energy instead of releasing energy. Thus, the star is unable to withstand the gravitation forces and the core collapses. If the mass of star's collapsed core is large enough to over come the degeneracy pressure of the electrons, a neutron star is formed. The neutron star consist of several layers like a regular star such as the atmosphere, crust and the core. The crust and the core are sub-divided into the outer and inner constituents as show in Figure 2. The density of the different layers are described in terms of the nuclear density, $\rho_o = 2.8 \times 10^{14}\ g\ cm^{-3}$. The atmosphere of a neutron is thought to be made up of thin plasma with a thickness of about ten centimeters in hot neutron star to a couple millimeter in cooler ones. The effective surface temperature ranges from $10^5 - 10^6\ K$. It is theorized that the outer envelope of neutron stars consists of free electrons with various levels of degeneracy. The radiation pressure can be mostly neglected and the EOS of envelopes are developed from the pressure exhibited by free electrons [1]. The crust of a neutron star is divided into two sections, the inner and the outer crust. The outer crust consists of non-degenerate to degenerate electron gas. As we go deeper into the outer crust non-degenerate electrons become degenerate due the stronger gravitational forces. The pressure in this region is generated by the electron degeneracy where they behave as a Fermi gas. Moreover, the Fermi energy of this system increases as the density of matter increases.

$$\begin{aligned} n &\to p + e + \overline{\nu_e} \\ n &\to p + \mu + \overline{\nu_\mu} \\ p + e &\to n + \nu_e \\ p + \mu &\to n + \nu_\mu \end{aligned} \tag{1}$$

This results in beta-capture processes occurring as we reach the inner crust. The inner crust consists of free neutron and neutron-rich atomic nuclei created by the beta-processes shown in Equation 1. At the crust-core interface, the density of nucleons is really high,$\approx 0.5\rho_o$, where they reach a degenerate state and can exist as a superfluid. The outer and inner core are of interest in this paper. The outer mainly is made up of neutrons with a small fraction of protons, electron and muons. Muons only exist if the Fermi energy

$$\epsilon_F = \frac{h^2}{2m_e}\left(\frac{3n}{8\pi}\right)^{2/3} \tag{2}$$

of the matter is greater than the rest mass energy of muons. The EOS of matter in this region is primarily determined by the equilibrium of these fundamental particles undergoing beta-decay and capture processes outlined in Equation 1. In more massive neutron stars, an inner core is formed which is made up of *exotic* matter. There are four hypotheses which describe the constituents in the inner core. Hyperons, Pion condensate, Kaon condensate, and quark matter. The Hyperons description has been highly studied experimentally and they provide some insights into the conditions of the inner core. In particular, the findings of heavy Λ hypernuclie, bound-Σ, and double-Λ hypernucleus [9]. The Pion and

Kaon condensate have not yet been detected in experiments. However, in some experiments quark decoupling has been observed [4].

III. DEVELOPMENT OF AN EQUATION OF STATE

The Equation of State of a particular matter refers to the dependence of pressure, P, on the temperature, T, and matter density, ρ, (or energy density) of the matter. Hence, we have

$$P \rightarrow P(\rho, T) \tag{3}$$

The matter within the core of neutron star are made up of degenerate fermions. Hence, the EOS is always evaluated at the $T = 0$ which also means the temperature is less than the Fermi temperature

$$T < T_F \tag{4}$$

Hence, the Equation of State only depends on the mass (or energy) density of the matter.

$$P(\epsilon) \tag{5}$$

In order to understand the interaction of the fundamental particles, we need utilise *Quantum Chromodynamics*(QCD) formulation to get acquire the energy density, ϵ.

The composition of a neutron star core is mainly baryons and exotic substance is more massive star such as Hyperons. Thus, the energy density is only a function of the baryon density n_b such that

$$\epsilon \rightarrow \epsilon(n_b) \tag{6}$$

The problem we need to solve is finding the ground state solutions for matter composed of hadrons and leptons. The charges in system are conserved, and the total charge is zero since neutrons have zero charge. The baryonic charge is A which determines the total number of baryons in the system. Once we determine Hamiltonian, we can calculate the state with the minimum energy per baryon which would only depend on the baryon density, n_b. This minimum energy state corresponds to the state at which the system is in equilibrium meaning the beta-capture and beta-decay processes are under equilibrium.

Once the equilibrium state and energy is determined as

$$\epsilon(n_b) = \epsilon_N(n_n, n_p) + \epsilon(n_e) + \epsilon(n_\mu) \tag{7}$$

we can use Lagrande multiplier to minimize our functional $\epsilon(n_b)$ with our constraints fixed baryon density and net zero charge which gives us

$$\begin{aligned} n_n + n_p - n_b &= 0 \\ n_e + n_\mu - n_p &= 0 \end{aligned} \tag{8}$$

Once we use Legrange multiplier to minimize our energy functional, we get the equilibrium relation between the chemical potentials

$$\begin{aligned} \mu_n &= \mu_p + \mu_e \\ \mu_\mu &= \mu_e \end{aligned} \tag{9}$$

This allows to then write the pressure as

$$P = n_b^2 \frac{\partial(\epsilon/n_b)}{\partial n_b} \tag{10}$$

which is the EOS. We note that P is only a function of baryon density, n_b. Each particular EOS is completely dependent on the QCD model used to determine the energy per baryon.

IV. THE CORE OF A NEUTRON STAR

The simplest formulation of nuclear matter in a core of a neutron star is determined by two body nucleon-nucleon interaction potential known as the Brueckner-Bethe-Goldstone Theory (BBG). The theory studies the nuclear matter in terms of non-relativistic two-nucleon interactions. The Hamiltonian can be written as

$$\hat{H} = \hat{H}_k + \hat{H}_{int} \tag{11}$$

where the kinetic and interaction potential terms are given, respectively, as

$$\begin{aligned} \hat{H}_k &= \sum_{i=1}^{A} \frac{\hbar^2 \nabla_i^2}{2m} \\ \hat{H}_{int} &= \frac{1}{2} \sum_{i,j=1}^{A} \hat{v}_{ij} \end{aligned} \tag{12}$$

The interaction potential between nucleon i and j is written as

$$\hat{v}_{ij} = \sum_{u=1}^{18} v_u(r_{ij}) \hat{O}_{ij}^u \tag{13}$$

where we split it into the radial component $v_u(r_{ij})$ and the angular component $\hat{O}_{ij}^u$ which has eighteen terms based on all the combinations of the isospin, τ_i, intrinsic spin components, σ_i, intrinsic spin, $\hat{S}_i$, angular momentum $\hat{L}$ and total spins $\hat{S}^2$, and $\hat{L}^2$. The first fourteen components are invariant with rotation in isospin space meaning exchange of isospin doesn't change the state. While the other four vary but are negligible because their contributions are less than the uncertainty.

In order to solve this Hamiltonian, we use perturbation theory. We can split up the Hamiltonian into unperturbed and the perturbed parts as

$$\hat{H} = \hat{H}_o + \hat{H}_1 \tag{14}$$

where $\hat{H}_o$, and $\hat{H}_1$ is written as

$$\begin{aligned}\hat{H}_o &= \hat{H}_{kin} + \hat{U} \\ \hat{H}_1 &= \hat{H}_{int} + \hat{U} \\ \hat{U} &= \sum_{j=1}^{A} \hat{U}_j\end{aligned} \tag{15}$$

The $\hat{U}_j$ term is the single particle potential acting on j^{th} nucleon. The unperturbed energy is

$$\epsilon_N = \frac{\hbar^2 p^2}{2m} + U_N \tag{16}$$

The problem now is calculating the single particle potential, $\hat{U}_j$. During the initial stages of BBG formulation, Brueckner represented the perturbation expansion in terms of *ladder diagrams* and formed the G-matrix which sums up all possible interaction between two nucleons. The G-matrix is

$$\hat{G}_{NN'} = \hat{v}_{NN'} + \hat{v}_{NN'} \frac{\hat{Q}_{NN'}}{z - \hat{h}_{NN'}} \hat{G}_{NN'} \tag{17}$$

where $\hat{Q}_{NN'}$ is known as the Pauli-Exclusion operator, and $\hat{h}_{NN'}$ is the uncorrelated two-particle states Hamiltonian operator [4]. In low-dense regime, we get

$$\begin{aligned}\hat{h}\ |p_1 p_2) &= \frac{\hbar^2}{2m}(p_1^2 + p_2^2)|p_1 p_2) \\ \hat{Q} &\to 1\end{aligned} \tag{18}$$

The lower-order solution to the Hamiltonian is given by Hartree-Fock expression which is

$$\epsilon = \epsilon_F + \sum_{ij=p}^{q} \frac{1}{2} \int_i dp_1 \int_q dp_2 (p_1 p_2 | G_{ij}(\epsilon_i + \epsilon_j) | p_1 p_2) \tag{19}$$

where ϵ_F is the energy density of Fermi gas and the second term in the sum of the interactions between proton-proton (p-p), neutron-neutron (n-n) and proton-neutron (p-n). The exact solution to this is performed through Partial-Wave expansion where we split the eiganstates into their angular components and solve using boundary conditions. Shang et al. showed that this can also be performed using an Angle-Average Approximation and the Total Momentum Approximation.

In the Angle-Average Approximation (AA), the

$$\hat{F} = \frac{\hat{Q}_{NN'}}{z - \hat{h}_{NN'}}$$

term is replaced with the average value of this quantity where

$$\overline{\hat{F}} = \int \hat{F} \frac{d\Omega}{4\pi} \tag{20}$$

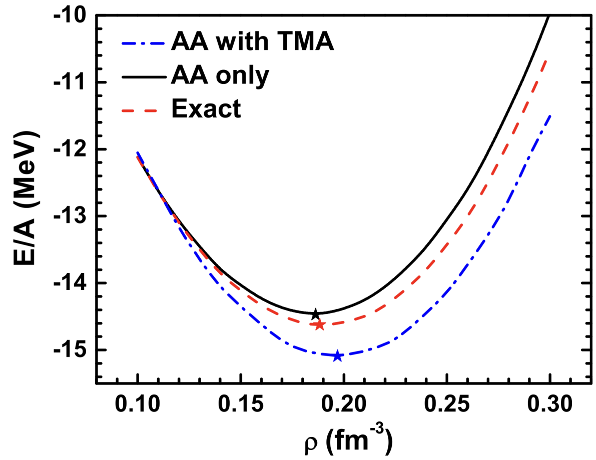

FIG. 3. The exact solutions for energy density per baryon using Partial-Wave expansion, and approximation solutions using Angle-Average Approximation and Total Momentum Approximation plotted against baryon density from Shang et al [5].

Similarly, the Total-Momentum Approximation (TMA) involves taking the average value of the total momentum over all momentum states [5]. Figure 3 shows the solutions to the BBG equations using the Partial-Wave expansion, Angle-Average and Total-Momentum Approximation method. This shows that the solutions to BBG equations can be approximated by AA and TMA which gives us relatively similar solution to the minimum energy. The three points indicated are the minimum energy solutions that represents the equilibrium state between beta-processes [5].

However, this approximation fails to reproduce experimental parameters of nuclear matter at saturation density [4]. This is can solved by considering higher order interactions in the which the interaction potential is given by a three-body nucleon potential

$$\hat{V}^{(3)} = \frac{1}{6} \sum_{i,j,k}^{A}{}'' \hat{V}_{ijk} \tag{21}$$

A simplification that we can do to solve this potential is consider the three-body potential as an effective two body potential by averaging over the positions of the third nucleon. We also cannot make the approximation shown in Equation 18 and we need three-body interactions G-matrix. Another issue with the BBG theory is that it is non-relativistic. As the density increases towards the center of the star's core, the matter become highly relativistic. Hence, we need to make an relativistic extension by solving the Dirac equations. The relativistic solutions are known as Dirac-Brueckner-Hartree-Fock Theory.

The approach outlined in this paper is one of many different approaches to getting the energy density per baryon. Some other methods includes Green's Function Theory, Variational Method, and Relativistic Mean-Field Model. Each differ in the method they solve the Hamiltonian or by a different interaction potential.

V. HYDROSTATIC EQUILIBRIUM EQUATIONS

Once we solve the QCD problem and obtain the energy density per nucleon, $\epsilon(n_b)$, we can get the EOS, $P(\epsilon)$. In order to study the structure and evolution, we need Hydrostatic Equilibrium equations from Einstein's General Theory of Relativity. The three relativistic equations of hydrostatic equilibrium for a static spherically symmetric neutron stars are derived as

$$\frac{dP}{dr} = -\frac{G\rho m}{r^2}\left(1+\frac{P}{\epsilon}\right)\left(1+\frac{4\pi r^3}{mc^2}\right)\left(1-\frac{2Gm}{c^2 r}\right)^{-1} \tag{22}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho \tag{23}$$

$$\frac{d\Phi}{dr} = -\frac{1}{\rho c^2}\frac{dP}{dr}\left(1+\frac{P}{\rho c^2}\right)^{-1} \tag{24}$$

Along with the EOS and these three Hydrostatic Equilibrium equations, we can derive observable parameters such as mass, and radius which allows us to constraint our EOS. Thus, allowing us to breakdown the large set of EOS into a few that precisely describe neutron stars.

VI. NUMERICAL SOLUTIONS TO EOS

We can numerically solve for the structure of the neutron star once the EOS are known. The three nuclear EOS used in the numerical solutions are AV14+UII, UV14+UVII, UV14+TN1 which were derived by Wiringa et al. They considered the two nucleon interaction along with an explicit three nucleon interaction potential discussed above and used variational method to calculate the energy as a function of baryon density [7]. The fits functions used for the numerical calculations was given by Kutschera et al. where they discussed the possibility of quark core for a neutron star [3]. We use a 4^{th} order Runga-Kutta to perform the numerical integration.

Figure 4 and Figure 5 show the solutions to the Hydrostatic Equilibrium equations for the three EOS. We can also study the mass of a neutron as a function of radius of a neutron predicted by the EOS shown in Figure 6.

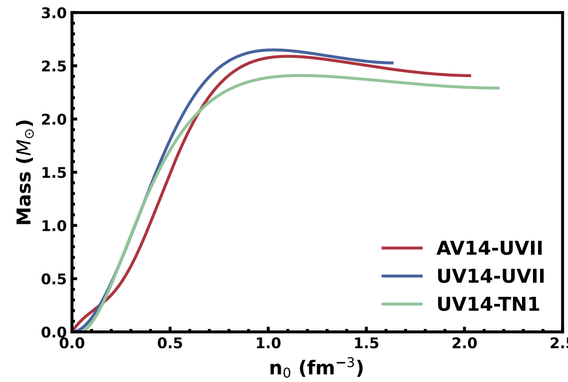

FIG. 4. The mass as a function of central density for three EOS solutions.

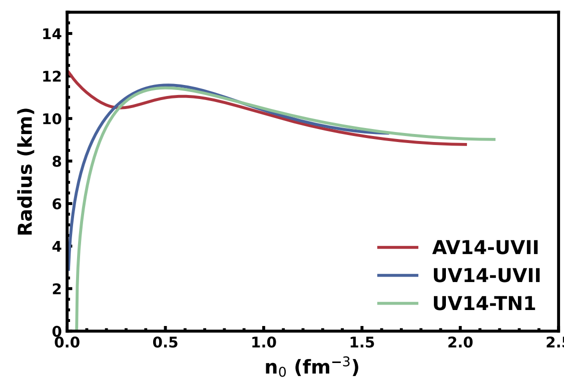

FIG. 5. The radius as a function of central density for three EOS solutions.

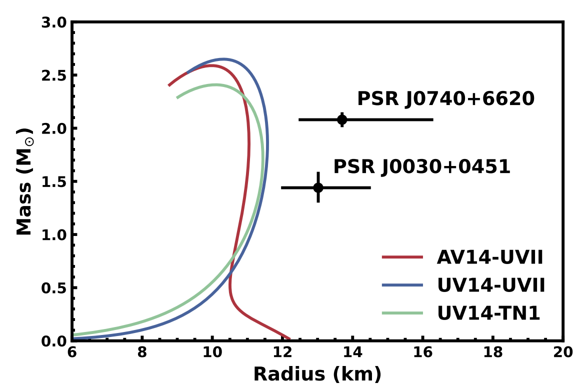

FIG. 6. The mass as a function of radius of a neutron star for three EOS solutions. Two observation data of the mass vs radius is plotted with the error bars.

We plot two observations, PSR J0740+6620 and PSR J0030+0451, of the mass and radius of a neutron star to determine the viability of the EOS. We see that neither of the data points lie close to the analytical result. The three EOS used here were based on an approximate variational method with Gaussian proton profile. Thus, the inconsistency suggests that further details about the nucleon interactions must be consider.

VII. CONCLUSION

Landau initially predicted the existence of high-dense nuclear core stars. Later after the prediction made by Baade and Zwicky, the journey to study the EOS of neutrons began. Solving different nuclear Hamiltonian gives the energy density per nucleon for matter within cores of neutron star. The various nuclear Hamiltonian differ based on the interaction potentials between the constituents of the core. These may include degenerate neutrons under beta-decays equilibrium or more exotic particles such as hyperons, quarks etc. Once we solve the QCD problem, we can studying the structure and evolution of the neutron star using the Relativistic Hydrostatic Equilibrium Equations. We can study the mass and radius of neutron which allow us to constraint the parameters of the EOS as shown in Figure 6. These constraints allow us to understand the accuracy of our descriptions of nuclear matter. The model discussed in the review are initial attempts at describing Neutron stars. Now, more complex more have been proposed which consider complex interactions between nucleons. However, one can still use the approach discussed in the review to derive any unique EOS and perform numerical simulations to test its validity. The variety in EOS of neutron stars arise from the chosen interaction potential.

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