

# Chapter 2

## Kinematics

### 2.1 Basic Concepts

Kinematics describes the *motion* of mechanical systems, without considering the forces that produce that motion. Kinematics deals with velocities and accelerations, which are defined for points of interest on the mechanical systems. The description of motion is relative in nature. Velocities and accelerations are therefore defined with respect to a reference frame.

### 2.2 Kinematics of a particle. Rectilinear and curvilinear motion

The particle is classically represented as a point placed somewhere in space. A *rectilinear motion* is a straight-line motion. A *curvilinear motion* is a motion along a curved path.

#### 2.2.1 Position vector. Velocity vector. Acceleration vector

The *position vector*  $\mathbf{r}(t)$  (see Fig. 2.1) of the particle P at a given instant of time  $t$  refers to its location relative to some reference point usually taken as the origin of a coordinate system. Note that every vector considered in section 2.2 may be projected onto the coordinate frame  $oxyz$ . As the particle moves along its straight-line path, its position changes with time. By definition the *displacement*  $\Delta\mathbf{r}$  of the particle during a time interval  $\Delta t$  is given by the change of its position during this time interval.

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t) \quad (2.1)$$

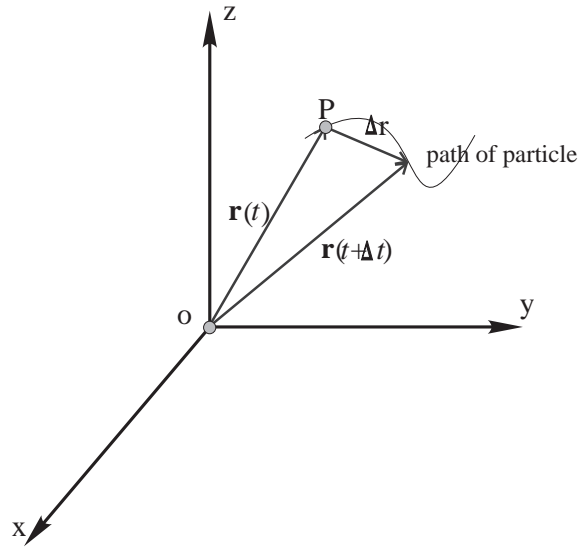


Figure 2.1: Position vector

### 2.2.2 Average and instantaneous velocities

The *average velocity* during the time interval  $\Delta t$  is defined as

$$\mathbf{v}_{av} = \frac{\Delta \mathbf{r}}{\Delta t} \quad (2.2)$$

The *instantaneous velocity* is given by

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt} = \dot{\mathbf{r}} \quad (2.3)$$

### 2.2.3 Average and instantaneous acceleration

We need to learn how the velocity varies with time, we define *average acceleration* by

$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t} \quad (2.4)$$

and the *instantaneous acceleration* is defined by letting the time interval  $\Delta t$  approach zero in the limit:

$$\mathbf{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{v}}{\Delta t} = \frac{d\mathbf{v}}{dt} = \dot{\mathbf{v}} \quad (2.5)$$

### 2.2.4 Absolute frame

Let us express the position vector  $\mathbf{r}_P$  to point P on the path of the particle in terms of  $x, y, z$  components (see Fig. 2.2)

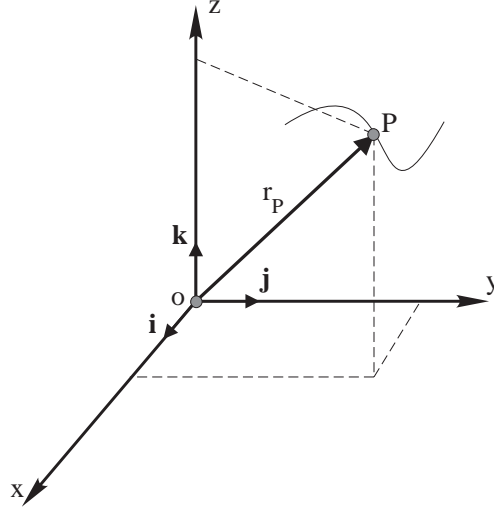


Figure 2.2: Absolute frame

$$\mathbf{r}_P(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad (2.6)$$

$$\mathbf{v}_P = \frac{d\mathbf{r}_P}{dt} = \dot{x}(t)\mathbf{i} + \dot{y}(t)\mathbf{j} + \dot{z}(t)\mathbf{k} \quad (2.7)$$

$$\mathbf{a}_P = \frac{d\mathbf{v}_P}{dt} = \ddot{x}(t)\mathbf{i} + \ddot{y}(t)\mathbf{j} + \ddot{z}(t)\mathbf{k} \quad (2.8)$$

thus we have the magnitudes  $v_P = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$  and  $a_P = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$ . The velocity and acceleration vectors are obtained by successive time differentiation of position vector. Let us state now some notations which will be used.  $\mathbf{r}_P$  and  $\mathbf{r}_Q$  being the position vectors of two points P and Q, we have;

$$\mathbf{r}_{P/Q} = \mathbf{r}_P - \mathbf{r}_Q \quad (2.9)$$

$$\mathbf{v}_{P/Q} = \mathbf{v}_P - \mathbf{v}_Q \quad (2.10)$$

### 2.2.5 Tangential and normal coordinates

In many plane problems dealing with the motion of a particle along a curve, it is convenient to express the acceleration in term of two components; one along the

tangent to the trajectory, and the second along the inward normal to the path. For this purpose we define two unit vectors  $\mathbf{n}$  and  $\mathbf{t}$  respectively along the inward normal and along the tangent to the path (see Fig. 2.3).

Consider a particle moving along a curved path in a plane shown in Fig. 2.3

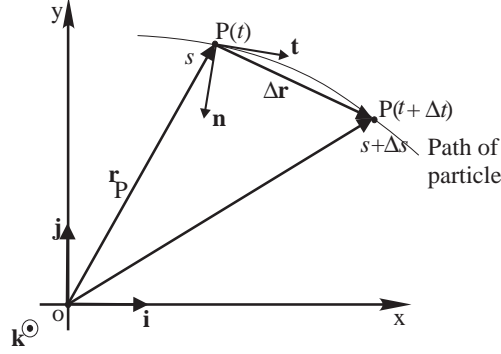


Figure 2.3: Plane motion of a particle

As we have seen above;

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta t} = \frac{d\mathbf{r}}{dt}$$

Thus we can write;

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \left( \frac{\Delta \mathbf{r}}{\Delta s} \right) \left( \frac{\Delta s}{\Delta t} \right)$$

and

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

As  $\Delta t$  goes to zero, the direction of  $\Delta \mathbf{r}$  approaches the tangent to the trajectory at position  $\mathbf{r}_p(t)$  and approaches  $\Delta s$  in magnitude. Consequently, in the limit,  $\Delta \mathbf{r} / \Delta s$  becomes the unit vector  $\mathbf{t}$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta \mathbf{r}}{\Delta s} = \mathbf{t}$$

thus

$$\mathbf{v} = \lim_{\Delta t \rightarrow 0} \left( \frac{\Delta \mathbf{r}}{\Delta s} \right) \left( \frac{\Delta s}{\Delta t} \right) = \frac{ds}{dt} \mathbf{t}$$

Note that  $ds/dt$  is the magnitude of the velocity,

$$v = \frac{ds}{dt}$$

Let calculate now the two components of the acceleration

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt} \left( \frac{ds}{dt} \mathbf{t} \right)$$

$$\mathbf{a} = \frac{d^2s}{dt^2} \mathbf{t} + \frac{ds}{dt} \frac{d\mathbf{t}}{dt} = \frac{d^2s}{dt^2} \mathbf{t} + \frac{ds}{dt} \frac{d\mathbf{t}}{ds} \frac{ds}{dt}$$

Let evaluate the derivation of  $\mathbf{t}$  with respect to  $s$ .

Consider now the unit vector  $\mathbf{t}$  at two positions  $s$  and  $s + \Delta s$  (see Fig. 2.4 (a))

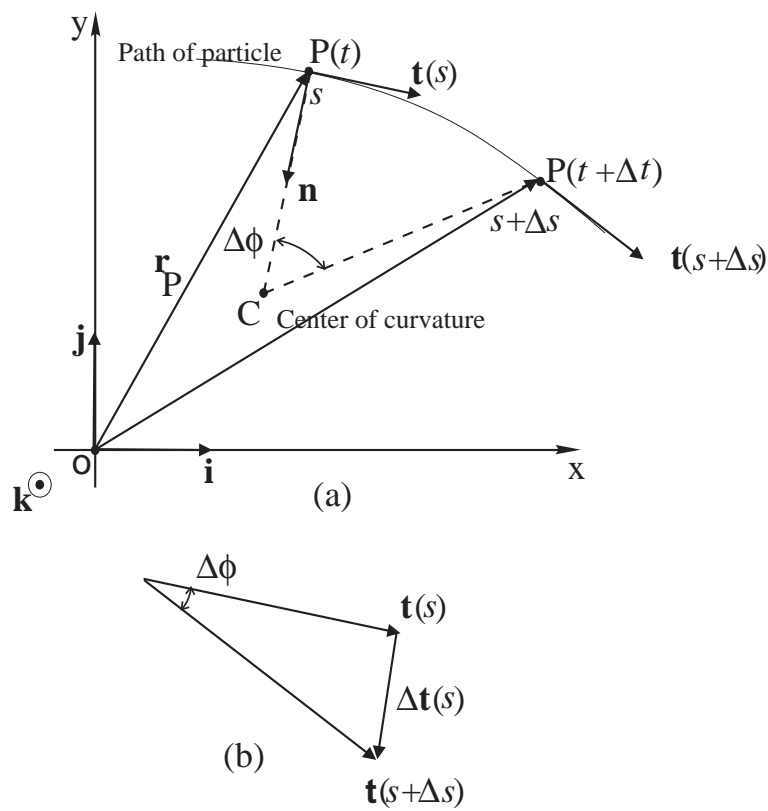


Figure 2.4: Plane motion of a particle

$$\frac{d\mathbf{t}}{ds} = \lim_{\Delta s \rightarrow 0} \left( \frac{\mathbf{t}(s + \Delta s) - \mathbf{t}(s)}{\Delta s} \right) = \lim_{\Delta s \rightarrow 0} \frac{\Delta \mathbf{t}}{\Delta s}$$

In the limit as  $\Delta s$  goes to zero, the vector  $\Delta \mathbf{t}$  ends up in the plane normal to the path at  $s$  and directed toward the center of curvature, it is the direction of the unit vector  $\mathbf{n}$  (see Fig. 2.4 (b)).

Knowing the limiting direction of  $\Delta \mathbf{t}$ , we next evaluate its limiting magnitude. According to Fig. 2.4 (b) we can say that, when  $\Delta s \rightarrow 0$ :

$$|\Delta \mathbf{t}| \rightarrow |\mathbf{t}| \Delta \phi = \Delta \phi \rightarrow \frac{\Delta s}{R}$$

thus the magnitude and the direction are established in an approximate manner.

$$\Delta \mathbf{t} \rightarrow \frac{\Delta s}{R} \mathbf{n}$$

and so

$$\frac{d\mathbf{t}}{ds} = \lim_{\Delta s \rightarrow 0} \left( \frac{\Delta \mathbf{t}}{\Delta s} \right) = \lim_{\Delta s \rightarrow 0} \left[ \frac{(\Delta s/R) \mathbf{n}}{\Delta s} \right] = \frac{\mathbf{n}}{R}$$

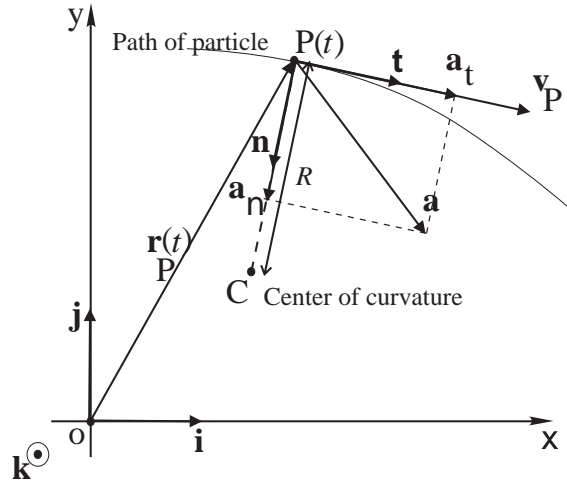


Figure 2.5: The acceleration components

then the acceleration can be evaluated by:

$$\mathbf{a} = \frac{d^2 s}{dt^2} \mathbf{t} + \frac{(ds/dt)^2}{R} \mathbf{n} \quad (2.11)$$

or

$$\mathbf{a} = \mathbf{a}_t + \mathbf{a}_n = a_t \mathbf{t} + a_n \mathbf{n} \quad (2.12)$$

where

$$a_t = \frac{d^2 s}{dt^2} = \frac{dv}{dt} \quad (2.13)$$

$$a_n = \frac{(ds/dt)^2}{R} = \frac{v^2}{R} \quad (2.14)$$

For a plane curve  $y = y(x)$ , the radius  $R$  of curvature is given by;

$$R = \frac{\left(1 + \left(\frac{dy}{dx}\right)^2\right)^{\frac{3}{2}}}{\left|\frac{d^2y}{dx^2}\right|} \quad (2.15)$$

### 2.2.6 Rotation around a fixed point in a plane

The center  $O$  of the fixed frame (see Fig. 2.6) is the center of rotation; the instantaneous position and velocity of the point  $P$  are given by;

$$\mathbf{r}_P = \mathbf{OP} = R \cos \theta \mathbf{i} + R \sin \theta \mathbf{j}$$

$$\mathbf{v}_P = R \left( \frac{d\theta}{dt} \right) [-\sin \theta \mathbf{i} + \cos \theta \mathbf{j}] \quad (2.16)$$

$$= \frac{d\theta}{dt} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} R \cos \theta \\ R \sin \theta \\ 0 \end{bmatrix} \quad (2.17)$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \frac{d\theta}{dt} \\ R \cos \theta & R \sin \theta & 0 \end{vmatrix} \quad (2.18)$$

$$= \frac{d\theta}{dt} \mathbf{k} \times \mathbf{r}_P \quad (2.19)$$

$$= \boldsymbol{\omega} \times \mathbf{r}_P \quad (2.20)$$

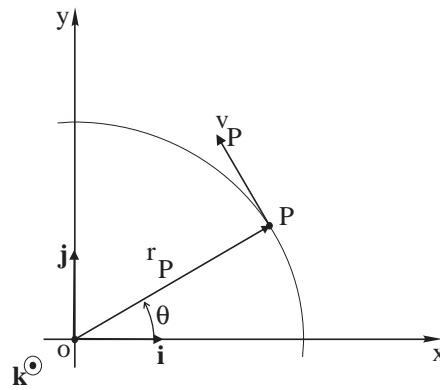


Figure 2.6: Rotation of a particle around a fixed point

## 2.3 Kinematics of a rigid body

The description of motion is *relative*. Any velocity or acceleration is expressed with regard to a specific reference frame. This fact induces specific notations that must be understood:  $\mathbf{v}_{P/S/s}$  denotes the instantaneous velocity of the point P attached to the body S, relatively to the body s.

A *rigid body* is considered to be composed of continuous of distribution of particles having fixed distances between each others. There are various types of rigid-body motion but the most important of them are *translations* and *rotations*.

### 2.3.1 Translation

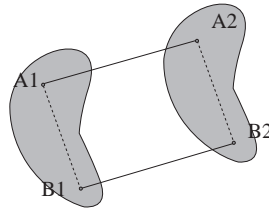


Figure 2.7: Rectilinear translation

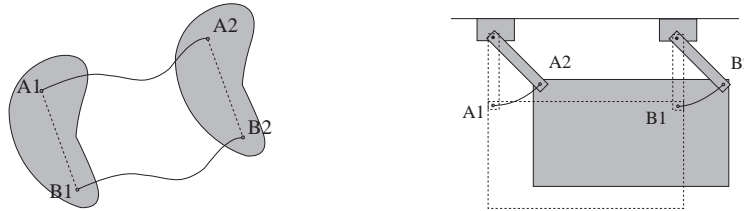


Figure 2.8: Curvilinear translation

A motion is said to be a *translation* if any straight line defined inside the body keeps the same direction during the motion. In translation all particles move along parallel paths. We have rectilinear translation when the paths are straight lines as in Fig. 2.7 in other cases it is a curvilinear translation as in Fig. 2.8. Referring to Eq. 2.9, we have;

$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \quad (2.21)$$

where  $\mathbf{r}_{B/A} = \mathbf{AB}$ .

Let us differentiate the relation with respect to  $t$ . Since the vector  $\mathbf{r}_{B/A} = \mathbf{AB}$  has a



constant direction and a constant magnitude, its time derivative is zero:

$$\begin{aligned}\mathbf{v}_B &= \mathbf{v}_A \\ \mathbf{a}_B &= \mathbf{a}_A\end{aligned}$$

In a translation all particles of the rigid body have same velocity and same acceleration.

### 2.3.2 Rotation about a fixed axis

If a part of a rigid body, or a hypothetical extension of the body, has zero velocity to some reference, the body is said to be in *rotation*. The *axis of rotation* is the line of stationary particles. Since the velocity of P is a vector perpendicular to the plane

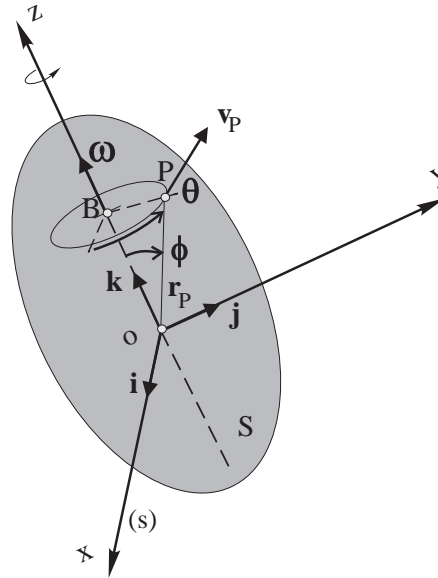


Figure 2.9: Rotation about fixed axis

(Fig. 2.9) containing the rotation axis and  $\mathbf{r}_P$ . We can write referring, to Eq. 2.19:

$$\mathbf{v}_{P_{S/s}} = \frac{d\mathbf{r}_P}{dt} \quad (2.22)$$

$$= \boldsymbol{\omega}_{S/s} \times \mathbf{r}_P \quad (2.23)$$

or in a condensed form

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \boldsymbol{\omega} \times \mathbf{r} \quad (2.24)$$

Note that the *vector product* can be computed as the determinant:

$$\mathbf{v} = \begin{vmatrix} v_x \\ v_y \\ v_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \omega_x & \omega_y & \omega_z \\ x & y & z \end{vmatrix} \quad (2.25)$$

And then

$$\begin{aligned} v_x \mathbf{i} &= \mathbf{i} \begin{vmatrix} \omega_y & \omega_z \\ y & z \end{vmatrix} = \mathbf{i} (\omega_y z - y \omega_z) \\ v_y \mathbf{j} &= -\mathbf{j} \begin{vmatrix} \omega_x & \omega_z \\ x & z \end{vmatrix} = -\mathbf{j} (\omega_x z - x \omega_z) \\ v_z \mathbf{k} &= \mathbf{k} \begin{vmatrix} \omega_x & \omega_y \\ x & y \end{vmatrix} = \mathbf{k} (\omega_x y - x \omega_y) \end{aligned}$$

Since

$$\boldsymbol{\omega} = \dot{\theta} \mathbf{k} \quad (2.26)$$

We have  $\omega_x = 0$ ,  $\omega_y = 0$ ,  $\omega_z = \dot{\theta}$  and the velocity is completely determined. The acceleration  $\mathbf{a}$  of P is now determined as

$$\mathbf{a}_{P/S/s} = \frac{d\mathbf{v}_{P/S/s}}{dt} \quad (2.27)$$

$$= \frac{d}{dt} (\boldsymbol{\omega}_{S/s} \times \mathbf{r}_P) \quad (2.28)$$

$$= \frac{d\boldsymbol{\omega}_{S/s}}{dt} \times \mathbf{r}_P + \boldsymbol{\omega}_{S/s} \times \frac{d\mathbf{r}_P}{dt} \quad (2.29)$$

$$= \ddot{\theta} \mathbf{k} \times \mathbf{r}_P + \boldsymbol{\omega}_{S/s} \times (\boldsymbol{\omega}_{S/s} \times \mathbf{r}_P) \quad (2.30)$$

### 2.3.3 Particular case: Motion in plane

A *Plane Motion* is a motion in which all particles of the body move in parallel planes.

#### Velocity in plane motion

Given two particles A and B of a rigid body in plane motion the velocity  $\mathbf{v}_B$  of B is obtained from the velocity formula (referring to Eq. 2.10)

$$\mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \quad (2.31)$$

In relative motion about A, A is fixed ( $\mathbf{v}_{A/A} = \mathbf{0}$ ). Thus  $\mathbf{v}_{B/A}$  can be associated with the rotation of the body about A and is measured with respect to axes centered at A

$$\mathbf{v}_{B/A} = \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (2.32)$$

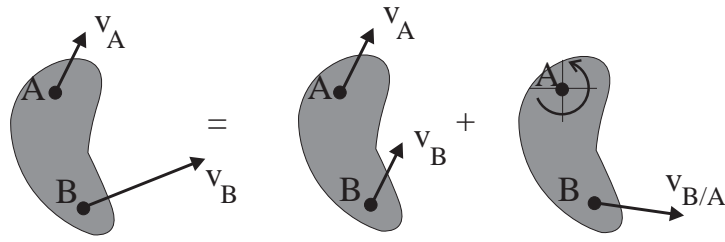


Figure 2.10: Plane motion

and

$$v_{B/A} = AB \, \omega \quad (2.33)$$

$\omega = \omega \mathbf{k}$  is the *angular velocity* of the body, we note that  $\mathbf{r}_{B/A} = \mathbf{AB}$

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{AB} \quad (2.34)$$

### Acceleration in plane motion

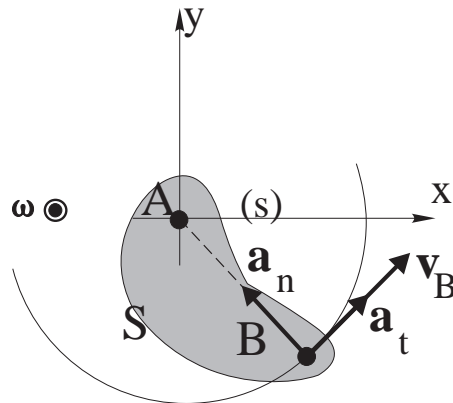


Figure 2.11: Plane motion

$$\mathbf{a}_B = \frac{d\mathbf{v}_B}{dt} \quad (2.35)$$

$$= \frac{d\mathbf{v}_A}{dt} + \frac{d(\boldsymbol{\omega} \times \mathbf{r}_{B/A})}{dt} \quad (2.36)$$

$$= \mathbf{a}_A + \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times \frac{d\mathbf{r}_{B/A}}{dt} \quad (2.37)$$

$$= \mathbf{a}_A + \dot{\boldsymbol{\omega}} \times \mathbf{r}_{B/A} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \quad (2.38)$$

If A is the center of a fixed frame  $s$  (see Fig. 2.11) we have  $\mathbf{a}_A = \mathbf{0}$  and then;

$$\mathbf{a}_B = \dot{\omega} \mathbf{k} \times \mathbf{r}_{B/A} - \omega^2 \mathbf{r}_{B/A} \quad (2.39)$$

$$= \dot{\omega} \mathbf{k} \times \mathbf{AB} - \omega^2 \mathbf{AB} \quad (2.40)$$

In the right hand of Eq. 2.40, the first term is perpendicular to  $\mathbf{AB}$  and the second is parallel.

### Equiprojectivity

For two points A and B of a given rigid body we can write

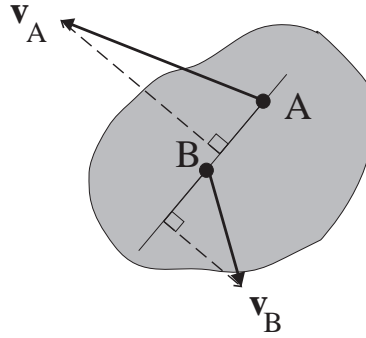


Figure 2.12: Equiprojectivity

$$\mathbf{v}_B = \mathbf{v}_A + \boldsymbol{\omega} \times \mathbf{r}_{B/A} \quad (2.41)$$

$$\mathbf{v}_B \cdot \mathbf{r}_{B/A} = \mathbf{v}_A \cdot \mathbf{r}_{B/A} + (\boldsymbol{\omega} \times \mathbf{r}_{B/A}) \cdot \mathbf{r}_{B/A} \quad (2.42)$$

$$\mathbf{v}_B \cdot \mathbf{r}_{B/A} = \mathbf{v}_A \cdot \mathbf{r}_{B/A} \quad (2.43)$$

$$\mathbf{v}_B \cdot \mathbf{AB} = \mathbf{v}_A \cdot \mathbf{AB} \quad (2.44)$$

### Instantaneous center of rotation

Considering a general plane motion of a body, at given instant, the velocities of various particles of the body could be expressed as the result of a rotation whose axis is perpendicular to the plane. This axis intersects the plane at a point called the *instantaneous center of rotation*. The position of this particular point can be defined in many ways. If the directions of the velocities of two particles A and B are known and if they are different, (Fig. 2.13, at left) the instantaneous center of rotation is obtained by drawing the perpendicular to  $\mathbf{v}_A$  through A and the perpendicular to  $\mathbf{v}_B$

through B and finding the point in which these two lines intersect.

If the velocities  $v_A$  and  $v_B$  are perpendicular to the line AB and if their magnitude are known, the instantaneous center of rotation can be found by intersecting AB with the line joining the extremities of the vector (Fig. 2.13, at right).

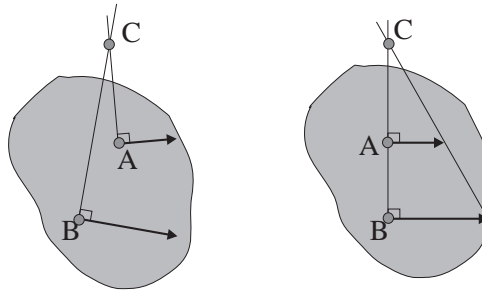


Figure 2.13: Instantaneous center of rotation

### Kennedy's theorem

The Kennedy's theorem states that the three instant centers shared by three rigid bodies in relative planar motion to another (whether or not connected) all lie on the same straight line.

#### *Application of Kennedy's theorem*

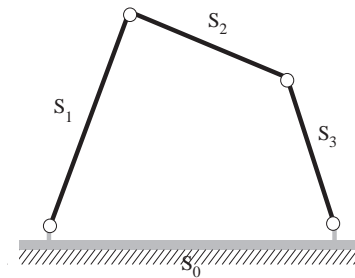


Figure 2.14: Four-bar linkage

The figure 2.14 shows four-bar linkage let us locate all instant centers. When the number of bodies is large, it is helpful to use some kind method to find the instant centers. Note that  $S_0$  represents the stationary frame.

1. By inspection ,determine as many centers as possible, in the exemple the instant centers  $I_{01}$ ,  $I_{12}$ ,  $I_{23}$ ,  $I_{03}$  are easily placed.

2. Using Kennedy's theorem with links  $S_0, S_1, S_2$  the instant center  $I_{02}$  must lie on the same straight line with  $I_{01}, I_{12}$  but it must also lie on the line through  $I_{23}$  and  $I_{03}$ . The location is defined by the intersection of the two lines.
3. The same reason can be used to locate the center  $I_{13}$ .

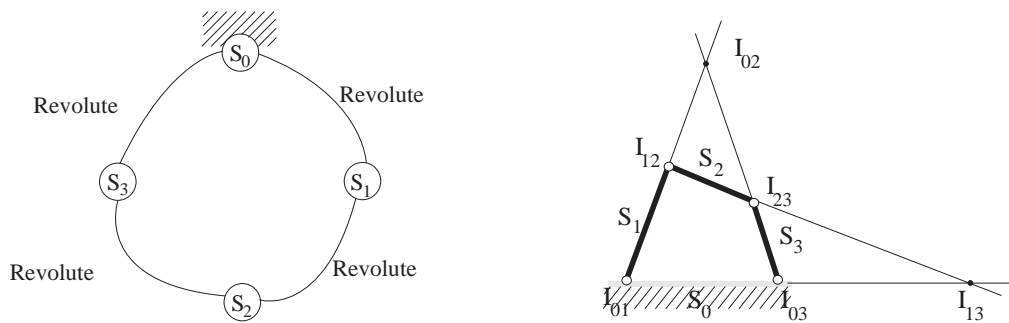


Figure 2.15: Locating instant center

### 2.3.4 General motion in space

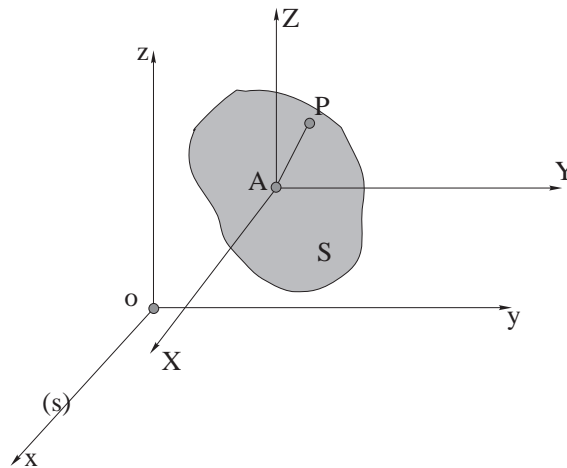


Figure 2.16: General motion in space without rotating axis

The most general motion of a rigid body in space is equivalent at any given instant to the combination of a translation and a rotation (as we have seen for plane

motion). Considering two particles A and P of the rigid body S, we have:

$$\mathbf{v}_P = \mathbf{v}_A + \mathbf{v}_{P/A} \quad (2.45)$$

Where  $\mathbf{v}_{P/A}$  is the velocity of P relative to a frame attached to A, thus  $\mathbf{v}_{P/A} = \boldsymbol{\omega}_{S/s} \times \mathbf{r}_{P/A}$  or  $\mathbf{v}_{P/A} = \boldsymbol{\omega}_{S/s} \times \mathbf{AP}$  where  $\boldsymbol{\omega}$  is the angular velocity of the body S relative to the fixed frame s. The absolute velocity of a particle P belong to S is given by from above:

$$\mathbf{v}_{P_{S/s}} = \mathbf{v}_{A_{S/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{AP} \quad (2.46)$$

The equation 2.46 allows the determination of the velocity of any point P of a body S with respect to another frame s,  $\mathbf{v}_{P_{S/s}}$ , if the following variables are known:

- $\mathbf{v}_{A_{S/s}}$ : velocity of a point A of the body S with respect to s.
- $\boldsymbol{\omega}_{S/s}$ : angular velocity of S with respect to s.
- AP: position of the particle P with respect to A.

The acceleration of P is obtained by differentiating the equation with respect to time.

$$\mathbf{a}_{P_{S/s}} = \frac{d\mathbf{v}_P}{dt} \quad (2.47)$$

$$= \frac{d\mathbf{v}_{A_{P/s}}}{dt} + \frac{d(\boldsymbol{\omega}_{S/s} \times \mathbf{AP})}{dt} \quad (2.48)$$

$$= \mathbf{a}_{A_{S/s}} + \dot{\boldsymbol{\omega}} \times \mathbf{AP} + \boldsymbol{\omega}_{S/s} \times \frac{d\mathbf{AP}}{dt} \quad (2.49)$$

$$= \mathbf{a}_{A/s} + \dot{\boldsymbol{\omega}}_{S/s} \times \mathbf{AP} + \boldsymbol{\omega}_{S/s} \times (\boldsymbol{\omega}_{S/s} \times \mathbf{AP}) \quad (2.50)$$

The equation 2.50 allows the determination of the acceleration of any point P of a body S with respect to another frame s,  $\mathbf{a}_{P_{S/s}}$ , if the following variables are known:

- $\mathbf{a}_{A_{S/s}}$ : acceleration of the point A of the body S with respect to s.
- $\boldsymbol{\omega}_{S/s}$ : angular velocity of S with respect to s.
- $\dot{\boldsymbol{\omega}}_{S/s}$ : angular acceleration of S with respect to s.
- AP: position of the particle P with respect to A.

In some cases, (Fig. 2.17) it is needed to express either the velocity either the acceleration in different frames. then the following equation can be used:

For velocity:

$$\mathbf{v}_{P/s} = \mathbf{v}_{P/S} + \mathbf{v}_{P_{S/s}} \quad (2.51)$$

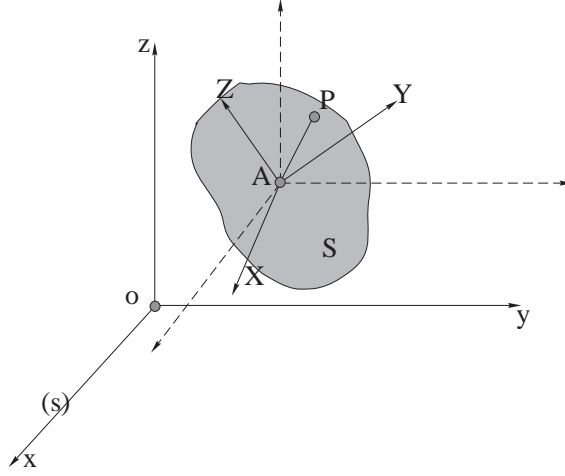


Figure 2.17: General motion of a rigid body in space with rotating axis

assume to  $S$  and  $s$  are two frames, note that here  $\mathbf{v}_{P_{S/s}}$  represents the velocity of the frame  $S$  with respect to the frame  $s$ .

The acceleration is then given by:

$$\mathbf{a}_{P/s} = \mathbf{a}_{P/S} + \mathbf{a}_{P_{S/s}} + \mathbf{a}_{\text{cor}} \quad (2.52)$$

where  $\mathbf{a}_{\text{cor}}$  is the Coriolis acceleration:

$$\mathbf{a}_{\text{cor}} = 2\boldsymbol{\omega}_{S/s} \times \mathbf{v}_{P/S} \quad (2.53)$$

The Coriolis acceleration has a zero value if:

- the point  $P$  has no relative velocity with respect to  $S$  ( $\mathbf{v}_{P/S} = \mathbf{0}$ );
- the relative velocity  $\mathbf{v}_{P/S} = \mathbf{0}$  is parallel to the angular velocity  $\boldsymbol{\omega}_{S/s}$ .

(see [2] for demonstration)

### 2.3.5 Rolling without slipping

The point of contact  $G$  between a cylinder and a the flat ground has instantaneously zero velocity ( $\mathbf{v}_G = \mathbf{0}$ ) if the cylinder rolls without slipping (Fig. 2.18).

$$\begin{aligned} \mathbf{v}_{P_{S/s}} &= \mathbf{v}_{G_{S/s}} + \boldsymbol{\omega} \times \mathbf{GP} \\ \mathbf{v}_{P_{S/s}} &= \mathbf{0} + \boldsymbol{\omega}_{S/s} \times \mathbf{GP} \end{aligned}$$



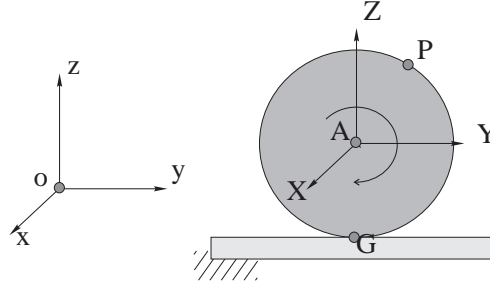


Figure 2.18: Rolling without slipping

In particular for the center A of the cylinder we get from above

$$\begin{aligned}\mathbf{v}_{A/S/s} &= \mathbf{v}_{G/S/s} - \omega_{S/s} \mathbf{x} \times \mathbf{GA} \\ \mathbf{v}_{A/S/s} &= \mathbf{0} - \omega_{S/s} \mathbf{x} \times \mathbf{GA}\end{aligned}$$

thus

$$\mathbf{v}_A = \begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \\ -\omega & 0 & 0 \\ 0 & 0 & R \end{vmatrix} = R\omega \mathbf{y}$$

## 2.4 Kinematics of systems of rigid bodies

### 2.4.1 Mechanism

A *mechanism* is an collection of rigid bodies connected together by joints. Mechanisms transfer motion and mechanical work from one or more members to others. When several links are connected by joints, they form a *kinematics chain* with one link possibly fixed. The joints permit relative motion in some directions while constraining motion in others.

### 2.4.2 Degrees of freedom

The types of motion permitted are related to the *degree of freedom* (dof) also called *mobility*. This represents the number of input parameters which can be controlled independently in order to bring the device in a particular position. It is possible to determine the mobility of a mechanism by counting the number of links (including the fixed one) and the degrees of freedom constrained by each joint. For a plan motion, we have:

$$\text{dof} = 3(n_b - n_j) + \sum f_j \quad (2.54)$$

where

- $n_b$  is the total number of rigid bodies including the fixed link;
- $n_j$  is the total number of joints possibly including the fixed link
- $f_j$  degree of freedom of relative motion between the bodies constrained by the kinematical joint.

For a three-dimensional motion

$$\text{dof} = 6 (n_b - n_j) + \sum f_j \quad (2.55)$$

### 2.4.3 Lower pairs and higher pairs

Name	Relative motion	Degree of freedom ( $f_j$ )	Skeeth symbol	Other view
Rigid joint	0 rotation 0 translation	0		
Revolute	1 rotation 0 translation	1		
Prismatic	0 rotation 1 translation	1		
Helical	1 rotation 1 translation	1		
Cylindrical	1 rotation 1 translation	2		
Spherical	3 rotations 0 translation	2		
Planar	1 rotation 2 translations	3		

Figure 2.19: Lower pair joints

We divide joints joints into two groups:

- A lower pair joint is one in which contact two rigid bodies occurs at every points of one or more surface segments (see Fig. 2.19).


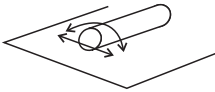
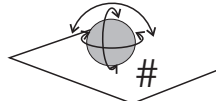
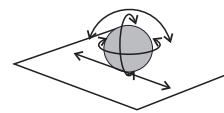
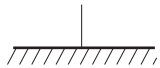
Description	Degree of freedom ( $f_j$ )	Typical form
Cylindrical surface on a plan without slipping	1	
Cylindrical surface on a plan with slipping	2	
Ball on a plan without slipping	3	
Point on a plan with slipping	5	

Figure 2.20: Higher pair joints

Name	Symbol
s Fixed body	

- A higher pair joint is one which contact occurs only at isolated points or along a line segments (see Fig. 2.20)

# represents that there is not slipping on the plane.

### 2.4.4 Kinematics exercises

The MATLAB file `kexx.m` can be executed by typing `kexx` in the interactive window of MATLAB. It provides an interface where the user may examine the numerical aspects of the exercises simply by pressing command buttons, corresponding to the various kinematics exercises. Each button calls the corresponding MATLAB file with an illustration of the exercise solution. It is also possible to see the solution of each exercise by calling the corresponding file, directly from the command line.

#### Exercise 2.4.1 Piston connected to a nut (see Fig. 2.21)

`Kex1.m`

A single-threaded screw  $S_0$ , defining a pitch  $p$  (translation of the nut along the screw axis for one turn), supports a nut  $S_1$ . A solid rod  $AB$  (length  $L$ ) connects the nut (point  $B$ ) and to a piston  $S_3$  (point  $A$ ). The piston  $S_3$  can only slide along the  $Az$  axis in a straight slot (this axis is parallel to the screw axis  $Oz$ ). The distance between the two axes is  $d$ .

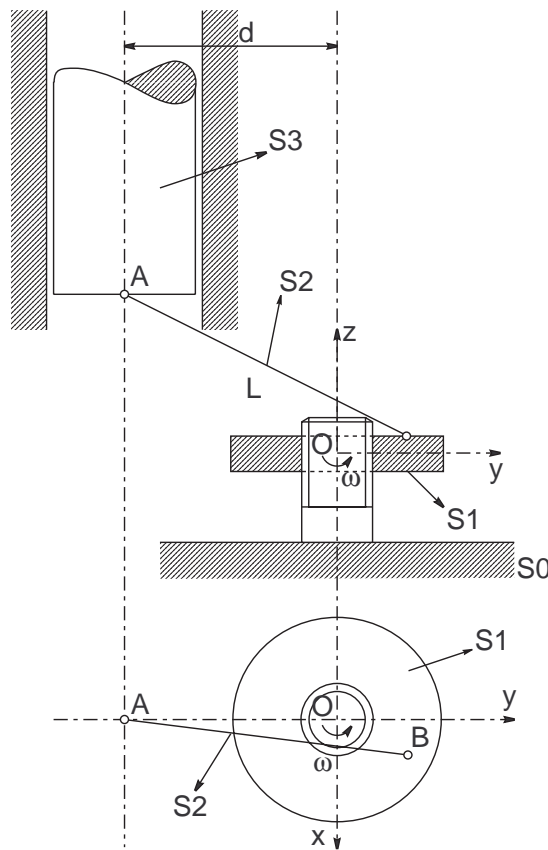


Figure 2.21: Piston connected to a nut.

The coordinates of A are  $(0; -d; z_A)$ .

The coordinates of B are  $(R \cos \alpha; R \sin \alpha; p/2\pi\alpha)$ .

If the nut rotates at the constant angular speed  $\omega = d\alpha/dt$ , what is the vertical velocity  $v_A$  of the piston?

For the following parameters :  $R = 30$  mm,  $d = 50$  mm,  $L = 10$  mm,  $p = 100$  mm,  $\omega = 1$  rad/s, compute the vertical velocity, with respect to  $\alpha$  and compare it with the results provided by the `Kex1.m` file.

### Solution

- The velocity  $v_A$  is the first time derivative of the coordinate  $z_A$ . We will first determine the expression of  $z_A$ .

$$\begin{aligned}
 \mathbf{AB} &= \mathbf{AO} + \mathbf{OB} = \mathbf{OB} - \mathbf{OA} \\
 &= \left( R \cos \alpha, R \sin \alpha + d, \frac{p}{2\pi}\alpha - z_A \right) \\
 AB^2 &= L^2 \\
 &= R^2 \cos^2 \alpha + R^2 \sin^2 \alpha + d^2 + 2Rd \sin \alpha + \left( \frac{p\alpha}{2\pi} - z_A \right)^2 \\
 \left( \frac{p\alpha}{2\pi} - z_A \right)^2 &= L^2 - R^2 - d^2 - 2Rd \sin \alpha \\
 z_A &= \pm \sqrt{L^2 - R^2 - d^2 - 2Rd \sin \alpha} + \frac{p}{2\pi}\alpha
 \end{aligned}$$

There are two different solutions but the only physically valid solution is  $z_A > p\alpha/2\pi$ .

- Determination of  $v_{Az}$

$$\begin{aligned}
 v_{Az} &= \frac{dz_A}{dt} = \frac{+(-2Rd \cos \alpha \frac{d\alpha}{dt})}{2\sqrt{L^2 - R^2 - d^2 - 2Rd \sin \alpha}} + \frac{p}{2\pi} \frac{d\alpha}{dt} \\
 v_{Az} &= \frac{-\omega R d \sin \alpha}{\sqrt{L^2 - R^2 - d^2 - 2Rd \sin \alpha}} + \frac{p}{2\pi} \omega
 \end{aligned}$$

Another way to solve the problem is presented here.

- First, the velocity  $\mathbf{v}_B$  is determined.

$$\begin{aligned}
 \mathbf{v}_B &= \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{OB} \\
 \Rightarrow \mathbf{v}_O &= \frac{p}{2\pi} \boldsymbol{\omega} \\
 \Rightarrow \mathbf{v}_B &= \left( 0, 0, \frac{p}{2\pi} \omega \right) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ R \cos \alpha & R \sin \alpha & 0 \end{vmatrix} \\
 \mathbf{v}_B &= \left( -\omega R \sin \alpha, \omega R \cos \alpha, \frac{p}{2\pi} \omega \right)
 \end{aligned}$$

- Equiprojectivity along AB is then used:

$$\begin{aligned}
 \mathbf{AB} \cdot \mathbf{v}_{A_{AB}/s} &= \mathbf{AB} \cdot \mathbf{v}_{B_{AB}/s} \\
 \text{with } \mathbf{v}_{B_{AB}/s} &= \mathbf{v}_{B_{S_1}/s} = \left( -R \sin \alpha \omega; R \cos \alpha \omega; \frac{p}{2\pi} \omega \right) \\
 \mathbf{AB} &= \left( R \cos \alpha; R \sin \alpha + d; \frac{p}{2\pi} - z_A \right) \\
 \mathbf{v}_{A_{AB}/s} &= (0, 0, v_{Az}) \\
 \Rightarrow v_{Az} \left( \frac{p}{2\pi} \alpha - z_A \right) &= -R^2 \sin \alpha \cos \alpha \omega + R^2 \sin \alpha \cos \alpha \omega \\
 &\quad + Rd \cos \alpha \omega + \frac{p}{2\pi} \omega \\
 \Rightarrow v_{Az} &= \frac{Rd \cos \alpha \omega}{\frac{p}{2\pi} \alpha - z_A} + \frac{p}{2\pi} \omega \quad (2.56) \\
 \Rightarrow v_{Az} &= \frac{Rd \cos \alpha \omega}{\frac{p}{2\pi} \alpha - \frac{p}{2\pi} \alpha - \sqrt{L^2 - R^2 - d^2} - 2Rd \sin \alpha} + \frac{p}{2\pi} \omega \quad (2.57) \\
 \Rightarrow v_{Az} &= \frac{p}{2\pi} \omega - \frac{Rd \cos \alpha \omega}{\sqrt{L^2 - R^2 - d^2} - 2Rd \sin \alpha} \quad (2.58)
 \end{aligned}$$

The file `Kex1.m` illustrates the exercise. First, the geometrical parameters of the system ( $R, d, L, p$ ), the angular velocity and the number of rotations for  $S_1$  have to be introduced (see Fig. 2.22). Then, the vertical coordinates and the velocity of the point A are calculated and plotted. An animation of the mechanism is performed (see Fig. 2.23). The mechanical system is shown in different configurations when the solid  $S_1$  turns around the  $Oz$  axis.

#### Exercise 2.4.2 Rolling trolley (see Fig. 2.24)

`Kex2.m`

A trolley  $S_2$ , supported by multiple rigid steel balls  $S_1$  (radius  $r$ ), can only have a rectilinear motion along the conductor rail  $s$  ( $z$  axis).

It is assumed that there is no sliding at the contact points M, N et P (see Fig. 2.24).

If the velocity  $\mathbf{v} = v\mathbf{k}$  of the trolley is known, and if O is considered as a point of  $S_1$ , determine, with respect to parameters  $v$  and  $r$ :

1. the relative velocity of the point O depending on the reference frame  $s$ ;
2. the relative velocity of the point O depending on the reference frame  $S_2$ .

For the following parameters:  $r = 10$  mm,  $v = 10$  m/s, compute the relative velocities of O depending on the reference frame  $s$ , and depending on the reference frame  $S_2$  compare them with the results provided by the `Kex2.m` file.

**Solution**

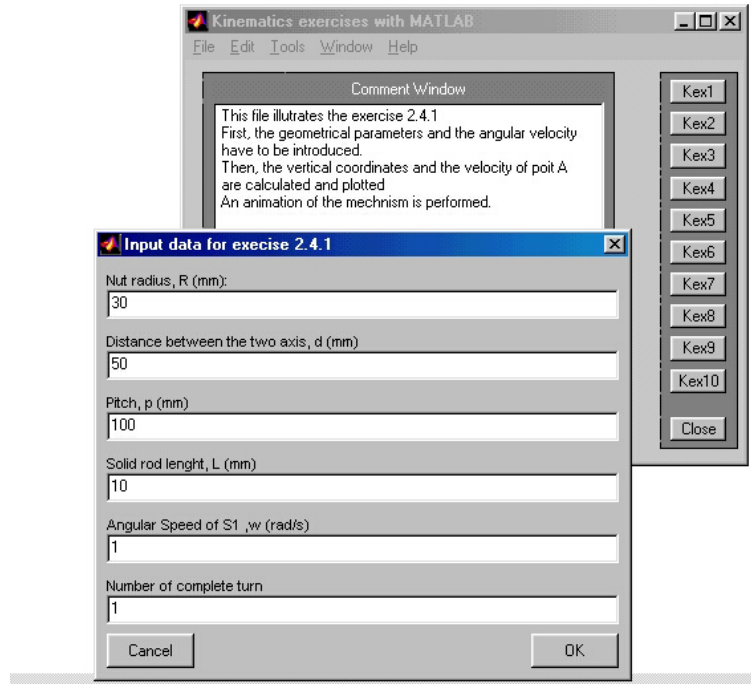


Figure 2.22: Introduce the parameters values or keep the default values.

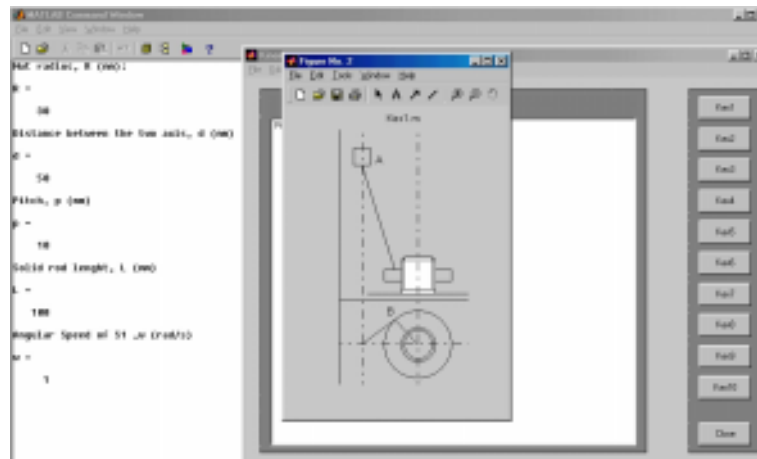


Figure 2.23: Animation of the piston in two views.

1. Motion of  $S_1$  depending on the reference frame  $s$ .  
As there is no sliding at points M and N:  $\mathbf{v}_{MS_1/s} = \mathbf{0}$  and  $\mathbf{v}_{NS_1/s} = \mathbf{0}$ .

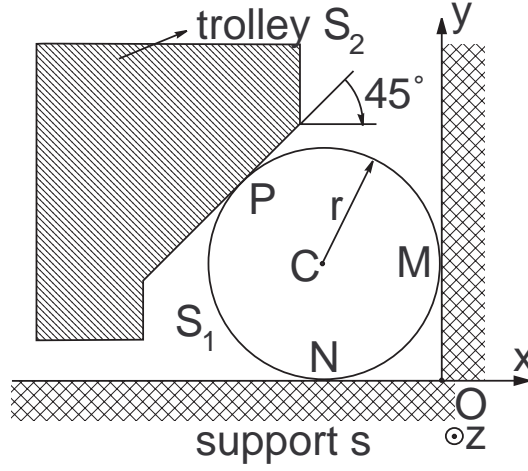


Figure 2.24: Rolling trolley.

With  $s$  as a reference, we note that  $MN$  defines the instantaneous axis of rotation for the motion of  $S_1$ . This implies that  $\omega_{S_1/s}$  lies on the axis  $MN$ .

$$\omega_{S_1/s} = \omega_1 \cdot \mathbf{u}_{NM} \text{ with } \omega_1 > 0.$$

Since there is no sliding at point P:  $\mathbf{v}_{P_{S_1}/S_2} = \mathbf{0} \Rightarrow \mathbf{v}_{P_{S_1}/s} = \mathbf{v}_{P_{S_2}/s}$

Now, we also have:  $\mathbf{v}_{P_{S_2}/s} = v\mathbf{k}$  and

$$\mathbf{v}_{P_{S_1}/s} = \omega_1 \underbrace{(\text{distance between P and the instantaneous axis of rotation MN})}_{r + r\frac{\sqrt{2}}{2}} \mathbf{k}$$

$$\Rightarrow v = \omega_1 \left( r + r\frac{\sqrt{2}}{2} \right) \Rightarrow \omega_1 = \frac{v}{r \left( 1 + \frac{\sqrt{2}}{2} \right)}$$

$$\text{Thus: } \omega_{S_1/s} = \frac{v}{r \left( 1 + \frac{\sqrt{2}}{2} \right)} \cdot \mathbf{u}_{NM} = \frac{v}{r \left( 1 + \frac{\sqrt{2}}{2} \right)} \left( \frac{1}{\sqrt{2}}; \frac{1}{\sqrt{2}}; 0 \right) = \left( \frac{v\sqrt{2}}{r(2+\sqrt{2})}; \frac{v\sqrt{2}}{r(2+\sqrt{2})}; 0 \right)$$

$$\mathbf{v}_{O_{S_1}/s} = -\omega_1 \underbrace{(\text{distance between O and the instantaneous axis of rotation MN})}_{r\frac{\sqrt{2}}{2}} \mathbf{k}$$

$$\Rightarrow \mathbf{v}_{O_{S_1}/s} = -\omega_1 r \frac{\sqrt{2}}{2} \mathbf{k} = -\frac{v}{r \left( 1 + \frac{\sqrt{2}}{2} \right)} r \frac{\sqrt{2}}{2} \mathbf{k} = -\frac{v\sqrt{2}}{2 + \sqrt{2}} \mathbf{k} \quad (2.59)$$

2. Motion of  $S_1$  depending on the reference frame  $S_2$ .



The relative velocities between  $S_1$  and  $s$  and between  $S_2$  and  $s$  are known.

Actually:  $\mathbf{v}_{0S_2/s} : (0; 0; v)$  and  $\boldsymbol{\omega}_{S_2/s} : (0; 0; 0)$  ( $S_2/s$  = translation).

and :  $\mathbf{v}_{0S_1/S_2} = \mathbf{v}_{0S_1/s} - \mathbf{v}_{0S_2/s} = \left(0; 0; -\frac{v\sqrt{2}}{2+\sqrt{2}} - v\right)$

Now, we also have:

$$\boldsymbol{\omega}_{S_1/S_2} = \boldsymbol{\omega}_{S_1/s} - \boldsymbol{\omega}_{S_2/s} = \boldsymbol{\omega}_{S_1/s} = \left( \frac{v\sqrt{2}}{r(2+\sqrt{2})}; \frac{v\sqrt{2}}{r(2+\sqrt{2})}; 0 \right) \quad (2.60)$$

The MATLAB file concerning this exercise is `Kex2.m`. After having introduced the numerical parameters for this exercise (radius of the balls and the velocity of the trolley), the relative velocity of point O depending on the reference frame  $s$  or  $S_2$  are computed and displayed on the screen.

### Exercise 2.4.3 Gear set (see Fig. 2.25)

`Kex3.m`

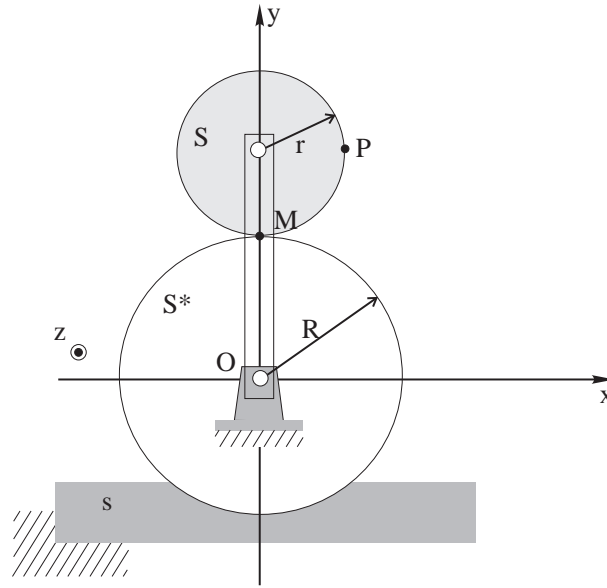


Figure 2.25: Gear set.

The gear set is defined by the two wheels  $s$  (centre O, radius R) and  $S$  (centre C, radius  $r$ ), the wheel  $s$  being fixed.

A solid rod  $S^*$  connects the two gears at points O and C. The assembly defines two revolute joints: between  $s$  and  $S^*$  at point O and between  $S$  and  $S^*$  at point C.

A pure rolling without sliding occurs between  $S$  and  $S^*$ .

If the velocity  $\boldsymbol{\omega}_{S^*/s}$  and the radius R and  $r$  are known, determine  $v_{P_{S/s}}$ , the relative velocity of P (attached to S) depending on the reference frame  $s$ .

For the following parameters:  $R = 50 \text{ mm}$ ,  $r = 10 \text{ mm}$ ,  $\omega = 1 \text{ rad/s}$ , compute the amplitude of the velocity of P give the graph with respect to the time  $t$  and compare it with the results provided by the `kex3.m` file.

**Solution**

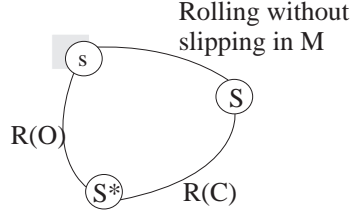


Figure 2.26: Kinematical chain.

The number of degrees of freedom can be determined quite directly (see Fig. 2.26):  
 $(2 \text{ bodies} \times 3) - (2 \text{ revolute joints} \times 2) - (1 \text{ pure rolling constraint} \times 1) = 1$ .

One can notice that the pure rolling without sliding only introduces one constraint (no sliding) since the contact is already defined by  $S^*$  (that cannot be warped).

$$\begin{aligned}
 \mathbf{v}_{P_{S/s}} &= \mathbf{v}_{C_{S/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{CP} \\
 &= \mathbf{v}_{C_{S^*/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{CP} \\
 &= \underbrace{\boldsymbol{\omega}_{S^*/s} \times \mathbf{OC}}_{=0} + \mathbf{v}_{O_{S^*/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{CP} \\
 \Rightarrow \mathbf{v}_{P_{S/s}} &= \boldsymbol{\omega}_{S^*/s} \times \mathbf{OC} + \boldsymbol{\omega}_{S/s} \times \mathbf{CP} \\
 &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{S^*/s} \\ 0 & R+r & 0 \end{vmatrix} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{S/s} \\ r & 0 & 0 \end{vmatrix} \\
 &= -\omega_{S^*/s}(R+r)\mathbf{i} + \omega_{S/s}r\mathbf{j} \\
 &= -\mathbf{i}(\omega_{S^*/s}(R+r)) + \underbrace{\omega_{S/s}r\mathbf{j}}_{?}
 \end{aligned}$$

One can deduce the relation between  $\omega_{S^*/s}$  et  $\omega_{S/s}$  from the pure rolling without sliding that occurs at M

$$\mathbf{v}_{M_{S/s}} = \mathbf{v}_{C_{S/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{CM} = 0 \quad (2.61)$$

$$= \mathbf{v}_{C_{S^*/s}} + \boldsymbol{\omega}_{S/s} \times \mathbf{CM} \quad (2.62)$$

$$= \boldsymbol{\omega}_{S^*/s} \times \mathbf{OC} + \boldsymbol{\omega}_{S/s} \times \mathbf{CM} \quad (2.63)$$

$$\Rightarrow \boldsymbol{\omega}_{S^*/s} \times \mathbf{OC} + \boldsymbol{\omega}_{S/s} \times \mathbf{CM} = 0 \quad (2.64)$$

$$-\omega_{S^*/s} \cdot |\mathbf{OC}| \cdot \mathbf{i} + \omega_{S/s} \cdot r \cdot \mathbf{i} = 0 \quad (2.65)$$

$$\omega_{S/s} = \omega_{S^*/s} \cdot \frac{R+r}{r} \quad (2.66)$$

$$\Rightarrow \mathbf{v}_{P_{S/s}} = -\omega_{S^*/s} (R+r) \mathbf{i} + \omega_{S^*/s} (R+r) \mathbf{j} \quad (2.67)$$

$$\Rightarrow \mathbf{v}_{P_{S/s}} \perp \mathbf{MP} \quad (2.68)$$

The MATLAB file `ke3.m` is divided into two parts. The first one is used to calculate the speed of P depending on the reference frames as function of the numerical parameters introduced by the user: the radii  $r$  and  $R$  and the angular velocity  $\omega$ . The scalar product of the velocity of P and the vector  $\mathbf{MP}$  is also computed to show that they are perpendicular. The amplitude of the velocity is plotted when the rod  $S_2$  makes a complete revolution around O.

The second part of the MATLAB file `ke3.m` is used to get an animated sketch of the mechanism and a plot of the velocity of P when the rod  $S_2$  covers a number of turns chosen by the user.

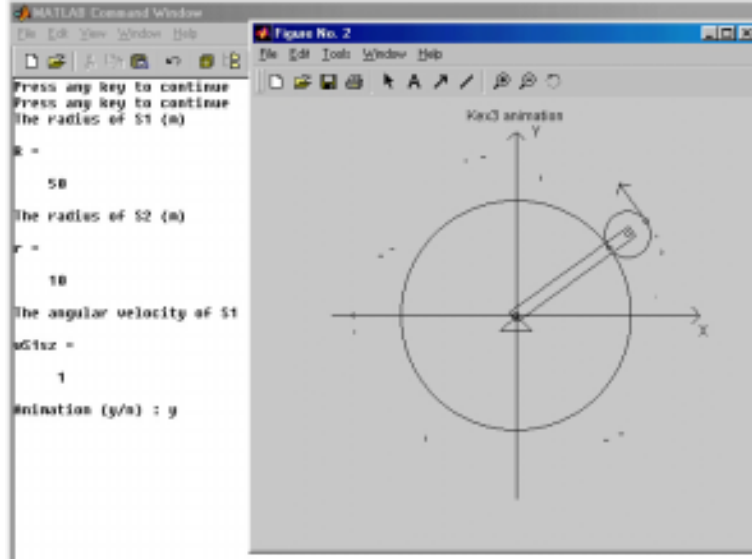


Figure 2.27: Animation of the gear set.

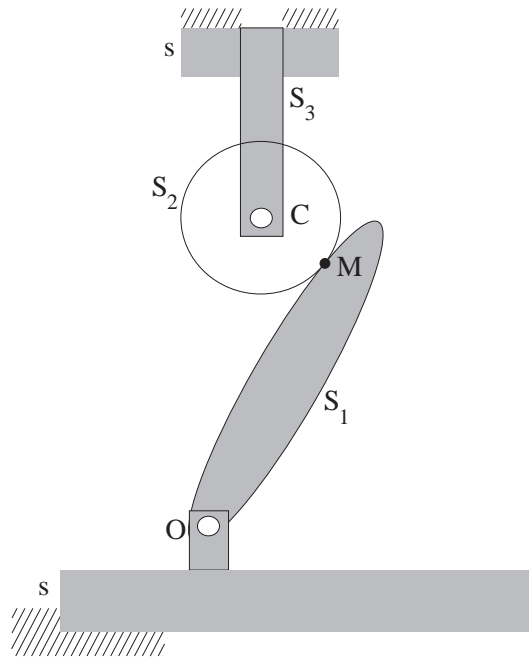


Figure 2.28: Cam

**Exercise 2.4.4 Cam (see Fig. 2.28)**

Kex4.m

The ground  $s$  being the reference frame, the assembly includes three different bodies:

- the cam  $S_1$  defining with the reference  $s$  a revolute joint at point  $O$ ;
- the cam wheel  $S_2$  defining with the piston  $S_3$  a revolute joint at point  $C$  (centre of the wheel), a pure rolling without sliding occurring at point  $M$  between the wheel  $S_2$  and the came  $S_1$ ;
- the piston  $S_3$  defining with the reference frame a prismatic joint.

To set things clear, we have, in terms of motion:

- a rotation of  $S_1$  with respect to  $s$ ,
- a rotation  $S_2$  with respect to  $S_3$  and a pure rolling without sliding of  $S_2$  on  $S_1$ ,
- a translation of  $S_3$  with respect to  $s$ .

We ask to determine  $\mathbf{v}_{C_{S_3}}$  as a function of  $\omega_{S_1/s}$ .

For the following parameters:  $\omega_{S_1/s} = 1 \text{ rad/s}$ ,  $r_1 = 0.2 \text{ m}$ ,  $r_2 = 0.1 \text{ m}$ , the  $S_3$  length =  $0.5 \text{ m}$ , the length  $L_e = 0.1 \text{ m}$ , compute the coordinates of the instant center

$I_{13}$  (see Fig. 2.4.4) at the initial time  $t = 0$ , and compare it with the results provided by the `Kex4.m` file.

### Solution

The number of degrees of freedom can be determined quite directly:

$(3 \text{ bodies} \times 3) - (2 \text{ revolute joints}) - (1 \text{ prismatic joint} \times 2) - (1 \text{ pure rolling without sliding} \times 2) = 1$ .

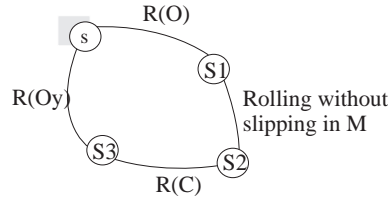


Figure 2.29: Kinematical chain relative to exercise 2.4.4.

One can notice that the pure rolling without sliding introduces two constraints: zero normal and tangential velocities.

We have:

$$\begin{aligned} \mathbf{v}_{M_{S_1/s}} &= \boldsymbol{\omega}_{S_1/s} \times \mathbf{OM} \\ \mathbf{v}_{M_{S_2/S_1}} &= 0 \end{aligned}$$

coming from the pure rolling without sliding that allows to write:

$$\begin{aligned} \mathbf{v}_{M_{S_2/s}} + \mathbf{v}_{M_{s/S_1}} &= 0 \Rightarrow \mathbf{v}_{M_{S_2/s}} = \mathbf{v}_{M_{S_1/s}} \\ \mathbf{v}_{C_{S_2/s}} &= \mathbf{v}_{M_{S_2/s}} + \boldsymbol{\omega}_{S_2/s} \times \mathbf{MC} \\ \mathbf{v}_{C_{S_3/s}} &= \mathbf{v}_{C_{S_2/s}} \\ \underbrace{\mathbf{v}_{C_{S_3/s}}}_{\text{vertical}} &= \underbrace{\boldsymbol{\omega}_{S_1/s} \times \mathbf{OM}}_{\text{known}} + \underbrace{\boldsymbol{\omega}_{S_2/s} \times \mathbf{MC}}_{\perp MC} \\ \Rightarrow \boldsymbol{\omega}_{S_1/s} \times \mathbf{OM} &= \mathbf{v}_{C_{S_3/s}} - \boldsymbol{\omega}_{S_2/s} \times \mathbf{MC} \end{aligned}$$

ICR	$S_3$	$S_2$	$S_1$	s
$S_3$	?	C	$I_{13}$	$P_x(\infty)$
$S_2$	C	?	M	$I_{02}$
$S_1$	$I_{13}$	M	?	O
s	$P_x(\infty)$	$I_{02}$	O	?

$I_{02}$  is the intersection between  $Cx$  (horizontal axis including C) and OM (Kennedy's theorem). We therefore can write:

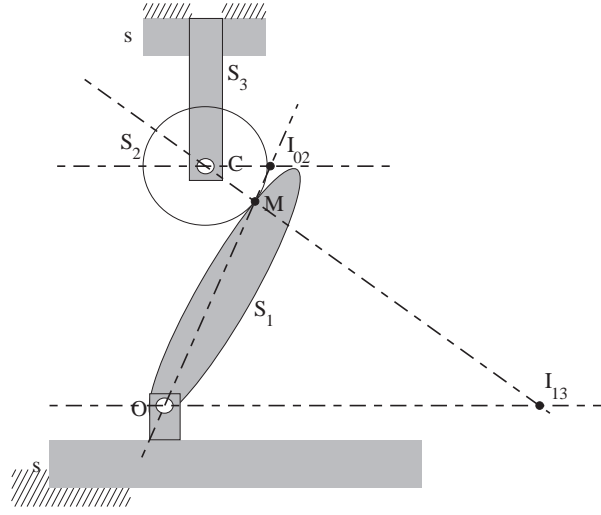


Figure 2.30: Kinematical sketch relative to exercise 2.4.4.

$$\begin{aligned}
 \left| \mathbf{v}_{C_{S_2/s}} \right| &= \omega_{S_2/s} \times I_{02}C \\
 \mathbf{v}_{M_{S_1/s}} &= \mathbf{v}_{M_{S_2/s}} \\
 \left| \omega_{S_1/s} \right| \cdot |OM| &= \left| \omega_{S_2/s} \right| \cdot |I_{02}M| \\
 \Rightarrow \left| \omega_{S_2/s} \right| &= \frac{\left| \omega_{S_1/s} \right| \cdot |OM|}{|I_{02}M|} \\
 \Rightarrow \left| \mathbf{v}_{C_{S_2/s}} \right| &= \left| \omega_{S_1/s} \right| \cdot \frac{|OM|}{|I_{02}M|} \cdot |I_{02}C|
 \end{aligned}$$

$I_{13}$  is the intersection between  $Ox$  (horizontal axis including  $O$ ) and  $CM$  (Kennedy theorem) and the instantaneous centre of rotation between  $S_3$  and  $S_1$ . We therefore can write:

$$\begin{aligned}
 \mathbf{v}_{I_{13}S_3/S_1} &= \mathbf{0} \\
 \mathbf{v}_{I_{13}S_3/s} &= \mathbf{v}_{I_{13}S_1/s} \\
 \mathbf{v}_{C_{S_3/s}} + \omega_{S_3/s} \times \mathbf{I}_{13} &= \mathbf{v}_{I_{13}S_1/s} \\
 \mathbf{v}_{C_{S_3/s}} &= \mathbf{v}_{I_{13}S_1/s} \\
 &= \underbrace{\mathbf{v}_{O_{S_1/s}}}_{=0} + \underbrace{\omega_{S_1/s} \times \mathbf{OI}_{13}}_{\perp \mathbf{OI}_{13}}
 \end{aligned}$$

The MATLAB file `kex4.m` supposes that the body  $S_1$  is a circle with a radius  $r_1$ . After giving the two radius  $r_1, r_2$  and the angular velocity  $\omega_{S_1/s}$ , we have an animation of the mechanism where the instantaneous centers of rotation are represented.

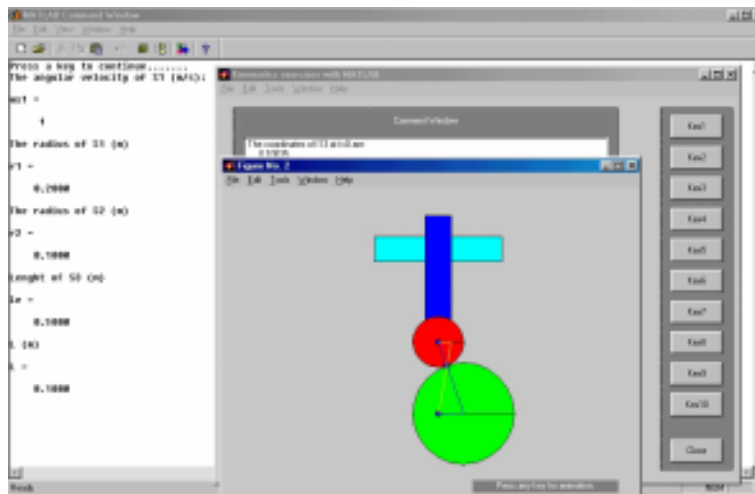


Figure 2.31: Animation of the system of cam.

**Exercise 2.4.5 Assembly of 3 bodies (see Fig. 2.32)**

Kex5.m

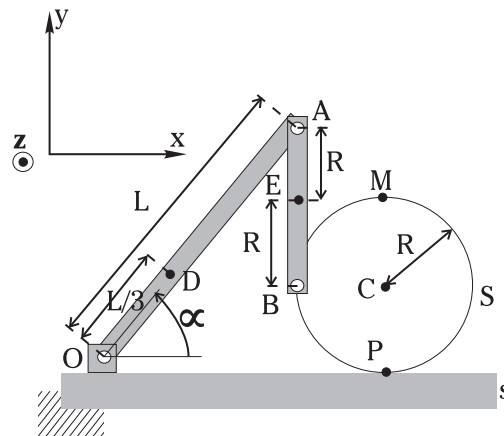


Figure 2.32: Assembly of bodies

The ground  $s$  being the reference frame, the assembly includes three different bodies:

- a solid rod  $OA$  (length  $L$ ) defining with the reference  $s$  a revolute joint at point  $O$ ;
- a wheel  $S$  (centre  $C$ ; radius  $R$ ) defining with the solid rod  $AB$  a revolute joint at point  $B$  ( $BC = R$ ), a pure rolling without sliding occurring at point  $P$

between the wheel S and the ground s;

- a solid rod AB (length  $2R$ ) defining with the solid OA a revolute joint at point A.

In the current state, the rod OA and the horizontal axis X define an angle  $\alpha$ , the rod AB being vertical and the segment BC being horizontal.

If the instantaneous angular velocity of the rod OA depending on the reference frame s is known ( $\dot{\alpha}$ ), determine:

1.  $\mathbf{v}_{D_{OA}/s}$
2.  $\mathbf{v}_{A_{OA}/s}$
3.  $\mathbf{v}_{B_S/s}$
4.  $\mathbf{v}_{E_{AB}/s}$
5.  $\mathbf{v}_{M_S/s}$

For the following parameters:  $OA = 50$  mm, the radius of S  $R = 10$  mm,  $\omega = 1$  rad/s,  $\alpha = 30^\circ$ , compute the velocities according to corresponding equations and compare them with the results provided by the `Kex5.m` file.

**Solution**

1.  $\mathbf{v}_{D_{OA}/s}$

$$\begin{aligned}
 \boldsymbol{\omega}_{OA/s} &= \dot{\alpha} \mathbf{k} \\
 \mathbf{v}_{D_{OA}/s} &= \mathbf{v}_{O_{OA}/s} + \boldsymbol{\omega}_{OA/s} \times \mathbf{OD}. \\
 &= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \dot{\alpha} \\ \frac{L}{3} \cos \alpha & \frac{L}{3} \sin \alpha & 0 \end{vmatrix} \\
 &= \left( -\dot{\alpha} \frac{L}{3} \sin \alpha; \dot{\alpha} \frac{L}{3} \cos \alpha; 0 \right)
 \end{aligned}$$

2.  $\mathbf{v}_{A_{OA}/s}$

$$\begin{aligned}
 \mathbf{v}_{A_{OA}/s} &= \mathbf{v}_{O_{OA}/s} + \boldsymbol{\omega}_{OA/s} \times \mathbf{OA}. \\
 &= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \dot{\alpha} \\ L \cos \alpha & L \sin \alpha & 0 \end{vmatrix} \\
 &= (-\dot{\alpha} L \sin \alpha; \dot{\alpha} L \cos \alpha; 0)
 \end{aligned}$$



3.  $\mathbf{v}_{B_{S/s}}$

P being the instantaneous centre of rotation between S and s, we have:

$$\begin{aligned}\mathbf{v}_{B_{S/s}} &= \underbrace{\mathbf{v}_{P_{S/s}}}_{=0} + \boldsymbol{\omega}_{S/s} \times \mathbf{PB} = \omega_{S/s} \mathbf{k} \times \mathbf{PB} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{S/s} \\ -R & R & 0 \end{vmatrix} \\ &= (-\omega_{S/s}R; -\omega_{S/s}R; 0)\end{aligned}$$

With the revolute joint at point A between OA and AB, we have:  $\mathbf{v}_{A_{OA/s}} = \mathbf{v}_{A_{AB/s}}$

With the revolute joint at point B between AB and S, we have:  $\mathbf{v}_{B_{AB/s}} = \mathbf{v}_{B_{S/s}}$

Using the property 2.44, we have:

$$\mathbf{v}_{A_{AB/s}} \cdot \mathbf{AB} = \mathbf{v}_{B_{AB/s}} \cdot \mathbf{AB}$$

$$\text{and then } \mathbf{v}_{A_{OA/s}} \cdot \mathbf{AB} = \mathbf{v}_{B_{S/s}} \cdot \mathbf{AB}$$

$$\begin{aligned}(-\dot{\alpha}L \sin \alpha; \dot{\alpha}L \cos \alpha) \cdot (0; -2R) &= (-\omega_{S/s}R; -\omega_{S/s}R) \cdot (0; -2R) \\ \dot{\alpha}L \cos \alpha 2R &= -\omega_{S/s}R 2R \\ \rightarrow \omega_{S/s} &= -\frac{\dot{\alpha}L \cos \alpha}{R}\end{aligned}$$

Finally, we get:  $\mathbf{v}_{B_{S/s}} = (\dot{\alpha}L \cos \alpha; \dot{\alpha}L \cos \alpha; 0)$

4.  $\mathbf{v}_{E_{AB/s}}$

$$\begin{aligned}\mathbf{v}_{E_{AB/s}} &= \frac{\mathbf{v}_{A_{AB/s}} + \mathbf{v}_{B_{AB/s}}}{2} \\ &= \left( \frac{\dot{\alpha}L}{2} (\cos \alpha - \sin \alpha); \dot{\alpha}L \cos \alpha; 0 \right)\end{aligned}$$

Other method (we solve the problem at once for  $\omega_{S/s}$  and  $\omega_{AB/s}$ )

$$\begin{aligned}
\mathbf{v}_{B_{S/s}} &= \mathbf{v}_{B_{AB/s}} = \mathbf{v}_{A_{AB/s}} + \omega_{AB/s} \mathbf{k} \times \mathbf{AB} \\
(-\omega_{S/s} R; -\omega_{S/s} R; 0) &= (-\dot{\alpha} L \sin \alpha; \dot{\alpha} L \cos \alpha; 0) + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_{AB/s} \\ 0 & -2R & 0 \end{vmatrix} \\
&= (-\dot{\alpha} L \sin \alpha + \omega_{AB/s} 2R; \dot{\alpha} L \cos \alpha; 0)
\end{aligned}$$

Projection along the X axis:  $\dot{\alpha} L \cos \alpha = -\dot{\alpha} L \sin \alpha + \omega_{AB/s} 2R$ .

Projection along the Y axis:  $-\omega_{S/s} R = \dot{\alpha} L \cos \alpha \rightarrow \omega_{S/s} = -\frac{\dot{\alpha} L \cos \alpha}{R} \rightarrow \omega_{AB/s} = \frac{\dot{\alpha} L}{2R} (\cos \alpha + \sin \alpha)$ .

$$\begin{aligned}
\mathbf{v}_{E_{AB/s}} &= \mathbf{v}_{A_{AB/s}} + \omega_{AB/s} \mathbf{k} \times \mathbf{AE} \\
&= (-\dot{\alpha} L \sin \alpha; \dot{\alpha} L \cos \alpha; 0) + \\
&\quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \frac{\dot{\alpha} L}{2R} (\cos \alpha + \sin \alpha) \\ 0 & -R & 0 \end{vmatrix} \\
&= \left( \frac{\dot{\alpha} L}{2} (\cos \alpha - \sin \alpha); \dot{\alpha} L \cos \alpha; 0 \right)
\end{aligned}$$

5.  $\mathbf{v}_{M_{S/s}}$

$$\begin{aligned}
\mathbf{v}_{M_{S/s}} &= \underbrace{\mathbf{v}_{P_{S/s}}}_{=0} + \omega_{S/s} \times \mathbf{PM} = \omega_{S/s} \mathbf{k} \times \mathbf{PM} \\
&= \mathbf{0} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & -\frac{\dot{\alpha} L \cos \alpha}{R} \\ 0 & 2R & 0 \end{vmatrix} \\
&= (2\dot{\alpha} L \cos \alpha; 0)
\end{aligned}$$

The file `Kex5.m` illustrates the exercise. First, the geometrical parameters of the system ( $R, d, L, p$ ), and the angular velocity of  $S_1$  have to be introduced. Then, the five velocities  $\mathbf{v}_{D_{OA/s}}, \mathbf{v}_{A_{OA/s}}, \mathbf{v}_{B_{S/s}}, \mathbf{v}_{E_{AB/s}}, \mathbf{v}_{M_{S/s}}$  are calculated.

#### Exercise 2.4.6 Wheel (see Fig. 2.33)

`Kex6.m`

The wheel  $S$  (centre  $C$ , radius  $R$ ) defines with the horizontal axis  $Ox$  (ground reference  $s$ ) a pure rolling without sliding at point  $M$ .

The displacement of point  $M$  is described by the law:  $x_M = f(t)$

Determine:

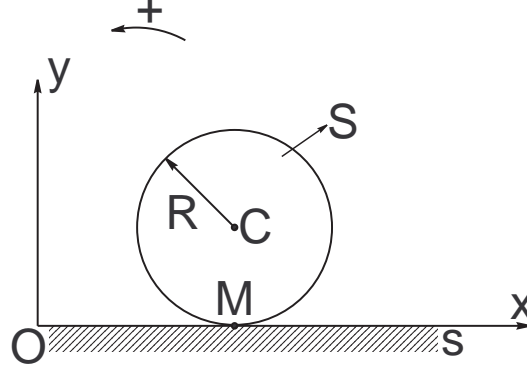


Figure 2.33: Wheel

1. the velocities of S;
2. the accelerations of S

For the following parameters:  $r = 0.7$  m,  $\dot{\omega} = -0.001$  rad/s, compute the velocity and the acceleration at the instant  $t = 125$  and compare it with the results provided by the `κex6.m` file.

**Solution**

Using the fundamental relationship 2.34  $\mathbf{v} = \mathbf{v}_O + \boldsymbol{\omega} \times \mathbf{e}$ , we have:

$$v_x = v_{0x} - \omega_z y \quad (2.69)$$

$$v_y = v_{0y} + \omega_z x \quad (2.70)$$

$$(2.71)$$

M being the instantaneous centre of rotation between S and s, we have  $|\mathbf{v}_C| = |\mathbf{v}_M| = |f'(t)| = |\omega_z| R$  and  $\omega_z = -\frac{f'(t)}{R}$ .

Since  $\mathbf{v}_{M/S} = \mathbf{0}$ , we have  $0 = v_{0x} - \omega_z \cdot 0$  and  $0 = v_{0y} + \omega_z f(t)$ .

Expressions that can be simplified in  $0 = v_{0x}$  and  $v_{0y} = \frac{f(t)f'(t)}{R}$ .

As a consequence, the velocities of S are:

$$\begin{aligned} v_x &= f'(t) \frac{y}{R}, \\ v_y &= \frac{f'(t)}{R} (f(t) - x). \end{aligned}$$

Coming from these, the accelerations are:

$$a_x = \frac{v_{0x}}{dt} - \frac{\omega_z}{dt}y - \omega_z v_{0y} - \omega_z^2 x, \quad (2.72)$$

$$a_y = \frac{v_{0y}}{dt} + \frac{\omega_z}{dt}x - \omega_z v_{0x} - \omega_z^2 y, \quad (2.73)$$

$$a_x = \frac{f''(t)}{R}y + \frac{f'(t)}{R} \frac{f(t)f'(t)}{R} - \frac{f'^2(t)}{R^2}x, \quad (2.74)$$

$$a_y = \frac{f'^2(t) + f(t)f''(t)}{R} - \frac{f''(t)}{R}x - \frac{f'^2(t)}{R^2}y, \quad (2.75)$$

$$a_x = \frac{f'^2(t)}{R^2}f - \frac{f'^2(t)}{R^2}x + \frac{f''(t)}{R}y, \quad (2.76)$$

$$a_y = \frac{f'^2 + f f''}{R} - \frac{f''}{R}x - \frac{f'^2}{R^2}y. \quad (2.77)$$

The file `Kex6.m` illustrates this exercise by giving an animation of the wheel rolling on a fixed ground. First, the geometrical parameters of the system ( $r$ ,  $\omega$ ) have to be introduced. Then, the user need to choose between a representation of the velocity vector or the acceleration vector during the wheel motion. We suppose in this exercise that, the angular acceleration of the body S is constant.

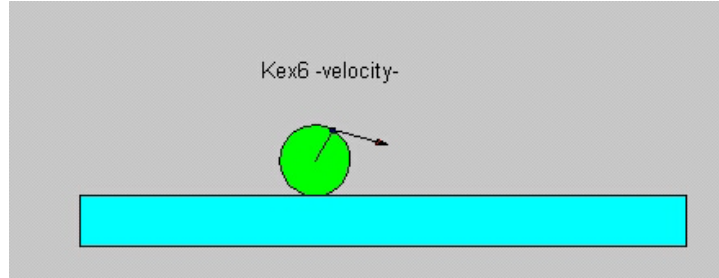


Figure 2.34: In the animation given by the `Kex6.m` file we have either the velocity vector or the acceleration vector of a point of the circle.

#### Exercise 2.4.7 Ship motion (see Fig. 2.35)

`Kex7.m`

A sailing O ship purely moves forward along its longitudinal axis (this axis can be seen as the intersection between the longitudinal symmetry plane of the ship and the surface of the sea). For such a motion to occur, the wind  $w$  should blow in the backward face of the sail.

If  $\alpha$  is the angle between the longitudinal axis and the wind direction and  $\gamma$  the angle between the longitudinal axis and the sail orientation (see Fig. 2.35), determine the limit angle  $\gamma_{\text{lim}}$  so that a longitudinal motion of the ship still occurs. Show

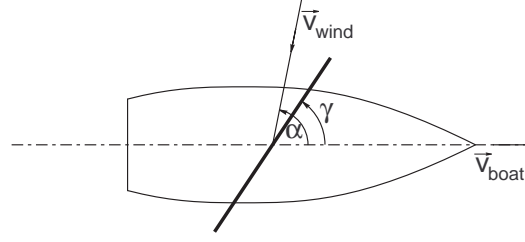


Figure 2.35: Ship motion

how it is possible to sail forward with dead winds.

For the following parameters:  $v_{w/s} = 10$  m/s,  $\alpha = 10^\circ$ ,  $v_{o/s} = 20$  m/s, draw the two functions  $y_1 = v_{w/s} \sin(\alpha - \gamma) = y_1(\gamma)$  and  $y_2 = v_{o/s} \sin(\gamma) = y_2(\gamma)$ , with respect to  $\gamma$  on the same figure and find  $\gamma_{lim}$ , compare it with the results provided by the `Kex7.m` file.

### Solution

If we take the ship O as a reference, it will still sail forward only if the relative velocity of the wind  $\mathbf{v}_{w/o}$  shows a positive component along the longitudinal axis:

$$\mathbf{v}_{w/o} \cdot \mathbf{u}_{no} > 0 \quad (2.78)$$

Using the angles  $\alpha$  and  $\gamma$ , we can write (for a ground reference s):

$$\begin{aligned} \mathbf{v}_{w/o} &= \mathbf{v}_{w/s} - \mathbf{v}_{o/s} \\ &= \mathbf{v}_{w/s} - \mathbf{v}_{o/s} \\ &= (-v_{w/s} \cos \alpha - v_{o/s})\mathbf{i} - v_{w/s} \sin \alpha \mathbf{j} \end{aligned} \quad (2.79)$$

$$\mathbf{u}_{no} = \sin \gamma \mathbf{i} - \cos \gamma \mathbf{j} \quad (2.80)$$

By replacing in Eq. 2.78:

$$\begin{aligned} \mathbf{v}_{w/o} \cdot \mathbf{u}_{no} &> 0 \\ -v_{w/s} \cos \alpha \sin \gamma - v_{o/s} \sin \gamma + v_{w/s} \cos \gamma \sin \alpha &> 0 \\ v_{w/s} \sin(\alpha - \gamma) - v_{o/s} \sin \gamma &> 0 \end{aligned} \quad (2.81)$$

For an unvarying wind ( $v_{w/s}$  and  $\alpha$ ), this last equation (2.81) gives the limit angle  $\gamma_{lim}$ . One can see that, even for dead winds ( $\alpha < \frac{\pi}{2}$ ), it is possible to sail forward.

The file `Kex7.m` gives a graphical solution of the exercise. First, the parameters of the system;  $v_{w/s}$ , the wind velocity,  $\alpha$  its angle with respect to the longitudinal axis and  $v_{o/s}$ , the ship velocity have to be introduced. According to the Eq. 2.81 two graphs with respect to  $\gamma$  are drawn on the same figure and the required  $\gamma_{lim}$  is

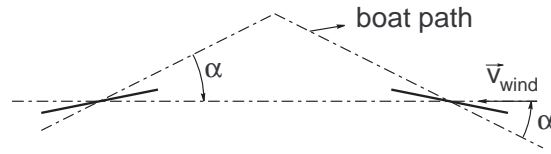


Figure 2.36: Ship motion.

given.

**Exercise 2.4.8** (see Fig. 2.37)

Kex8.m

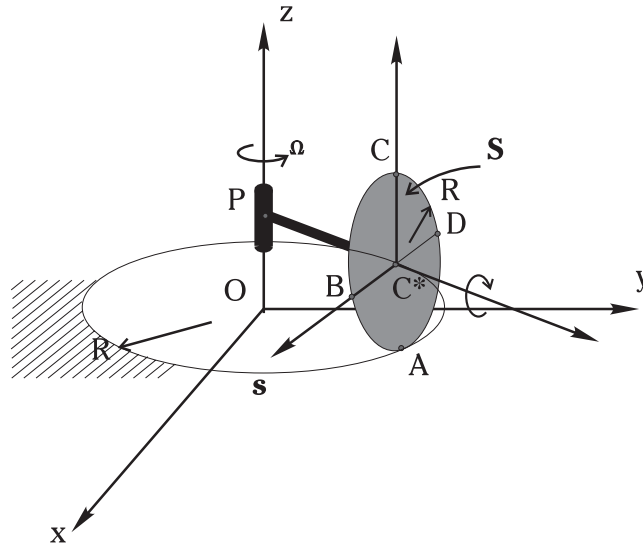


Figure 2.37: Circle around horizontal axis

The circle  $s$  (centre  $O$ , radius  $R$ ) being the reference frame, a wheel  $S$  (center  $C^*$ , radius  $R$ ) revolves around the horizontal axis  $PC^*$  (this axis being perpendicular to the wheel reference plane) and rolls without sliding on the ground at point  $A$ , describing the abovementioned circle  $s$ .

The axis  $PC^*$  revolves around this vertical axis  $OP$  with a known angular velocity  $\Omega$ .

The points  $A$ ,  $B$ ,  $C$  and  $D$  being clearly located by the figure:

1. determine  $\mathbf{v}_{A_{S/s}}$ ,  $\mathbf{v}_{B_{S/s}}$ ,  $\mathbf{v}_{C_{S/s}}$  and  $\mathbf{v}_{D_{S/s}}$ ;
2. considering a constant velocity  $\Omega$ , determine the accelerations  $\mathbf{a}_{A_{S/s}}$ ,  $\mathbf{a}_{B_{S/s}}$ ,  $\mathbf{a}_{C_{S/s}}$  and  $\mathbf{a}_{D_{S/s}}$ .

For the following parameters:  $R = 30$  m,  $\Omega = 0.1$  rad/s compute all required velocities and accelerations according to the adequate formulas and compare the values with the results provided by the `κex8.m` file.

**Solution**

1. Since a pure rolling without sliding occurs at point A between S and s, we have:

$$\begin{aligned}\mathbf{v}_{A_{S/s}} &= 0 \\ \mathbf{v}_{A_{S/s}} &= \mathbf{v}_{C_{S/s}^*} + \boldsymbol{\omega} \times \mathbf{C}^* \mathbf{A}\end{aligned}$$

We know that:

$$\mathbf{v}_{C_{S/s}^*} = \mathbf{v}_{C_{PC^*/s}^*} = -R\Omega \mathbf{i}$$

Combining the last two equations we have:

$$\begin{aligned}\mathbf{v}_{A_{S/s}} &= -R\Omega \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\omega_{S/s} \frac{\sqrt{2}}{2} & \omega_{S/s} \frac{\sqrt{2}}{2} \\ 0 & 0 & -R \end{vmatrix} \\ &= -R\Omega \mathbf{i} + R\omega_{S/s} \frac{\sqrt{2}}{2} \mathbf{i} = \mathbf{0}\end{aligned}$$

And, as a consequence,

$$\omega_{S/s} = \sqrt{2}\Omega$$

The other velocities are easily determined as shown here below:

$$\begin{aligned}
 \mathbf{v}_{B_{S/s}} &= \mathbf{v}_{C_{S/s}}^* + \boldsymbol{\omega} \times \mathbf{C}^* \mathbf{B} \\
 &= -R\Omega \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\Omega & \Omega \\ R & 0 & 0 \end{vmatrix} \\
 &= -R\Omega \mathbf{i} + R\Omega \mathbf{j} + R\Omega \mathbf{k} \\
 \mathbf{v}_{C_{S/s}} &= \mathbf{v}_{C_{S/s}}^* + \boldsymbol{\omega} \times \mathbf{C}^* \mathbf{C} \\
 &= -R\Omega \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\Omega & \Omega \\ 0 & 0 & R \end{vmatrix} \\
 &= -R\Omega \mathbf{i} - R\Omega \mathbf{i} = -2R\Omega \mathbf{i} \\
 \mathbf{v}_{D_{S/s}} &= \mathbf{v}_{C_{S/s}}^* + \boldsymbol{\omega} \times \mathbf{C}^* \mathbf{D} \\
 &= -R\Omega \mathbf{i} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -\Omega & \Omega \\ -R & 0 & 0 \end{vmatrix} \\
 &= -R\Omega \mathbf{i} - R\Omega \mathbf{j} - R\Omega \mathbf{k}
 \end{aligned}$$

2. Quite logically, we have:

$$\mathbf{a}_{P_{S/s}} = \mathbf{0}$$

M being a point randomly selected on S, we have:

$$\mathbf{a}_{M_{S/s}} = \frac{d\boldsymbol{\omega}}{dt} \times \mathbf{PM} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{PM}), M \in S$$

We also have:

$$\boldsymbol{\omega} = \boldsymbol{\omega}_{S/s} = \Omega \sqrt{2} \frac{\mathbf{AP}}{R\sqrt{2}} = \Omega \frac{\mathbf{AP}}{R}$$

If  $\alpha$  is the instantaneous angle between PC and Ox during the motion, we can write:

$$\begin{aligned}
 \mathbf{AP} &= R\mathbf{k} - R\cos\alpha \mathbf{i} - R\sin\alpha \mathbf{j} \\
 \boldsymbol{\omega} &= \Omega (\mathbf{k} - \cos\alpha \mathbf{i} - \sin\alpha \mathbf{j}) . \\
 \frac{d\boldsymbol{\omega}}{dt} &= \Omega (\sin\alpha \mathbf{i} - \cos\alpha \mathbf{j}) \frac{d\alpha}{dt} = \Omega^2 (\sin\alpha \mathbf{i} - \cos\alpha \mathbf{j}) .
 \end{aligned}$$

For the current position, we have:

$$\alpha = \frac{\pi}{2},$$



$$\boldsymbol{\omega} = \Omega (\mathbf{k} - \mathbf{j}),$$

$$\frac{d\boldsymbol{\omega}}{dt} = \Omega^2 \mathbf{i},$$

$$\text{and } \mathbf{a}_{M_{S/s}} = \Omega^2 \mathbf{i} \times \mathbf{PM} + \Omega^2 ((\mathbf{k} - \mathbf{j}) \cdot \mathbf{PM}) (\mathbf{k} - \mathbf{j}) - 2\Omega^2 \mathbf{PM}.$$

We finally get:

$$\mathbf{a}_{A_{S/s}} = \Omega^2 R (\mathbf{j} + \mathbf{k})$$

$$\mathbf{a}_{B_{S/s}} = -\Omega^2 R (2\mathbf{i} + \mathbf{j})$$

$$\mathbf{a}_{C_{S/s}} = -\Omega^2 R (3\mathbf{j} + \mathbf{k})$$

$$\mathbf{a}_{D_{S/s}} = \Omega^2 R (2\mathbf{i} - \mathbf{j})$$

The MATLAB file `Kex8.m` illustrates this exercise by calculating all required velocities and accelerations assume to  $R$  the radius of the body  $S$ , and the angular velocity of  $\mathbf{PC}^*$  around the vertical axis  $\mathbf{OP}$ ,  $\Omega$  are known.

#### Exercise 2.4.9 Slider-crank mechanism (see Fig. 2.38)

`Kex9.m`

The mechanism includes:

- a solid rod  $S_1$  of length  $L$  defining with the reference frame  $s$  a revolute joint at point  $O$ ,
- a solid rod  $S_2$  of length  $L$  defining with the  $S_1$  a revolute joint at point  $P$ ,
- a piston  $S_3$  defining with the reference frame a prismatic joint.

In the configuration represented at Fig. 2.38, we want to determine the relative velocity of point  $B$  of  $S_3$  depending on the reference frame  $s$  as a function of  $\omega_{S_1/s}$ .

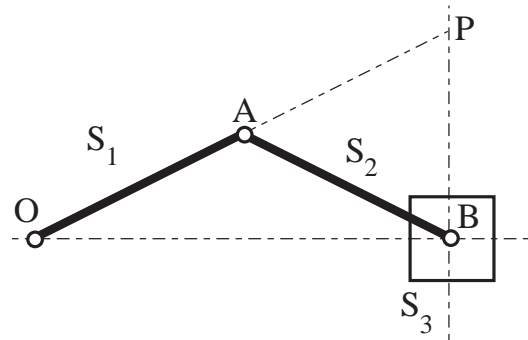


Figure 2.38: Slider-crank mechanism.

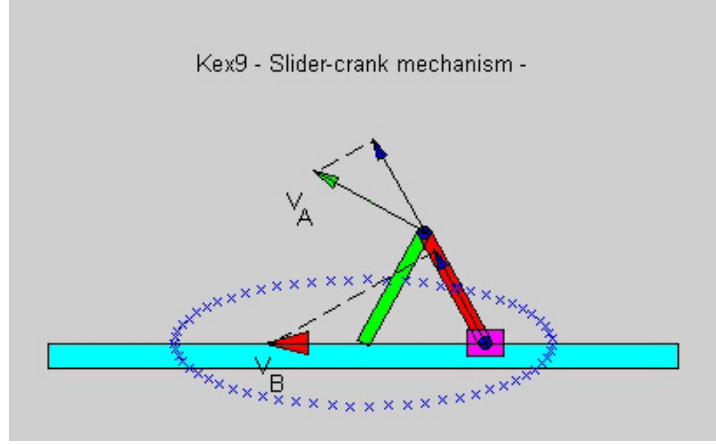


Figure 2.39: In the animation you can see two velocity vectors  $\mathbf{v}_A$  and  $\mathbf{v}_B$ , their projections and the center of AB trajectory.

For the following parameters:  $\omega_{S_1/s} = 1 \text{ rad/s}$ ,  $L = 1 \text{ m}$  plot the magnitude of  $\mathbf{v}_{B_{S_3/s}}$  with respect to  $\alpha$  and compare it with the results provided by the `Kex9.m` file.

**Solution**

The velocity of the point A of  $S_1$  depending on the reference frame  $s$  is determined by:

$$\begin{aligned}\mathbf{v}_{A_{S_1/s}} &= \mathbf{v}_{O_{S_1/s}} + \boldsymbol{\omega}_{S_1/s} \times \mathbf{OA} \\ \mathbf{v}_{A_{S_1/s}} &= (-L\omega_{S_1/s} \sin \alpha; L\omega_{S_1/s} \cos \alpha; 0)\end{aligned}$$

The centre of rotation of  $S_2$  depending on the reference frame  $s$  is the point P which is at the intersection between OA and the vertical issued from B.

The angular velocity of  $S_2$  can be deduced of:

$$\begin{aligned}\mathbf{v}_{A_{S_1/s}} &= \mathbf{v}_{A_{S_2/s}} = \boldsymbol{\omega}_{S_2/s} \times \mathbf{PA} = \boldsymbol{\omega}_{S_2/s} \times \mathbf{OA} \\ \boldsymbol{\omega}_{S_2/s} &= -\boldsymbol{\omega}_{S_1/s} \text{ since } \mathbf{PA} = -\mathbf{OA}\end{aligned}$$

And finally, the velocity of B is determined by:

$$\mathbf{v}_{B_{S_3/s}} = \mathbf{v}_{B_{S_2/s}} \quad (2.82)$$

$$= \boldsymbol{\omega}_{S_2/s} \times \mathbf{PB} \quad (2.83)$$

$$= \boldsymbol{\omega}_{S_2/s} \times (0; -2L \sin \alpha; 0) \quad (2.84)$$

$$= (-2L\omega_{S_1/s} \sin \alpha; 0; 0) \quad (2.85)$$

The file `Kex9.m` illustrates the exercise. Considering as parameters of the system  $\omega_{S_1/s}$ , the angular velocity of the rod  $S_1$  around the point O and  $L$  the length of the

two rods  $S_1$  and  $S_2$  have been introduced, the velocities of the points A and B are calculated. An animation of the mechanism is performed (see Fig. 2.39). The mechanical system is shown in different configurations when the solid  $S_1$  turns around the  $Oz$  axis. The two velocity vectors and their projection on the AB axis are represented. Thus the user can see that the relation of equiprojectivity (see Eq. 2.44) is always verified.

#### Exercise 2.4.10 Excavator (see Fig. 2.40)

Kex10.m

An excavator works with two handles  $S_1$  and  $S_2$  of respective lengths  $L'$  and  $L''$ . As shown on the Fig. 2.40, the problem includes two revolute joints: one at point A and the other at point B.  $\alpha$  is the angle between  $S_1$  and the vertical axis and  $\beta$  is the angle between the two handles  $S_1$  and  $S_2$ .

The ground  $s$  being the reference, find the ratio  $(d\alpha/dt)/(d\beta/dt)$  so that the claws of the shovel (point C) describes a horizontal motion.

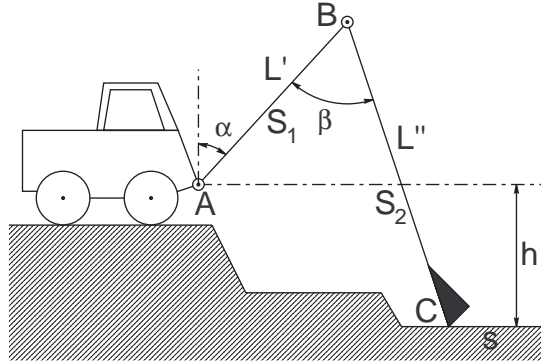


Figure 2.40: Excavator.

For the following parameters:  $L' = 3$  m,  $L'' = 5$  m,  $\alpha = 30^\circ$  plot for five given  $\alpha$  the ratio  $\omega_1/\omega_2$  versus  $\beta$  and compare it with the results provided by the Kex10.m file.

#### Solution

We successively get the following equations:

$$\begin{aligned}
 \mathbf{v}_{C_{S_2/s}} &= \mathbf{v}_{B_{S_2/s}} + \boldsymbol{\omega}_{S_2/s} \times \mathbf{BC} \\
 \mathbf{v}_{B_{S_2/s}} &= \boldsymbol{\omega}_{S_1/s} \times \mathbf{AB} \\
 \mathbf{v}_{C_{S_2/s}} &= \boldsymbol{\omega}_{S_1/s} \times \mathbf{AB} + \boldsymbol{\omega}_{S_2/s} \times \mathbf{BC} \\
 &= \boldsymbol{\omega}_{S_1/s} \times \mathbf{AB} + (\boldsymbol{\omega}_{S_2/S_1} + \boldsymbol{\omega}_{S_1/s}) \times \mathbf{BC} \\
 &= \boldsymbol{\omega}_{S_1/s} \times \underbrace{(\mathbf{AB} + \mathbf{BC})}_{=\mathbf{AC}} + \boldsymbol{\omega}_{S_2/S_1} \times \mathbf{BC}
 \end{aligned}$$

$$\begin{aligned}\mathbf{AC} &= [L' \sin \alpha + L'' \sin (\beta - \alpha)] \mathbf{i} + [L' \cos \alpha - L'' \cos (\beta - \alpha)] \mathbf{j} \\ \mathbf{BC} &= [-L'' \sin (\beta - \alpha)] \mathbf{i} + [-L'' \cos (\beta - \alpha)] \mathbf{j}\end{aligned}$$

$$\begin{aligned}\omega_{S_1/s} \times \mathbf{AC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_1 \\ L' \sin \alpha + L'' \sin (\beta - \alpha) & L' \cos \alpha - L'' \cos (\beta - \alpha) & 0 \end{vmatrix} \\ &= -\omega_1 [L' \cos \alpha - L'' \cos (\beta - \alpha)] \mathbf{i} + \omega_1 [L' \sin \alpha + L'' \sin (\beta - \alpha)] \mathbf{j} \\ \omega_{S_2/S_1} \times \mathbf{BC} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega_2 \\ -L'' \sin (\beta - \alpha) & -L'' \cos (\beta - \alpha) & 0 \end{vmatrix} \\ &= \omega_2 L'' \cos (\beta - \alpha) \mathbf{i} - \omega_2 L'' \sin (\beta - \alpha) \mathbf{j}\end{aligned}$$

Still  $\mathbf{v}_{C_{S_2/s}}$  has to be parallel to the horizontal axis  $\mathbf{i}$ , we can write:

$$\begin{aligned}\omega_2 L'' \sin (\beta - \alpha) &= \omega_1 [L' \sin \alpha + L'' \sin (\beta - \alpha)] \\ \frac{\omega_2}{\omega_1} &= \frac{[L' \sin \alpha + L'' \sin (\beta - \alpha)]}{L'' \sin (\beta - \alpha)} \\ \frac{\frac{d\alpha}{dt}}{\frac{d\beta}{dt}} &= \frac{\omega_1}{\omega_2} = \frac{1}{1 + \frac{L' \sin \alpha}{L'' \sin (\beta - \alpha)}}\end{aligned}$$

The file `Kex10.m` gives a graphical solution of the exercise. The parameters of the system are  $L'$ , the length of the body  $S_1$ ,  $L''$ , the length of the body  $S_2$ . We plot for five given  $\alpha$  the ratio  $\omega_1/\omega_2$  with respect to  $\beta$ .

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