MA 423 – Matrix Computations

Lab – 3

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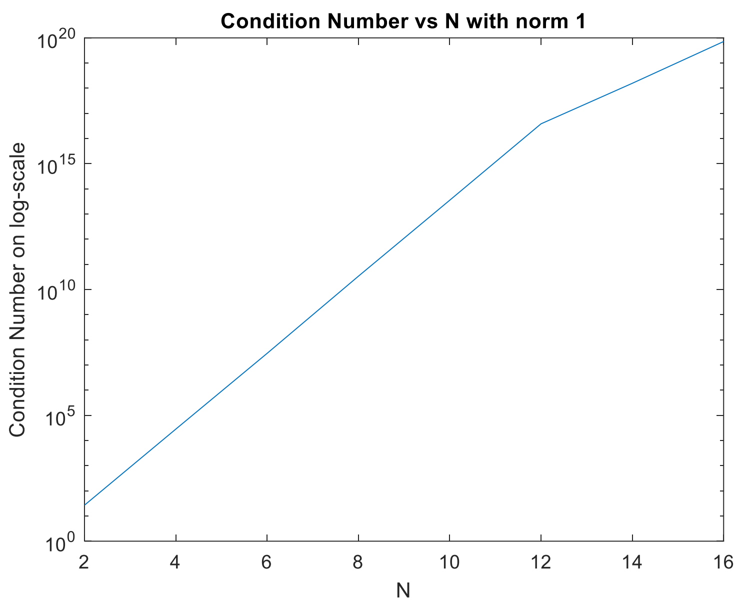
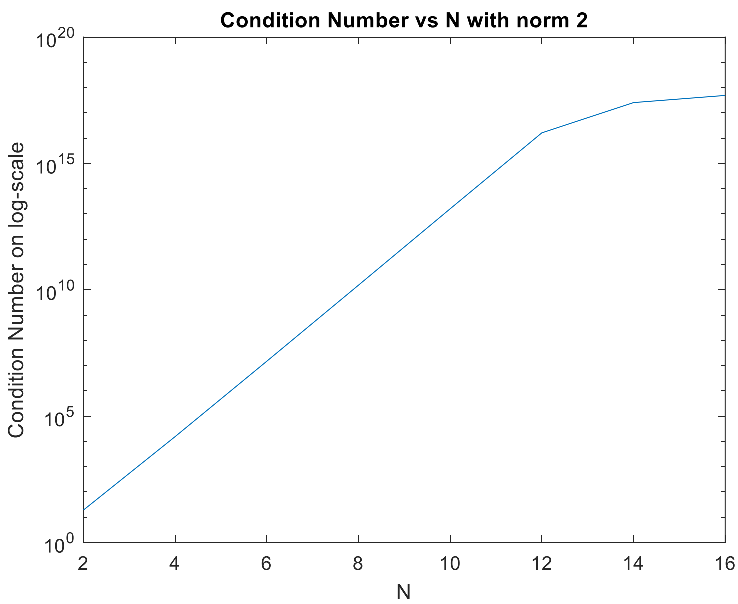
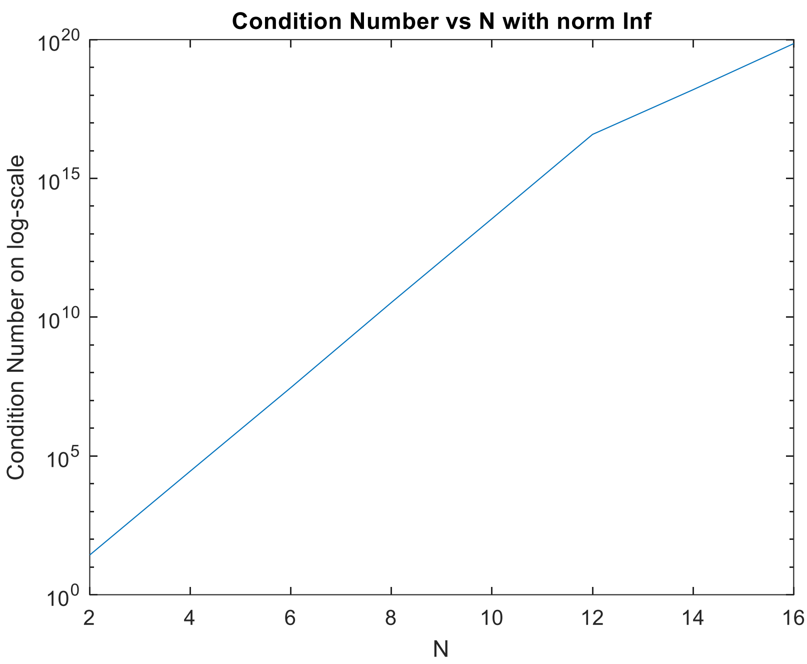
1 QUESTION - 1:

Workspace variables: *The workspace is stored in the file* ***'Q1_workspace.mat'*** *which contains following variables:*

i. ***C_1*** *– It stores the 1-norm condition number.*
ii. ***C_2*** *– It stores the 2-norm condition number.*
iii. ***C_3*** *– It stores the infinity-norm condition number.*
iv. ***H*** *– It contains the Hilbert matrix.*







Observations:

According to the above graphs, there is a linear relationship between N and log of condition number of H. This implies that the condition number of the Hilbert matrix grows exponentially with size. Beyond 10^{15} and somewhere around 10^{17}, the matrix becomes extremely ill-conditioned and behaves like a singular matrix numerically.
Also we can observe that there is a change in the nature of curve beyond this point and linearity is somewhat lost.

2 QUESTION - 2:

Workspace variables: *The workspace is stored in the file* ***'Q2_workspace.mat'*** *which contains following variables:*

1. ***cond_arr –*** *It has the condition number for H for different values of n stored as array.*
2. ***norm1 –*** *It has norm(x-x1)/norm(x) for different values of n stored as array.*
3. ***norm2 –*** *It has norm(x-x2)/norm(x) for different values of n stored as array.*
4. ***norm3 –*** *It has norm(x-x3)/norm(x) for different values of n stored as array.*

The random vector x generated is stored in the workspace file with format as: ***'Q2_workspace_x_8.mat'*** *for n = 8. Same follows for other values of n too.*

The tabulated data for n = 8, 10 and 12 is:

n	cond(H)	norm(x-x1)/norm(x)	norm(x-x2)/norm(x)	norm(x-x3)/norm(x)
8.00000000000000e+00	1.52575755666280e+10	3.05580461568675e-07	3.76336447702450e-07	1.78608874255886e-07
1.00000000000000e+01	1.60250281681132e+13	2.38362989584385e-04	2.24541584191680e-04	1.82482728853915e-04
1.20000000000000e+01	1.62116390474750e+16	1.38594079326903e-01	7.01006850023350e-02	2.07961099494942e-01

a) 10 digits are lost for n = 8, 13 digits are lost for n = 10 and for n = 12, all 16 digits are lost in computing x1, x2 and x3.

b) There isn't much a difference since for each n the error is of the same order and the mantissa doesn't vary much or with a specific pattern.

c) Yes, the loss of accuracy agrees with the value predicted by the ***Rule-of-Thumb***.

   The value of ||x – x'||/||x|| is converted to the form s.t its value <= 0.5 * 10^{-p}, then x and x' agree at least p significant digits for all entries (indices). The corresponding result is clearly shown by the tabulated results, hence there is agreement in the loss of accuracy.

3 QUESTION - 3:

Workspace variables: *The workspace is stored in the file* ***'Q3_workspace.mat'*** *which contains the variables given in the question.*

$$\frac{||r||}{||b||} = 2.073493948461995e - 16$$

$$\frac{||x - x'||}{||x||} = 1.112970979544248e - 04$$

Observations:

As we can observe that a small ||r||/||b|| does not imply a small ||x – x'||/||x||. The former is of the order 10^{-16} while the latter is of order 10^{-4}. So there is a huge difference between them and we can't deduce that x is sufficiently close to x' by simple looking at ||r||/||b||.

4 Question - 4:

Workspace variables: *The workspace is stored in the file* ***'Q3_workspace.mat'*** *which contains the variables:*

i. ***X_gepp –*** *It has the solution of Wx = b using GEPP.*
ii. ***X_qr –*** *It has the solution of Wx = b using QR decomposition.*
iii. ***error_gepp –*** *It has the value of forward error for GEPP method.*
iv. ***error_qr –*** *It has the value of forward error for QR method.*
v. ***ratio_gepp –*** *It has the value of* $||r||_\infty/||b||_\infty$ *for GEPP method.*
vi. ***ratio_qr –****It has the value of* $||r||_\infty/||b||_\infty$ *for QR method.*

The random vector x generated is stored in the workspace file with format as: ***'Q4_workspace_x_32.mat'*** *for n = 32. Same follows for other values of n too.*

The tabulated result is:

n	Condition Number	Error (GEPP)	Error (QR)	||r||/||b|| (GEPP)	||r||/||b|| (QR)
3.20000000000000e+01	3.20000000000000e+01	2.17198379876610e-08	2.35973619719469e-15	2.19642017622045e-09	3.95640513030762e-16
6.40000000000000e+01	6.40000000000018e+01	9.09870946616053e-01	1.16635032406997e-14	1.74021164211673e-01	7.29783385698314e-16

a) The solution using QR decomposition appears to give a lower error in the computed solution. The error using QR decomposition is of the order $10^{-15} – 10^{-14}$ which is very less.

b) The solution using QR decomposition has a lower value of $||r||_\infty/||b||_\infty$ for both n = 32 and 64.

c) For QR decomposition, the ***Rule-of-Thumb*** is predicting the correct answer. Here, t = 1 for both values of n. Hence s – t = 15, which is also demonstrated by the forward error computed.

   For GEPP, the error is of the order 10^{-9} for n = 32, which is no-where near 10^{-16}.

d) QR algorithm is behaving in a backward stable manner while GEPP is not backwards stable as seen by the computed data and the ***Rule-of-Thumb*** analysis.

5 QUESTION - 5:

Workspace variables: *The entire workspace is stored in the file* ***'Q5_workspace.mat'*** *which contains all the variables. The random matrix A generated is stored in the workspace file with format as: '****Q5_workspace_A_20.mat****' for n = 20. Same follows for other values of n too.*

The tabulated data is as follows:

*********** For norm-1 ***********

<table>
<tr><th>n</th><th>||L|| (genp)</th><th>||U|| (genp)</th><th>||LU - A||/||A|| (genp)</th><th>||L|| (gepp)</th><th>||U|| (gepp)</th><th>||LU - PA||/||A|| (gepp)</th></tr>
<tr><td>2.00000000000000e+01</td><td>2.69755306155194e+15</td><td>1.62673240601601e+13</td><td>1.79515759299038e-02</td><td>9.21094910133498e+00</td><td>9.97541014158957e+00</td><td>1.73574183910912e-16</td></tr>
<tr><td>4.00000000000000e+01</td><td>3.25209087143941e+16</td><td>2.86457338000178e+14</td><td>8.42490812755382e-02</td><td>2.12325008258141e+01</td><td>2.78238776091343e+01</td><td>2.68549541423157e-16</td></tr>
<tr><td>6.00000000000000e+01</td><td>2.96461165751991e+16</td><td>8.69450162367106e+14</td><td>5.90927300434871e-02</td><td>2.79208792056337e+01</td><td>4.80308464561421e+01</td><td>3.76563197062273e-16</td></tr>
<tr><td>8.00000000000000e+01</td><td>1.30020596720834e+16</td><td>5.37187021560055e+13</td><td>1.91167537876821e-02</td><td>3.86827884426436e+01</td><td>7.69365171215641e+01</td><td>4.88184068059972e-16</td></tr>
<tr><td>1.00000000000000e+02</td><td>5.56119366706232e+15</td><td>3.15353211771000e+12</td><td>1.48423624016886e-02</td><td>5.09009154376691e+01</td><td>9.11237717305383e+01</td><td>6.63694908378090e-16</td></tr>
<tr><td>1.20000000000000e+02</td><td>1.28719009187374e+16</td><td>8.33979966814124e+13</td><td>1.05160706616309e-02</td><td>6.13779179486077e+01</td><td>1.08875159234020e+02</td><td>6.52886945072753e-16</td></tr>
<tr><td>1.40000000000000e+02</td><td>1.34334944479578e+16</td><td>1.90349925604027e+14</td><td>1.04242106275450e-02</td><td>6.63970488719594e+01</td><td>1.67957664388914e+02</td><td>7.20958451492130e-16</td></tr>
</table>

*********** For norm-2 ***********

<table>
<tr><th>n</th><th>||L|| (genp)</th><th>||U|| (genp)</th><th>||LU - A||/||A|| (genp)</th><th>||L|| (gepp)</th><th>||U|| (gepp)</th><th>||LU - PA||/||A|| (gepp)</th></tr>
<tr><td>2.00000000000000e+01</td><td>7.36783420940594e+14</td><td>4.17135554792193e+13</td><td>1.12776572245002e-02</td><td>3.90480343973056e+00</td><td>4.60814048654704e+00</td><td>9.17519526158580e-17</td></tr>
<tr><td>4.00000000000000e+01</td><td>5.79926809934753e+15</td><td>1.01466607281011e+15</td><td>3.05382560145351e-02</td><td>6.64921457612838e+00</td><td>1.12674937264551e+01</td><td>1.08502590960501e-16</td></tr>
<tr><td>6.00000000000000e+01</td><td>4.57374415685849e+15</td><td>3.75186307798135e+15</td><td>2.74159450063284e-02</td><td>6.38160452064947e+00</td><td>1.29507718195430e+01</td><td>1.20344397689758e-16</td></tr>
<tr><td>8.00000000000000e+01</td><td>1.70982180412189e+15</td><td>2.59170388769634e+14</td><td>4.40945351949584e-03</td><td>8.55089914270813e+00</td><td>2.20629854018941e+01</td><td>1.26508629340891e-16</td></tr>
<tr><td>1.00000000000000e+02</td><td>6.47576974577728e+14</td><td>1.72912676653221e+13</td><td>3.74443941360387e-03</td><td>9.55501422498558e+00</td><td>2.77111503681743e+01</td><td>1.39603907787173e-16</td></tr>
<tr><td>1.20000000000000e+02</td><td>1.34981062660149e+15</td><td>5.61480192609726e+14</td><td>2.42099347477746e-03</td><td>1.12352801172563e+01</td><td>2.94046426480035e+01</td><td>1.42902586632913e-16</td></tr>
<tr><td>1.40000000000000e+02</td><td>1.31761377553729e+15</td><td>1.32093972459816e+15</td><td>2.05674129479588e-03</td><td>1.15359213050674e+01</td><td>3.58772156806973e+01</td><td>1.43700151039392e-16</td></tr>
</table>

*********** For Infinity norm ***********

<table>
<tr><th>n</th><th>||L|| (genp)</th><th>||U|| (genp)</th><th>||LU - A||/||A|| (genp)</th><th>||L|| (gepp)</th><th>||U|| (gepp)</th><th>||LU - PA||/||A|| (gepp)</th></tr>
<tr><td>2.00000000000000e+01</td><td>3.29328719922129e+14</td><td>1.58760350415669e+14</td><td>2.50234757011846e-02</td><td>8.95612758070650e+00</td><td>1.00804278519113e+01</td><td>1.47309541805434e-16</td></tr>
<tr><td>4.00000000000000e+01</td><td>1.53165700927969e+15</td><td>5.27625142290003e+15</td><td>8.64539656863863e-02</td><td>1.67135339821747e+01</td><td>2.65046353329478e+01</td><td>2.76087459375300e-16</td></tr>
<tr><td>6.00000000000000e+01</td><td>1.06179022361222e+15</td><td>2.44256859625238e+16</td><td>9.25556765110766e-02</td><td>2.02274556170820e+01</td><td>3.36726798450738e+01</td><td>3.42703804699596e-16</td></tr>
<tr><td>8.00000000000000e+01</td><td>3.36120021217356e+14</td><td>1.98452064338054e+15</td><td>1.61028015455658e-02</td><td>3.13148005492245e+01</td><td>5.51663168714809e+01</td><td>4.76192417993479e-16</td></tr>
<tr><td>1.00000000000000e+02</td><td>1.09255278009214e+14</td><td>1.45050110962126e+14</td><td>1.97183768209468e-02</td><td>3.42818298327155e+01</td><td>8.32974697560835e+01</td><td>5.48776911624117e-16</td></tr>
<tr><td>1.20000000000000e+02</td><td>2.17315526102499e+14</td><td>5.40399285259273e+15</td><td>1.16657654640016e-02</td><td>4.09573472383205e+01</td><td>1.05107115133460e+02</td><td>6.09487066834019e-16</td></tr>
<tr><td>1.40000000000000e+02</td><td>2.02320655453862e+14</td><td>1.35697027039521e+16</td><td>1.02699862802082e-02</td><td>4.57276110836191e+01</td><td>1.26554691328891e+02</td><td>5.95972811429866e-16</td></tr>
</table>

Observations:

We can observe that gepp performs better than the genp since the relative error in gepp is of the order 10^{-16} while in genp it is 10^{-3}. So the genp produces a larger norm (for relative error) as compared to the gepp method.

So it demonstrates that the better numerical properties of Gaussian Elimination with Partial Pivoting over no pivoting in dealing with matrices with small entries in pivotal positions.