FINITE DIFFERENCE METHOD FOR THE MULTI-ASSET BLACK-SCHOLES EQUATIONS

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Chapter 1

Introduction

The following discussion sustains to solve the Black-Scholes (BS) partial differential equation (PDE) in a multi-asset setup using finite difference methods. This differs from our analysis of BS PDE in the case of a single underlying asset. Let $u(s_1, s_2, \ldots, s_n, t)$ be the value of the option, where s_i is the underlying i^{th} asset value. Then the BS PDE can be generalized for n-assets as in equation 1.1.

$$\frac{\partial u(\boldsymbol{s},t)}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} s_i s_j \frac{\partial^2 u(\boldsymbol{s},t)}{\partial s_i \partial s_j} + r\sum_{i=1}^{n} s_i \frac{\partial u(\boldsymbol{s},t)}{\partial s_i} = ru(\boldsymbol{s},t) \tag{1.1}$$

for $(\boldsymbol{s}, t) = (s_1, s_2, \ldots, s_n, t) \in \boldsymbol{R}_+^n \times [0, T)$ with the final condition as

$$u(\boldsymbol{s}, T) = u_T(\boldsymbol{s}). \tag{1.2}$$

Here r is an interest rate, σ_i is a constant volatility of the i^{th} asset, and ρ_{ij} is the correlation coefficient between i^{th} and j^{th} underlying assets. In the following section, we will try to compute the solution of three-asset BS PDE using a finite difference numerical scheme.

Chapter 2

Numerical Solution using Finite Difference Method

Since we know the final condition (Equation 1.2), we will make a transformation in the BS PDE and change the time variable t to τ where $\tau = T - t$. Then Equation 1.1 becomes

$$\frac{\partial u(\boldsymbol{s},\tau)}{\partial \tau} = \frac{1}{2}\sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} s_i s_j \frac{\partial^2 u(\boldsymbol{s},\tau)}{\partial s_i \partial s_j} + r\sum_{i=1}^{n} s_i \frac{\partial u(\boldsymbol{s},\tau)}{\partial s_i} - ru(\boldsymbol{s},\tau) \quad (2.1)$$

with the initial condition

$$u(\boldsymbol{s},0) = u_T(\boldsymbol{s}). \quad (2.2)$$

2.1 Three-asset Cash or Nothing Option

Now consider a three-asset setup, and let $x = s_1, y = s_2$, and $z = s_3$. Also, let $\mathcal{L}_{BS}$ be the following operator

$$\mathcal{L}_{BS}u = \frac{1}{2}\sigma_x^2 x^2 \frac{\partial^2 u}{\partial x^2} \ + \ \frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2 u}{\partial y^2} \ + \ \frac{1}{2}\sigma_z^2 z^2 \frac{\partial^2 u}{\partial z^2} + \\ \rho_{xy}\sigma_x\sigma_y xy \frac{\partial^2 u}{\partial x \partial y} + \rho_{yz}\sigma_y\sigma_z yz \frac{\partial^2 u}{\partial y \partial z} + \\ \rho_{zx}\sigma_z\sigma_x zx \frac{\partial^2 u}{\partial z \partial x} + rx\frac{\partial u}{\partial x} + ry\frac{\partial u}{\partial y} + rz\frac{\partial u}{\partial z} - ru \quad (2.3)$$

Then the BS equation can be rewritten as:

$$\frac{\partial u(\boldsymbol{s}, \tau)}{\partial \tau} = \mathcal{L}_{BS}u \quad (2.4)$$

for $(x, y, z, \tau) \in \Omega \times (0, T]$, with $u(x, y, z, 0) = u_T(x, y, z)$.

In order to solve the PDE numerically we truncate the original infinite domain into a finite domain $\Omega = (0, L) \times (0, M) \times (0, N)$. Once we have restricted our domain, we will discretize Ω, but unlike our usual method of equidistant discretization in space, we will use non-uniform space steps given by

$$h_i^x = x_{i+1} - x_i \ \forall \ i = 0, \ldots, N_x - 1, \quad (2.5)$$

$$h_j^y = y_{j+1} - y_j \ \forall \ j = 0, \ldots, N_y - 1, \quad (2.6)$$

$$h_k^z = z_{k+1} - z_k \ \forall \ k = 0, \ldots, N_z - 1 \quad (2.7)$$

where $x_0 = y_0 = z_0 = 0$ and $x_{Nx} = L, y_{Ny} = M, z_{Nz} = N$, and time step is given by $\Delta\tau = T/N_\tau$.

Let $u_{ijk}^n = u(x_i, y_j, z_k, n\Delta\tau)$, where $i = 0, \ldots, N_x$, $j = 0, \ldots, N_y$, $k = 0, \ldots, N_z$ and $n = 0, \ldots, N_\tau$. The boundary conditions to be used are:

1. **Zero Dirichlet boundary condition**:

$$u_{0jk}^n = 0 \text{ for } j = 0 \ldots N_y \ \& \ k = 0 \ldots N_z \quad (2.8)$$

$$u_{i0k}^n = 0 \text{ for } i = 0 \ldots N_x \ \& \ k = 0 \ldots N_z \quad (2.9)$$

$$u_{ij0}^n = 0 \text{ for } i = 0 \ldots N_x \ \& \ j = 0 \ldots N_y \quad (2.10)$$

2. **Homogenous Neumann boundary condition**:

$$u^n_{N_x+1,jk} = u^n_{N_x,jk} \text{ for } j = 0 \dots N_y \;\&\; k = 0 \dots N_z \tag{2.11}$$
$$u^n_{i,N_y+1,k} = u^n_{i,N_y,k} \text{ for } i = 0 \dots N_x \;\&\; k = 0 \dots N_z \tag{2.12}$$
$$u^n_{ij,N_z+1} = u^n_{ij,N_z} \text{ for } i = 0 \dots N_x \;\&\; j = 0 \dots N_y \tag{2.13}$$

To solve the Equation 2.4, we use the operator splitting methods in the following way:

$$\begin{aligned}
\frac{u^{n+\frac{1}{3}}_{ijk} - u^n_{ijk}}{\Delta\tau} &= (\mathcal{L}^x_{BS}u)^{n+\frac{1}{3}}_{ijk} \\
\frac{u^{n+\frac{2}{3}}_{ijk} - u^{n+\frac{1}{3}}_{ijk}}{\Delta\tau} &= (\mathcal{L}^y_{BS}u)^{n+\frac{2}{3}}_{ijk} \\
\frac{u^{n+1}_{ijk} - u^{n+\frac{2}{3}}_{ijk}}{\Delta\tau} &= (\mathcal{L}^z_{BS}u)^{n+1}_{ijk}
\end{aligned} \tag{2.14}$$

where the discrete difference operators are given by Eqn 2.15.

$$\begin{aligned}
(\mathcal{L}^x_{BS}u)^{n+\frac{1}{3}}_{ijk} =& \frac{(\sigma_x x_i)^2}{2} D_{xx} u^{n+\frac{1}{3}}_{ijk} + r x_i D_x u^{n+\frac{1}{3}}_{ijk} + \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy} u^n_{ijk} \\
&+ \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz} u^n_{ijk} + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_k x_i D_{zx} u^n_{ijk} - \frac{1}{3} r u^{n+\frac{1}{3}}_{ijk} \\
(\mathcal{L}^y_{BS}u)^{n+\frac{2}{3}}_{ijk} =& \frac{(\sigma_y y_j)^2}{2} D_{yy} u^{n+\frac{2}{3}}_{ijk} + r y_j D_y u^{n+\frac{2}{3}}_{ijk} + \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy} u^{n+\frac{1}{3}}_{ijk} \\
&+ \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz} u^{n+\frac{1}{3}}_{ijk} + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_k x_i D_{zx} u^{n+\frac{1}{3}}_{ijk} - \frac{1}{3} r u^{n+\frac{2}{3}}_{ijk} \\
(\mathcal{L}^z_{BS}u)^{n+1}_{ijk} =& \frac{(\sigma_z z_k)^2}{2} D_{zz} u^{n+1}_{ijk} + r z_k D_z u^{n+1}_{ijk} + \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy} u^{n+\frac{2}{3}}_{ijk} \\
&+ \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz} u^{n+\frac{2}{3}}_{ijk} + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_k x_i D_{zx} u^{n+\frac{2}{3}}_{ijk} - \frac{1}{3} r u^{n+1}_{ijk}
\end{aligned} \tag{2.15}$$

The numerical scheme is given in Eqn 2.16 through Eqn 2.24.

$$D_x u_{ijk} = -\frac{h_i^x}{h_{i-1}^x\left(h_{i-1}^x + h_i^x\right)} u_{i-1,jk} + \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} u_{ijk} + \frac{h_{i-1}^x}{h_i^x\left(h_{i-1}^x + h_i^x\right)} u_{i+1,jk} \tag{2.16}$$

$$D_y u_{ijk} = -\frac{h_j^y}{h_{j-1}^y\left(h_{j-1}^y + h_j^y\right)} u_{i,j-1,k} + \frac{h_j^y - h_{j-1}^y}{h_{j-1}^y h_j^y} u_{ijk} + \frac{h_{j-1}^y}{h_j^y\left(h_{j-1}^y + h_j^y\right)} u_{i,j+1,k} \tag{2.17}$$

$$D_z u_{ijk} = -\frac{h_k^z}{h_{k-1}^z\left(h_{k-1}^z + h_k^z\right)} u_{ij,k-1} + \frac{h_k^z - h_{k-1}^z}{h_{k-1}^z h_k^z} u_{ijk} + \frac{h_{k-1}^z}{h_k^z\left(h_{k-1}^z + h_k^z\right)} u_{ij,k+1} \tag{2.18}$$

$$D_{xx} u_{ijk} = \frac{2}{h_{i-1}^x\left(h_{i-1}^x + h_i^x\right)} u_{i-1,jk} - \frac{2}{h_{i-1}^x h_i^x} u_{ijk} + \frac{2}{h_i^x\left(h_{i-1}^x + h_i^x\right)} u_{i+1,jk} \tag{2.19}$$

$$D_{yy} u_{ijk} = \frac{2}{h_{j-1}^y\left(h_{j-1}^y + h_j^y\right)} u_{i,j-1,k} - \frac{2}{h_{j-1}^y h_j^y} u_{ijk} + \frac{2}{h_j^y\left(h_{j-1}^y + h_j^y\right)} u_{i,j+1,k} \tag{2.20}$$

$$D_{zz} u_{ijk} = \frac{2}{h_{k-1}^z\left(h_{k-1}^z + h_k^z\right)} u_{ij,k-1} - \frac{2}{h_{k-1}^z h_k^z} u_{ijk} + \frac{2}{h_k^z\left(h_{k-1}^z + h_k^z\right)} u_{ij,k+1} \tag{2.21}$$

$$D_{xy} u_{ijk} = \frac{u_{i+1,j+1,k} - u_{i-1,j+1,k} - u_{i+1,j-1,k} + u_{i-1,j-1,k}}{h_i^x h_j^y + h_{i-1}^x h_j^y + h_i^x h_{j-1}^y + h_{i-1}^x h_{j-1}^y} \tag{2.22}$$

$$D_{yz} u_{ijk} = \frac{u_{i,j+1,k+1} - u_{i,j-1,k+1} - u_{i,j+1,k-1} + u_{i,j-1,k-1}}{h_j^y h_k^z + h_{j-1}^y h_k^z + h_j^y h_{k-1}^z + h_{j-1}^y h_{k-1}^z} \tag{2.23}$$

$$D_{zx} u_{ijk} = \frac{u_{i+1,j,k+1} - u_{i+1,j,k-1} - u_{i-1,j,k+1} + u_{i-1,j,k-1}}{h_k^z h_i^x + h_{k-1}^z h_i^x + h_k^z h_{i-1}^x + h_{k-1}^z h_{i-1}^x} \tag{2.24}$$

Now the numerical algorithm for each equation in Eqn 2.14 will yield an implicit system of equations on solving which we will obtain the value at the next time step. So in total, we will have three systems to solve, and the equations can be written in matrix form as shown below.

The first equation can be written in matrix form in the following way.

$$\alpha_i u_{i-1,jk}^{n+\frac{1}{3}} + \beta_i u_{ijk}^{n+\frac{1}{3}} + \gamma_i u_{i+1,jk}^{n+\frac{1}{3}} = f_{ijk} \tag{2.25}$$

where

$$\begin{aligned}
\alpha_i &= -\frac{(\sigma_x x_i)^2}{h_{i-1}^x \left(h_{i-1}^x + h_i^x\right)} + r x_i \frac{h_i^x}{h_{i-1}^x \left(h_{i-1}^x + h_i^x\right)} \\
\beta_i &= \frac{1}{\Delta\tau} + \frac{(\sigma_x x_i)^2}{h_{i-1}^x h_i^x} - r x_i \frac{h_i^x - h_{i-1}^x}{h_{i-1}^x h_i^x} + \frac{r}{3}, \quad \gamma_i = -\frac{(\sigma_x x_i)^2}{h_i^x \left(h_{i-1}^x + h_i^x\right)} - r x_i \frac{h_{i-1}^x}{h_i^x \left(h_{i-1}^x + h_i^x\right)}, \\
f_{ijk}^n &= \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy} u_{ijk}^n + \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz} u_{ijk}^n + \frac{1}{3}\sigma_x\sigma_z\rho_{zx}x_i z_k D_{zx} u_{ijk}^n - \frac{1}{\Delta\tau}u_{ijk}^n
\end{aligned} \tag{2.26}$$

for fixed indices j and k; and $f_{1:N_x,jk}^n = [f_{1jk}^n \ f_{2jk}^n \ \dots \ f_{N_xjk}^n]$.
The tridiagonal systems pertaining to the first equation on performing the operator splitting method can be given by

$$A_x u_{1:N_x,jk}^{n+\frac{1}{3}} = f_{1:N_x,jk}^n \tag{2.27}$$

with zero Dirichlet (i.e., $u_{0jk}^{n+\frac{1}{3}} = 0$ at $x = 0$) and the one-sided difference for the homogeneous Neumann (i.e., $u_{N_x+1,jk}^{n+\frac{1}{3}} = u_{N_x,jk}^{n+\frac{1}{3}}$ at $x = L$) boundary condition which generates the following tridiagonal matrix

$$A_x = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{N_x-1} & \gamma_{N_x-1} \\ 0 & 0 & 0 & \dots & \alpha_{N_x} & \beta_{N_x} \end{bmatrix}. \tag{2.28}$$

The second equation can be written in matrix form in the following way.

$$\alpha_j u_{i,j-1,k}^{n+\frac{2}{3}} + \beta_j u_{ijk}^{n+\frac{2}{3}} + \gamma_j u_{i,j+1,k}^{n+\frac{2}{3}} = f_{ijk}^{n+\frac{1}{3}},$$

where

$$\begin{aligned}
\alpha_j &= -\frac{(\sigma_y y_j)^2}{h_{j-1}^y(h_{j-1}^y + h_j^y)} + r y_j \frac{h_j^y}{h_{j-1}^y(h_{j-1}^y + h_j^y)}, \\
\beta_j &= \frac{1}{\Delta\tau} + \frac{(\sigma_y y_j)^2}{h_{j-1}^y h_j^y} - r y_j \frac{h_j^y - h_{j-1}^y}{h_{j-1}^y h_j^y} + \frac{r}{3}, \quad \gamma_j = -\frac{(\sigma_y y_j)^2}{h_j^y(h_{j-1}^y + h_j^y)} - r y_j \frac{h_{j-1}^y}{h_j^y(h_{j-1}^y + h_j^y)}, \\
f_{ijk}^{n+\frac{1}{3}} &= \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy} u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz} u_{ijk}^{n+\frac{1}{3}} + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_k x_i D_{zx} u_{ijk}^{n+\frac{1}{3}} - \frac{1}{\Delta\tau} u_{ijk}^{n+\frac{1}{3}}.
\end{aligned}$$

Figure 2.1: Matrix form of second equation in Eqn 2.14 for fixed indices i and k; and $f_{i,1:N_y,k}^n = [f_{i1k}^n \; f_{i2k}^n \; \dots \; f_{iN_yk}^n]$

The tridiagonal systems pertaining to the second equation on performing the operator splitting method can be given by

$$A_y u_{i,1:N_y,k}^{n+\frac{2}{3}} = f_{i,1:N_y,k}^{n+\frac{1}{3}} \tag{2.29}$$

with zero Dirichlet (i.e., $u_{i0k}^{n+\frac{2}{3}} = 0$ at $y = 0$) and the one-sided difference for the homogeneous Neumann (i.e., $u_{i,N_y+1,k}^{n+\frac{2}{3}} = u_{i,N_y,k}^{n+\frac{2}{3}}$ at $y = M$) boundary condition which generates the following tridiagonal matrix

$$A_y = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{N_y-1} & \gamma_{N_y-1} \\ 0 & 0 & 0 & \dots & \alpha_{N_y} & \beta_{N_y} \end{bmatrix}. \tag{2.30}$$

The third equation can be written in matrix form in the following way.

$$\alpha_k u_{ij,k-1}^{n+1} + \beta_k u_{ijk}^{n+1} + \gamma_k u_{ij,k+1}^{n+1} = f_{ijk}^{n+\frac{2}{3}} \tag{2.31}$$

where

$$\alpha_k = -\frac{(\sigma_z z_k)^2}{h^z_{k-1}\left(h^z_{k-1}+h^z_k\right)} + r z_k \frac{h^z_k}{h^z_{k-1}\left(h^z_{k-1}+h^z_k\right)}$$

$$\beta_k = \frac{1}{\Delta\tau} + \frac{(\sigma_z z_k)^2}{h^z_{k-1}h^z_k} - r z_k \frac{h^z_k - h^z_{k-1}}{h^z_{k-1}h^z_k} + \frac{r}{3}, \quad \gamma_k = -\frac{(\sigma_z z_k)^2}{h^z_k\left(h^z_{k-1}+h^z_k\right)} - r z_k \frac{h^z_{k-1}}{h^z_k\left(h^z_{k-1}+h^z_k\right)}'$$

$$f^{n+\frac{2}{3}}_{ijk} = \frac{1}{3}\sigma_x\sigma_y\rho_{xy}x_i y_j D_{xy}u^{n+\frac{2}{3}}_{ijk} + \frac{1}{3}\sigma_y\sigma_z\rho_{yz}y_j z_k D_{yz}u^{n+\frac{2}{3}}_{ijk} + \frac{1}{3}\sigma_z\sigma_x\rho_{zx}z_k x_i D_{zx}u^{n+\frac{2}{3}}_{ijk} - \frac{1}{\Delta\tau}u^{n+\frac{2}{3}}_{ijk} \tag{2.32}$$

for fixed indices i and j; and $f^n_{ij,1:N_z} = [f^n_{ij1}\ f^n_{ij2}\ \dots\ f^n_{ijN_z}]$.
The tridiagonal systems pertaining to the third equation on performing the operator splitting method can be given by

$$A_z u^{n+1}_{ij,1:N_z} = f^{n+\frac{2}{3}}_{ij,1:N_z} \tag{2.33}$$

with zero Dirichlet (i.e., $u^{n+1}_{ij0} = 0$ at $z = 0$) and the one-sided difference for the homogeneous Neumann (i.e., $u^{n+1}_{ij,N_z+1} = u^{n+1}_{ij,N_z}$ at $z = N$) boundary condition which generates the following tridiagonal matrix

$$A_z = \begin{bmatrix} \beta_1 & \gamma_1 & 0 & \dots & 0 & 0 \\ \alpha_2 & \beta_2 & \gamma_2 & \dots & 0 & 0 \\ 0 & \alpha_3 & \beta_3 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \beta_{N_z-1} & \gamma_{N_z-1} \\ 0 & 0 & 0 & \dots & \alpha_{N_z} & \beta_{N_z} \end{bmatrix}. \tag{2.34}$$

To solve the tridiagonal systems found in Equations 2.27, 2.29 and 2.33, we used the backslash operator in MATLAB.

Chapter 3

Numerical Experiments and Analysis

For performing numerical experiments, three types of options are considered:

1. one-asset cash or nothing
2. two-asset cash or nothing
3. three-asset cash or nothing

The experiments are done on following 3 types of non-uniform grids:

1. $\Omega_1 = [0, 1.5, 5.5, 9.5, \ldots, 77.5, 80.5, 83.5, \ldots, 122.5, 126.5, 130.5, \ldots, 298.5, 300]$,
2. $\Omega_2 = [0, 1, 4, 7, \ldots, 79, 81, 83, \ldots, 121, 124, 127, \ldots, 298, 300]$
3. $\Omega_3 = [0, 0.5, 2.5, 4.5, \ldots, 80.5, 81.5, 82.5, \ldots, 120.5, 122.5, 124.5, \ldots, 298.5, 300]$

The error analysis of the numerical solutions is carried out using the respective closed-form solutions and corresponding plots are drawn.

The relative error is defined as:

$$e_{L^2} = \sqrt{\frac{1}{\aleph} \sum_i \sum_j \sum_k \left(\frac{u_{ijk}^{N_\tau} - u\left(x_i, y_j, z_k, T\right)}{u\left(x_i, y_j, z_k, T\right)} \right)^2} \tag{3.1}$$

$$\text{for } (x_i, y_j, z_k) \in (80, 120) \times (80, 120) \times (80, 120), (39)$$

where $\aleph$, u_{ijk}, N_τ and $u(x_i, y_j, z_k, T)$ are the number of grid points within the interval $(80, 120) \times (80, 120) \times (80, 120)$, the numerical and the closed-form solutions of the cash-or-nothing option, respectively.

3.1 One-asset Cash or Nothing Option

The payoff is :

$$u_T(x) = \begin{cases} c, & \text{if } x \geq K \\ 0, & \text{otherwise} \end{cases}$$

The closed-form formula for the cash-or-nothing option is as follows:

$$u(x,\tau) = ce^{-r\tau}N(d), d = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$$

where the cumulative normal distribution $N(d) = (1/\sqrt{2\pi})\int_{-\infty}^{d} e^{-0.5\bar{\zeta}^2} d\xi$.

3.1.1 Derivation of Closed Form Solution

The Payoff is given by:

$$u_T(x) = \begin{cases} c, & \text{if } x \geq K \\ 0, & \text{otherwise} \end{cases}$$

Here 'x' follows a Geometric Brownian Motion i.e

$$x(T) = x(t)e^{\sigma(W(T)-W(t)) \, + \, (r-\frac{\sigma^2}{2})(\tau)}$$

where W(t)$(0 \leq t \leq T)$ is a Brownian Motion and W(t) follows N(0,t) Normal Distribution. Also $x(t)$ is value of the asset at time $T = t$.

Using the above equation the payoff can be rewritten as:

$$u_T(x') = \begin{cases} c, & \text{if } x' \geq ln(K) \\ 0, & \text{otherwise} \end{cases}$$

Here $x' = ln(x)$

Using 'Risk-Neutral Pricing' the price of the option at time T = t can be written as

$$u_t(x') = E[u_T(x')|F_t]$$

where F_t is a filtration w.r.t time.

Using *Independence Lemma* we can say that $'u_T(x')|F_t'$ has $N(ln(x(t)) \ + \ (r - \frac{\sigma^2}{2})t \ , \ \sigma^2 t)$ distribution.

Now, taking $\frac{x' - ln(x(t)) \ + \ (r - \frac{\sigma^2}{2})\tau}{\sigma\sqrt{\tau}}$ as 'y' and using the above results and PDF of 'N(0,1)' Random Variable:

$$u_t(x') = e^{-r\tau} \times \int_{-\infty}^{d} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

Here $d = \frac{\ln\left(\frac{x}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}$
The same technique can be followed to derive closed form solutions for 2 and 3 asset Cash or Nothing Options.

3.1.2 Error Analysis

The error analysis is shown in Table 3.1:

Table 3.1: Numerical solution and errors for one-asset option at x = 100

Grid	Numerical Solution	Relative error
Ω_1	46.579	0.00096356
Ω_2	46.585	0.00049427
Ω_3	46.592	0.00024844

In Table 3.1, the closed form solution at x = 100, that is, u(100, T) = 46.5873.

3.1.3 Plots

The comparison between the numerical solution and the closed form solution using plots is as follows:

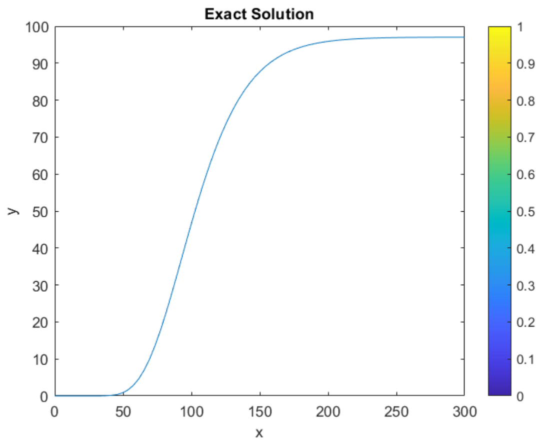

Figure 3.1: Exact Solution

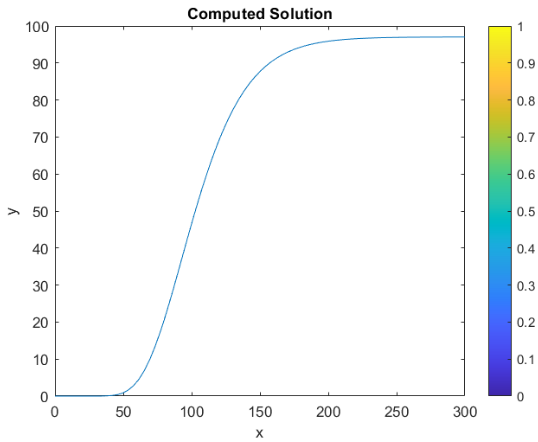

Figure 3.2: Computed Solution

3.2 Two-asset Cash or Nothing Option

The payoff is :

$$u_T(x,y) = \begin{cases} c, & \text{if } x \geq K_1, y \geq K_2 \\ 0, & \text{otherwise} \end{cases}$$

The closed-form formula for the cash-or-nothing option is

$$u(x,y,\tau) = ce^{-r\tau} B\left(d_x, d_y; \rho\right),$$

$$d_x = \frac{\ln\left(\frac{x}{K_1}\right) + \left(r - 0.5\sigma_x^2\right)\tau}{\sigma_x\sqrt{\tau}}, d_y = \frac{\ln\left(\frac{y}{K_2}\right) + \left(r - 0.5\sigma_y^2\right)\tau}{\sigma_y\sqrt{\tau}}$$

where the bivariate cumulative normal distribution function $B(\alpha, \beta; \beta)$[26] is

$$B\left(d_x, d_{y;}\rho\right) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{d_x} \int_{-\infty}^{d_y} e^{-\frac{\xi_1^2 - 2\rho\tilde{\imath}\tilde{\imath}^2 2 + \xi_2^2}{2(1-\rho^2)}} d\tilde{\xi}_2 d\tilde{\xi}_1$$

3.2.1 Error Analysis

The error analysis is shown in Table 3.2:

Table 3.2: Numerical solution and errors for one-asset option at x = y = 100

Grid	Numerical Solution	Relative error
Ω_1 x Ω_1	30.4	0.0013688
Ω_2 x Ω_2	30.424	0.00066143
Ω_3 x Ω_3	30.446	0.00034155

In Table 3.2, the closed form solution at x = y = 100, that is, u(100, 100, T) = 30.4355.

3.2.2 Plots

The comparison between the numerical solution and the closed form solution using plots is as follows:

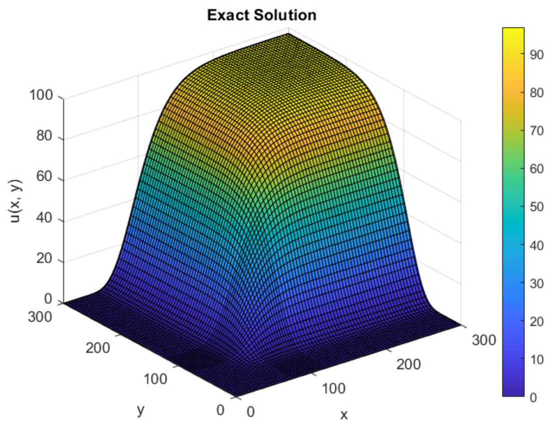

Figure 3.3: Exact Solution

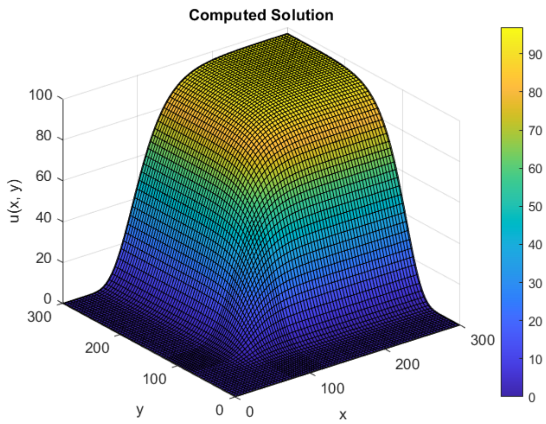

Figure 3.4: Computed Solution

3.3 Three-asset Cash or Nothing Option

The payoff is :

$$u_{\mathrm{T}}(x, y, z)=\begin{cases} c & \text{if } x \geq K_1, y \geq K_2, z \geq K_3 \\ 0, & \text{otherwise} \end{cases}$$

The closed-form formula for the cash-or-nothing option with three underlying assets is as follow:

$$u(x, y, z, \tau)=c e^{-r \tau} M\left(d_x, d_y, d_z ; R\right)$$

where,

$$d_x=\frac{\ln \left(\frac{x}{K_1}\right)+\left(r-0.5 \sigma_x^2\right) \tau}{\sigma_x \sqrt{\tau}}, d_y=\frac{\ln \left(\frac{y}{K_2}\right)+\left(r-0.5 \sigma_y^2\right) \tau}{\sigma_y \sqrt{\tau}}, d_z=\frac{\ln \left(\frac{z}{K_3}\right)+\left(r-0.5 \sigma_z^2\right) \tau}{\sigma_z \sqrt{\tau}}$$

and the trivariate cumulative normal distribution function $M\left(d_x, d_y, d_z ; R\right)$ is defined in by

$$M\left(d_x, d_y, d_z ; R\right)=\frac{1}{\sqrt{|R|(2 \pi)^3}} \int_{-\infty}^{d_x} \int_{-\infty}^{d_y} \int_{-\infty}^{d_z} e^{-0.5 d R^{-1} d^T} d \xi_3 d \xi_2 d \xi_1$$

where $\mathbf{d}=\left(d_X, d_y, d_z\right)$ and a correlation matrix R is

$$R=\left(\begin{array}{ccc} 1 & \rho_{xy} & \rho_{zx} \\ \rho_{xy} & 1 & \rho_{yz} \\ \rho_{zx} & \rho_{yz} & 1 \end{array}\right)$$

3.3.1 Error Analysis

The error analysis is shown in Table 3.3:

Table 3.3: Numerical solution and errors for three-asset option at x = y = z = 100

Grid	Numerical Solution	Relative error
Ω_1 x Ω_1 x Ω_1	22.484	0.0017075
Ω_2 x Ω_2 x Ω_2	22.515	0.0142
Ω_3 x Ω_3 x Ω_3	22.535	0.0058

In Table 3.3, the closed form solution at x = y = z = 100, that is, u(100, 100, 100, T) = 22.5292.

3.3.2 Plots

The comparison between the numerical solution and the closed form solution using plots (taking 2 assets at a time) is as follows:

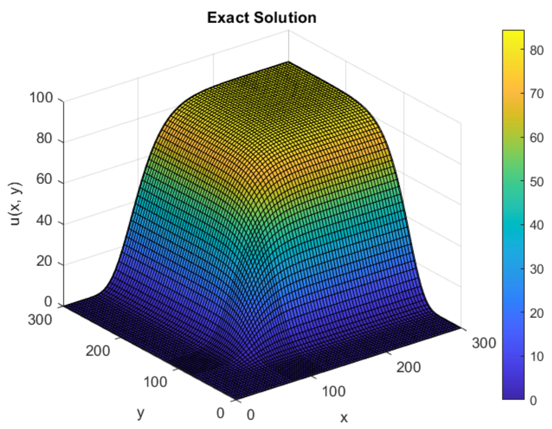

Figure 3.5: Exact Solution (XY plane)

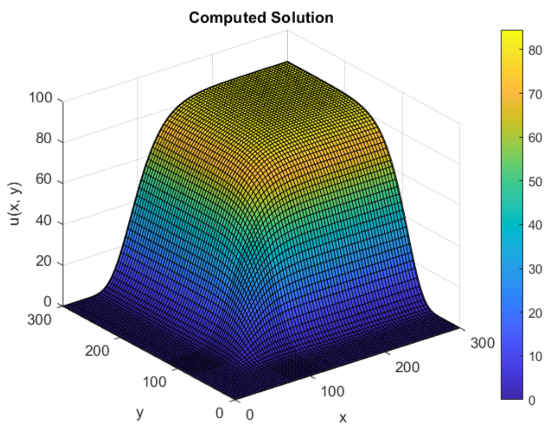

Figure 3.6: Computed Solution (XY plane)

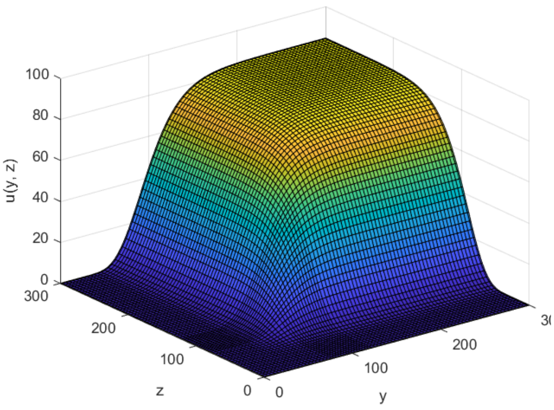

Figure 3.7: Exact Solution
(YZ plane)

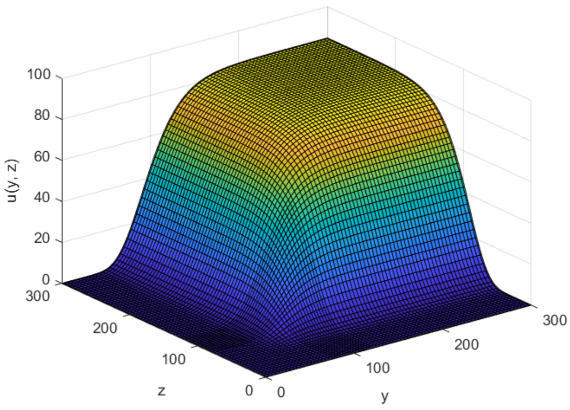

Figure 3.8: Computed Solution
(YZ plane)

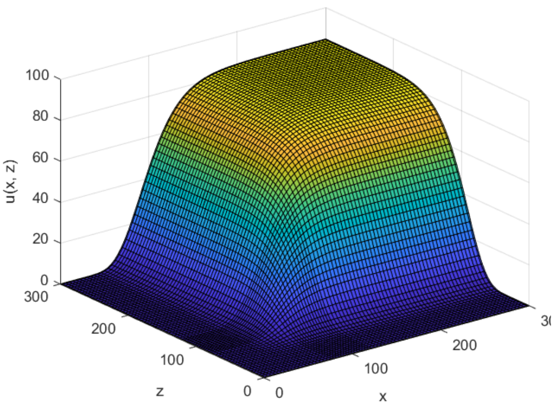

Figure 3.9: Exact Solution
(XZ plane)

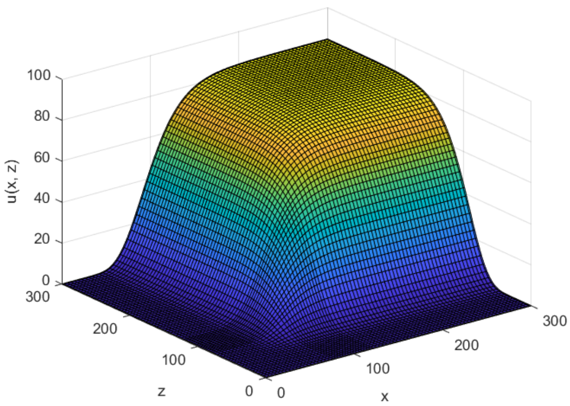

Figure 3.10: Computed Solution
(XZ plane)

Chapter 4

Conclusion

The paper mainly focused on solving multi-dimensional Black-Scholes Equation using operator splitting method. The BS Equations were discretized non-uniformly in space and implicitly in time(i.e backward).

In Numerical Analysis, we performed experiment on characteristic examples like 'Cash-or-Nothing Options'. The computational results were in good agreement to the closed form solutions of the black-scholes equation.

Bibliography

[1] Sangkwon Kim, Darae Jeong, Chaeyoung Lee, and Junseok Kim. Finite difference method for the multi-asset black–scholes equations. *Mathematics*, 8(3), 2020.