Optimal scale-up of HIV treatment programs in resource-limited settings under supply uncertainty

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Problem Definition: HIV clinics in sub-Saharan Africa face an important challenge of allocating scarce and unreliable supply of antiretroviral drugs (ARVs) among new patients (treatment initiation) and patients already on treatment (treatment continuation). The key trade-off underlying this allocation is between the marginal health benefit obtained by initiating an untreated patient on treatment and that obtained by avoiding treatment interruption of a treated patient.

Academic / Practical Relevance: Existing national level policies on ARV allocation, based on socioeconomic and demographic criteria, are qualitative and of limited utility in providing quantitative guidance on scaling up of treatment programs at individual clinics. Moreover, the trade-off involved in clinic level allocation decisions has not been studied in the clinical literature, which focuses either on the incremental value obtained from initiating treatment (over no treatment) or on the value of providing continuous treatment (over interrupted treatment) but not on the difference of the two.

Methodology: We cast the clinic's problem as a stochastic dynamic program, derive the optimal policy structure for some special cases and use it to derive a practically relevant *Two-Period* heuristic. We conduct extensive numerical analysis to compare the performance of this heuristic with *Safety-Stock* heuristic widely used in practice.

Results: Not serving higher value patients (those already on treatment), to avoid interrupting their treatment in the future, might be optimal under some conditions. Performance of the *Two-Period* heuristic (within 5% of the upper bound) is significantly better than that of the *Safety-Stock* heuristic (as much as 20% lower than the upper bound) and is robust to misspecification of key problem parameters, which might be difficult to estimate.

Managerial Implications: Our model can serve as a basis for developing a decision-support tool for clinics to design their ARV treatment program scale-up plans.

Key words: inventory rationing, supply uncertainty, global HIV pandemic, antiretroviral treatment

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1. Introduction

Many organizations strive to attain a balance between expanding their services to new customers and maintaining quality of service for existing customers. This trade-off becomes particularly acute when the organization faces uncertainty in the availability of a key resource. In this paper, we study this trade-off in a specific operational context, that of scaling up HIV treatment programs in sub-Saharan Africa. HIV clinics in this context receive extremely limited and uncertain supply of antiretroviral drugs (ARVs) that needs to be used for initiating untreated patients on treatment and for continuing treatment for patients who have been previously treated. On the one hand, clinics can focus on ensuring continuous treatment to their previously treated patients (Schouten et al. 2011) by being conservative in their scale-up and capping the number of new enrolments. However, this can lead to delay in treatment initiation for untreated patients, which in turn can lead to significant reduction in their quality of life and even death due to disease progression (Doherty et al. 2005, Farnham et al. 2013, Ford et al. 2010). On the other hand, clinics can be aggressive by enrolling many new patients but increase the risk of their treatment interruption in future periods due to drug stockouts (IRINNews.org 2013a,b, VoANews.com 2013b), which in turn can lead to adverse clinical outcomes such as treatment failure (Bartlett 2006, Hamers et al. 2012), drug resistance (WHO et al. 2013, Curran 2004) and increased mortality (El-Sadr et al. 2006), and which can necessitate transitioning patients to a much more expensive second line of therapy (Oyugi et al. 2007, Van Oosterhout et al. 2005).

We model this trade-off embedded in the scale-up of ARV treatment programs using a stochastic dynamic programming framework. We classify HIV patients from a clinic's catchment area into the following categories: (i) patients who are not yet clinically eligible to be initiated on treatment based on national guidelines (*ineligible*), (ii) patients who are clinically eligible but have not yet been initiated on treatment (*eligible and untreated*), (iii) patients who have been initiated on treatment earlier and are still responsive to it (*eligible and treated*), and (iv) patients who had been previously initiated on treatment but are now resistant to it due to previous treatment interruptions (*resistant*). Each period, based on the inventory of ARVs and the number of patients in each of the above four categories, the clinic decides the number of *treated* patients and *untreated* patients to treat and sets aside any remaining inventory to be carried over to the next period. Then the following transitions occur:

- New infections are added to the category of *ineligible* patients.

- A fraction of *ineligible* patients move to the category of eligible patients due to natural progression of their disease.
- Some of the *untreated* patients move to the category of *treated* patients upon enrollment.
- A fraction of *treated* patients whose treatment is interrupted in this period move to the category of resistant patients.

Following these transitions, a fraction of patients in each category die and the surviving patients obtain a reward, which is equal to the per-period quality of life (QOL) utility associated with their category. At the end of the period, the clinic receives a shipment of ARVs of an uncertain quantity. The objective of the clinic is to maximize the total expected quality-adjusted life years (QALYs) of its patients over the planning horizon subject to availability of drugs and the number of patients in different segments.

The resulting decision problem is related to but substantially different from the traditional models of multi-product and/or multi-customer inventory rationing (Evans 1967, DeCroix and Arreola-Risa 1998) due to non-stationarity of rewards and endogenous movement of patients across segments depending on whether they received the product in the current period. Consequently, unlike those papers, static prioritization of one (high value) segment over the other (low value) is not optimal for our problem. In contrast, the optimal policy is state-dependent and is characterized by thresholds and switching curves that define regions in the state space such that allocation decisions are different across these regions. However, analytical difficulties preclude complete characterization of the optimal policy for the most general version of the problem and make it an unlikely candidate for implementation. Hence, we on few special cases to understand the underlying trade-offs. These include the two-period problem as well as two extreme cases of the general multi-period problem: (i) none of the patients with treatment interruption develop drug resistance, and (ii) when all patients with treatment interruption develop drug resistance.

Insights from these special cases indicate that the optimal policy for the general formulation, under some conditions, prescribes denying treatment to *treated* patients (i.e., patients already on treatment) to avoid future treatment interruptions. This structure presents an important dilemma pertaining to the social responsibility of the ARV program and is unlikely to be tenable in practice. Hence, based on the structure of the optimal policy for the three special cases described above, we develop a *Two-Period* heuristic which has a far simpler structure that makes it amenable to implementation. Extensive computational experiments show that the average optimality gap of this heuristic is less than 4% over a wide range of parameter values. In contrast, a *Safety-Stock* heuristic based on current practice (WHO et al. 2013, Schouten et al. 2011) yields average optimality gaps of around 20%. Beyond better overall performance, the *Two-Period* heuristic is also much more robust to misspecification of the parameter values and is no more difficult to implement than the *Safety-Stock* heuristic.

The remainder of the paper is organized as follows. In Section 2, we describe the operational challenges of scaling up ART programs in resource-constrained settings in greater detail. Section 3 outlines our contribution to various streams of related literature. Section 4 provides the model formulation and Section 5 presents a partial characterization of the optimal policy and its properties. Section 6 includes a formal description of the two heuristics which are either used in practice or have practical appeal and a procedure for obtaining the upper bound. Section 7 contains extensive numerical illustrations to compare the performance of these heuristics with the optimal policy. Finally, Section 8 provides concluding remarks.

2. Operational Challenges in HIV Drug Supply

Benefits of antiretroviral therapy (ART) in terms of reduced mortality, morbidity and hospitalization are well established (Ford et al. 2010, Palella et al. 1998, Walensky et al. 2006). Yet the coverage of ART in sub-Saharan Africa, the epicentre of the epidemic, is still abysmally low due to insufficient funding from international donors after the global financial crisis in 2009 (Leach-Kemon et al. 2012, Serieux et al. 2012). To ensure that limited funds are used most effectively, WHO guidelines stipulate strict eligibility criteria such that only individuals with severe or advanced HIV clinical disease (defined as WHO clinical stage 3 or 4 and individuals with CD4 count ≤ 350 cells/mm^3) can be initiated on ART (WHO et al. 2013). Many countries further restrict eligibility based on clinical, demographic, and socioeconomic factors (Bennett and Chanfreau 2005, Rosen et al. 2005a). However, of the 21.2 million patients considered eligible even by these stringent criteria, only 7.5 million were on treatment in 2013 (UNAIDS 2013).

Beyond insufficient funding, logistical constraints further add to the inefficiency in the ARV supply chain. Distribution systems for ARVs in resource-constrained countries consist of central medical depots, typically located in the national capital, from where the drugs are *pushed* to the sites of health care delivery (WHO 2005, 2003, Harries et al. 2007). Inadequate inventory management skills at the clinics make it very difficult to implement a *pull* system, where clinics place orders for drugs with the central medical depots (JSI 2006, WHO 2003). Thus, due to weak physical infrastructure, poor supply chain management and lack of adequate information and transport systems in the supply chain (Curran 2004, Schouten et al. 2011, Bateman 2013, Georgeu et al. 2012), the supply actually received at an HIV clinic is highly variable. This aggregate shortage and variability of supply, when combined with sub-optimal allocation policies, leads to periodic stock-outs of ARVs (VoANews.com 2013a, Ekong et al. 2004, IRINNews.org 2013b, Wangu and Osuga 2014, Windisch et al. 2011).

A common response to supply uncertainty in practice is to maintain a safety stock of ARVs equivalent to several months of consumption by the *treated* patients and imposing a cap on the number of new patients that can be enrolled in every period (AllAfrica.com 2013, Boateng and Yameogo 2013, Schouten et al. 2011). Yet the effectiveness of these rules of thumb has not been rigorously studied and there is an acknowledgment among practitioners that more formal models are needed to understand the underlying trade-offs (Daniel 2006).

3. Literature Review

The primary contribution of this paper is to provide quantitative insights on how HIV clinics in resource-limited settings should scale up their treatment programs when faced with uncertain and limited drug supply. A secondary contribution, in doing so, is to extend models of multi-item inventory management by incorporating endogenous customer dynamics between different segments and supply uncertainty. We discuss these contributions in the context of three relevant streams of literature related to our paper.

3.1. HIV Treatment Rationing

Early literature on rationing of ARVs (Rosen et al. 2005a, Bennett and Chanfreau 2005, Macklin 2004, McGough et al. 2005, Sharif and Noroozi 2010) is dominated by discussion of prioritization schemes based on socioeconomic and demographic variables (i.e., “which” new patients to enroll) so as to meet ethical criteria such as equity and fairness. Rosen et al. (2005b) extends these to include clinical effectiveness, implementation feasibility, cost, economic efficiency, social equity and provides a qualitative evaluation of various national policies.

However, this discussion has limited utility in making operational decisions on how to scale up treatment programs at the level of individual clinics because of a lack of quantitative framework. Moreover, it adopts a static perspective, i.e., the rationing or prioritization decision is treated as a one-time decision and focuses only on the aggregate shortage of drugs but ignores the uncertainty in the supply of ARVs received at the health facilities.

We address these limitations in a quantitative model which captures the trade-off between initiating new patients on treatment and ensuring treatment continuation of patients already on treatment. This trade-off has not been explored in the clinical literature, which either shows that initiating treatment is better than no treatment through cost-effectiveness studies (Palella et al. 1998, Badri et al. 2006, Cleary et al. 2006) or that continuous treatment is better than interrupted treatment through studies on structured treatment interruptions and impact of treatment adherence (Paterson et al. 2000, Danel et al. 2006, Lawrence et al. 2003).

3.2. Global Health Supply Chains

Recent work on global health supply chains represents an important part of the emerging literature on socially responsible operations. Kraiselburd and Yadav (2013) highlight the lack of coordination across multiple donors and the recent reduction in budgetary allocations toward humanitarian aid after the financial crisis as the major challenges at the macro level for these supply chains. Taylor and Xiao (2014) compare the effectiveness of purchase vs. sales subsidy in private distribution channels to improve the availability of products from the perspective of donors. Gallien et al. (2014) build a quantitative model to predict the joint impact of procurement and funding delays in global health programs on drug availability and stock-outs at the country level. Natarajan and Swaminathan (2014) focus on managing inventory levels of a nutritional product and determining optimal procurement policy by taking into account the uncertainty associated with funding amount and schedule.

We focus on the tactical decisions with the clinic as the decision-making unit and study the impact of operational decisions directly on health outcomes. Similar to our approach, McCoy and Johnson (2014) also consider a multi-period problem faced by a clinic in deciding the number of new patients to enroll in each period so as to minimize the number of infected patients over the problem horizon. However, the main trade-off in their model is between reduced disease transmission due to initiation of new patients on treatment and reduced treatment adherence due to every subsequently enrolled patient living farther from the clinic. They do not consider supply uncertainty and treatment interruption induced by stock-outs; the main driver of treatment interruption and drug resistance in their model is reduced adherence of patients to follow-up visits if they live farther from the clinic. Another recent paper that is related to ours is Natarajan and Swaminathan (2017), which analyzes the problem of allocating limited inventory of drugs among patients with different health states. The main point of distinction here is that we model a chronic disease where the patients who receive treatment continue to return in subsequent periods and thus compete for drugs along with new patients whereas Natarajan and Swaminathan (2017) model an acute condition where the patients in the less severe health state are cured after treatment and exit the system. This leads to significant differences in the state transition equations and consequently the structure of the optimal policy. A secondary point of distinction is that Natarajan and Swaminathan (2017) focus on the impact of the timing and the variability in donor funding that is used to procure the drugs whereas our focus is on designing clinic-level program scale-up policies in the presence of an uncertain exogenous supply of drugs.

3.3. Inventory Management

Our model is related to the broader literature on periodic review multi-item inventory models with resource constraint (Evans 1967, DeCroix and Arreola-Risa 1998). These models typically include an exogenous and deterministic constraint on the procurement budget in each period and the underage and overage costs of multiple products are also exogenously specified and stationary, which allows ranking them in the order of importance. For continuous time formulations, see Ha (1997) and de Véricourt et al. (2003).

Our model differs from these in two key aspects. On the demand side, the customer segments are inherently related as customers from one pool (*untreated*) are moved permanently to another pool (*treated*) as a result of the treatment decisions. Such dynamics in the product space, corresponding to cross-product substitution, have not been studied previously. Combined with non-stationary cost of not serving each of the two segments, these dynamics yield an interesting result. In some cases, the high-value patient segment is not fully served despite the availability of adequate inventory. This can be interpreted as inter-temporal rationing between patients of the same segment, a feature that is absent from the existing models. In a recent paper, Deng et al. (2014) also consider an inventory model where the demand in future periods depends, among other things, on past service experience of customers. However, in their models, customers from the two segments cannot be distinguished from each other and the focus is on deciding the optimal inventory levels in each period and not the allocation of available inventory among the two segments. On the supply side, the key point of departure from the literature is the uncertainty in the supply and the absence of an ordering decision for the clinics.

4. Model Preliminaries

In this section, we present a formal model for the decision problem faced by a resource constrained individual clinic. To reflect the finite planning horizon of such clinics, we consider N discrete periods, where $n = 1$ denotes the last period and $n = N$ denotes the first period. The clinic receives an exogenous uncertain supply of first-line ARVs every period and needs to decide how to allocate the available supply of drugs between *untreated* and *treated* patients so as to maximize the expected quality-adjusted survival of its patients. We first describe various building blocks of our model, and then combine them to formulate the clinic's dynamic decision problem.

4.1. Patient Dynamics

We divide the patient population in each period into four broad segments depending on their health status, subsequent clinical eligibility for treatment initiation and sensitivity to first-line ARVs.

Let $y_{n,i}$ be the number of patients who are *ineligible* for treatment according to the national treatment eligibility guidelines at the beginning of period n. Let $y_{n,u}$ denote the number of previously untreated eligible patients, henceforth referred to as *untreated* patients. Similarly, let $y_{n,t}$ denote the number of previously treated patients, who are still responsive to first-line treatment and are henceforth referred to as *treated* patients. Finally, let $y_{n,r}$ be the number of previously treated patients who have failed the first-line of therapy and are henceforth referred to as *resistant* patients.

Next, we describe the transitions between these different segments, which are represented schematically in Figure 1. Note that the demand for first-line ARVs, which is the focus of our analysis, consists of *treated* and *untreated* patients, $y_{n,t}$ and $y_{n,u}$. Let $x_{n,t}$ and $x_{n,u}$ denote the number of *treated* and *untreated* patients that the clinic decides to treat in period n.

The pool of *ineligible* patients increases by a factor α_i due to diagnosis of new infections and decreases by a factor α_e due to patients becoming eligible for treatment due to clinical disease progression. A fraction β_i of the *ineligible* patients in period n survives to the next period $n-1$. Then, the number of *ineligible* patients at the beginning of period $n-1$ as a result of these transitions is given by:

$$y_{n-1,i} = \beta_i \left(y_{n,i} \left(1 - \alpha_e + \alpha_i \right) \right) \tag{1}$$

The pool of *untreated* patients increases by $\alpha_e y_{n,i}$ as *ineligible* patients become eligible due to disease progression and reduces by $x_{n,u}$ due to initiation of patients on treatment. A fraction β_u of the *untreated* patients survives to the next period. Thus, the number of eligible *untreated* patients at the beginning of period $n-1$ is given by:

$$y_{n-1,u} = \beta_u \left(y_{n,u} - x_{n,u} + \alpha_e y_{n,i} \right) \tag{2}$$

At the beginning of period n, the clinic decides to treat $x_{n,t}$ of the $y_{n,t}$ previously treated patients and enroll $x_{n,u}$ new patients. Of the remaining $(y_{n,t} - x_{n,t})$ patients, who were treated previously but remain untreated in this period, a fraction γ_t develop resistance to first-line treatment, which we refer to as the coefficient of resistance. Let $\bar{y}_{n-1,t}$ denote the number of previously treated patients who receive treatment and $\underline{y}_{n-1,t}$ denote the number of patients who do not receive the treatment in period n but are still sensitive to treatment. Then, the number of *treated* patients at the beginning of period $n-1$ is given by:

$$\begin{aligned} y_{n-1,t} &= \bar{y}_{n-1,t} + \underline{y}_{n-1,t} \\ &= \beta_t \left(x_{n,t} + x_{n,u} \right) + \beta_t \left(1 - \gamma_t \right) \left(y_{n,t} - x_{n,t} \right) \\ &= \beta_t \left(y_{n,t} - \gamma_t \left(y_{n,t} - x_{n,t} \right) + x_{n,u} \right) \end{aligned} \tag{3}$$

where β_t denotes the survival rate of *treated* patients. We assume that $\beta_t > \beta_u$ as treatment initiation is expected to improve the survival rate of patients.

On the flip side, the pool of resistant patients increases by $\gamma_t \left(y_{n,t} - x_{n,t} \right)$ due to treatment interruption of some of the *treated* patients. Thus, the number of resistant patients at the beginning of period $n-1$ is given by:

$$y_{n-1,r} = \beta_r \left(y_{n,r} + \gamma_t \left(y_{n,t} - x_{n,t} \right) \right), \tag{4}$$

where the survival rate of resistant patients is given by β_r. Again, we assume that $\beta_r < \beta_t$ because resistance to first-line treatment is expected to reduce the survival rate of patients.

Thus, the system of equations (1)–(4) represent the patient dynamics in the clinic's decision problem.

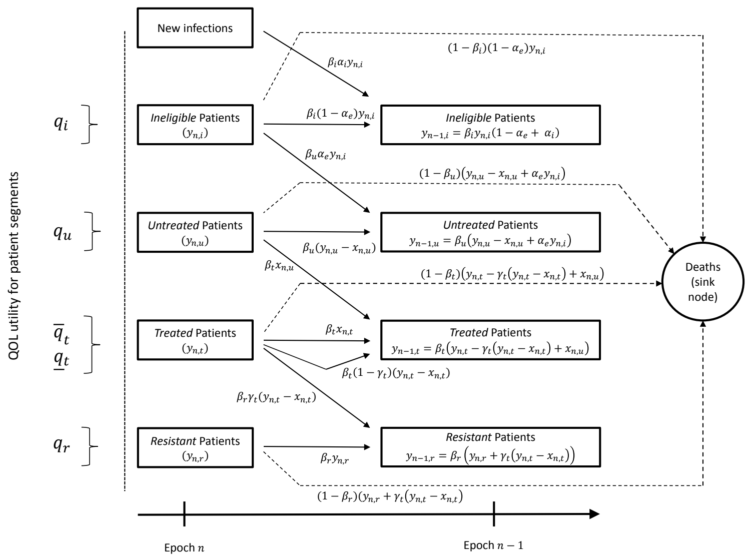


Figure 1 **Flow of HIV patients through various stages of disease progression and treatment**

Remark 1. *It is plausible that the development of resistance depends not only on whether the patients' treatment is interrupted in the current period but also on the duration and frequency of their treatment interruption in the past. However, explicitly modelling these dynamics will require dividing the segment of current patients further into subsegment of patients based on the number of doses received in the previous periods thereby resulting in a significantly expanded state space and loss of analytical tractability. Viewed from this angle, our model parameter γ_t can be considered as the effect of treatment interruption averaged over these underlying subsegments. Thus, the patient dynamics specified above implicitly assume that the composition of the* treated *patient pool (in terms of number of periods over which their treatment is interrupted in the past) does not change significantly over the problem horizon. This assumption is likely to be reasonable in the long run as patients continuously enter and leave the pool of treated patients.*

4.2. Inventory Dynamics

Let w_n denote the opening stock of drugs at the beginning of period n, where one unit of drug corresponds to one period of treatment for one patient. Of these, the clinic uses $x_{n,u}$

doses to initiate *untreated* patients on treatment and $x_{n,t}$ doses to treat *treated* patients. To reflect the practical reality that clinics often do not have control over how many doses they receive, we use z_{n-1} to denote the exogenous random supply that the clinic receives at the end of period n. Thus, the amount of inventory at the beginning of period $n-1$ is given by:

$$w_{n-1} = w_n - x_{n,t} - x_{n,u} + z_{n-1} = I_{n-1} + z_{n-1}, \tag{5}$$

where I_n denotes the inventory of drugs after the allocation but before the receipt of supply in period n. We assume that z_n are independent, but not necessarily identical, random variables with cumulative distribution $F_n(\cdot)$ and support on $[0, z_n^U]$. Given the sufficiently long shelf lives of first-line ARVs, we do not explicitly model the perishability of drugs.

4.3. Reward Structure

The objective of the clinic administration is to maximize the total discounted quality-adjusted life years (QALYs) of the entire patient population (treated and untreated) in the catchment area. It is calculated as the sum of quality of life (QOL) utilities of all patients over the planning horizon, discounted to account for the fact that an additional year of life today is worth more than that in the future (Vergel and Sculpher 2008, Whitehead and Ali 2010). This objective function is routinely used in cost-effectiveness studies (Brennan et al. 2006) and resource allocation problems (Zenios et al. 2000, Richter et al. 1999, Brandeau et al. 2003, Deo et al. 2013).

Let q_i and q_u denote the QOL utility for *ineligible* and *untreated* patients. For the pool of *treated* patients, we assume that the patients who continue to receive treatment in that period (given by $\bar{y}_{n-1,t}$ in period n) enjoy a QOL utility of $\bar{q}_t$ while the patients whose treatment has been interrupted in that period but are still sensitive to treatment (given by $\underline{y}_{n-1,t}$ in period n) get a QOL utility of $\underline{q}_t < \bar{q}_t$. Further, the patients who do not receive treatment and develop resistance to first-line treatment move to the pool of resistant patients and receive a QOL utility of q_r. QOL utilities associated with each of the patient segments are indicated on the left-hand side of Figure 1.

Then the reward collected by the clinic at the end of period n is given by:

$$h_n(\mathbf{y}_{n-1}) = \bar{q}_t \bar{y}_{n-1,t} + \underline{q}_t \underline{y}_{n-1,t} + q_u y_{n-1,u} + q_r y_{n-1,r} + q_i y_{n-1,i}$$

where $\mathbf{y}_{n-1} \triangleq [\bar{y}_{n-1,t}\ \underline{y}_{n-1,t}\ y_{n-1,u}\ y_{n-1,r}\ y_{n-1,i}]^t$. Substituting the expressions for each of the variables in terms of the state variables and decisions from (1) to (4) and similarly defining $\mathbf{x}_n \triangleq [x_{n,t}\ x_{n,u}]^t$, we obtain:

$$h_n(\mathbf{x}_n, \mathbf{y}_n) = (\bar{q}_t\beta_t - q_u\beta_u)\,x_{n,u} + \left((\bar{q}_t - \underline{q}_t(1-\gamma_t))\beta_t - \gamma_t q_r\beta_r\right)x_{n,t} + y_{n,u}(q_u\beta_u) + y_{n,r}(q_r\beta_r)$$
$$+ y_{n,t}\left(\underline{q}_t(1-\gamma_t)\beta_t + q_r\gamma_t\beta_r\right) + y_{n,i}(q_i\beta_i(1-\alpha_e+\alpha_i) + q_u\alpha_e\beta_u) \tag{6}$$

4.4. Model Formulation

Using the above components, now we can state the decision problem of the clinic in period n as follows:

$$\max_{x_{n,t}\geq 0, x_{n,u}\geq 0} E\left[\sum_{n=1}^{N}\delta^{N-n}h_n(\mathbf{x}_n,\mathbf{y}_n)\right] \tag{7}$$
$$\text{s.t.}(1),(2),(3),(4) \text{ and } (5)$$
$$x_{n,t} + x_{n,u} \leq w_n \tag{8}$$
$$x_{n,t} \leq y_{n,t} \tag{9}$$
$$x_{n,u} \leq y_{n,u} \tag{10}$$

Equations (1)–(5) are the patient and inventory dynamics described earlier. Constraint (8) states that the total number of treatments delivered in period n is limited by the available inventory. Constraint (9) and (10) state that the number of *treated* and *untreated* patients treated cannot be more than the total number of *treated* and *untreated* patients in that period respectively.

Next, we note from the above formulation that the decisions $\mathbf{x}_n$ do not depend on $y_{n,i}$ and $y_{n,r}$. Hence, we use the recursive equations in (1) and (4) to derive expressions for these state variables in terms of the initial conditions in period N and subsequent treatment decisions. Further, to reflect the extreme resource-constrained nature of our application setting, we assume that number of *untreated* patients outstrips the available supply of drugs throughout the problem horizon for all feasible allocation policies, i.e., $y_{n,u} > w_n\ \forall n$. This allows us to reformulate an equivalent dynamic program with reduced state space, which is formalized in Proposition 1 (ii) below.

PROPOSITION 1. ***[Problem Reformulation]*** *(i) The equations for state variables $y_{n,r}$, $y_{n,i}$ and $y_{n,u}$ $\forall$ n are as follows:*

$$y_{n,r} = y_{N,r}\beta_r^{N-n} + \gamma_t \sum_{j=1}^{N-n} \left(y_{n+j,t} - x_{n+j,t}\right)\left(\beta_r\right)^j \tag{11}$$

$$y_{n,i} = y_{N,i}\left(\beta_i\left(1-\alpha_e+\alpha_i\right)\right)^{N-n} \tag{12}$$

$$y_{n,u} = \beta_u^{N-n} y_{N,u} - \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u} + \beta_i\alpha_e \sum_{j=1}^{N-n} \beta_u^{j-1} y_{n+j,i} \tag{13}$$

(ii) If $y_{n,u} > w_n$ $\forall n$, then the decision problem as stated in (7) *can be equivalently reformulated as:*

$$V_n\left(y_{n,t}, w_n\right) = \max_{x_{n,t}\geq 0, x_{n,u}\geq 0} \left\{\hat{h}_n\left(\mathbf{x}_n, y_{n,t}\right) + \delta\mathbb{E}_z\left[V_{n-1}\left(y_{n-1,t}, w_{n-1}\right)\right]\right\} \tag{14}$$

$$s.t.\ (3), (5)$$

$$x_{n,t} + x_{n,u} \leq w_n$$

$$x_{n,t} \leq y_{n,t}$$

$$and\ V_0(y_{0,t}, w_0) = 0$$

$$\text{where} \qquad \hat{h}_n\left(\mathbf{x}_n, y_{n,t}\right) = \Delta_{n,t}x_{n,t} + \Delta_{n,u}x_{n,u} + \tilde{\Delta}_{n,t}y_{n,t} \tag{15}$$

$$\Delta_{1,u} = \left(\bar{q}_t\beta_t - q_u\beta_u\right) \tag{16}$$

$$\Delta_{1,t} = \left(\bar{q}_t - \underline{q}_t(1-\gamma_t)\right)\beta_t - \gamma_t q_r\beta_r \tag{17}$$

$$\Delta_{n,u} = \begin{cases} \Delta_{1,u} - q_u\beta_u \sum_{j=1}^{n-1} \left(\delta\beta_u\right)^j & n \geq 2 \\ \left(\bar{q}_t\beta_t - q_u\beta_u\right) & n = 1 \end{cases} \tag{18}$$

$$\Delta_{n,t} = \begin{cases} \Delta_{1,t} - \gamma_t q_r\beta_r \sum_{j=1}^{n-1} \left(\delta\beta_r\right)^j & n \geq 2 \\ \left(\bar{q}_t - \underline{q}_t(1-\gamma_t)\right)\beta_t - \gamma_t q_r\beta_r & n = 1 \end{cases} \tag{19}$$

$$\tilde{\Delta}_{n,t} = \bar{q}_t\beta_t - \Delta_{n,t} \qquad n \geq 1 \tag{20}$$

In other words, under the assumption of extreme resource constraint $y_{n,u} > w_n$, any optimal solution to (7) *is an optimal solution to* (14) *and vice versa.*

Note that the immediate marginal social benefits of treating a patient from the two segments in period n, $\Delta_{n,t}$ and $\Delta_{n,u}$, are non-stationary and are functions of the patient transition parameters because of our focus on population-level outcomes in the presence of constrained resources. In contrast, the focus of most of the clinical literature has been on comparing individual marginal benefits of treating *treated* and *untreated* patients ($\Delta_{1,t}$ and $\Delta_{1,u}$) by implicitly ignoring the resource constraint and the effect of current treatment decisions on the pool of patients in the future periods (e.g. Granich et al. 2009).

Further, the single period marginal benefits of treating a patient from the two segments do not have a clear and stable ranking over the problem horizon, unlike existing models of multi-product inventory management under budget constraint (Evans 1967, DeCroix and Arreola-Risa 1998) where these rewards are essentially the single period shortage costs for the two segments. In other words, in our context, a segment that is more important in the current period could become less important in a future period and vice versa–a feature that is absent from those existing models. Since the main underlying driver for this effect is the coefficient of resistance γ_t as seen from (19)–(20), we solve special instances of the formulation (14) corresponding to extreme values of γ_t to obtain further insights.

5. Partial Characterization of the Optimal Policy

A complete characterization of the dynamic program (14) would require distinguishing between many cases corresponding to different relative rankings of the QOL utility parameters $\left(\bar{q}_t, \underline{q}_t, q_u, q_r\right)$. To make the analysis more manageable, we restrict our attention to QOL utilities that are consistent with the clinical definitions of various patient segments and that yield nontrivial inventory allocation decisions. We begin by assuming that the *treated* patients who received the treatment in a given period earn the highest QOL utility among all patient segments who are eligible for HIV treatment ($\bar{q}_t > \underline{q}_t, q_u, q_r$). Further, we assume that $\underline{q}_t > q_r$, i.e. the patients who do not receive the treatment but are sensitive to first-line therapy enjoy a better QOL utility than the patients who have developed resistance to treatment. Finally, some cases are trivial: When $q_u < \underline{q}_t, q_r$ i.e., when *untreated* patients enjoy a lower QOL utility than *treated* patients who do not receive treatment and resistant patients, it is optimal to prioritize *treated* patients in all periods and exhaust the available inventory of drugs, i.e., $x^*_{n,u} = w_n$ and $x^*_{n,t} = 0$ because $y_n > w_n \ \forall \ n$. Hence, to focus on nontrivial and interesting cases, we assume that $\bar{q}_t > \underline{q}_t, q_u > q_r$ throughout our analysis.

5.1. Two-Period Problem

In this section, we analyze a two-period problem, which is the smallest nontrivial problem instance that captures the trade-off between initiating *untreated* patients on treatment now vs. reducing the chance of treatment interruption for *treated* patients later. The structure of the optimal policy depends on the relative values of the QOL utilities of various patient segments and the coefficient of resistance, which is formally stated in Proposition 2. In Section 6.2 below, we use this structure to develop a heuristic for the more general multi-period problem that is computationally tractable and performs well.

PROPOSITION 2. *There exist $0 \le \gamma_1, \gamma_2 \le 1$ (defined in the Appendix such that for period $n = 2$,*

(i) If $\underline{q}_t > q_u$ and $0 \le \gamma_t < \min\{\gamma_1, \gamma_2\}$, the optimal policy is to prioritize untreated patients over treated patients and exhaust the available inventory of drugs, i.e., $x^*_{2,u} = w_2$ and $x^*_{2,t} = 0$.

(ii) If $\underline{q}_t > q_u$ and $0 < \gamma_2 < \gamma_t \le \gamma_1$, the optimal policy is to prioritize treated patients over untreated patients and exhaust the available inventory of drugs, i.e., $x^*_{2,u} = [w_2 - y_{2,t}]^+$ and $x^*_{2,t} = \min\{y_{2,t}, w_2\}$.

(iii) If $\underline{q}_t > q_u$ and $\gamma_1 < \gamma_t \le 1$ or if $\underline{q}_t < q_u$ and $0 \le \gamma_t \le 1$ the optimal policy is to prioritize treated patients over untreated patients and to keep some drugs in inventory (See Table 1 and Figure 2.).

Table 1 Two-period optimal policy structure for conditions state in Proposition 2 (iii)

Region	State Space	Optimal Policy		
		$x^*_{2,u}$	$x^*_{2,t}$	$w_2 - x^*_{2,u} - x^*_{2,t}$
TP_A	$0 \le y_{2,t} \le w_2 < \frac{\theta_{2,u}}{\beta_t}$	$w_2 - y_{2,t}$	$y_{2,t}$	0
TP_B	$0 \le y_{2,t} \le w_2 \cap w_2 \ge \max\{\frac{\theta_{2,u}}{\beta_t}, y_{2,t}(1+\beta_t) - \theta_{2,u}\}$	$\frac{w_2+\theta_{2,u}}{1+\beta_t} - y_{2,t}$	$y_{2,t}$	$\frac{w_2\beta_t - \theta_{2,u}}{1+\beta_t}$
TP_C	$y_{2,t} \ge \frac{\theta_{2,u}}{\beta_t} \cap y_{2,t} \le w_2 \le y_{2,t}(1+\beta_t) - \theta_{2,u}$	0	$y_{2,t}$	$w_2 - y_{2,t}$
TP_D	$0 \le w_2 \le y_{2,t}$	0	w_2	0

Begin by considering the case when $\underline{q}_t > q_u$, i.e., a *treated* patient with interrupted treatment has higher QOL utility than an *untreated* patient. Intuitively, one would think that it must be optimal to prioritize *untreated* patients over *treated* patients under this scenario because the immediate marginal benefit obtained from treating an *untreated* patient is greater than that obtained from treating a *treated* patient who stopped receiving treatment, i.e., $\overline{q}_t - \underline{q}_t < \overline{q}_t - q_u$. However, this comparison does not account for the likelihood of developing

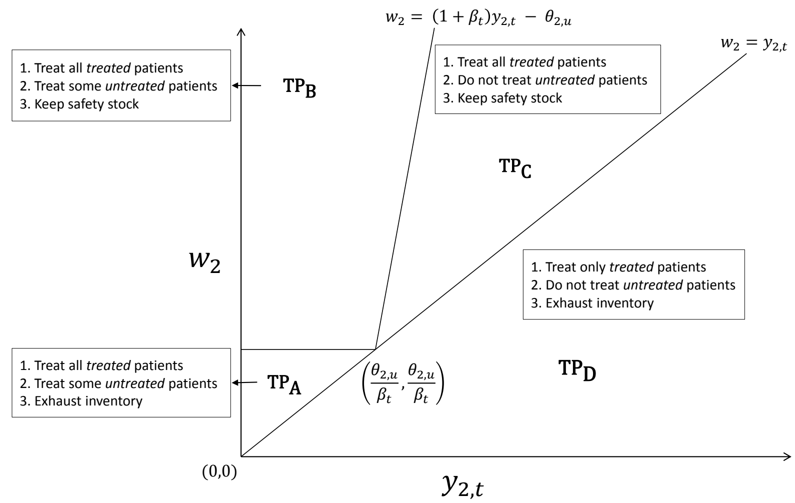


Figure 2 **Characterization of two-period optimal policy for conditions stated in Proposition 2 (iii)**

resistance upon treatment interruption and the QOL for resistant patients q_r. Proposition 2 shows that, after accounting for these effects, prioritizing *untreated* patients is optimal only when the coefficient of resistance is sufficiently low, i.e., when $0 \le \gamma_t < \min\{\gamma_1, \gamma_2\}$ but the prioritization reverses when it is sufficiently large ($\gamma_t > \gamma_1$ or $0 < \gamma_2 < \gamma_t \le \gamma_1$).

Since the pool of *untreated* patients is infinitely large, prioritizing that segment (Proposition 2 (i)) results in all inventory being utilized. Similarly, when prioritizing *treated* patients, it is optimal to exhaust all the inventory when coefficient of resistance is sufficiently low (Proposition 2 (ii)). However, for sufficiently high values of the coefficient of resistance ($\gamma_t > \gamma_1$), Table 1 and Figure 2 show that whether it is optimal to utilize all available inventory depends on the relative magnitudes of available inventory and *treated* patient pool.

Specifically, for any given value of the *treated* patient pool, it is optimal to carry a safety stock by restricting the enrollment of *untreated* patients, if the available inventory is greater than a certain threshold (Region TP_B). In this region, the optimal safety stock level is $\frac{w_2\beta_t - \theta_{2,u}}{1+\beta_t}$ so as to protect against future treatment interruptions. However, when the *treated* patient pool is sufficiently large (Region TP_C), there might not be enough inventory available to hold this level of safety stock. In such cases, it is optimal to not initiate any *untreated* patient on treatment even though there is inventory on hand as drug stockouts in future periods may lead to patients developing resistance.

Finally, when $\underline{q}_t < q_u$, treating a *untreated* patient yields a lower marginal benefit than treating a *treated* patient, irrespective of the value of the coefficient of resistance (γ_t). In such cases, the optimal policy coincides with that for $\underline{q}_t > q_u$ and high coefficient of resistance (Table 1, Figure 2).

5.2. N-Period Problem: Special cases

Next, we turn our attention to the more general problem of $N > 2$ periods. Characterizing the optimal policy for this general problem is analytically challenging. Specifically, the structure of the optimal policy seems to depend critically on the coefficient of resistance. Consequently, we focus on instances of the problem at extreme values of the coefficient of resistance, i.e. $\gamma_t = 0$ and $\gamma_t = 1$, for several reasons. First, these cases are analytically tractable. Moreover, as we show later, the insights from the structure of the optimal policy for these special cases allow us to construct a heuristic that performs well for more realistic problems ($0 < \gamma_t < 1$).

PROPOSITION 3. *Assume treatment interruption never leads to drug resistance, i.e.,* $\gamma_t = 0$. *Then:*

(i) If $\underline{q}_t > q_u$, *then it is optimal to prioritize untreated patients over treated patients in all periods and exhaust the available inventory of drugs i.e.* $x^*_{n,u} = w_n$ *and* $x^*_{n,t} = 0$.

(ii) If $q_u > \underline{q}_t$, *then it is optimal to prioritize treated patients over untreated patients and the optimal policy is given by Table 2, where* $\phi^{-1}_{n,u}(y_{n,t})$ *is a monotonically increasing function such that* $\phi^{-1}_{n,u}(\theta_{n,u}) = \theta_{n,u}$.

Region	State Space	Optimal Policy		
		$x^*_{n,u}$	$x^*_{n,t}$	$w_n - x^*_{n,t} - x^*_{n,u}$
Z_A	$0 \le y_{n,t} \le w_n < \theta_{n,u}$	$w_n - y_{n,t}$	$y_{n,t}$	0
Z_B	$0 \le y_{n,t} \le w_n \cap w_n \ge \max\{\theta_{n,u}, \phi^{-1}_{n,u}(y_{n,t})\}$	$\phi_{n,u}(w_n) - y_{n,t}$	$y_{n,t}$	$w_n - \phi_{n,u}(w_n)$
Z_C	$y_{n,t} \ge \theta_{n,u} \cap y_{n,t} \le w_n \le \phi^{-1}_{n+1,u}(y_{n+1,t})$	0	$y_{n,t}$	$w_n - y_{n,t}$
Z_D	$0 \le w_n \le y_{n,t}$	0	w_n	0

Table 2 N-period optimal policy structure for $\gamma_t = 0$

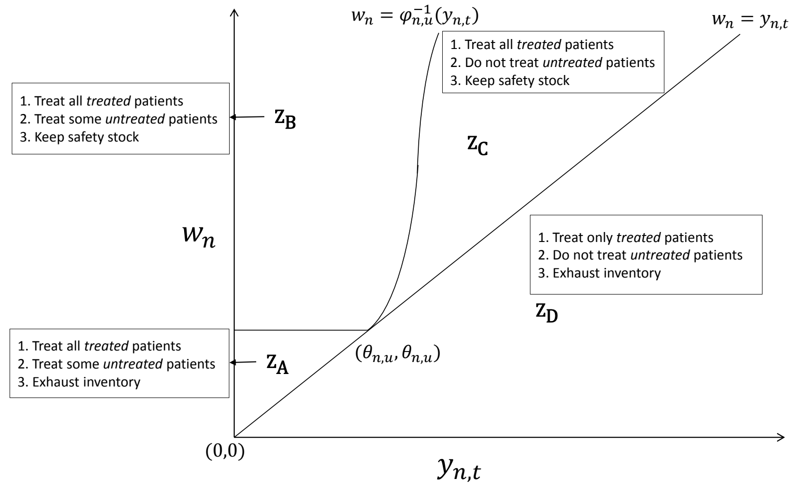


Figure 3 Characterization of N-period optimal policy for $\gamma_t = 0$

It is clear that the structure of the optimal policy for this problem instance is consistent with that for the two-period problem discussed above with some differences as highlighted in Figure 3. First, note that the linear function $w_2 = (1+\beta_t)y_{2,t} - \theta_{2,u}$ that separates region TP_B and TP_C (in Figure 2) is replaced by a nonlinear monotone function $\phi_{n,u}^{-1}(y_{n,t})$ that separates the regions Z_B and Z_C (in Figure 3). Second, the threshold $\theta_{n,u}$ cannot be characterized using a closed-form expression.

However, the key aspects of the optimal policy remain unchanged. In particular, it is optimal to maintain a safety stock of drugs when the inventory is sufficiently high and to restrict enrollment of *untreated* patients (region Z_B) to avoid treatment interruption in subsequent periods. Note that, although in this scenario treatment interruption does not lead to resistance, the patients whose treatment has been interrupted enjoy a lower quality of life than *untreated* patients (as $q_u > \underline{q}_t$) for the rest of the horizon. Further, $\phi_{n,u}(w_n)$ can be interpreted as the optimal *treat up-to level*, i.e., the total number of treatments to be disbursed in period n for both *untreated* and *treated* patients. Equivalently, $(w_n - \phi_{n,u}(w_n))$ represents the optimal amount of safety stock to be carried over to the next period. Finally, when the supply is insufficient to treat all *treated* patients and reach this level of safety stock then it is optimal to not treat any *untreated* patient (Regions Z_C and Z_D).

PROPOSITION 4. *Assume treatment interruption always leads to drug resistance, i.e.,* $\gamma_t = 1$*. Then, the optimal policy is given by Table 3, where* $\phi_{n,t}(\cdot)$ *and* $\phi_{n,u}(\cdot)$ *are mono-*

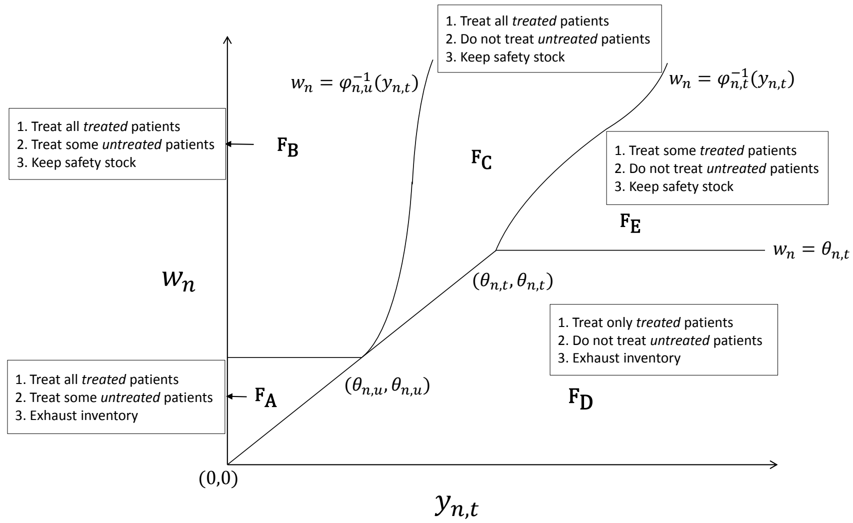


Figure 4 **Characterization of N-period optimal policy for** $\gamma_t = 1$ **and** $N > 2$

tonically increasing functions and pass through the points $(\theta_{n,t}, \theta_{n,t})$ *and* $(\theta_{n,u}, \theta_{n,u})$ *in the* $(y_{n,t}, w_n)$ *state-space.*

Region	State Space	Optimal Policy		
		$x_{n,u}^*$	$x_{n,t}^*$	$w_2 - x_{n,t}^* - x_{n,u}^*$
F_A	$0 \le y_{n,t} \le w_n < \theta_{n,u}$	$w_n - y_{n,t}$	$y_{n,t}$	0
F_B	$0 \le y_{n,t} \le w_n \cap w_n \ge \max\{\theta_{n,u}, \phi_{n,u}^{-1}(y_{n,t})\}$	$\phi_{n,u}(w_n) - y_{n,t}$	$y_{n,t}$	$w_n - \phi_{n,u}(w_n)$
F_C	$0 \le y_{n,t} \le w_n \cap \phi_{n,t}^{-1}(y_{n,t}) \le w_n \le \phi_{n,u}^{-1}(y_{n,t})$	0	$y_{n,t}$	$w_n - y_{n,t}$
F_D	$0 \le w_n \le y_{n,t} \cap w_n < \theta_{n,t}$	0	w_n	0
F_E	$y_{n,t} \ge \theta_{n,t} \cap \theta_{n,t} \le w_n \le \phi_{n,t}^{-1}(y_{n,t})$	0	$\phi_{n,t}(w_n)$	$w_n - \phi_{n,t}(w_n)$

Table 3 **Optimal policy structure for** $\gamma_t = 1$ **and** $N > 2$

The structure of the optimal policy for $\gamma_t = 1$ is very similar to that for $\gamma_t = 0$ but with one point of distinction. The region corresponding to Z_D is split into two subregions F_D and F_E. The structure of the optimal policy in region F_D is similar to that in Z_D but that in F_E is significantly different. Table 3 shows that, even though it is optimal to prioritize *treated* patients over *untreated* patients in this region, it is not optimal to utilize all available inventory for *treated* patients but to leave some *treated* patients untreated. This result is different from those in the traditional inventory rationing literature, where it is always optimal to satisfy the entire demand from the high value segment. The intuition behind this result is that treating all *treated* patients might be beneficial in the current period but increases the pool of *treated* patients and can result in more treatment interruptions and, consequently, more resistant patients, in the event of supply shortage in the subsequent periods. We clarify this intuition using an illustrative sample path in the example below.

Example 1. *Consider a decision problem over 4 periods, i.e., $N=4$. Suppose at the beginning of the problem horizon $y_{4,t}=y_{4,u}=2$ and $I_4=0$. Further, assume that $z_1=z_3=2$ and $z_2=z_4=0$. Now consider a "No Buffer" policy wherein, if possible, available drugs are used to satisfy the entire demand from treated patients. According to this policy, $x_{4,t}=2$ and $x_{4,u}=0$ leading to $w_3=0$. Consequently, $x_{3,t}=x_{3,u}=0$ and both treated patients will turn resistant from $n=3$ due to stockout induced treatment interruption. The next drug supply arrives in $n=2$ so that $w_2=2, y_{2,t}=0, y_{2,u}=2$. Then, clearly $x_{n,u}=2$, i.e., the two untreated patients are initiated on treatment but they turn resistant in the last period because of stock-out induced treatment interruption. The QOL utilities earned by the patients is shown in Table 4. Assuming $\delta\approx 1$, the total QALYs for this "No Buffer" policy are $4\bar{q}_t+8q_r+4q_u$. Now consider an alternative "Buffer" policy that allocates 1 unit of drug to 1 treated patient in all periods and thus keeps a buffer stock of 1 in $n=4,2$. As a result, one of the treated patients turns resistant from $n=3$ and the two untreated patients never receive any treatment. The total QALYs for the "Buffer" policy are $4\bar{q}_t+4q_r+8q_u$. Since $q_u>q_r$, it is clear that the "Buffer" policy outperforms the "No Buffer" policy.*

	"No Buffer" policy				"Buffer" policy			
Period	4	3	2	1	4	3	2	1
Supply z	2	0	2	0	2	0	2	0
QALY - *treated* patient 1	$\bar{q}_t$	q_r	q_r	q_r	$\bar{q}_t$	$\bar{q}_t$	$\bar{q}_t$	$\bar{q}_t$
QALY - *treated* patient 2	$\bar{q}_t$	q_r	q_r	q_r	q_r	q_r	q_r	q_r
QALY - *untreated* patient 3	q_u	q_u	$\bar{q}_t$	q_r	q_u	q_u	q_u	q_u
QALY - *untreated* patient 4	q_u	q_u	$\bar{q}_t$	q_r	q_u	q_u	q_u	q_u

Table 4 Example to illustrate that it need not be optimal to utilize all drugs when prioritizing treated patients for $\gamma_t=1$

It is worth noting that the above policy, which is optimal from the perspective of the overall population quality of life, can present an ethical dilemma for the clinic administration if some *treated* patients are to be denied treatment even when inventory of drugs is available. To help address this dilemma in practice, we propose two heuristics in the next section, which impose the constraint that all *treated* patients should be treated in each period as long as sufficient inventory is available in that period. Administrators can then qualitatively trade-off the loss in optimality by following these heuristics with the ethical difficulty of holding back treatment from *treated* patients in the presence of sufficient inventory.

6. Heuristics and Upper Bound

As discussed above, analytical difficulties prevent us from characterizing the optimal policy structure for the general multi-period problem for intermediate values of the coefficient of resistance ($0 < \gamma_t < 1$). Hence, we develop two heuristic approaches to obtain feasible solutions for the more general problem. The first approach is similar to that taken by practitioners in the field (*Safety-Stock*) and the second is motivated by the structure of the optimal policy derived for the special cases above (*Two-Period*). Finally, we also construct an upper bound on the optimal objective function in (14) to evaluate the performance of these heuristics.

6.1. *Safety-Stock Heuristic*

A common approach recommended in practice to manage the scale-up of ART programs is to maintain a safety stock equivalent to few months of demand to buffer against supply uncertainty and consequent treatment interruptions in the future. An equivalent approach is to designate enrollment caps, i.e., a maximum number of *untreated* patients that can be enrolled in every period. For instance, Schouten et al. (2011) describe how Malawi's Ministry of Health along with UNICEF has used this approach to scale up their ART programme since 2004. We abstract from the implementation details and formalize the heuristic using our modeling framework as follows.

First prioritize the treatment of *treated* patients. Obviously, no *untreated* patients can be enrolled if the supply is already exhausted, i.e., $x^*_{n,u} = 0$ if $y_{n,t} \geq w_n$. However, if the supply is in excess of the *treated* patients, i.e., $y_{n,t} < w_n$, then treat all the *treated* patients first, i.e., $x^*_{n,t} = y_{n,t}$ and enroll *untreated* patients such that $y_{n-1,t} = y_{n,t}$. The safety stock carried to the next period is proportional to the number of *treated* patients in the next period. More formally, $I_{n-1} = Ay_{n-1,t}$, where the proportionality constant A can be interpreted as the number of months of demand that is carried over as safety stock. From (5), we have $I_{n-1} = w_n - x_{n,t} - x_{n,u}$. Consequently, the number of *untreated* patients to be treated is calculated as $x^*_{n,u} = [w_n - x^*_{n,t} - I_{n-1}]^+$. In summary,

$$x^*_{n,t} = \begin{cases} w_n & \text{if } w_n \le y_{n,t} \\ y_{n,t} & \text{if } w_n > y_{n,t} \end{cases}$$

$$x^*_{n,u} = \begin{cases} 0 & \text{if } w_n \le y_{n,t} \\ \left[\frac{w_n}{A+1} - y_{n,t}\right]^+ & \text{if } w_n > y_{n,t} \end{cases}$$

Note that the *Safety-Stock* heuristic does not depend on the coefficient of resistance, γ_t. The state-space is partitioned into two regions depending on the relative values of w_n and $y_{n,t}$. Clearly, the performance of this heuristic will depend on the value of A. Lower values of A correspond to lower safety stock and increased risk of treatment interruptions in the future periods whereas higher values of A correspond to greater safety buffer and fewer *untreated* patients being initiated on treatment in the current period. In our numerical experiments, we perform line search over a sufficiently large interval to compute the best value of A which we denote by A^*.

6.2. *Two-Period* Heuristic

We use the structure of the of the N-period optimal policy for $\gamma_t = 1$ and make two modifications to it using the structure two-period optimal policy as described in Proposition 2 (iii). First, we replace $\theta_{n,u}, n > 2$, for which we do not have a closed-form analytical expression, with $\theta_{2,u} = F^{-1}\left(1 + \frac{\Delta_{2,u} + \delta(\tilde{\Delta}_{1,t} - \Delta_{1,t})}{2\delta(\Delta_{1,t} - \Delta_{1,u})}\right)$. Second, based on the numerical characterization of N-period optimal policy for $\gamma_t = 1$ and $\beta_t = 1$, we replace the curve $\phi^{-1}_{n,u}(y_{n,t})$ with $w_n = ny_{n,t} - (n-1)\theta_{2,u}$. This allows the regions to change over different periods and consequently provides a better performance as compared to replacement of $\phi^{-1}_{n,u}(y_{n,t})$ directly with the curve from the two-period optimal policy, $w_2 = (1 + \beta_t)y_{2,t} - \theta_{2,u}$. The resulting structure of the heuristic is shown in Figure 5. Under this *Two-Period* heuristic:

$$x^*_{n,t} = \min\{y_{n,t}, w_n\}$$

$$x^*_{n,u} = \begin{cases} w_n - y_{n,t} & \text{if } y_{n,t} < \min\{w_n, \theta_{2,u}\} \ \&\ w_n < \theta_{2,u} \\ \frac{w_n + (n-1)\theta_{2,u}}{n} - y_{n,t} & \text{if } w_n \ge \max\{\theta_{2,u}, ny_{n,t} - (n-1)\theta_{2,u}\} \\ 0 & \text{otherwise} \end{cases}$$

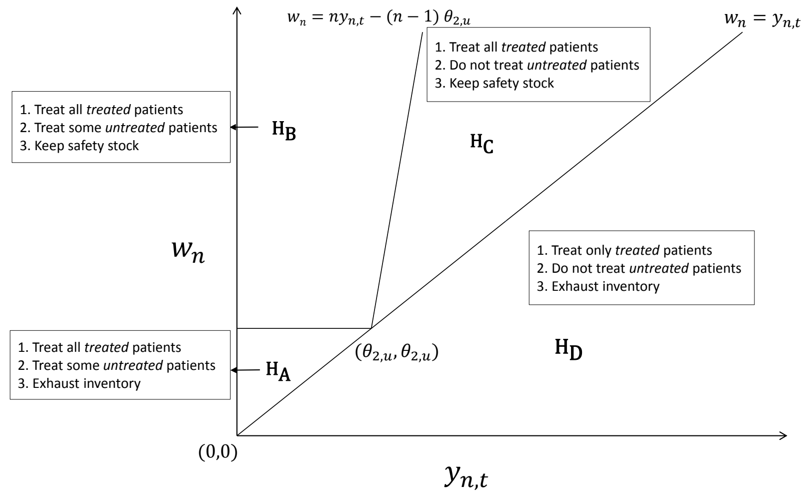


Figure 5 Characterization of Two-Period heuristic

We conclude this subsection by comparing the structure of the two heuristics. The *Safety-Stock* heuristic is conceptually quite simple but difficult to implement as it requires computation of the optimal value of A numerically. The *Two-Period* heuristic, on the other hand, has a seemingly complex structure but it can be implemented easily as it is characterized by a single threshold which can be computed using a spreadsheet if a user can provide the underlying parameter values.

6.3. Upper bound

Due to the computational challenges involved in calculating the optimal policy through backward induction for longer horizons and intermediate values of $0 < \gamma_t < 1$, we construct a simple perfect information upper bound on the objective function to evaluate and compare the performance of the heuristics (e.g. Bertsekas 1999, Chapter 6). In particular, we generate K sample paths corresponding to realizations of drug supplies over the problem horizon denoted by $\vec{z}^i = [z_1^i, z_2^i, ..., z_N^i]$, where z_n^i is drawn from the distribution $F_n(\cdot)$. Then, the inventory dynamics equation (5) for the i^{th} sample path is given by:

$$w_n^i = w_{n+1}^i - x_{n+1,t}^i - x_{n+1,u}^i + z_n^i \qquad \forall n = 1, 2, ..., N. \tag{21}$$

We calculate the optimal decisions for the i^{th} sample path by solving the following linear program:

$$U_N^i(y_{N,t}, w_N) = \max_{x_{n,t}^i, x_{n,u}^i \forall n} \sum_{n=1}^{N} (\Delta_{n,t} x_{n,t}^i + \Delta_{n,u} x_{n,u}^i + \tilde{\Delta}_{n,t} y_{n,t}^i) \tag{22}$$

$$\text{s.t. } (3),(21) \text{ and}$$
$$x^i_{n,t} + x^i_{n,u} \leq w^i_n \ \ \forall n$$
$$x^i_{n,t} \leq y^i_{n,t} \ \ \forall n$$
$$x^i_{n,u} \geq 0 \ \ \forall n$$
$$x^i_{n,t} \geq 0 \ \ \forall n$$

The upper bound, $U_N(y_{N,t}, w_N)$, is then calculated as a sample average of K perfect information value functions $U^i_N(y_{N,t}, w_N)$:

$$U_N(y_{N,t}, w_N) = \frac{1}{K}\sum_{i=1}^{K} U^i_N(y_{N,t}, w_N). \tag{23}$$

7. Numerical Illustrations

In this section, we evaluate the performance of the two heuristics using the objective value measured in QALYs and understand how it is affected by the magnitude of supply uncertainty and the coefficient of resistance. We divide our numerical experiments in two parts: (i) comparison of upper bound with the optimal policy (Section 7.2) for small sized problems and (ii) comparison of the heuristics with the upper bound (Section 7.3) for large sized problems. We begin by describing the parameter values used in the numerical experiments (Section 7.1).

7.1. Parameter values

To the extent possible, we base our parameter values on published literature and vary them over reasonable ranges to conduct sensitivity analysis. All parameter values and their corresponding sources are listed in Table 5.

As noted earlier (Section 5), we restrict our numerical experiments to nontrivial cases, i.e. $\bar{q}_t > \underline{q}_t, q_u > q_r$. Since the QOL utility values reported in the literature depend on the underlying health state of the patients, which is determined by a combination of the CD4+ count and viral load, we make certain assumptions to map them onto the treatment categories as required in our model.

As per WHO guidelines for resource limited settings, patients become eligible for ART when the CD4+ count drops below 350 cells per cubic millimeter (WHO et al. 2013). Hence, we assumed that the *untreated* patients have an average CD4+ count of 200–350 cells per cubic millimeter. We estimated the QOL utility for these patients (q_u) to be 0.84

Table 5 Parameter Values for Numerical Experiments

Parameter	Nominal value	Range of values	Source
δ	0.99	-	(Shepard and Thompson 1979, Drummond 1989)
$\beta_t, \beta_u, \beta_r$	1	-	
$\bar{q}_t$	0.93	-	(Weinstein et al. 2001, Tengs and Lin 2002, Sanders et al. 2005)
q_r	0.73	-	(Tengs and Lin 2002, Sanders et al. 2005)
$\underline{q}_t$	0.83	-	
q_u	0.84	{0.74, 0.76, 0.78, 0.80, 0.82, 0.84, 0.86, 0.88, 0.90, 0.92}	(Weinstein et al. 2001, Tengs and Lin 2002)
γ_t	1	{0,0.2,0.4,0.6,0.8,1}	Oyugi et al. (2007), Parienti et al. (2008)
z	$U(1,10)$	{U(1,10), U(2,9), U(3,8), U(4,7), U(5,6)}	Model Assumption
N	24	{12,18,24}	Model Assumption

based on the results of a meta-analysis of more than 25 individual studies (Tengs and Lin 2002) and varied it from 0.74 to 0.92 for sensitivity analysis. Further, we assumed that *treated* patients who are sensitive to treatment are asymptomatic and accordingly estimated their QOL utility value ($\bar{q}_t$) to be 0.93 based on the values reported in the literature (Weinstein et al. 2001, Tengs and Lin 2002, Sanders et al. 2005) for asymptomatic patients. Further, we assumed that the QOL utility for *treated* patients who did not receive treatment in a particular period but are still sensitive to treatment to be approximately 10% lower ($\underline{q}_t = 0.83$) than the *treated* patients who are receiving treatment to reflect the effect of short-term treatment interruption. The QOL utility values for patients resistant to treatment was the most difficult to estimate. The typical clinical outcome of failing therapy is rebound of viral load, which then results in rapid decline in CD4+ count and development of opportunistic infections. Some patients might also develop clinical AIDS. Since the range of outcomes is quite large and varied, we used a wide range of estimates and center it around 0.73 for this category.

Oyugi et al. (2007) report that about 13% patients in a HAART program in Uganda developed resistance during prolonged interruption of treatment due to drug shortages. In contrast, Parienti et al. (2008) found that sustained interruptions are more likely to result in failure of therapy than intermittent interruptions stemming from behavioral noncompliance to treatment. Using statistical analysis, they estimated that almost 100% of the patients would fail therapy if the interruption lasted about 30 days. Hence we considered the entire range of 0 to 1 for coefficient of resistance γ_t.

A period in our model is taken to be one month to reflect the typical frequency of shipment of drugs to the clinics. We choose $N = 24$ reflecting a time horizon of 2 years, which is typical of the funding cycles in global health context (Natarajan and Swaminathan 2014). We also consider alternate time horizons of 12 and 18 months in our sensitivity analysis. We set the single period (monthly) discount rate $\delta = 0.99$, which is approximately equivalent to an annual discount rate of 5% (Shepard and Thompson 1979, Drummond 1989). However, due to the short horizon our results are not sensitive to the actual choice of the discount rate. A meta-analysis of more than 13 cohort studies (Egger et al. 2002) found that the annual mortality rate for HIV patients ranges from 1% to 5 % for CD4+ counts of our interest. These correspond to a monthly survival rate of 99.5% to 99.9%. Hence, we assume that the average monthly survival rates β_t, β_u and β_r are constant and equal to 1 for ease of computation. The initial *treated* patient pool and the ARV inventory level are set to 0, i.e. $y_{N,t} = 0$ and $w_N = 0$, to reflect the situation faced by a new HIV clinic. We do not have access to operational data on the distributions of drug supply received by clinics. Hence, we vary the support of the supply distribution such that the mean is held constant while changing the variability.

7.2. Comparison of upper bound with optimal policy

In this experiment, we perform simulations to evaluate the tightness of the upper bound. We compute the optimal policy and the upper bound for the parameter values mentioned in Table 5 and measure the tightness of the bound as $T = \frac{QALY_{ub} - QALY_{opt}}{QALY_{opt}}$, where $QALY_{ub}$ is the average QALYs collected over the planning horizon under deterministic supply and

Table 6 **Tightness of upper bound with respect to optimal policy for** $\gamma_t = 1$

q_u	Tightness		
	$N = 12$	$N = 18$	$N = 24$
0.74	0.71%	0.88%	1.08%
0.76	0.97%	1.24%	1.53%
0.78	1.08%	1.40%	1.74%
0.80	1.15%	1.51%	1.88%
0.82	1.20%	1.59%	1.99%
0.84	1.27%	1.74%	2.05%
0.86	1.32%	1.82%	2.15%
0.88	1.37%	1.89%	2.25%
0.90	1.42%	1.99%	2.39%
0.92	1.52%	2.15%	2.59%

$QALY_{opt}$ is the optimal objective function value computed through backward induction. We set the coefficient of resistance, $\gamma_t = 1$ for computational ease and because we observed numerically that the gap between the two is the largest for this case. We use $K = 100000$ sample paths in equation (23).

Table 6 shows that the average tightness of the upper bound is below 1.6% for a 12 period problem and below 2.6% for a 24 period problem. In general, the upper bound is tighter for lower values of q_u for a fixed horizon. The reason for this is that optimal solution enrolls fewer *untreated* patients compared to the upper bound solution due to supply uncertainty. The impact of this under-enrollment is lower for higher values of q_u. Further, the upper bound is looser for longer problem horizons as the gaps in QALYs due to suboptimal decisions accumulate. These experiments indicate that the upper bound is reasonably close to the optimal policy and hence we use it as a benchmark to compare the performance of our heuristics for large problem sizes, where numerical characterization of the optimal policy is computationally cumbersome.

7.3. Performance of Heuristics

In this experiment, we first compare the performance of the two heuristics against the upper bound for different values of q_u and γ_t. We chose to vary these two parameters because their estimates based on the published literature are the least certain. Next, we investigate the impact of supply uncertainty on the performance of the heuristics.

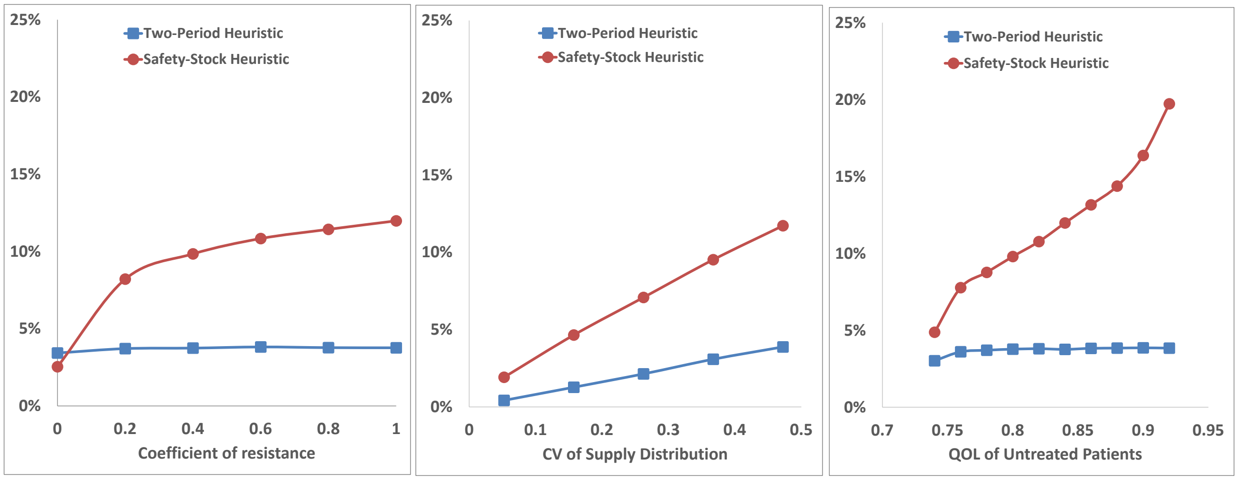
7.3.1. Impact of the coefficient of resistance (γ_t) and QOL of *untreated* patients (q_u): Figure 6a shows that the average performance gap of the *Two-Period* and the *Safety-Stock* heuristics increases with the coefficient of resistance, γ_t, reflecting that the trade-off becomes progressively more expensive. Further, the performance gap for the *Two-Period* heuristic is slightly greater than that for the *Safety-Stock* heuristic for $\gamma_t = 0$ but is substantially lower for all values of $\gamma_t > 0$. While the latter increases from 2.54% (for $\gamma_t = 0$) to 11.98% (for $\gamma_t = 1$), the former only increases from 3.43% to 3.76%.

Figure 6b shows that both heuristics become increasingly suboptimal (their performance gap increases) for increasing values of q_u. This is because, for fixed values of $\bar{q}_t, \underline{q}_t$ and q_r, higher values of q_u represent cases with higher relative penalty of treatment interruption as *untreated* patients are progressively healthier than patients whose treatment has been interrupted. However, the impact on *Safety-Stock* heuristic is much more substantial whereas the performance of *Two-Period* heuristic is relatively stable over different values of q_u.

In summary, from the first experiment, it is noteworthy that the *Two-Period* heuristic is robust to variations in γ_t and q_u. This is primarily because the value of the threshold in the *Two-Period* heuristic ($\theta_{2,u}$) adjusts based on the changes in the underlying problem parameters (γ_t and q_u) allowing it to better capture the changes in the underlying trade-off of the decision problem. However, such dependence is not explicitly built into the estimation of the parameter A^* of the *Safety-Stock* heuristic, which depends only on the pool of the *treated* patients.

7.3.2. Impact of supply uncertainty on heuristics: To investigate the impact of supply uncertainty on the performance of heuristic, we vary the coefficient of variation of supply distribution by keeping the mean fixed and varying the support of the distribution as described in Section 7.1 and shown in Table 5.

Figure 6c shows that the performance of *Safety-Stock* and *Two-Period* heuristic worsens with increase in supply variability. For supply distribution with low variability (CV = 0.05), the average performance gap of *Safety-Stock* and *Two-Period* heuristic is below 2% and 0.5%, respectively. However, for supply distributions with higher variability (CV = 0.47),



(a) (b) (c)

Figure 6 **Comparison of the performance of Two-Period and Safety-Stock heuristics**

the performance gap of *Safety-Stock* and *Two-Period* heuristic worsens and is around 12% and 4%, respectively. The *Two-Period* heuristic consistently outperforms the *Safety-Stock* heuristic and the gap between the two is increasingly greater for higher levels of supply uncertainty.

To investigate whether the superior performance of the *Two-Period* heuristic is because of the small scale of the problem, we also conduct experiments under two additional supply distributions, $U(1,50)$ and $U(1,100)$. We find that the main qualitative insights remain unchanged: (i) The *Safety-Stock* heuristic becomes increasingly sub-optimal for greater values of q_u and the performance gap increases upto 25% for $U(1,100)$, and (ii) the performance of the *Two-Period* heuristic is fairly stable for different values of q_u as well as the supply distribution (around 5%). These results imply that the performance of the heuristics depends largely on the CV of the underlying supply distribution; which is nearly the same for $U(1,10), U(1,50)$ and $U(1,100)$. In the interest of brevity, we report these results in the Appendix.

In summary, we note that the performance gap for the *Two-Period* heuristic is consistently below 4% for all realistic combinations of the parameter values considered whereas that for *Safety-Stock* heuristic can be as high as 20%, especially for high values of q_u and γ_t, i.e., when the likelihood and impact of treatment interruption is very high. Further, its performance is much more robust to changes in parameter values compared to that of the *Safety-Stock* heuristic. These observations suggest that a simplified two period problem is able to capture the essence of the dynamic trade-off between enrolling a *untreated* patient now and increasing the risk of not being able to provide uninterrupted treatment to her in the future. On the contrary, the heuristic in which the safety stock depends only on the size of the *treated* patient pool but not the current inventory is unable to do so and hence performs poorly. Finally, it is worth noting that we numerically compute the "optimal" level of safety stock for our experiments, which is unlikely to be the case in practice. Hence, the potential benefit of switching to a *Two-Period* heuristic might be even higher.

8. Conclusions

The primary contribution of our paper is the development of a parsimonious model, which highlights the fundamental trade-off faced by HIV clinics in resource-limited settings arising from limited and uncertain supply of drugs. Unlike previous qualitative discussions on ARV rationing in the literature, our model is more suited for operational planning decisions at the clinic level as it accounts for the inventory level and the size of *treated* patient pool. Despite making simplifying assumptions, the analytical structure of the optimal policy is quite complex to be immediately applied in practice. However, it helps us to design a simpler heuristic that performs much better than the type of heuristic currently used in practice. A secondary contribution of our paper is to the inventory rationing literature by explicitly including dynamics of customers across segments as a result of past decisions. Introduction of these dynamics yields an optimal policy that is structurally different from that of the conventional inventory models involving multiple products or customer segments.

In practice, any decision related to treatment rationing at the operational level is intricately linked to other aspects of the HIV epidemic such as clinical guidelines on eligibility and impact of treatment on prevention. Deo (2007) presents a framework for broader issues involved in treatment scale up including how treatment, prevention and diagnoses are interlinked via patient behavior and disease epidemiology. However, here we focus only on the impact of supply uncertainty on the aggregate health outcomes of HIV positive patients for a given set of clinical eligibility guidelines as a starting point since it has not been studied before. We believe that the insights generated from our analysis, in turn, can be used to build a simulation model that can allow for more detailed disease dynamics and a more accurate calculation of the QALYs for various enrollment policies.

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Appendix: Proofs of Theoretical Results

The dynamic programming problem is given by

$$V_n\left(y_{n,t}, w_n\right) = \max_{x_{n,t}\geq 0, x_{n,u}\geq 0} \left\{\hat{h}_{n,t}\left(\mathbf{x}_n, y_{n,t}\right) + \delta\mathbb{E}_z\left[V_{n-1}\left(y_{n-1,t}, w_{n-1}\right)\right]\right\}$$

$$\text{s.t. } (3), (5)$$

$$x_{n,t} + x_{n,u} \leq w_n$$

$$x_{n,t} \leq y_{n,t}$$

$$\text{and } V_0(y_{0,t}, w_0) = 0$$

Define $J_n(y_{n,t}, I_n) \triangleq E_{z_n}[V_n(y_{n,t}, w_n)]$ and $G_n(x_{n,t}, x_{n,u}) \triangleq \hat{h}_{n,t}\left(\mathbf{x}_n, y_{n,t}\right) + \delta\mathbb{E}_z\left[V_{n-1}\left(y_{n-1,t}, w_{n-1}\right)\right]$ $\forall n$. For period n, we denote the constraint set by C_n i.e.

$C_n = \{(\mathbf{x}_n, (y_{n,t}, w_n)) \mid x_{n,t} + x_{n,u} \leq w_n, x_{n,t} \leq y_{n,t}, x_{n,u} \geq 0, x_{n,t} \geq 0\}$. Therefore, the dynamic program is compactly represented as:

$$V_n(y_{n,t}, w_n) = \max_{x_{n,t}, x_{n,u}} G_n(x_{n,t}, x_{n,u}) \;\text{ s.t. }\; (\mathbf{x}_n, (y_{n,t}, w_n)) \in C_n \tag{EC.1}$$

EC.1. Proof of Proposition 1

EC.1.1. Proof of Proposition 1- (i) (Elimination of state variables)

We want to eliminate $y_{n,r}$, $y_{n,i}$ and $y_{n,u}$ from the objective function as described in equation (6). We proceed by backward induction on n.

Resistant patients: For $n = N-1$, Equation (11) reduces to $y_{N-1,r} = \beta_r\left(y_{N,r} + \gamma_t(y_{N,t} - x_{N,t})\right)$ which is true. Assume $y_{n,r} = y_{N,r}\beta_r^{N-n} + \gamma_t \sum_{j=1}^{N-n}\left(y_{n+j,t} - x_{n+j,t}\right)\left(\beta_r\right)^j$ holds for period n. Then for the period $n-1$, from equation (4), we get:

$$\begin{aligned}
y_{n-1,r} &= \beta_r\left(y_{n,r} + \gamma_t(y_{n,t} - x_{n,t})\right)\\
&= \beta_r\left(y_{N,r}\beta_r^{N-n} + \gamma_t\sum_{j=1}^{N-n}\left(y_{n+j,t} - x_{n+j,t}\right)\left(\beta_r\right)^j + \gamma_t(y_{n,t} - x_{n,t})\right)\\
&= y_{N,r}\beta_r^{N-(n-1)} + \gamma_t\sum_{j=1}^{N-(n-1)}\left(y_{n+j-1,t} - x_{n+j-1,t}\right)\left(\beta_r\right)^j
\end{aligned}$$

Thus, equation (11) holds true for $n = N-1, N-2, ..., 1$ ■

Ineligible patients: For $n = N-1$, Equation (12) reduces to $y_{N-1,i} = \beta_i\left(y_{N,i}(1 - \alpha_e + \alpha_i)\right)$ which is true. Assume $y_{n,i} = y_{N,i}\left(\beta_i\left(1 - \alpha_e + \alpha_i\right)\right)^{N-n}$ holds for period n. Then for period $n-1$, from equation (1), we get:

$$y_{n-1,i} = \beta_i\left(y_{n,i}(1 - \alpha_e + \alpha_i)\right)$$

$$
\begin{aligned}
&= \beta_i \left(y_{N,i} \left(\beta_i \left(1-\alpha_e+\alpha_i\right)\right)^{N-n} \left(1-\alpha_e+\alpha_i\right)\right) \\
&= y_{N,i} \left(\beta_i \left(1-\alpha_e+\alpha_i\right)\right)^{N-(n-1)}
\end{aligned}
$$

Thus, equation (12) holds true for $n = N-1, N-2, ..., 1$ ■

***untreated* eligible patients:** For $n = N-1$, Equation (13) reduces to $y_{N-1,u} = \beta_u \left(y_{N,u} - x_{N,u} + \alpha_e y_{N,i}\right)$ which is true. Assume $y_{n,u} = \beta_u^{N-n} y_{N,u} - \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u} + \alpha_e \sum_{j=1}^{N-n} \beta_u^j y_{n+j,i}$ holds for period n. Then, for period $n-1$, from equation (2), we get:

$$
\begin{aligned}
y_{n-1,u} &= \beta_u \left(y_{n,u} - x_{n,u} + \alpha_e y_{n,i}\right) \\
&= \beta_u \left(\beta_u^{N-n} y_{N,u} - \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u} + \alpha_e \sum_{j=1}^{N-n} \beta_u^j y_{n+j,i} - x_{n,u} + \alpha_e y_{n,i} \right) \\
&= y_{N,u} \beta_u^{N-(n-1)} - \sum_{j=1}^{N-(n-1)} \beta_u^j x_{n+j-1,u} + \alpha_e \sum_{j=1}^{N-(n-1)} \beta_u^j y_{n+j-1,i}
\end{aligned}
$$

Thus, equation (13) holds true for $n = N-1, N-2, ..., 1$ ■

EC.1.2. Proof of Proposition 1 - (ii) (Model reformulation)

The equations for state variable $y_{n,r}$, $y_{n,i}$ and $y_{n,u}$ $\forall$ n are as follows:

$$
\begin{aligned}
y_{n,r} &= y_{N,r} \beta_r^{N-n} + \gamma_t \sum_{j=1}^{N-n} \left(y_{n+j,t} - x_{n+j,t}\right) \left(\beta_r\right)^j \\
y_{n,i} &= y_{N,i} \left(\beta_i \left(1-\alpha_e+\alpha_i\right)\right)^{N-n} \\
y_{n,u} &= \beta_u^{N-n} y_{N,u} - \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u} + \beta_i \alpha_e \sum_{j=1}^{N-n} \beta_u^{j-1} y_{n+j,i}
\end{aligned}
$$

Define $\Delta_{1,u} \triangleq \bar{q}_t \beta_t - q_u \beta_u$ and $\Delta_{1,t} \triangleq (\bar{q}_t - \underline{q}_t(1-\gamma_t))\beta_t - \gamma_t q_r \beta_r$. Substituting the value of $y_{n,r}$, $y_{n,i}$ and $y_{n,u}$ in equation (6), we get:

$$
\begin{aligned}
h_n\left(\mathbf{x}_n, \mathbf{y}_n\right) &= \left(\bar{q}_t \beta_t - q_u \beta_u\right) x_{n,u} + \left(\left(\bar{q}_t - \underline{q}_t (1-\gamma_t)\right) \beta_t - \gamma_t q_r \beta_r \right) x_{n,t} + y_{n,u} \left(q_u \beta_u\right) + y_{n,r} \left(q_r \beta_r\right) \\
&\quad + y_{n,t} \left(\underline{q}_t \left(1-\gamma_t\right) \beta_t + q_r \gamma_t \beta_r \right) + y_{n,i} \left(q_i \beta_i \left(1-\alpha_e+\alpha_i\right) + q_u \alpha_e \beta_u\right) \\
&= \Delta_{1,u} x_{n,u} + \Delta_{1,t} x_{n,t} + q_u \beta_u \left(\beta_u^{N-n} y_{N,u} - \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u} + \alpha_e \sum_{j=1}^{N-n} \beta_u^j y_{n+j,i} \right) \\
&\quad + q_r \beta_r \left(y_{N,r} \beta_r^{N-n} + \gamma_t \sum_{j=1}^{N-n} \left(y_{n+j,t} - x_{n+j,t}\right) \left(\beta_r\right)^j \right) + y_{n,t} \left(\underline{q}_t \left(1-\gamma_t\right) \beta_t + q_r \gamma_t \beta_r \right) \\
&\quad + y_{N,i} \left(\beta_i \left(1-\alpha_e+\alpha_i\right)\right)^{N-n} \left(q_i \beta_i \left(1-\alpha_e+\alpha_i\right) + q_u \alpha_e \beta_u\right) \\
&= \Delta_{1,u} x_{n,u} + \Delta_{1,t} x_{n,t} + q_u \beta_u^{N-n+1} y_{N,u} - q_u \beta_u \sum_{j=1}^{N-n} \beta_u^j x_{n+j,u}
\end{aligned}
$$

$$
\begin{aligned}
&+q_u\beta_u\alpha_e\sum_{j=1}^{N-n}\beta_u^j y_{N,i}\left(\beta_i(1-\alpha_e+\alpha_i)\right)^{N-n-j}+q_r\beta_r^{N-n+1}y_{N,r}\\
&+q_r\beta_r\gamma_t\sum_{j=1}^{N-n}\left(y_{n+j,t}-x_{n+j,t}\right)\left(\beta_r\right)^j+y_{n,t}\left(\underline{q}_t\left(1-\gamma_t\right)\beta_t+q_r\gamma_t\beta_r\right)\\
&+y_{N,i}\left(\beta_i\left(1-\alpha_e+\alpha_i\right)\right)^{N-n}\left(q_i\beta_i\left(1-\alpha_e+\alpha_i\right)+q_u\alpha_e\beta_u\right)
\end{aligned}
$$

Now, $y_{N,r}$ and $y_{N,i}$ are constants and from equation (7) we see that they do not affect the decision variables $x_{n,t}$ and $x_{n,u}$. Thus, we eliminate these terms and rewrite the single period objective function as:

$$
\begin{aligned}
h_n\left(\mathbf{x}_n,y_{n,t}\right)=\;&\Delta_{1,u}x_{n,u}-q_u\beta_u\sum_{j=1}^{N-n}\beta_u^j x_{n+j,u}+\Delta_{1,t}x_{n,t}-\gamma_t q_r\beta_r\sum_{j=1}^{N-n}\beta_r^j x_{n+j,t}\\
&+y_{n,t}\left(\underline{q}_t\left(1-\gamma_t\right)\beta_t+q_r\gamma_t\beta_r\right)+\gamma_t q_r\beta_r\sum_{j=1}^{N-n}\beta_r^j y_{n+j,t}+q_u\beta_u^{N-n+1}y_{N,u}
\end{aligned}
\tag{EC.2}
$$

Substituting equation (EC.2) in equation (7), we get:

$$
\begin{aligned}
\sum_{n=1}^{N}\delta^{N-n}h_n\left(\mathbf{x}_n,y_{n,t}\right)=\;&\Delta_{1,u}\sum_{n=1}^{N}x_{n,u}\delta^{N-n}-q_u\beta_u\sum_{n=1}^{N}\sum_{j=1}^{N-n}x_{n+j,u}\beta_u^j\delta^{N-n}\\
&+\Delta_{1,t}\sum_{n=1}^{N}x_{n,t}\delta^{N-n}-\gamma_t q_r\beta_r\sum_{n=1}^{N}\sum_{j=1}^{N-n}x_{n+j,t}\beta_r^j\delta^{N-n}\\
&+\left(\underline{q}_t\left(1-\gamma_t\right)\beta_t+q_r\gamma_t\beta_r\right)\sum_{n=1}^{N}y_{n,t}\delta^{N-n}\\
&+\gamma_t q_r\beta_r\sum_{n=1}^{N}\sum_{j=1}^{N-n}y_{n+j,t}\beta_r^j\delta^{N-n}+\sum_{n=1}^{N}\delta^{N-n}q_u\beta_u^{N-n+1}y_{N,u}
\end{aligned}
\tag{EC.3}
$$

Let us first evaluate individual terms from equation (EC.3)

$$
\begin{aligned}
\sum_{n=1}^{N}\sum_{j=1}^{N-n}x_{n+j,u}\beta_u^j\delta^{N-n}&=\delta^{N-1}\sum_{j=1}^{N-1}x_{j+1,u}\beta_u^j+\delta^{N-2}\sum_{j=1}^{N-2}x_{j+2,u}\beta_u^j+\cdots+\delta\beta_u x_{N,u}\\
&=\sum_{j=1}^{N-1}x_{N,u}(\delta\beta_u)^j+\delta\sum_{j=1}^{N-2}x_{N-1,u}(\delta\beta_u)^j+\cdots+\delta^{N-2}(\delta\beta_u)x_{2,u}\\
&=\sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_u)^j x_{n,u}
\end{aligned}
\tag{EC.4}
$$

Similarly,

$$\sum_{n=1}^{N}\sum_{j=1}^{N-n} x_{n+j,t}\beta_r^j\delta^{N-n} = \sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_r)^j x_{n,t} \tag{EC.5}$$

$$\text{and } \sum_{n=1}^{N}\sum_{j=1}^{N-n} y_{n+j,t}\beta_r^j\delta^{N-n} = \sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_r)^j y_{n,t} \tag{EC.6}$$

Substituting equations (EC.4), (EC.5) and (EC.6) in (EC.3), we get:

$$\begin{aligned}
\sum_{n=1}^{N}\delta^{N-n}h_n(\mathbf{x}_n, y_{n,t}) &= \Delta_{1,u}\sum_{n=1}^{N}x_{n,u}\delta^{N-n} - q_u\beta_u\sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_u)^j x_{n,u} \\
&\quad +\Delta_{1,t}\sum_{n=1}^{N}x_{n,t}\delta^{N-n} - \gamma_t q_r\beta_r\sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_r)^j x_{n,t} \\
&\quad +\left(\underline{q}_t(1-\gamma_t)\beta_t + q_r\gamma_t\beta_r\right)\sum_{n=1}^{N}y_{n,t}\delta^{N-n} \\
&\quad +\gamma_t q_r\beta_r\sum_{n=2}^{N}\delta^{N-n}\sum_{j=1}^{n-1}(\delta\beta_r)^j y_{n,t} + \sum_{n=1}^{N}\delta^{N-n}q_u\beta_u^{N-n+1}y_{N,u} \\
&= \Delta_{1,u}x_{1,u}\delta^{N-1} + \sum_{n=2}^{N}\delta^{N-n}\left(\Delta_{1,u} - q_u\beta_u\sum_{j=1}^{n-1}(\delta\beta_u)^j\right)x_{n,u} \\
&\quad +\Delta_{1,t}x_{1,t}\delta^{N-1} + \sum_{n=2}^{N}\delta^{N-n}\left(\Delta_{1,t} - \gamma_t q_r\beta_r\sum_{j=1}^{n-1}(\delta\beta_r)^j\right)x_{n,t} \\
&\quad +\left(\underline{q}_t(1-\gamma_t)\beta_t + q_r\gamma_t\beta_r\right)y_{1,t}\delta^{N-1} \\
&\quad +\sum_{n=2}^{N}\delta^{N-n}\left(\left(\underline{q}_t(1-\gamma_t)\beta_t + q_r\gamma_t\beta_r\right) + \gamma_t q_r\beta_r\sum_{j=1}^{n-1}(\delta\beta_r)^j\right)y_{n,t} \\
&\quad +\sum_{n=1}^{N}\delta^{N-n}q_u\beta_u^{N-n+1}y_{N,u} \\
&= \sum_{n=1}^{N}\delta^{N-n}\left(\Delta_{n,u}x_{n,u} + \Delta_{n,t}x_{n,t} + \tilde{\Delta}_{n,t}y_{n,t}\right) + \sum_{n=1}^{N}\delta^{N-n}q_u\beta_u^{N-n+1}y_{N,u}
\end{aligned}$$

where $\Delta_{n,u}$, $\Delta_{n,t}$ and $\tilde{\Delta}_{n,t}$ are as defined in equations (18), (19) and (20) respectively.

Now, by the assumption $y_{n,u} > w_n\ \forall n$ and the inventory constraint $x_{n,t} + x_{n,u} \le w_n$ we have $x_{n,u} < y_{n,u}\ \forall n$. Thus, the constraint $x_{n,u} \le y_{n,u}$ in (7) is never tight and the feasible set of $\mathbf{x_n}$ does not depend on $y_{n,u}$. Therefore, we can eliminate the constraint $y_{n,u} > x_{n,u}$ and the term involving $y_{N,u}$ from the objective function under the assumption $y_{n,u} > w_n\ \forall n$.

$$\begin{aligned}
\therefore\ \sum_{n=1}^{N}\delta^{N-n}h_n(\mathbf{x}_n, y_{n,t}) &= \sum_{n=1}^{N}\delta^{N-n}\left(\Delta_{n,u}x_{n,u} + \Delta_{n,t}x_{n,t} + \tilde{\Delta}_{n,t}y_{n,t}\right) \\
&= \sum_{n=1}^{N}\delta^{N-n}\hat{h}_n(\mathbf{x}_n, y_{n,t})
\end{aligned} \tag{EC.7}$$

Thus, the model formulation in (7) reduces to the model stated in (14).

We now show that both the formulations are equivalent. We have already shown in (EC.7) that the objective function in (14) is equivalent to the objective function in (7) under the assumption $y_{n,u} > w_n \ \forall n$. Consider any feasible solution, $\tilde{\mathbf{x}}_{\mathbf{n}} = [\tilde{x}_{n,t} \ \tilde{x}_{n,u}]^t$ that satisfies (7). Clearly, $\tilde{\mathbf{x}}_{\mathbf{n}}$ satisfies all the constraints in (14). Therefore a feasible solution to (7) is also feasible in (14). Similarly, if $\tilde{\mathbf{x}}_{\mathbf{n}}$ is any feasible to (14), then $\tilde{x}_{n,t} + \tilde{x}_{n,u} \leq w_n$ and by assumption $y_{n,u} > w_n \ \forall n$, we get $x_{n,u} \leq y_{n,u}$. From Proposition 1 - (i), $\tilde{\mathbf{x}}_{\mathbf{n}}$ satisfies the patient dynamics constraints (1), (2) and (4) required in (7). Thus $\tilde{\mathbf{x}}_{\mathbf{n}}$ satisfies all the constraints in (7) and hence the two formulations are equivalent. ∎

EC.2. Proof of Proposition 2 (Two Period Optimal Policy)

Define $\gamma_1 \triangleq \frac{q_t\beta_t - q_u\beta_u}{\underline{q}_t\beta_t - q_r\beta_r}$. Consider the case $0 \leq \gamma_t < \gamma_1$ and $\underline{q}_t > q_u$. Then, for period $n = 1$, $\gamma_t < \gamma_1 \Rightarrow \Delta_{1,u} - \Delta_{1,t} > 0$. Thus, $x^*_{1,t} = 0$ and $x^*_{1,u} = w_1$. The value function is given by $V_1(y_{1,t}, w_1) = \Delta_{1,u} w_1 + \tilde{\Delta}_{1,t} y_{1,t}$. $J_1(y_{1,t}, I_1) = \Delta_{1,u}(w_2 - x_{2,t} - x_{2,u}) + \Delta_{1,u}\mu_1 + \tilde{\Delta}_{1,t}(y_{2,t}(1-\gamma_t) + x_{2,t}\gamma_t + x_{2,u})$

For $n = 2$, $G_2(x_{2,t}, x_{2,u}) = \Delta_{2,t} x_{2,t} + \Delta_{2,u} x_{2,u} + \tilde{\Delta}_{2,t} y_{2,t} + J_1(y_{1,t}, I_1)$. Next, $G_2^{(1)}(x_{2,t}, x_{2,u}) = \Delta_{2,t} - \delta\Delta_{1,u} + \delta\gamma_t\beta_t\tilde{\Delta}_{1,t}$. Similarly $G_2^{(2)}(x_{2,t}, x_{2,u}) = \Delta_{2,u} - \delta\Delta_{1,u} + \delta\beta_t\tilde{\Delta}_{1,t} = (\bar{q}_t\beta_t - q_u\beta_u)(1 - \delta) + \delta\left(\underline{q}_t\beta_t^2 - q_u\beta_u^2\right) - \delta\gamma_t\left(\underline{q}_t\beta_t - q_r\beta_r\right)$. As $\gamma_t < \gamma_1$, we get $G_2^{(2)}(x_{2,t}, x_{2,u}) > 0$.

Now, $G_2^{(2)}(x_{2,t}, x_{2,u}) - G_2^{(1)}(x_{2,t}, x_{2,u}) = \Delta_{2,u} - \Delta_{2,t} + \delta\beta_t(1-\gamma_t)\tilde{\Delta}_{1,t} = \gamma_t^2\left(\delta\underline{q}_t\beta_t^2 - \delta q_r\beta_r\beta_t\right) + \gamma_t\left((q_r\beta_r - \underline{q}_t\beta_t) + (\delta q_r\beta_r^2 - \delta\underline{q}_t\beta_t^2) + (\delta q_r\beta_r\beta_t - \delta\underline{q}_t\beta_t^2)\right) + (\underline{q}_t\beta_t - q_u\beta_u) + \delta(\underline{q}_t\beta_t^2 - q_u\beta_u^2)$, which is quadratic in γ_t. Note that the value of the quadratic expression is greater than 0 at $\gamma_t = 0$ and less than 0 at $\gamma_t = 1$. Thus, there is exactly one root lying between 0 and 1. The root of this quadratic expression is given by:

$$\begin{aligned}\gamma_2 = \Bigg[\Bigg(&\left(4\delta^2\beta_t^2\beta_u^2 + 4\delta\beta_t^2\beta_u\right)\underline{q}_t + 4\left(\left(-4\delta^2\beta_r\beta_t\beta_u^2\right) - 4\delta\beta_r\beta_t\beta_u\right)q_r\Bigg)q_u + \left(4\delta^2\beta_t^4 + \beta_t^2\right)\underline{q}_t^2 \\ &+ \Bigg(\left(\left(-4\beta_r\beta_t^3\right) - 8\beta_r^2\beta_t^2\right)\delta^2 + \left(\left(-2\beta_r\beta_t^2\right) - 2\beta_r^2\beta_t\right)\delta - 2\beta_r\beta_t\Bigg)q_r\underline{q}_t + \Bigg(\beta_r^2\beta_t^2 + 6\beta_r^3\beta_t + \beta_r^4\delta^2 + \\ &\left(2\beta_r^2\beta_t + 2\beta_r^3\right)\delta + \beta_r^2\Bigg)q_r^2 + \left(\left(-2\beta_t^2\delta\right) - \beta_t\right)\underline{q}_t + \left(\left(\beta_r\beta_t + \beta_r^2\right)\delta + \beta_r\right)q_r\Bigg] \Bigg/ \Bigg[2 * \delta\beta_t(\underline{q}_t\beta_t - q_r\beta_r)\Bigg]\end{aligned} \tag{EC.8}$$

Now, if $\gamma_2 > \gamma_1$, we get $G_2^{(2)}(x_{2,t}, x_{2,u}) - G_2^{(1)}(x_{2,t}, x_{2,u}) > 0 \ \forall 0 < \gamma_t < \gamma_1$. The optimal solution is then given by $x^*_{2,t} = 0$ and $x^*_{2,u} = w_2$.

On the other hand, if $0 < \gamma_t < \gamma_2 < \gamma_1$, then $G_2^{(2)}(x_{2,t}, x_{2,u}) - G_2^{(1)}(x_{2,t}, x_{2,u}) > 0$ and the optimal solution is given by $x^*_{2,t} = 0$ and $x^*_{2,u} = w_2$. If $0 < \gamma_2 < \gamma_t < \gamma_1$, then $G_2^{(2)}(x_{2,t}, x_{2,u}) - G_2^{(1)}(x_{2,t}, x_{2,u}) < 0$ and the optimal solution is given by $x^*_{2,t} = \min\{y_{2,t}, w_2\}, x^*_{2,u} = [w_2 - y_{2,t}]^+$.

Now consider the case when $\gamma_1 < \gamma_t < 1$. For n = 1, $\Delta_{1,t} - \Delta_{1,u} > 0$. Thus, $x^*_{1,t} = \min\{y_{1,t}, w_1\}$ and $x^*_{1,u} = [w_1 - y_{1,t}]^+$. $J_1(y_{1,t}, I_1) = (\Delta_{1,t} - \Delta_{1,u})\left((y_{1,t} - I_1) - \int_0^{y_{1,t}-I_1} F_1(z_1)dz_1\right) + \tilde{\Delta}_{1,t}y_{1,t} + \Delta_{1,t}I_1 + \Delta_{1,u}\mu_1$. The partials of $J_1(y_{1,t}, I_1)$ are given by: $J_1^{(1)}(y_{1,t}, I_1) = (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1)) + \tilde{\Delta}_{1,t}$ and $J_1^{(2)}(y_{1,t}, I_1) = -(\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1)) + \Delta_{1,t}$.

For n = 2, $G_2(x_{2,t}, x_{2,u}) = \Delta_{2,t}x_{2,t} + \Delta_{2,u}x_{2,u} + \tilde{\Delta}_{2,t}y_{2,t} + J_1(y_{1,t}, I_1)$. The partials of $G_2(x_{2,t}, x_{2,u})$ are given by: $G_2^{(1)}(x_{2,t}, x_{2,u}) = \Delta_{2,t} + \delta\beta_t\gamma_t J_1^{(1)}(y_{1,t}, I_1) - \delta J_1^{(2)}(y_{1,t}, I_1)$ and $G_2^{(2)}(x_{2,t}, x_{2,u}) = \Delta_{2,u} + \delta\beta_t J_1^{(1)}(y_{1,t}, I_1) - \delta J_1^{(2)}(y_{1,t}, I_1)$.

We rewrite $G_2^{(1)}(x_{2,t}, x_{2,u})$ and $G_2^{(2)}(x_{2,t}, x_{2,u})$ as $\tilde{G}_2^{(1)}(y_{1,t} - I_1)$ and $\tilde{G}_2^{(2)}(y_{1,t} - I_1)$ respectively.

Now, $\tilde{G}_2^{(1)}(y_{1,t} - I_1) = \left[(\bar{q}_t - \underline{q}_t)\beta_t + \gamma_t(\underline{q}_t\beta_t - q_r\beta_r)\right](1 - \delta) + \delta\gamma_t\left[\underline{q}_t\beta_t^2(1-\gamma_t) - q_r\beta_r(\beta_r - \gamma_t\beta_t)\right] + \delta(\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))(1 + \beta_t\gamma_t) \Rightarrow \tilde{G}_2^{(1)}(y_{1,t} - I_1) > 0$.

Also, $\tilde{G}_2^{(1)}(y_{1,t} - I_1) - \tilde{G}_2^{(2)}(y_{1,t} - I_1) > 0$. The solution to $\tilde{G}_2^{(2)}(y_{1,t} - I_1) = 0$ is given by $y_{1,t} - I_1 = F_1^{-1}\left(1 + \frac{\Delta_{2,u} + \delta\beta_t\tilde{\Delta}_{1,t} - \delta\Delta_{1,t}}{\delta(\Delta_{1,t} - \Delta_{1,u})(1+\beta_t)}\right) = \theta_{2,u}(say)$.

Now, we need to solve the stochastic dynamic program as stated in (EC.1). The Lagrangian of the optimization problem is given by:

$$\begin{aligned}\Lambda(x_{2,t}, x_{2,u}, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= G_2(x_{2,t}, x_{2,u}) + \lambda_1(w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\ &\quad + \lambda_2(y_{n+1,t} - x_{n+1,t}) + \lambda_3(x_{n+1,t}) + \lambda_4(x_{n+1,u})\end{aligned}$$

The stationarity conditions are given by $\tilde{G}_2^{(1)}(y_{1,t} - I_1) - \lambda_1 - \lambda_2 + \lambda_3 = 0$ and $\tilde{G}_2^{(2)}(y_{1,t} - I_1) - \lambda_1 + \lambda_4 = 0$. The conditions for primal feasibility are - (i) $x^*_{2,t} + x^*_{2,u} \leq w_2$ (ii) $x^*_{2,t} \leq y_{2,t}$(iii) $x^*_{2,t} \geq 0$ and (iv) $x^*_{2,u} \geq 0$. The complementary slackness conditions are (i) $\lambda_1(w_2 - x^*_{2,t} - x^*_{2,u}) = 0$ (ii) $\lambda_2(y_{2,t} - x^*_{2,t}) = 0$ (iii) $\lambda_3(x^*_{2,t}) = 0$ and (iv) $\lambda_4(x^*_{2,u}) = 0$. The dual feasibility conditions are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$. The state space is partitioned by considering permutations on the values taken by KKT multipliers.

Case 1: $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$

$x^*_{2,u} = x^*_{2,t} = 0,\ y_{2,t} = 0,\ w_2 = 0$

Case 2: $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 = 0$

$x^*_{2,t} = y_{2,t} = 0,\ x^*_{2,u} = w_2,\ w_2 < \frac{\theta_{2,u}}{\beta_t}$

Case 3: $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 > 0$

$w_2 = y_{2,t} = x^*_{2,t},\ x^*_{2,u} = 0$

Case 4: $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 > 0$

$x^*_{2,t} = x^*_{2,u} = w_2 = 0$

Case 5: $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 > 0$

$x^*_{2,t} = x^*_{2,u} = y_{2,t} = 0,\ \tilde{G}^{(2)}_2(y_{1,t} - I_1) < 0 \Rightarrow y_{1,t} - I_1 > \theta_{2,u} \Rightarrow w_2 < -\theta_{2,u}$. Thus, no feasible solution exists for this permutation of KKT multipliers.

Case 6: $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 > 0$

As $\tilde{G}^{(1)}_2(y_{1,t} - I_1) < 0$, this case yields infeasible solution.

Case 7: $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 > 0$

$x^*_{2,t} = w_2,\ x^*_{2,u} = 0,\ 0 \le x^*_{2,t} \le y_{2,t},\ w_2 \le y_{2,t}$

Case 8: $\lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 = 0$

$x^*_{2,t} = y_{2,t},\ x^*_{2,u} = w_2 - y_{2,t},\ y_{2,t} \le w_2 \le \frac{\theta_{2,u}}{\beta_t}$

Case 9: $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 > 0, \lambda_4 = 0$

$x^*_{2,t} = y_{2,t} = 0,\ x^*_{2,u} = \frac{w_2 + \theta_{2,u}}{1 + \beta_t},\ w_2 \ge \frac{\theta_{2,u}}{\beta_t}$

Case 10: $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 > 0$

$x^*_{2,t} = y_{2,t},\ x^*_{2,u} = 0,\ y_{2,t}(1 + \beta_t) - w_2 \ge \theta_{2,u},\ y_{2,t} \le w_2$.

Case 11: $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 = 0$

As $\tilde{G}^{(1)}_2(y_{1,t} - I_1) - \tilde{G}^{(2)}_2(y_{1,t} - I_1) < 0$, this case is infeasible.

Case 12: $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 > 0$

As $\tilde{G}^{(1)}_2(y_{1,t} - I_1) = 0$, this case is infeasible.

Case 13: $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 > 0, \lambda_4 = 0$

As $\tilde{G}^{(1)}_2(y_{1,t} - I_1) - \tilde{G}^{(2)}_2(y_{1,t} - I_1) < 0$, this case is infeasible.

Case 14: $\lambda_1 = 0, \lambda_2 > 0, \lambda_3 = 0, \lambda_4 = 0$

$x^*_{2,t} = y_{2,t},\ x^*_{2,u} = \frac{w_2 + \theta_{2,u}}{1 + \beta_t} - y_{2,t},\ w_2 \ge \frac{\theta_{2,u}}{\beta_t},\ w_2 \ge y_{2,t}(1 + \beta_t) - \theta_{2,u}$

Case 15: $\lambda_1 > 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$

As $\tilde{G}^{(1)}_2(y_{1,t} - I_1) - \tilde{G}^{(2)}_2(y_{1,t} - I_1) < 0$, this case is infeasible.

Case 16: $\lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 0$
As $\tilde{G}_2^{(1)}(y_{1,t} - I_1) = 0$, this case is infeasible.

Next, consider the case $\underline{q}_t < q_u$. For this case, we have $\Delta_{1,t} > \Delta_{1,u}\ \forall \gamma_t$. Thus, for n = 1, the optimal solution is given by $x^*_{1,t} = \min\{y_{1,t}, w_1\}$ and $x^*_{1,u} = [w_1 - y_{1,t}]^+$.

Now, $J_1(y_{1,t}, I_1) = (\Delta_{1,t} - \Delta_{1,u})\left((y_{1,t} - I_1) - \int\limits_0^{y_{1,t}-I_1} F_1(z_1)dz_1\right) + \tilde{\Delta}_{1,t} y_{1,t} + \Delta_{1,t} I_1 + \Delta_{1,u}\mu_1$. The partials of $J_1(y_{1,t}, I_1)$ are given by: $J_1^{(1)}(y_{1,t}, I_1) = (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1)) + \tilde{\Delta}_{1,t}$ and $J_1^{(2)}(y_{1,t}, I_1) = -(\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1)) + \Delta_{1,t}$.

For n = 2, $G_2(x_{2,t}, x_{2,u}) = \Delta_{2,t}x_{2,t} + \Delta_{2,u}x_{2,u} + \tilde{\Delta}_{2,t}y_{2,t} + J_1(y_{1,t}, I_1)$. The partials of $G_2(x_{2,t}, x_{2,u})$ are given by: $G_2^{(1)}(x_{2,t}, x_{2,u}) = \Delta_{2,t} + \delta\beta_t\gamma_t J_1^{(1)}(y_{1,t}, I_1) - \delta J_1^{(2)}(y_{1,t}, I_1)$ and $G_2^{(2)}(x_{2,t}, x_{2,u}) = \Delta_{2,u} + \delta\beta_t J_1^{(1)}(y_{1,t}, I_1) - \delta J_1^{(2)}(y_{1,t}, I_1)$.

We rewrite $G_2^{(1)}(x_{2,t}, x_{2,u})$ and $G_2^{(2)}(x_{2,t}, x_{2,u})$ as $\tilde{G}_2^{(1)}(y_{1,t} - I_1)$ and $\tilde{G}_2^{(2)}(y_{1,t} - I_1)$ respectively.

Note that $\tilde{G}_2^{(1)}(y_{1,t} - I_1) = \Delta_{2,t} + \delta\beta_t\gamma_t\tilde{\Delta}_{1,t} - \delta\Delta_{1,t} + \delta(\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))(1 + \beta_t\gamma_t)$. Thus, to show that $\tilde{G}_2^{(1)}(y_{1,t} - I_1) > 0$, it is sufficient to show that $\Delta_{2,t} + \delta\beta_t\gamma_t\tilde{\Delta}_{1,t} - \delta\Delta_{1,t} > 0$.

Now, $\Delta_{2,t} + \delta\beta_t\gamma_t\tilde{\Delta}_{1,t} - \delta\Delta_{1,t} = \Delta_{1,t}(1-\delta) - \delta\gamma_t q_r\beta_r^2 + \delta\beta_t\gamma_t\tilde{\Delta}_{1,t} = \gamma_t^2\left(-\delta\beta_t(\underline{q}_t\beta_t - q_r\beta_r)\right) + \gamma_t\left(\delta(\underline{q}_t\beta_t^2 - q_r\beta_r^2) + (1-\delta)(\underline{q}_t\beta_t - q_r\beta_r)\right) + (\bar{q}_t - \underline{q}_t)\beta_t(1-\delta)$, which is a quadratic in γ_t. $\tilde{G}_2^{(1)}(y_{1,t} - I_1)|_{\gamma_t=0} = (\bar{q}_t - \underline{q}_t)\beta_t(1-\delta) > 0$. $\tilde{G}_2^{(1)}(y_{1,t} - I_1)|_{\gamma_t=1} = \left(\underline{q}_t\beta_t - q_r\beta_r\right)(1 - \delta - \delta\beta_t) + \delta(\underline{q}_t\beta_t^2 - q_r\beta_r^2) + (\bar{q}_t - \underline{q}_t)\beta_t(1-\delta)$. Now, $\tilde{G}_2^{(1)}(y_{1,t} - I_1)|_{\gamma_t=1,\delta=0} = \left(\underline{q}_t\beta_t - q_r\beta_r\right) + \beta_t(\bar{q}_t - \underline{q}_t) > 0$ and $\tilde{G}_2^{(1)}(y_{1,t} - I_1)|_{\gamma_t=1,\delta=1} = \delta q_r\beta_r(\beta_t - \beta_r) > 0$. As the coefficients of the quadratic term are linear in δ, we get $\tilde{G}_2^{(1)}(y_{1,t} - I_1)|_{\gamma_t=1} > 0$. Also, the coefficient of γ_t^2 is less than 0 and thus the quadratic is concave in γ_t. Thus, $\tilde{G}_2^{(1)}(y_{1,t} - I_1) > 0\ \forall 0 \leq \gamma_t \leq 1$.

Next, $\tilde{G}_2^{(2)}(y_{1,t} - I_1) = 0 \Rightarrow y_{1,t} - I_1 = F_1^{-1}\left[1 + \frac{\Delta_{2,u} + \delta\beta_t\tilde{\Delta}_{1,t} - \delta\Delta_{1,t}}{\delta(\Delta_{1,t} - \Delta_{1,u})(1+\beta_t)}\right] = \theta_{2,u}$ (say). Also, we have $\tilde{G}_2^{(1)}(y_{1,t} - I_1) - \tilde{G}_2^{(2)}(y_{1,t} - I_1) > 0$.

Similarly, $\tilde{G}_2^{(1)}(y_{1,t} - I_1) - \tilde{G}_2^{(2)}(y_{1,t} - I_1) = 0 \Rightarrow y_{1,t} - I_1 = F_1^{-1}\left[1 + \frac{\Delta_{2,t} - \Delta_{2,u} - \delta\beta_t(1-\gamma_t)\tilde{\Delta}_{1,t}}{\delta\beta_t(1-\gamma_t)(\Delta_{1,t} - \Delta_{1,u})}\right] = \theta_2$ (say).

Now, we need to solve the stochastic dynamic program as stated in (EC.1). The Lagrangian of the optimization problem is given by:

$$\begin{aligned}\Lambda(x_{2,t}, x_{2,u}, \lambda_1, \lambda_2, \lambda_3, \lambda_4) &= G_2(x_{2,t}, x_{2,u}) + \lambda_1(w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\ &\quad + \lambda_2(y_{n+1,t} - x_{n+1,t}) + \lambda_3(x_{n+1,t}) + \lambda_4(x_{n+1,u})\end{aligned}$$

The stationarity conditions are given by $\tilde{G}_2^{(1)}(y_{1,t} - I_1) - \lambda_1 - \lambda_2 + \lambda_3 = 0$ and $\tilde{G}_2^{(2)}(y_{1,t} - I_1) - \lambda_1 + \lambda_4 = 0$. The conditions for primal feasibility are - (i) $x^*_{2,t} + x^*_{2,u} \leq w_2$ (ii) $x^*_{2,t} \leq y_{2,t}$(iii) $x^*_{2,t} \geq$

0 and (iv) $x^*_{2,u} \geq 0$. The complementary slackness conditions are (i) $\lambda_1(w_2 - x^*_{2,t} - x^*_{2,u}) = 0$ (ii) $\lambda_2(y_{2,t} - x^*_{2,t}) = 0$ (iii) $\lambda_3(x^*_{2,t}) = 0$ and (iv) $\lambda_4(x^*_{2,u}) = 0$. The dual feasibility conditions are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$. The state space is partitioned by considering permutations on the values taken by KKT multipliers.

Note that this problem is equivalent to the case $\underline{q}_t > q_u, \gamma_t > \gamma_1$ and hence they have the same characterization of optimal policy. This completes the proof for Proposition 2. ■

EC.3. Proof of Proposition 3

EC.3.1. Proof of Proposition 3 - (i) (Optimal policy for $\gamma_t = 0$)

We prove the following claims through induction:

(1) $V_n(y_{n,t}, w_n) = K_n + y_{n,t}\left[\underline{q}_t\beta_t \sum_{j=0}^{n-1}(\delta\beta_t)^j\right] + w_n\left[\Delta_{n,u} + \underline{q}_t\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j\right]$ $\forall n \geq 2$, where K_n is a constant and (2) $x^*_{n,t} = 0, x^*_{n,u} = w_n$ $\forall n$.

For $n = 1\&2$, the claim follows from EC.2. Assume (1) holds for some $n \geq 2$. Then, $\delta J_n(y_{n,t}, I_n) = K_{n+1} + \delta\beta_t(y_{n+1,t} + x_{n+1,u})\left[\underline{q}_t\beta_t \sum_{j=0}^{n-1}(\delta\beta_t)^j\right] + \delta(w_{n+1} - x_{n+1,u} - x_{n+1,t})\left[\Delta_{n,u} + \underline{q}_t\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j\right]$. Next,

$$\begin{aligned}
G_{n+1}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,u}x_{n+1,u} + \Delta_{n+1,t}x_{n+1,t} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n) \\
&= x_{n+1,u}\left[\Delta_{n+1,u} + \underline{q}_t\beta_t \sum_{j=1}^{n}(\delta\beta_t)^j - \delta\Delta_{n,u} - \underline{q}_t\delta\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j\right] \\
&\quad + x_{n+1,t}\left[\Delta_{n+1,t} - \delta\Delta_{n,u} - \underline{q}_t\delta\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j\right] \\
&\quad + \underline{q}_t\beta_t y_{n+1,t} + y_{n+1,t}\left[\underline{q}_t\beta_t \sum_{j=1}^{n}(\delta\beta_t)^j\right] + w_{n+1}\left[\delta\Delta_{n,u} + \underline{q}_t\delta\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j\right]
\end{aligned}$$

$G^{(1)}_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,t} - \delta\Delta_{n,u} - \underline{q}_t\delta\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j$. Similarly, $G^{(2)}_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} - \delta\Delta_{n,u} - \underline{q}_t\delta\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j$. Simplifying, we get $G^{(2)}_{n+1}(x_{n+1,t}, x_{n+1,u}) = \underline{q}_t\beta_t)(\delta\beta_t)^n - q_u\beta_u(\delta\beta_u)^n + (1-\delta)(\bar{q}_t\beta_t - q_u\beta_u) + (1-\delta)\left[\underline{q}_t\beta_t \sum_{j=1}^{n-1}(\delta\beta_t)^j - q_u\beta_u \sum_{j=1}^{N-1}(\delta\beta_u)^j\right]$. Thus, $G^{(2)}_{n+1}(x_{n+1,t}, x_{n+1,u}) > 0$.. Also, $G^{(2)}_{n+1}(x_{n+1,t}, x_{n+1,u}) - G^{(1)}_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} + \underline{q}_t\beta_t \sum_{j=1}^{n}(\delta\beta_t)^j - (\bar{q}_t - \underline{q}_t)\beta_t = (\underline{q}_t\beta_t - q_u\beta_u) + \left[\underline{q}_t\beta_t \sum_{j=1}^{n}(\delta\beta_t)^j - q_u\beta_u \sum_{j=1}^{n}(\delta\beta_u)^j\right]$. Thus, $G^{(2)}_{n+1}(x_{n+1,t}, x_{n+1,u}) - G^{(1)}_{n+1}(x_{n+1,t}, x_{n+1,u}) > 0 \Rightarrow x^*_{n+1,t} = 0, x^*_{n+1,u} = w_{n+1}$. Substituting the optimal values, we get $V_{n+1}(y_{n+1,t}, w_{n+1}) = K_{n+1} + y_{n+1,t}\left[\underline{q}_t\beta_t \sum_{j=0}^{n}(\delta\beta_t)^j\right] + w_{n+1}\left[\Delta_{n+1,u} + \underline{q}_t\beta_t \sum_{j=1}^{n+1}(\delta\beta_t)^j\right]$. ■

EC.3.2. Proof of Proposition 3 - (ii) (Optimal policy for $\gamma_t = 0$)

We assume a structure on $G_n(x_{n,t}, x_{n,u}), J_{n-1}(y_{n-1,t}, I_{n-1})$ and the optimal policy for period n. We then show that the same structure holds for period $n+1$ and derive the optimal policy. The structure for period n is stated as follows:

(M1) $G_n(x_{n,t}, x_{n,u})$ and $J_{n-1}(y_{n-1,t}, I_{n-1})$ are twice differentiable and jointly concave in their arguments

(M2) $J^{(1,2)}_{n-1}(y_{n-1,t}, I_{n-1}) \geq 0$

(M3) $\Delta_{n,t} - \delta J^{(2)}_{n-1}(y_{n-1,t}, I_{n-1}) > 0$

(M4) The solution to $G^{(2)}_n(y_{n,t}, x_{n,u}) = 0$ is given by $x^*_{n,u} = \phi_{n,u}(w_n) - y_{n,t}$ where $\phi_{n,u}(w_n)$ is differentiable and satisfies $0 < \phi'_{n,u}(w_n) < 1$

(M5) The optimal policy for period n is as described in Table 2

(M6) Under the optimal policy, we further claim the following:

(a) $G^{(1)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = y_{n,t})} = G^{(1)}_n(y_{n,t}, 0) = \Delta_{n,t} - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, 0) > 0 \;\; \forall y_{n,t} \geq \theta_{n,u}$

(b) $G^{(2)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = \phi^{-1}_{n,u}(y_{n,t}))} = G^{(2)}_n(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J^{(1)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) > 0 \;\; \forall y_{n,t} \geq \theta_{n,u}$

We initiate the induction by showing that the claim holds true for $N = 2$. Then assuming the above claims hold true for period n, we show that $J_n(y_{n,t}, I_n)$ is jointly concave in its arguments and is twice differentiable. From this, we show that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is twice differentiable. Next, we show that **(M2)** holds for period $n+1$ and consequently that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is concave. After showing **(M1)** and **(M2)** hold for period $n+1$, we show that **(M3)** and **(M4)** hold true for period $n+1$. Finally, using components **(M1) - (M4)**, we complete the induction by deriving the optimal policy for period $n+1$.

The objective function for $n = 1$ is given by $G_1(x_{1,t}, x_{1,u}) = \Delta_{1,u} x_{1,u} + \Delta_{1,t} x_{1,t} + \tilde{\Delta}_{1,t} y_{1,t}$. As $G^{(1)}_1(x_{1,t}, x_{1,u}) = \Delta_{1,t} > 0$ and $G^{(2)}_1(x_{1,t}, x_{1,u}) = \Delta_{1,u} > 0$ and as $q_u > \underline{q}_t \Rightarrow \Delta_{1,t} > \Delta_{1,u}$, we get $x^*_{1,u} = [w_1 - y_{1,t}]^+$ and $x^*_{1,t} = \min\{y_{1,t}, w_1\}$. $\therefore V_1(y_{1,t}, w_1) = G_1(x^*_{1,t}, x^*_{1,u}) = \Delta_{1,u}[w_1 - y_{1,t}]^+ + \Delta_{1,t}\min\{y_{1,t}, w_1\} + \tilde{\Delta}_{1,t} y_{1,t}$. Thus,

$$\begin{aligned}
J_1(y_{1,t}, I_1) &= E_{z_1}[V_1(y_{1,t}, w_1)] = E_{z_1}[G_1(x^*_{1,t}, x^*_{1,u})] \\
&= E_{z_1}\left[\Delta_{1,u}[w_1 - y_{1,t}]^+ + \Delta_{1,t}\min\{y_{1,t}, w_1\} + \tilde{\Delta}_{1,t} y_{1,t}\right] \\
&= \tilde{\Delta}_{1,t} y_{1,t} + \Delta_{1,t} I_1 + \Delta_{1,u}\mu_1 + (\Delta_{1,t} - \Delta_{1,u})\left((y_{1,t} - I_1) - \int_0^{y_{1,t} - I_1} F_1(z_1)\mathrm{d}z_1\right)
\end{aligned}$$

Note that $J_1(y_{1,t}, I_1)$ is twice differentiable and concave in its arguments. Also, $J^{(1)}_1(y_{1,t}, I_1) = \tilde{\Delta}_{1,t} + (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))$. Thus, $J^{(1,2)}_1(y_{1,t}, I_1) =$

$(\Delta_{1,u}+\Delta_{1,t})f_1(z_1)\geq 0$. As $J_1(y_{1,t},I_1)$ is twice differentiable and concave, we conclude that $G_2(x_{2,t},x_{2,u})$ is twice differentiable and concave in its argument. Thus **(M1)** and **(M2)** holds true for $n=2$. Now, $G_2(x_{2,t},x_{2,u})=\Delta_{2,t}x_{2,t}+\Delta_{2,u}x_{2,u}+\tilde{\Delta}_{2,t}y_{2,t}+\delta J_1(y_{1,t},I_1)$.

The partials of $G_2(x_{2,t},x_{2,u})$ are given by: $G_2^{(1)}(x_{2,t},x_{2,u})=\Delta_{2,t}-\delta J_1^{(2)}(y_{1,t},I_1)$ and $G_2^{(2)}(x_{2,t},x_{2,u})=\Delta_{2,u}+\delta\beta_t J_1^{(1)}(y_{1,t},I_1)-\delta J_1^{(2)}(y_{1,t},I_1)$.

We rewrite $G_2^{(1)}(x_{2,t},x_{2,u})$ and $G_2^{(2)}(x_{2,t},x_{2,u})$ as $\tilde{G}_2^{(1)}(y_{1,t}-I_1)$ and $\tilde{G}_2^{(2)}(y_{1,t}-I_1)$ respectively.

Now, $\tilde{G}_2^{(1)}(y_{1,t}-I_1)=\left[(\bar{q}_t-\underline{q}_t)\beta_t\right](1-\delta)+\delta(\Delta_{1,t}-\Delta_{1,u})(1-F_1(y_{1,t}-I_1))\Rightarrow\tilde{G}_2^{(1)}(y_{1,t}-I_1)>0$. This implies $\Delta_{2,t}-\delta J_1^{(2)}(y_{1,t},I_1)>0$. Thus, **(M3)** holds for $n=2$

Also, $\tilde{G}_2^{(1)}(y_{1,t}-I_1)-\tilde{G}_2^{(2)}(y_{1,t}-I_1)>0$. The solution to $\tilde{G}_2^{(2)}(y_{1,t}-I_1)=0$ is given by $y_{1,t}-I_1=F_1^{-1}\left(1+\frac{\Delta_{2,u}+\delta\beta_t\tilde{\Delta}_{1,t}-\delta\Delta_{1,t}}{\delta(\Delta_{1,t}-\Delta_{1,u})(1+\beta_t)}\right)=\theta_{2,u}(say)$.

Thus, the solution to $G_2^{(2)}(y_{2,t},x_{2,u})=0$ is given by $x_{2,u}=\frac{w_2+\theta_{2,t}}{(1+\beta_t)}-y_{2,t}=\phi_{2,u}(w_2)-y_{2,t}$. Also, $0<\phi'_{2,u}(w_2)=\frac{1}{(1+\beta_t)}<1$. Thus, **(M4)** holds for $n=2$. The optimal policy for $n=2$ can be derived from EC.2 by taking the limit $\gamma_t\rightarrow 0$.

Lemma 1 *If $J_{n-1}(y_{n-1,t},I_{n-1})$ is jointly concave in its arguments and $y_{n-1,t}=\beta_t(y_{n,t}+x_{n,u})$, $I_{n-1}=w_n-x_{n,t}-x_{n,u}$ then the function $J_{n-1}(\beta_t(y_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u})$ is jointly concave in $(\beta_t(y_{n,t}+x_{n,u}),w_n)$*

Proof. Let $0\leq\theta\leq 1$ and $\bar{\theta}=1-\theta$ and let $(y_{n-1,t},I_n)$, $(\bar{y}_{n-1,t},\bar{I}_{n-1})$ be two points in domain of $J_{n-1}(y_{n-1,t},I_{n-1})$. As $J_{n-1}(y_{n-1,t},I_{n-1})$ is concave, we have $J_{n-1}(\theta y_{n-1,t}+\bar{\theta}\bar{y}_{n-1,t},\theta I_n+\bar{\theta}\bar{I}_{n-1})\geq\theta J_{n-1}(y_{n-1,t},I_n)+\bar{\theta}J_{n-1}(\bar{y}_{n-1,t},\bar{I}_{n-1})$. Thus,

$$\begin{aligned}J_{n-1}&\left((\theta\beta_t(y_{n,t}+x_{n,u})+\bar{\theta}\beta_t(\bar{y}_{n,t}+\bar{x}_{n,u})),\theta w_{n+1}+\bar{\theta}\bar{w}_n-\theta x_{n,t}-\bar{\theta}\bar{x}_{n,t}-\theta x_{n,u}-\bar{\theta}\bar{x}_{n,u}\right)\\&\geq\theta J_{n-1}\left(\beta_t(y_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u}\right)+\bar{\theta}J_{n-1}(\beta_t(\bar{y}_{n,t}+\bar{x}_{n,u}),\bar{w}_n-\bar{x}_{n,t}-\bar{x}_{n,u})\end{aligned}$$

Thus $J_{n-1}(\beta_t(y_{n,t}+x_{n,u}),w_{n+1}-x_{n,t}-x_{n,u})$ is jointly concave in $(\beta_t(y_{n,t}+x_{n,u}),w_{n+1})$. ■

Now, $G_n(\mathbf{x}_n,(y_{n,t},w_n))=\Delta_{n,t}x_{n,t}+\Delta_{n,u}x_{n,u}+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t(y_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u})$. From Lemma 1 and the assumption that $J_{n-1}(y_{n-1,t},I_{n-1})$ is concave in its argument we get that $G_n(\mathbf{x}_n,(y_{n,t},w_n))$ jointly concave in $((y_{n,t}+x_{n,u}),(y_{n,t},w_n))$.

Lemma 2 *(Heyman and Sobel, 1984:525) If the set C_n is convex and $G_n(\mathbf{x}_n,(y_{n,t},w_n))$ is a concave function on C_n, then $V_n(y_{n,t},w_n)$ is concave in $(y_{n,t},w_n)$.*

Proof. Refer 'Stochastic Inventory Theory by Evan L. Porteus - Theorem A.4'

As $J_n(y_{n,t}, I_n) = E_{z_n}[V_n(y_{n,t}, w_n)]$, we conclude from Lemma 1 and 2 that $J_n(y_{n,t}, I_n)$ is jointly concave in its argument. From **(M1)**, $G_n(x_{n,t}, x_{n,u})$ is twice differentiable and by the definition of $J_n(y_{n,t}, I_n)$, we get that $J_n(y_{n,t}, I_n)$ is twice differentiable. Now $G_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n)$. As $J_n(y_{n,t}, I_n)$ is twice differentiable and all other terms are linear, we conclude that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is twice differentiable. From **(M5)**, we have the optimal policy in period n. We now show that, $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0$.

To calculate the partials of $J_n(y_{n,t}, I_n)$, we partition the state space in period n, $(y_{n,t}, w_n)$, into two regions viz., S_A and S_B. The regions are defined as : $S_A = \{(y_{n,t}, w_n) \mid 0 \leq y_{n,t} \leq \theta_{n,u}\}$ and $S_B = \{(y_{n,t}, w_n) \mid \theta_{n,u} \leq y_{n,t}\}$.

The region S_A is further divided into 3 sub-regions - (i) $S_A^1 : 0 \leq z_n \leq y_{n,t} - I_n$ (ii) $S_A^2 : y_{n,t} - I_n \leq z_n \leq \theta_{n,u} - I_n$ and (iii) $S_A^3 : \theta_{n,u} - I_n \leq z_n \leq z_n^U$.

In sub-region S_A^1, we have $x_{n,t}^* = w_n$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = 0$. In sub-region S_A^2, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = w_n - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = 0$. In sub-region S_A^3, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = \phi_{n,u}(w_n) - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t \phi_{n,u}(w_n)$ and $I_{n-1} = w_n - \phi_{n,u}(w_n)$. We now evaluate $V_n(y_{n,t}, w_n)$ in each of the above sub-regions. In sub-region S_A^1,

$$V_n(y_{n,t}, w_n) = \Delta_{n,t}w_n + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0) = \Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)$$

In sub-region S_A^2,

$$\begin{aligned} V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t} + \Delta_{n,u}(w_n - y_{n,t}) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0) \\ &= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + (I_n + z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t y_{n,t}, 0) \end{aligned}$$

In sub-region S_A^3,

$$\begin{aligned} V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t} + \Delta_{n,u}(\phi_{n,u}(w_n) - y_{n,t}) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t(\phi_{n,u}(w_n)), w_n - \phi_{n,u}(w_n)) \\ &= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \phi_{n,u}(I_n + z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t \phi_{n,u}(I_n + z_n), I_n + z_n - \phi_{n,u}(I_n + z_n)) \end{aligned}$$

Now, $J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\, dz_n$. Thus in regions S_A,

$$J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$+ \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$

We evaluate the individual terms in the above equation to calculate $J_n^{(1)}(y_{n,t}, I_n)$ in region S_A

$$\frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} \left[\Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + (\tilde{\Delta}_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)) F_n(y_{n,t} - I_n)$$

$$\frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} \left[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + (I_n + z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= -f_n(y_{n,t} - I_n)\left[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \Delta_{n,u} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right]$$
$$+ \left[\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)\right](F_n(\theta_{n,u} - I_n) - F_n(y_{n,t} - I_n))$$

$$\frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} \Big[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \phi_{n,u}(I_n + z_n)\Delta_{n,u}$$
$$+ \delta J_{n-1}(\beta_t \phi_{n,u}(I_n + z_n), I_n + z_n - \phi_{n,u}(I_n + z_n))\Big] f_n(z_n)\,\mathrm{d}z_n$$
$$= \left[\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}\right](1 - F_n(\theta_{n,u} - I_n))$$

$$\begin{aligned}
\therefore J_n^{(1)}(y_{n,t}, I_n) &= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] \\
&\quad + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) F_n(y_{n,t} - I_n) + \tilde{\Delta}_{n,t} F_n(y_{n,t} - I_n) \\
&\quad - f_n(y_{n,t} - I_n)\left[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \Delta_{n,u} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] \\
&\quad + \left[\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}\right](F_n(\theta_{n,u} - I_n) - F_n(y_{n,t} - I_n)) \\
&\quad + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)(F_n(\theta_{n,u} - I_n) - F_n(y_{n,t} - I_n)) \\
&\quad + \left[\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}\right](1 - F_n(\theta_{n,u} - I_n)) \\
&= \tilde{\Delta}_{n,t} F_n(y_{n,t} - I_n) + (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u})(1 - F_n(y_{n,t} - I_n)) + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) F_n(y_{n,t} - I_n) \\
\therefore J_n^{(1,2)}(y_{n,t}, I_n) &= (\Delta_{n,t} - \Delta_{n,u} - \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)) f_n(y_{n,t} - I_n) \\
&= \left(G_n^{(1)}(x_{n,t}, x_{n,u})|_{y_{n,t}, w_n - y_{n,t}} - G_n^{(2)}(x_{n,t}, x_{n,u})|_{y_{n,t}, w_n - y_{n,t}}\right) f_n(y_{n,t} - I_n) > 0
\end{aligned}$$

The region S_B is further divided into 3 sub-regions - (i) $S_B^1 : 0 \le z_n \le y_{n,t} - I_n$ (ii) $S_B^2 : y_{n,t} - I_n \le z_n < \phi_{n,u}^{-1}(y_{n,t}) - I_n$ and (iii) $S_B^3 : \phi_{n,u}^{-1}(y_{n,t}) - I_n \le z_n \le z_n^U$. In sub-region S_B^1, we have $x_{n,t}^* = w_n$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = 0$. In sub-region S_B^2, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = w_n - y_{n,t}$. In sub-region S_B^3, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = \phi_{n,u}(w_n) - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t \phi_{n,u}(w_n)$ and $I_{n-1} = w_n - \phi_{n,u}(w_n)$. We now evaluate $V_n(y_{n,t}, w_n)$ in each of the above sub-regions. In sub-region S_B^1,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t} w_n + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0) \\
&= \Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)
\end{aligned}$$

In sub-region S_B^2,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t} y_{n,t} + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, w_n - y_{n,t}) \\
&= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t}) + \delta J_{n-1}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})
\end{aligned}$$

In sub-region S_B^3,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t} y_{n,t} + \Delta_{n,u}(\phi_{n,u}(w_n) - y_{n,t}) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t \phi_{n,u}(w_n), w_n - \phi_{n,u}(w_n)) \\
&= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \Delta_{n,u}(\phi_{n,u}(I_n + z_n)) + \delta J_{n-1}(\beta_t \phi_{n,u}(w_n), I_n + z_n - \phi_{n,u}(w_n))
\end{aligned}$$

Now, $J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$. Thus in regions S_B,

$$J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$+ \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$

We evaluate the individual terms in the above equation to calculate $J_n^{(1)}(y_{n,t}, I_n)$ in region S_B

$$\frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} \left[\Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + (\tilde{\Delta}_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)) F_n(y_{n,t} - I_n)$$

$$\frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \left[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t}) + \delta J_{n-1}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= \phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$
$$- f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + (\Delta_{n,t} + \tilde{\Delta}_{n,t})(F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - F_n(y_{n,t} - I_n))$$
$$+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$

$$\frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} \Bigg[(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u}(\phi_{n,u}(I_n + z_n)) + \delta J_{n-1}(\beta_t \phi_{n,u}(w_n), I_n + z_n - \phi_{n,u}(w_n))\Bigg] f_n(z$$

$$= -\phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\Bigg[(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u} y_{n,t}$$

$$+ \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\Bigg] + \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) f_n(z_n)\,\mathrm{d}z_n$$

$$= -\phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$

$$+ (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u})(1 - F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n))$$

$$\therefore\ J_n^{(1)}(y_{n,t}, I_n)$$

$$= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + (\tilde{\Delta}_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0)) F_n(y_{n,t} - I_n)$$

$$+ \phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$

$$- f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right]$$

$$+ (\Delta_{n,t} + \tilde{\Delta}_{n,t})(F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - F_n(y_{n,t} - I_n))$$

$$+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$

$$- \phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$

$$+ (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u})(1 - F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n))$$

$$= (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \Delta_{n,u} F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - \Delta_{n,t} F_n(y_{n,t} - I_n) + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) F_n(y_{n,t} - I_n)$$

$$+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$

$$\therefore\ J_n^{(1,2)}(y_{n,t}, I_n)$$

$$= -\Delta_{n,u} f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) + \Delta_{n,t} f_n(y_{n,t} - I_n) - \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) f_n(y_{n,t} - I_n)$$

$$- \delta f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$

$$+ \delta f_n(y_{n,t} - I_n)\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) - J_{n-1}^{(2)}(\beta_t y_{n,t}, 0)\right]$$

$$+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$

Rearranging the terms, we get :

$$
\begin{aligned}
&J_n^{(1,2)}(y_{n,t}, I_n) \\
&= f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\Big[-\Delta_{n,u} - \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) + \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\Big] \\
&+ f_n(y_{n,t} - I_n)\Big[\Delta_{n,t} - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, 0)\Big] \\
&+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

From **(M6)**-(a), we have $G_n^{(1)}(y_{n,t}, 0) = \Delta_{n,t} - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, 0) > 0$ and from **(M6)**-(b), we have $G_n^{(2)}(x_{n,t}^*, x_{n,u}^*) = G_n^{(2)}(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) > 0$. As $J_{n-1}(y_{n-1,t}, I_{n-1})$ is concave, we have $J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) < 0$ and as we have assumed $J_{n-1}^{(1,2)}(y_{n-1,t}, I_{n-1}) \geq 0$, we have $J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) \geq 0$. Using these components, we get

$$
\begin{aligned}
&J_n^{(1,2)}(y_{n,t}, I_n) \\
&= f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\Big[-G_n^{(2)}(y_{n,t}, 0)\Big] + f_n(y_{n,t} - I_n)\Big[G_n^{(1)}(y_{n,t}, 0)\Big] \\
&+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n \\
&> 0
\end{aligned}
$$

Thus, $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0 \quad \forall (y_{n,t}, I_n)$ ■

Next, we show that $\Delta_{n+1,t} - \delta J_n^{(2)}(y_{n,t}, I_n) > 0$. In region S_A, we have:

$$
\begin{aligned}
J_n^{(2)}(y_{n,t}, I_n) = {}& \frac{\partial}{\partial I_n}\int_0^{y_{n,t}-I_n} V_n(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial I_n}\int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n \\
&+ \frac{\partial}{\partial I_n}\int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

We evaluate individual terms:

$$
\frac{\partial}{\partial I_n}\int_0^{y_{n,t}-I_n} V_n(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n = -V_n(y_{n,t}, y_{n,t}) f_n(y_{n,t} - I_n) + \int_0^{y_{n,t}-I_n} V_n^{(2)}(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n
$$

$$
\frac{\partial}{\partial I_n}\int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t}, I_n + z_n) f_n(z_n)\,\mathrm{d}z_n = V_n(y_{n,t}, y_{n,t}) f_n(y_{n,t} - I_n) - V_n(y_{n,t}, \theta_{n,u}) f_n(\theta_{n,u} - I_n)
$$

$$+\int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

$$\frac{\partial}{\partial I_n}\int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n = V_n(y_{n,t},\theta_{n,u}) f_n(\theta_{n,u}-I_n) + \int_{\theta_{n,u}-I_n}^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

$$\therefore J_n^{(2)}(y_{n,t}, I_n) = \int_0^{y_{n,t}-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n + \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n + \int_{\theta_{n,u}-I_n}^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

Since $V_n(y_{n,t}, w_n)$ is a concave function, we have $V_n^{(2,2)}(y_{n,t}, w_n) < 0$. Also, in sub-region S_A^1, $V_n^{(2)}(y_{n,t}, I_n+z_n) = \Delta_{n,t}$. Therefore,

$$J_n^{(2)}(y_{n,t}, I_n) \le \int_0^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n)|_{S_A^1} f_n(z_n)\,\mathrm{d}z_n = \Delta_{n,t} = \Delta_{n+1,t} \le \frac{\Delta_{n+1,t}}{\delta}$$

Similarly, in region S_B, we have:

$$J_n^{(2)}(y_{n,t}, I_n) = \frac{\partial}{\partial I_n}\int_0^{y_{n,t}-I_n} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial I_n}\int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial I_n}\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

We evaluate individual terms:

$$\frac{\partial}{\partial I_n}\int_0^{y_{n,t}-I_n} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n = -V_n(y_{n,t}, y_{n,t}) f_n(y_{n,t}-I_n) + \int_0^{y_{n,t}-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

$$\frac{\partial}{\partial I_n}\int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n = V_n(y_{n,t}, y_{n,t}) f_n(y_{n,t}-I_n) - V_n(y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})) f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n) + \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

$$\frac{\partial}{\partial I_n}\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n = V_n(y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})) f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)$$

$$+\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n$$

$$\begin{aligned}\therefore J_n^{(2)}(y_{n,t}, I_n) = &\int_0^{y_{n,t}-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n + \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n\\ &+\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n) f_n(z_n)\,\mathrm{d}z_n\end{aligned}$$

Since $V_n(y_{n,t}, w_n)$ is a concave function, we have $V_n^{(2,2)}(y_{n,t}, w_n) < 0$. Also, in sub-region S_B^1, $V_n^{(2)}(y_{n,t}, I_n + z_n) = \Delta_{n,t}$. Therefore,

$$J_n^{(2)}(y_{n,t}, I_n) \le \int_0^{z_n^U} V_n^{(2)}(y_{n,t}, I_n+z_n)|_{S_A^1} f_n(z_n)\,\mathrm{d}z_n = \Delta_{n,t} = \Delta_{n+1,t} \le \frac{\Delta_{n+1,t}}{\delta}$$

Therefore, $\Delta_{n+1,t} - \delta J_n^{(2)}(y_{n,t}, I_n) \ge 0 \ \forall (y_{n,t}, I_n)$ ■

Now in period $n+1$,

$$\begin{aligned}G_{n+1}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t}\\ &\quad +\delta J_n(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ \Rightarrow G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t} - \delta J_n^{(2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ \Rightarrow G_{n+1}^{(1,1)}(x_{n+1,t}, x_{n+1,u}) &= \delta J_n^{(2,2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\end{aligned}$$

Similarly,

$$\begin{aligned}G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ &\quad -\delta J_n^{(2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ \Rightarrow G_{n+1}^{(2,2)}(x_{n+1,t}, x_{n+1,u}) &= \delta\beta_t^2 J_n^{(1,1)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ &\quad -2\delta\beta_t J_n^{(1,2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ &\quad +\delta J_n^{(2,2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\end{aligned}$$

and

$$\begin{aligned}G_{n+1}^{(1,2)}(x_{n+1,t}, x_{n+1,u}) &= -\delta\beta_t J_n^{(1,2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\\ &\quad +\delta J_n^{(2,2)}(\beta_t(y_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})\end{aligned}$$

As $J_n(y_{n,t}, I_n)$ is jointly concave and $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0$, we have

$$J_n^{(1,1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) < 0$$
$$J_n^{(2,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) < 0$$
$$J_n^{(1,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \geq 0$$

Thus, $G_{n+1}^{(1,1)}(x_{n+1,t}, x_{n+1,u}) < 0, G_{n+1}^{(1,2)}(x_{n+1,t}, x_{n+1,u}) < 0$, and $G_{n+1}^{(2,2)}(x_{n+1,t}, x_{n+1,u}) < 0$. Thus, $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is jointly concave as its Hessian matrix is negative semidefinite.

We now show that **(M4)** holds for period $n+1$. In period $n+1$, we have $G_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u}x_{n+1,u} + \Delta_{n+1,t}x_{n+1,t} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n)$. Therefore, $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t(y_{n+1,t} + x_{n+1,u}), w_{n+1} - y_{n+1,t} - x_{n+1,u}) - \delta J_n^{(2)}(\beta_t(y_{n+1,t} + x_{n+1,u}), w_{n+1} - y_{n+1,t} - x_{n+1,u})$. Define $z_{n+1,u} := x_{n+1,u} + y_{n+1,t}$. Thus, $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - \delta J_n^{(2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})$. Thus $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = 0 \Rightarrow z_{n+1,u} = \phi_{n+1,u}(w_{n+1})$. Thus, $x_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$. Now, $\phi_{n+1,u}(w_{n+1})$ is differentiable as $J_n(y_{n,t}, I_n)$ is differentiable. From implicit function theorem, we have

$$\begin{aligned}
&\phi'_{n+1,u}(w_{n+1}) \\
&= -\frac{\delta\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - \delta J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}{\delta\beta_t^2 J_n^{(1,1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - 2\delta\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) + \delta J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})} \\
&= \frac{\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}{\beta_t^2 J_n^{(1,1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - 2\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) + J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}
\end{aligned}$$

In shorthand, $\phi'_{n+1,u}(w_{n+1}) = -\frac{\beta_t J_n^{(1,2)}(\cdot) - J_n^{(2,2)}(\cdot)}{\beta_t^2 J_n^{(1,1)}(\cdot) - 2\beta_t J_n^{(1,2)}(\cdot) + J_n^{(2,2)}(\cdot)}$. As $J_n(y_{n,t}, I_n)$ is concave, we have $J_n^{(1,1)}(\cdot) < 0$ and $J_n^{(2,2)}(\cdot) < 0$. Also we have $J_n^{(1,2)}(\cdot) \geq 0$. Thus, using these components we get $0 < \phi'_{n+1,u}(w_{n+1}) < 1$ ∎

Next, we derive the optimal policy for period $n+1$. Note that $q_u > \underline{q}_t$ implies $\Delta_{n+1,t} > \Delta_{n+1,u}$ for $\gamma_t = 0$. Thus, $G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) - G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u}) > 0$.

Now, we need to solve the stochastic dynamic program as stated in (EC.1). The Lagrangian of the optimization problem is given by:

$$\begin{aligned}
\Lambda(x_{n+1,t}, x_{n+1,u}, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = {} & G_{n+1}(x_{n+1,t}, x_{n+1,u}) + \lambda_1(w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
& + \lambda_2(y_{n+1,t} - x_{n+1,t}) + \lambda_3(x_{n+1,t}) + \lambda_4(x_{n+1,u})
\end{aligned}$$

The stationarity conditions are given by $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - \lambda_1 - \lambda_2 + \lambda_3 = 0$ and $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - \lambda_1 + \lambda_4 = 0$. The conditions for primal feasibility are - (i) $x^*_{n+1,t} + x^*_{n+1,u} \le w_{n+1}$ (ii) $x^*_{n+1,t} \le y_{n+1,t}$(iii) $x^*_{n+1,t} \ge 0$ and (iv) $x^*_{n+1,u} \ge 0$. The complementary slackness conditions are (i) $\lambda_1(w_{n+1} - x^*_{n+1,t} - x^*_{n+1,u}) = 0$ (ii) $\lambda_2(y_{n+1,t} - x^*_{n+1,t}) = 0$ (iii) $\lambda_3(x^*_{n+1,t}) = 0$ and (iv) $\lambda_4(x^*_{n+1,u}) = 0$. The dual feasibility conditions are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \ge 0$. The state space is partitioned by considering permutations on the values taken by KKT multipliers. We only show the solution to non-trivial permutation below.

EC.3.2.1. Case 1: $\lambda_1 > 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0 \quad \lambda_4 > 0$

From stationarity conditions, we get $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. From complementary slackness, we get $x^*_{n+1,t} = w_{n+1}$ and $x^*_{n+1,u} = 0$. Primal feasibility leads to $0 \le x^*_{n+1,t} \le y_{n+1,t}$. Thus, the feasible state space is given by Z_D : $\{(y_{n+1,t}, w_{n+1}) \mid 0 \le w_{n+1} \le y_{n+1,t}\}$ and the optimal policy in this region is given by $x^*_{n+1,t} = w_{n+1}$ and $x^*_{n+1,u} = 0$.

Define $g_{n+1,A}(y_{n+1,t}, w_{n+1}) \triangleq G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = G^{(2)}_{n+1}(w_{n+1}, 0)$. Note that arguments of $g_{n+1,A}(y_{n+1,t}, w_{n+1})$ satisfy $0 \le w_{n+1} \le y_{n+1,t}$. Let $q_{n+1,A}(y_{n+1,t})$ be a function that solves $g_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) = 0$. By Implicit Function Theorem, $q'_{n+1,A}(y_{n+1,t}) = -\frac{g^{(1)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t}))}{g^{(2)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t}))}$. Now,

$$
\begin{aligned}
g^{(1)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})\frac{\partial x^*_{n+1,t}}{\partial y_{n+1,t}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})\frac{\partial x^*_{n+1,u}}{\partial y_{n+1,t}} = 0 \\
g^{(2)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})\frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})\frac{\partial x^*_{n+1,u}}{\partial w_{n+1}} \\
&= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \\
\Rightarrow q'_{n+1,A}(y_{n+1,t}) &= 0 \\
\therefore q_{n+1,A}(y_{n+1,t}) &= \theta_{n+1,u} \quad (say) \\
\therefore g_{n+1,A}(y_{n+1,t}, \theta_{n+1,u}) &= 0 \;\forall\; 0 \le w_{n+1} \le y_{n+1,t}
\end{aligned}
$$

As $(\theta_{n+1,u}, \theta_{n+1,u}) \in g_{n+1,A}(y_{n+1,t}, w_{n+1})$, we get:

$$
q_{n+1,A}(\theta_{n+1,u}) = \theta_{n+1,u} \tag{EC.9}
$$

EC.3.2.2. Case 2: $\lambda_1 > 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 = 0$

From stationarity conditions, we get (i) $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and (ii) $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ which leads to (iii) $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$.

From complementary slackness, we get (i) $x^*_{n+1,t} = y_{n+1,t}$ and (ii) $x^*_{n+1,u} = w_{n+1} - y_{n+1,t}$. Substituting this in the primal feasibility condition gives $0 \le y_{n+1,t} \le w_{n+1}$. Define $g_{n+1,B}(y_{n+1,t}, w_{n+1}) \triangleq G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = G^{(2)}_{n+1}(y_{n+1,t}, w_{n+1} - y_{n+1,t})$. Note that arguments $g_{n+1,B}(y_{n+1,t}, w_{n+1})$ satisfy $0 \le y_{n+1,t} \le w_{n+1}$. Define $q_{n+1,B}(y_{n+1,t})$ as a function that solves $g_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) = 0$. By Implicit Function Theorem, $q'_{n+1,B}(y_{n+1,t}) = -\frac{g^{(1)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t}))}{g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t}))}$. Now,

$$\begin{aligned} g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial w_{n+1}} \\ &= G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \end{aligned}$$

From **Case 1**, we have $g_{n+1,A}(y_{n+1,t}, \theta_{n+1,u}) = G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$. Note that the point $(\theta_{n+1,u}, \theta_{n+1,u})$ satisfy the condition $0 \le y_{n+1,t} \le w_{n+1}$ and thus belong to the domain of $g_{n+1,B}(y_{n+1,t}, w_{n+1})$. Put $y_{n+1,t} = w_{n+1} = \theta_{n+1,u}$ to get $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$. As $g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$ and $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = 0$, therefore $g_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) = 0 \ \forall \ 0 \le y_{n+1,t} \le w_{n+1} \Rightarrow q_{n+1,B}(y_{n+1,t}) = \theta_{n+1,u}$.

As $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = 0, g^{(1)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$ and $g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$, therefore $g_{n+1,B}(y_{n+1,t}, w_{n+1}) > 0 \Rightarrow y_{n+1,t} < \theta_{n+1,u}$ and $w_{n+1} < \theta_{n+1,u}$. Therefore, $y_{n+1,t} < \min\{\theta_{n+1,t}, \theta_{n+1,u}\} = \theta_{n+1,u}$ and $w_{n+1} < \min\{\theta_{n+1,t}, \theta_{n+1,u}\} = \theta_{n+1,u}$. Thus, the feasible state space is given by $Z_A : \{(y_{n+1,t}, w_{n+1}) : 0 \le y_{n+1,t} \le w_{n+1} \cap w_{n+1} < \theta_{n+1,u}\}$ and the optimal policy is given by : $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = w_{n+1} - y_{n+1,t}$

EC.3.2.3. Case 3: $\lambda_1 = 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 = 0$

From stationarity conditions, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = 0$. From complementary slackness, $x^*_{n+1,t} = y_{n+1,t}$. From **(M4)** and primal feasibility, we get $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$ and $0 \le y_{n+1,t} \le \phi_{n+1,u}(w_{n+1}) \le w_{n+1}$. From **Case 1**, $G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$ and from **(M4)**, $\phi_{n+1,u}(w_{n+1})$ is continuous. Thus $\lim_{w_{n+1} \to \theta_{n+1,u}} \phi_{n+1,u}(w_{n+1}) = \phi_{n+1,u}(\theta_{n+1,u})$. As $G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})$ is strictly monotonous and as $G^{(2)}_{n+1}(y_{n+1,t}, \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}) = 0$, putting $y_{n+1,t} = \theta_{n+1,u}$, we get $\phi_{n+1,u}(\theta_{n+1,u}) = \theta_{n+1,u}$. Also, as $\phi'_{n+1,u}(w_{n+1}) < 1$, $\phi_{n+1,u}(w_{n+1}) \le w_{n+1}$ and $\phi_{n+1,u}(\theta_{n+1,u}) = \theta_{n+1,u}$, we get $w_{n+1} \ge \theta_{n+1,u}$. As $\phi_{n+1,u}(w_{n+1})$ is strictly monotonous, we get $y_{n+1,t} \le \phi_{n+1,u}(w_{n+1}) \Rightarrow w_{n+1} \ge \phi^{-1}_{n+1,u}(y_{n+1,t})$. Thus, the feasible state space is given by $Z_B : \left\{(y_{n+1,t}, w_{n+1}) : 0 \le y_{n+1,t} \le w_{n+1} \cap w_{n+1} \ge \max\{\theta_{n+1,u}, \phi^{-1}_{n+1,u}(y_{n+1,t})\}\right\}$ and the optimal policy is given by $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$

EC.3.2.4. Case 4: $\lambda_1 = 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 > 0$

From stationarity conditions, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$. Thus, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. From complementary slackness, $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = 0$. From primal feasibility, $0 \le y_{n+1,t} \le w_{n+1}$. From **(M4)**, the solution to $G^{(2)}_{n+1}(y_{n+1,t}, x_{n+1,u}) = 0$ is given by $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$. As $G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$, $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \Rightarrow G^{(2)}_{n+1}(y_{n+1,t}, 0) < 0 \Rightarrow y_{n+1,t} > \phi_{n+1,u}(w_{n+1}) \Rightarrow w_{n+1} < \phi^{-1}_{n+1,u}(y_{n+1,t})$. Thus, the feasible state space is given by Z_C : $\left\{(y_{n+1,t}, w_{n+1}) : 0 \le y_{n+1,t} \le w_{n+1} \cap \phi^{-1}_{n+1,t}(y_{n+1,t}) < w_{n+1} < \phi^{-1}_{n+1,u}(y_{n+1,t})\right\}$ and the optimal policy is given by $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = 0$. Thus Proposition 3 - ii holds true. ∎

EC.4. Proof of Proposition 4 (Optimal policy for $\gamma_t = 1$)

To prove this proposition, we assume certain structure on $G_n(x_{n,t}, x_{n,u})$, $J_{n-1}(y_{n-1,t}, I_{n-1})$ and the optimal policy for period n. Next, assuming this structure for period n, we show that the same structure holds through for period $n+1$ and derive the optimal policy. This structure for period n is stated as follows:

(L1) $G_n(x_{n,t}, x_{n,u})$ and $J_{n-1}(y_{n-1,t}, I_{n-1})$ are twice differentiable and jointly concave in their arguments

(L2) $J^{(1,2)}_{n-1}(y_{n-1,t}, I_{n-1}) \ge 0$

(L3) The solution to $G^{(1)}_n(x_{n,t}, 0) = 0$ is given by $x^*_{n,t} = \phi_{n,t}(w_n)$ where $\phi_{n,t}(w_n)$ is differentiable and satisfies $0 < \phi'_{n,t}(w_n) < 1$

(L4) The solution to $G^{(2)}_n(y_{n,t}, x_{n,u}) = 0$ is given by $x^*_{n,u} = \phi_{n,u}(w_n) - y_{n,t}$ where $\phi_{n,u}(w_n)$ is differentiable and satisfies $0 < \phi'_{n,u}(w_n) < 1$

(L5) The optimal policy for period n is as described in Table 3.

(L6) Under the optimal policy, we further claim the following:

(a) $G^{(1)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = y_{n,t})} = G^{(1)}_n(y_{n,t}, 0) = \Delta_{n,t} + \delta\beta_t J^{(1)}_{n-1}(\beta_t y_{n,t}, 0) - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, 0) > 0 \ \forall \theta_{n,u} \le y_{n,t} \le \theta_{n,t}$

(b) $G^{(2)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = \phi^{-1}_{n,u}(y_{n,t}))} = G^{(2)}_n(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J^{(1)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) > 0 \ \forall \theta_{n,u} \le y_{n,t} \le \theta_{n,t}$

(c) $G^{(1)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = \phi^{-1}_{n,t}(y_{n,t}))} = G^{(1)}_n(y_{n,t}, 0) = \Delta_{n,t} + \delta\beta_t J^{(1)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,t}(y_{n,t}) - y_{n,t}) - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,t}(y_{n,t}) - y_{n,t}) = 0 \ \forall \theta_{n,t} \le y_{n,t}$

(d) $G^{(2)}_n(x^*_{n,t}, x^*_{n,u})|_{(y_{n,t}, w_n = \phi^{-1}_{n,u}(y_{n,t}))} = G^{(2)}_n(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J^{(1)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) - \delta J^{(2)}_{n-1}(\beta_t y_{n,t}, \phi^{-1}_{n,u}(y_{n,t}) - y_{n,t}) = 0 \ \forall \theta_{n,t} \le y_{n,t}$

We initiate the induction by showing that the claim holds true for $N = 2$. Then, assuming the above claims hold true for period n, we show that $J_n(y_{n,t}, I_n)$ is jointly concave in its arguments and is twice differentiable. From this, we show that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is twice differentiable. Next, we show that **(L2)** holds for period $n+1$ and consequently that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is concave. After showing that **(L1)** and **(L2)** holds for period $n+1$, we show that **(L3)** and **(L4)** holds true for period $n+1$. Finally, using the components **(L1) - (L4)**, we complete the induction by deriving the optimal policy for period $n+1$.

The objective function for $n = 1$ is given by $G_1(x_{1,t}, x_{1,u}) = \Delta_{1,u}x_{1,u} + \Delta_{1,t}x_{1,t} + \tilde{\Delta}_{1,t}y_{1,t}$. As $G_1^{(1)}(x_{1,t}, x_{1,u}) = \Delta_{1,t} > 0$ and $G_1^{(2)}(x_{1,t}, x_{1,u}) = \Delta_{1,u} > 0$ and $\Delta_{1,t} > \Delta_{1,u}$, we get $x^*_{1,u} = [w_1 - y_{1,t}]^+$ and $x^*_{1,t} = \min\{y_{1,t}, w_1\}$. $\therefore V_1(y_{1,t}, w_1) = G_1(x^*_{1,t}, x^*_{1,u}) = \Delta_{1,u}[w_1 - y_{1,t}]^+ + \Delta_{1,t}\min\{y_{1,t}, w_1\} + \tilde{\Delta}_{1,t}y_{1,t}$. Thus,

$$
\begin{aligned}
J_1(y_{1,t}, I_1) &= E_{z_1}[V_1(y_{1,t}, w_1)] = E_{z_1}[G_1(x^*_{1,t}, x^*_{1,u})] \\
&= E_{z_1}\left[\Delta_{1,u}[w_1 - y_{1,t}]^+ + \Delta_{1,t}\min\{y_{1,t}, w_1\} + \tilde{\Delta}_{1,t}y_{1,t}\right] \\
&= \tilde{\Delta}_{1,t}y_{1,t} + \Delta_{1,t}I_1 + \Delta_{1,u}\mu_1 + (\Delta_{1,t} - \Delta_{1,u})\left((y_{1,t} - I_1) - \int_0^{y_{1,t}-I_1} F_1(z_1)\mathrm{d}z_1\right)
\end{aligned}
$$

Note that $J_1(y_{1,t}, I_1)$ is twice differentiable and concave in its arguments. Also, $J_1^{(1)}(y_{1,t}, I_1) = \tilde{\Delta}_{1,t} + (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))$. Thus, $J_1^{(1,2)}(y_{1,t}, I_1) = (\Delta_{1,u} + \Delta_{1,t})\, f_1(z_1) \geq 0$. As $J_1(y_{1,t}, I_1)$ is twice differentiable and concave, we conclude that $G_2(x_{2,t}, x_{2,u})$ is twice differentiable and concave in its argument. Thus **(L1)** and **(L2)** holds true for $n = 2$. Now, $G_2(x_{2,t}, x_{2,u}) = \Delta_{2,t}x_{2,t} + \Delta_{2,u}x_{2,u} + \tilde{\Delta}_{2,t}y_{2,t} + \delta J_1(y_{1,t}, I_1)$. From (3), $y_{1,t} = \beta_t(x_{2,t} + x_{2,u})$ and from (5), $I_1 = w_2 - x_{2,t} - x_{2,u}$.

$$
\begin{aligned}
\therefore G_2^{(1)}(x_{2,t}, x_{2,u}) &= \Delta_{2,t} + \delta\left(J_1^{(1)}(y_{1,t}, I_1) - J_1^{(2)}(y_{1,t}, I_1)\right) \\
&= \Delta_{2,t} + \delta\left(\tilde{\Delta}_{1,t} + (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))\right) \\
&\quad -\delta\left(\Delta_{1,t} - (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))\right) \\
\text{and } G_2^{(2)}(x_{2,t}, x_{2,u}) &= \Delta_{2,u} + \delta\left(J_1^{(1)}(y_{1,t}, I_1) - J_1^{(2)}(y_{1,t}, I_1)\right) \\
&= \Delta_{2,u} + \delta\left(\tilde{\Delta}_{1,t} + (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))\right) \\
&\quad -\delta\left(\Delta_{1,t} - (\Delta_{1,t} - \Delta_{1,u})(1 - F_1(y_{1,t} - I_1))\right) \\
\therefore G_2^{(1)}(x_{2,t}, x_{2,u}) - G_2^{(2)}(x_{2,t}, x_{2,u}) &= \Delta_{2,t} - \Delta_{2,u} > 0
\end{aligned}
$$

Thus, it is optimal to prioritize treating *treated* patients over enrolling *untreated* patients. Note that $G_2^{(1)}(x_{2,t}, x_{2,u})$ and $G_2^{(2)}(x_{2,t}, x_{2,u})$ can be written as functions with argument

$(y_{1,t}-I_1)$. Thus we rewrite $G_2^{(1)}(x_{2,t},x_{2,u})$ as $\tilde{G}_2^{(1)}(y_{1,t}-I_1)$ and $G_2^{(2)}(x_{2,t},x_{2,u})$ as $\tilde{G}_2^{(2)}(y_{1,t}-I_1)$. Now, $\tilde{G}_2^{(1)}(y_{1,t}-I_1)=0$

$$\Rightarrow\ \Delta_{2,t}+\delta\left(\tilde{\Delta}_{1,t}+(\Delta_{1,t}-\Delta_{1,u})(1-F_1(y_{1,t}-I_1))\right)-\delta\left(\Delta_{1,t}-(\Delta_{1,t}-\Delta_{1,u})(1-F_1(y_{1,t}-I_1))\right)=0$$

$$\begin{aligned}\Rightarrow y_{1,t}-I_1 &= F_1^{-1}\left(1+\frac{\Delta_{2,t}+\delta(\tilde{\Delta}_{1,t}-\Delta_{1,t})}{2\delta(\Delta_{1,t}-\Delta_{1,u})}\right)\\ &=\theta_{2,t}\geq 0 \text{ (say)}\end{aligned}$$

Thus, the solution to $G_2^{(1)}(x_{2,t},0)=0$ is given by $x_{2,t}=\frac{w_2+\theta_{2,t}}{(1+\beta_t)}=\phi_{2,t}(w_2)$. Also, $0<\phi'_{2,t}(w_2)=\frac{1}{(1+\beta_t)}<1$. Thus, **(L3)** holds for $n=2$. Similarly, $\tilde{G}_2^{(2)}(y_{1,t}-I_1)=0$

$$\Rightarrow\ \Delta_{2,u}+\delta\left(\tilde{\Delta}_{1,t}+(\Delta_{1,t}-\Delta_{1,u})(1-F_1(y_{1,t}-I_1))\right)-\delta\left(\Delta_{1,t}-(\Delta_{1,t}-\Delta_{1,u})(1-F_1(y_{1,t}-I_1))\right)=0$$

$$\begin{aligned}\Rightarrow y_{1,t}-I_1 &= F_1^{-1}\left(1+\frac{\Delta_{2,u}+\delta(\tilde{\Delta}_{1,t}-\Delta_{1,t})}{2\delta(\Delta_{1,t}-\Delta_{1,u})}\right)\\ &=\theta_{2,u}\geq 0 \text{ (say)}\end{aligned}$$

Thus, the solution to $G_2^{(2)}(y_{2,t},x_{2,u})=0$ is given by $x_{2,u}=\frac{w_2+\theta_{2,t}}{(1+\beta_t)}-y_{2,t}=\phi_{2,u}(w_2)-y_{2,t}$. Also, $0<\phi'_{2,u}(w_2)=\frac{1}{(1+\beta_t)}<1$. Thus, **(L4)** holds for $n=2$. The optimal policy for $n=2$ can be derived from EC.2 by taking the limit $\gamma_t\to 1$.

Lemma 3 *If $J_{n-1}(y_{n-1,t},I_{n-1})$ is jointly concave in its arguments and $y_{n-1,t}=\beta_t(x_{n,t}+x_{n,u})$, $I_{n-1}=w_n-x_{n,t}-x_{n,u}$ then the function $J_{n-1}(\beta_t(x_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u})$ is jointly concave in $(\beta_t(x_{n,t}+x_{n,u}),w_n)$*

Proof. Let $0\leq\theta\leq 1$ and $\bar{\theta}=1-\theta$ and let $(y_{n-1,t},I_n)$, $(\bar{y}_{n-1,t},\bar{I}_{n-1})$ be two points in domain of $J_{n-1}(y_{n-1,t},I_{n-1})$. As $J_{n-1}(y_{n-1,t},I_{n-1})$ is concave, we have $J_{n-1}(\theta y_{n-1,t}+\bar{\theta}\bar{y}_{n-1,t},\theta I_n+\bar{\theta}\bar{I}_{n-1})\geq \theta J_{n-1}(y_{n-1,t},I_n)+\bar{\theta}J_{n-1}(\bar{y}_{n-1,t},\bar{I}_{n-1})$. Thus,

$$\begin{aligned}&J_{n-1}\left((\theta\beta_t(x_{n,t}+x_{n,u})+\bar{\theta}\beta_t(\bar{x}_{n,t}+\bar{x}_{n,u})),\theta w_{n+1}+\bar{\theta}\bar{w}_n-\theta x_{n,t}-\bar{\theta}\bar{x}_{n,t}-\theta x_{n,u}-\bar{\theta}\bar{x}_{n,u}\right)\\ &\quad\geq\theta J_{n-1}\left(\beta_t(x_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u}\right)+\bar{\theta}J_{n-1}(\beta_t(\bar{x}_{n,t}+\bar{x}_{n,u}),\bar{w}_n-\bar{x}_{n,t}-\bar{x}_{n,u})\end{aligned}$$

Thus $J_{n-1}(\beta_t(x_{n,t}+x_{n,u}),w_{n+1}-x_{n,t}-x_{n,u})$ is jointly concave in $(\beta_t(x_{n,t}+x_{n,u}),w_{n+1})$. ■

Now, $G_n(\mathbf{x}_n,(y_{n,t},w_n))=\Delta_{n,t}x_{n,t}+\Delta_{n,u}x_{n,u}+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t(x_{n,t}+x_{n,u}),w_n-x_{n,t}-x_{n,u})$. From Lemma 3 and the assumption that $J_{n-1}(y_{n-1,t},I_{n-1})$ is concave in its argument we get that $G_n(\mathbf{x}_n,(y_{n,t},w_n))$ jointly concave in $((x_{n,t}+x_{n,u}),(y_{n,t},w_n))$.

Lemma 4 *(Heyman and Sobel, 1984:525) If the set C_n is convex and $G_n(\mathbf{x}_n, (y_{n,t}, w_n))$ is a concave function on C_n, then $V_n(y_{n,t}, w_n)$ is concave in $(y_{n,t}, w_n)$.*

Proof. Refer 'Stochastic Inventory Theory by Evan L. Porteus - Theorem A.4'

As $J_n(y_{n,t}, I_n) = E_{z_n}[V_n(y_{n,t}, w_n)]$, we conclude from Lemma 3 and 4 that $J_n(y_{n,t}, I_n)$ is jointly concave in its argument. From **(L1)**, $G_n(x_{n,t}, x_{n,u})$ is twice differentiable and by the definition of $J_n(y_{n,t}, I_n)$, we get that $J_n(y_{n,t}, I_n)$ is twice differentiable. Now, $G_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n)$. As $J_n(y_{n,t}, I_n)$ is twice differentiable and all other terms are linear, we conclude that $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is twice differentiable. From **(L5)**, we have the optimal policy in period n. We now show that $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0$.

To calculate the partials of $J_n(y_{n,t}, I_n)$, we partition the state space in period n, $(y_{n,t}, w_n)$, into three regions viz., S_A, S_B and S_C. The regions are defined as : $S_A = \{(y_{n,t}, w_n) \mid 0 \leq y_{n,t} \leq \theta_{n,u}\}$, $S_B = \{(y_{n,t}, w_n) \mid \theta_{n,u} \leq y_{n,t} \leq \theta_{n,t}\}$ and $S_C = \{(y_{n,t}, w_n) \mid \theta_{n,t} \leq y_{n,t}\}$.

The region S_A is further divided into 3 sub-regions - (i) $S_A^1 : 0 \leq z_n \leq y_{n,t} - I_n$ (ii) $S_A^2 : y_{n,t} - I_n \leq z_n \leq \theta_{n,u} - I_n$ and (iii) $S_A^3 : \theta_{n,u} - I_n \leq z_n \leq z_n^U$. In sub-region S_A^1, we have $x_{n,t}^* = w_n$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t w_n$ and $I_{n-1} = 0$. In sub-region S_A^2, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = w_n - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t w_n$ and $I_{n-1} = 0$. In sub-region S_A^3, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = \phi_{n,u}(w_n) - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t \phi_{n,u}(w_n)$ and $I_{n-1} = w_n - \phi_{n,u}(w_n)$. We now evaluate $V_n(y_{n,t}, w_n)$ in each of the above sub-regions. In sub-region S_A^1,

$$V_n(y_{n,t}, w_n) = \Delta_{n,t}w_n + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t w_n, 0) = \Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t(I_n + z_n), 0)$$

In sub-region S_A^2,

$$\begin{aligned} V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t} + \Delta_{n,u}(w_n - y_{n,t}) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t w_n, 0) \\ &= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + (I_n + z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t(I_n + z_n), 0) \end{aligned}$$

In sub-region S_A^3,

$$\begin{aligned} V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t} + \Delta_{n,u}(\phi_{n,u}(w_n) - y_{n,t}) + \tilde{\Delta}_{n,t}y_{n,t} + \delta J_{n-1}(\beta_t(\phi_{n,u}(w_n)), w_n - \phi_{n,u}(w_n)) \\ &= y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \phi_{n,u}(I_n + z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t \phi_{n,u}(I_n + z_n), I_n + z_n - \phi_{n,u}(I_n + z_n)) \end{aligned}$$

Now, $J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\, \mathrm{d}z_n$. Thus in regions S_A,

$$J_n^{(1)}(y_{n,t},I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$+ \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n$$

We evaluate the individual terms in the above equation to calculate $J_n^{(1)}(y_{n,t},I_n)$ in region S_A

$$\frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} \left[\Delta_{n,t}(I_n+z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t(I_n+z_n),0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= f_n(y_{n,t}-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t},0)\right] + \tilde{\Delta}_{n,t} F_n(y_{n,t}-I_n)$$

$$\frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\theta_{n,u}-I_n} \left[y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}) + (I_n+z_n)\Delta_{n,u} + \delta J_{n-1}(\beta_t(I_n+z_n),0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= -f_n(y_{n,t}-I_n)\left[y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}) + \Delta_{n,u} y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t},0)\right]$$
$$+ \left[\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}\right](F_n(\theta_{n,u}-I_n) - F_n(y_{n,t}-I_n))$$

$$\frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} V_n(y_{n,t},w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,u}-I_n}^{z_n^U} \Big[y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}) + \phi_{n,u}(I_n+z_n)\Delta_{n,u}$$
$$+ \delta J_{n-1}(\beta_t \phi_{n,u}(I_n+z_n), I_n+z_n-\phi_{n,u}(I_n+z_n))\Big] f_n(z_n)\,\mathrm{d}z_n$$
$$= \left[\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}\right](1-F_n(\theta_{n,u}-I_n))$$

$$\begin{aligned}
\therefore J_n^{(1)}(y_{n,t}, I_n) &= f_n(y_{n,t}-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},0)\right]+\tilde{\Delta}_{n,t}F_n(y_{n,t}-I_n)\\
&\quad -f_n(y_{n,t}-I_n)\left[y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})+\Delta_{n,u}y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},0)\right]\\
&\quad +\left[\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}\right](F_n(\theta_{n,u}-I_n)-F_n(y_{n,t}-I_n))\\
&\quad +\left[\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u}\right](1-F_n(\theta_{n,u}-I_n))\\
&= \tilde{\Delta}_{n,t}F_n(y_{n,t}-I_n)+(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})(1-F_n(y_{n,t}-I_n))\\
\therefore J_n^{(1,2)}(y_{n,t}, I_n) &= (\Delta_{n,t}-\Delta_{n,u})f_n(y_{n,t}-I_n)>0
\end{aligned}$$

The region S_B is further divided into 3 sub-regions - (i) $S_B^1 : 0 \le z_n \le y_{n,t}-I_n$ (ii) $S_B^2 :$ $y_{n,t}-I_n \le z_n < \phi_{n,u}^{-1}(y_{n,t})-I_n$ and (iii) $S_B^3 : \phi_{n,u}^{-1}(y_{n,t})-I_n \le z_n \le z_n^U$. In sub-region S_B^1, we have $x_{n,t}^* = w_n$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t w_n$ and $I_{n-1} = 0$. In sub-region S_B^2, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = w_n - y_{n,t}$. In sub-region S_B^3, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = \phi_{n,u}(w_n) - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t \phi_{n,u}(w_n)$ and $I_{n-1} = w_n - \phi_{n,u}(w_n)$. We now evaluate $V_n(y_{n,t}, w_n)$ in each of the above sub-regions. In sub-region S_B^1,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t}w_n+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t w_n,0)\\
&= \Delta_{n,t}(I_n+z_n)+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t(I_n+z_n),0)
\end{aligned}$$

In sub-region S_B^2,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t}+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},w_n-y_{n,t})\\
&= y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t})+\delta J_{n-1}(\beta_t y_{n,t},I_n+z_n-y_{n,t})
\end{aligned}$$

In sub-region S_B^3,

$$\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t}y_{n,t}+\Delta_{n,u}(\phi_{n,u}(w_n)-y_{n,t})+\tilde{\Delta}_{n,t}y_{n,t}+\delta J_{n-1}(\beta_t\phi_{n,u}(w_n),w_n-\phi_{n,u}(w_n))\\
&= y_{n,t}(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})+\Delta_{n,u}(\phi_{n,u}(I_n+z_n))+\delta J_{n-1}(\beta_t\phi_{n,u}(w_n),I_n+z_n-\phi_{n,u}(w_n))
\end{aligned}$$

Now, $J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$. Thus in regions S_B,

$$\begin{aligned}
J_n^{(1)}(y_{n,t}, I_n) &= \frac{\partial}{\partial y_{n,t}}\int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}}\int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n\\
&\quad +\frac{\partial}{\partial y_{n,t}}\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n
\end{aligned}$$

We evaluate the individual terms in the above equation to calculate $J_n^{(1)}(y_{n,t}, I_n)$ in region S_B

$$
\begin{aligned}
&\frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n \\
&= \frac{\partial}{\partial y_{n,t}} \int_0^{y_{n,t}-I_n} \left[\Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t(I_n + z_n), 0)\right] f_n(z_n)\,\mathrm{d}z_n \\
&= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + \tilde{\Delta}_{n,t} F_n(y_{n,t} - I_n)
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n \\
&= \frac{\partial}{\partial y_{n,t}} \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \left[y_{n,t}(\Delta_{n,t} + \tilde{\Delta}_{n,t}) + \delta J_{n-1}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n \\
&= {\phi_{n,u}^{-1}}'(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right] \\
&- f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + (\Delta_{n,t} + \tilde{\Delta}_{n,t})(F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - F_n(y_{n,t} - I_n)) \\
&+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n \\
&= \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} \Bigg[(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u}(\phi_{n,u}(I_n + z_n)) + \delta J_{n-1}(\beta_t \phi_{n,u}(w_n), I_n + z_n - \phi_{n,u}(w_n))\Bigg] f_n(z \\
&= -{\phi_{n,u}^{-1}}'(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\Bigg[(\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u} y_{n,t} \\
&+ \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\Bigg] + \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) f_n(z_n)\,\mathrm{d}z_n \\
&= -{\phi_{n,u}^{-1}}'(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right] \\
&+ (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u})(1 - F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n))
\end{aligned}
$$

$$
\begin{aligned}
&\therefore\ J_n^{(1)}(y_{n,t}, I_n)\\
&= f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right] + \tilde{\Delta}_{n,t} F_n(y_{n,t} - I_n)\\
&+ {\phi_{n,u}^{-1}}'(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]\\
&- f_n(y_{n,t} - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, 0)\right]\\
&+ (\Delta_{n,t} + \tilde{\Delta}_{n,t})(F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - F_n(y_{n,t} - I_n))\\
&+ \int_{y_{n,t} - I_n}^{\phi_{n,u}^{-1}(y_{n,t}) - I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n\\
&- {\phi_{n,u}^{-1}}'(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t})y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]\\
&+ (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u})(1 - F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n))\\
&= (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) + \Delta_{n,u} F_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) - \Delta_{n,t} F_n(y_{n,t} - I_n)\\
&+ \int_{y_{n,t} - I_n}^{\phi_{n,u}^{-1}(y_{n,t}) - I_n} \delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

$$
\begin{aligned}
&\therefore\ J_n^{(1,2)}(y_{n,t}, I_n)\\
&= -\Delta_{n,u} f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n) + \Delta_{n,t} f_n(y_{n,t} - I_n)\\
&- \delta f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]\\
&+ \delta f_n(y_{n,t} - I_n)\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) - J_{n-1}^{(2)}(\beta_t y_{n,t}, 0)\right]\\
&+ \int_{y_{n,t} - I_n}^{\phi_{n,u}^{-1}(y_{n,t}) - I_n} \delta\left[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

Rearranging the terms, we get :

$$
\begin{aligned}
&J_n^{(1,2)}(y_{n,t}, I_n)\\
&= f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[-\Delta_{n,u} - \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) + \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]\\
&+ f_n(y_{n,t} - I_n)\left[\Delta_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, 0)\right]\\
&+ \int_{y_{n,t} - I_n}^{\phi_{n,u}^{-1}(y_{n,t}) - I_n} \delta\left[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}
$$

From **(L6)**-(a), we have $G_n^{(1)}(x_{n,t}^*, x_{n,u}^*) = G_n^{(1)}(y_{n,t}, 0) = \Delta_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, 0) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, 0) > 0$ and from **(L6)**-(b), we have $G_n^{(2)}(x_{n,t}^*, x_{n,u}^*) = G_n^{(2)}(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) > 0$. As $J_{n-1}(y_{n-1,t}, I_{n-1})$ is concave, we have $J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) < 0$ and as we have assumed $J_{n-1}^{(1,2)}(y_{n-1,t}, I_{n-1}) \geq 0$, we have $J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) \geq 0$. Using these components, we get

$$
\begin{aligned}
&J_n^{(1,2)}(y_{n,t}, I_n) \\
&= f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[-G_n^{(2)}(y_{n,t}, 0)\right] + f_n(y_{n,t} - I_n)\left[G_n^{(1)}(y_{n,t}, 0)\right] \\
&+ \int_{y_{n,t}-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\left[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n \\
&> 0
\end{aligned}
$$

The region S_C is further divided into 4 sub-regions - (i) $S_C^1 : 0 \leq z_n < \theta_{n,t} - I_n$ (ii) $S_C^2 : \theta_{n,t} - I_n \leq z_n \leq \phi_{n,t}^{-1}(y_{n,t}) - I_n$ (iii) $S_C^3 : \phi_{n,t}^{-1}(y_{n,t}) - I_n < z_n < \phi_{n,u}^{-1}(y_{n,t}) - I_n$ and (iv) $S_C^4 : \phi_{n,u}^{-1}(y_{n,t}) - I_n \leq z_n \leq z_n^U$. In sub-region S_C^1, we have $x_{n,t}^* = w_n$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t w_n$ and $I_{n-1} = 0$. In sub-region S_C^2, we have $x_{n,t}^* = \phi_{n,t}(w_n)$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t\phi_{n,t}(w_n)$ and $I_{n-1} = w_n - \phi_{n,t}(w_n)$. In sub-region S_C^3, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = 0$. Thus, $y_{n-1,t} = \beta_t y_{n,t}$ and $I_{n-1} = w_n - y_{n,t}$. In sub-region S_C^4, we have $x_{n,t}^* = y_{n,t}$ and $x_{n,u}^* = \phi_{n,u}(w_n) - y_{n,t}$. Thus, $y_{n-1,t} = \beta_t\phi_{n,u}(w_n)$ and $I_{n-1} = w_n - \phi_{n,u}(w_n)$. We now evaluate $V_n(y_{n,t}, w_n)$ in each of the above sub-regions. In sub-region S_C^1,

$$V_n(y_{n,t}, w_n) = \Delta_{n,t} w_n + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t w_n, 0) = \Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t(I_n + z_n), 0)$$

In sub-region S_C^2,

$$
\begin{aligned}
V_n(y_{n,t}, w_n) &= \Delta_{n,t}\phi_{n,t}(w_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t\phi_{n,t}(w_n), w_n - \phi_{n,t}(w_n)) \\
&= \Delta_{n,t}\phi_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t\phi_{n,t}(I_n + z_n), I_n + z_n - \phi_{n,t}(I_n + z_n))
\end{aligned}
$$

In sub-region S_C^3,

$$V_n(y_{n,t}, w_n) = (\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, w_n - y_{n,t}) = (\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})$$

In sub-region S_C^4,

$$
\begin{aligned}
V_n(y_{n,t}, w_n) &= (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u}\phi_{n,u}(w_n) + \delta J_{n-1}(\beta_t\phi_{n,u}(w_n), w_n - \phi_{n,u}(w_n)) \\
&= (\Delta_{n,t} + \tilde{\Delta}_{n,t} - \Delta_{n,u}) y_{n,t} + \Delta_{n,u}\phi_{n,u}(I_n + z_n) + \delta J_{n-1}(\beta_t\phi_{n,u}(I_n + z_n), I_n + z_n - \phi_{n,u}(I_n + z_n))
\end{aligned}
$$

Now, $J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$. Thus in Region S_C,

$$J_n^{(1)}(y_{n,t}, I_n) = \frac{\partial}{\partial y_{n,t}} \int_0^{\theta_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,t}-I_n}^{\phi_{n,t}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$+ \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n + \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$

We evaluate the individual terms in the above equation to calculate $J_n^{(1)}(y_{n,t}, I_n)$ in region S_C

$$\frac{\partial}{\partial y_{n,t}} \int_0^{\theta_{n,t}-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_0^{\theta_{n,t}-I_n} \left[\Delta_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t(I_n + z_n), 0)\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= \tilde{\Delta}_{n,t} F_n(\theta_{n,t} - I_n)$$

$$\frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,t}-I_n}^{\phi_{n,t}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{\theta_{n,t}-I_n}^{\phi_{n,t}^{-1}(y_{n,t})-I_n} \left[\Delta_{n,t}\phi_{n,t}(I_n + z_n) + \tilde{\Delta}_{n,t} y_{n,t} + \delta J_{n-1}(\beta_t \phi_{n,t}(I_n + z_n), I_n + z_n - \phi_{n,t}(I_n + z_n))\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= \phi_{n,t}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,t}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t}) - y_{n,t})\right]$$
$$+ \tilde{\Delta}_{n,t}(F_n(\phi_{n,t}^{-1}(y_{n,t}) - I_n) - F_n(\theta_{n,t} - I_n))$$

$$\frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} V_n(y_{n,t}, w_n) f_n(z_n)\,\mathrm{d}z_n$$
$$= \frac{\partial}{\partial y_{n,t}} \int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, I_n + z_n - y_{n,t})\right] f_n(z_n)\,\mathrm{d}z_n$$
$$= \phi_{n,u}^{-1\,\prime}(y_{n,t}) f_n(\phi_{n,u}^{-1}(y_{n,t}) - I_n)\left[(\Delta_{n,t} + \tilde{\Delta}_{n,t}) y_{n,t} + \delta J_{n-1}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t})\right]$$

$$-\phi_{n,t}^{-1\prime}(y_{n,t})f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,t}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+(\Delta_{n,t}+\tilde{\Delta}_{n,t})\left[F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)-F_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\right]$$
$$+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n}\delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t},I_n+z_n-y_{n,t})-J_{n-1}^{(2)}(\beta_t y_{n,t},I_n+z_n-y_{n,t})\right]f_n(z_n)\,\mathrm{d}z_n$$

$$\frac{\partial}{\partial y_{n,t}}\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U}V_n(y_{n,t},w_n)f_n(z_n)\,\mathrm{d}z_n$$
$$=\frac{\partial}{\partial y_{n,t}}\int_{\phi_{n,u}^{-1}(y_{n,t})-I_n}^{z_n^U}\Big[(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})y_{n,t}+\Delta_{n,u}\phi_{n,u}(I_n+z_n)$$
$$+\delta J_{n-1}(\beta_t\phi_{n,u}(I_n+z_n),I_n+z_n-\phi_{n,u}(I_n+z_n))\Big]f_n(z_n)\,\mathrm{d}z_n$$
$$=-\phi_{n,u}^{-1\prime}(y_{n,t})f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})y_{n,t}+\Delta_{n,u}y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})(1-F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n))$$
$$=-\phi_{n,u}^{-1\prime}(y_{n,t})f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})(1-F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n))$$

$$\therefore\ J_n^{(1)}(y_{n,t},I_n)$$
$$=\phi_{n,t}^{-1\prime}(y_{n,t})f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,t}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+\tilde{\Delta}_{n,t}(F_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)-F_n(\theta_{n,t}-I_n))+\tilde{\Delta}_{n,t}F_n(\theta_{n,t}-I_n)$$
$$+\phi_{n,u}^{-1\prime}(y_{n,t})f(\phi_{n,u}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$-\phi_{n,t}^{-1\prime}(y_{n,t})f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,t}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+(\Delta_{n,t}+\tilde{\Delta}_{n,t})\left[F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)-F_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\right]$$
$$+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n}\delta\left[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t},I_n+z_n-y_{n,t})-J_{n-1}^{(2)}(\beta_t y_{n,t},I_n+z_n-y_{n,t})\right]f_n(z_n)\,\mathrm{d}z_n$$
$$-\phi_{n,u}^{-1\prime}(y_{n,t})f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\left[(\Delta_{n,t}+\tilde{\Delta}_{n,t})y_{n,t}+\delta J_{n-1}(\beta_t y_{n,t},\phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\right]$$
$$+(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})(1-F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n))$$
$$=(\Delta_{n,t}+\tilde{\Delta}_{n,t}-\Delta_{n,u})+\Delta_{n,u}F_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)-\Delta_{n,t}F_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)$$

$$+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n$$

$$\begin{aligned}
\therefore\ & J_n^{(1,2)}(y_{n,t}, I_n)\\
&= -\Delta_{n,u} f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n) + \Delta_{n,t} f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\\
&- \delta f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\Big[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})-y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\Big]\\
&+ \delta f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\Big[\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t})-y_{n,t}) - J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t})-y_{n,t})\Big]\\
&+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}$$

Rearranging the terms, we get :

$$\begin{aligned}
& J_n^{(1,2)}(y_{n,t}, I_n)\\
&= f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\Big[-\Delta_{n,u} - \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})-y_{n,t}) + \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t})-y_{n,t})\Big]\\
&+ f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\Big[\Delta_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t})-y_{n,t}) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t})-y_{n,t})\Big]\\
&+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}$$

From **(L6)**-(c), we have $G_n^{(1)}(x_{n,t}^*, x_{n,u}^*) = G_n^{(1)}(y_{n,t}, 0) = \Delta_{n,t} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t}) - y_{n,t}) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,t}^{-1}(y_{n,t}) - y_{n,t}) = 0$ and from **(L6)**-(c), we have $G_n^{(2)}(x_{n,t}^*, x_{n,u}^*) = G_n^{(2)}(y_{n,t}, 0) = \Delta_{n,u} + \delta\beta_t J_{n-1}^{(1)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) - \delta J_{n-1}^{(2)}(\beta_t y_{n,t}, \phi_{n,u}^{-1}(y_{n,t}) - y_{n,t}) = 0$. As $J_{n-1}(y_{n-1,t}, I_{n-1})$ is concave, we have $J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) < 0$ and as we have assumed $J_{n-1}^{(1,2)}(y_{n-1,t}, I_{n-1}) \geq 0$, we have $J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n + z_n - y_{n,t}) \geq 0$. Using these components, we get

$$\begin{aligned}
& J_n^{(1,2)}(y_{n,t}, I_n)\\
&= f_n(\phi_{n,u}^{-1}(y_{n,t})-I_n)\Big[-G_n^{(2)}(y_{n,t},0)\Big] + f_n(\phi_{n,t}^{-1}(y_{n,t})-I_n)\Big[G_n^{(1)}(y_{n,t},0)\Big]\\
&+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n\\
&+\int_{\phi_{n,t}^{-1}(y_{n,t})-I_n}^{\phi_{n,u}^{-1}(y_{n,t})-I_n} \delta\Big[\beta_t J_{n-1}^{(1,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t}) - J_{n-1}^{(2,2)}(\beta_t y_{n,t}, I_n+z_n-y_{n,t})\Big] f_n(z_n)\,\mathrm{d}z_n
\end{aligned}$$

$$\geq 0$$

Thus $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0 \ \forall \ (y_{n,t}, I_n)$ ■

Now in period $n+1$,

$$\begin{aligned}
G_{n+1}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t} \\
&\quad + \delta J_n(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
\Rightarrow G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t} + \delta\beta_t J_n^{(1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
&\quad - \delta J_n^{(2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
\Rightarrow G_{n+1}^{(1,1)}(x_{n+1,t}, x_{n+1,u}) &= \delta\beta_t^2 J_n^{(1,1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
&\quad - 2\delta\beta_t J_n^{(1,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
&\quad + \delta J_n^{(2,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u})
\end{aligned}$$

Note that $G_{n+1}^{(1,1)}(x_{n+1,t}, x_{n+1,u}) = G_{n+1}^{(1,2)}(x_{n+1,t}, x_{n+1,u}) = G_{n+1}^{(2,2)}(x_{n+1,t}, x_{n+1,u})$. As $J_n(y_{n,t}, I_n)$ is jointly concave and $J_n^{(1,2)}(y_{n,t}, I_n) \geq 0$, we have

$$\begin{aligned}
J_n^{(1,1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) &< 0 \\
J_n^{(2,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) &< 0 \\
J_n^{(1,2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) &\geq 0
\end{aligned}$$

Thus, $G_{n+1}^{(1,1)}(x_{n+1,t}, x_{n+1,u}) = G_{n+1}^{(1,2)}(x_{n+1,t}, x_{n+1,u}) = G_{n+1}^{(2,2)}(x_{n+1,t}, x_{n+1,u}) < 0$ and $G_{n+1}(x_{n+1,t}, x_{n+1,u})$ is jointly concave as its Hessian matrix is negative semidefinite.

We now show that **(L3)** holds for period $n+1$. In period $n+1$, we have $G_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u}x_{n+1,u} + \Delta_{n+1,t}x_{n+1,t} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n)$. Therefore, $G_{n+1}^{(1)}(x_{n+1,t}, 0) = \Delta_{n+1,t} + \delta\beta_t J_n^{(1)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) - \delta J_n^{(2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t})$. Thus, $G_{n+1}^{(1)}(x_{n+1,t}, 0) = 0 \Rightarrow x_{n+1,t}^* = \phi_{n+1,t}(w_{n+1})$. Now, $\phi_{n+1,t}(w_{n+1})$ is differentiable as $J_n(y_{n,t}, I_n)$ is differentiable. From implicit function theorem, we have

$$\begin{aligned}
&\phi_{n+1,t}'(w_{n+1}) \\
&= -\frac{\delta\beta_t J_n^{(1,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) - \delta J_n^{(2,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t})}{\delta\beta_t^2 J_n^{(1,1)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) - 2\delta\beta_t J_n^{(1,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) + \delta J_n^{(2,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t})} \\
&= -\frac{\beta_t J_n^{(1,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) - J_n^{(2,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t})}{\beta_t^2 J_n^{(1,1)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) - 2\beta_t J_n^{(1,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t}) + J_n^{(2,2)}(\beta_t x_{n+1,t}, w_{n+1} - x_{n+1,t})}
\end{aligned}$$

In shorthand, $\phi'_{n+1,t}(w_{n+1}) = -\frac{\beta_t J_n^{(1,2)}(\cdot)-J_n^{(2,2)}(\cdot)}{\beta_t^2 J_n^{(1,1)}(\cdot)-2\beta_t J_n^{(1,2)}(\cdot)+J_n^{(2,2)}(\cdot)}$. As $J_n(y_{n,t}, I_n)$ is concave $\forall n$, we have $J_n^{(1,1)}(\cdot) < 0$ and $J_n^{(2,2)}(\cdot) < 0$. Also we have $J_n^{(1,2)}(\cdot) \geq 0$. Thus, using these components we get $0 < \phi'_{n+1,t}(w_{n+1}) < 1$ ■

We now show that **(L4)** holds for period $n+1$. In period $n+1$, we have $G_{n+1}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u}x_{n+1,u} + \Delta_{n+1,t}x_{n+1,t} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n)$. Therefore, $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t(y_{n+1,t} + x_{n+1,u}), w_{n+1} - y_{n+1,t} - x_{n+1,u}) - \delta J_n^{(2)}(\beta_t(y_{n+1,t} + x_{n+1,u}), w_{n+1} - y_{n+1,t} - x_{n+1,u})$. Define $z_{n+1,u} := x_{n+1,u} + y_{n+1,t}$. Thus, $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - \delta J_n^{(2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})$. Thus $G_{n+1}^{(2)}(y_{n+1,t}, x_{n+1,u}) = 0 \Rightarrow z_{n+1,u} = \phi_{n+1,u}(w_{n+1})$. Thus, $x_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$. Now, $\phi_{n+1,u}(w_{n+1})$ is differentiable as $J_n(y_{n,t}, I_n)$ is differentiable. From implicit function theorem, we have

$$
\begin{aligned}
&\phi'_{n+1,u}(w_{n+1}) \\
&= -\frac{\delta\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - \delta J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}{\delta\beta_t^2 J_n^{(1,1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - 2\delta\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) + \delta J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})} \\
&= \frac{\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}{\beta_t^2 J_n^{(1,1)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) - 2\beta_t J_n^{(1,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u}) + J_n^{(2,2)}(\beta_t z_{n+1,u}, w_{n+1} - z_{n+1,u})}
\end{aligned}
$$

In shorthand, $\phi'_{n+1,u}(w_{n+1}) = -\frac{\beta_t J_n^{(1,2)}(\cdot)-J_n^{(2,2)}(\cdot)}{\beta_t^2 J_n^{(1,1)}(\cdot)-2\beta_t J_n^{(1,2)}(\cdot)+J_n^{(2,2)}(\cdot)}$. As $J_n(y_{n,t}, I_n)$ is concave, we have $J_n^{(1,1)}(\cdot) < 0$ and $J_n^{(2,2)}(\cdot) < 0$. Also we have $J_n^{(1,2)}(\cdot) \geq 0$. Thus, using these components we get $0 < \phi'_{n+1,u}(w_{n+1}) < 1$ ■

Next, we derive the optimal policy for period $n+1$. Note that $q_u > q_r$ implies $\Delta_{n+1,t} > \Delta_{n+1,u}$ for $\gamma_t = 1$. To show that it is optimal to prioritize the treatment of *treated* patients over the enrollment of *untreated* patients, we evaluate the partials $G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u})$ and $G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u})$

$$
\begin{aligned}
G_{n+1}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t} + \delta J_n(y_{n,t}, I_n) \\
&= \Delta_{n+1,t}x_{n+1,t} + \Delta_{n+1,u}x_{n+1,u} + \tilde{\Delta}_{n+1,t}y_{n+1,t} \\
&\quad + \delta J_n(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
\therefore G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,t} + \delta\beta_t J_n^{(1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
&\quad - \delta J_n^{(2)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u}) \\
\text{and } G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u}) &= \Delta_{n+1,u} + \delta\beta_t J_n^{(1)}(\beta_t(x_{n+1,t} + x_{n+1,u}), w_{n+1} - x_{n+1,t} - x_{n+1,u})
\end{aligned}
$$

$$-\delta J_n^{(2)}(\beta_t(x_{n+1,t}+x_{n+1,u}), w_{n+1}-x_{n+1,t}-x_{n+1,u})$$

$$\therefore G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) - G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u}) = \Delta_{n+1,t} - \Delta_{n+1,u} > 0$$

$$\Rightarrow G_{n+1}^{(1)}(x_{n+1,t}, x_{n+1,u}) > G_{n+1}^{(2)}(x_{n+1,t}, x_{n+1,u})$$

Thus it is optimal to prioritize the treatment of *treated* patients. Now, we need to solve the stochastic dynamic program as stated in (EC.1). The Lagrangian of the optimization problem is given by:

$$\Lambda(x_{n+1,t}, x_{n+1,u}, \lambda_1, \lambda_2, \lambda_3, \lambda_4) = G_{n+1}(x_{n+1,t}, x_{n+1,u}) + \lambda_1(w_{n+1} - x_{n+1,t} - x_{n+1,u})$$
$$+\lambda_2(y_{n+1,t} - x_{n+1,t}) + \lambda_3(x_{n+1,t}) + \lambda_4(x_{n+1,u})$$

The stationarity conditions are given by $G_{n+1}^{(1)}(x^*_{n+1,t}, x^*_{n+1,u}) - \lambda_1 - \lambda_2 + \lambda_3 = 0$ and $G_{n+1}^{(2)}(x^*_{n+1,t}, x^*_{n+1,u}) - \lambda_1 + \lambda_4 = 0$. The conditions for primal feasibility are - (i) $x^*_{n+1,t} + x^*_{n+1,u} \leq w_{n+1}$ (ii) $x^*_{n+1,t} \leq y_{n+1,t}$(iii) $x^*_{n+1,t} \geq 0$ and (iv) $x^*_{n+1,u} \geq 0$. The complementary slackness conditions are (i) $\lambda_1(w_{n+1} - x^*_{n+1,t} - x^*_{n+1,u}) = 0$ (ii) $\lambda_2(y_{n+1,t} - x^*_{n+1,t}) = 0$ (iii) $\lambda_3(x^*_{n+1,t}) = 0$ and (iv) $\lambda_4(x^*_{n+1,u}) = 0$. The dual feasibility conditions are $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$. The state space is partitioned by considering permutations on the values taken by KKT multipliers. We only show the solution to non-trivial permutation below.

EC.4.0.1. Case 1: $\lambda_1 > 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0 \quad \lambda_4 > 0$

From stationarity conditions, we get $G_{n+1}^{(1)}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G_{n+1}^{(1)}(x^*_{n+1,t}, x^*_{n+1,u}) - G_{n+1}^{(2)}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. From complementary slackness, we get $x^*_{n+1,t} = w_{n+1}$ and $x^*_{n+1,u} = 0$. Primal feasibility leads to $0 \leq x^*_{n+1,t} \leq y_{n+1,t}$. Thus, $0 \leq w_{n+1} \leq y_{n+1,t}$.

Define $f_{n+1,A}(y_{n+1,t}, w_{n+1}) \triangleq G_{n+1}^{(1)}(x^*_{n+1,t}, x^*_{n+1,u}) = G_{n+1}^{(1)}(w_{n+1}, 0)$. Note that arguments of $f_{n+1,A}(y_{n+1,t}, w_{n+1})$ satisfy $0 \leq w_{n+1} \leq y_{n+1,t}$. We want to find all the values of $(y_{n+1,t}, w_{n+1})$ that satisfy $f_{n+1,A}(y_{n+1,t}, w_{n+1}) > 0$. Let $p_{n+1,A}(y_{n+1,t})$ be a function that solves $f_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t})) = 0$. By Implicit Function Theorem, $p'_{n+1,A}(y_{n+1,t}) = -\frac{f^{(1)}_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t}))}{f^{(2)}_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t}))}$. Now,

$$f^{(1)}_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t})) = G_{n+1}^{(1,1)}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial y_{n+1,t}} + G_{n+1}^{(1,2)}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial y_{n+1,t}} = 0$$

$$f^{(2)}_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t})) = G_{n+1}^{(1,1)}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G_{n+1}^{(1,2)}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial w_{n+1}}$$
$$= G_{n+1}^{(1,1)}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$$

$$
\begin{aligned}
\Rightarrow p'_{n+1,A}(y_{n+1,t}) &= 0 \\
\therefore p_{n+1,A}(y_{n+1,t}) &= \theta_{n+1,t} \quad (say) \\
\therefore f_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) &= 0 \ \forall \ 0 \le w_{n+1} \le y_{n+1,t}
\end{aligned}
$$

Similary, define $g_{n+1,A}(y_{n+1,t}, w_{n+1}) \triangleq G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = G^{(2)}_{n+1}(w_{n+1}, 0)$. Note that arguments of $g_{n+1,A}(y_{n+1,t}, w_{n+1})$ satisfy $0 \le w_{n+1} \le y_{n+1,t}$. Let $q_{n+1,A}(y_{n+1,t})$ be a function that solves $g_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) = 0$. By Implicit Function Theorem, $q'_{n+1,A}(y_{n+1,t}) = -\frac{g^{(1)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t}))}{g^{(2)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t}))}$. Now,

$$
\begin{aligned}
g^{(1)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial y_{n+1,t}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial y_{n+1,t}} = 0 \\
g^{(2)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial w_{n+1}} \\
&= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \\
\Rightarrow q'_{n+1,A}(y_{n+1,t}) &= 0 \\
\therefore q_{n+1,A}(y_{n+1,t}) &= \theta_{n+1,u} \quad (say) \\
\therefore g_{n+1,A}(y_{n+1,t}, \theta_{n+1,u}) &= 0 \ \forall \ 0 \le w_{n+1} \le y_{n+1,t}
\end{aligned}
$$

As $(\theta_{n+1,t}, \theta_{n+1,t})$ and $(\theta_{n+1,u}, \theta_{n+1,u}) \in$ domain of $f_{n+1,A}(y_{n+1,t}, w_{n+1})$ and $g_{n+1,A}(y_{n+1,t}, w_{n+1})$ respectively, we get:

$$
p_{n+1,A}(\theta_{n+1,t}) = \theta_{n+1,t} \tag{EC.10}
$$

$$
q_{n+1,A}(\theta_{n+1,u}) = \theta_{n+1,u} \tag{EC.11}
$$

Also from stationarity conditions, we have $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. Thus, $f_{n+1,A}(y_{n+1,t}, w_{n+1}) - g_{n+1,A}(y_{n+1,t}, w_{n+1}) > 0 \ \forall \ 0 \le w_{n+1} \le y_{n+1,t}$. Put $w_{n+1} = \theta_{n+1,t}$ and fix $y_{n+1,t}$ to get, $f_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) - g_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) > 0$. As $f_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) = 0$, we get $g_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) < 0$. Also as $g_{n+1,A}(y_{n+1,t}, \theta_{n+1,u}) = 0$ and $g^{(2)}_{n+1,A}(y_{n+1,t}, q_{n+1,A}(y_{n+1,t})) < 0$, we get $\theta_{n+1,u} < \theta_{n+1,t}$. As $f_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) = 0$ and $f^{(2)}_{n+1,A}(y_{n+1,t}, p_{n+1,A}(y_{n+1,t})) < 0$, $f_{n+1,A}(y_{n+1,t}, w_{n+1}) > 0 \Rightarrow w_{n+1} < \theta_{n+1,t}$. Thus, the feasible state space is given by R_E : $\{(y_{n+1,t}, w_{n+1}) \mid 0 \le w_{n+1} \le y_{n+1,t} \cap w_{n+1} < \theta_{n+1,t}\}$ and the optimal policy in this region is given by $x^*_{n+1,t} = w_{n+1}$ and $x^*_{n+1,u} = 0$.

EC.4.0.2. Case 2: $\lambda_1 > 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 = 0$

From stationarity conditions, we get (i) $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and (ii) $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ which leads to (iii) $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. From complementary slackness, we get (i) $x^*_{n+1,t} = y_{n+1,t}$ and (ii) $x^*_{n+1,u} = w_{n+1} - y_{n+1,t}$. Substituting this in the primal feasibility condition gives $0 \le y_{n+1,t} \le w_{n+1}$.

Define $f_{n+1,B}(y_{n+1,t}, w_{n+1}) \triangleq G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = G^{(1)}_{n+1}(y_{n+1,t}, w_{n+1} - y_{n+1,t})$ and $g_{n+1,B}(y_{n+1,t}, w_{n+1}) \triangleq G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = G^{(2)}_{n+1}(y_{n+1,t}, w_{n+1} - y_{n+1,t})$. Note that arguments of $f_{n+1,B}(y_{n+1,t}, w_{n+1})$ and $g_{n+1,B}(y_{n+1,t}, w_{n+1})$ satisfy $0 \le y_{n+1,t} \le w_{n+1}$. Define $p_{n+1,B}(y_{n+1,t})$ and $q_{n+1,B}(y_{n+1,t})$ as the functions that solve $f_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) = 0$ and $g_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) = 0$ respectively. By Implicit Function Theorem, $p'_{n+1,B}(y_{n+1,t}) = -\frac{f^{(1)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t}))}{f^{(2)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t}))}$ and $q'_{n+1,B}(y_{n+1,t}) = -\frac{g^{(1)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t}))}{g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t}))}$. Now,

$$
\begin{aligned}
f^{(1)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) &= G^{(1,1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial y_{n+1,t}} + G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial y_{n+1,t}} \\
&= G^{(1,1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = 0 \\
f^{(2)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) &= G^{(1,1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial w_{n+1}} \\
&= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \\
\Rightarrow p'_{n+1,B}(y_{n+1,t}) &= 0 \\
\therefore p_{n+1,B}(y_{n+1,t}) &= \tilde{\theta}_{n+1,t} \quad (say) \\
\therefore f_{n+1,B}(y_{n+1,t}, \tilde{\theta}_{n+1,t}) &= 0 \quad \forall \ 0 \le y_{n+1,t} \le w_{n+1}
\end{aligned}
$$

From **Case 1**, we have $f_{n+1,A}(y_{n+1,t}, \theta_{n+1,t}) = G^{(1)}_{n+1}(\theta_{n+1,t}, 0) = 0$. Note that the points $(\theta_{n+1,t}, \theta_{n+1,t})$ satisfy the condition $0 \le y_{n+1,t} \le w_{n+1}$ and thus belong to the domain of $f_{n+1,B}(y_{n+1,t}, w_{n+1})$. Put $y_{n+1,t} = w_{n+1} = \theta_{n+1,t}$ to get $f_{n+1,B}(\theta_{n+1,t}, \theta_{n+1,t}) = G^{(1)}_{n+1}(\theta_{n+1,t}, 0) = 0$. As $f^{(2)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) < 0$ and $f_{n+1,B}(\theta_{n+1,t}, \theta_{n+1,t}) = 0$, therefore $f_{n+1,B}(y_{n+1,t}, \tilde{\theta}_{n+1,t}) = 0 \ \forall \ 0 \le y_{n+1,t} \le w_{n+1} \Rightarrow \tilde{\theta}_{n+1,t} = \theta_{n+1,t}$. Thus, $p_{n+1,B}(y_{n+1,t}) = \theta_{n+1,t}$.

Similarly,

$$
\begin{aligned}
g^{(1)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial y_{n+1,t}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial y_{n+1,t}} \\
&= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = 0 \\
g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) &= G^{(1,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,t}}{\partial w_{n+1}} + G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) \frac{\partial x^*_{n+1,u}}{\partial w_{n+1}} \\
&= G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0
\end{aligned}
$$

$$\Rightarrow q'_{n+1,B}(y_{n+1,t}) = 0$$

$$\therefore q_{n+1,B}(y_{n+1,t}) = \tilde{\theta}_{n+1,u} \quad (say)$$

$$\therefore g_{n+1,B}(y_{n+1,t}, \tilde{\theta}_{n+1,u}) = 0 \quad \forall \ 0 \leq y_{n+1,t} \leq w_{n+1}$$

Again, from **Case 1**, we have $g_{n+1,A}(y_{n+1,t}, \theta_{n+1,u}) = G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$. Note that the point $(\theta_{n+1,u}, \theta_{n+1,u})$ satisfy the condition $0 \leq y_{n+1,t} \leq w_{n+1}$ and thus belong to the domain of $g_{n+1,B}(y_{n+1,t}, w_{n+1})$. Put $y_{n+1,t} = w_{n+1} = \theta_{n+1,u}$ to get $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$. As $g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$ and $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = 0$, therefore $g_{n+1,B}(y_{n+1,t}, \tilde{\theta}_{n+1,u}) = 0 \ \forall \ 0 \leq y_{n+1,t} \leq w_{n+1} \Rightarrow \tilde{\theta}_{n+1,u} = \theta_{n+1,u}$. Thus, $q_{n+1,B}(y_{n+1,t}) = \theta_{n+1,u}$.

As $f_{n+1,B}(\theta_{n+1,t}, \theta_{n+1,t}) = 0, f^{(1)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) < 0$ and $f^{(2)}_{n+1,B}(y_{n+1,t}, p_{n+1,B}(y_{n+1,t})) < 0$, $f_{n+1,B}(y_{n+1,t}, w_{n+1}) > 0 \Rightarrow y_{n+1,t} < \theta_{n+1,t}$ and $w_{n+1} < \theta_{n+1,t}$. Similarly, as $g_{n+1,B}(\theta_{n+1,u}, \theta_{n+1,u}) = 0, g^{(1)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$ and $g^{(2)}_{n+1,B}(y_{n+1,t}, q_{n+1,B}(y_{n+1,t})) < 0$, therefore $g_{n+1,B}(y_{n+1,t}, w_{n+1}) > 0 \Rightarrow y_{n+1,t} < \theta_{n+1,u}$ and $w_{n+1} < \theta_{n+1,u}$. Therefore, $y_{n+1,t} < \min\{\theta_{n+1,t}, \theta_{n+1,u}\} = \theta_{n+1,u}$ and $w_{n+1} < \min\{\theta_{n+1,t}, \theta_{n+1,u}\} = \theta_{n+1,u}$. Thus, the feasible state space is given by $R_A : \{(y_{n+1,t}, w_{n+1}) : 0 \leq y_{n+1,t} \leq w_{n+1} \cap w_{n+1} < \theta_{n+1,u}\}$ and the optimal policy is given by : $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = w_{n+1} - y_{n+1,t}$

EC.4.0.3. Case 3: $\lambda_1 = 0 \quad \lambda_2 = 0 \quad \lambda_3 = 0 \quad \lambda_4 > 0$

From stationarity conditions, we get (i) $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = 0$ and (ii) $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$. From complementary slackness, we get $x^*_{n+1,u} = 0$. From **(L3)**, $x^*_{n+1,t} = \phi_{n+1,t}(w_{n+1})$. From primal feasibility, we have $\phi_{n+1,t}(w_{n+1}) \leq w_{n+1}$, $\phi_{n+1,t}(w_{n+1}) \leq y_{n+1,t}$ and $\phi_{n+1,t}(w_{n+1}) \geq 0$.

From **Case 1**, $G^{(1)}_{n+1}(\theta_{n+1,u}, 0) = 0$ and from **(L3)**, $\phi_{n+1,t}(w_{n+1})$ is continuous. Thus $\lim_{w_{n+1} \to \theta_{n+1,t}} \phi_{n+1,t}(w_{n+1}) = \phi_{n+1,t}(\theta_{n+1,t})$. As $G^{(1,1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})$ is strictly monotonous and as $G^{(1)}_{n+1}(\phi_{n+1,t}(w_{n+1}), 0) = 0$, we get $\phi_{n+1,t}(\theta_{n+1,t}) = \theta_{n+1,t}$.

Also, as $\phi_{n+1,t}(w_{n+1}) \leq w_{n+1}$, $\phi'_{n+1,t}(w_{n+1}) < 1$ and $\phi_{n+1,t}(\theta_{n+1,t}) = \theta_{n+1,t}$, we get $w_{n+1} \geq \theta_{n+1,t}$. Similarly, as $\phi_{n+1,t}(w_{n+1}) \leq y_{n+1,t}$, $\phi'_{n+1,t}(w_{n+1}) > 0$, $\phi_{n+1,t}(\theta_{n+1,t}) = \theta_{n+1,t}$ and $w_{n+1} \geq \theta_{n+1,t}$, we get $y_{n+1,t} \geq \theta_{n+1,t}$. As $\phi_{n+1,t}(w_{n+1}) \leq y_{n+1,t}$ and $\phi_{n+1,t}(w_{n+1})$ is strictly monotonous, we get $w_{n+1} \leq \phi^{-1}_{n+1,t}(y_{n+1,t})$. Thus, the feasible state space is given by $R_D :$
$\left\{(y_{n+1,t}, w_{n+1}) : y_{n+1,t} \geq \theta_{n+1,t} \cap \theta_{n+1,t} \leq w_{n+1} \leq \phi^{-1}_{n+1,t}(y_{n+1,t})\right\}$
and the optimal policy is given by $x^*_{n+1,t} = \phi_{n+1,t}(w_{n+1})$ and $x^*_{n+1,u} = 0$

EC.4.0.4. Case 4: $\lambda_1 = 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 = 0$

From stationarity conditions, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) = 0$. From complementary slackness, $x^*_{n+1,t} = y_{n+1,t}$. From **(L4)** and primal feasibility, we get $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$ and $0 \le y_{n+1,t} \le \phi_{n+1,u}(w_{n+1}) \le w_{n+1}$. From **Case 1**, $G^{(2)}_{n+1}(\theta_{n+1,u}, 0) = 0$ and from **(L4)**, $\phi_{n+1,u}(w_{n+1})$ is continuous. Thus $\lim_{w_{n+1} \to \theta_{n+1,u}} \phi_{n+1,u}(w_{n+1}) = \phi_{n+1,u}(\theta_{n+1,u})$. As $G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u})$ is strictly monotonous and as $G^{(2)}_{n+1}(y_{n+1,t}, \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}) = 0$, putting $y_{n+1,t} = \theta_{n+1,u}$, we get $\phi_{n+1,u}(\theta_{n+1,u}) = \theta_{n+1,u}$. Also, as $\phi'_{n+1,u}(w_{n+1}) < 1$, $\phi_{n+1,u}(w_{n+1}) \le w_{n+1}$ and $\phi_{n+1,u}(\theta_{n+1,u}) = \theta_{n+1,u}$, we get $w_{n+1} \ge \theta_{n+1,u}$. As $\phi_{n+1,u}(w_{n+1})$ is strictly monotonous, we get $y_{n+1,t} \le \phi_{n+1,u}(w_{n+1}) \Rightarrow w_{n+1} \ge \phi^{-1}_{n+1,u}(y_{n+1,t})$. Thus, the feasible state space is given by $R_B : \left\{(y_{n+1,t}, w_{n+1}) : 0 \le y_{n+1,t} \le w_{n+1} \cap w_{n+1} \ge \max\{\theta_{n+1,u}, \phi^{-1}_{n+1,u}(y_{n+1,t})\}\right\}$ and the optimal policy is given by $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$

EC.4.0.5. Case 5: $\lambda_1 = 0 \quad \lambda_2 > 0 \quad \lambda_3 = 0 \quad \lambda_4 > 0$

From stationarity conditions, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$ and $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$. Thus, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) - G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0$. From complementary slackness, $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = 0$. From primal feasibility, $0 \le y_{n+1,t} \le w_{n+1}$. From **(L3)**, the solution to $G^{(1)}_{n+1}(x_{n+1,t}, 0) = 0$ is given by $x^*_{n+1,t} = \phi_{n+1,t}(w_{n+1})$. As $G^{(1,1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$, $G^{(1)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) > 0 \Rightarrow G^{(1)}_{n+1}(y_{n+1,t}, 0) > 0 \Rightarrow y_{n+1,t} < \phi_{n+1,t}(w_{n+1}) \Rightarrow w_{n+1} > \phi^{-1}_{n+1,t}(y_{n+1,t})$. From **(L4)**, the solution to $G^{(2)}_{n+1}(y_{n+1,t}, x_{n+1,u}) = 0$ is given by $x^*_{n+1,u} = \phi_{n+1,u}(w_{n+1}) - y_{n+1,t}$. As $G^{(2,2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0$, $G^{(2)}_{n+1}(x^*_{n+1,t}, x^*_{n+1,u}) < 0 \Rightarrow G^{(2)}_{n+1}(y_{n+1,t}, 0) < 0 \Rightarrow y_{n+1,t} > \phi_{n+1,u}(w_{n+1}) \Rightarrow w_{n+1} < \phi^{-1}_{n+1,u}(y_{n+1,t})$. Thus, the feasible state space is given by $R_C : \left\{(y_{n+1,t}, w_{n+1}) : 0 \le y_{n+1,t} \le w_{n+1} \cap \phi^{-1}_{n+1,t}(y_{n+1,t}) < w_{n+1} < \phi^{-1}_{n+1,u}(y_{n+1,t})\right\}$ and the optimal policy is given by $x^*_{n+1,t} = y_{n+1,t}$ and $x^*_{n+1,u} = 0$. Thus Proposition 4 holds true. ∎

EC.5. Supplementary Results from Numerical Experiments

	U(1,10)		U(1,50)		U(1,100)	
q_u	*Two-Period*	*Safety-Stock*	*Two-Period*	*Safety-Stock*	*Two-Period*	*Safety-Stock*
0.74	3.79%	5.66%	4.48%	6.59%	4.57%	6.73%
0.76	4.41%	8.61%	5.21%	9.87%	5.33%	10.01%
0.78	4.51%	9.61%	5.34%	11.38%	5.45%	11.49%
0.8	4.58%	10.65%	5.42%	12.56%	5.54%	12.66%
0.82	4.61%	11.63%	5.46%	13.78%	5.57%	13.99%
0.84	4.57%	12.85%	5.41%	14.97%	5.52%	15.23%
0.86	4.63%	14.04%	5.49%	16.33%	5.60%	16.51%
0.88	4.65%	15.26%	5.51%	17.84%	5.62%	18.08%
0.9	4.67%	17.27%	5.53%	21.90%	5.64%	20.92%
0.92	4.65%	20.65%	5.50%	40.93%	5.62%	25.09%

Table EC.1 Performance gap of heuristics for varying levels of supply distribution for $\gamma_t = 1$ and $N = 24$.