

Assignment 3

Problem 56 UGC Math 2018

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Question

Consider a linear model $Y_i = \theta_1 + \theta_2 + \epsilon_i$ for $i = 1, 2$ and $Y_i = \theta_1 - \theta_3 + \epsilon_i$ for $i = 3, 4$, where ϵ_i are independent with $R(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2 > 0$ for $i = 1, \dots, 4$, and $\theta_1, \dots, \theta_3 \in R$. Which of the following parametric function is estimable?

- (A) $\theta_1 + \theta_3$
- (B) $\theta_2 - \theta_3$
- (C) $\theta_2 + \theta_3$
- (D) $\theta_1 + \theta_2 + \theta_3$

Answer: C, $\theta_2 + \theta_3$

Solution

Given,

$$Y_1 = \theta_1 + \theta_2 + \epsilon_1$$

$$Y_2 = \theta_1 + \theta_2 + \epsilon_2$$

$$Y_3 = \theta_1 - \theta_3 + \epsilon_3$$

$$Y_4 = \theta_1 - \theta_3 + \epsilon_4$$

Matrix Notation

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \epsilon_2 \\ \epsilon_3 \\ \epsilon_4 \end{bmatrix}$$

Which can be represented as

$$Y = X\beta + \epsilon_i$$

Given, $R(\epsilon_i) = 0$, ϵ_i is independent, $\text{Var}(\epsilon_i) = \sigma^2 > 0$ for all $i = 1, 2, 3, 4$

1. Independence of ϵ_i implies that all of them are un-correlated.
2. $\text{Var}(\epsilon_i) = \sigma^2$. So co-variance matrix is of form $\sigma^2 I$.
3. Which means the co-variance is spherical.

Estimable parametric function

Gauss-Markov property of estimable parameters

Whenever

$$X\beta_1 = X\beta_2 \quad \beta_1, \beta_2 \in R^k$$

Then If,

$$\varphi(\beta_1) = \varphi(\beta_2)$$

$\varphi(\beta)$ is estimable

Evaluating Options

$$\text{For } \beta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } \beta_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$X\beta_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X\beta_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

Option A $\theta_1 + \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\varphi(\beta_2) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

Option B $\theta_2 - \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$$

$$\varphi(\beta_2) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

Option D $\theta_1 + \theta_2 + \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$$

$$\varphi(\beta_2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

$$\text{Let } \beta_1 = \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} \quad \beta_2 = \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix}$$

$$X\beta_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} i_1 + j_1 \\ i_1 + j_1 \\ i_1 - k_1 \\ i_1 - k_1 \end{bmatrix}$$

$$X\beta_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} i_2 + j_2 \\ i_2 + j_2 \\ i_2 - k_2 \\ i_2 - k_2 \end{bmatrix}$$

If β_1, β_2 satisfy $X\beta_1 = X\beta_2$

then, $i_1 + j_1 = i_2 + j_2, \quad i_1 - k_1 = i_2 - k_2 \implies j_1 + k_1 = j_2 + k_2$

$$\varphi(\beta_1) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = j_1 + k_1$$

$$\varphi(\beta_2) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \\ k_3 \end{bmatrix} = j_2 + k_2$$

$\varphi(\beta_1) = \varphi(\beta_2)$ for any β_1, β_2 so **Option C** is correct answer