Assignment 1 - Problem 53

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Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

Answer: 4, X+Y and X-Y

1 Solution:

1.1 Definition of Bivariate Gaussian and its independency

Bi-variate random variables are distribution of normal distribution to two co-ordinates. are said to be bivariate normal or jointly normal, if $aX + bY$ has normal distribution $\forall\ a, b \in R$.

Random normal vector Z = $\begin{bmatrix} X \\ Y \end{bmatrix}$ is Bi-variate when it is jointly normal
Joint PDF of Z is given as

$$f_z(Z) = \frac{1}{2\pi\sqrt{det\Sigma}} \quad \exp\left\{\frac{-1}{2}(z-m)^T\Sigma^{-1}(z-m)\right\}$$

$$Where,$$

$$m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, \quad ,\Sigma = [[\mathbf{Z} - \mathbf{E}(\mathbf{Z})][\mathbf{Z} - \mathbf{E}(\mathbf{Z})]^{\mathbf{T}}]$$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix}$$

If X, Y, which are independent, then they are un-correlated or their co-variances are $\rho\sigma_Y\sigma_X = 0$ then co-variance matrix becomes a diagonal matrix

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$$

1.2 Co-variance Matrix

$\Sigma = [[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T]$

$\textit{When,} \quad \mathbf{Z} = \mathbf{TZ}, \quad \textit{Where } T \textit{ is a transformation}$

$$\begin{aligned} \Sigma_{TZ} &= [[\mathbf{TZ} - E(\mathbf{TZ})][\mathbf{TZ} - E(\mathbf{TZ})]^T] \\ &= [[T\mathbf{Z} - TE(\mathbf{Z})][T\mathbf{Z} - TE(\mathbf{Z})]^T] \\ &= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\ &= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\ &= [T\Sigma T^T] \end{aligned}$$

$$\textit{Where}[[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T] = \Sigma$$

$$\Sigma_{TZ} = [T\Sigma T^T] \tag{1}$$

1.3 Evaluating option 1

: Given $\sigma_x = \sigma_y$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_X \\ \rho\sigma_X\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X^2 \\ \rho\sigma_X^2 & \sigma_X^2 \end{bmatrix} = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Σ is not a diagonal matrix so components of Z in option 1 are not independent

1.4 Evaluating option 2

X, X-Y can be written as $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

Where T = $\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$, Z = $\begin{bmatrix} X \\ Y \end{bmatrix}$

Co-variance matrix Σ for X, X-Y From Eq. 1 $\Sigma_{TZ} = [T\Sigma T^T]$

$$\Sigma_{TZ} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \qquad = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ 1-\rho & \rho - 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$
$$= \sigma^2 \begin{bmatrix} 1 & 1-\rho \\ 1-\rho & 2-\rho \end{bmatrix}$$

Σ_{TZ} is not a diagonal matrix, so components of TZ are not independent

1.5 Evaluating option 3

X+Y and Y can be written as $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}$

Where T = $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}, Z = \begin{bmatrix}X\\Y\end{bmatrix}$

Co-variance matrix Σ for X+Y, Y From Eq. 1 $\Sigma_{TZ} = [T\Sigma T^T]$

$$\begin{aligned}\Sigma_{TZ} &= \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\sigma_X^2\begin{bmatrix}1 & \rho\\\rho & 1\end{bmatrix}\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\\&= \sigma_x^2\begin{bmatrix}2+2\rho & \rho+1\\\rho+1 & 1\end{bmatrix}\end{aligned}$$

Σ_{TZ} is not a diagonal matrix, so components of TZ are not independent

1.6 Evaluating option 4

X+Y and X-Y can be written as $\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}X\\Y\end{bmatrix}$

From Eq. 1 $\Sigma_{TZ} = [T\Sigma T^T]$

$$\begin{aligned}\Sigma_{TZ} &= \begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\sigma_X^2\begin{bmatrix}1 & \rho\\\rho & 1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\\&= \sigma_x^2\begin{bmatrix}1+\rho & \rho+1\\1-\rho & \rho-1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}\\&= \begin{bmatrix}2+2\rho & 0\\0 & 2-2\rho\end{bmatrix}\end{aligned}$$

Hence option 4 is correct