Assignment 1 - Problem 53

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January 2022

Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

Answer: 4, X+Y and X-Y
Solution: Given X and Y are bi-variate random variables. Bivariate random variables are distribution of normal distribution to two coordinates. If any two random variables, $X_1 and X_2$ are said independent, then it follows that the co-relation co-efficient $\rho[X_1, X_2]$ is equal to 0.

$$\rho[X_1, X_2] = 0$$

We know that, for a bi-variate random variable correlation is given as follows:

$$\rho[X, Y] = \frac{\sigma_{[X,Y]}}{\sqrt{\sigma_X^2 \times \sigma_Y^2}}$$

For for testing independence of $[X + Y, X - Y]$, we need to see if $\rho[X + Y, X - Y]$ becomes 0.
if $\rho[X + Y, X - Y] = 0$, then it follows from the bivariate random distribution that $\sigma_{[X+Y,X-Y]}$ equates to 0

We know that

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$
$$\sigma_{[X,X]} = \sigma^2_{[X]} \tag{2}$$
$$if \quad \sigma[X, Y] = \sigma[X]^2 \quad then \quad X = Y \tag{3}$$
$$\sigma_{X,Y} = 0 \implies Y, X\, are\ dependant \tag{4}$$

(Proofs for above equations are given in appendix)

1. Testing for the independence of X,Y

   (a)

$$\begin{aligned}\sigma[X,Y] &= E((X-\mu_X)(Y-\mu_Y)) \\ &= E[(XY) - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\ &= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y \\ &= E(XY) - E(X)E(Y) \\ \sigma[X,Y] = 0 &\implies E(XY) = E(X)E(Y)\end{aligned}$$

   (b)

$$\begin{aligned}\sigma[X,X] &= \sigma[Y,Y] \\ &\implies E[(X-\mu_X)^2] = E[(Y-\mu_Y)^2] \\ &\implies E[X^2] - E[X]^2 = E[Y^2] - E[Y]^2\end{aligned}$$

> **Argument**: Both equations can not be solved with the given hypothesis, so can not comment on the in dependency

2. Testing independence of X,X-Y

   (a)

$$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$

$$\sigma_{[X-Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$

   (b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = X$ From Eq, **??**

> **Argument**: $X-Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

3. Testing for independence of X,X+Y

(a)

$$\begin{aligned}\sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]}\end{aligned}$$

$$\sigma_{[X+Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$

(b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = -X$ which means they are dependant

> **Argument**: $X+Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = -X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

4. Testing for independence of X+Y,X-Y

   (a)

$$\begin{aligned}&\sigma_{[X+Y,X-Y]} \\ &= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\ &= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_Y^2\end{aligned}$$

   (b) Now testing for $\sigma_1 = \sigma_2 \implies \sigma_{[X+Y,X-Y]} = 0$

   (c) Hence testing for $\sigma_1 = \sigma_2 \implies X+Y, X-Y$ are independent.

Appendix

Co variance is a measure of how much two random variables vary together

1. $\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]}$

$$\begin{aligned}
\sigma_{[X+Y,Z]} &= E((X+Y-\mu_{X+Y}) \times (Z-\mu_Z)) \\
&= E((X+Y-\mu_X-\mu_Y) \times (Z-\mu_Z)) \\
&= E(XZ - X\mu_z + YZ - Y\mu_Z - Z\mu_X + \mu_X\mu_Z - \mu_Y Z + \mu_Y\mu_Z) \\
&= E((XZ - \mu_Z X - \mu_X Z + \mu_z\mu_x) + (YZ - Y\mu_Z - \mu_Y Z + \mu_Y\mu_Z)) \\
&= E((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z)) \\
&= E((X-\mu_X)(Z-\mu_Z)) + E((Y-\mu_Y)(Z-\mu_Z)) \\
&= \sigma_{[X,Z]} + \sigma_{[Y,Z]}
\end{aligned}$$

2. $\sigma_{[X,Y]} = \sigma^2_{[X]}$

$$\begin{aligned}
\sigma_{[X,Y]} &= E[(X-\mu_X)(Y-\mu_Y)], \quad \mathit{if} \quad Y = X \\
\sigma_{[X,X]} &= E[(X-\mu_X)(X-\mu_X)] \\
&= E[(X-\mu_X)^2] \\
&= \sigma^2_X
\end{aligned}$$

3. if $\sigma[X,Y] = \sigma[X]^2 \quad \mathit{then} \quad X = Y$
   From 2 it follows that $\sigma[X,Y] = \sigma[X]^2$ when X=Y

4. $\sigma_{X,Y} = 0 \implies Y, X$ are dependant

$$\begin{aligned}
\sigma[X,Y] &= E((X-\mu_X)(Y-\mu_Y)) \\
&= E[(XY) - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\
&= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_y \\
&= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y
\end{aligned}$$

$\sigma[X,Y] = 0 \implies E(XY) = E(X)E(Y)$

if $E(XY) = E(X)E(Y) then \rho(X,Y) = 1 \implies X, Y$ are dependant