Assignment 6

Problem 119 UGC Math Dec-2017

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Question

119.) Arrival of customers in a shop is a Poisson process with intensity $\lambda = 2$. Let X be the number of customers entering during the time interval (1, 2) and let Y be the number of customers entering during the time interval (5,10). Which of the following are true?

1. $P(X = 0|(X + Y = 12)) = (\frac{5}{6})^{12}$
2. X and Y are in-dependant
3. $X + Y$ is a Poisson with parameter 6
4. $X - Y$ is a Poisson with parameter 8

Answer: 1,2

Definitions

1. Poisson distribution is limiting Bernoulli distribution Bern(n,p), where $n- > \infty, p- > 0$
2. Since all Bernoulli trails are in-dependant RVs in each disjoint intervals are also disjoint
3. MGF of Poisson distribution is given by

$$M_X(t) = e^{(e^t - 1)\lambda} \tag{1}$$

Poisson as limiting Bernoulli Distribution

$$
\begin{aligned}
X &\sim Bin(n, p), n \to \infty, p \to 0, \lambda = np \\
P(X = k) &= \binom{n}{k}(p^k)(1-p)^{(n-k)} \\
&= \frac{n!}{(n-k)!k!}\frac{\lambda^k}{n^k}(1-\frac{\lambda}{k})^{-k}(1-\frac{\lambda}{n})^n \\
&= \frac{\lambda^k}{k!}e^{-\lambda}
\end{aligned}
$$

MGF of Poisson

$$
\begin{aligned}
P(X = k) &= \frac{\lambda^k e^{\lambda}}{k!} \\
M_x(t) &= E(e^{tX}) \\
&= \sum_{n=0}^{\infty} \frac{\lambda^n e^{-\lambda}}{n!} e^{tn} \\
&= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda e^t)^n}{n!} \\
&= e^{(e^t - 1)\lambda}
\end{aligned}
$$

MGF of two Independent Poisson distributions

$M_X(t) = E(e^{tX})$, $M_Y(t) = E(e^{tY})$

$$\begin{aligned} M_{X+Y}(t) &= E(e^{t(X+Y)}) \\ &= E(e^{tX})E(e^{tY}) \\ &= e^{(e^t-1)\lambda_1} e^{(e^t-1)\lambda_2} \\ &= e^{e^t(\lambda_1+\lambda_2)-(\lambda_1+\lambda_2)} \end{aligned}$$

Option 1

$P(X=0|(X+Y=12)) = (\frac{5}{6})^{12}4$

$$\begin{aligned}
P(X=0|(X+Y=12)) &= P(X=0, X+Y=12)/P(X+Y=12) \\
&= P(X=0, Y=12)/P(X+Y=12) \\
&= P(X=0)P(Y=12)/P(X+Y=12) \\
&= \frac{e^{-2}2^0}{0!}\frac{e^{-2(5)}5^{12}}{12!} \times (10)^{12} \times \frac{12!}{(2+10)^{12}e^{-12}} \\
&= (\frac{10}{12})^{12} \\
&= (\frac{5}{6})^{12}
\end{aligned}$$

Option 2

X, Y are Poisson distributions of a disjoint intervals so they are in-dependant

Option 3

From the MGF of two Poisson distributions we know that X+Y is Poisson and the rate is $\lambda_1 + \lambda_2 = 2 + 10 = 12$
So Option 3 is incorrect

Option 4

From the MGF of two Poisson distributions we know that X+Y is Poisson and the rate is $\lambda_1 + \lambda_2 = 2 - 10 = -8$
So Option 4 is incorrect