Assignment 1 - Problem 53

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Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

Answer: 4, X+Y and X-Y
Solution: Given X and Y are bi-variate random variables. Bivariate random variables are distribution of normal distribution to two coordinates. If any two random variables, $X_1 and X_2$ are said independent, then it follows that the co-relation co-efficient $\rho[X_1, X_2]$ is equal to 0.

$$\rho[X_1, X_2] = 0$$

We know that, for a bi-variate random variable correlation is given as follows:

$$\rho[X, Y] = \frac{\sigma_[X, Y]}{\sqrt{\sigma_[X]^2 \times \sigma_[Y]^2}}$$

For for testing independence of $[X + Y, X - Y]$, we need to see if $\rho[X + Y, X - Y]$ becomes 0.
if $\rho[X + Y, X - Y] = 0$, then it follows from the bivariate random distribution that $\sigma_{[X+Y,X-Y]}$ equates to 0

We know that

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$

$$\sigma_{[X,X]} = \sigma^2_{[X]} \tag{2}$$

Testing for the independence of X,Y
$\sigma_{[X,Y]}$ is always not equal to 0 when $\sigma_1^2 = \sigma_2^2$.
Also if $\sigma_{[X,Y]}$ is equal to 0, it means that XY are in the form of

$Y = C \times X$ or $Y = X^2$ or any $Y = f(X)$, which means both of them are dependent. So, $\sigma_1^2 = \sigma_2^2$ does not imply X,Y are in-dependant.

Testing independence of X,X-Y

$$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$

$\sigma_X^2 \neq \sigma_{[X,Y]}$ which $\implies$ X,X-Y are not independent

Testing for independence of X,X+Y

$$\begin{aligned}\sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]}\end{aligned}$$

$\sigma_X^2 \neq -\sigma_{[X,Y]}$ which $\implies$ X,X-Y are not independent

Testing for independance of X+Y,X-Y

$$\begin{aligned}&COV[X+Y, X-Y] \\ &= COV[(X+Y), X] - COV[(X+Y), Y] \\ &= COV[(X, X)] + COV[X, Y] - COV[X, Y] - COV[Y, Y] \\ &= Var[X] - Var[Y]\end{aligned}$$

Now testing for $\sigma_1 = \sigma_2 \implies$
$COV[X+Y, X-Y] = 0 \Rightarrow \rho[X+Y, X-Y] = 0$
Hence testing for $\sigma_1 = \sigma_2 \implies X+Y, X-Y$ are independent.