

Assignment 1 - Problem 53

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January 2022

Question 53) Suppose (X, Y) follows bivariate normal distribution with means μ_1, μ_2 , standard deviations σ_1, σ_2 and correlation coefficient ρ , where all parameters are un-known. Then, testing $H_0: \sigma_1 = \sigma_2$ is equivalent to testing the independence of

- 1.) X and Y
- 2.) X and $X - Y$
- 3.) $X + Y$ and Y
- 4.) $X + Y$ and $X - Y$

Answer: 4, $X + Y$ and $X - Y$

Solution: Given X and Y are bi-variate random variables. Bivariate random variables are distribution of normal distribution to two coordinates. If any two random variables, X_1 and X_2 are said independent, then it follows that the correlation co-efficient $\rho[X_1, X_2]$ is equal to 0.

$$\rho[X_1, X_2] = 0$$

We know that, for a bi-variate random variable correlation is given as follows:

$$\rho[X, Y] = \frac{\sigma_{[X, Y]}}{\sqrt{\sigma_X^2 \times \sigma_Y^2}}$$

For for testing independence of $[X + Y, X - Y]$, we need to see if $\rho[X + Y, X - Y]$ becomes 0.

if $\rho[X + Y, X - Y] = 0$, then it follows from the bivariate random distribution that $\sigma_{[X + Y, X - Y]}$ equates to 0

We know that

$$\sigma_{[X + Y, Z]} = \sigma_{[X, Z]} + \sigma_{[Y, Z]} \quad (1)$$

$$\sigma_{[X, X]} = \sigma_X^2 \quad (2)$$

$$\text{if } \sigma_{[X, Y]} = \sigma_X^2 \text{ then } X = Y \quad (3)$$

(Proofs for above equations are given in appendix)

1. Testing for the independence of X,Y

- (a) If $\sigma_{[X,Y]}$ is equal to 0, it means that XY are in the form of $Y = C \times X$, which means both of them are dependent.
- (b) So, $\sigma_1^2 = \sigma_2^2$ does not imply X,Y are in-dependant.

2. Testing independence of X,X-Y

(a)

$$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$

- (b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = X$ which means they are dependant
- (c) $\sigma_X^2 \neq \sigma_{[X,Y]}$ which \implies X,X-Y are not independent

3. Testing for independence of X,X+Y

(a)

$$\begin{aligned}\sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]}\end{aligned}$$

- (b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = -X$ which means they are dependant
- (c) $\sigma_X^2 \neq -\sigma_{[X,Y]}$ which \implies X,X-Y are not independent

4. Testing for independence of X+Y,X-Y

(a)

$$\begin{aligned}\sigma_{[X+Y,X-Y]} &= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\ &= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_Y^2\end{aligned}$$

- (b) Now testing for $\sigma_1 = \sigma_2 \implies \sigma_{[X+Y, X-Y]} = 0$
(c) $\Rightarrow \rho[X+Y, X-Y] = 0$
(d) Hence testing for $\sigma_1 = \sigma_2 \implies X+Y, X-Y$ are independent.

Appendix

Covariance is a measure of how much two random variables vary together

$$1. \sigma_{[X+Y, Z]} = \sigma_{[X, Z]} + \sigma_{[Y, Z]}$$

$$\begin{aligned} \sigma_{[X+Y, Z]} &= E((X+Y - \mu_{X+Y}) \times (Z - \mu_Z)) \\ &= E((X+Y - \mu_X - \mu_Y) \times (Z - \mu_Z)) \\ &= E(XZ - X\mu_Z + YZ - Y\mu_Z - Z\mu_X + \mu_X\mu_Z - \mu_Y Z + \mu_Y\mu_Z) \\ &= E((XZ - \mu_Z X - \mu_X Z + \mu_Z\mu_X) + (YZ - Y\mu_Z - \mu_Y Z + \mu_Y\mu_Z)) \\ &= E((X - \mu_X)(Z - \mu_Z) + (Y - \mu_Y)(Z - \mu_Z)) \\ &= E((X - \mu_X)(Z - \mu_Z)) + E((Y - \mu_Y)(Z - \mu_Z)) \\ &= \sigma_{[X, Z]} + \sigma_{[Y, Z]} \end{aligned}$$

$$2. \sigma_{[X, Y]} = \sigma_X^2$$

$$\begin{aligned} \sigma_{[X, Y]} &= E[(X - \mu_X)(Y - \mu_Y)], \quad \text{if } Y = X \\ \sigma_{[X, X]} &= E[(X - \mu_X)(X - \mu_X)] \\ &= E[(X - \mu_X)^2] \\ &= \sigma_X^2 \end{aligned}$$

3. if $\sigma[X, Y] = \sigma[X]^2$ then $X = Y$
From 2 it follows that $\sigma[X, Y] = \sigma[X]^2$ when $X=Y$