Assignment 3 - Problem 56 Dec-2018 Paper

Consider a linear model $Y_i = \theta_1 + \theta_2 + \epsilon_i$ for $i = 1, 2$ and $Y_i = \theta_1 - \theta_3 + \epsilon_i$ for i = 3,4, where ϵ_i are independent with $R(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2 > 0$ for i = 1, ... 4, and $\theta_1, ...\theta_3 \in R$. Which of the following parametric function is estimable?
(A) $\theta_1 + \theta_3$
(B) $\theta_2 - \theta_3$
(C) $\theta_2 + \theta_3$
(D) $\theta_1 + \theta_2 + \theta_3$

Answer: C, $\theta_2 + \theta_3$

1 Solution:

Given,

$$Y_i = \theta_1 + \theta_2 + \epsilon_l \quad for\ i{=}1{,}2 \tag{1}$$

$$Y_i = \theta_1 - \theta_3 + \epsilon_l \quad for\ i{=}3{,}4 \tag{2}$$

Which means,

$$\begin{aligned} Y_1 &= \theta_1 + \theta_2 + \epsilon_1 \\ Y_2 &= \theta_1 + \theta_2 + \epsilon_2 \\ Y_3 &= \theta_1 - \theta_3 + \epsilon_3 \\ Y_4 &= \theta_1 - \theta_3 + \epsilon_4 \end{aligned}$$

Matrix Notation

$$\begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{bmatrix}$$

Which can be represented as

$$Y = X\beta + \epsilon_i$$

Given R(ϵ_i) = 0, ϵ_i is In dependant, Var(ϵ_i) = $\sigma^2 > 0$ for all $i = 1, 2, 3, 4$

In dependence implies that all of them un-correlated and given Var(ϵ_i) = σ^2. So co-variance matrix is of form σ^2 I. Which means the co-variance is spherical.

1.1 Estimable parametric function

A parametric functional $\varphi(\beta)$ is estimable if it is uniquely determined by $X\beta$ in the sense that $\varphi(\beta_1) = \varphi(\beta_2)$ whenever $\beta_1, \beta_2 \in R^k$ satisfy $X\beta_1 = X\beta_2$. This is known as Gauss-Markov property of estimable parameters

1.2 Evaluating Options

For $\beta_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\beta_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$X\beta_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$X\beta_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

1. **Option A** $\theta_1 + \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

$$\varphi(\beta_2) = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

2. **Option B** $\theta_2 - \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$$

$$\varphi(\beta_2) = \begin{bmatrix} 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = -1$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

3. **Option D** $\theta_1 + \theta_2 + \theta_3$

$$\varphi(\beta_1) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 1$$

$$\varphi(\beta_2) = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 2$$

$$\varphi(\beta_1) \neq \varphi(\beta_2)$$

4. **Option C** $\theta_2 + \theta_3$

   Let $\beta_1 = \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} \beta_2 = \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix}$

$$X\beta_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = \begin{bmatrix} i_1 + j_1 \\ i_1 + j_1 \\ i_1 - k_1 \\ i_1 - k_1 \end{bmatrix}$$

$$X\beta_2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = \begin{bmatrix} i_2 + j_2 \\ i_2 + j_2 \\ i_2 - k_2 \\ i_2 - k_2 \end{bmatrix}$$

   If β_1, β_2 satisfy $X\beta_1 = X\beta_2$

then, $i_1 + j_1 = i_2 + j_2, \quad i_1 - k_1 = i_2 - k_2 \implies j_1 + k_1 = j_2 + k_2$

$$\varphi(\beta_1) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_1 \\ j_1 \\ k_1 \end{bmatrix} = j_1 + k_1$$

$$\varphi(\beta_2) = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} i_2 \\ j_2 \\ k_2 \end{bmatrix} = j_2 + k_2$$

$\varphi(\beta_1) = \varphi(\beta_2)$ for any β_1, β_2 so **Option C** is correct answer

2 References

1. The Coordinate-Free Approach to Linear Models. Section 4.8