Assignment 3 - Problem 56 Dec-2018 Paper

Consider a linear model $Y_1 = \theta_1 + \theta_2 + \epsilon_i$ for $l = 1, 2$ and $Y_1 = \theta_1 - \theta_3 + \epsilon$ for l = 3,4, where ϵ_i are independent with $R(\epsilon_i) = 0$, $Var(\epsilon_i) = \sigma^2 > 0$ for i = 1, ... 4, and $\theta_1, ...\theta_3 \in R$. Which of the following parametric function is estimable?
(A) $\theta_1 + \theta_3$
(B) $\theta_2 - \theta_3$
(C) $\theta_2 + \theta_3$
(D) $\theta_2 + \theta_2 + \theta_3$

Answer: C, $\theta_2 + \theta_3$

1 Solution:

Given,

$$Y_1 = \theta_1 + \theta_2 + \epsilon \tag{1}$$

$$Y_1 = \theta_1 - \theta_3 + \epsilon \tag{2}$$

To estimate the parameters we try to find the minimum of squared errors ϵ^2

1.1 Generalised equation for least squares

Let X be input matrix, $\bar{\theta}$ be the parameter vector to be estimated. Then

$$Y = X\bar{\theta} \tag{3}$$

Let $\bar{t}$ be estimated targets.
Then error

$$\bar{e} = [\bar{t} - Y]$$

$$\bar{e}^2 = \frac{1}{2}Tr[[\bar{t} - Y][\bar{t} - Y]^T]$$

Taking derivative w.r.t $\bar{\theta}$ and equating it to 0 to minimise the error and find best value of $\bar{\theta}$. Let the objective function be represented as $J(\theta)$

By chain rule

$$\nabla J(\bar{\theta}) = \frac{\partial J(\bar{\theta})}{\partial \bar{\theta}}$$
$$= \Big[\frac{\partial Y)}{\partial \bar{\theta}}\Big]\Big[\frac{\partial \bar{e}}{\partial Y}\Big]\Big[\frac{\partial J(\bar{\theta})}{\partial \bar{e}}\Big]$$

$$\frac{\partial Y}{\partial \bar{\theta}} = \frac{\partial X\bar{\theta}}{\partial \bar{\theta}} = X^T$$
$$\frac{\partial \bar{e}}{\partial Y} = \frac{\partial}{\partial Y}[\bar{t} - Y] = -I$$
$$\frac{\partial J(\bar{\theta})}{\partial \bar{e}} = \frac{\partial}{\partial \bar{e}}[\bar{e}\bar{e}^T] = 2\bar{e}$$

$$\nabla J(\bar{\theta}) = X^T\bar{e} = 0$$
$$\implies X^T[\bar{t} - Y] = 0$$
$$\implies X^T[\bar{t}] = X^T Y$$
$$\implies \bar{\theta} = [X^T X]^{-1}[X^T\bar{t}]$$

1.2 Evaluating options

Subtracting equation (2) from equation (1), we get parametric function

$$\theta_2 + \theta_3$$

Whose Parameters can be estimated by finding least squares estimation of (1) - (2). But for all the other options parametric estimation can not be done given (1) and (2), as we can not write them as linear combinations of (1) and (2).
Hence the answer is C