Assignment 1,2

Problem 53, 58 of UGC Math 2019

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Assignment 1

Question

Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

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Answer: 4
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Definition

Bivariate Normal Distribution:

Distribution XY in 2 dimensions is said to be Bi-variate normal distribution if X and Y are independently normal distributions and are jointly normal in 2 dimensions.

Correlation of of X,Y $\rho[X, Y] = \frac{\sigma_{[X,Y]}}{\sqrt{\sigma_X^2 \times \sigma_Y^2}}$

If X,Y are independent then $\sigma_{XY} = 0$

So, for testing the independence, if we assume they are independent and then try to prove if $\sigma_{XY} = 0$, for all the options, we can find out the answer.

Known equations

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$

$$\sigma_{[X,X]} = \sigma^2_{[X]} \tag{2}$$

$$if \quad \sigma[X,Y] = \sigma[X]^2 \quad then \quad X = Y \tag{3}$$

(Proofs for above equations are given in the end)

Testing Independence of X,Y

$$\begin{aligned}
\sigma[X,Y] &= E((X-\mu_X)(Y-\mu_Y)) \\
&= E[(XY) - \mu_X Y - \mu_Y X + \mu_X\mu_Y] \\
&= E(XY) - \mu_X\mu_Y - \mu_Y\mu_X + \mu_X\mu_Y \\
&= E(XY) - E(X)E(Y) \\
\sigma[X,Y] = 0 &\implies E(XY) = E(X)E(Y)
\end{aligned}$$

$$\begin{aligned}
\sigma[X,X] &= \sigma[Y,Y] \\
&\implies E[(X-\mu_X)^2] = E[(Y-\mu_Y)^2] \\
&\implies E[X^2] - E[X]^2 = E[Y^2] - E[Y]^2
\end{aligned}$$

Argument: Both equations can not be solved with the given hypothesis, so can not comment on the in dependency

Testing Independence of X and X-Y

1.

$$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$

$$\sigma_{[X-Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$

2. if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = X$ From Eq, 3

Argument: $X - Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

Testing independence of X+Y, Y

1.

$$\begin{aligned} \sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]} \end{aligned}$$

$$\sigma_{[X+Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$

2. if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = -X$ which means they are dependant

> **Argument**: $X + Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = -X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

Testing for Independence of X+Y and X-Y

1.

$$
\begin{aligned}
&\sigma_{[X+Y,X-Y]} \\
&= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\
&= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\
&= \sigma_X^2 - \sigma_Y^2
\end{aligned}
$$

2. Now testing for $\sigma_X = \sigma_Y \implies \sigma_{[X+Y,X-Y]} = 0$
3. Hence testing for $\sigma_X = \sigma_Y \implies X+Y, X-Y$ are independent.

$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]}$

$$
\begin{aligned}
\sigma_{[X+Y,Z]} &= E((X+Y-\mu_{X+Y}) \times (Z-\mu_Z)) \\
&= E((X+Y-\mu_X-\mu_Y) \times (Z-\mu_Z)) \\
&= E(XZ - X\mu_z + YZ - Y\mu_Z - Z\mu_X + \mu_X\mu_Z - \mu_Y Z + \mu_Y\mu_Z) \\
&= E((XZ - \mu_Z X - \mu_X Z + \mu_z\mu_x) + (YZ - Y\mu_Z - \mu_Y Z + \mu_Y\mu_Z)) \\
&= E((X-\mu_X)(Z-\mu_Z) + (Y-\mu_Y)(Z-\mu_Z)) \\
&= E((X-\mu_X)(Z-\mu_Z)) + E((Y-\mu_Y)(Z-\mu_Z)) \\
&= \sigma_{[X,Z]} + \sigma_{[Y,Z]}
\end{aligned}
$$

$\sigma_{[X,Y]} = \sigma^2_{[X]}$

$$\begin{aligned}
\sigma_{[X,Y]} &= E[(X - \mu_X)(Y - \mu_Y)], \quad if \quad Y = X \\
\sigma_{[X,X]} &= E[(X - \mu_X)(X - \mu_X)] \\
&= E[(X - \mu_X)^2] \\
&= \sigma^2_X
\end{aligned}$$

Assignment 2

Question

A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement sampling scheme, where the probabilities proportional to size are
The probability of inclusion of unit 1 in the sample is

i	1	2	3	4
Pi	0.4	0.2	0.2	0.2

1. 0.4
2. 0.6
3. 0.7
4. 0.75

Answer : Option - 3 - (0.7)

Definition of PPS without replacement

1. **Sampling**: Selecting smaller units from a large population
2. **Simple Sampling scheme**: Probability of selecting every unit is uniformly distributed
   $P(U_1) = P(U_2).... = P(U_n)$
3. **PPS WOR**: When P selection of every unit is not same, repetition is not allowed

Solution

For finding the inclusion probability of unit 1 in a sample of 2, we need find in how many ways unit 1 can be included in a sample of 2.

1. $P_1(1) = P_1$
2. $P_1(2)$

P $= P_1(1) + P_1(2)$

contd ..

$P_{i(2)}$ can occur in following possible ways:

- U_2 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_3 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_4 is selected in 1^{st} draw and U_1 is selected at 2 draw

$P_{i(2)} = \sum_2^4 P_i(1) \times P_1(2)$

contd ..)

$$P_i(1) \times P_1(2) = P_i(1) \times P_1(U_1/U_i)$$
$$P_1(U_1/U_i) = P_1/1 - P_i$$

P_1 over $1 - P_i$, because U_i can not be selected again

contd ..

$$
\begin{aligned}
P_{i(2)} &= \sum_{i=2}^{4} P_i \times \frac{P_1}{1 - P_i(0)} \\
&= 0.4 \times \left(\frac{0.2}{1 - 0.2} + \frac{0.2}{1 - 0.2} + \frac{0.2}{1 - 0.2} \right) \\
&= 0.4 \times \left(\frac{0.6}{0.8} \right) \\
&= 0.3
\end{aligned}
$$

contd ..

- The probability of inclusion of unit 1 in the sample is $P_{i(1)} + P_{i(2)}$
- 0.3$+$0,4 $=$ 0.7