Assignment 1,2

Problem 53, 58 of UGC Math 2019

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Assignment 1

Question

Suppose (X,Y) follows bivariate normal distribution with means $\mu1\mu2$, standard deviations $\sigma1$,$\sigma2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma1 = \sigma2$ is equivalent to testing the independence of

1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

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Answer: 4
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Definition

Bivariate Normal Distribution:

Random vector A $= \begin{bmatrix} X \\ Y \end{bmatrix}$ is Bi-variate when it is jointly normal

Joint PDF of A is given as

$$f_a(A) = \frac{1}{(2\pi)}\sqrt{detC} \quad exp\left\{ \frac{-1}{2}(a-m)^T C^{-1}(a-m) \right\}$$

$$Where, m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

$$C = \begin{bmatrix} \rho\sigma_x^2 & \sigma_{xy} \\ \rho\sigma_{yx} & \sigma_y^2 \end{bmatrix}$$

Known equations

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$

$$\sigma_{[X,X]} = \sigma^2_{[X]} \tag{2}$$

$$if \quad \sigma[X,Y] = \sigma[X]^2 \quad then \quad X = Y \tag{3}$$

(Proofs for above equations are given in the end)

Testing Independence of X,Y

1. X,Y can be proven as indepndant if we can prove $\sigma_{XY} = 0$
2. Or if joint PDF can be written as product of PDFs of X,Y
3. With given $\sigma_X = \sigma_X$, both of them can not be proved, hence H_0 in dependency can not be tested.

Testing Independence of X and X-Y

1.

$$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]}\\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$

2. Given $\sigma_X^2 = \sigma_Y^2$ we can not say if $\sigma_{[X-Y,X]} = 0$ or not

Testing independence of X+Y, Y

1. $$\begin{aligned}\sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]}\\ &= \sigma_X^2 + \sigma_{[X,Y]}\end{aligned}$$

2. Given $\sigma_X^2 = \sigma_Y^2$ we can not say if $\sigma_{[X+Y,Y]} = 0$ or not

Testing for Independence of X+Y and X-Y

1.

$$
\begin{aligned}
&\sigma_{[X+Y,X-Y]} \\
&= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\
&= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\
&= \sigma_X^2 - \sigma_Y^2
\end{aligned}
$$

2. Now testing for $\sigma_X = \sigma_Y \implies \sigma_{[X+Y,X-Y]} = 0$
3. Hence testing for $\sigma_X = \sigma_Y \implies X + Y, X - Y$ are independent.

$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]}$

$$
\begin{aligned}
\sigma_{[X+Y,Z]} &= \mathbf{E}[((\mathbf{X}+\mathbf{Y}) - \mathbf{E}(\mathbf{X}+\mathbf{Y}))(\mathbf{Z} - \mathbf{EZ})^{\mathbf{T}}] \\
&= \mathbf{E}((\mathbf{X}+\mathbf{Y} - \mu_{\mathbf{X}} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}(\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{z}}^{\mathbf{T}} + \mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{X}}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}}) \\
&= \mathbf{E}((\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{x}}\mu_{\mathbf{z}}^{\mathbf{T}}) + (\mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}}) + (\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) + \mathbf{E}((\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) \\
&= \sigma_{[\mathbf{X},\mathbf{Z}]} + \sigma_{[\mathbf{Y},\mathbf{Z}]}
\end{aligned}
$$

$\sigma_{[X,Y]} = \sigma^2_{[X]}$

$$
\begin{aligned}
\sigma_{[X,Y]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Y} - \mu_{\mathbf{Y}})^{\mathsf{T}}], \quad \textit{if} \quad Y = X \\
\sigma_{[X,X]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\mathsf{T}}] \\
&= \sigma^2_X
\end{aligned}
$$

Assignment 2

Question

A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement sampling scheme, where the probabilities proportional to size are
The probability of inclusion of unit 1 in the sample is

i	1	2	3	4
Pi	0.4	0.2	0.2	0.2

1. 0.4
2. 0.6
3. 0.7
4. 0.75

Answer : Option - 3 - (0.7)

Definition of PPS without replacement

1. **Sampling**: Selecting smaller units from a large population
2. **Simple Sampling scheme**: Probability of selecting every unit is uniformly distributed
   $P(U_1) = P(U_2).... = P(U_n)$
3. **PPS WOR**: When P selection of every unit is not same, repetition is not allowed

Solution

For finding the inclusion probability of unit 1 in a sample of 2, we need find in how many ways unit 1 can be included in a sample of 2.

1. $P_1(1) = P_1$
2. $P_1(2)$

P $= P_1(1) + P_1(2)$

contd ..

$P_{i(2)}$ can occur in following possible ways:

- U_2 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_3 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_4 is selected in 1^{st} draw and U_1 is selected at 2 draw

$P_{1(2)} = \sum_2^4 P_i(1) \times P_1(2)$

contd ..)

$$P_i(1) \times P_1(2) = P_i(1) \times P_1(U_1/U_i)$$
$$P_1(U_1/U_i) = \frac{P_1}{(1 - P_i)}$$

P_1 over $1 - P_i$, because U_i can not be selected again

contd ..

$$
\begin{aligned}
P_{i(2)} &= \sum_{i=2}^{4} P_i \times \frac{P_1}{1 - P_i(1)} \\
&= 0.4 \times \left(\frac{0.2}{1 - 0.2} + \frac{0.2}{1 - 0.2} + \frac{0.2}{1 - 0.2} \right) \\
&= 0.4 \times \left(\frac{0.6}{0.8} \right) \\
&= 0.3
\end{aligned}
$$

contd ..

- The probability of inclusion of unit 1 in the sample is $P_{i(1)} + P_{i(2)}$
- $0.3+0.4 = 0.7$