Assignment 1,2

Problem 53, 58 of UGC Math 2019

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Assignment 1

Question

Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

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Answer: 4
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Definition

Bivariate Normal Distribution:

Random normal vector $Z = \begin{bmatrix} X \\ Y \end{bmatrix}$ is Bi-variate when it is jointly normal. Z = aX + bY

Joint PDF of Z is given as

$$f_z(Z) = \frac{1}{2\pi\sqrt{detC}} \quad \exp\left\{\frac{-1}{2}(z-m)^T C^{-1}(z-m)\right\}$$

Where,

$$m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}, C = \begin{bmatrix} \sigma_X^2 & \rho\sigma_{XY} \\ \rho\sigma_{YX} & \sigma_Y^2 \end{bmatrix}$$

Definition of Independence

If X, Y, which are independent, then they are un-corelated or their co-variances are $\sigma_{XY} = \sigma_{YX} = 0$ then covariance matrix becomes a diagonal matrix

$$C = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_Y\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & 0 \\ 0 & \sigma_Y^2 \end{bmatrix}$$

Co-variance Matrix

$$\Sigma = [[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T]$$
$$\textit{When,} \quad \mathbf{Z}^{'} = \mathbf{TZ}, \quad \textit{Where } T \textit{ is a transformation}$$

$$\begin{aligned}
\Sigma_{TZ} &= [[\mathbf{TZ} - E(\mathbf{TZ})][\mathbf{TZ} - E(\mathbf{TZ})]^T] \\
&= [[T\mathbf{Z} - TE(\mathbf{Z})][T\mathbf{Z} - TE(\mathbf{Z})]^T] \\
&= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\
&= [T[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T T^T] \\
&= [T\Sigma T^T]
\end{aligned}$$

$$\textit{Where}[[\mathbf{Z} - E(\mathbf{Z})][\mathbf{Z} - E(\mathbf{Z})]^T] = \Sigma$$

$$\Sigma_{TZ} = [T\Sigma T^T] \tag{1}$$

Evaluating option 1

Given $\sigma_x = \sigma_y$

$$\Sigma = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X\sigma_X \\ \rho\sigma_X\sigma_X & \sigma_Y^2 \end{bmatrix} = \begin{bmatrix} \sigma_X^2 & \rho\sigma_X^2 \\ \rho\sigma_X^2 & \sigma_X^2 \end{bmatrix} = \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

Σ is not a diagonal matrix so components of Z in option 1 are not independent

Evaluating option 2

X, X-Y can be written as $\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}X\\ Y\end{bmatrix}$

Where T = $\begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}$, Z = $\begin{bmatrix}X\\ Y\end{bmatrix}$

Co-variance matrix Σ for X, X-Y From Eq. 1 $\Sigma_{TZ} = [T\Sigma T^T]$

$$\Sigma_{TZ} = \begin{bmatrix}1 & 0\\ 1 & -1\end{bmatrix}\sigma_X^2\begin{bmatrix}1 & \rho\\ \rho & 1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix} \quad = \sigma_X^2\begin{bmatrix}1 & \rho\\ 1-\rho & \rho-1\end{bmatrix}\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}$$

$$= \sigma_X^2\begin{bmatrix}1 & 1-\rho\\ 1-\rho & 2-\rho\end{bmatrix}$$

Σ_{TZ} is not a diagonal matrix, so components of TZ are not independent

Evaluating option 3

X+Y and Y can be written as $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

Where T = $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, Z = \begin{bmatrix} X \\ Y \end{bmatrix}$

Co-variance matrix Σ for X+Y, Y From Eq. 1 $\Sigma_{TZ} = [T \Sigma T^T]$

$$\Sigma_{TZ} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$
$$= \sigma_x^2 \begin{bmatrix} 2 + 2\rho & \rho + 1 \\ \rho + 1 & 1 \end{bmatrix}$$

Σ_{TZ} is not a diagonal matrix, so components of TZ are not independent

Evaluating option 4

X+Y and X-Y can be written as $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$

From Eq. 1 $\Sigma_{TZ} = [T\Sigma T^T]$

$$\Sigma_{TZ} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \sigma_X^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \sigma_x^2 \begin{bmatrix} 1+\rho & \rho+1 \\ 1-\rho & \rho-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$
$$= \begin{bmatrix} 2+2\rho & 0 \\ 0 & 2-2\rho \end{bmatrix}$$

Hence option 4 is correct

Assignment 2

Question

A sample of size $n = 2$ is drawn from a population of size $N = 4$ using probability proportional to size without replacement sampling scheme, where the probabilities proportional to size are
The probability of inclusion of unit 1 in the sample is

i	1	2	3	4
Pi	0.4	0.2	0.2	0.2

1. 0.4
2. 0.6
3. 0.7
4. 0.75

Answer : Option - 3 - (0.7)

Definition of PPS without replacement

1. **Sampling**: Selecting smaller units from a large population
2. **Simple Sampling scheme**: Probability of selecting every unit is uniformly distributed
   $P(U_1) = P(U_2).... = P(U_n)$
3. **PPS WOR**: When P selection of every unit is not same, repetition is not allowed

Solution

For finding the inclusion probability of unit 1 in a sample of 2, we need find in how many ways unit 1 can be included in a sample of 2.

1. $P_1(1) = P_1$
2. $P_1(2)$

P $= P_1(1) + P_1(2)$

contd ..

$P_{i(2)}$ can occur in following possible ways:

- U_2 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_3 is selected in 1^{st} draw and U_1 is selected at 2 draw
- U_4 is selected in 1^{st} draw and U_1 is selected at 2 draw

$P_{1(2)} = \sum_2^4 P_i(1) \times P_1(2)$

contd ..)

$$P_i(1) \times P_1(2) = P_i(1) \times P_1(U_1/U_i)$$
$$P_1(U_1/U_i) = \frac{P_1}{(1 - P_i)}$$

P_1 over $1 - P_i$, because U_i can not be selected again

contd ..

$$
\begin{aligned}
P_{i(2)} &= \sum_{i=2}^{4} P_i \times \frac{P_1}{1 - P_i(1)} \\
&= 0.4 \times \left(\frac{0.2}{1-0.2} + \frac{0.2}{1-0.2} + \frac{0.2}{1-0.2} \right) \\
&= 0.4 \times \left(\frac{0.6}{0.8} \right) \\
&= 0.3
\end{aligned}
$$

contd ..

- The probability of inclusion of unit 1 in the sample is $P_{i(1)} + P_{i(2)}$
- $0.3+0.4 = 0.7$