Assignment 1 - Problem 53

Vishnu Gollamudi

January 2022

Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

Answer: 4, X+Y and X-Y
Solution:
Bi-variate random variables are distribution of normal distribution to two co-ordinates. are said to be bivariate normal or jointly normal, if $aX + bY$ has normal distribution $\forall\ a, b \in R$.

Random normal vector A = $\begin{bmatrix} X \\ Y \end{bmatrix}$ is Bi-variate when it is jointly normal
Joint PDF of A is given as

$$f_a(A) = \frac{1}{(2\pi)}\sqrt{detC} \quad exp\left\{\frac{-1}{2}(a-m)^T C^{-1}(a-m)\right\}$$
$$Where, m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$
$$C = \begin{bmatrix} \rho\sigma_x^2 & \sigma_{xy} \\ \rho\sigma_{yx} & \sigma_y^2 \end{bmatrix}$$

We know that

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$
$$\sigma_{[X,X]} = \sigma^2_{[X]} \tag{2}$$
$$if \quad \sigma[X,Y] = \sigma[X]^2 \quad then \quad X = Y \tag{3}$$

(Proofs for above equations are given in appendix)

1. Testing for the independence of X,Y
   (a) X,Y can be proven as indepndant if we can prove $\sigma_{XY} = 0$
   (b) Or if joint PDF can be written as product of PDFs of X,Y
   (c) With given $\sigma_X = \sigma_X$, both of them can not be proved, hence H_0 in dependency can not be tested.
   (d) In NUll hypothesis if we can not prove X,Y are independant then it does not mean they are independant
2. Testing independence of X,X-Y
   (a)
   $$\begin{aligned} \sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]} \end{aligned}$$
   $$\sigma_{[X-Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$
   (b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = X$

   > **Argument**: $X - Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

3. Testing for independence of X,X+Y
   (a)
   $$\begin{aligned} \sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]} \end{aligned}$$
   $$\sigma_{[X+Y,X]} = 0 \implies \sigma_X^2 = \sigma_{[X,Y]}$$
   (b) if $\sigma_X^2 = \sigma_{[X,Y]}$ then it means $Y = -X$ which means they are dependant

   > **Argument**: $X + Y, X$ can be independent if and only if $\sigma_{[X-Y,X]} = 0$. But $\sigma_{[X-Y,X]} = 0 \implies Y = -X$, hence they are dependant irrespective of $\sigma_X = \sigma_Y$

4. Testing for independence of X+Y,X-Y

   (a)

$$\begin{aligned}
&\sigma_{[X+Y,X-Y]} \\
&= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\
&= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\
&= \sigma_X^2 - \sigma_Y^2
\end{aligned}$$

   (b) Now testing for $\sigma_1 = \sigma_2 \implies \sigma_{[X+Y,X-Y]} = 0$

   (c) Hence testing for $\sigma_1 = \sigma_2 \implies X+Y, X-Y$ are independent.

Appendix

Co variance is a measure of how much two random variables vary together

1. $\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]}$

$$\begin{aligned}
\sigma_{[X+Y,Z]} &= \mathbf{E}[((\mathbf{X}+\mathbf{Y}) - \mathbf{E}(\mathbf{X}+\mathbf{Y}))(\mathbf{Z} - \mathbf{EZ})^{\mathbf{T}}] \\
&= \mathbf{E}((\mathbf{X}+\mathbf{Y} - \mu_{\mathbf{X}} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}(\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{z}}^{\mathbf{T}} + \mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{X}}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}}) \\
&= \mathbf{E}((\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{x}}\mu_{\mathbf{z}}^{\mathbf{T}}) + (\mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}}) + (\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) + \mathbf{E}((\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) \\
&= \sigma_{[\mathbf{X},\mathbf{Z}]} + \sigma_{[\mathbf{Y},\mathbf{Z}]}
\end{aligned}$$

2. $\sigma_{[X,Y]} = \sigma_{[X]}^2$

$$\begin{aligned}
\sigma_{[X,Y]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Y} - \mu_{\mathbf{Y}})^{\mathbf{T}}], \quad \textit{if} \quad Y = X \\
\sigma_{[X,X]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\mathbf{T}}] \\
&= \sigma_X^2
\end{aligned}$$

3. if $\sigma[X,Y] = \sigma[X]^2 \quad then \quad X = Y$
   From 2 it follows that $\sigma[X,Y] = \sigma[X]^2$ when X=Y