Assignment 1 - Problem 53

Vishnu Gollamudi

January 2022

Question 53) Suppose (X,Y) follows bivariate normal distribution with means $\mu 1 \mu 2$, standard deviations $\sigma 1$,$\sigma 2$ and correlation coefficient ρ, where all parameters are un-known. Then, testing Ho: $\sigma 1 = \sigma 2$ is equivalent to testing the independence of
1.) X and Y
2.) X and X-Y
3.) X+Y and Y
4.) X+Y and X-Y

Answer: 4, X+Y and X-Y
Solution:
Bi-variate random variables are distribution of normal distribution to two co-ordinates. are said to be bivariate normal or jointly normal, if $aX + bY$ has normal distribution $\forall\ a, b \in R$.

Random normal vector A = $\begin{bmatrix} X \\ Y \end{bmatrix}$ is Bi-variate when it is jointly normal
Joint PDF of A is given as

$$f_a(A) = \frac{1}{(2\pi)} \sqrt{detC} \quad exp\left\{ \frac{-1}{2}(a-m)^T C^{-1} (a-m) \right\}$$

$$Where, m = \begin{bmatrix} \mu_x \\ \mu_y \end{bmatrix}$$

$$C = \begin{bmatrix} \rho\sigma_x^2 & \sigma_{xy} \\ \rho\sigma_{yx} & \sigma_y^2 \end{bmatrix}$$

We know that

$$\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]} \tag{1}$$

$$\sigma_{[X,X]} = \sigma_{[X]}^2 \tag{2}$$

$$if \quad \sigma[X,Y] = \sigma[X]^2 \quad then \quad X = Y \tag{3}$$

(Proofs for above equations are given in appendix)

1. Testing for the independence of X,Y
   (a) X,Y can be proven as indepndant if we can prove $\sigma_{XY} = 0$
   (b) Or if joint PDF can be written as product of PDFs of X,Y
   (c) With given $\sigma_X = \sigma_X$, both of them can not be proved, hence H_0 in dependency can not be tested.
2. Testing independence of X,X-Y
   (a)
   $$\begin{aligned}\sigma_{[X-Y,X]} &= \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_{[X,Y]}\end{aligned}$$
   (b) Given $\sigma_X^2 = \sigma_Y^2$ we can not say if $\sigma_{[X-Y,X]} = 0$ or not
3. Testing for independence of X,X+Y
   (a)
   $$\begin{aligned}\sigma_{[X+Y,X]} &= \sigma_{[X,X]} + \sigma_{[X,Y]} \\ &= \sigma_X^2 + \sigma_{[X,Y]}\end{aligned}$$
   (b) Given $\sigma_X^2 = \sigma_Y^2$ we can not say if $\sigma_{[X+Y,Y]} = 0$ or not
4. Testing for independence of X+Y,X-Y
   (a)
   $$\begin{aligned}&\sigma_{[X+Y,X-Y]} \\ &= \sigma_{[(X+Y),X]} - \sigma_{[(X+Y),Y]} \\ &= \sigma_{[X,X]} + \sigma_{[X,Y]} - \sigma_{[X,X]} - \sigma_{[X,Y]} \\ &= \sigma_X^2 - \sigma_Y^2\end{aligned}$$
   (b) Now testing for $\sigma_1 = \sigma_2 \implies \sigma_{[X+Y,X-Y]} = 0$
   (c) Hence testing for $\sigma_1 = \sigma_2 \implies X+Y, X-Y$ are independent.

Appendix

Co variance is a measure of how much two random variables vary together

1. $\sigma_{[X+Y,Z]} = \sigma_{[X,Z]} + \sigma_{[Y,Z]}$

$$
\begin{aligned}
\sigma_{[X+Y,Z]} &= \mathbf{E}[((\mathbf{X}+\mathbf{Y}) - \mathbf{E}(\mathbf{X}+\mathbf{Y}))(\mathbf{Z} - \mathbf{E}\mathbf{Z})^{\mathbf{T}}] \\
&= \mathbf{E}((\mathbf{X}+\mathbf{Y} - \mu_{\mathbf{X}} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}(\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{z}}^{\mathbf{T}} + \mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{X}}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}}) \\
&= \mathbf{E}((\mathbf{X}\mathbf{Z}^{\mathbf{T}} - \mathbf{X}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{X}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{x}}\mu_{\mathbf{z}}^{\mathbf{T}}) + (\mathbf{Y}\mathbf{Z}^{\mathbf{T}} - \mathbf{Y}\mu_{\mathbf{Z}}^{\mathbf{T}} - \mu_{\mathbf{Y}}\mathbf{Z}^{\mathbf{T}} + \mu_{\mathbf{Y}}\mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}}) + (\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z}^{\mathbf{T}} - \mu_{\mathbf{Z}}^{\mathbf{T}})) \\
&= \mathbf{E}((\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) + \mathbf{E}((\mathbf{Y} - \mu_{\mathbf{Y}})(\mathbf{Z} - \mu_{\mathbf{Z}})^{\mathbf{T}}) \\
&= \sigma_{[\mathbf{X},\mathbf{Z}]} + \sigma_{[\mathbf{Y},\mathbf{Z}]}
\end{aligned}
$$

2. $\sigma_{[X,Y]} = \sigma^2_{[X]}$

$$
\begin{aligned}
\sigma_{[X,Y]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{Y} - \mu_{\mathbf{Y}})^{\mathbf{T}}], \quad \mathit{if} \quad Y = X \\
\sigma_{[X,X]} &= \mathbf{E}[(\mathbf{X} - \mu_{\mathbf{X}})(\mathbf{X} - \mu_{\mathbf{X}})^{\mathbf{T}}] \\
&= \sigma^2_X
\end{aligned}
$$

3. if $\sigma[X,Y] = \sigma[X]^2 \quad \mathit{then} \quad X = Y$
   From 2 it follows that $\sigma[X,Y] = \sigma[X]^2$ when X=Y