ANALYTICAL PLANE GEOMETRY

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PREFACE

This book is designed primarily for the use of the B. A. (Pass and Hons.) students of the Punjab University : but it will also be found useful for the B. A. students of other Indian Universities. In preparing the book, several works have been freely consulted, to the authors of which our obligations are gratefully acknowledged. We are also indebted to Prof. A. N. Ganguli for suggesting the problem of the bisectors of acute and obtuse angles between two lines.

No fresh notations have been introduced except that *w. r.* to stands for 'with respect to' or 'with regard to'; and 'st. line' stands for 'straight line'. $(\alpha, \beta\ \gamma)$ and (x, y, z) are sometimes indifferently used for Trilinear co-ordinates.

Chap. I deals with 'one-dimensional extent', and 'straight line' is treated of in Chaps. II and III. A slight modification of the normal form of the equation of a st. line has enabled us to discuss the topics of 'angle between two lines', 'bisector of the angle' etc. in greater detail than is customary. 'Circle' forms the topic of Chaps. IV and V, and is discussed as a particular case of the general 2nd degree equation in x, y. The usual definition of a conic (Chap. VII) by the focus-directrix property has been discarded. To avoid repetition, some general properties of conics have been established in this Chapter. Conormality of four points on a conic has been discussed at some length. An attempt has been made to make the Trilinears independent of the Cartesians. The last Chapter on systems of conics deals with both the systems of co-ordinates.

There are plenty of examples at theend of each chapter. Some have been selected from the works mentioned in the bibliography, while others have been taken from the Tripos and Scholarship Examinations of the University of Cambridge. To all these writers whose works we have consulted, we tender our thanks. Our thanks are also due to the Syndics of the Cambridge University Press for permission to include questions from the Tripos and Scholarship Examination-papers, as well as from Radford's Problem Papers.

It is hoped that the book will be found useful and adapted to the purpose for which it is intended. Corrections and suggestions for improvement will be welcomed.

Lahore, 1-9-1938. AUTHORS.

CONTENTS

CHAPTER I

INTRODUCTION. ONE DIMENSIONAL EXTENT

1. Plane Geometry studies the properties and mutual relations of points and lines in a plane. It is sometimes called two dimensional Geometry. Co-ordinate Geometry relates together Geometry which started as the science of space, and Algebra which has its origin in the science of numbers. Our first object will, therefore, be to explain how the position of a point on a straight line can be represented by algebraic symbols.

1.1. Wherever our notion of a st. line may be derived from, we assume that the system of real numbers can be uniquely and reversibly represented by the points of a real st. line. This statement enables us to assign to each point of the line a particular number of the real aggregate. Take a line L produced indefinitely in both directions. Then in order to obtain points on L corresponding to real numbers, we fix a point O on L (called the origin); and to facilitate the measurements, we choose a definite scale representing a certain real number, unity for instance. Let OA denote a unit of length.

To the point O we assign the number 0 and to A the number 1. The position of any other point on L will be determined by its distance from O and this distance will be expressed in terms of the chosen scale; the length being regarded positive in one direction (when measured to the right of O) and negative in the other (when measured to the left of O).

A real number x to which corresponds a point P on L is called the x co-ordinate of P, and is denoted by P(x).

Since one co-ordinate fixes a point on L, a st. line is called One Dimensional Extent.

When the point Q is situated so far away from O, that its co-ordinate cannot be represented by any real number, however large, we say that the point Q is at infinity and we denote its position by the symbol $+\infty$ or $-\infty$ according as Q is to the right or left of O. It will be shown later on

that in Euclidean Geometry the points $+\infty$ and $-\infty$ are not distinct. Thus every st. line has a unique point at infinity. Since there is no number infinity (∞) in the number system, the position of Q on the st. line does, therefore, appear to be vague and indefinite.

It should be borne in mind that the symbol ∞ does not denote any number. Unlike P(2) which means that P is at a distance of two units from O to the right of O, $Q(\infty)$ simply denotes that there is no point on L which will not ultimately be passed by Q.

2. Principle of Continuity. Imaginary Points. Modern geometry is characterized by generality both of its methods and its results. This is, to a large extent, due to the assumption of the principle of continuity. It asserts that "if we once demonstrate a property for any figure in any one of its general states, and if we then suppose the figure to change its form, subject, of course, to the conditions in accordance with which it was first traced, the property we have proved, though it may become unmeaning, can never become untrue, even if every point and every line by means of which it was originally proved should disappear." For instance, a st. line can be drawn to cut a circle in two points, hence we say that every st. line cuts a circle in two points, real, coincident or imaginary. Similarly, from an external point two tangents can be drawn to a circle. In virtue of this principle, the statement remains true if the point be on or inside the circle, these tangents being now coincident or imaginary. Algebraical Geometry comes to our help to prove the general statements.

Thus "the Principle of continuity asserts that if from the nature of a particular problem we should expect a certain number of solutions and if in any particular case we find this number of solutions, then there will be the same number of solutions in all cases. In fact it asserts that theorems concerning real points or lines may be extended to imaginary points or lines."

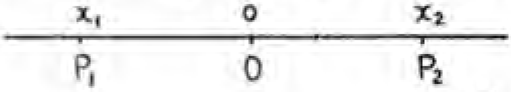
The existence of imaginary points, is, therefore, a natural consequence of the principle of continuity. With every imaginary point we associate a complex number x which we call its co-ordinate, and conversely. In this sense, a point is nothing more than a concise expression for a value of x. We may, therefore, state as follows :—

With every point P on a line is associated a number x called its co-ordinate, and conversely.

This has been established for real points in Art. **1.1** and postulated in the present Art. for imaginary points.

It may be pointed out, that the change from a real to an imaginary state takes place only when some elements of the figure have coincided. For instance, the intersections of a st. line and a circle become imaginary after the st. line has become a tangent, *i.e.*, after both the points of intersection fall together. Imagine a st. line cutting the circle in two points and let it move parallel to itself. During its motion, it will be seen that the two points of intersection approach nearer and nearer until they tend to coincide, *i.e.*, until the line touches the circle. After this position of the st. line, the line will cut the circle in two imaginary points.

3. *To find the distance between two points* $P_1(x_1)$, $P_2(x_2)$



The distance P_1P_2

$$= OP_2 - OP_1$$

$$= x_2 - x_1 \qquad \text{......(1)}$$

This proposition, like many others that follow, is partly a theorem and partly a postulate. This is a theorem as far as real points are concerned and this has been proved above. But it is a definition for imaginary points.

4. Section Formula. *To find the co-ordinate of the point* P *that divides the segment* P_1P_2 *in a given ratio.*

Let $P_1(x_1)$, $P_2(x_2)$ be the extremities of the segment, and λ the given ratio. If the co-ordinate of P be x.

$$\frac{P_1P}{PP_2} = \lambda,$$

$$\therefore \qquad \frac{x - x_1}{x_2 - x} = \lambda,$$

or

$$x = \frac{x_1 + \lambda x_2}{1 + \lambda} \qquad \text{......(2)}$$

The ratio λ is positive or negative according as P divides the segment internally or externally.

5. Definitions. A number of collinear points are said to form a *range*.

If A, B, C, D be a range of points, the ratio $\frac{AB.CD}{AD.CB}$ is called a *cross-ratio* of the four points and is represented by (ABCD), in which the order of the letters is the same as their order in the numerator of the cross-ratio. According to the cyclic arrangement the letters are taken clockwise for the numerator and counter-clockwise for the denominator.

The cross-ratio is also represented by (AC, BD).

Let AC be a segment and B, D two other points which divide AC internally and externally in the ratios of λ, μ. Then

$$\frac{AB}{BC}=\lambda,\quad \frac{AD}{DC}=\mu.$$

The ratio $\frac{\lambda}{\mu}$ is said to be the cross-ratio of the range ABCD.

Evidently $$(AC,\ BD)=\frac{\lambda}{\mu}=\frac{AB.CD}{AD.CB}=(ABCD).$$

Four collinear points A, B, C, D are said to form a *harmonic range* if $(ABCD)=-1$, *i.e.*, if $\lambda=-\mu$.

The points B, D are called the *harmonic conjugates* to the points A, C. Also A, C are harmonic conjugates to B, D. Four collinear points A, B, C, D are said to form an *equi-anharmonic cross-ratio* if $(ABCD)=-w$, or $-w^2$, w, w^2 being the cube roots of unity.

5.1. There are 24 arrangements of A, B, C, D, being the number of permutations of 4 given letters, *viz.* $\underline{|4}$.

The following equalities among the cross-ratios are easily proved:

$$(ABCD)=(BADC)=(CDAB)=(DCBA)$$
$$(ADCB)=(BCDA)=(CBAD)=(DABC)$$
$$(ACBD)=(BDAC)=(CADB)=(DBCA)$$
$$(ADBC)=(BCAD)=(CBDA)=(DACB)$$
$$(ABDC)=(BACD)=(CDBA)=(DCAB)$$
$$(ACDB)=(BDCA)=(CABD)=(DBAC)$$

6. *If* P_1, P_2, P_3, P_4, *be four points forming a harmonic range, i.e.,* $(P_1\,P_2\,P_3\,P_4)=-1$, *required the relation between* x_1, x_2, x_3, x_4.

Since $(P_1\,P_2\,P_3\,P_4)=-1$

$$\frac{P_1P_2.P_3P_4}{P_1P_4.P_3P_2}=-1$$

$\therefore\quad (x_2-x_1)(x_4-x_3)=-(x_4-x_1)(x_2-x_3)$

or $\quad (x_1+x_3)(x_2+x_4)=2(x_1x_3+x_2x_4) \qquad \ldots\ldots(3)$

If the origin O be taken at P_1, $x_1=0$

$\therefore\quad$ (3) becomes $2x_2x_4=x_3x_2+x_3x_4$

i.e., $$\frac{2}{OP_3}=\frac{1}{OP_2}+\frac{1}{OP_4}. \qquad \ldots\ldots(4)$$

6.1. *If the roots of the equations* $ax^2+2hx+b=0$ *and* $a'x^2+2h'x+b'=0$, *form harmonic pairs of points, show that* $ab'+a'b=2hh'$.

Let x_1, x_3 be the roots of $ax^2+2hx+b=0$ and x_2, x_4 those of $a'x^2+2h'x+b'=0$.

Then $\quad x_1+x_3=-\dfrac{2h}{a},\ x_1x_3=\dfrac{b}{a}$,

$$x_2+x_4=\frac{-2h'}{a'},\ x_2x_4=\frac{b'}{a'},$$

From (3) we get the required result

$$ab'+a'b=2hh' \qquad \ldots\ldots(4)$$

6.2. The *cross-ratio* $(P_1\,P_2\,P_3\,P_4)$ of four collinear points $P_i(x_i)$ is

$$\frac{(x_1-x_2)\,(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}.$$

6.3. *If* $(P_1P_2P_3P_4)=-1$ *and* O *the mid-point of* P_1P_3.

$$OP_2.OP_4=OP_1^2=OP_3^2;$$

and conversely.

Take the origin at O.

Then from (3), since $x_1+x_3=0$, we have $x_1x_3+x_2x_4=0$

$\therefore\quad x_2x_4=x_1^2=x_3^2$

i.e., $\quad OP_2.OP_4=OP_1^2=OP_3^2 \qquad \ldots\ldots(5)$

Conversely $OP_1=-OP_3$

$\therefore\quad OP_1.OP_3+OP_2.OP_4=0.$

Let O′ be the origin and x_1, x_2, x_3, x_4 ξ the co-ordinates of P_1, P_2, P_3, P_4 and O respectively.

$$(\xi-x_1)(\xi-x_3)+(\xi-x_2)(\xi-x_4)=0$$
$$2\xi^2-\xi(x_1+x_3)-\xi(x_2+x_4)+x_1x_3+x_2x_4=0.$$

But $\xi=\dfrac{x_1+x_3}{2}$

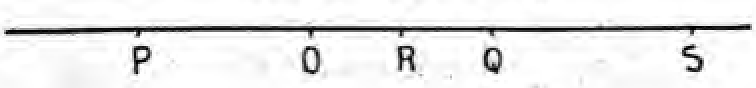
$\therefore\quad (x_1+x_3)(x_2+x_4)=2(x_1x_3+x_2x_4)$

i.e., $\quad \dfrac{(x_1-x_2)(x_3-x_4)}{(x_1-x_4)(x_3-x_2)}=-1$ *i.e.*, $\dfrac{P_1P_2.P_3P_4}{P_1P_4.P_3P_2}=-1.$

$\therefore\quad (P_1P_2P_3P_4)=-1.$

7. Point at Infinity. Let PQ be any segment of a st. line, and R, S be harmonic conjugates *w. r.* to P, Q. If O be the mid-point of PQ,

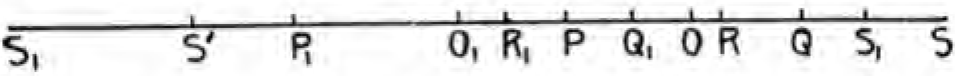
$$OR.OS=OP^2=OQ^2.$$



If R approaches O, S recedes from O. Thus the fourth harmonic conjugate of the mid-point of an arbitrary segment of a st. line *w. r.* to its extremities is said to be a point at infinity on the line.

Now S tends to $+\infty$ or $-\infty$ according as R approaches O from right or from left of O. But for a given R, S is unique. Thus the point at $(+\infty)$ and the point at $(-\infty)$ on a st. line are not distinct.

It follows that the mid-points of all segments of a st. line have the same harmonic conjugate *w. r.* to their respective extremities.



If the statement be false, there will be an infinity of points at infinity on a st. line. But since every segment gives rise to opposite infinities on a st. line, which coincide, all these points at infinity must, therefore, also coincide. (Take arbitrary equal segments, PQ, P_1Q_1'. S, S' coincide and S_1, S_1' coincide and hence all the four coincide).

Hence there is one and only one point at infinity on a st. line.

A st. line may be regarded as a closed curve.

8. Def. Involution. Two co-basal ranges $[P(x)]$, $[P'(x')]$, referred to the same origin, are said to belong to an involution, if there exists a point C, such that

$$CP_1.CP_1' = CP_2.CP_2' \ldots\ldots = k^2 (\text{constant}) \qquad \ldots\ldots(6)$$

The point C is called the *centre of involution.*

The point pairs (P_1, P_1') etc., are called corresponding points or conjugate pairs.

From result (6) it follows that if we take two points F and F_1 on the line L on the two sides of C at a distance k from it, we may write the result (6) as

$$CP.CP' = CF^2 = CF_1^2,$$

where P and P′ are two arbitrary corresponding points. This result in conjunction with 6.3 shows that

$$(FF_1, PP') = -1 \qquad \ldots\ldots(7)$$

The points F and F_1 are called the *double points, foci,* or *self-corresponding points* of the involution.

bud

8.1. *An involution is completely determined by two pairs of corresponding points.*

Let $P_1(x_1)$ and $P_1'(x'_1)$, $P_2(x_2)$ and $P_2'(x_2')$ be two pairs of corresponding points referred to the same origin and $C(\xi)$ the centre of involution, then

$$(x_1-\xi)(x_1'-\xi)=(x_2-\xi)(x_2'-\xi)=k^2 \qquad \ldots\ldots(8)$$

$$\therefore \quad \xi(x_1+x_1'-x_2-x_2')=x_1x_1'-x_2x_2' \qquad \ldots\ldots(9)$$

This determines only one finite value of ξ, and consequently only *one finite position* of the centre, provided $x_1+x_1'-x_2-x_2' \neq 0$.

If $x_1+x_1'-x_2-x_2'=0$ *i.e.*, the segments P_1P_1', P_2P_2' have the same mid-point. C is at infinity and conversely. Now, one of the two double points is also at infinity and therefore, the other double point coincides with the common mid-point of the segments. Thus *the extremities of all segments with a common mid-point belong to an involution, the common mid-point being one of the double points, the other being at infinity.*

Oral

8.2. If $P(x)$, $P'(x')$ belong to an involution, relation (8) shows that $(x-\xi)(x'-\xi)=k^2$ which is of the form

$$pxx'+q(x+x')+r=0 \qquad \ldots\ldots(9A)$$

Since the double points are self-corresponding points they are given by the equation

$$px^2+2qx+r=0 \qquad \ldots\ldots(9B)$$

9. Homogeneous Co-ordinates. The position of a point at infinity which is represented vaguely and indefinitely in the one co-ordinate system can be represented by definite algebraic numbers by a simple device adopted below. The new system then has the advantage that we can deal with the point at infinity with as much ease as with points in the finite part of the line.

For this purpose, we locate the position of a point on a line by two numbers, whose ratio should be known, the precise values of the numbers being of no consequence. We will represent these two numbers by x and t, and define the ratio by the equation

$$X=\frac{x}{t},$$

where X is the one co-ordinate of the point, so that the point $P(X)$ is written as $P(x, t)$. In this system the point $P(x, t)$ is the same as the point (kx, kt), $k \neq 0$.

As P recedes indefinitely from O, $t \to 0$. Hence we make the convention that the point at infinity on the line L is represented by $(x, 0)$, $x \neq 0$.

There is no point on the line for which $x=0$, $t=0$, as the ratio of these values is indeterminate, and thus the point is indeterminate.

9.1. Section Formula. Let (x_1, t_1), (x_2, t_2) be the homogeneous co-ordinates of the points P, Q, whose non-homogeneous co-ordinates are X_1, X_2, so that

$$X_1 = \frac{x_1}{t_1}, \quad X_2 = \frac{x_2}{t_2}.$$

If X be the co-ordinate of the point R whose homogeneous co-ordinates are (x, t) and $p : q$ the ratio in which the segment PQ is divided at R,

$$\frac{x}{t} = X = \frac{qX_1 + pX_2}{p+q} = \frac{q\dfrac{x_1}{t_1} + p\dfrac{x_2}{t_2}}{p+q}$$

$$= \frac{x_1 + \dfrac{p}{q} \cdot \dfrac{t_1}{t_2} x_2}{t_1 + \dfrac{p}{q} \cdot \dfrac{t_1}{t_2} t_2} = \frac{x_1 + \lambda x_2}{t_1 + \lambda t_2} \text{ where } \lambda = \frac{p}{q} \cdot \frac{t_1}{t_2}$$

We can, therefore, set

$$x = x_1 + \lambda x_2, \quad t = t_1 + \lambda t_2 \qquad \text{......(10)}$$

which are the co-ordinates of R.

Note that $\lambda \neq \dfrac{PR}{RQ}$, but $\lambda = \dfrac{PR}{RQ} \cdot \dfrac{t_1}{t_2}$.

Exercises I

1. If A, B, C, D, be four collinear points, show that
$$AB.CD + BC.AD + CA.BD = 0.$$

2. If O, A, B. C be points on a st. line and if P, Q, R be the mid-points of BC, CA, AB respectively, prove that
$$OP.BC + OQ.CA + OR.AB = 0.$$

3. If A, B, C, D be collinear, prove that
$$BC.AD^2 + CA.BD^2 + AB.CD^2 = -BC.CA.AB.$$

4. If [AA′BB′] be a range of points and P and Q respectively the mid-points of AA′, BB′, show that
$$2PQ.AA' = AB.AB' - A'B.A'B'.$$

5. If (AB, CD) $= -1$, prove that $\dfrac{1}{AC} + \dfrac{1}{AD} = \dfrac{2}{AB}$.

6. If (AB, CD) = -1 and P be any point on the line AB,

$$2\frac{PB}{AB}=\frac{PC}{AC}+\frac{PD}{AD}$$

7. If $(P_1P_2, P_3P_4)=k$, show that $\dfrac{1-k}{P_1P_2}=\dfrac{1}{P_1P_4}-\dfrac{k}{P_1P_3}$.

8. If $(P_1P_2, P_3P_4)=\lambda$, show that

$$(P_1P_2, P_4P_3)=\frac{1}{\lambda}, \ (P_1P_3, P_2P_4)=1-\lambda$$

[*Hint.* The first can be easily verified. To prove the second, we use Euler's Theorem (Ex. 1) *viz.*,

$$P_1P_3.P_2P_4+P_3P_2.P_1P_4+P_2P_1.P_3P_4=0$$

or $$\frac{P_1P_3.P_2P_4}{P_1P_4.P_2P_3}-1+\frac{P_1P_2.P_3P_4}{P_1P_4.P_3P_2}=0, \text{ or } \lambda-1+(P_1P_3, P_2P_4)=0].$$

incl 9. Show that the equation

$$ax^2+2hx+b+\lambda(a'x^2+2h'x+b')=0,$$

for different values of λ, defines point pairs of an involution.

incl 10. Find the double points of the involution defined by the point-pairs $3x^2-28x+60=0$, $5x^2-32x+56=0$

incl 11. Show that the point pairs

$$a_ix^2+2h_ix+b_i=0 \ (i=1, 2, 3)$$

belong to an involution range if

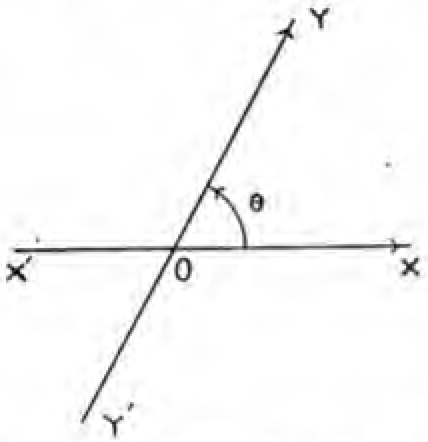
$$\begin{vmatrix} a_1 & h_1 & b_1 \\ a_2 & h_2 & b_2 \\ a_3 & h_3 & b_3 \end{vmatrix}=0$$

Imp 12. If (AB, CD) = -1 what is the position of D when C coincides with (*i*) A, (*ii*) the mid-point of AB, (*iii*) B, (*iv*) the point at infinity ?

CHAPTER II

CO-ORDINATES

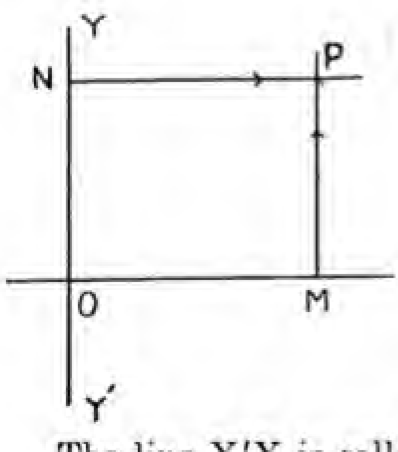
10. Angle between two directed lines

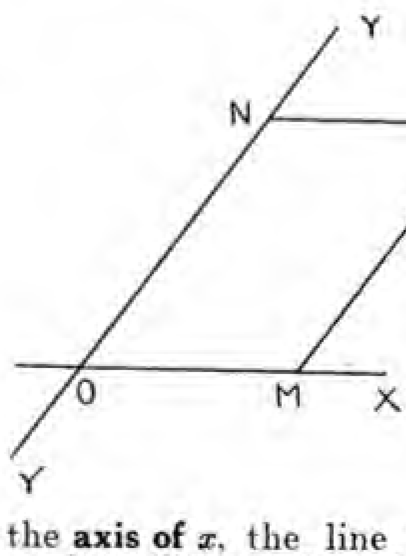


Def. A directed line is a st. line upon which an origin, a unit of length and positive direction have been chosen.

The angle θ that the directed line $Y'OY$ makes with the directed line $X'OX$ is defined to be the angle that the positive direction of $Y'OY$ makes with $X'OX$ measured in the counter-clockwise direction from the positive direction of $X'OX$.

11. Co-ordinate System. Let $X'X$ and $Y'Y$ be two directed lines with the common point O as the zero point.





The line $X'X$ is called the **axis of x**, the line $Y'Y$ the **axis of y** and the point O **the origin.** The two lines $X'X$, $Y'Y$ together we call the **co-ordinate axes** or **axes of reference.** The positive direction on the x-axis is *from* X' *towards* X and on the y-axis, it is *from* Y' *towards* Y.

An arbitrary point P in the plane of the axes fixes two numbers h and k the measures of its distances $OM=NP$

and ON=MP from the y-axis and the x-axis respectively measured along or parallel to the x-axis and y-axis respectively. These numbers are called the co-ordinates of P and written in a definite order as (h, k). Conversely, a given ordered pair of numbers (h, k) defines a unique point. For cut off $OM=h$, $ON=k$ along the x-axis and y-axis respectively and draw parallels to the axes through M and N. These lines intersect in a unique point P.

The point P whose co-ordinates are (h, k) is briefly written as $P(h, k)$.

Thus there is a unique and reversible correspondence between the real points of a plane and ordered pairs of real numbers.

The co-ordinates of an arbitrary point P are denoted by (x, y).

The abscissa (x) of a point is its distance from the y-axis measured parallel to the x-axis (in the same direction).

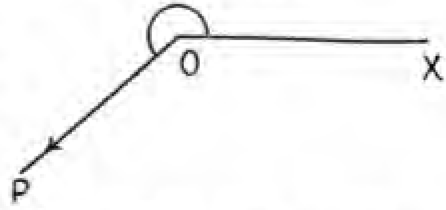
The ordinate y of a point is its distance from the x-axis measured parallel to the direction of the y-axis (in the same direction).

We now introduce the conception of imaginary points by saying that every imaginary point is defined by a pair of numbers (x, y), one or both of which may be complex. With this assumption, we establish a unique and reversible correspondence between all points (real or imaginary) of a plane and ordered pairs of numbers. In this sense, a 'point' is nothing but a concise expression for a pair of numbers (x, y).

If the angle $XOY=\frac{\pi}{2}$ the frame of reference is called **rectangular**, otherwise **oblique**. If the axes are oblique, $\angle XOY$ is usually denoted by ω.

The system of co-ordinates described above is called *cartesian system*, after Descartes with whom originated this branch of mathematics *viz.*, 'Co-ordinate Geometry."

11.1. Polar Co-ordinates. In this system of co-ordinates, the position of a point is fixed by (*i*) its distance OP (called the *radius vector*) from a fixed O called the **pole** or



origin, the positive direction of the radius vector being from O to P: and (*ii*) the angle θ which the positive direction of the radius vector makes with the positive direc-

tion of a line OX (called the **initial line**) through O measured in the counter clockwise direction. The angle θ is called the **vectorial angle**.

Usually an angle is positive if it is measured in the counter clockwise direction, but occasionally the clockwise rotation is taken as positive.

The radius vector is usually denoted by r and the vectorial angle by θ. The position of the point P whose radius vector is r and vectorial angle θ is denoted by the symbol $P(r, \theta)$.

Conversely, the given r and θ determine a unique point P.

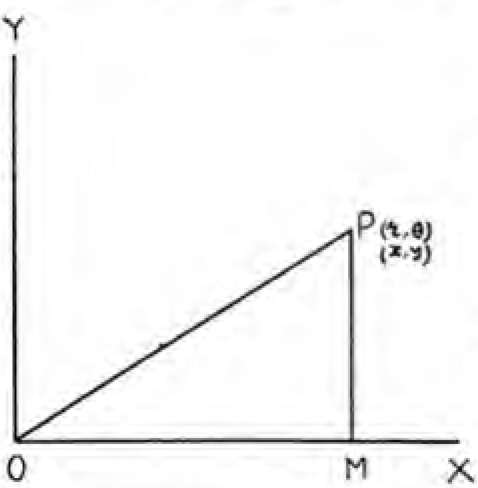
Note 1. It will be seen that the points $(r, \pi+\theta)$ and $(-r, \theta)$ are identical.

2. We assume that imaginary points can be represented in this system of co-ordinates for imaginary values of r or θ or both.

11.2. Relation connecting the polar and cartesian co-ordinates. Let the x-axis OX of the cartesian system coincide with the initial line of the polar system, the origin in both cases being the same. Suppose (r, θ), (x, y) the polar and cartesian co-ordinates of a point P referred to the two systems. Then

$$x = OM,\ y = MP \text{ where } MP \parallel OY,\ \angle XOP = \theta,\ OP = r.$$

(*a*) **Rectangular Axes.**



Obviously from the figure

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\} \qquad \ldots\ldots(1)$$

Conversely, if x and y are given, we have

$$\left.\begin{aligned} r^2 &= x^2 + y^2 \\ \tan\theta &= \frac{y}{x} \end{aligned}\right\} \qquad \ldots\ldots(2)$$

These equations do not determine r and θ uniquely, for $r = \pm\sqrt{x^2+y^2}$ and θ has infinite number of values. If we agree to put

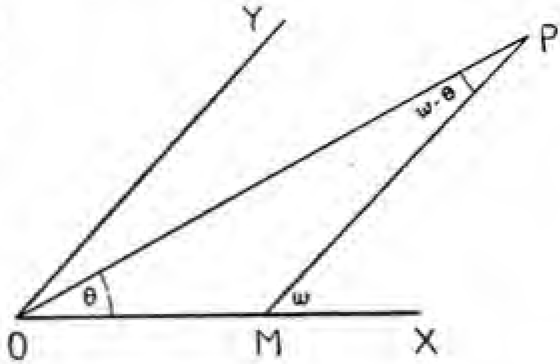
$$r = \sqrt{x^2+y^2},$$

then the two equations $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$ determine

$$\theta = 2n\pi + \alpha$$

where α is any one value of the angle. The value of θ is taken to be that value that lies between 0 and 2π.

(*b*) **Oblique axes.** Let the axes be inclined at an angle w. From the triangle OMP



$$\frac{OM}{\sin(w-\theta)} = \frac{MP}{\sin\theta} = \frac{OP}{\sin w} \quad \ldots\ldots(A)$$

$$\left.\begin{aligned} \therefore \quad x &= \frac{\sin(w-\theta)}{\sin w}\, r \\ y &= \frac{\sin\theta}{\sin w}\, r \end{aligned}\right\} \quad \ldots\ldots\ldots(3)$$

From relations (A)

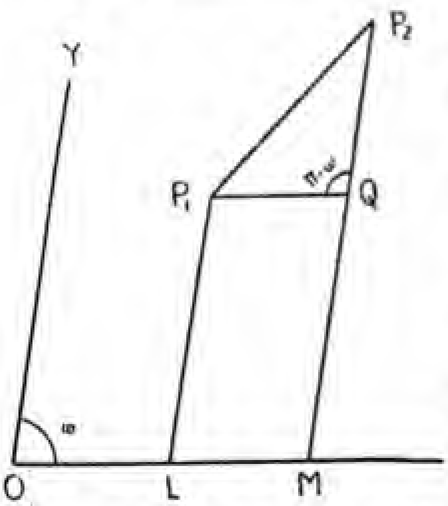
$$x\sin\theta = y\sin(w-\theta)$$
$$= y(\sin w\cos\theta - \cos w\sin\theta)$$

$$\left.\begin{aligned} \therefore \tan\theta &= \frac{y\sin w}{x + y\cos w} \\ \text{Also from the figure } r^2 &= x^2 + y^2 + 2xy\cos w. \end{aligned}\right\} \quad \ldots\ldots(4)$$

Projecting the sides of the triangle OMP on the x-axis and y-axis respectively, we also get the relations

$$x + y\cos w = r\cos\theta$$
$$x\cos w + y = r\cos(w-\theta) \quad \ldots\ldots(5).$$

12. Distance between two points. Cartesian system.



Let $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ be two points referred to OX, OY as axes inclined at an angle w. Draw LP_1, MP_2 parallel to OY and P_1Q parallel to OX, meeting MP_2 in Q. Then

$$P_1Q = LM = x_2 - x_1$$
$$QP_2 = MP_2 - MQ = MP_2 - LP_1$$
$$= y_2 - y_1,$$

and therefore

$$P_1P_2^2$$
$$= P_1Q^2 + QP_2^2 + 2P_1Q.QP_2\cos w$$
$$= (x_1-x_2)^2 + (y_1-y_2)^2 +$$
$$\cancel{-}\,2(x_1-x_2)(y_1-y_2)\cos w. \quad \ldots\ldots(6)$$

If the axes are rectangular $w = \dfrac{\pi}{2}$,

$$P_1P_2^2=(x_1-x_2)^2+(y_1-y_2)^2 \qquad \ldots\ldots (7)$$

12.1. Polar System.

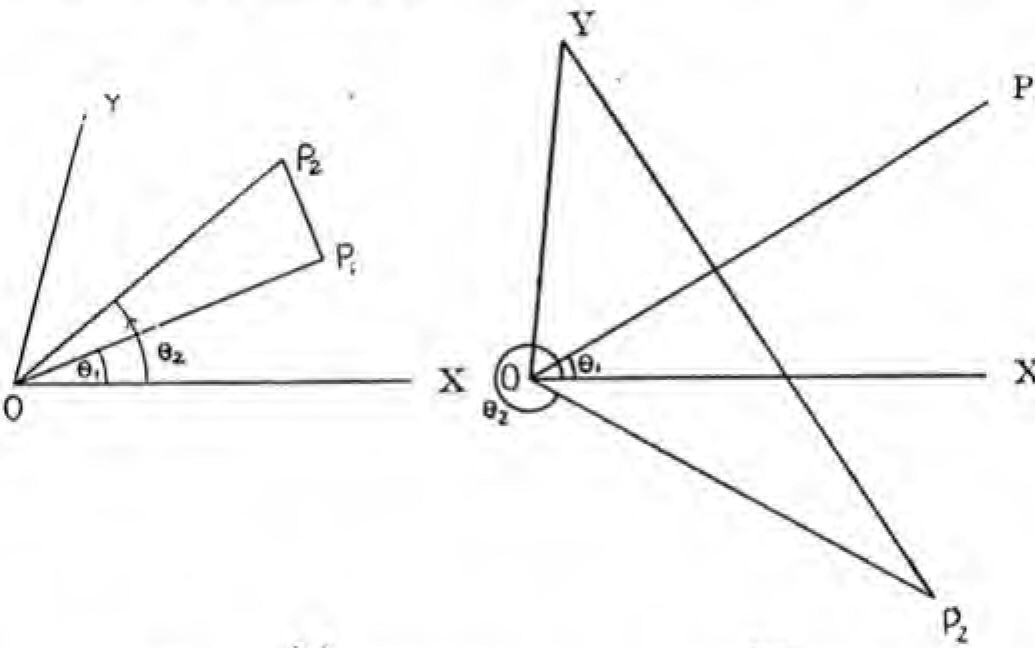
(For figure see next Art.)

Let $P_1(r_1, \theta_1)$, $P_2(r_2, \theta_2)$ be the co-ordinates of the points referred to O as origin and OX as the initial line, then

$$P_1P_2^2=OP_1^2+OP_2^2-2OP_1.OP_2 \cos P_1OP_2$$
$$=r_1^2+r_2^2-2r_1r_2 \cos(\theta_2-\theta_1) \qquad \ldots\ldots (8)$$

13. Area of a triangle.

To find the area of the triangle formed by joining the origin to the points P_1, P_2.



(*a*) (*b*)

Let (r_1, θ_1), (r_2, θ_2) be the polar co-ordinates of the points P_1 and P_2 respectively. Suppose $\theta_2 > \theta_1$.

The area of the triangle

$$OP_1P_2=\tfrac{1}{2}OP_1.OP_2 \sin P_1OP_2$$
$$=\tfrac{1}{2}r_1r_2 \sin(\theta_2-\theta_1) \qquad \text{(Fig. } a\text{)}$$
$$=\tfrac{1}{2}r_1r_2 \sin(2\pi-\overline{\theta_2-\theta_1}) \qquad \text{(Fig. } b\text{)}$$

Hence the area of the triangle is

$$\pm\tfrac{1}{2}r_1r_2 \sin(\theta_2-\theta_1) \qquad \ldots\ldots (9)$$

An area is considered as positive when it lies to the left of the tracing point, whereas it would be regarded as negative if it is to the right.

13.1. Area of a triangle (cartesian system).

(*a*) *Rectangular Co-ordinates.* Let the cartesian co-ordinates of the points P_1 and P_2 be (x_1, y_1), (x_2, y_2).

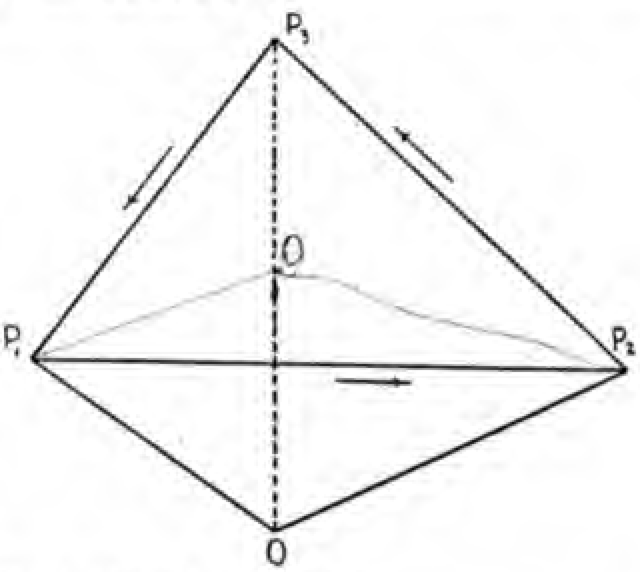
The area of the triangle

$$= \pm \tfrac{1}{2}\, r_1 r_2 \sin(\theta_2 - \theta_1)$$
$$= \pm \tfrac{1}{2}(r_2 \sin\theta_2 . r_1 \cos\theta_1 - r_2 \cos\theta_2 . r_1 . \sin\theta_1)$$
$$= \pm \tfrac{1}{2}(y_2 x_1 - x_2 y_1) \qquad \ldots\ldots(10)$$

(*b*) *Oblique Axes.* If the axes be inclined at an angle w, and the co-ordinates of P_1, P_2 be (x_1, y_1), (x_2, y_2), the area of the triangle

$$= \pm \tfrac{1}{2} r_1 r_2 \sin(\theta_2 - \theta_1)$$
$$= \pm \tfrac{1}{2}(r_2 \sin\theta_2 . r_1 \cos\theta_1 - r_2 \cos\theta_2 . r_1 \sin\theta_1)$$
$$= \pm \tfrac{1}{2}\{y_2 \sin w\,(x_1 + y_1 \cos w) - y_1 \sin w(x_2 + y_2 \cos w)\}$$
$$= \pm \tfrac{1}{2}(y_2 x_1 - y_1 x_2) \sin w. \quad \text{[Art. (11. 2b, 5)]} \qquad \ldots\ldots(11)$$

13.2. *To find the area of the triangle formed by three arbitrary points* $P_1\, P_2\, P_3$.



Join the vertices to the origin.

Whether the origin is within or without the triangle.

$$\Delta P_1P_2P_3 = \Delta OP_1P_2 + \Delta OP_2P_3 + \Delta OP_3P_1$$

(proper regard being had to signs of areas).

(*a*) Let the polar co-ordinates of the vertices

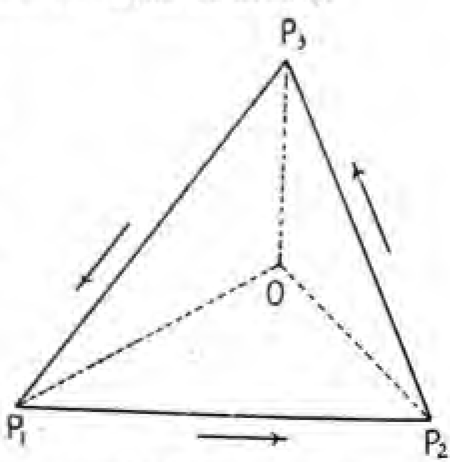
P_i be (r_i, θ_i), $(i = 1, 2, 3)$, then

$$\Delta\, P_1P_2\,P_3 = \pm \tfrac{1}{2}\{r_1 r_2 \sin(\theta_2 - \theta_1) + r_2 r_3 \sin(\theta_3 - \theta_2) + r_3 r_1 \sin(\theta_3 - \theta_1)\} \quad \ldots\ldots(12)$$

(*b*) Let the co-ordinates of vertices be

$$(x_1, y_1), (x_2, y_2), (x_3, y_3).$$

If the axes are rectangular, by Art. (13.1, (*a*)



$$\Delta\, P_1\, P_2\, P_4 = \pm \tfrac{1}{2}\{(x_1 y_2 - x_2 y_1) + (x_2 y_3 - x_3 y_2) + (x_3 y_1 - x_1 y_3)\} \ldots(13)$$

If the axes are oblique, by Art. (13.1, (*b*)).

$$\Delta P_1 P_2 P_3 = \pm\tfrac{1}{2}\{x_1y_2 - x_2y_1 + x_2y_3 - x_2y_3 + x_3y_1 - x_1y_3\}\sin w \quad \ldots(14)$$

These results can be written in the form

$$\pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \quad \text{and} \quad \pm\tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\sin w.$$

13.3. *Area of a polygon.* Let $P_i(x_i, y_i), (r_i, \theta_i)(i=1, 2 \ldots n)$ be the vertices of the polygon. Join O to the vertices.

The area of the polygon

$$= \Delta OP_1P_2 + \Delta OP_2P_3 + \ldots\ldots + \Delta OP_nP_1$$

$$= \tfrac{1}{2}[r_1r_2\sin(\theta_2-\theta_1) + r_2r_3\sin(\theta_3-\theta_2)\ldots\ldots + r_nr_1\sin(\theta_1-\theta_n)]$$

(Polar co-ordinates)

$$= \tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ldots\ldots + x_ny_1 - x_1y_n)] \text{ (rectangular axes)}$$

$$= \tfrac{1}{2}[x_1y_2 - x_2y_1 + x_2y_3 - x_3y_2 + \ldots\ldots + x_ny_1 - x_1y_n]\sin w \text{ (oblique axes)}$$

13.4. Condition of collinearity of three points.

Let the given points be $P_i(x_i, y_i)$ or (r_i, θ_i) $(i=1, 2, 3)$. These points are collinear if the area of the triangle formed by the three points is zero and conversely. Hence the conditions are

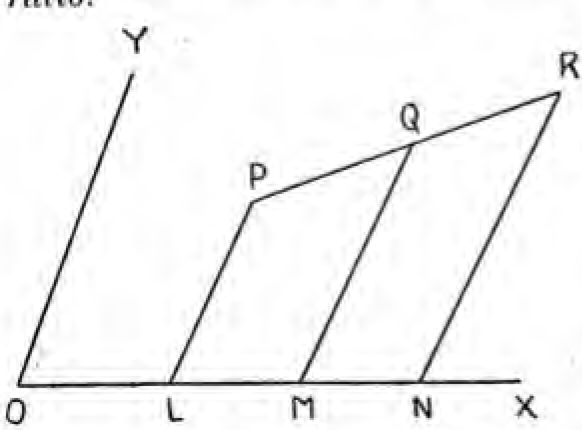
$$r_1r_2\sin(\theta_2-\theta_1) + r_2r_3\sin(\theta_3-\theta_2) + r_3r_1\sin(\theta_1-\theta_3) = 0 \quad \ldots\ldots(15)$$

(Polar co-ordinates)

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0 \quad \ldots\ldots\ldots(16)$$

(Cartesian co-ordinates)

14. Section Formula. *To find the co-ordinates of a point which divides the join of two given points in a given ratio.*



Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be the given points and $R(x, y)$ the point which divides PQ in the ratio $l : m$ so that

$$\frac{PR}{RQ} = \frac{l}{m}.$$

Draw LP, MQ, NR parallel to OY, then

$$\frac{LN}{NM} = \frac{PR}{RQ} = \frac{l}{m}$$

or $\dfrac{x-x_1}{x_2-x}=\dfrac{l}{m}$

$$\left.\begin{aligned} \therefore \qquad x&=\frac{lx_2+mx_1}{l+m} \\ \text{Similarly} \quad y&=\frac{ly_2+my_1}{l+m} \end{aligned}\right\} \qquad \ldots\ldots(17)$$

If the point R divides PQ internally, the segments PR, RQ have the same sign, and therefore, the ratio PR : RQ $=l:m$ is positive. If the division is external, PR, RQ are of opposite signs, therefore PR : RQ $=l:m$ is negative.

Note. The points (x_1, y_1), $\left(\dfrac{lx_2+mx_1}{l+m}, \dfrac{ly_2+my_1}{l+m}\right)$,

$$(x_2, y_2)\left(\frac{lx_2-mx_1}{l-m}, \frac{ly_2-my_1}{l-m}\right)$$

form a harmonic range.

(*a*) If $l+m=0$, x and y become infinite, but in this case the ratio $1:-1$ is unique : it follows that there is one and only one point at infinity on a st. line.

(*b*) If we consider the ratio $l:m$ as arbitrary, we have the co-ordinates of any point R on the line PQ.

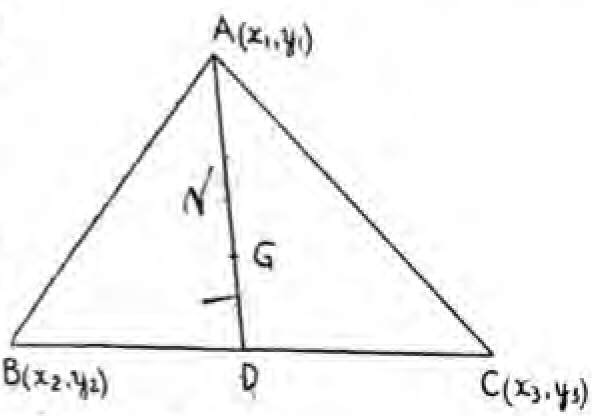
Eliminating l/m from (17), we get a relation of the 1st degree in x, y which denotes the condition of collinearity of three points P, Q, R and which also represents the st. line through P, Q.

Illustrative Examples

1. *Find the centroid of the triangle* $A(x_1, y_1)$, $B(x_2, y_2)$ $C(x_3, y_3)$.

The co-ordinates of D, the mid-point of BC are

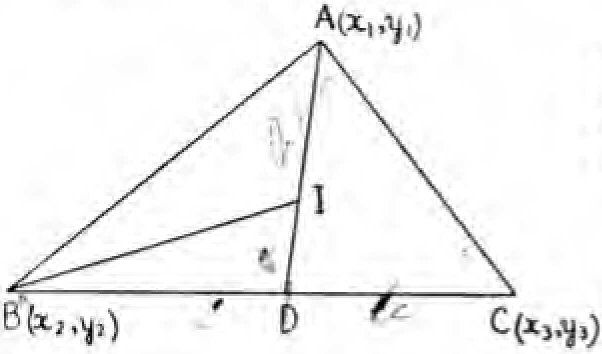
$$\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$$



The centroid G divides the median AD such that AG : GD $=2:1$. Hence the co-ordinates of G are

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_2}{3}\right)$$

2. *Find the in-centre and e-centres of the triangle* $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$.



Let the bisector of the angle A meet the base BC in D, and the bisector of the angle B meet the bisector AD in the in-centre I.

Now $\dfrac{BD}{DC}=\dfrac{c}{b}$,

therefore the co-ordinates of D are

$$\left(\frac{bx_2+cx_3}{b+c},\ \frac{by_2+cy_3}{b+c}\right)$$

As $\dfrac{BD}{c}=\dfrac{DC}{b}=\dfrac{BD+DC}{b+c}=\dfrac{a}{b+c}$

$\therefore \quad BD=\dfrac{ac}{b+c}$.

In the triangle ABD, BI is the bisector of the angle B,

$\therefore \quad \dfrac{DI}{IA}=\dfrac{BD}{c}=\dfrac{a}{b+c}$.

Hence the abscissa of I is

$$\frac{ax_1+(b+c)\dfrac{bx_2+cx_3}{b+c}}{a+b+c}=\frac{ax_1+bx_2+cx_3}{a+b+c}$$

and similarly the ordinate is $\dfrac{ay_1+by_2+cy_3}{a+b+c}$.

Thus the co-ordinates of I are

$$\left(\frac{ax_1+bx_2+cx_3}{a+b+c},\ \frac{ay_1+by_2+cy_3}{a+b+c}\right).$$

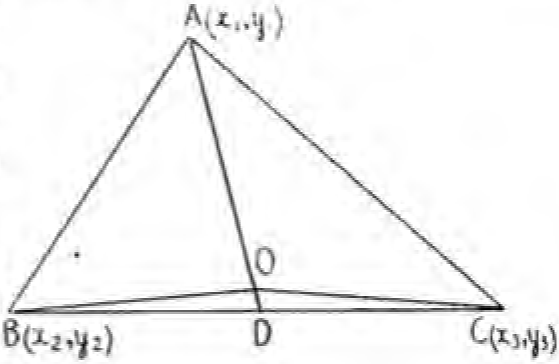
Proceeding as above, it will be found that the co-ordinates of the centres of *e*-circles opposite to A, B, C are

$$\left(\frac{-ax_1+bx_2+cx_3}{-a+b+c},\ \frac{-ay_1+by_2+cy_3}{-a+b+c}\right)$$

$$\left(\frac{ax_1-bx_2+cx_3}{a-b+c},\ \frac{ay_1-by_2+cy_3}{a-b+c}\right)$$

$$\left(\frac{ax_1+bx_2-cx_3}{a+b-c},\ \frac{ay_1+by_2-cy_3}{a+b-c}\right).$$

3. *Find the circum-centre of the triangle ABC.*



Let O be the circum-centre.

Join BO, CO and let AO meet BC in D.

The angles BOC, COA, AOB are respectively 2A, 2B, 2C.

$$\text{Now } \frac{BD}{DC} = \frac{\triangle BDO}{\triangle ODC} = \frac{\triangle BDA}{\triangle ADC} = \frac{\triangle BDA - \triangle BDO}{\triangle ADC - \triangle ODC}$$

$$= \frac{\triangle BOA}{\triangle AOC} = \frac{R^2 \sin 2C}{R^2 \sin 2B} = \frac{\sin 2C}{\sin 2B}$$

where R is the circum-radius.

Hence the co-ordinates of D are

$$\left(\frac{\sin 2B.x_2 + \sin 2C.x_3}{\sin 2B + \sin 2C}, \frac{\sin 2B.y_2 + \sin 2C.y_3}{\sin 2B + \sin 2C}\right)$$

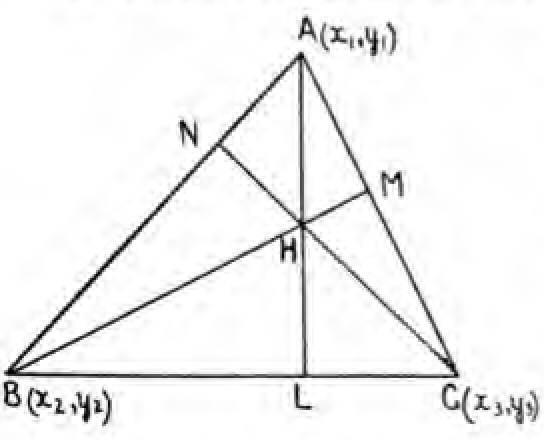
$$\text{Again } \frac{DO}{OA} = \frac{\triangle OBD}{\triangle OAB} = \frac{\triangle ODC}{\triangle OCA} = \frac{\triangle OBC}{\triangle OAB + \triangle OCA}$$

$$= \frac{\frac{1}{2} R^2 \sin 2A}{\frac{1}{2} R^2 \sin 2C + \frac{1}{2} R^2 \sin 2B} = \frac{\sin 2A}{\sin 2B + \sin 2C}$$

Hence the co-ordinates of O are

$$\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}.$$

4. *Find the orthocentre of the triangle* ABC.



Let the altitudes AL, BM, CN meet in H.

$$\text{Now } \frac{\tan C}{\tan B} = \frac{BL}{LC}$$

∴ The co-ordinates of L are

$$\frac{x_2 \tan B + x_3 \tan C}{\tan B + \tan C},$$

$$\frac{y_2 \tan B + y_3 \tan C}{\tan B + \tan C}.$$

$$\text{Also } \frac{\tan C}{\tan B + \tan C} = \frac{BL}{BL + LC} = \frac{BL}{a}$$

i.e., $BL=\dfrac{a\tan C}{\tan B+\tan C}$

Again $\dfrac{LH}{HA}=\dfrac{\Delta\,LBH}{\Delta\,HBA}=\dfrac{\frac{1}{2}\,BL.BH\sin CBM}{\frac{1}{2}\,HB.BA\sin MBA}$

$$=\frac{BL.\cos C}{c.\cos A}=\frac{a\tan C}{\tan B+\tan C}\cdot\frac{\cos C}{c\cos A}$$

$\therefore\quad \dfrac{LH}{HA}=\dfrac{\tan A}{\tan B+\tan C}$.

Consequently the co-ordinates of H are

$$\left(\frac{x_1\tan A+x_2\tan B+x_3\tan C}{\tan A+\tan B+\tan C},\ \frac{y_1\tan A+y_2\tan B+y_3\tan C}{\tan A+\tan B+\tan C}\right).$$

5. The vertices of a triangle are $(a\cos\alpha,\ a\sin\alpha)$, $(a\cos\beta,\ a\sin\beta)$, $(a\cos\gamma,\ a\sin\gamma)$; show that the co-ordinates of the orthocentre are

$$\{a(\cos\alpha+\cos\beta+\cos\gamma),\ a(\sin\alpha+\sin\beta+\sin\gamma)\}.$$

It is obvious that the three vertices are equidistant from the origin. The origin, therefore, is the circum-centre of the triangle. The co-ordinates of the centroid C are

$$\left(\frac{a\,\Sigma\cos\alpha}{3},\ \frac{a\,\Sigma\sin\alpha}{3}\right).$$

If H be the orthocentre, it is known that

$\dfrac{OG}{GH}=\frac{1}{2}$, hence $\dfrac{OH}{HG}=-\dfrac{3}{2}$. Hence the co-ordinates of H are

$$\frac{3\,\dfrac{a\,\Sigma\cos\alpha}{3}-2\times 0}{3-2},\quad \frac{3\,\dfrac{a\,\Sigma\sin\alpha}{3}-2\times 0}{3-2}$$

or $(a\,\Sigma\cos\alpha,\ a\,\Sigma\sin\alpha)$.

Examples II

(Axes are rectangular unless otherwise stated).

1. Find the distance between the following pairs of points:

(*i*) (2, 3), (6, 6); (*ii*) (11, 16), (23, 21),

(*iii*) (66, 25), (99, 69).

2. Find the distance between the following pairs of points, the angle between the axes being 60°:

(*i*) (7, 6), (4, 5); (*ii*) (−13, −3), (−4, −15).

3. Show that the following triads of points form right angled triangles:

(*i*) $(1,-\frac{3}{2})$, $(-3,-\frac{7}{2})$, $(-4,-\frac{5}{2})$, (*ii*) (2, 2), (6, 3), (4, 11).

4. Show that the following triads of points form equilateral triangles :

(*i*) $(6, 4), (3, 4+3\sqrt{3}), (9, 4+3\sqrt{3})$,

(*ii*) $(a, 0), (0, 2a), (2a, a)$, axes being inclined at an angle of $60°$.

(*iii*) $(0, 0), \left(4, \frac{\pi}{3}\right), \left(4, \frac{2\pi}{3}\right)$.

5. Show that the origin $(0, 0)$, and the points $(a \cos \alpha, a \sin \alpha)$, $(b \cos \beta, b \sin \beta)$ will form an isosceles triangle of perimeter $a+2b$ if $\cos(\alpha-\beta)=a/2b$.

6. Prove that the points $(2, 4), (3, 2), (8, 4)$ and $(7, 6)$ are the vertices of a parallelogram.

7. Prove that the points $(0, 1), (1, 4), (4, 3)$, and $(3, 0)$ are the vertices of a square.

8. Find the areas of the triangles formed by the triads of points $(4, 3), (1, -3), (-3, 1)$ and $(4, 3), (-3, 1), (1, -3)$ and explain the difference of signs in the two cases.

9. Find the areas of the quadrilaterals formed by the points

(*i*) $(1, 1), (3, 5), (-2, 4)$ and $(-1, -5)$

(*ii*) $(2, 1)$ $(3, 5), (-3, 4)$ $(-2, -2)$.

10. An isosceles triangle has the extremities of its base at $(2, 5)$ and $(-2, 2)$. Find the two possible positions of the vertex if its area is 25 sq. units.

11. Given the following pairs of opposite vertices of a square, find the other pairs :

(*i*) $(2, 6), (5, 2)$, (*ii*) $(-3, 5), (9, 10)$.

12. Find the centre and radius of the circle that passes through the points $(2, 2), (6, 0)$ and $(3, -1)$.

13. In what ratios do the diagonals of the following quadrilaterals divide each other :

(*i*) $(2, 3), (4, 7)$; $(6, 0), (0, 8)$

(*ii*) $(1, 3), (5, 9)$; $(7, 0), (0, 6)$.

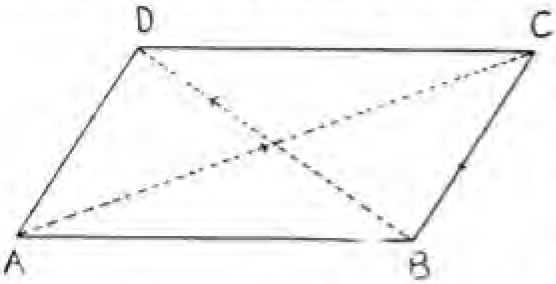
14. Find the in-centres of the triangles whose vertices are as follows :

(*i*) $(6, -3), (6, 18), (-\frac{3}{4}, 2)$

(*ii*) $(5, 3), (5, -1), (-7, -6)$.

15. Given two pairs of points P, Q and R, S on a line, show that there are two points (real, coincident or imaginary) whose position ratios *w. r.* to P, Q and R, S have a given ratio. Explain the special case when the ratio is equal to 1.

16. The points A, B, C, D are $(-2,-1)$, $(1, 0)$, $(4, 3)$ $(1, 2)$. By taking the points in the order A, C, B, D and A, B, D, C, prove that AB is parallel to CD and BC to DA.



Area of the figure ACBD $(-2,-1)$, $(4, 3)$, $(1,0)$, $(1,2)$ is

$=\frac{1}{2}\{-6+4-3+2-1+4\}$
$=0.$

$\therefore$ the triangles ABC and ABD are of equal area and consequently AB is parallel to DC.

Again, area of the quadrilateral ABDC

$$(-2,-1), (1,0), (1, 2), (4, 3).$$
$$=\tfrac{1}{2}[1+2+3-8-4+6]=0.$$

$\therefore$ The triangles DAB and DAC are equal in area, consequently AD is parallel to BC.

Thus ABCD is a parallelogram.

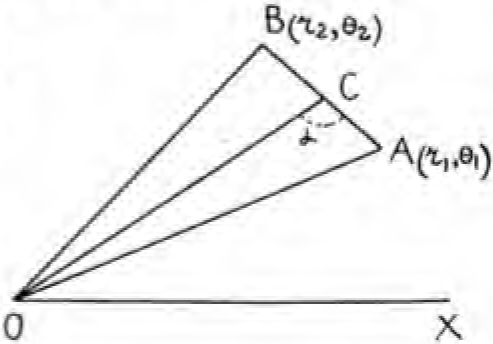
17. Show that $(2, 4)$, $(3, 0)$, $(5, 3)$ and $4, 7)$ are the vertices of a parallelogram.

18. Show that the area of the triangle whose vertices are $(a \tan \theta, b \cot \theta)$, $(a \tan \phi, b \cot \phi)$, $(a \tan \psi, b \cot \psi)$ is

$$\frac{4\,ab \sin(\theta-\phi)\sin(\phi-\psi)\sin(\psi-\theta)}{\sin 2\theta \sin 2\phi \sin 2\psi}.$$

19. *Find the section formula for polar co-ordinates.*

Let $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ be the given points and (r, θ) the co-ordinates of C which divides AB in the ratio $l : m$, *i.e.*,



$$\frac{AC}{CB}=\frac{l}{m}$$

$$\therefore \quad AC=\frac{l}{l+m}\ AB$$

$$\text{Now } \frac{l}{m}=\frac{AC}{CB}=\frac{\Delta\ OAC}{\Delta\ OCB}$$

$$=\frac{r_1 \sin(\theta-\theta_1)}{r_2 \sin(\theta_2-\theta)}$$

$$\therefore \quad lr_2(\sin\theta_2\cos\theta-\cos\theta_2\sin\theta)= mr_1(\sin\theta\cos\theta_1-\cos\theta\sin\theta_1)$$

$$\text{or} \quad \cos\theta\,(lr_2\sin\theta_2+mr_1\sin\theta_1)= \sin\theta\,(lr_2\cos\theta_2+mr_1\cos\theta_1)$$

$$\therefore \quad \theta = \tan^{-1}\frac{lr_2 \sin\theta_2 + mr_1 \sin\theta_1}{lr_2\cos\theta_2 + mr_1\cos\theta_1}$$

Again if $\angle ACO = \alpha$,

$$r_1^2 = r^2 + AC^2 - 2AC.\, r\cos\alpha$$

$$r_2^2 = r^2 + CB^2 + 2.BC.r\cos\alpha$$

$$\therefore \quad mr_1^2 + lr_2^2 = (l+m)r^2 + m.AC^2 + l.BC^2$$

$$= (l+m)r^2 + \left[\frac{m.l^2}{(l+m)^2} + \frac{lm^2}{(l+m)^2}\right]AB^2$$

$$= (l+m)r^2 + \frac{lmAB^2}{l+m}$$

or $\quad m(l+m)r_1^2 + l(l+m)r_2^2 = (l+m)^2r^2$

$$+ lm[r_1^2 + r_2^2 - 2r_1r_2\cos(\theta_2 - \theta_1)]$$

$$\therefore \quad m^2r_1^2 + l^2r_2^2 + 2lmr_1r_2\cos(\theta_2-\theta_1) = (l+m)r^2$$

or $\quad r = \dfrac{\sqrt{[m^2r_1^2 + l^2r_2^2 + 2\,lm\;r_1r_2\cos(\theta_2 - \theta_1)]}}{l+m}$

20. The co-ordinates of the points A, B are (r_1, θ_1), (r_2, θ_2) referred to O as pole. The internal bisector of the angle AOB meets the line AB in D. Find the co-ordinates of D.

21. Show that the diameter of the circum-circle formed by the points $A(r, \theta)$, $B(\rho, \phi)$ and the pole is

$$\frac{\sqrt{r^2 + \rho^2 - 2r\rho\cos(\theta - \phi)}}{\sin(\phi - \theta)}.$$

22. Find the conditions that the four points (x_1, y_1), (x_2, y_2), (x_3, y_3) (x_4, y_4) may be the vertices of (i) a square (ii) a rectangle (iii) a rhombus (iv) a parallelogram.

CHAPTER III

STRAIGHT LINE

15. The **Locus** of a point is the path traced out by the point under certain geometrical conditions.

The equation which expresses the invariable relation that exists between the co-ordinates of every point of a locus (curve) traced out in a plane is called the equation to the locus (curve).

We assume that an arbitrary relation between x and y always represents a **curve**.

16. The equation of a st. line is of the first degree and conversely.

It is shown that if (x_1, y_1), (x_2, y_2) be two points on a st. line, the co-ordinates of an arbitrary point on it can be expressed in the form

$$\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}.$$

As λ varies, the point traces out the st. line, which is the join of (x_1, y_1), (x_2, y_2). Calling the above point (x, y), we have

$$\begin{aligned}(1+\lambda)x - x_1 - \lambda x_2 &= 0\\(1+\lambda)y - y_1 - \lambda y_2 &= 0\\(1+\lambda) \;\; - 1 \;\; - \lambda \;\; &= 0.\end{aligned}$$

Eliminating $1+\lambda$, λ, the required equation is found to be

$$\begin{vmatrix} x & x_1 & x_2 \\ y & y_1 & y_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

which can be written in the form

$$\frac{x-x_1}{x_1-x_2} = \frac{y-y_1}{y_1-y_2} \qquad \text{..... (1)}$$

which is obviously an equation of the first degree.

Conversely, *every equation of the first degree represents a st. line.*

Let the given equation be

$$ax+by+c=0$$

and $P_i(x_i, y_i)$ $(i=1, 2)$ two arbitrary points on the locus, represented by the equation.

$$\therefore \quad ax_1+by_1+c=0$$
$$ax_2+by_2+c=0.$$

Adding λ times the second relation to the first and dividing by $1+\lambda$, we have

$$a\,\frac{x_1+\lambda x_2}{1+\lambda}+b\,\frac{y_1+\lambda y_2}{1+\lambda}+c=0$$

which shows that the point $\left(\frac{x_1+\lambda x_2}{1+\lambda}, \frac{y_1+\lambda y_2}{1+\lambda}\right)$ is also on the locus; $(x_1, y_1), (x_2, y_2)$ being arbitrary. Thus the line joining two arbitrary points on the locus lies wholly on the locus. Hence the locus is a st. line.

The proposition will be assumed to be true as far as it refers to imaginary points and to equations with complex coefficients. In this sense, a st. line means simply the totality of pairs of values of x and y which satisfy the relation

$$ax+by+c=0.$$

16.1. The result obtained in Art. 16 can be deduced from Euclid's definition of a st. line who defines a ***segment of a st. line to be the shortest distance between its extremities.***

Let $A(x_1, y_1)$, $B(x_2, y_2)$ be the extremities of a segment AB and P (x, y) any arbitrary point of the segment AB.

Let $F(x, y)=U^{\frac{1}{2}}+V^{\frac{1}{2}}$
where $U=(x-x_1)^2+(y-y_1)^2$; $\quad V=(x-x_2)^2+(y-y_2)^2$
and $\sqrt{U}$, $\sqrt{V}$ are taken with proper signs.

$$F_x=U^{-\frac{1}{2}}(x-x_1)+V^{-\frac{1}{2}}(x-x_2)$$
$$F_y=U^{-\frac{1}{2}}(y-y_1)+V^{-\frac{1}{2}}(y-y_2)$$

For the stationary values of F, F_x and F_y should vanish, consequently the co-ordinates of the points satisfy the relation

$$\frac{x-x_1}{y-y_1}=\frac{x-x_2}{y-y_2}$$

which is of the 1st degree.

$$\text{Again}\quad F_{xx}=U^{-\frac{3}{2}}(y-y_1)^2+V^{-\frac{3}{2}}(y-y_2)^2$$
$$F_{yy}=U^{-\frac{3}{2}}(x-x_1)^2+V^{-\frac{3}{2}}(x-x_2)^2$$
$$F_{xy}=U^{-\frac{3}{2}}(x-x_1)(y-y_1)+V^{-\frac{3}{2}}(x-x_2)(y-y_2)$$
$$F_{xx}F_{yy}-(F_{xy})^2=U^{-\frac{3}{2}}V^{-\frac{3}{2}}[(x-x_1)(y-y_2)-(x-x_2)(y-y_1)]^2$$
$$=0$$

and $F_{xx}>0$. Thus the conditions for a minimum are satisfied.

Hence the equation of a st. line is of the 1st degree in x and y.

To prove the converse, we note that if A, B, P be three points on the locus $ax+by+c=0$

Then
$$ax_1+by_1+c=0$$
$$ax_2+by_2+c=0$$
$$ax+by+c=0.$$

$$\therefore \quad a(x_1-x_2)+b(y_1-y_2)=0, \quad a(x-x_1)+b(y-y_1)=0,$$
$$a(x-x_2)+b(y-y_2)=0.$$

Now $\quad \text{AB}=\left\{(x_2-x_1)^2+(y_2-y_1)^2\right\}^{\frac{1}{2}}=\dfrac{\sqrt{a^2+b^2}}{b}\,(x_2-x_1).$

Similarly $\text{BP}=(a^2+b^2)^{\frac{1}{2}}\,(x-x_2)\div b$

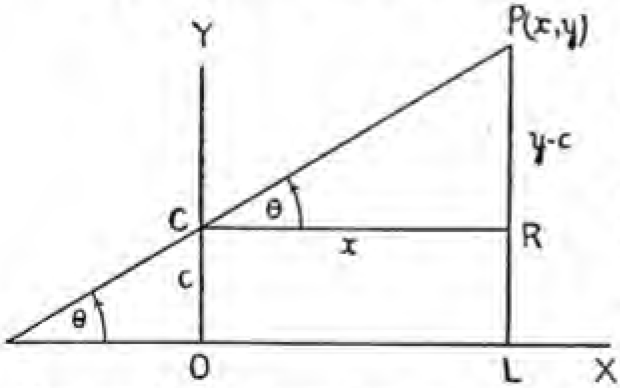
$\text{AP}=(a^2+b^2)(x-x_1)\div b$

$\therefore \quad \text{AB}=\text{PB}-\text{PA}.$

Hence A, B, P lie on a st. line and thus the locus of the equation $ax+by+c=0$ is a st. line.

Note. It should be observed that the first degree equation $ax+by+c=0$ in x and y contains, in fact, two arbitrary constants, *i.e.*, the ratio of any two to the third. These constants can be determined from two independent linear equations which will arise from two independent conditions to which the line is subjected. In the next articles will be found the equation of a line under different conditions.

17. Special forms of the equation of a st. line.



Definition. The tangent of the angle which a st. line makes with the positive direction of the x-axis is called the gradient or slope of the line.

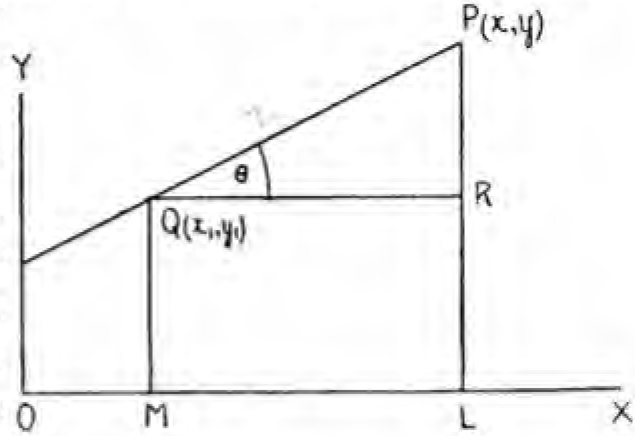
Gradient Forms. *Rectangular axes.* Let θ be the angle which the line makes with the positive direction of x-axis and c the intercept on y-axis, *i.e.*, $\text{OC}=c$. Let P(x, y) be any point on the line. Draw the ordinate LP and CR $\parallel$ OX to meet LP in R. Then $\text{CR}=x$, $\text{RP}=y-c$, and $\text{RP}=\text{CR}\tan\theta$. Therefore the equation of the line is

$$y-c=x\tan\theta$$

i.e., $$y=mx+c, \qquad \text{......(2)}$$

where $m=\tan\theta$

17.1. The st. line will be fixed if in addition to θ, the co-ordinates of any point $Q(x_1, y_1)$ on the line be given. Let $P(x, y)$ be any point on the line. Draw the ordinates MQ, LP and QR ∥ OX meeting LP in R. From the triangle QRP, if $QP = r$,



$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r, \qquad \ldots\ldots(3)$$

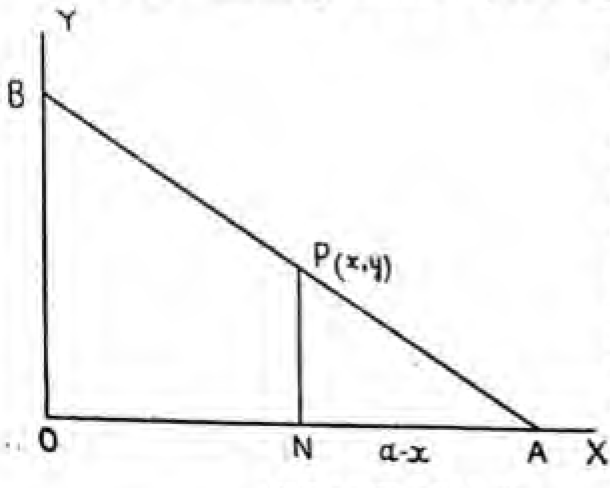
where r does not form part of the equation.

Or
$$\left.\begin{array}{l} x = x_1 + r\cos\theta \\ y = y_1 + r\sin\theta \end{array}\right\} \qquad \ldots\ldots(4)$$

Equation (3) is the *constraint equation* of the line whose *parametric or freedom equations* are (4). The equations (4) express the co-ordinates of any point on the line in terms of a variable r called the parameter.

Cor. From equation (3) $\tan\theta = \dfrac{y - y_1}{x - x_1}$, *i.e., the slope of a line joining two points is equal to the ratio of the difference of the ordinates to the difference of the abscissæ taken in the same order.*

17. 2. Intercept Form. (Rectangular or oblique axes)



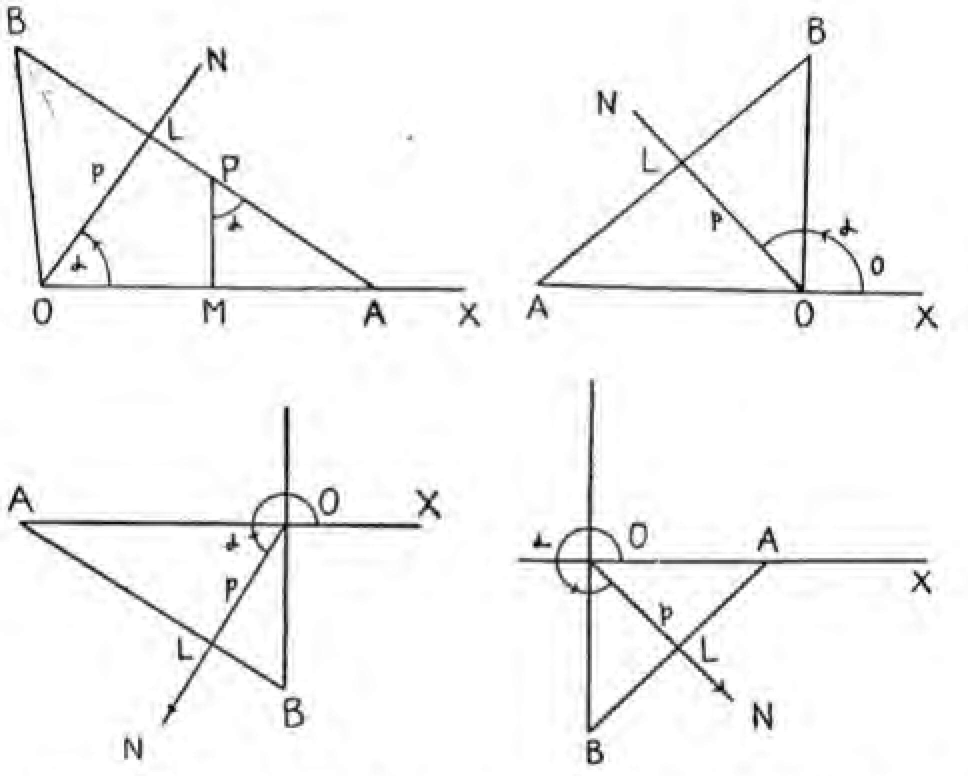
Let the given line meet the axes in A and B. The two points A and B are sufficient to determine the line. Let $OA = a$, $OB = b$ and $P(x, y)$ be any point on the line. Draw the ordinate NP. From similar triangles NAP and OAB,

$$NP : OB = NA : OA$$

i.e., $$\frac{y}{b}=\frac{a-x}{a}$$

or $$\frac{x}{a}+\frac{y}{b}=1 \qquad \ldots\ldots(5)$$

17.3. Normal Form. *Given the measure of the perpendicular from the origin on a st. line and the angle which this perpendicular makes with the x-axis, to find the equation of the st. line.*



Let AB be any line and ON the perpendicular from O to AB. The positive direction on ON is taken from O towards N, *i.e.*, from the origin towards the line, and the angle which the positive direction of ON makes with the positive direction of the x-axis is measured in the counter-clockwise direction. If p be the measure of the directed length OL and α the angle which OL makes with OX, the position of the line is determined by the values of p and α, both p and α being positive and $\alpha<2\pi$; and conversely, every line determines p and α.

If the line AB passes through the origin, the upward direction of the perpendicular is taken to be positive and α is measured from 0 to π.

Let $P(x, y)$ be an arbitrary point on the line. Draw the ordinate MP.

$$\begin{aligned} p &= OA \cos\alpha = (OM + MA)\cos\alpha \\ &= OM\cos\alpha + MA\cos\alpha \\ &= OM\cos\alpha + y\tan\alpha.\cos\alpha \\ &= x\cos\alpha + y\sin\alpha \end{aligned}$$

i.e. $p - x\cos\alpha - y\sin\alpha = 0$ ......(6)

This we shall regard as the *standard normal form*.

17.4. *Polar equation of a st. line* (See fig Art. 17.3).

Let the co-ordinates of P be (r, θ), with the conditions of Art. 17.3, $\angle PON = \alpha - \theta$, we have

$$p = r\cos(\theta - \alpha) \qquad \text{......(7)}$$

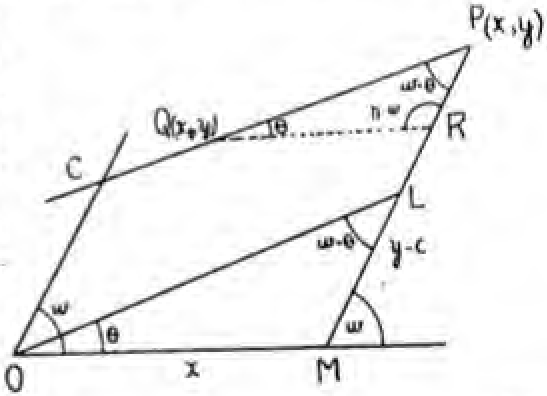
as the required equation.

Ex. Deduce (6) from (7).

18. Special forms of the equations of a st. line (Oblique axes)

Gradiant Forms

Let θ be the angle that the line CP makes with x-axes, and c the intercept OC on y-axis. Suppose $P(x, y)$ any point on the line. Draw the ordinate MP and OL ∥ AB meeting MP in L, then $OM = x$, $ML = y - c$, $\angle MLO = w - \theta$, where w is the angle between the axes. From the $\triangle OML$,



$$\frac{x}{\sin(w - \theta)} = \frac{y - c}{\sin\theta}$$

$\therefore \quad y = \mu x + c$ ......(8)

$$\text{where } \mu = \frac{\sin\theta}{\sin(w - \theta)},$$

$$\text{whence } \tan\theta = \frac{\mu \sin w}{1 + \mu\cos w}. \qquad \text{......(9)}$$

Imp **18.1.** The straight line is fixed if, in addition to θ, the co-ordinates of a point $Q(x_1, y_1)$ on the line be given.

Through Q draw QR ∥ OX to meet the ordinate MP of the point $P(x, y)$ in R. Then $QR = x - x_1$, $RP = y - y_1$, and if $QP = r$,

$$\frac{x - x_1}{\sin(w - \theta)} = \frac{y - y_1}{\sin\theta}\left(= \frac{r}{\sin w}\right) \qquad \text{......(10)}$$

which is the *constraint equation* of the given line, the *freedom equations* being

$$\left.\begin{aligned} x&=x_1+\frac{r}{\sin w}\sin(w-\theta)\\ y&=y_1+\frac{r}{\sin w}\sin\theta \end{aligned}\right\} \qquad \ldots\ldots(11)$$

r being the variable parameter.

Note. If $w-\theta=\phi$, $\quad \theta+\phi=w$

$\therefore\quad \cos(\theta+\phi)=\cos w$

$\cos\theta\cos\phi-\sin\theta\sin\phi=\cos w$

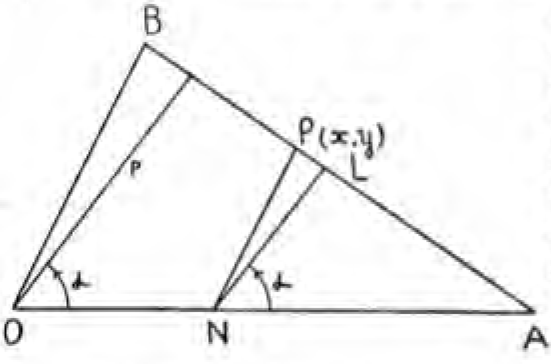
$\therefore\quad \cos^2\theta\cos^2\phi=(\sin\theta\sin\phi+\cos w)^2$

$=\sin^2\theta\sin^2\phi+\cos^2 w+2\sin\theta\sin\phi\cos w$

or $(1-\sin^2\theta)(1-\sin^2\phi)=\sin^2\theta+\sin^2\phi+\cos^2 w+2\sin\theta\sin\phi\cos w$

$\therefore\quad \sin^2\theta+\sin^2\phi+2\sin\theta\sin\phi\cos w=\sin^2 w.$

18.2. Normal Form. Let p be the measure of the perpendicular on the line AB and α the angle which this perpendicular makes with OX. Let $\alpha+\beta=w$. Draw the ordinate NP of a point $P(x, y)$ on the line and NL ∥ to p to meet AB in L.



$p=\text{OA}\cos\alpha=(\text{ON}+\text{NA})\cos\alpha$

$=\text{ON}\cos\alpha+\text{NL}$

$=\text{ON}\cos\alpha+\text{NP}\cos\beta$

$=x\cos\alpha+y\cos\beta.$

The required equation is, therefore,

$$p-x\cos\alpha-y\cos\beta=0 \qquad \ldots\ldots(12)$$

19. *To reduce the equation $ax+by+c=0$ to the normal form.*

Case I. (*Rectangular axes*).

If the normal form for $ax+by+c=0$ be

$$p-x\cos\alpha-y\sin\alpha=0,$$

on identifying the equation, we get

$$\frac{p}{c}=\frac{-\cos\alpha}{a}=\frac{-\sin\alpha}{b}$$

$$=\pm\frac{1}{\sqrt{a^2+b^2}}.$$

Hence the required equation is

$$\frac{c}{\pm\sqrt{a^2+b^2}}+\frac{ax}{\pm\sqrt{a^2+b^2}}+\frac{by}{\pm\sqrt{a^2+b^2}}=0$$

This leaves an ambiguity *w. r.* to the sign of the radical. If the standard normal form be assumed to be

$$p - x\cos\alpha - y\sin\alpha = 0,$$

the sign of the radical must be chosen so as to make the absolute term positive.

In theoretical work, this may be ensured by considering the equation $c(ax+by+c)=0$. The *absolute term being now positive*, the required normal form will be

$$\frac{c^2}{\sqrt{c^2(a^2+b^2)}} - \frac{-acx}{\sqrt{c^2(a^2+b^2)}} - \frac{-bcy}{\sqrt{c^2(a^2+b^2)}} = 0 \qquad \ldots\ldots(13)$$

If $c=0$, $p=0$, α must be measured from 0 to π, consequently $\sin\alpha$ is positive, so the sign of the radical must be chosen to make the co-efficient of y *positive*.

Note. $\cos\alpha = \dfrac{-ac}{\sqrt{c^2(a^2+b^2)}}$, $\sin\alpha = \dfrac{-bc}{\sqrt{c^2(a^2+b^2)}}$

Case II (*Oblique axes*).

If w be the angle between the axes, then on identifying the equation $c(ax+by+c)=0$ with $p - x\cos\alpha - y\cos\beta = 0$, we get

$$\frac{p}{c^2} = \frac{-\cos\alpha}{ac} = \frac{-\cos\beta}{bc} = \lambda \text{ (say.)}$$

Since $\alpha+\beta=w$,

$\therefore \cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos w = \sin^2 w.$

Hence $\qquad \lambda^2c^2(a^2+b^2-2ab\cos w) = \sin^2 w.$

or $\quad \lambda = \dfrac{\sin w}{\sqrt{c^2(a^2+b^2-2ab\cos w)}}.$

The required normal form is

$$c^2\lambda - (-ac\lambda)x - (-bc\lambda)y = 0.$$

Ex. Reduce the following equations to the normal forms :—

(*i*) $3x+4y+10=0$ (*ii*) $5x+12y-26=0$

(*iii*) $5x+12y=0$ (*iv*) $3x-4y=0$

20. To find the angle between two given straight lines.

Let the equations of the two lines be

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0.$$

(*a*) Let the axes be *Rectangular axes*.

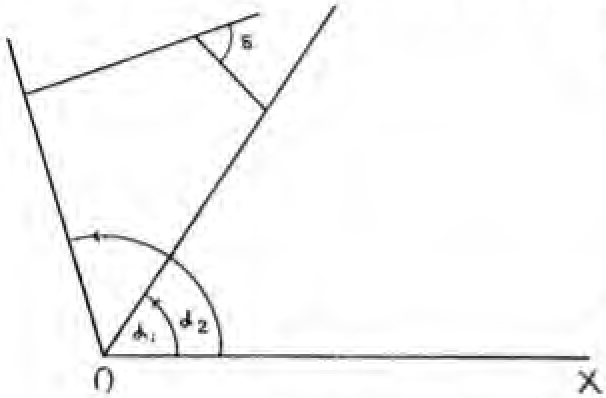
Suppose the normal forms of the given lines are

$$p_i - x\cos\alpha_i - y\sin\alpha_i = 0,\ i=1, 2.$$

$\therefore \quad \cos \alpha_i = -a_i c_i \div d_i, \ \sin \alpha_i = -b\ c_i \div d_i$

where $\quad d_i = \sqrt{c_i^2(a_i^2 + b_i^2)}$

If the origin does not lie within the angle δ between the lines



$$\cos \delta = \cos(\alpha_2 - \alpha_1)$$
$$= \frac{c_1 c_2 (a_1 a_2 + b_1 b_2)}{d_1 d_2} \quad \text{......(14)}$$

Similarly

$$\sin \delta = \frac{c_1 c_2 (a_1 b_2 - a_2 b_1)}{d_1 d_2} \quad \text{......(15)}$$

Consequently,

$$\tan \delta = (a_1 b_2 - a_2 b_1)/(a_1 a_2 + b_1 b_2) \quad \text{......(16)}$$

If the lines be given by the equations

$$y = m_1 x + c_1, \ y = m_2 x + c_2,$$

then $\quad \tan \delta = \dfrac{m_2 - m_1}{1 + m_1 m_2}$ ......(17)

(*b*) Let the axes be oblique and inclined at an angle w.

Set $D_i \sin w = \sqrt{c_i^2(a_i^2 + b_i^2 - 2a_i b_i \cos w}$ $\qquad i = 1,2.$

If $p_1 - x\cos\alpha_1 - y\cos\beta_1 = 0, \ p_2 - x\cos\alpha_2 - y\cos\beta_2 = 0$ be the normal forms of the lines

$$a_i x + b_i y + c_i = 0 \qquad i = 1,2$$

$$\cos \alpha_i = -\frac{a_i c_i}{D_i} \cos \beta_i = -\frac{b_i c_i}{D_i} (i = 1,2).$$

Also $\cos \beta_i = \cos(w - \alpha_i) = \cos w \cos \alpha_i + \sin w \sin \alpha_i$,

$$\therefore \quad \sin \alpha_i = \frac{a_i c_i \cos w - b_i c_i}{D_i \sin w}.$$

Hence $\cos \delta = \cos(\alpha_2 - \alpha_1) = \cos \alpha_2 \cos \alpha_1 + \sin \alpha_2 \sin \alpha_1$

$$= c_1 c_2 [a_1 a_2 \sin^2 w + (a_1 \cos w - b_1)(a_2 \cos w - b_2)] \div D_1 D_2 \sin^2 w$$
$$= c_1 c_2 [a_1 a_2 + b_1 b_2 - (a_1 b_2 + a_2 b_1) \cos w]/D_1 D_2 \sin^2 w \quad \text{......(18)}$$

Also $\sin \delta = \sin \alpha_2 \cos \alpha_1 - \cos \alpha_2 \sin \alpha_1$

$$= c_1 c_2 [a_1 (b_2 - a_2 \cos w) - a_2 (b_1 - a_1 \cos w)] \div D_1 D_2 \sin^2 w$$
$$= c_1 c_2 (a_1 b_2 - a_2 b_1)/D_1 D_2 \sin^2 w, \quad \text{......(19)}$$

$$\therefore \quad \tan \delta = \frac{(a_1 b_2 - a_2 b_1) \sin w}{a_1 a_2 + b_1 b_2 - (a_1 b_2 + a_2 b_1) \cos w} \quad \text{......(20)}$$

If the lines be given by the equations

$$y=m_1x+c_1,\ y=m_2x+c_2.$$

then $\tan\delta=\dfrac{(m_2-m_1)\sin w}{1+m_1m_2+(m_1+m_2)\cos w}$ ......(21)

20.1 Alternative method.

Write the equations of the lines

$$a_1x+b_1y+c_1=0,\ a_2x+b_2y+c_2=0$$

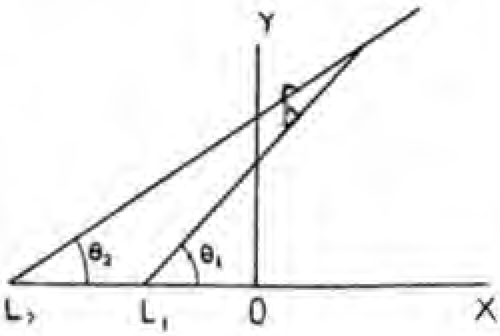
in the form $y=m_1x+k_1,\ y=m_2x+k_2$.

If δ be the angle between the lines,

$\delta=\theta_1-\theta_2$

$$\tan\delta=\tan(\theta_1-\theta_2)$$

$$=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}.$$



For rectangular axes,

$$\tan\delta=\frac{m_1-m_2}{1+m_1m_2}$$

$$=\frac{\left(-\dfrac{a_1}{b_1}\right)-\left(-\dfrac{a_2}{b_2}\right)}{1+\dfrac{a_1a_2}{b_1b_2}}.$$

$$=\frac{a_2b_1-a_1b_2}{a_1a_2+b_1b_2}.$$

For oblique axes,

$$\tan\delta=\frac{\dfrac{m_1\sin w}{1+m_1\cos w}-\dfrac{m_2\sin w}{1+m_2\cos w}}{1+\dfrac{m_1m_2\sin^2 w}{(1+m_1\cos w)(1+m_2\cos w)}}$$ [Art. 18.]

$$=\frac{(m_1-m_2)\sin w}{1+(m_1+m_2)\cos w+m_1m_2}$$

$$=\frac{(a_2b_1-a_1b_2)\sin w}{(a_1a_2+b_1b_2)-(a_1b_2+a_2b_1)\cos w}.$$

20.2. Conditions of parallelism and perpendicularity.

Let the equations of the lines be

$$\left.\begin{array}{rl} & a_ix+b_iy+c_i=0 \\ \text{or} & y=m_ix+k_i=0\end{array}\right\}(i=1,\ 2).$$

The lines will be parallel if the angle δ between them is zero or π. Consequently $\tan\delta$ (or $\sin\delta$) is zero. Hence

$$\left.\begin{array}{rl} & a_1b_2-a_2b_1=0 \\ \text{or} & m_1-m_2=0\end{array}\right\}$$ ......(22)

whether the axes be oblique or rectangular.

On the other hand, if $\delta = \frac{\pi}{2}$, the lines are at right angles, consequently $\cos\delta = 0$ or $\tan\delta \to \infty$. Hence the condition of perpendicularity for rectangular axes is

$$\left.\begin{array}{l} a_1a_2 + b_1b_2 = 0 \\ \text{or} \quad 1 + m_1m_2 = 0 \end{array}\right\} \quad \text{......(23)}$$

For oblique axes, the corresponding condition is

$$\left.\begin{array}{l} a_1a_2 + b_1b_2 = (a_1b_2 + a_2b_1)\cos w \\ 1 + m_1m_2 = -(m_1 + m_2)\cos w \end{array}\right\} \quad \text{......(24)}$$

In particular, the line $ax + by + c = 0$ is parallel to

$$ax + by + d = 0$$

whether the axes be oblique or rectangular, and perpendicular to

$$bx - ay + k = 0 \text{ or } (b - a\cos w)\,x - (a - b\cos w)y + k = 0,$$

according as the axes are rectangular or oblique, whatever k may be.

Illustrative Examples

(1) *The hypotenuse BC of a right angled isosceles triangle ABC is given by the equation $3x + 4y = 12$. If A is the point $(4, 3)$, find the equations of AB and AC.*

The equations of the sides will be of the form

$$a(x - 4) + b(y - 3) = 0,$$

this makes an angle 45° with the line

$$3x + 4y = 12,$$

$$\therefore \quad \tan\frac{\pi}{4} = 1 = \pm\frac{4a - 3b}{3a + 4b},$$

so $7b = a$ or $7a + b = 0$.

Hence the required lines are given by the equations

$$7x + y - 31 = 0, \quad x - 7y + 17 = 0.$$

(2) *Prove that the co-ordinates of the point which is the reflection of the point (h, k) in the line $lx + my + n = 0$ are $(h - lt, k - mt)$ where $t = 2(lh + mk + n)/(l^2 + m^2)$.*

The reflection of a point P in a line l is a point Q, such that PQ is bisected at right angles by l. Let $P(h, k)$ be the given point, and Q its reflection in the line. The equation of PQ, which is perpendicular to the given line is

$$\frac{x - h}{l} = \frac{y - k}{m} (= -t \text{ say}).$$

So the co-ordinates of Q we can suppose to be

$$(h - lt, k - mt).$$

The mid-point of PQ is $(h - \frac{1}{2}lt, k - \frac{1}{2}mt)$ and lies on the given line,

$\therefore \quad l(h-\frac{1}{2}lt)+m(k-\frac{1}{2}mt)+n=0$

$\therefore \quad t=2(lh+mk+n)\div(l^2+m^2).$

Examples III

1. In what ratio is the join of $(2, 3,)$ $(3, -2)$ divided by the line $5x+12y-13=0$

2. In what ratio is the join of $(2, 3)$ and $(3, 4)$ divided by the join of $(1, 1)$ and $(5, -2)$.

3. An isosceles triangle has the extremities of its base at $(-6, 5)$, $(6, 10)$. Find the two possible positions of the vertex if the area is 26 sq. units.

4. ABC is an isoceles triangle, right angled at B $(3, 0)$. If the equation of AC is $2x=y+17$, find the equations of AB and BC.

5. The opposite vertices of a square are $(0, -1)$, $(0, 3)$; find the equations of the four sides.

6. *Show that if a real st. line contains an imaginary point, it also contains its conjugate imaginary.*

7. *Show that the join of two conjugate imaginary points is a real st. line.*

8. One side of an equilateral triangle is $x+y\sqrt{3}=1$ and the opposite vertex is at the origin. Find the equations of the other sides.

9. Find the angle between the lines

$$\frac{l}{r}=2\cos\theta+3\sin\theta,\ \frac{l'}{r}=3\cos\theta-2\sin\theta.$$

10. ABCD is a parallelogram. Taking A as pole, and AB as initial line, find the polar equations of the four sides and of the two diagonals.

11. Find the equations of st. lines, which pass through a given point and which are inclined to a given line at a given angle.

21. The distance of a point from a line.

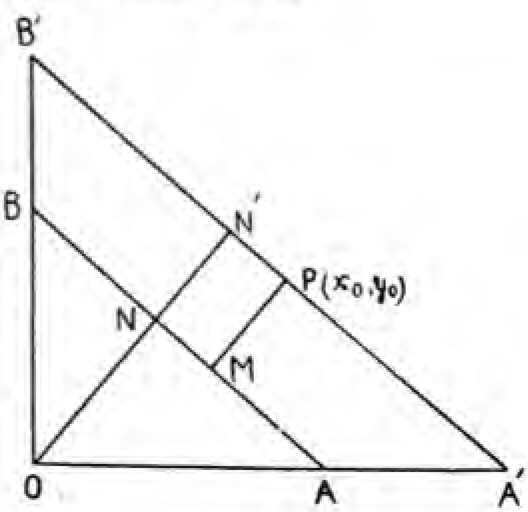
Let the equation of the line be given in the normal form, *viz.*,

$p-x\cos\alpha-y\sin\alpha=0$,

and let $P(x_0, y_0)$ be a given point, from which the perpendicular PM is drawn on the given line AB. Through P draw $A'B' \parallel AB$, the equation of $A'B'$ is, therefore

$ON'-x\cos\alpha-y\sin\alpha=0$.

The line passes through $P(x_0, y_0)$.



$\therefore\quad ON' = x_0 \cos\alpha + y_0 \sin\alpha.$

Now $PM = -MP = -(ON' - ON)$
$= ON - ON',$

whether P be on the origin or the non-origin side of the line AB. Hence

$$PM = p - x_0 \cos\alpha - y_0 \sin\alpha. \qquad \text{......(25)}$$

The above expression for the perpendicular involves its own sign. If P is on the same side of the line AB as the origin ON>ON', the expression for PM will be positive. If P is on the side of the line other than the origin, the expression $p - x_0 \cos\alpha - y_0 \sin\alpha$ will be found to be negative.

The perpendiculars from two given points on a given line are of the same or opposite signs according as the points are on the same or opposite sides of the line.

21.1. Let the equation of the line be given in the general form

$$ax + by + c = 0,$$

which when reduced to the normal form, becomes

$$\frac{c^2}{d} - \frac{acx}{-d} - \frac{bcy}{-d} = 0$$

where $\quad d = \sqrt{c^2(a^2+b^2)}.$

Applying to this the result of Art. 21, the expression for the perpendicular from (x_0, y_0) on the line is seen to be

$$\frac{c(c + ax_0 + by_0)}{\sqrt{c^2(a^2+b^2)}} \qquad \text{......(26)}$$

If the axes are oblique, the results corresponding to (25) and (26) are

$$p - x_0 \cos\alpha - y_0 \cos\beta \qquad \text{......(25A)}$$

$$c(c + ax_0 + by_0) \div D \qquad \text{......(26A)}$$

where $\quad D = \sqrt{c^2(a^2+b^2-2ab\cos w)} \div \sin w.$

21.2. Alternative Method.

Let the line $\dfrac{x-x_0}{\cos\theta} = \dfrac{y-y_0}{\sin\theta} = r$ be perpendicular to the line

$$ax + by + c = 0.$$

If r be the length of the perp. from (x_0, y_0) on the 2nd line, then $(x_0 + r\cos\theta,\ y_0 + r\sin\theta)$ lies on it

$\therefore\quad a(x_0 + r\cos\theta) + b(y_0 + r\sin\theta) + c = 0$

$\therefore\quad r = -\dfrac{ax_0 + by_0 + c}{a\cos\theta + b\sin\theta}.$

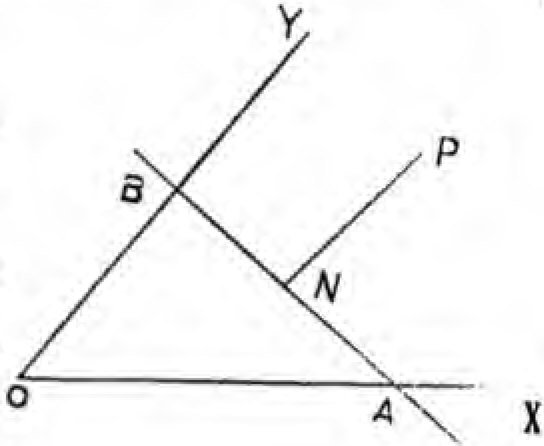
But $a \sin\theta - b\cos\theta = 0$

i.e., $$\frac{\sin\theta}{b} = \frac{\cos\theta}{a} = \frac{\pm 1}{\sqrt{a^2+b^2}}$$

$$\therefore \quad r = \pm\frac{ax_0+by_0+c}{\sqrt{(a^2+b^2)}}.$$

Similarly we may get the result for oblique axes,

Or We may proceed thus:—



Let the line
$$ax+by+c=0$$
cut the axes at A, B. Then A is $\left(-\frac{c}{a}, 0\right)$, B$\left(0, -\frac{c}{b}\right)$.

P is (x_0, y_0).

Now $\triangle$ OAB + $\triangle$ APB
= $\triangle$ OAP + $\triangle$ OPB.

This gives the perpendicular
$$\text{PN} = \pm\frac{(ax_0+by_0+c)\sin w}{\sqrt{(a^2+b^2-2ab\cos w)}}$$

22. *To interpret the inequation*
$$ax+by+c \gtrless 0$$

Let the expression $ax+by+c=\cancel{0}$ be denoted by U so that the equation of the st. line can be written as $U=0$, Set $U_i \equiv ax_i+by_i+c$. It will now be shown that the expression U changes sign as the point (x, y) crosses the line $U=0$.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points whose join meets the line $U=0$ in a point R which divides PQ in the ratio $\lambda : 1$. The co-ordinates of R are therefore
$$\frac{x_1+x_2\lambda}{1+\lambda}, \quad \frac{y_1+y_2\lambda}{1+\lambda},$$
which lies on the line $U=0$, hence
$$a(x_1+\lambda x_2)+b(y_1+\lambda y_2)+c(1+\lambda)=0.$$
$$\therefore \quad \lambda = -\frac{U_1}{U_2}.$$

If the points P and Q are on the same side of the line $U=0$, the point R is external to the segment PQ, consequently λ is negative and hence U_1 and U_2 have the same sign; and conversely.

If the points P and Q are on different sides of the line, the point R is internal to the segment PQ, the ratio λ, is, therefore, positive. Thus U_1 and U_2 have opposite signs and conversely.

The line $U=0$, therefore, divides the plane in which it is drawn into two regions, such that for all points of one region $U>0$ and for all points of the other region $U<0$. These regions are respectively called the *positive* and *negative* regions.

It is obvious that a point which is in the positive region of the line $U=0$ is in the negative region of the line $-U=0$ and *vice versa*, and the lines $U=0$ and $-U=0$ are identical. The choice of positive and negative regions is, thus, arbitrary. Consequently, the relative position of a point shall have to be determined completely with reference to another point which is usually chosen to be the origin. The sign of the expression cU does not change when U is replaced by $-U$, for such a change also changes c into $-c$ and $cU>0$ for $x=0$, $y=0$. *Hence* $cU>0$ *represents the origin side* (**O-side**) *of the line* and $cU<0$ *the non-origin side* (**N-O-side**) of the line $U=0$.

Def. The expression cU will be called the *power* of $P(x, y)$ *w. r.* to the line $U=0$.

23. The point of intersection of two given lines.

Let the lines be given by the equations

$$a_1x+b_1y+c_1=0$$
$$a_2x+b_2y+c_2=0.$$

A point of intersection of two loci is a point common to both. Its co-ordinates, therefore, satisfy the equations of both. The co-ordinates of the points of intersection are, consequently, found by solving the equations of the loci simultaneously. Thus solving the above two equations simultaneously, we have

$$\frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}=\frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \qquad \ldots(27)$$

(*i*) If $a_1b_2-a_2b_1\neq0$, the values of x and y are finite and unique. The st. lines, then, meet in a finite point.

(*ii*) If $a_1b_2-a_2b_1=0$, but $b_1c_2-b_2c_1$ and $c_1a_2-a_1c_2$ do not vanish simultaneously, the values of x and y are infinite. The st. lines are, then, said to meet in a unique point at infinity ; and are, therefore, parallel.

(*iii*) If $a_1b_2-a_2b_1=0$, $b_1c_2-b_2c_1=0$, $c_1a_2-a_1c_2=0$, but the four co-efficients a_1, b_1, a_2, b_2 do not vanish simultaneously, the two lines are identical, and a single infinity (∞^1) of values of x and y, *viz*., the co-ordinates of the points of the line, satisfy the equations. The values of x and y can be determined in terms of a single parameter.

If $a_i^2+b_i^2 \neq 0$, $(i=1, 2)$, we consider the additional equation

$$-b_ix+a_iy+\rho=0,$$

which combined with $a_ix+b_iy+c_i=0$, gives the system of values of x and y in terms of a single parameter ρ, *viz*.,

$$x:y:1=-a_ic_i+b_i\rho : -b_ic_i-a_i\rho : a_i^2+b_i^2 \qquad \text{......(28)}$$

If $a_i^2+b_i^2=0$, but $a_i \neq 0$ $b_i \neq 0$, the equation

$$a_ix+b_iy+c_i=0$$

may be combined with the equation

$$-a_ix+b_iy+\sigma=0,$$

where σ is a parameter. The values for x and y are given by the equations

$$x:y:1=b_i(\sigma-c_i) : -a_i\ (c_i+\sigma) : 2a_i\ b_i \qquad \text{......(29)}$$

(*iv*) $a_1=a_2=b_1=b_2=0$, but all the six co-efficients do not vanish simultaneously, there exist no finite roots of the equations. The values of both x and y are infinite. There exist ∞^1 of pair of values of x and y which satisfy the equations. The case will be dealt with later on.

(*v*) $a_1=a_2=b_1=b_2=c_1=c_2=0$. There exist double infinity (∞^2) of roots ; x and y can be chosen arbitrarily.

24. Equation of a st. line through the intersection of two given st. lines. Pencil of lines.

Let the equations of the lines be

$$U_1 \equiv a_1x+b_1y+c_1=0, \quad U_2 \equiv a_2x+b_2y+c_2=0.$$

The first method that suggests itself is to find the point (x_0, y_0) of intersection of the lines U_1 and U_2, then the equation of an arbitrary st. line through (x_0, y_0) is

$$l(x-x_0)+m(y-y_0)=0,$$

where l and m can have any values, but do not vanish simultaneously. The ratio $l:m$ is to be determined by one more condition.

There is, however, an easier solution of the problem. The required equation must be of the first degree in x and y and must be satisfied by $x=x_0$, $y=y_0$, where (x_0, y_0) is the point of intersection of the lines U_1 and U_2. Also, as two conditions fix a st. line and the problem assigns only one condition, the equation must contain one arbitrary para-

meter to be determined by another condition, *i.e., the equation must have one degree of freedom.*

These conditions are satisfied by the equation

$$U_1+kU_2\equiv a_1x+b_1y+c_1+k(a_2x+b_2y+c_2)=0 \qquad \text{......(30)}$$

That it is of the first degree and has one degree of freedom are obvious. The line also passes through the intersection (x_0, y_0) of U_1 and U_2 for $a_1x_0+b_1y_0+c_1=0$, $a_2x_0+b_2y_0+c_2=0$, and consequently

$$a_1x_0+b_1y_0+c_1+k(a_2x_0+b_2y_0+c_2)=0,$$

and this is precisely the condition that the point (x_0, y_0) lies on the line $U_1+kU_2=0$.

Conversely, *the equation of any st. line through the point of intersection of* U_1 and U_2 *is represented by equation* (30). For every such line joins (x_0, y_0) with another point (x', y'). We have, therefore, to determine k, from the condition that U_1+kU_2 must vanish for $x=x'$, $y=y'$.

$\therefore \quad U_1'+kU_2'=0$, where $U_i'=a_ix'+b_iy'+c_i$.

Substituting this value of k in (30), the required equation is found to be

$$U_1U_2'-U_2U_1'=0.$$

The equation (30) represents an infinity of st. lines obtained by giving different values to k, but all passing through (x_0, y_0). Such a system, which has one degree of freedom is called a **pencil** (**Buschel**). Thus equation (30) represents a **pencil of lines.**

24.1. Condition that three lines may belong to a pencil, i.e., the condition of concurrence of three lines.

Let the equations of the lines be

$$U_i\equiv a_ix+b_iy+c_i=0,\ i=1, 2, 3.$$

The necessary and sufficient condition that the lines U_1, U_2, U_3 may be concurrent is that

$$\Delta\equiv\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0 \qquad \text{......(31)}$$

The condition is necessary, for, if the three lines are concurrent, the point of intersection of any two of them lies on the third.

The point of intersection of U_1 and U_2 is by Art. 23.

$$x:y:1=\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} : -\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} : \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \text{......(31 A)}$$

This lies on the line $U_3=0$, if

$$a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} - b_3\begin{vmatrix} a_1 & c_1 \\ a_2 & b_2 \end{vmatrix} + c_3\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0 \quad \ldots\ldots(31\text{ B})$$

which is precisely $\Delta=0$.

Conversely, if $\Delta=0$, the three lines are concurrent. For, the condition $\Delta=0$ is equivalent to (31 B). This shows that the line U_3 passes through the point given by (31 A) which is the intersection of U_1 and U_2.

The above discussion is based on the assumption, that
$a_1b_2-a_2b_1\neq 0$. If $a_1b_2-a_2b_1=0$, let

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=k\ (\text{say})\neq\frac{c_1}{c_2},$$

then $\Delta\equiv+(kc_1-c_2)(a_1b_3-a_3b_1)$,
and the lines U_1 and U_2 are parallel. They have a unique common point at infinity, such that

$$x:y=b_1:-a_1.$$

This point lies on U_3, *viz.*,

$$a_3\frac{x}{y}+b_3+\frac{c_3}{y}=0;$$

$$\therefore\quad a_3b_1-b_3a_1=0 \text{ as } y\to\infty \text{ and } \frac{x}{y}\to-\frac{b_1}{a_1}$$

which differs from Δ by a non-zero multiple: hence $\Delta=0$. Conversely, $\Delta=0$ implies that $a_3b_1-b_3a_1=0$, since $kc_1-c_2\neq 0$. This shows that the line U_3 is parallel to U_1 or U_2. Hence the three lines meet in a unique point at infinity.

24.2. A test, which is more convenient in practice, for determining whether three lines belong to a pencil, can be stated as follows :—

The necessary and sufficient condition that three lines $U_i\equiv a_ix+b_iy+c_i=0$ $(i=1, 2, 3)$ *may belong to a pencil is, that there exist three constants* l_1, l_2, l_3, *such that*

$$l_1U_1+l_2U_2+l_3U_3\equiv 0 \qquad \ldots\ldots(32)$$

Necessity. The equation $l_1U_1+l_2U_2=0$ represents a system of lines through the intersection P of U_1 and U_2, and so by proper choice of $l_1:l_2$ can be made to represent any line through P and in particular U_3. Consequently, $l_1U_1+l_2U_2=0$ is identical with U_3.

$$\text{Hence}\quad l_1U_1+l_2U_2\equiv-l_3U_3$$
$$\text{or}\quad l_1U_1+l_2U_2+l_3U_3\equiv 0.$$

Sufficiency. The equation $l_1U_1+l_2U_2=0$, which is identical with $U_3=0$ in virtue of relation $l_1U_1+l_2U_2+l_3U_3\equiv 0$, represents a st. line through the intersection P of U_1 and U_2, hence $U_3=0$ also passes through that point.

The co-efficients $l_1:l_2:l_3$ may be easily calculated. In virtue of relation (32),

$$l_1a_1+l_2a_2+l_3a_3=0$$
$$l_1b_1+l_2b_2+l_3b_3=0$$
$$l_1c_1+l_2c_2+l_3c_3=0.$$

$$\therefore\ l_1:l_2:l_3=A_1:A_2:A_3=B_1:B_2:B_3=C_1:C_2:C_3,$$

where each capital letter denotes the co-factor of the corresponding small letter in the determinant Δ (31).

25. The equations of four st. lines, no three of which belong to a pencil, are linearly connected.

Let $U_i\equiv a_ix+b_iy+c_i=0$ $\qquad i=1, 2, 3, 4.$
be the equations of four st. lines no three of which are concurrent. The determinant :

$$\begin{vmatrix} U_1 & U_2 & U_3 & U_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \end{vmatrix}$$

vanishes identically, for, the first row vanishes identically, if from it be subtracted the sum of x-times the second, y-times the third and the fourth row. Now expanding *w. r.* to the first row (with the usual determinantal notation) the required linear relation is

$$U_1(a_2b_3c_4)-U_2(a_1b_3c_4)+U_3(a_1b_2c_4)-U_4(a_1b_2c_3)\equiv 0 \qquad \text{......(33)}$$

This shows that U_4 can be expressed linearly in terms of U_1, U_2, U_3. *Hence if three lines* $U_1=0$, $U_2=0$, $U_3=0$ *form a triangle, the equation of any fourth line can be expressed in the form*

$$\lambda_1U_1+\lambda_2U_2+\lambda_3U_3=0,$$

where λ_1, λ_2, λ_3 *are properly chosen.*

25.1. Diagonals of a quadrilateral.

Let $U_i=a_ix+b_iy+c_i=0$ be the equations of the four lines no three of which belong to a pencil. Hence they form a quadrilateral. The equations of these lines are connected by the relation (33). Now

$$U_1(a_2b_3c_4)-U_2(a_1b_3c_4)=0$$

represents a line through the intersection of U_1 and U_2, and this in virtue of relation (33) is identical with

$$U_3(a_1b_2c_4) - U_4(a_1b_2c_3) = 0,$$

which is a line passing through the intersection of U_3 and U_4, and is, therefore, one of the diagonals.

The equations of the other two diagonals can, similarly, be determined. The equation of the three diagonals are

$$\left.\begin{array}{l} U_1\,(a_2b_3c_4) - U_2\,(a_1b_3c_4) = 0 \\ U_1\,(a_2b_3c_4) + U_3\,(a_1b_2c_4) = 0 \\ U_1\,(a_2b_3c_4) - U_4\,(a_1b_2c_3) = 0 \end{array}\right\} \quad \ldots\ldots\ldots\ldots\ldots (34)$$

Illustrative Examples

(1) *Show that the medians of a triangle are concurrent.*

Let $A(a_1, a_2)$, $B(b_1, b_2)$, $C(c_1, c_2)$ be the vertices of a triangle. The co-ordinates of the mid-point of BC are $\left(\dfrac{b_1+c_1}{2}, \dfrac{b_2+c_2}{2}\right)$

and the equation of the line which joins this point to A is

$$\frac{x-a_1}{2a_1-b_1-c_1} = \frac{y-a_2}{2a_2-b_2-c_2}$$

or $\quad x(2a_2-b_2-c_2) - y(2a_1-b_1-c_1) = b_1a_2 - b_2a_1 + c_1a_2 - a_1c_2.$

The equations of the other two medians are similarly found to be

$$x(2b_2-a_2-c_2) - y(2b_1-a_1-c_1) = b_2(a_1+c_1) - b_1(a_2+c_2)$$
$$x(2c_2-b_2-a_2) - y(2c_1-a_1-b_1) = c_2(a_1+b_1) - c_1(a_2+b_2).$$

These three equations when added vanish identically. Hence the three lines meet in a point.

(2) *The co-ordinates of the vertices of two triangles A, B, C. $A'B'C'$ are respectively (a_1, b_1), (a_2, b_2), (a_3, b_3); (α_1, β_1), (α_2, β_2), (α_3, β_3). The joins of the corresponding vertices are divided similarly in the points D_1, D_2, D_3. If the perpendiculars from D_1, D_2, D_3 on the sides of either triangle be concurrent, prove that*

$$\begin{vmatrix} a_1 & \alpha_1 & 1 \\ a_2 & \alpha_2 & 1 \\ a_3 & \alpha_3 & 1 \end{vmatrix} + \begin{vmatrix} b_1 & \beta_1 & 1 \\ b_2 & \beta_2 & 1 \\ b_3 & \beta_3 & 1 \end{vmatrix} = 0$$

Let $\lambda : 1$ be the ratio in which AA′, BB′, CC′ are divided at D_1, D_2, D_3. The co-ordinates of D_1 are

$$\left(\frac{a_1+\lambda\alpha_1}{1+\lambda}, \frac{b_1+\lambda\beta_1}{1+\lambda}\right),$$

and the slope of B′C′ is $\dfrac{\beta_3-\beta_2}{\alpha_3-\alpha_2}$. Hence, the equation of the perpendicular to B′C′ through D_1 is

$$y-\frac{b_1+\lambda\beta_1}{1+\lambda}=-\frac{\alpha_3-\alpha_2}{\beta_3-\beta_2}\left(x-\frac{a_1+\lambda\alpha_1}{1+\lambda}\right),$$

or $$(1+\lambda)x(\alpha_2-\alpha_3)+(1+\lambda)(\beta_2-\beta_3)y$$
$$=(\alpha_2-\alpha_3)(a_1+\lambda\alpha_1)+(\beta_2-\beta_3)(b_1+\lambda\beta_1)\ldots(i)$$

Similarly the perpendiculars from D_2, D_3 on C′A′, A′B′ are respectively given by the equations

$$(1+\lambda)(\alpha_3-\alpha_1)x+(1+\lambda)(\beta_3-\beta_1)y=(\alpha_3-\alpha_1)(a_2+\lambda\alpha_2)$$
$$+(\beta_3-\beta_1)(b_2+\lambda\beta_2) \qquad\ldots\ldots(ii)$$
$$(1+\lambda)(\alpha_1-\alpha_2)x+(1+\lambda)(\beta_1-\beta_2)y=(\alpha_1-\alpha_2)(a_3+\lambda\alpha_3)$$
$$+(\beta_1-\beta_2)(b_3+\lambda\beta_3) \qquad\ldots\ldots(iii)$$

Since these three lines are concurrent, there exist constants l_1, l_2, l_3, such that the sum of l_1 times the (i) l_2 times (ii), l_3 times (iii) vanishes identically ;

$$\therefore \qquad l_1(\alpha_2-\alpha_3)+l_2(\alpha_3-\alpha_1)+l_3(\alpha_1-\alpha_2)=0$$
$$l_1(\beta_2-\beta_3)+l_2(\beta_3-\beta_1)+l_3(\beta_1-\beta_2)=0$$
and $\Sigma l_1(a_1+\lambda\alpha_1)(\alpha_2-\alpha_3)+\Sigma(b_1+\lambda\beta_1)(\beta_2-\beta_3)=0.$

The first two of these three relations give $l_1=l_2=l_3$. With these values of l_1, l_2, l_3, the third relation becomes

$$\Sigma a_1(\alpha_2-\alpha_3)+\Sigma b_1(\beta_2-\beta_3)=0$$

which is the required condition. The symmetry of the result shows that the perpendiculars from D_1, D_2, D_3 on BC, CA, AB are also concurrent.

(3) *The sides of a triangle are* $a_rx+b_ry+c_r=0$, $r=1, 2, 3.$ *The feet of the perpendiculars from* A, B, C, *on the opposite sides are respectively* L, M, N. *The lines* MN, BC *meet in* P ; NL, AC *in* Q ; LM, AB *in* R. *Prove that the points* P, Q, R, *lie on the line*

$$\Sigma(a_2a_3+b_2b_3)(a_1x+b_1y+c_1)=0.$$

Denote by U_r the expression $a_rx+b_ry+c_r$; $(r=1, 2, 3)$. The equation of AL is of the form $U_2+\lambda U_3=0$; since this is perpendicular to U_1,

$$\therefore \qquad a_1(a_2+\lambda a_3)+b_1(b_2+\lambda b_3)=0$$
or $$(a_1a_2+b_1b_2)+\lambda(a_1a_3+b_1b_3)=0.$$

Hence the equation of AL is

$$(a_1a_3+b_1b_3)U_2-(a_1a_2+b_1b_2)U_3=0.$$

Similarly, the equations of BM and CN are

$$\text{(BM)} \quad (a_2a_1+b_2b_1)U_3-(a_2a_3+b_2b_3)U_1=0.$$
$$\text{(CN)} \quad (a_3a_2+b_3b_2)U_1-(a_3a_1+b_3b_1)U_2=0.$$

Now MN joins the intersections of BM, AC and CN, AB; its equations will be of the forms

$$(a_2a_1+b_2b_1)U_3-(a_2a_3+b_2b_3)U_1+\lambda U_2=0$$
$$(a_3a_2+b_3b_2)U_1-(a_3a_1+b_3b_1)U_2+\mu U_3=0.$$

These lines will be the same if $\lambda=(a_3a_1+b_3b_1)$, and $\mu=-(a_2a_1+b_2b_1)$. The equation of MN is, therefore,

$$(\text{MN})\quad -(a_2a_3+b_2b_3)U_1+(a_1a_3+b_1b_3)U_2+(a_1a_2+b_1b_2)U_3=0.$$

This meets $U_1=0$ at a point which lies on

$$(a_2a_3+b_2b_3)U_1+(a_1a_3+b_1b_3)U_2+(a_1a_2+b_1b_2)U_3=0$$

Remarks. The equations of the lines AL, BM, CN when added vanish identically. The three lines, therefore, meet in a point. Hence *the altitudes of a triangle are concurrent.*

Examples IV

1. The sides of a triangle are given by the equations

$$2x-3y+120=0,\ x+y=0,\ 3x+4y-6=0,$$

find the equations of the altitudes and show that they meet in a point.

2. Show that the diagonals of the parallelogram formed by the lines $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\dfrac{x}{b}+\dfrac{y}{a}=1$, $\dfrac{x}{a}+\dfrac{y}{b}=2$, $\dfrac{x}{b}+\dfrac{y}{a}=2$ are at right angles.

3. Find the diagonals of a parallelogram formed by the lines

$$x-6y=5,\ x-6y=11,\ 3x+2y=12,\ 3x+2y=6.$$

4. Show that the lines $(b+c)x-bcy=a(b^2+bc+c^2)$, $(c+a)x-cay=b(c^2+ac+a^2)$, $(a+b)x-aby=c(a^2+ab+b^2)$ meet at the point $(\Sigma bc,\ \Sigma a)$.

5. Find the condition that the lines $y+t_ix=2at_i+at_i^3=0$ $i=1, 2, 3$, may be concurrent.

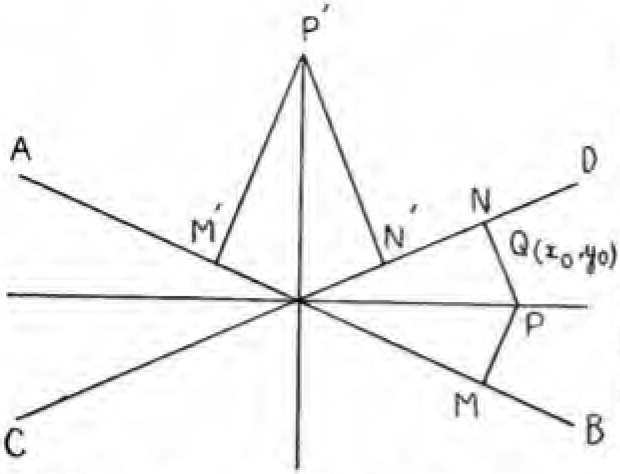
6. A, B, C, are the points $(-2, 6)$, $(-4, 4)$ and $(8, 2)$ respectively. Find the co-ordinates of the points D and E so that ABCD, ABEC are both parallelograms.

7. Find the diagonals of the parallelogram formed by the lines $U=0$, $U=a$, $V=0$, $V=b$.

26. Bisectors of the angles between lines.

Let
$$l_1x+m_1y+n_1=0$$
$$l_2x+m_2y+n_2=0$$

be the equations of the given lines, AB, CD.



Now the perpendiculars drawn on the two lines from any point of the bisector of an angle between them are equal. Hence P and P′ being points on the bisectors and PM, PN, P′M′, P′N′ perpendiculars on the lines,

$/PM/=/PN/, /P'M'/ =/P'N'/$. Again PM and P′M′ are drawn in the same sense, they have, therefore, the same sign. Similarly PN and P′N′ are of opposite signs. Consequently, if PM=PN both in magnitude and sign, P′M′ = − P′N′ and *vice versa*. Hence the equations of the two bisectors are

$$\frac{l_1x+m_1y+n_1}{\sqrt{l_1^2+m_1^2}}=\pm\frac{l_2x+m_2y+n_2}{\sqrt{l_2^2+m_2^2}} \qquad \ldots\ldots(35)$$

26.1. To find the bisector of the angle in which a particular point lies.

Let $Q(x_0, y_0)$ be the particular point which lies in the plane of the given lines

$$U\equiv l_1x+m_1y+n_1=0,\ V\equiv l_2x+m_2y+n_2=0,$$

and denote by U_0, V_0 the expressions $l_1x_0+m_1y_0+n_1$ and $l_2x_0+m_2y_0+n_2$. Let $P(x, y)$ be a point of the bisector of the angle between the lines in which Q lies.

Since P and Q are on the same side of U, it being supposed that P lies in the same angle as Q, the expressions U and U_0 have the same sign, and U_0U is therefore positive. Consequently, the measure of the perpendicular from P on U, *viz.*,

$$\frac{UU_0}{\sqrt{U_0^2(l_1^2+m_1^2)}}$$

is positive. For a similar reason, the expression

$$\frac{VV_0}{\sqrt{V_0^2(l_2^2+m_2^2)}},$$

which denotes the measure of the perpendicular from P on V, is also positive. But these perpendiculars being

equal, we get on equating, the equation of the *required bisector*, *viz.*

$$\frac{UU_0}{\sqrt{U_0^2(l_1^2+m_1^2)}}=\frac{VV_0}{\sqrt{V_0^2(l_2^2+m_2^2)}} \qquad \ldots\ldots(36)$$

The same expression is obtained if P be taken in the vertically opposite angle.

If U_0 and V_0 *be of the same sign*, the equation (35) reduces to

$$\frac{U}{\sqrt{l_1^2+m_1^2}}=\frac{V}{\sqrt{l_2^2+m_2^2}} \qquad \ldots\ldots(36\text{ A})$$

Similarly, *the bisector of the angle in which* $Q(x_0, y_0)$ *does not lie is given by the equation*

$$\frac{UU_0}{\sqrt{U_0^2(l_1^2+m_1^2)}}=-\frac{VV_0}{\sqrt{V_0^2(l_2^2+m_2^2)}} \qquad \ldots\ldots(37)$$

or simply by the equation

$$\frac{U}{\sqrt{l_1^2+m_1^2}}=-\frac{V}{\sqrt{l_2^2+m_2^2}} \qquad \ldots\ldots(37\text{ A})$$

if U_0 and V_0 have the same sign.

In particular, for the origin, the corresponding equations take the form

$$\frac{Un_1}{\sqrt{n_1^2(l_1^2+m_1^2)}}=\pm\frac{Vn_2}{\sqrt{n_2^2(l_2^2+m_2^2)}} \qquad \ldots\ldots(38)$$

the positive sign being taken for the bisector of the angle in which the origin lies, and negative for the other bisector.

If n_1 and n_2 have the same sign, the result (37) takes a simpler form

$$\frac{U}{\sqrt{l_1^2+m_1^2}}=\pm\frac{V}{\sqrt{l_2^2+m_2^2}} \qquad \ldots\ldots(38\text{ A})$$

the positive sign being taken for the bisector in which the origin lies.

26.2. To determine the bisector of an acute or obtuse angle between the two given lines.

This case we shall reduce to that of the preceding Article by determining whether the origin lies in the acute or obtuse angle between the lines.

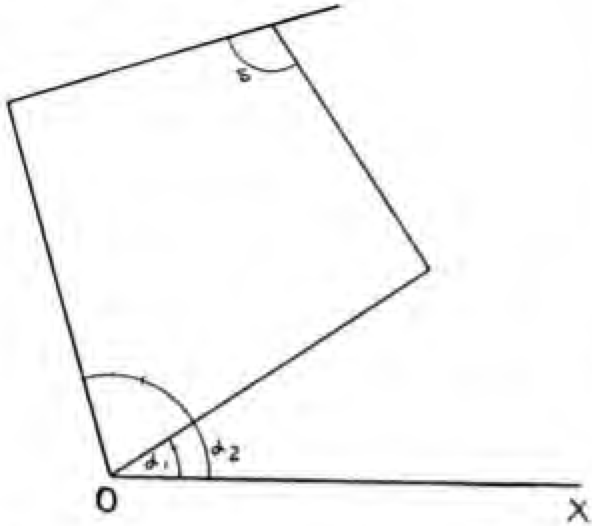
If the equations of the lines be

$$U_i \equiv l_i x+m_i y+n_i=0, \qquad i=1,2$$

and

$$p_i-x\cos\alpha_i-y\sin\alpha_i=0, \qquad i=1,2$$

their normal forms, then



$$\cos \alpha_1 = \frac{-n_1 l_1}{d_1},$$
$$\cos \alpha_2 = \frac{-n_2 l_2}{d_2}$$
$$\sin \alpha_1 = \frac{-n_1 m_1}{d_1},$$
$$\sin \alpha_2 = \frac{-n_2 m_2}{d_2},$$

where

$$d_1 = \sqrt{n_1^2(l_1^2 + m_1^2)}$$
$$d_2 = \sqrt{n_2^2(l_2^2 + m_2^2)}$$

Let δ be the angle between the lines in which the origin lies, then

$$\delta = \pi - (\alpha_2 - \alpha_1).$$

$$\therefore \cos \delta = -\cos(\alpha_2 - \alpha_1) = -(\cos \alpha_2 \cos \alpha_1 + \sin \alpha_2 \sin \alpha_1)$$

or $n_1 n_2 (l_1 l_2 + m_1 m_2) \div d_1 d_2 = -\cos \delta$.

If δ is obtuse, the expression $n_1 n_2 (l_1 l_2 + m_1 m_2) \div d_1 d_2$ is positive, and negative if δ is acute. Since $d_1 d_2$ is positive, the sign of $n_1 n_2$ $(l_1 l_2 + m_1 m_2)$ may only be considered. Hence: *the origin lies in the obtuse or acute angle between the lines according as*

$$n_1 n_2 (l_1 l_2 + m_1 m_2) \qquad \text{......(39)}$$

is positive or negative.

But the bisector of the angle, in which the origin does or does not lie, can be found by Art. 26.1.

26.3. The problem of Art. 26.2 admits of another solution. The bisector of the acute angle makes with either of the lines an angle θ less than $\frac{\pi}{4}$ or greater than $\frac{3\pi}{4}$, so that $/\tan \theta/ < 1$, and the bisector of the obtuse angle makes with either of the lines an angle ϕ which is greater than $\frac{\pi}{4}$ or less than $\frac{3\pi}{4}$, so that $/\tan \phi/ > 1$.

If the equations of the two lines be $l_r x + m_r y + n_r = 0$, the equations of the two bisectors are Art. 26. (35)

$$x(l_1 d_2 - l_2 d_1) + y(m_1 d_2 - m_2 d_1) + (n_1 d_2 - n_2 d_1) = 0 \qquad \text{......}(i)$$
$$x(l_1 d_2 + l_2 d_1) + y(m_1 d_2 + m_2 d_1(+ (n_1 d_2 + n_2 d_1) = 0 \qquad \text{......}(ii)$$

and $\tan\theta \; \dfrac{l_1m_2-l_2m_1}{d_1d_2-l_1l_2-m_1m_2}$, $\tan\varphi = -\dfrac{l_1m_2-l_2m_1}{d_1d_2+l_1l_2+m_1m_2}$

where $d_1=\sqrt{l_1^2+m_1^2}$, $d_2=\sqrt{l_2^2+m_2^2}$

Hence, if the line (i) bisects the acute angle between the lines

$$\left|l_1m_2-l_2m_1\right|<\left|d_1d_2-l_1l_2-m_1m_2\right|; \qquad \text{......(40)}$$

otherwise it will bisect the obtuse angle.

Ex. Obtain the result of Art. 26.3 from the fact, that if (i) is the bisector of the acute angle, the length of the perpendicular from any point on any one of the given lines on (i) is less than the perpendicular on (ii).

26.4. The incentre and e-centres of a triangle.

Let the equations of the sides BC, CA, AB be

$$U_i \equiv l_ix+m_iy+n_i=0 \qquad i=1, 2, 3.$$

Suppose that the value of the expression U_1 for the co-ordinates (x', y') of A is k,

$$\therefore \quad l_1x'+m_1y'+n_1-k=0$$
$$l_2x'+m_2y'-n_2=0$$
$$l_3x'+m_3y'-n_3=0,$$

since A lies on AB and AC. Hence

$$\begin{vmatrix} l_1 & m_1 & n_1-k \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0 \text{ or } 0 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} - k \begin{vmatrix} l_1 & m_1 & 1 \\ l_2 & m_2 & 0 \\ l_3 & m_3 & 0 \end{vmatrix}$$

i.e., $\Delta - kN_1=0$ where N_i is the co-factor of n_i in the determinant

$$\Delta \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Thus $k=\Delta/N_1$.

Similarly the values of the expressions U_2, U_3 when the co-ordinates of the opposite vertices B and C are respectively substituted in U_2 and U_3 are Δ/N_2 and Δ/N_3. Denote the sign of Δ/N_i by ω_i. ϵ

Let (x, y) be the co-ordinates of the incentre. As A and (x, y) are on the same side of BC, the perpendicular on $U_1=0$ from (x, y) has the sign ϵ_1, and its value is therefore

$$\frac{\epsilon_1 U_1}{\sqrt{l_1^2+m_1^2}}.$$

Similarly, the measures of the perpendiculars from (x, y) on the sides CA, AB are

$$\epsilon_2 U_2/\sqrt{l_2^2+m_2^2},\ \epsilon_3 U_3/\sqrt{l_3^2+m_3^2}.$$

Since these perpendiculars are equal, the co-ordinates of the incentre are given by the common solutions of the equations

$$\epsilon_1 U_1/\sqrt{l_1^2+m_1^2}=\epsilon_2 U_2/\sqrt{l_2^2+m_2^2}=\epsilon_3 U_3/\sqrt{l_3^2+m_3^2}.$$

If the equations $U_i=0$ be so written that N_1, N_2, N_3 have the same sign, $\epsilon_1=\epsilon_2=\epsilon_3$, the co-ordinates of the incentre are given by the common solutions of the equations

$$U_1 : \sqrt{l_1^2+m_1^2}=U_2 : \sqrt{l_2^2+m_2^2}=U_3 : \sqrt{l_3^2+m_3^2} \quad \ldots\ldots(41)$$

For the e-centres opposite to A, B, C, U_1, U_2, U_3 have negative signs respectively.

Illustrative Examples

(1) *Find the bisectors of the angles between the lines* $4x+3y+10=0$ *and* $12x-5y+2=0$ *in which the points* (*i*) $(-1, 1)$ (*ii*) $(0, 0)$ lie.

If the co-ordinates $(-1, 1)$ be substituted in the left-hand sides of the equations of the lines, we get 9 and -15 which have opposite signs. Thus the perpendiculars from any point (x, y) of the bisectors on the lines are equal and have the signs positive and negative respectively. Hence the equation of the required bisector is

$$\frac{4x+3y+10}{5}=\frac{-12x+5y-2}{13}$$

i.e., $\qquad 8x+y+10=0$

The equation of the bisector of the angle which contains the origin is

$$\frac{4x+3y+10}{5}=\frac{12x-5y+2}{13}$$

i.e., $\qquad x-8y-15=0.$

(2) *Find the bisector of the obtuse angle between the lines* $3x+2y+2=0$ and $18x-y-1=0$

The value of the expression $n_1n_2(l_1l_2+m_1m_2)$ is $-2(54-2)$ which is negative, hence the origin lies in the acute angle between the lines. Thus it is required to find the bisector of the angle in which the *origin does not lie*, and the equation of the required bisector is

$$\frac{3x+2y+2}{\sqrt{13}}=\frac{18x-y-1}{5\sqrt{13}}$$

i.e., $\qquad 3x-11y-11=0.$

Second Method. The equations of the two bisectors are

$$3x - 11y - 11 = 0 \qquad \ldots\ldots(i)$$
$$11x + 3y + 3 = 0 \qquad \ldots\ldots(ii)$$

The angle between the bisector (i) and any one of the given lines is $\tan^{-1}3$. Hence (i) is equation of the bisector of the obtuse angle and (ii) that of acute angle.

Third Method. Take any point on any one of the lines *e.g.*, $(-2, 2)$ is on the first line. The lengths of the perpendiculars from this on the two bisectors are respectively $\dfrac{39}{\sqrt{130}}$ and $\dfrac{13}{\sqrt{130}}$.

The first perpendicular being greater than the second, equation (i) represents the bisector of the obtuse angle and equation (ii) that of the acute angle.

Remark. The first method has the advantage that we actually find *one bisector* while in second and third methods, both the bisectors have to be found and then they are distinguished.

(3) *Find the co-ordinates of the incentre of the triangle formed by the lines* $x+y+11=0$, $x-y+1=0$, $x-7y+7=0$. [*Math. Trip. I 1919*].

The equations of the sides being

$$x + y + 11 = 0 \qquad \ldots\ldots(i)$$
$$x - y + 1 = 0 \qquad \ldots\ldots(ii)$$
$$x - 7y + 7 = 0, \qquad \ldots\ldots(iii)$$

the values of N_1, N_2, N_3 are respectively $-6, 8, -2$ which are not of the same sign. Change the sign of equation (ii) which becomes

$$-x + y - 1 = 0.$$

The new values of N_1, N_2, N_3 are 6, 8, 2, which have the same sign. The co-ordinates of the in-centre are, therefore, given by the equations

$$\frac{x+y+11}{\sqrt{2}} = \frac{-x+y-1}{\sqrt{2}} = \frac{x-7y+7}{\sqrt{50}}$$

or $$x+y+11 = -x+y-1 = \frac{x-7y+7}{5}.$$

$\therefore$ $$x = -6, \; y = -2$$

are the co-ordinates of the in-centre.

omit

27. Points at infinity. Cartesian Homogeneous Equations.

Let the equations of two lines be

$$U \equiv a_1x + b_1y + c_1 = 0$$
$$V \equiv a_2x + b_2y + c_2 = 0$$

If $a_1b_2-a_2b_1=0$ the two lines are parallel, and no finite set of values of x and y satisfy the equations. The lines, therefore have no finite point of intersection, unless they coincide.

Let $\dfrac{a_1}{a_2}=\dfrac{b_1}{b_2}=k$, the equation of the line V can then be written in the form

$$V\equiv a_1x+b_1y+kc_2=0 \qquad kc_2\neq c_1$$

Displace this line a little and consider the equation

$$W\equiv(a_1+\epsilon)x+(b_1+\epsilon)y+kc_2=0.$$

The lines U and W intersect at the point given by the equations

$$x=\frac{kb_1c_2-c_1(b_1+\epsilon)}{(a_1-b_1)\epsilon},\ y=\frac{c_1(a_1+\epsilon)-ka_1c_2}{(a_1-b_1)\epsilon}.$$

As $\epsilon\rightarrow 0$, the line W approaches V and each co-ordinate of the point of intersection tends to infinity. We may, therefore, say that parallel lines meet at infinity.

The infinite co-ordinates can be avoided, if instead of two co-ordinates x and y we use the ratio of three co-ordinates $x:y:z$. These are called homogeneous co-ordinates of a point. The homogeneous co-ordinates $x:y:z$ of a point are defined by the equations

$$X=\frac{x}{z},\quad Y=\frac{y}{z},$$

where (X, Y) are the non-homogeneous co-ordinates of the same point. From the definition, it follows, that x, y, z can be replaced by kx, ky, kz, where k is any number different from zero, *e.g.*, the point $\left(\frac{3}{4},\ \frac{5}{6}\right)$ *i.e.*, $\left(\frac{9}{12},\ \frac{10}{12}\right)$ can be written as $(9:10:12)$ or $(18:20:24)$ or $(3:\frac{10}{3}:4)$ etc.

With this notation the co-ordinates of the point of intersection of U and W will be written as

$$x:y:z=kb_1c_2-c_1(b_1+\epsilon):c_1(a_1+\epsilon)-ka_1c_2:\epsilon(a_1-b_1)$$

As $\epsilon\rightarrow 0$, the point recedes to infinity and its co-ordinates can be written in the form

$$x:y:z=b_1(kc_2-c_1):a_1(c_1-kc_2):0$$

or $\quad x:y:z=b_1:-a_1:0.$

Thus $(b_1:-a_1:0)$ is the point at infinity on U or V. It is in fact on every line parallel to U. Thus all parallel lines have the same point at infinity.

Note (*i*) There is no point $(0:0:0)$.

Note (*ii*) The homogeneous equation of the st. line $ax+by+c=0$ is obtained by writing $\frac{x}{z}$, $\frac{y}{z}$ in place of x and y, and is therefore $ax+by+cz=0$. Thus *every homogenous equation of the first degree in* x, y, z *represents a st. line and conversely.*

27.1. To interpret the equation $z=o$. Line at infinity.

The equation $z=0$ is the limiting form of the equation $ax+by+cz=0$ when $a:c$ and $b:c$ both tend to zero. We may, therefore, legitimately call the equation $z=0$ as the equation of a st. line.

The equation $z=0$ *i.e.*, $ox+oy+z=0$ is not satisfied by finite values of x, y, z unless $z=0$ *i.e.*, the line does not contain any finite point. Thus every point on the line is at infinity.

Conversely, every point at infinity is on this line. For, let P be a point at infinity and A and B two points in the finite part of the plane. The lines AP, BP meet at infinity, and are therefore parallel. Hence their equations are of the form

$$a_1x+b_1y+c_1z=0,\ a_1x+b_1y+c_2z=0,\ c_1 \neq c_2.$$

These equations show, that these two lines and the line $z=0$ meet at the same point *i.e.*, P which is at infinity.

Hence the line $z=0$ embraces all the points at infinity. We shall call this line *the line at infinity*.

The line at infinity has evidently an indeterminate direction.

The existence of the line at infinity may be inferred from the following consideration.

Since there is only one point at infinity on a real st. line, whatever be the locus of the points at infinity in a plane, every st. line cuts this locus in one and only one point. Hence the locus is of the first degree and is therefore a st. line.

We may note here that the points at infinity defined by the homogeneous system of co-ordinates are points precisely in the same sense as (x, y, z) $z \neq 0$ is a point and the line at infinity $z=0$ is a line precisely in the same sense as $ax+by+cz=0$ is a line.

Examples V

1. Find the bisectors of the angles between the following pairs of lines

(*i*)	$3x-4y+13=0$	$12x+5y-32=0$
(*ii*)	$12x+5y=4$	$24y-7x=9$
(*iii*)	$x-2y+1=0$	$x-3y+3=0$

Point out in each case the bisector of the angles which contain (*i*) the origin (*ii*) the point (2, 2).

2. Find the bisectors of the obtuse angles between the following pairs of lines.

(*i*) $12x+5y-4=0$ $3x+4y+7=0$
(*ii*) $4x+3y-7=0$ $24x+7y-31=0$

3. Find the bisectors of acute angles between the pairs of lines :—

(*i*) $3x+4y-9=0$ $12x+5y-3=0$
(*ii*) $4x-3y-5=0$ $12x+5y-3=0$
(*iii*) $x+y+2=0$ $x-7y-2=0.$

4. Find if the point (1, 1) lies in the acute or obtuse angle between the following pairs of lines :—

(*i*) $13x+15y=26$, $13x+25y=52$
(*ii*) $2x+5y+3=0$, $3x-y+4=0.$

5. The sides BC, CA, AB of a triangle are given by the equations $x-5y+7=0$, $x-2y+1=0$, $5x-13y-1=0$, find the regions in which the following points lie.

(*i*) (2, 1), (*ii*) (3, 1), (*iii*) (−1, 1), (*iv*) (−3, 1).

6. Find the in-centres of the triangles formed by the following trios of lines

(*i*) $5x-12y=0$, $5x+12y+60=0$, $12x-5y-60=0$
(*ii*) $4x-3y-12=0$, $5x-12y-4=0$, $12x-5y-13=0$
(*iii*) $3x+4y-1=0$, $12x+5y-8=0$, $4x-3y+2=0$
(Math. Trip 1921)
(*iv*) $11x+2y=13$, $22x-19y=3$, $x-2y-119=0.$

7. Sketch roughly the lines

$$x+y=4,\ x-y=2,\ 17x+7y=28.$$

Show that the point $(-4\frac{1}{2}, 1)$ is the centre of a circle which touches the three lines, and find the co-ordinates of the centre of the circle inscribed in the triangle formed. (Math. Trip 1915)

Miscellaneous Examples VI

1. Show that the equation of the line that joins the points (r_1, θ_1), (r_2, θ_2) is

$$\frac{1}{r}\sin(\theta_1-\theta_2) = \frac{1}{r_1}\sin(\theta-\theta_2)+\frac{1}{r_2}\sin(\theta-\theta_1).$$

2. A triangle is formed by the three lines

$$U \equiv 8x+y-7=0,\ V \equiv 4x-7y=0,\ W \equiv 5x-3y-8=0,$$

prove that $U-kV=0$ (*i*) passes through its orthocentre if $41k=37$, (*ii*) bisects one of its angles if $k=1$, (*iii*) passes through the circumcentre if $37k=41$. [Math Trip I 1916]

3. The equation of the bisector of the angle between two lines is $7x-4y+1=0$. The equation of one of the lines is $3x+4y=11$, find the equation of the other line.
[Math. Trip 1913]

4. One side of a square of length p has one of its extremities at the origin and is inclined to the x-axis at an angle α, prove that the equations of its diagonals are

$$y(\cos\alpha-\sin\alpha)=x(\cos\alpha+\sin\alpha)$$
$$y(\cos\alpha+\sin\alpha)+x(\cos\alpha-\sin\alpha)=p.$$

5. The sides of a quadrilateral are given by the equations $U_1\equiv x+y-2=0$, $U_2\equiv x-y+6=0$, $U_3\equiv 2x-y+3=0$, $U_4\equiv x-3y+2=0$. Show that $11U_1+8U_2-12U_3+5U_4\equiv0$, find the equations of the three diagonals.

6. Find the area of the parallelogram formed by the lines $5y=12x-7a$, $5y=12x-126a$, $12y=5x+7a$, $12y=5x+126a$.

7. If (a, b), (c, d) are opposite vertices of a parallelogram, and (c, b) is a third vertex, find the co-ordinates of the fourth vertex.

8. Show that the area of the triangle formed by the st. lines $y=x\tan\alpha$, $y=x\tan\beta$, $y=x\tan\gamma+c$ is

$$\frac{c^2}{2}\frac{\sin(\alpha-\beta)\cos^2\gamma}{\sin(\alpha-\gamma)\sin(\beta-\gamma)}.$$

9. If the lines $y=x\tan\dfrac{11\pi}{24}$, $y=x\tan\dfrac{19\pi}{24}$ be at right angles, show that the angle between the axes is $\dfrac{\pi}{4}$.

10. If the axes be inclined at an angle $\dfrac{\pi}{6}$, show that the equations of the st. lines through the origin which with the line $x+y=3$ make a right angled isosceles triangle are $y+x\sqrt{3}=0$ $y\sqrt{3}+x=0$.

11. Show that the lines $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ will be equally inclined in the opposite direction if $\dfrac{m_1}{l_1}+\dfrac{m_2}{l_2}=2\cos\omega$, where ω is the angle between the axes.

12. Show that the reflection of the line $px+qy+r=0$ in the line $lx+my+n=0$ is

$$(px+qy+r)(l^2+m^2)-2(lp+mq)(lx+my+n)=0.$$

13. Two sides of a parallelogram are formed by the st. lines (*i*) $3x-4y=4$, (*ii*) $y=mx$ and the other two sides are formed by two right lines through the point $(5, -1)$ which are parallel to the lines (*i*) and (*ii*). Find the two values of m for which the area of the parallelogram is 12. [Math. Trip 1914.]

14. Show that the co-ordinates of the foot of the perpendicular from the point (h, k) on the line $lx+my+n=0$ is given by the equations

$$\frac{x-h}{l}=\frac{y-k}{m}=-\frac{lh+mk+n}{\sqrt{l^2+m^2}}.$$

15. The four lines

$$\left.\begin{array}{l} x-a=l_1r \\ y-b=m_1r \end{array}\right\} \quad \left.\begin{array}{l} x-a=l_2r \\ x-b=m_2r \end{array}\right\} \quad \left.\begin{array}{l} x-a'=l_1'r \\ y-b'=m_1'r \end{array}\right\} \quad \left.\begin{array}{l} x-a'=l_2'r \\ y-b'=m_2'r \end{array}\right\}$$

where $(l^2{}_k+m^2{}_k=1,\ l'^2{}_k+m'^2{}_k=1,\ k=1, 2)$ intersect each other in four points other than (a, b) and (a', b'). Show that a necessary condition that these four points should lie on a circle is

$$(l_1l'_1+m_1m_1')^2=(l_2l_2'+m_2m_2')^2. \qquad \text{[Math. Trip I, 1927)}$$

16. The equations of the sides BC, CA, AB of a triangle are $U_i \equiv a_ix+b_iy+c_i=0$ $i=1, 2, 3$. Show that the line $U_2-kU_3=0$ (*i*) passes through the orthocentre of ABC if $(a_1a_3+b_1b_3)k =(a_1a_2+b_1b_2)$, (*ii*) is parallel to BC if $(a_3b_1-a_1b_3)k+(a_1b_2-a_2b_1)=0$, (*iii*) passes through the mid-point of BC if $(a_3b_1-b_3a_1)k=(a_1b_2-a_2b_1)$.

17. The equations of the sides of a triangle are $U=0$, $V=0$, $W=0$, where $U=x\cos\alpha+y\sin\alpha-p$, $V=x\cos\beta+y\sin\beta-q$, $W=x\cos\gamma+y\sin\gamma-r$. Prove that the orthocentre of the triangle is given by the equations

$U\cos(\beta-\gamma)=V\cos(\gamma-\alpha)=W\cos(\alpha-\beta)$ [Math. Trip I 1920].

18. If the distances of a certain point from the lines

$$x\cos\alpha+y\sin\alpha-p=0,\quad x\cos\beta+y\sin\beta-q=0,$$

$x\cos\gamma+y\sin\gamma-r=0$, be d_1, d_2, d_3 respectively and if

$$\lambda_1=p+d_1,\ \lambda_2=q+d_2\ \alpha_3=r+d_3 \text{ prove that}$$

$$\lambda_1\sin(\beta-\gamma)+\lambda_2\sin(\gamma-\alpha)+\lambda_3\sin(\alpha-\beta)=0.$$

19. Show that the locus of the centroids of the triangles of which the three altitudes lie along the lines

$$y=m_ix=0 \text{ is } y(3+\Sigma m_2m_3)=x(\Sigma m_1+3m_1m_2m_3).$$

20. The sides of a parallelogram are $U=a_1$, $U=a_2$, $V=b_1$, $V=b_2$. Show that the diagonals are

$$\begin{vmatrix} U & V & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} U & V & 1 \\ a_1 & b_2 & 1 \\ a_2 & b_1 & 1 \end{vmatrix}=0 \text{ [Math. Trip 1928]}$$

21. Show that the co-ordinates of the point of intersection of the join of the points (x_1,y_1), (x_2,y_2) with the line $ax+by+c=0$ are

$$x=\frac{U_1x_2-U_2x_1}{U_1-U_2},\ y=\frac{U_1y_2-U_2y_1}{U_1-U_2}$$

where $U_1 \equiv ax_1+by_1+c$, $U_2=ax_2+by_2+c$.

22. Prove that the conditions that the line $lx+my+n=0$ may cut externally all the sides of the triangle formed by the lines $l_rx+m_ry+n_r=0$, $(r=1, 2, 3)$ are that all the expressions

$$\begin{vmatrix} l & m & n \\ l_r & m_r & n_r \\ l_s & m_s & n_s \end{vmatrix} \div \begin{vmatrix} l_r & m_r \\ l_s & m_s \end{vmatrix}$$

should have the same sign. [Math. Trip I 1929]

(*Hint.* All the three vertices lie on the same side of the line $lx+my+n=0$).

23. Show that the area of the triangle formed by the lines whose equations $a_sx+b_sy+c_s=0$. $(s=1, 2, 3)$ is $\Delta^2 \div 2C_1C_2C_3$

where $$\Delta \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \quad \text{and } C_i = \frac{\partial \Delta}{\partial c_i} \ (i=1, 2, 3)$$

24. Show that the area of the triangle formed by the lines $x \cos \alpha + y \sin \alpha = p$, $x \cos \beta + y \sin \beta = q$, $x \cos \gamma + y \sin \gamma = r$ is

$$\frac{[p \sin(\beta-\gamma)+q \sin(\gamma-\alpha)+r \sin(\alpha-\beta)]^2}{2 \sin(\beta-\gamma) \sin(\gamma-\alpha) \sin(\alpha-\beta)}.$$

25. Show that the area of the triangle formed by the lines $y=m_rx+c_r$, $r=1, 2, 3$ is

$$\tfrac{1}{2}\frac{(c_2-c_3)^2}{m_2-m_3}+\tfrac{1}{2}\frac{(c_3-c_1)^2}{m_3-m_1}+\tfrac{1}{2}\frac{(c_1-c_2)^2}{m_1-m_2}.$$

Note. The angle between the axes in each of the following questions is ω.

26. From a point P (h, k) are drawn perpendiculars to the axes. Prove that the length of the line that joins the feet of the perpendiculars is $\sin \omega \sqrt{h^2+k^2+2hk \cos \omega}$.

27. Perpendiculars PL, PM are drawn from P (h, k) on the axes OL, OM. Show that the length of the perpendicular from P on the line LM is $hk \sin \omega \div \sqrt{h+k^2+2hk \cos \omega}$ and that the equation of the perpendicular is

$$h(x-h)=k(y-k).$$

28. The vertices A, B, C of a triangle ABC lie on the three concurrent lines and the sides CA, CB pass through two fixed points, show that the line AB also passes through a fixed point. (Hint. Take two of the lines as axes).

29. A and B are points on the axes of x and y respectively, such that OA$=a$, OB$=b$, AC$=c$. Prove that the equation of the line that joins the mid-point of AB with the in-centre of triangle OAB is

$$b(b+c-a)\,x-a(c+a-b)y+ab(a-b)=0.$$

CHAPTER IV

STRAIGHT LINES, TRANSFORMATION OF AXES, INVARIANTS

28. A homogeneous equation of the nth degree in x and y represents n st. lines through the origin.

Let the equation be

$$a_0y^n + a_1y^{n-1}x + a_2y^{n-2}x^2 + \ldots\ldots + a_{n-1}yx^{n-1} + a_nx^n = 0.$$

On dividing by x^n, the equation takes the form

$$a_0\left(\frac{y}{x}\right)^n + a_1\left(\frac{y}{x}\right)^{n-1} + \ldots\ldots + a_{n-1}\left(\frac{y}{x}\right) + a_n = 0.$$

It is an equation of the nth degree in $\frac{y}{x}$, and has, therefore, exactly n roots, say $m_1, m_2, \ldots\ldots m_n$. Hence the equation can be written as

$$a_0\left(\frac{y}{x} - m_1\right)\left(\frac{y}{x} - m_2\right)\ldots\ldots\left(\frac{y}{x} - m_n\right) = 0.$$

This equation is satisfied by all points which satisfy any one of the equations

$$y = m_1x,\ y = m_2x \ldots\ldots y = m_nx$$

and each one of these represents a st. line through the origin.

29. To find the angle between the lines represented by the equation

$$ax^2 + 2hxy + by^2 = 0$$

Let $by^2 + 2hxy + ax^2 \equiv b(y - m_1x)(y - m_2x)$.

$\therefore$ $b(m_1 + m_2) = -2h,\ bm_1m_2 = a.$

Suppose θ is the angle between the lines.

(*i*) *If the axes are rectangular.*

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1m_2} = \frac{[(m_1 + m_2)^2 - 4m_1m_2]^{\frac{1}{2}}}{1 + m_1m_2}$$

$$= \pm\frac{2\sqrt{h^2 - ab}}{a + b} \qquad \ldots\ldots(1)$$

(*ii*) *If the axes are inclined at an angle* ω,

$$\tan\theta = \frac{(m_1 - m_2)\sin\omega}{1 + (m_1 + m_2)\cos\omega + m_1m_2}$$

$$= \pm\frac{[(m_1 + m_2)^2 - 4m_1m_2]^{\frac{1}{2}}}{1 + (m_1 + m_2)\cos\omega + m_1m_2}\sin\omega$$

$$= \pm\frac{2\sqrt{h^2 - ab}}{a + b - 2h\cos\omega}\sin\omega \qquad \ldots\ldots(2)$$

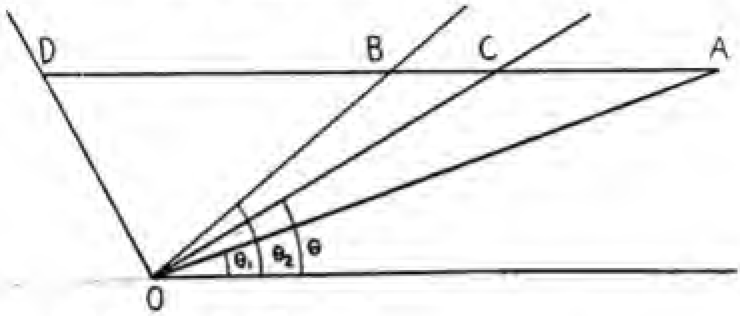
If the lines coincide, $\tan\theta=0$, $\therefore\ h^2=ab$ ......(3)

If the lines be at right angles, $\tan\theta\to\infty$

$\therefore\ a+b=0$ (rectangular axes) ......(4)

and $a+b-2h\cos\omega=0$ (oblique axes)

30. To find the bisectors of the angles between the st. lines $ax^2+2hxy+by^2=0$



Let $by^2+2hxy+ax^2$
$\equiv b(y-m_1x)(y-m_2x)$

If θ_1, θ_2 be the angles which OA, OB make with OX and θ the angle that any one of the bisectors (OC, OD) makes with OX, then

$$\theta=\frac{\theta_1+\theta_2}{2}\text{ or }\frac{\theta_1+\theta_2}{2}\pm\frac{\pi}{2}.$$

In either case $\tan 2\theta=\tan(\theta_1+\theta_2)$,

or $$\frac{2\tan\theta}{1-\tan^2\theta}=\frac{\tan\theta_1+\tan\theta_2}{1-\tan\theta_1\tan\theta_2} \quad \text{......(A)}$$

$$=\frac{2h}{a-b}.$$

If (x, y) be any point on either bisector, $\frac{y}{x}=\tan\theta$,

hence the joint equation of the bisectors is

$$\frac{2xy}{x^2-y^2}=\frac{2h}{a-b}$$

or $$\frac{x^2-y^2}{a-b}=\frac{xy}{h} \quad \text{......(5)}$$

30.1. If the axes be oblique and inclined at an angle ω,

$$\tan\theta_1=\frac{m_1\sin\omega}{1+m_1\cos\omega},\ \tan\theta_2=\frac{m_2\sin\omega}{1+m_2\cos\omega}$$

and the equation of one of the bisectors which makes an angle θ with the x-axis is

$$y=x\frac{\sin\theta}{\sin(\omega-\theta)}$$

$$\therefore \qquad \tan\theta = \frac{y\sin\omega}{x+y\cos\omega}.$$

Hence substituting in relation ......(A)

$$\frac{2y(x+y\cos\omega)}{(x+y\cos\omega)^2-y^2\sin^2\omega} = \frac{(m_1+m_2)+2m_1m_2\cos\omega}{1+(m_1+m_2)\cos\omega+m_1m_2\cos 2\omega}$$

$$\therefore \quad \frac{2y(x+y\cos\omega)}{x^2-y^2} = \frac{(m_1+m_2)+2m_1m_2\cos\omega}{1-m_1m_2}$$

$$= \frac{2a\cos\omega-2h}{b-a}$$

or $$\frac{y(x+y\cos\omega)}{x^2-y^2} = \frac{a\cos\omega-h}{b-a}$$

i.e. $h(x^2-y^2)+(b-a)xy=(ax^2-by^2)\cos\omega$ ......(6)

Second Method. Let the equation of the bisectors OC, OD be

$$Ax^2+2Hxy+By^2=0 \qquad ......(i)$$

Let an arbitrary line $y=\lambda$ meet the given lines OA and OB in A and B and the bisectors in C and D. The points C and D then divide internally and externally the base AB of the triangle OAB in the ratio of the sides. Consequently (AB, CD) $=-1$. The abscissae of the points A, B, C, D are given by the equations

$$ax^2+2hx\lambda+b\lambda^2=0$$
$$Ax^2+2Hx\lambda+B\lambda^2=0$$

$$\therefore \qquad Ab-2Hb+B=0 \qquad ......(ii)$$

Since the bisectors OC, OD are at right angles

$$\therefore \qquad A-2H\cos\omega+B=0 \qquad ......(iii)$$

The elimination of A, H, B gives the required equation, viz.,

$$\begin{vmatrix} x^2 & -xy & y^2 \\ b & h & a \\ 1 & \cos\omega & 1 \end{vmatrix} = 0$$

30.2. Harmonic Pencil.

Let OA, OC, OB, OD be four concurrent lines which are met by a fifth line in points A, C, B, D, if (AB, CD) $=-1$, the lines are said to form a harmonic pencil and this is asymbolically expressed by the relation

$$O(AB, CD) = -1$$

The point D is called the vertex of the pencil and OC, OD are harmonic conjugates *w.r.* to OA OB. The two line-pairs are said to be *apolar*.

Let O be the origin and suppose that the line pairs OA, OB ; OC, OD are represented by the equations

$$ax^2+2hxy+by^2=0 \qquad a'x^2+2h'xy+b'y^2=0.$$

Let the co-ordinates of A and B be respectively (x_1, y_1) and (x_2, y_2), then equations $ax^2+2hxy+by^2=0$ and $(xy_1-yx_1)(xy_2-yx_2)=0$ are identical.

$$\therefore \quad \frac{y_1y_2}{a}=\frac{x_1y_2+x_2y_1}{-2h}=\frac{x_1x_2}{b}.$$

The co-ordinates of an arbitrary point on AB are

$$\left(\frac{x_1+\lambda x_2}{1+\lambda},\ \frac{y_1+\lambda y_2}{1+\lambda}\right)$$

This will be the point C or D if

$$\lambda^2(a'x_2^2+2h'x_2y_2+b'y_2^2)+2\lambda[a'x_1x_2+h'(x_1y_2+x_2y_1)+b'y_1y_2] \\ +(a'x_1^2+2h'x_1y_1+b'y_1^2)=0$$

This quadratic gives the ratios in which C and D divide the segment AB. Since (AB, CD) = − 1

$$a'x_1x_2+h'(x_1y_2+x_2y_1)+b'y_1y_2=0$$

i.e. $\qquad ab'+a'b=2hh'$

Note.—If a, h, b are real, the discriminant of (5) is easily seen to be positive. The discriminant of (6) is $(b-a)^2+4(h-a\cos\omega)(h-b\cos\omega)$

or $\quad [(a+b)\cos\omega-2h]^2+(a-b)^2\sin^2\omega$,

which is also positive. It is, therefore, remarkable that the lines given by equations (5) and (6) are always real even if the lines $ax^2+2hxy+by^2=0$ be imaginary. This leads to the curious result that a pair of imaginary lines given by a real equation has a pair of real lines bisecting the angles between them. "It is the existence of such relations between real and imaginary lines which makes the consideration of the latter profitable." (Salmon).

31. Notation. Unless otherwise stated, the following notations will be used in this book.

The general equation of the second degree will be taken to be

$$ax^2+2hxy+by^2+2fy+2gx+c=0.$$

The left-hand member of the equation will be denoted by $\phi(x, y)$ or simply ϕ. The expression ϕ_1 will stand for the value of ϕ for $x=x_1$, $y=y_1$. A similar significance is attached to ϕ', and so on. We also set

$$X \equiv \tfrac{1}{2}\frac{\partial\phi}{\partial x}=ax+hy+g, \quad Y\equiv\tfrac{1}{2}\frac{\partial\phi}{\partial y}=hx+by+f$$

$$Z\equiv gx+fy+c.$$

and X_0, Y_0, Z_0 will mean the value of X, Y, Z for $x=x_0$, $y=y_0$. The determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \equiv abc+2fgh-af^2-bg^2-ch^2$$

will be denoted by Δ and is called the discriminant of ϕ. The cofactors of the small letters will be denoted by the corresponding capital letters. It can be easily verified that

$$BC-F^2 \equiv a\Delta \qquad AC-G^2 \equiv b\Delta \qquad AB-H^2 \equiv c\Delta$$

$$GH-AF \equiv f\Delta \quad HF-BG \equiv g\Delta \qquad FG-CH = h\Delta$$

In particular if $\Delta=0$,

$$A:H:G=H:B:F=G:F:C.$$

31.1. If the equation $\phi=0$ be written in the homogeneous form

$$\phi(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

then we set

$$X \equiv \tfrac{1}{2}\frac{\partial \phi}{\partial x}=ax+hy+gz, \quad Y \equiv \tfrac{1}{2}\frac{\partial \phi}{\partial y}=hx+by+fz$$

$$Z \equiv \tfrac{1}{2}\frac{\partial \phi}{\partial z} \equiv gx+fy+cz,$$

and X_0, Y_0, Z_0 will be the values of X, Y, Z for $x=x_0$, $y=y_0$, $z=z_0$.

It can easily be verified that

$$\phi \equiv xX+yY+zZ.$$

We also put $T \equiv x_1X+y_1Y+z_1Z$
$\equiv xX_1+yY_1+zZ_1$

32. *The necessary and sufficient condition that a general equation of the second degree in x and y may represent a pair of st. lines is that its discriminant should be zero.*

Necessity. Let the general second degree equation be

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

If it represents a pair of st. lines, it is identical with

$$UV \equiv (lx+my+n)(l'x+m'y+n')=0.$$

Equating the co-efficients of the above two equations it is found that

$$ll'=a,\ mm'=b,\ nn'=c,\ mn'+m'n=2f$$
$$nl'+n'l=2g,\ lm'+l'm=2h.$$

Now the lines given by the equations

$$l'U+lV\equiv 2(ax+hy+g)=0$$
$$m'U+mV\equiv 2(hx+by+f)=0$$
$$n'U+nV\equiv 2(gx+fy+c)=0$$

all meet at the intersection of U and V. Hence

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

Sufficiency. Suppose that a and b do not vanish simultaneously. Suppose $a\neq 0$, the given equation can be written as

$$a^2x^2+2x\,(hy+g)\;a+a(by^2+2fy+c)=0$$

or $\quad (ax+hy+g)^2=-Cy^2+2Fy-B$
$$=(y\sqrt{-C}+\sqrt{-B})^2$$

$\because\quad BC-F^2\equiv a\Delta=0$ as $\Delta=0$. Thus
$$ax+hy+g=\pm(y\sqrt{-C}+\sqrt{-B})$$
which represents two st. lines.

If $a=0$, $b=0$, $h\neq 0$. Δ reduces to
$$2fgh-ch^2=0$$

The equation of the locus takes the form
$$h^2xy+ghx+fhy+fg=0$$
or $\quad (hx+f)(hy+g)=0$

and this represents a pair of st. lines.

Remarks. 1. The elimination of l, m, n, l', m', n' in the first part can be effected by the identity

$$0=\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}\begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}=8\Delta$$

2. If the equation $\phi=0$ represents a pair of st. lines, the point of their intersection is the same as those of the lines $l'U+lV=0$ and $m'U+mV=0$ *i.e.* of the lines

$$ax+hy+g=0,$$
$$hx+by+f=0.$$

$$\frac{x}{A}=\frac{y}{F}=\frac{1}{C}$$

$$\therefore\quad \left.\begin{aligned} x&=\frac{G}{C}=\sqrt{\frac{A}{C}} \\ y&=\frac{F}{C}=\sqrt{\frac{B}{C}} \end{aligned}\right\} \qquad \ldots\ldots(7)$$

3. The lines $(lx+my)(l'x+m'y)=0$ are parallel to UV through the origin. *i.e.* $ax^2+2hxy+by^2=0$ represents a pair of st. lines through the origin parallel to $\phi=0$ when ϕ represents a pair of st. lines.

33. *To find the equation to the two lines joining the origin to the points in which the st. line*

$$U \equiv lx+my+n=0$$

meets the locus given by the equation

$$\phi(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The co-ordinates of the points of intersection of U and ϕ satisfy the equation

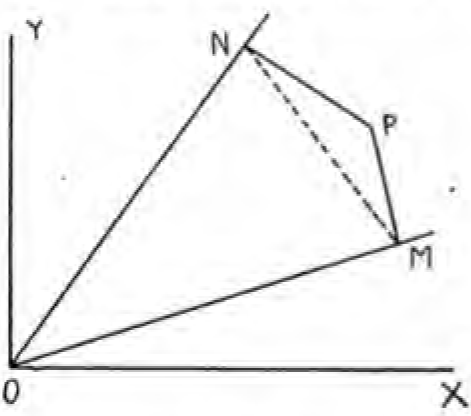
$$\psi(x, y) \equiv ax^2+2hx+by^2-2\frac{(gx+fy)(lx+my)}{n}+c\frac{(lx+my)^2}{n^2}=0.$$

It is also a homogeneous equation of the second degree in x and y. Hence $\psi(x, y)=0$ represents the required pair of st. lines.

Illustrative Examples.

(1) *From a point* $P(x_0, y_0)$ *perpendiculars* PM, PN *are drawn on the st. lines* $ax^2+2hxy+by^2=0$. *Show that if* O *is the origin, the area of the triangle* OMN *is*

$$(ay_0^2-2hx_0y_0+bx_0^2)(h^2-ab)^{\frac{1}{2}} \div \{(a-b)^2+4h^2\}$$



Let $by^2+2hxy+ax^2$

$$\equiv b(y-mx)(y-m'x)$$

$$\therefore \quad b(m+m')=-2h,$$

$$bmm'=a.$$

If OM and ON be represented by the equations $y=mx$, $y=m'x$, the equations of PM and PN are respectively

$$m(y-y_0)+(x-x_0)=0$$

$$m'(y-y_0)+(x-x_0)=0.$$

$\because$ M is $\left(\dfrac{x_0+my_0}{1+m^2}, \dfrac{m(n_0+my_0)}{1+m^2}\right)$ and N is

$$\left(\frac{x_0+m'y_0}{1+m'^2}, \frac{m'(n_0+m'y_0)}{1+m'^2}\right)$$

$$\therefore \quad \triangle OMN=\frac{(x_0+my_0)(x_0+m'y_0)(m-m')}{(1+m^2)(1+m'^2)}$$

$$=\frac{(ay_0^2-2hx_0y_0+bx_0^2)\sqrt{h^2-ab}}{(a-b)^2+4h^2}.$$

Or the measures of the perpendiculars OM, ON on PM, PN are

$$-\frac{my_0+x_0}{\sqrt{1+m^2}} \text{ and } -\frac{m'y_0+x_0}{\sqrt{1+m'^2}}.$$

$$\therefore \quad \text{OM.ON} = \frac{(my_0+x_0)(m'y_0+x_0)}{\sqrt{(1+m^2)(1+m'^2)}}$$

$$= \frac{mm'y_0^2+x_0y_0(m+m')+x_0^2}{\sqrt{1+(m^2+m'^2)+m^2m'^2}}$$

$$= \frac{ay_0^2-2hx_0y_0+bx_0^2}{\sqrt{(a-b)^2+4h^2}}.$$

If θ be the angle MON, $\tan\theta = \dfrac{2(h^2-ab)^{\frac{1}{2}}}{a+b}$

$$\therefore \quad \sin\theta = 2(h^2-ab)^{\frac{1}{2}} \div \{(a-b)^2+4h^2\}^{\frac{1}{2}}$$

Hence $\triangle\text{MON} = \frac{1}{2}\,\text{OM.ON}\sin\theta$

$$= (ay_0^2-2hx_0y_0+ax_0^2)(h^2-ab)^{\frac{1}{2}} \div \{(a-b)^2+4b^2\}.$$

(2) *Show that the conditions that the st. lines* $px^2-2qxy+ry^2=0$ *should form an equilateral triangle with* $x\cos\theta+y\sin\theta=k$ *are*

$$\frac{p}{1-2\cos 2\theta} = \frac{q}{2\sin 2\theta} = \frac{r}{1+2\cos 2\theta}.$$

The line $x\cos\theta+y\sin\theta=k$ is inclined to the x-axis at an angle $\left(\theta+\frac{\pi}{2}\right)$. The other two lines, therefore, make with the x-axis angles $\theta+\frac{\pi}{2}+\frac{2\pi}{3}$, $\theta+\frac{\pi}{2}+\frac{4\pi}{3}$. Hence

$$ry^2-2qxy+px^2 \equiv r\left[y-x\tan\left(\theta+\frac{7\pi}{6}\right)\right]\left[y-x\tan\left(\theta+\frac{11\pi}{6}\right)\right]$$

$$\therefore \quad r\left[\tan\left(\theta+\frac{7\pi}{6}\right)+\tan\left(\theta+\frac{11\pi}{6}\right)\right]=2q$$

$$r.\tan\left(\theta+\frac{7\pi}{6}\right)\tan\left(\theta+\frac{11\pi}{6}\right)=p$$

$$\therefore \quad \frac{p}{\sin\left(\theta+\frac{7\pi}{6}\right)\sin\left(\theta+\frac{11\pi}{6}\right)}$$

$$= \frac{2q}{\sin\left(\theta+\frac{7\pi}{6}\right)\cos\left(\theta+\frac{11\pi}{6}\right)+\cos\left(\theta+\frac{7\pi}{6}\right)\sin\left(\theta+\frac{11\pi}{6}\right)}$$

$$= \frac{r}{\cos\left(\theta+\frac{7\pi}{6}\right)\cos\left(\theta+\frac{11\pi}{6}\right)}$$

or $$\frac{p}{\cos\frac{2\pi}{3}-\cos(2\theta+3\pi)}=\frac{q}{\sin(2\theta+3\pi)}$$

$$=\frac{r}{\cos(2\theta+3\pi)+\cos\frac{2\pi}{3}}$$

i.e., $$\frac{p}{1-2\cos 2\theta}=\frac{q}{2\sin 2\theta}=\frac{r}{1+2\cos 2\theta}.$$

(3) *If the same st. line occurs in each of the two pairs* $ax^2+2hxy+by^2=0$, $a'x^2+2h'xy+b'y^2=0$, *and* θ *is the angle between the other two, then*

$$\pm 2\cot\theta=\frac{aa'}{ha'-h'a}+\frac{bb'}{h'b-hb'}.$$

Let $$by^2+2hxy+ax^2\equiv(y-mx)(y-m_1x)$$
$$b'y^2+2h'xy+a'x^2\equiv(y-mx)(y-m_2x)$$

$\therefore$ $$\left.\begin{aligned}b(m+m_1)&=-2h\\ b'(m+m_2)&=-2h'\end{aligned}\right\}\ldots(i)\qquad \left.\begin{aligned}bmm_1&=a\\ b'mm_2&=a'\end{aligned}\right\}\qquad\ldots(ii)$$

and $$\pm\cot\theta=\frac{1+m_1m_2}{m_1-m_2}.\qquad\ldots\ldots(iii)$$

Hence $$bb'(m_1-m_2)=-2(hb'-h'b)\qquad\ldots\ldots(iv)$$

Since m is the slope of the common line,

$$\begin{aligned}bm^2+2hm+a&=0\\ b'm^2+2h'm+a'&=0\\ bb'm^2m_1m_2\qquad\qquad -aa'&=0,\end{aligned}\qquad\text{from }(ii)$$

$$\therefore\quad\begin{vmatrix}b & h & a\\ b' & h' & a'\\ bb'm_1m_2 & 0 & -aa'\end{vmatrix}=0$$

or $$bb'm_1m_2(ha'-h'a)=aa'(bh'-b'h)\qquad\ldots\ldots(v)$$

Hence from (*iii*)

$$\pm 2\cot\theta=\frac{2}{m_1-m_2}+\frac{2m_1m_2}{m_1-m_2}$$

$$=\frac{bb'}{h'b-hb'}+\frac{aa'}{ha'-h'a}$$

Cor. If $\theta=\frac{\pi}{2}$, relation (*ii*) gives $m^2=-aa'\div bb'$.

Whence from $bm^2+2hm+a=0$, $b'm^2+2h'm+a'=0$, the following relations are obtained :

$$\frac{ha'b'}{b'-a'}=\frac{h'ab}{b-a}=\sqrt{-aa'bb'}.$$

(4) *Show that the equation*

$$(ab-h^2)\ (ax^2+by^2+2hxy+2gx+2fy)+af^2+bg^2-2fgh=0$$

represents a pair of st. lines and they form a rhombus with $ax^2+2hxy+by^2=0$ *if* $(a-b)fg+h(f^2-g^2)=0$.

It can easily be verified that the equation

$$(ab-h^2)(ax^2+by^2+2hxy+2gx+2fy)+af^2+bg^2-2fgh=0$$

represents a pair of st. lines and can be written in the form

$$ax^2+by^2+2hxy+2gx+2fy+c=0 \text{ where } c=\frac{af^2+bg^2-2fgh}{ab-h^2}.$$

The point of intersection of these two lines is $\left(\sqrt{\frac{A}{C}}, \sqrt{\frac{B}{C}}\right)$.

Also the origin is one of the vertices. Hence the line $\frac{y}{x}=\sqrt{\frac{B}{A}}$ is one of the diagonals and therefore bisects the angle between the lines $ax^2+2hxy+by^2=0$. The equation of the other bisector is $\frac{y}{x}=-\sqrt{\frac{A}{B}}$. Thus the joint equation of the two bisectors is

$$(y\sqrt{A}-x\sqrt{B})\ (y\sqrt{B}+x\sqrt{A})=0$$

or $$(x^2-y^2)\sqrt{AB}-xy(A-B)=0$$

or $$H(x^2-y^2)-xy(A-B)=0 \qquad \because AB=H^2.$$

But the equation of the bisectors of the angles between the lines $ax^2+2hxy+by^2=0$ is

$$h(x^2-y^2)-(a-b)xy=0.$$

Hence identifying the two equations of the bisectors, we get

$$\frac{H}{h}=\frac{A-B}{a-b}$$

$$\therefore \qquad (a-b)(fg-ch)=h(bc-f^2-ac+g^2)$$

or $$(a-b)fg=h(g^2-f^2).$$

The condition may be written as

$$\frac{a}{h}+\frac{f}{g}=\frac{b}{h}+\frac{g}{f}.$$

read **34**. *If the equation*

$$\phi(x,y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

represents a pair of st. lines, this equation can be written in the form

$$bX^2-2hXY+aY^2=0.$$

and the equation of the bisectors between these lines can be put in the form

$$h(X^2 - Y^2) - XY(a-b) = 0,$$

where $X \equiv ax + hy + g$ and $Y = hx + by + f$.

It has been proved (Art. 32) that if $\phi \equiv UV$, the lines $X=0$, $Y=0$ pass through the intersection of U and V. Consequently, the equation of a line through the intersection of U and V can be written in the form

$$Y = \lambda X \qquad \ldots\ldots(i)$$

The equation $ax^2 + 2hxy + by^2 = 0$ represents lines parallel to $\phi = 0$ *i.e.* $UV = 0$, hence their slopes are given by the equation

$$bm^2 + 2hm + a = 0 \qquad \ldots\ldots(ii)$$

The line $Y = \lambda X$ will be the line U or V if its slope $(h - \lambda a) \div (\lambda h - b)$ satisfies equation *(ii)* *i.e.* if

$$a\lambda^2 - 2h\lambda + b = 0.$$

The elimination of λ between this equation and *(i)* gives, therefore, the equation of the lines U and V in the form

$$bY^2 - 2hXY + aX^2 = 0 \qquad \ldots\ldots(6)$$

The bisectors of the angles between the lines U and V are parallel to the lines

$$h(y^2 - x^2) + (a-b)xy = 0$$

which equation represents the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$.

The slopes of these bisectors are given by the equation

$$h(m^2 - 1) + (a-b)m = 0 \qquad \ldots\ldots(iii)$$

The line *(i)* will coincide with one of the bisectors of angles between $\phi = 0$ if its slope satisfies equation *(iii)*

i.e. $$h(\lambda^2 - 1) + (a-b)\lambda = 0.$$

Eliminating λ, we get the equation of the bisectors, *i.e.*

$$h(Y^2 - X^2) + (a-b)XY = 0 \qquad \ldots\ldots(7)$$

34.1. *Find the intercept made by the lines* $\phi(x,y) = 0$ *on the line* $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m}$. *Deduce the equation of the line if the mid-point of the intercept is the point* (x_0, y_0). *Deduce also the equation of the bisectors of the angles between the lines* ϕ.

The equation of the line can be written as

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{\sqrt{(x - x_0)^2 + (y - y_0)^2}}{\sqrt{l^2 + m^2}} = R.\,\rho. \qquad \ldots\ldots(i)$$

where $\rho\sqrt{l^2 + m^2} = 1$ and R is the distance of an arbitrary point (x, y) from the point (x_0, y_0). The co-ordinates of

any point on the line are $(l\rho R + x_0, m\rho R + y_0)$ and at the point of intersection of this line with $\phi = 0$, we shall have

$$\phi(l\rho R + x_0, m\rho R + y_0) = 0$$

or $R^2\rho^2(al^2 + 2hlm + bm^2) + 2R\rho[lX_0 + mY_0] + \phi(x_0, y_0) = 0$ ...(ii)

The roots of this equation give the distances of (x_0, y_0) from the points where the line (i) meets the lines $\phi = 0$. If R_1, R_2 be the two values of R, the required intercept is

$$R_1 \sim R_2 = \sqrt{(R_1 + R_2)^2 - 4R_1R_2}$$

$$= 2\left[(lX_0 + mY_0)^2 - (al^2 + 2hlm + bm^2) \times \phi(x_0, y_0)\right]^{\frac{1}{2}}$$
$$\div \rho(al^2 + 2hlm + bm^2) \quad \text{......}(iii)$$

If the point (x_0, y_0) is the mid-point of the intercept the two values of R are equal and opposite *i.e.*, $R_1 + R_2 = 0$.

$$\therefore \quad lX_0 + mY_0 = 0 \quad \text{......}(iv)$$

Eliminating l, m with the help of (i), we get the equation of the required line *i.e.*

$$(x - x_0)X_0 + (y - y_0)Y_0 = 0 \quad \text{......}(v)$$

Suppose (x_0, y_0) is a point on the bisector, and since (x_0, y_0) is the mid-point of the line (v), this line is perpendicular to the line that joins (x_0, y_0) to the intersection of $\phi = 0$. The equation of the latter line is

$$\frac{X}{X_0} = \frac{Y}{Y_0} \quad i.e., \ XY_0 - YX_0 = 0.$$

or $x(aY_0 - hX_0) + y(hY_0 - bX_0) + gY_0 - fX_0 = 0$.

This is at right angles to (v)

$\therefore \quad X_0(aY_0 - hX_0) - Y_0(hY_0 - bX_0) = 0$.

Hence the locus of (x_0, y_0) *i.e.* the bisectors are represented by the equation

$$h(X^2 - Y^2) - XY(a - b) = 0.$$

The equation of the bisectors may be found otherwise.

The relation (iv) gives the condition that (x_0, y_0) be the mid-point of the intercept made on line (i) by lines $\phi = 0$.

If the direction of the lines (i) be fixed, as (x_0, y_0) varies, we shall have a system of parallel lines whose mid-points lie on the locus

$$lX + mY = 0, \text{..............}(vi)$$

If this line represents one of the bisectors of the angles between the lines ϕ, it would be perpendicular to (i).

$$\therefore \quad (la + mh)m - (lh + mb)l = 0$$

or $$h(m^2 - l^2) + (a - b)lm = 0$$

But for points on the bisectors $lX + mY = 0$

or $$\frac{X}{-m} = \frac{Y}{l}.$$

Hence the equation of the bisectors is obtained by eliminating l, m and the required equation is

$$h(X^2 - Y^2) - (a-b)XY = 0.$$

Examples VII

1. Find the angles between the following pairs of straight lines :—

(*i*) $5x^2 + 8xy + 3y^2 = 0$

(*ii*) $(1 + \sin\theta)x^2 + 2xy + (1 - \sin\theta)y^2 = 0.$

2. Show that the lines $ax^2 + 2hxy + by^2 = \lambda(x^2 + y^2)$ have the same bisectors for all values of λ.

3. If pairs of straight lines $x^2 - 2\lambda xy - y^2 = 0$ and $x^2 - 2\mu xy - y^2 = 0$ be such that each bisects the angles between the other, then $\lambda\mu = -1$.

4. Show that the pair of lines $a^3x^2 + 2h(a^2 + ab + b^2)xy + b^3y^2 = 0$ is equally inclined to the pair $ax^2 + 2hxy + by^2 = 0$.

5. Show that the lines $bx^2 - 2hxy + ay^2 = 0$ are perpendicular to the lines $ax^2 + 2hxy + by^2 = 0$.

6. Show that the equation $6x^2 + 5xy - 6y^2 + 13(x + y) + 5 = 0$ represents perpendicular straight lines.

7. Find the values of λ for which the following equations represent pairs of lines :—

(*i*) $x^2 + 2xy + y^2 + x + y + \lambda = 0.$

(*ii*) $3x^2 - 10xy + 7y^2 + 2\lambda x - 7y - 42 = 0.$

(*iii*) $\lambda x^2 + 10xy + 6y^2 + 16x + 14y + 8 = 0.$

(*iv*) $12x^2 + 2\lambda xy + 2y^2 + 11x - 5y + 2 = 0.$

(*v*) $x^2 + 2xy + 2y^2 + 8x + 8y + \lambda(x^2 + 2y^2 + 8y) = 0.$

8. Find the equations of the lines which join the intersections of the conic $x^2 + 2xy + 2y^2 - 6x - 2y + 9 = 0$ and $x = 3$ with the origin.

9. Show that the straight lines joining the origin to the points common to $(x-h)^2 + (y-k)^2 = c^2$ and $\dfrac{x}{h} + \dfrac{y}{k} = 2$ will be at right angles if $h^2 + k^2 = c^2$.

10. Show the lines joining the origin to the points of intersection of the curves

$$ax^2 + 2hxy + by^2 + 2gx = 0$$

and $$a'x^2 + 2h'xy + b'y^2 + 2g'x = 0$$

will be at right angles to one another, if

$$g'(a+b) = g(a'+b').$$

11. Show that the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my + n = 0$ form a right angled isosceles triangle either if

$$a + b = 0 \text{ and } h(l^2 - m^2) = (a-b)lm$$

or if $$al^2 + 2hlm + bm^2 = 0 \text{ and } (a+b)^2 + 4(ab - h^2) = 0.$$

12. Find the condition that one of the lines $ax^2+2hxy+by^2=0$ should (1) coincide with, (b) perpendicular to, one of the lines $a'x^2+2h'xy+b'y^2=0$.

13. Find λ so that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0$$

may represent a pair of straight lines. Find also the condition that the origin may be on one of the lines.

14. Find the angle between the lines

$$x^2+y^2=4(x\cos\theta+y\sin\theta)^2.$$

15. Prove that the equation

$$(a+2h+b)x^2+2(a-b)xy+(a-2h+b)y^2=0$$

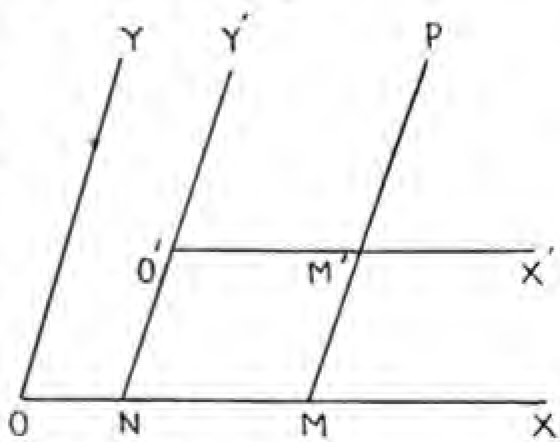
denotes a pair of straight lines each inclined at $\dfrac{\pi}{4}$ to one or other of the lines given by $ax^2+2hxy+by^2=0$.

35. Transformation of Co-ordinates.

It is often necessary, for the sake of simplification to pass from one set of axes to another. The operation of changing from one pair of axes to a second pair is known as **transformation of axes.** The relations which connect the co-ordinates of a point referred to the two systems of axes are called **equations of transformation.**

35.1. Translation of axes. To transform to new axes parallel to the old with a different origin.

Let P be a point whose co-ordinates referred to the old axes OX and OY and to new axes O′X′, O′Y′ (parallel to OX and OY) are (x,y) and (x',y') respectively. Suppose that the co-ordinates of O′ are (h, k). Then



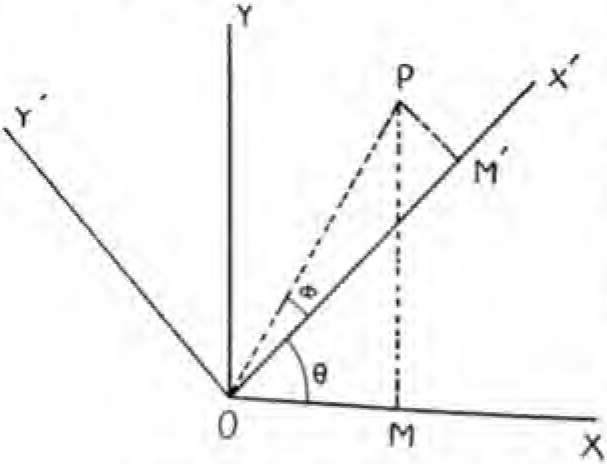
$$x=OM=ON+O'M'=h+x'$$
$$y=MP=NO'+M'P=k+y'.$$

So we obtain

$$\left.\begin{array}{ll} \left.\begin{array}{l} x=h+x' \\ y=k+y' \end{array}\right\} & (T) \\ \text{and} \quad \left.\begin{array}{l} x'=x-h \\ y'=y-k \end{array}\right\} & (T') \end{array}\right\} \qquad \ldots\ldots(8)$$

The transformations (T) and (T′) are called *inverse* of each other. By (T) we pass from old to the new axes while by (T′) we return from the new to the old axes.

35.2. Rotation of axes. To change from one rectangular system to another with the same origin.



Let the new x-axis OX′ make an angle θ with the old x-axis OX. If (x, y), (x', y') be the co-ordinates of a point P referred to the old and the new axes, and ϕ the angle that OP makes with OX′, then

$$x = \text{OM} = \text{OP} \cos(\theta + \phi) = \text{OP}(\cos\theta \cos\phi - \sin\theta \sin\phi)$$
$$= x' \cos\theta - y' \sin\theta, \text{ where OM}' = x', \text{ M}'\text{P} = y'$$

and $$y = \text{MP} = \text{OP} \sin(\theta + \phi) = \text{OP}(\sin\theta \cos\phi + \cos\theta \sin\phi)$$
$$= x' \sin\theta + y' \cos\theta.$$

Hence the equations of rotation are

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta \\ y &= x' \sin\theta + y' \cos\theta \end{aligned}\right\} \text{(R)}$$

The inverse transformation will be found to be

$$\left.\begin{aligned} x' &= x \cos\theta + y \sin\theta \\ y' &= -x \sin\theta + y \cos\theta \end{aligned}\right\} \text{(R}'\text{)}$$

The equations (R′) may be obtained from the equations (R) or directly from the figure, *e.g.*,

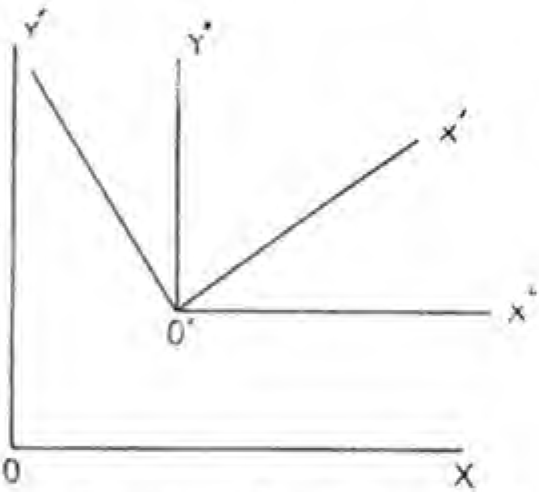
$$x' = \text{OP} \cos\phi = \text{OP} \cos(\phi + \theta - \theta) = \text{OP}[\cos(\theta + \phi)\cos\theta$$
$$+ \sin(\theta + \phi) \sin\theta] = x \cos\theta + y \sin\theta.$$

Similarly $y' = -x \sin\theta + y \cos\theta$.

The two sets of equations may conveniently be represented as follows:—

	x'	y'
x	$\cos\theta$	$-\sin\theta$
y	$\sin\theta$	$\cos\theta$

35.3. The general transformation from one set of rectangular axes to another is best carried out in two steps. The complete transformation is called the **resultant** of the two transformations called the **components.** Different results will be obtained by changing the order of the components.



Let the axes be first translated to a new origin O′ (h, k). If O′X″, O′Y″ be the position of the new axes and (x, y), (x'', y'') the co-ordinates of a point P referred to the two systems of co-ordinates old and new

$$x = x'' + h,\ y = y'' + k.$$

Now rotate the axes through an angle θ, so that the new axes are OX′, OY′ and (x', y') the co-ordinates of the same point P with respect OX′, OY′, then

$$x'' = x' \cos\theta - y' \sin\theta,\ y'' = x' \sin\theta + y' \cos\theta.$$

Hence

$$\left.\begin{aligned} x &= x' \cos\theta - y' \sin\theta + h \\ y &= x' \sin\theta + y' \cos\theta + k \end{aligned}\right\} \quad (G)$$

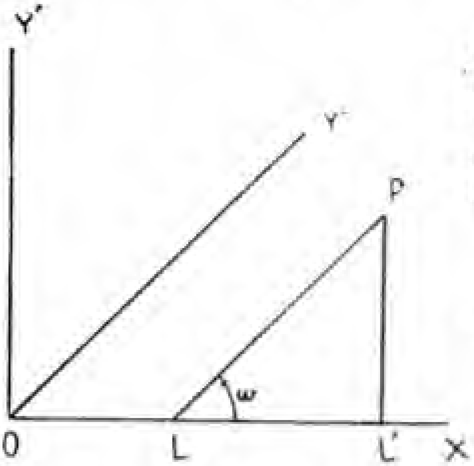
and
$$\left.\begin{aligned} x' &= \ \ (x-h)\cos\theta + (y-k)\sin\theta \\ y' &= -(x-h)\sin\theta + (y-k)\cos\theta \end{aligned}\right\} \quad (G')$$

35.4. Let the new axes be given by their equations. In this case (x', y') the co-ordinates of the point with respect to the new axes are the measures of the perpendiculars from the new axes. If the equations of the new axes O′Y′, O′X′ are respectively

$$lx + my + n = 0 \quad mx - ly + n' = 0,$$

then
$$x' = \frac{lx + my + n}{\sqrt{l^2 + m^2}}, \quad y' = \frac{mx - ly + n'}{\sqrt{l^2 + m^2}}$$

35.5. To change from oblique axes to rectangular axes with the same axis of x.



Let (x,y) be the co-ordinates of P referred to OX, OY as the axes inclined at an angle ω and (x', y') the co-ordinates of the same point referred to rectangular axes OX, OY'.

If $LP=y$, $L'P=y'$.

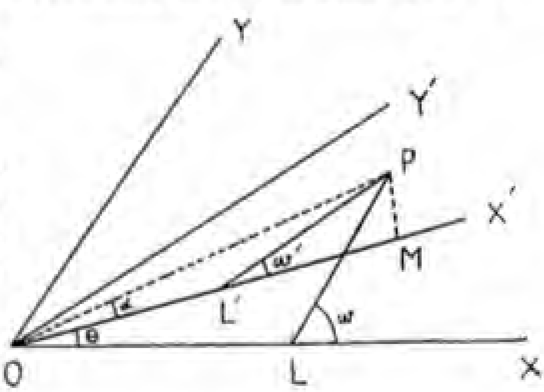
$$x=OL=OL'-LL'=x'-y'\cot\omega,$$
$$y=y'\operatorname{cosec}\omega \quad \ldots\ldots(A)$$

Similarly,

$$x'=x+y\cos\omega,$$
$$y'=y\sin\omega \quad \ldots\ldots(A')$$

35.6. Change from one set of oblique axes to another set of oblique axes with the same origin.

Let the co-ordinates of P, referred to sets of axes OX, OY : OX', OY' containing angles ω, ω' respectively be (x, y), (x', y').



Draw the ordinates LP, L'P of P *w.r.* to the two sets of axes. Let $\angle XOX'$ be θ and suppose $\angle X'OP=\alpha$.

Now

$$\frac{OP}{\sin\omega}=\frac{OL}{\sin(\omega-\theta-\alpha)}=\frac{LP}{\sin(\theta+\alpha)}$$

$$\begin{aligned}
\therefore\quad OL\sin\omega=x\sin\omega&=OP\sin(\omega-\theta-\alpha)\\
&=OP\sin(\omega-\theta)\cos\alpha-OP\cos(\omega-\theta)\sin\alpha\\
&=OM\sin(\omega-\theta)-MP\cos(\omega-\theta)
\end{aligned}$$

where $MP\perp OX'$

$$\begin{aligned}
&=(x'+y'\cos\omega')\sin(\omega-\theta)-y'\sin\omega'\cos(\omega-\theta)\\
&=x'\sin(\omega-\theta)+y'\sin(\omega-\omega'-\theta);
\end{aligned}$$

$$\begin{aligned}
\text{and } LP\sin\omega=y\sin\omega&=OP\sin(\theta+\alpha)\\
&=OP\sin\theta\cos\alpha+OP\cos\theta\sin\alpha\\
&=(x'+y'\cos\omega')\sin\theta+y'\sin\omega'\cos\theta\\
&=x'\sin\theta+y'\sin(\omega'+\theta)
\end{aligned}$$

Similarly $x'\sin\omega'=x\sin(\omega'+\theta)-y\sin(\omega-\omega'-\theta)$

$$y'\sin\omega'=-x\sin\theta+y\sin(\omega-\theta)$$

36. We have seen that the equations of transformation are of the form

$$x = lx' + my' + n$$
$$y = l'x' + m'y' + n',$$

which are of the first degree in the set of variables x, y : x', y'. Hence if $lx' + my' + n$, $l'x' + m'y' + n'$ replace x and y in any equation, the degree of the equation cannot be *raised*. Neither can the degree be lowered, for otherwise on transforming back, the degree of the equation will be raised. Hence :—

A linear transformation leaves the degree of an equation unaltered.

37. In the preceding sections we have looked upon a transformation as an operation that changes the position of the axes, the position of the point remaining fixed. We may look upon a transformation as an operation which transforms a given point into another, the position of the axes remaining unaltered. From this point of view, the points of one figure transform to other points so as to form a new figure called the transform of the first. It will be found that certain points or curves will remain unchanged during the operation of transformation. They are called invariant points or curves.

37.1. Invariants. In the process of transformation of one equation to another, there are found certain expressions, involving the co-efficients of the equation, which remain unaltered by the change of axes. Such expressions are called **invariants.**

37.2. If the co-ordinates of a point P referred to two sets of axes containing angles ω and ω' be (x, y), (x', y') and if the expression $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ transforms into $a'x'^2 + 2h'x'y' + b'y'^2 + 2g'x' + 2f'y' + c'$, where the co-efficients of x, y, x', y' are independent of these variables, then

$$\frac{a + b - 2h\cos\omega}{\sin^2\omega} = \frac{a' + b' - 2h'\cos\omega}{\sin^2\omega'}$$

$$\frac{ab - h^2}{\sin^2\omega} = \frac{a'b' - h'^2}{\sin^2\omega'}.$$

The general linear transformation is of the form

$$x = lx' + my' + n$$
$$y = l'x' + m'y' + n'.$$

This can be resolved into two transformations by the equations

$$x = \xi + n,\ y = \eta + n' \qquad \ldots\ldots(i)$$
$$\xi = lx' + my',\ \eta = l'x' + m'y' \qquad \ldots\ldots(ii)$$

The transformation (*i*) is a translation and (*ii*) of the type 35.6.

It is easily seen that a translation of axes does not alter the co-efficients a, b, h. The theorem is, therefore true under the translation of axes.

We may now consider the effect of transformation (*ii*). This transformation does not alter the degree of the expression $2gx+2fy+c$, hence these terms do not affect the co-efficients of the second degree terms. We have, therefore, only to consider the effect of the transformation on the terms $ax^2+2hxy+by^2$.

Let the expression

$ax^2+2hxy+by^2$ change to $a'x'^2+2h'x'y'+by'^2$ then

$x^2+y^2+2xy\cos\omega$ transforms into $x'^2+y'^2+2x'y'\cos\omega'$ as both these expressions give the square of the distance of P from the origin. Thus

$$ax^2+2hxy+by^2+\lambda(x^2+y^2+2xy\cos\omega) \qquad \ldots\ldots(iii)$$

will become

$$a'x'^2+2h'x'y'+b'y'^2+\lambda(x'^2+y'^2+2x'y'\cos\omega'). \qquad \ldots\ldots(iv)$$

If for any value of λ, expression (*iii*) be a perfect square, the same values of λ will make expression (*iv*) a perfect square.

Now (*iii*) is a perfect square

$$(a+\lambda)(b+\lambda)=(h+\lambda\cos\omega)^2$$

and (*iv*) is a perfect square if

$$(a'+\lambda)(b'+\lambda)=(h'+\lambda\cos\omega')^2.$$

These two equations, *viz.*,

$$\lambda^2\sin^2\omega+(a+b-2h\cos\omega)\lambda+(ab-h^2)=0$$
$$\lambda^2\sin^2\omega'+(a'+b'-2h'\cos\omega')\lambda+(a'b'-h'^2)=0$$

have the same roots and are therefore identical. Hence

$$\frac{a+b-2h\cos\omega}{\sin^2\omega}=\frac{a'+b'-2h'\cos\omega'}{\sin^2\omega'}$$

$$\frac{ab-h^2}{\sin^2\omega}=\frac{a'b'-h'^2}{\sin^2\omega'}.$$

37.3. If by any change of axes the expression

$$\phi\equiv ax^2+2hxy+by^2+2gx+2fy+c$$

becomes

$$\phi'\equiv a'x'^2+2h'x'y'+b'y'^2+2g'x'+2f'y'+c',$$

then

$$\frac{abc+2fgh-af^2-bg^2-ch^2}{\sin^2\omega}=\frac{a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2}{\sin^2\omega'}$$

i. e., $\Delta/\sin^2\omega=\Delta'/\sin^2\omega'$

where ω, ω' are the angles between the axes in the two systems of co-ordinates.

The proposition will be proved in two steps. The general linear transformation

$$x = l_1x' + m_1y' + n_1$$
$$y = l_2x' + m_2y' + n_2$$

is resolved into two component parts by the equations

$$\left.\begin{aligned} x &= x_1 + n_1 \\ y &= y_1 + n_2 \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(i)$$

and

$$\left.\begin{aligned} x_1 &= l_1x' + m_1y' \\ y_1 &= l_2x' + m_2y' \end{aligned}\right\} \qquad \ldots\ldots\ldots\ldots(ii)$$

By the transformation (i) the expression ϕ becomes

$$\phi_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2g_1x_1 + 2f_1y_1 + c_1$$

where
$$g_1 = an_1 + hn_2 + g, \quad f_1 = hn_1 + bn_2 + f$$
$$c_1 = an_1^2 + 2hn_1n_2 + bn_2^2 + 2gn_1 + 2fn_2 + c.$$

Let the discriminant of ϕ_1 be Δ_1.

$$\Delta_1 = \begin{vmatrix} a & h & an_1 + hn_2 + g \\ h & b & hn_1 + bn_2 + f \\ an_1 + hn_2 + g & hn_1 + bn_2 + f & \begin{aligned} &an_1^2 + 2hn_1n_2 + bn_2^2 \\ &+ 2gn_1 + 2fn_1 + c \end{aligned} \end{vmatrix}$$

From the third column subtract n_1 times the first and n_2 times the second, then

$$\Delta_1 = \begin{vmatrix} a & h & g \\ h & b & f \\ an_1 + hn_2 + g & hn_1 + bn_2 + f & gn_1 + fn_2 + c \end{vmatrix}$$

$$= \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \Delta$$

But the transformation (i) is a translation and does not affect the angle between the axes. Hence

$$\frac{\Delta}{\sin^2\omega} = \frac{\Delta_1}{\sin^2\omega} \qquad \ldots\ldots\ldots\ldots(iii)$$

Now apply to ϕ_1 the transformation (ii) then ϕ_1 becomes ϕ'. It may be pointed out in passing, that since the transformation in question does not contain an absolute term, the absolute term of ϕ_1 will not be affected. Hence

$$c_1 = c'.$$

Let (x_1, y_1), (x', y') be the co-ordinates of a point with respect to the two systems of co-ordinates, then the expression $x_1^2 + y_1^2 + 2x_1y_1 \cos\omega$ becomes $x'^2 + y'^2 + 2x'y' \cos\omega'$, as each gives the square of the distance of the point from the common origin. Hence

$$\equiv \phi_1 + \lambda(x_1^2 + y_1^2 + 2x_1y_1 \cos\omega)$$
$$\equiv \phi' + \lambda(x'^2 + y'^2 + 2x'y' \cos\omega).$$

If the left-hand side is the product of two linear factors for some values of λ so is the right-hand side for the same value of λ. Thus the equations

$$\begin{vmatrix} a+\lambda & h+\lambda\cos\omega & g_1 \\ h+\lambda\cos\omega & b+\lambda & f_1 \\ g_1 & f_1 & c_1 \end{vmatrix}=0,\quad \begin{vmatrix} a'+\lambda & h'+\lambda\cos\omega' & g' \\ h'+\lambda\cos\omega' & b'+\lambda & f' \\ g' & f' & c' \end{vmatrix}=0$$

have the same roots *i. e.*, the equations

$$\lambda^2 c_1\sin^2\omega+\lambda[c_1(a+b-2h\cos\omega)-(f_1^2+g_1^2-2f_1g_1\cos\omega)] + \Delta_1=0$$

and

$$\lambda^2 c'\sin^2\omega'+\lambda[c'(a'+b'-2h'\cos\omega')-(f'^2+g'^2-2f'g'\cos\omega')] + \Delta'=0$$

are identical, and since $c_1=c'$.

$$\therefore\qquad \frac{\Delta'}{\sin^2\omega'^2}=\frac{\Delta_1}{\sin^2\omega}=\frac{\Delta}{\sin^2\omega}\quad \text{(by } iii\text{)}.$$

Ex. 1. *Show that Δ is invariant under a linear orthognal transformation.*

Ex. 2. Show that the expressions

$$\frac{f^2+g^2-2fg\cos\omega}{\sin^2\omega},\quad \frac{af^2+bg^2-2fgh}{\sin^2\omega}$$

are invariants under linear transformation.

Note. In general, the vanishing of an invariant signifies a geometrical property. For example, if $\Delta=0$, the equation $\phi=0$ represents two st. lines and $ab-h^2=0$, $a+b=0$ (or $a+h-2h\cos\omega=0$) are respectively the conditions of parallelism and perpendicularity of these lines. The geometrical significance of the invariants $ab-h^2=0$ and $a+b=0$ (or $a+b-2b\cos\omega)=0$, when $\Delta\neq0$, as will appear later on, are respectively the conditions for a parabola and rectangular hyperbola.

Illustrative Examples

(1) *Transform to parallel axes through* $\left(\frac{-4}{5},-\frac{1}{10}\right)$ *the equation*

$$17x^2-12xy+8y^2+26x-8y+2=0.$$

We substitute $x-\frac{4}{5}$ for x and $y-\frac{1}{10}$ for y, and the equation becomes

$$17\left(x-\frac{4}{5}\right)^2-12\left(x-\frac{4}{5}\right)\left(y-\frac{1}{10}\right)+8\left(y-\frac{1}{10}\right)^2$$
$$+26\left(x-\frac{4}{5}\right)-8\left(y-\frac{1}{10}\right)+2=0$$

or $17x^2-12xy+8y^2-8=0.$

(2) *Show that by a rotation of axes, the equation*

$$17x^2-12xy+8y^2-8=0$$

can be deprived of its $x\ y$ term. Find the reduced equation.

Let the rotation be

$$x=x'\cos\theta-y'\sin\theta$$
$$y=x'\sin\theta+y'\cos\theta,$$

then the given equation becomes

$$17(x'\cos\theta-y'\sin\theta)^2-12(x'\cos\theta-y'\sin\theta)(x'\sin\theta+y'\cos\theta)$$
$$+8(x'\sin\theta+y'\cos\theta)^2-8=0 \qquad \ldots\ldots(i)$$

The co-efficient of $xy=0$

$\therefore\quad -18\cos\theta\sin\theta-12(\cos^2\theta-\sin^2\theta)=0$

or $\quad 2\tan^2\theta-3\tan\theta-2=0$

$\therefore\quad \tan\theta=2$ or $-\frac{1}{2}$

Taking $\tan\theta=2$, the equation (i) becomes

$$17(x'-2y')^2-12(x'-2y')(2x'+y')+8(2x'+y')^2-8\sec^2\theta=0$$

or $\qquad 25x'^2+100y'^2-40=0$

i.e., $\qquad 5x'^2+20y'^2-8=0.$

(3) *Show that the equation of the lines bisecting the angles between the bisectors of the angles between the lines* $ax^2+2hxy+by^2=0$ *is* $(a-b)(x^2-y^2)+4\ hxy=0$, *and that, if this pair of st. lines be taken as new axes of reference OX, OY, the equation of the pair of lines* $ax^2+2hxy+by=0$ *may be written as*

$$(a+b)(X^2+Y^2)+2\mu XY=0$$

where $\qquad \mu^2=(a-b)^2+4h^2$ \[*Math. Trip : I 1915* \]

The equation of the bisectors of the angles between the lines $\qquad ax^2+2hxy+by^2=0 \qquad \ldots\ldots(i)$

is $\qquad h(x^2-y^2)-(a-b)xy$

and the bisectors of the angles between these lines are given by the equation

$$\frac{x^2-y^2}{2h}=-\frac{2xy}{a-b}$$

or $\quad (a-b)(x^2-y^2)+4hxy=0 \qquad \ldots\ldots(ii)$

Suppose now with the new axes, the equation (i) becomes

$$AX^2+2HXY+BY^2=0 \qquad \ldots\ldots(iii)$$

$$\therefore \qquad A+B=a+b \qquad \ldots\ldots(iv)$$

$$AB-H^2=ab-h^2 \qquad \ldots\ldots(v)$$

Now the bisectors of the angles between the bisectors of the angles between the lines (iii) are given by the equation

$$(A-B)(X^2-Y^2)+4HXY=0$$

which are the new axes and must, therefore, be identical with

$$XY=0.$$

$$\therefore \qquad A=B=\frac{a+b}{2}$$

$$H^2=AB-(ab-h^2)=\frac{(a+b)^2}{4}-(ab-h^2)$$

$$=\tfrac{1}{4}[(a-b)^2+4h^2]=\tfrac{1}{4}\mu^2$$

Hence, the equation (*iii*) takes the form

$$(a+b)(X^2+Y^2)+2\mu XY=0.$$

Examples VIII

1. Show that the expression x^2+y^2 is an invariant under a rotation of axes.

2. If the lines $ax+by=0$ $bx-ay=0$ are taken as the axes, find the transforms of the lines given by the equations

$$ax+by+c=0,\ (ax+by)^2-3(bx-ay)^2=0,$$

and show that they form the sides of an equilateral triangle.

3. Find the equation of the lines $x^2+8\sqrt{2}xy+5y^2=0$ referred to the bisectors of the angles between them as axes.

4. Find the equation of the lines

$$3x^2-8xy-3y^2+30y-27=0$$

referred to the bisectors of the angles between them as axes.

5. Two sides of a rectangle are given by the equation

$$3x^2-8xy-3y^2+30y-27=0.$$

Find the equation of the other two sides, if the diagonals of the rectangle intersect at the origin.

6. The point $P(x_0, y_0)$ lies in the acute or obtuse angle between the lines $a_ix+b_iy+c_i=0$ $(i=1,2)$, according as

$$(a_1x_0+b_1y_0+c_1)(a_2x_0+b_2y_0+c_2)(a_1a_2+b_1b_2)\lessgtr 0.$$

[*Hint.*—Shift the origin to $P(x_0, y_0)$, then apply Art. 26.2].

7. Transform the following equations, first by shifting the origin to a suitable point so as to deprive the equations of the terms containing x and y and then rotating the axes so as to remove the xy term.

(*i*) $3x^2-8xy-3y^2+30y-43=0.$

(*ii*) $45x^2+85y^2-30xy-270x+234y+261=0.$

(*iii*) $27x^2-77y^2-78xy-162x-90y+81=0.$

8. Find the equation of the lines

$$(x^2+y^2)(\cos^2\theta.\sin^2\alpha+\sin^2\theta)=(x\tan\alpha-y\sin\theta)^2$$

referred to the lines $x\tan\alpha-y\sin\theta=0$ $x\sin\theta+y\tan\alpha=0$ as axes. Hence find the angle between the lines.

Miscellaneous Examples IX

1. The equation of a pair of st. lines through the origin is $ax^2+2hxy+by^2=0$; show that the equations of the lines through (p, q) (i) parallel, (ii) perpendicular to them are respectively given by the equations

$$a(x-p)^2+2h(x-p)(y-q)+b(y-q)^2=0$$
$$a(y-q)^2-2h(x-p)(y-q)+b(x-p)^2=0.$$

2. Two points A and B move, A along x-axis and B along y-axis, so that the algebraic sum OA+OB$=c$ (O being the origin), show that the locus traced out by a point P, so that AP : BP$=h:k$, is given by the equation

$$\frac{x}{k}+\frac{y}{h}=\frac{c}{h+k}.$$

3. Through a fixed point (p, q) two st. lines AB, AC are drawn at right angles to meet the x-axis and y-axis respectively in B and C, show that the locus of the mid-point of BC is

$$y-\frac{k}{2}+\frac{h}{k}\left(x-\frac{h}{2}\right)=0.$$

4. A st. line is drawn parallel to the base of a given triangle and its extremities are joined transversally to those of the base; show that the locus of the point of intersection of the joining lines is a st. line.

5. The st. lines joining a variable point P to two fixed points (x_1, y_1), (x_2, y_2) meet the axis of x in M and N respectively. Show that the locus of P, if the ratio OM : ON is given to be λ, O being the origin, is

$$(y-y_2)(x_1y-y_1x)=\lambda(y-y_1)(x_2y-y_2x).$$

6. Prove that two perpendicular st. lines passing through the points (x_1, y_1), (x_2, y_2) can be represented by the freedom equations

$$x=x_1+t,\ y=y_1+\lambda t\ ;\ x=x_2+u,\ y=y_2-u/\lambda$$

respectively. Hence show that the locus of the intersection of these lines is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0.$$

7. Show that the equation

$$(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)+\lambda xy=0$$

will represent a pair of st. lines when $\lambda=0$, $\lambda\to\infty$ and

$$n_1n_2\lambda=(l_1n_2-l_2n_1)(m_1n_2-m_2n_1).$$

8. Show that the two st. lines given by the equation

$$x^2(\tan^2\theta+\cos^2\theta)-2xy\sin\theta+y^2\sin^2\theta=0$$

make with the axis of x angles such that the difference of their tangents is 2.

9. Show that the st. lines

$$x^2 \sin^2 \alpha \cos^2\theta + 4xy \sin \alpha \sin \theta + y^2[4 \cos \alpha - (1 + \cos \alpha)^2 \cos^2\theta] = 0$$

meet at an angle α.

10. If the axes are oblique and inclined at an angle ω, show that the lines $ax^2 + 2hxy + by^2 = 0$ also include an angle ω, if $\quad 4ab \cos^2\omega - 4h(a+b) \cos \omega + (a+b)^2 = 0.$

11. Prove that the pair of st. lines represented by the equation

$$y^2(\cos \alpha + \sqrt{3} \sin \alpha) \cos \alpha - xy (\sin 2\alpha - \sqrt{3} \cos 2\alpha) + x^2 (\sin \alpha - \sqrt{3} \cos \alpha) \sin \alpha = 0$$

make with the line

$$(\cos \alpha - \sqrt{3} \sin \alpha)\, y - (\sin \alpha + \sqrt{3} \cos \alpha)\, x + a = 0$$

an equilateral triangle of area $a^2/4\sqrt{3}$.

12. Show that if $\lambda\mu = -1$, each pair of st. lines represented by the equations

$$x^2 - 2\lambda xy - y^2 = 0, \; x^2 - 2\mu xy - y^2 = 0$$

bisect the angles between the other pair.

13. Prove that the pairs of lines $a^2x^2 + 2h(a+b)xy + b^2y^2 = 0$ is equally inclined to the pair $ax^2 + 2hxy + by^2 = 0$.

✓14. Show that the angle between one of the lines given by $ax^2 + 2hxy + by^2 = 0$, and one of the lines

$$ax^2 + 2hxy + by^2 + \lambda(x^2 + y^2) = 0$$

is equal to the angle between the other lines of the system.

15. The diagonals of a quadrilateral are $x = c$, $y = d$ and a pair of opposite sides are $ax^2 + by^2 = 0$. Show that the other two sides intersect at the point

$\left(\dfrac{2bc}{b-a}, \dfrac{2ac}{a-b}\right)$ and are parallel to $(ax - by)^2 + ab(x+y)^2 = 0$.

16. If the sides of a parallelogram be parallel to the lines $ax^2 + 2hxy + by^2 = 0$ and one diagonal be parallel to $lx + my + n = 0$, show that the other diagonal is parallel to the line

$$y(bl - hm) = x(am - hl).$$

17. Prove that if the circle $a(x^2 + y^2) + 2gx + 2fy + c = 0$ intercepts on the line $lx + my + n = 0$ a length which subtends a right angle at the origin, then

$$c(l^2 + m^2) + 2n(gl + fm + an) = 0.$$

✓18. Find the length of the intercept on the line $lx + my + n = 0$ made by the lines $ax^2 + 2hxy + by^2 = 0$. Find also the area of the triangle formed by the lines.

Miscellaneous Examples X

1. Show that the product of the perpendiculars from the (x_0, y_0) on the straight lines $ax^2+2hxy+by^2=0$ is

$$\frac{ax_0^2+2hx_0y_0+by_0^2}{\sqrt{(a-b)^2+4h^2}}. \text{ (rectangular axes.)}$$

2. A straight line of length $2l$ has its extremities one on each of the straight lines $ax^2+2hxy+by^2=0$. Show that the locus of its mid-point is

$$(ax+hy)^2+(hx+by)^2+(ab-h^2)l^2=0.$$

3. Show that the ortho-centre of the triangle formed by the st. lines $ax^2+2hxy+by^2=1$, $lx+my=1$ is given by the equations

$$\frac{x}{l}=\frac{y}{m}=\frac{a+b}{am^2-2hlm+bl^2}.$$

Deduce the distance of the ortho-centre from the origin.

4. Two sides of a triangle are fixed and are given by the equation $ax^2+2hxy+by^2=0$, the third variable side passes through the fixed point (x_0, y_0), show that the locus of the ortho-centre of the triangle is

$$bx^2+2hxy+ay^2-(a+b)(xx_0+yy_0)=0.$$

5. A point moves so that the distance between the feet of the perpendiculars from it on the lines $ax^2+2hxy+by^2=0$ is a constant $2k$. Show that the equation of its locus is
$(x^2+y^2)(h^2-ab)=k^2\{(a-b)^2+4h^2\}$. [Math. Trip. 1925.]

6. If the sum of the squares of the perpendiculars from a point P on the lines $ax^2+2hxy+by^2=0$ is constant and equal to c^2, show that the locus of P is the conic

$$4h^2(x^2+y^2)+4h(a+b)xy+2(a-b)(ax^2-by^2)$$
$$=c^2\{(a-b)^2+4h^2\}.$$

7. Prove that one of the lines given by the equation

$$2x^3+3(1-k^2)xy^2+k(k^2-3)y^3=0$$

bisects the angle between the other two. (Selwyn, 1928.)

8. Show that one of the lines
$x^3\tan\theta+x^2y(2\tan 2\theta\tan\theta-1)-xy^2(2\tan 2\theta+\tan\theta)+y^3=0$
bisects the angle between the other two perpendicular lines.

9. Find the condition that the three lines represented by the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$ may be equally inclined to each other.

[*Hint.* If $\theta_1, \theta_2, \theta_3$ be the inclinations of the three lines, $\frac{2\pi}{3}=\theta_2-\theta_1=\theta_3-\theta_2$. The conditions found will be $a+c=0$, $b+d=0$].

10. Show that the line $ax+by+c=0$ forms with the lines $(ax+by)^2-3(ay-bx)^2=0$ an equilateral triangle of area

$$\frac{c^2}{\sqrt{3(a^2+b^2)}}.$$

11. Show that the equation

$$(x^3-3xy^2)\cos 3a+(y^3-3x^2y)\sin 3a+3a(x^2+y^2)-4a^3=0$$

represents three lines forming an equilateral triangle of area $3a^2\sqrt{3}$.

[*Hint.* Rotate the axes through $-\alpha$. Change the new set of Cartesian co-ordinates to Polar form. The two transformations can simultaneously be affected by $x=r\cos(\theta-a)$, $y=r\sin(\theta-a)$. The equation then reduces to $r^3\cos 3\theta+3ar^2-4a^3=0$ or

$r^3(4\cos^3\theta-3\cos\theta)+3ar^2-4a^3=0$ or

$(r\cos\theta-a).(r.\cos\theta+\sqrt{3}\sin\theta+2a)\times(r.\cos\theta-\sqrt{3}\sin\theta+2a)=0$.

So the equations of the lines in the new position are $x=a$, $x+\sqrt{3}y+2a=0$, $x-\sqrt{3}y+2a=0$].

12. Show that the equation $ax^3+3bx^2y+3cxy^2+dy^3=0$ represents

(*i*) three coincident st. lines if $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$,

(*ii*) two coincident st. lines and an other if

$$(bc-ad)^2=4(bd-c^2)(ac-b^2).$$

13. If two of the lines $ax^3-3bx^2y+3cxy^2-dy^3=0$ are perpendicular to each other, then $a^2+d^2+3(ac+bd)=0$.

14. If the lines $ax^2+2hxy+by^2=0$ are inclined to the line $x\cos\alpha+y\sin\alpha-p=0$ at angles ϕ_1, ϕ_2, show that $\tan\phi_1$, $\tan\phi_2$ are the roots of the equation

$$(a+2ht+bt^2)\lambda^2-2\lambda[ht^2-(a-b)t-h]+(at^2-2ht+b)=0$$

where $t=\tan\alpha$.

15. Show that if the st. lines given by the equation $ax^2+2hxy+by^2=0$ are turned through an angle α, the equation in the new position will be

$$ax^2+2hxy+by^2-2\{(b-a)xy+h(x^2-y^2)\}\tan\alpha$$
$$+(bx^2-2hxy+ay^2)\tan^2\alpha=0.$$

16. If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents a pair of parallel st. lines, show that

$$\frac{a+b}{d^2}=\frac{H}{h}=\frac{-B}{a}=\frac{-A}{b},$$

where $2d$ is the distance between them.

17. Show that the area of the parallelogram formed by the st. lines $6x^2-5xy+y^2=0$, $12x^2-10xy+2y^2+11x-5y+2=0$ is equal to unity.

18. Show that the four lines given by the equations

$$(y-mx)^2=c^2(1+m^2)$$
$$(y-nx)^2=c^2(1+n^2)$$

form a rhombus.

19. Show that the equations

$$(ab-h^2)(ax^2+by^2+2hxy+2gx+2fy)+af^2+bg^2-2fgh=0$$
$$(ab-h^2)(ax^2+by^2+2hxy-2gx-2fy)+af^2+bg^2-2fgh=0$$

represent lines which together form a parallelogram whose diagonals intersect in the origin.

[*Hint.* If (x, y) be a point on the first pair $(-x, -y)$ lies on the second pair.]

20. Show that the equation

$$p_0x^4+p_1x^3y+p_2x^2y^2+p_3xy^3+p_4y^4=0$$

represents two pairs of perpendicular st. lines if

$$p_1+p_3=0, \quad p_0=p_4.$$

21. Show that the equation

$$6(x^4+y^4)-5xy(x^2-y^2)-12x^2y^2=0$$

represents two pairs of st. lines at right angles.

22. The vertices of a triangle lie on the lines $y=x\tan\alpha$, $y=x\tan\beta$, $y=x\tan\gamma$, the circumcentre being at the origin, prove that the locus of the orthocentre is the line

$$x(\sin\alpha+\sin\beta+\sin\gamma)=y(\cos\alpha+\cos\beta+\cos\gamma).$$

23. A variable line through the fixed point (α, β) cuts the lines $ax^2+2hxy+by^2=0$ in P and Q. Prove that the locus of the mid-point of PQ is the conic

$$(ax+hy)(x-\alpha)+(hx+by)(y-\beta)=0$$

[King's etc., 1931].

24. Through the fixed point (α, β) a variable line is drawn cutting the fixed lines $ax^2+2hxy+by^2=0$ in the points A, B and the parallelogram OAPB is completed, O being the origin of the Cartesian co-ordinates. Prove that the equation of the locus of P is

$$a(x-\alpha)^2+2h(x-\alpha)(y-\beta)+b(y-\beta)^2=a\alpha^2+2h\alpha\beta+b\beta^2.$$

25. Prove that any pair of lines through the origin making equal angles with the line $lx+my+n=0$ is given by

$$(l^2-m^2)x^2+2lmxy+\lambda\{(l^2-m^2)y^2-2lmxy\}=0$$

where λ is a parameter, and explain on geometrical grounds why the equation of any such pair of lines can be written in the form

$$(lx+my)^2+\mu(mx-ly)^2=0,$$

where μ is a parameter.

[Math. Trip : 1930]

26. Show that the st. lines joining the origin to the points of intersection of the two curves

$$ax^2+2hxy+by^2+2gx=0 \text{ and } a'x^2+2h'xy+b'y^2+2g'x=0$$

will be at right angles to one another if $g'(a+b)=g(a'+b')$.

27. Show that the expression

$$\left(\frac{x}{a_1}+\frac{y}{b_2}-1\right)\left(\frac{x}{a_2}+\frac{y}{b_3}-1\right)\left(\frac{x}{a_3}+\frac{y}{b_1}-1\right)$$
$$-\left(\frac{x}{a_2}+\frac{y}{b_1}-1\right)\left(\frac{x}{a_1}+\frac{y}{b_3}-1\right)\left(\frac{x}{a_3}+\frac{y}{b_2}-1\right)$$

contains xy as a factor; and hence prove that if A_1, A_2, A_3 are three points on the axis of x, and B_1, B_2, B_3 are three points on the axis of y, then the three points of intersection of $A_1 B_2$ with $A_2 B_1$; A_2, B_3 with $A_3 B_2$ and $A_3 B_1$ with $A_1 B_3$ lie on a st. line.

28. If $F(x, y)=0$ represents the equation of n st. lines through the origin, show that the equations of the lines (*i*) parallel, (*ii*) perpendicular to these through the point (α, β) are respectively.

$$F(x-\alpha,\ y-\beta)=0,\quad F(y-\beta,\ \alpha-x)=0.$$

29. If (x, y) and (x', y') be the co-ordinates of the same point referred to two sets of axes with the same origin and $lx+my$ be transformed into $l'x+m'y$, then

$$\frac{l^2+m^2-2lm\cos\omega}{\sin^2\omega}=\frac{l'^2+m'^2-2l'm'\cos\omega'}{\sin^2\omega'}.$$

30. Prove that the transformation of rectangular axes which transforms

$$\frac{x'^2}{p}+\frac{y'^2}{q}\ \text{into}\ ax^2+2hxy+by^2,\ \text{will transform}$$
$$\frac{x'^2}{p-\lambda}+\frac{y'^2}{q-\lambda}\ \text{into}\ \frac{ax^2+2hxy+by^2-\lambda(ab-h^2)(x^2+y^2)}{1-\lambda(a+b)+(ab-h^2)\lambda^2}.$$

CHAPTER V

CIRCLE

38. A circle is the locus of a point which moves in a plane so that its distance from a fixed point remains constant.

The fixed point is called the **centre** and the constant distance the **radius.**

38.1. *To find the equation of a circle, given the centre and radius.* (*Rectangular axes*).

Let C (h, k) be the centre, and R the radius of the circle. If P(x, y) be the co-ordinates of a point of the circle, by definition

$$(x-h)^2+(y-k)^2=\mathrm{R}^2.$$

This is the required equation.

The equation of the circle is therefore of the form

$$x^2+y^2+2gx+2fy+c=0. \qquad \ldots\ldots(1)$$

Every equation of this form represents a circle. For, such an equation can be written as

$$(x+g)^2+(y+f)^2=g^2+f^2-c$$

which expresses the fact, that the point (x, y) remains at a constant distance $\sqrt{g^2+f^2-c}$ from the fixed point $(-g, -f)$; hence, by definition, the locus is a circle with $(-g, -f)$ as the centre and $\sqrt{g^2+f^2-c}$ as radius. The circle is real when both its centre and radius are real.

The previous results may be stated thus:—

The axes being rectangular, the necessary and sufficient condition that the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent a circle are

$$a=b, \; h=0.$$

The centre of the circle is $\left(\dfrac{-g}{a}, \dfrac{-f}{a}\right)$ *and radius is* $\dfrac{\sqrt{g^2+f^2-ac}}{a}$

38.2. Alternative proof. For the sake of generality, we consider oblique axes.

The necessary and sufficient conditions that the general second degree equation

$$\phi(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0$$

may represent a circle are

$$a=b,\ h=a\cos\omega,$$

ω being the angle between the axes.

Let $Q(x_0, y_0)$ be a point in the plane of the locus. The equation of a st. line through (x_0, y_0) with direction-sines l and m is

$$\frac{x-x_0}{l}=\frac{y-y_0}{m}=\frac{r}{\sin\omega}$$

where $\sin^{-1} l+\sin^{-1} m=\omega$ and r is the distance of an arbitrary point (x, y) from Q. Suppose in particular r is distance of the point of intersection of the line with the circle. Its co-ordinates are therefore $(rkl+x_0, rkm+y_0)$, where $k=\operatorname{cosec}\omega$. The point lies on the locus $\phi(x, y)=0$ if $\phi(rkl+x_0, rkm+y_0)=0$,

or $r^2k^2(al^2+2hlm+bm^2)+2rk(lX_0+mY_0)+\phi(x_0, y_0)=0.$

Suppose that every chord, *i.e.*, the line joining the two intersections is bisected at (x_0, y_0), then ......(*i*)

$$X_0=0,\quad Y_0=0$$

i.e. $\qquad ax_0+hy_0+g=0 \qquad$ ......(*ii*)

$\qquad hx_0+by_0+f=0 \qquad$ ......(*iii*)

or $$\frac{x_0}{hf-bg}=\frac{y_0}{gh-af}=\frac{1}{ab-h^2}. \qquad \text{......(iv)}$$

Thus Q is uniquely known, provided that $ab \neq h^2$.

And $\phi(x_0, y_0)\equiv x_0X_0+y_0Y_0+Z_0=gx_0+fy_0+c$
$= \Delta/C$ [from (*iv*).]

Thus the equation giving the values of r becomes

$$r^2(al^2+2hlm+bm^2)+\frac{\Delta}{c}\sin^2\omega=0.$$

i.e., $$l^2\left(ar^2+\frac{\Delta}{c}\right)+2\,lm\left(hr^2+\frac{\Delta}{c}\cos\omega\right)$$
$$+m^2\left(br^2+\frac{\Delta}{c}\right)=0.$$

$\because \sin^2\omega=l^2+m^2+2lm\cos\omega$ [Art. 18.1 Note]

The equation, for all values of $l:m$ gives two equal and opposite values of r.

In order that the given locus may be a circle, the values of r obtained from the above equation should be the same for every value of $l:m$. The equation should, therefore, be independent of l and m; and it must, therefore, be an identity.

This will be independent of $l : m$ if

$$ar^2 + \frac{\Delta}{c} = 0 \quad hr^2 + \frac{\Delta}{c}\cos\omega = 0, \quad br^2 + \frac{\Delta}{c} = 0$$

or $a = -\dfrac{\Delta}{cr^2} = b$, $h = -\dfrac{\Delta}{cr^2}\cos\omega = a\cos\omega$,

which are the required conditions for a circle with centre at Q.

Conversely, if these conditions are satisfied, the r-giving equation becomes

$$ar^2 + \frac{\Delta}{c} = 0$$

which shows that r is constant. Hence, it is seen that under these conditions $c \neq 0$.

38.3. From the discussion in Art 38.2 it follows that the equation of the circle in oblique co-ordinates is

$$a(x^2 + y^2 + 2xy\cos\omega) + 2gx + 2fy + c = 0,$$

which can be written as

$$x^2 + y^2 + 2xy\cos\omega + 2gx + 2fy + c = 0,$$

when $\dfrac{g}{a}$, $\dfrac{f}{a}$, $\dfrac{c}{a}$ are replaced by g, f and c.

The co-ordinates of the centre of this circle are given by the equation

$$x_0 : y_0 : 1 = f\cos\omega - g : g\cos\omega - f : \sin^2\omega, \qquad \ldots\ldots(2)$$

and the square of the radius R is given by the equation

$$R^2\sin^2\omega + \begin{vmatrix} 1 & \cos\omega & g \\ \cos\omega & 1 & f \\ g & f & c \end{vmatrix} = 0 \qquad \ldots\ldots(3)$$

39. Freedom Equations.

When the co-ordinates of any point of a curve can be expressed in terms of another variable (called *parameter*) the equations connecting the co-ordinates x and y with the parameter are called the freedom or parametric equations of a curve. For example, if the equation of the curve be $f(x, y) = 0$ and if there exists a parameter t, such that the equation $f(x, y) = 0$ is satisfied by the equations

$$x = \phi(t) \qquad y = \psi(t);$$

these equations are called the parametric equations of the locus whose *constraint* equation is $f(x, y) = 0$.

39.1. Freedom Equations of a circle.

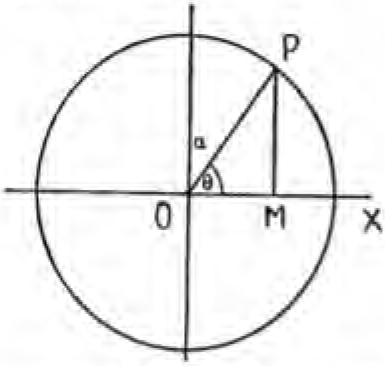
Suppose that the constraint equation of the circle is

$$x^2 + y^2 = a^2.$$

The equation is obviously satisfied by the equations

$$x=a\cos\theta \quad y=a\sin\theta.$$

where θ is a variable. These equations are freedom equations of the circle whose constraint equation is $x^2+y^2=a^2$. If we put $t=\tan\dfrac{\theta}{2}$, the equations may be written as



$$x=a\frac{1-t^2}{1+t^2},\ y=\frac{2at}{1+t^2}$$

or writing the equation in the form

$$\frac{y}{a+x}=\frac{a-x}{y}=t$$

we get $x=\dfrac{a(1-t^2)}{1+t^2}$, $y=\dfrac{2at}{1+t^2}$.

Geometrical representation of θ.

Let $P(x, y)$ be a point on the circle $x^2+y^2=a^2$, the axes of co-ordinates being two perpendicular diameters of the circle. Draw the ordinate MP. Obviously

$$x=a\cos MOP.\ y=a\sin MOP.$$

$\therefore\ \angle MOP=\theta.$

Freedom equations of the circle

$$(x-h)^2+(y-k)^2=a^2$$

are easily seen to be

$$x=h+a\cos\theta \quad y=k+a\sin\theta.$$

Ex. Prove that the parametric equations

$$x=a\sin\theta+b\cos\theta$$
$$y=a\cos\theta-b\sin\theta$$

represent a circle, θ being the parameter.

40. The points of intersection of a circle and a st. line.

Let the equations of the circle and the st. line be

$$S\equiv x^2+y^2+2gx+2fy+c=0$$

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}\left(=r\right)$$

Case I.

If $l^2+m^2\neq 0$, we shall suppose $l^2+m^2=1$, then r will denote the distance of the point (x, y) from the fixed point (x_1, y_1). The co-ordinates of an arbitrary point on the line are $(lr+x_1,\ mr+y_1)$. If this be the point of intersection of the two loci,

$$(lr+x_1)^2+(mr+y_1)^2+2g(lr+x_1)+2f(mr+y_1)+c=0$$

or $$r^2(l^2+m^2)+2r[l(x_1+g)+m(y_1+f)]+S_1=0 \quad \ldots\ldots(4)$$

where $$S_1\equiv x_1^2+y_1^2+2gx_1+2fy_1+c.$$

The equation is a quadratic in r which gives two values of r, corresponding to which we get two points U and V as the intersections of the two loci. Thus every st. line cuts a circle in two and only two points.

The discriminant of this equation is

$$[l(x_1+g)+m(y_1+f)]^2-(l^2+m^2)S_1$$

or $\quad (g^2+f^2-c)(l^2+m^2)-[m(x_1+g)-l(y_1+f)]^2$

i.e. $\quad (l^2+m^2)[R^2-p^2]$

where R is the radius and p the perpendicular from the centre $(-g, -f)$ on the line.

1. (*i*) If $l^2+m^2 \neq 0$ and $R^2-p^2>0$, the points U and V are real and distinct.

(*ii*) If $l^2+m^2 \neq 0$ and $R^2-p^2=0$, the points U and V coincide, the line is then said to be a tangent to the circle.

(*iii*) If $l^2+m^2 \neq 0$ and $R^2-p^2<0$, the points U and V are imaginary.

Case II.

(*i*) Let $l^2+m^2=0$, $m(x_1+g)-l(y_1+f) \neq 0$ *i.e.*, the given st. line does not pass through the centre of the circle, thus the co-efficient *viz.*, $l(x_1+g)+m(y_1+f) \neq 0$. Hence only one value of r is infinite. Only one intersection is therefore at infinity.

Note.—That the lines $m(x-x_1)-l(y-y_1)=0$ and $l(x-x_1)+m(y-y_1)=0$ are perpendicular to each other through (x_1, y_1). When $l^2+m^2=0$, both represent the same line. Thus the lines $y-y_1=\pm i(x-x_1)$ may each be regarded as perpendicular to itself.

(*ii*) If $l^2+m^2=0$, $m(x_1+g)-l(y_1+f)=0$,

i.e., the st. line passes through the centre of the circle, both the roots of the quadratic in r are infinite. The points U and V coincide at infinity, the line is a tangent, the point of contact being at infinity.

$$l \neq 0, \; m \neq 0.$$

If $l^2+m^2=0$, the equation of the line takes either of the form

$$y-y_1=i(x-x_1)$$
$$y-y_1=-i(x-x_1).$$

These lines are called *isotropic* or *circular lines* through (x_1, y_1).

41. Circular Points.

It has been proved in the preceding Art. that each of the st. lines

$$y=\pm ix+k,$$

meets every circle $x^2+y^2+2gx+2fy+c=0$ in a point at infinity. Now every st. line has one and only one point at infinity. Hence *the line $y=\pm ix+k$ meets every circle in its plane in the same point at infinity. Similarly the line $y=-ix+k$ meets every circle in the same point at infinity.*

Again, all the lines obtained by giving different values to k in the equation $y=ix+k$ have the same point at infinity. For if they be different, they lie on the circle and the line at infinity *i.e.*, the line at infinity meets the circle in more than two points which is impossible.

The equation

$$y=ix+k$$

where k varies may be regarded as representing as a system of parallel lines.

Thus every circle passes through two fixed points at infinity. These points are called the *circular points at infinity.*

It appears that concentric circles have double contact at the circular points.

The ratio of the co-ordinates of these points can easily be determined. By the equation,

$$\frac{y}{x}=\pm i+\frac{k}{x}. \quad \text{As } x\to\infty,\ y\to\infty, \quad \frac{y}{x}\to\pm i.$$

$$\therefore \quad x:y=\pm i:1.$$

The preceding results can be more easily obtained by the use of homogeneous co-ordinates. Making the equation of the circle homogeneous by writing $\frac{x}{z}$, $\frac{y}{z}$ for x and y, the equation becomes

$$x^2+y^2+cz^2+2fyz+2gzx=0.$$

This meets the line at infinity $z=0$ in points which lie on the locus $x^2+y^2=0$ or

$$y=ix,\ y=-ix.$$

and the points at infinity on these lines are

$$(1, i, 0),\ (1, -i, 0) \qquad \ldots\ldots(5)$$

which are the same for every circle.

It may be noted that an idea of direction cannot be associated with the circular lines. For if two lines OP, OQ make equal angles in the same sense with a third line OR, we say that OP and OQ coincide. But this ceases to be true if OR be given by the equation $y=\pm ix+k$, for $\tan\theta=\pm i$, which is independent of m, θ being the angle between $y=mx+c$ and $y=ix+k$ or $y=-ix+k$.

Omit

41.1. Perpendicularity. Let the lines OA, OB meet the line at infinity in points A and B, and suppose that I and J are the circular points at infinity. If (AB, IJ) = −1, the lines OA, OB are called perpendicular to each other.

Let the equation of the pair of lines PA, PB be

$$ax^2+2hxy+by^2+2gxz+2fyz+cz^2=0.$$

These lines meet the line at infinity $z=0$ in points A and B and the joint equation of OA, OB (O being the origin) is

$$ax^2+2hxy+by^2=0.$$

The equation of OI, OJ is

$$x^2+y^2=0.$$

and these are apolar if

$$a+b=0$$

If the axes are oblique, the equation of the circular lines through the origin is

$$x^2+2xy\cos\omega+y^2=0$$

and the condition of apolarity becomes

$$a+b=2h\cos\omega.$$

42. Equation of the chord whose mid-point is given.

Let $P(x_1, y_1)$ be the mid-point of the segment UV intercepted on the line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta} \qquad \ldots\ldots(i)$$

by the circle $\quad S\equiv x^2+y^2+2gx+2fy+c=0$,

then PU, PV are the roots of the equation

$$r^2+2r\,[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+S_1=0.$$

Since P is the mid-point of UV, PU + PV = 0.

$$\therefore \quad (x_1+g)\cos\theta+(y_1+f)\sin\theta=0 \qquad \ldots\ldots(ii)$$

Hence equation (*i*) will represent the required line if condition (*ii*) be satisfied, consequently its equation is

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0$$

or $\quad T-S_1=0 \qquad \ldots\ldots(6)$

where $\quad T\equiv x(x_1+g)+y(y_1+f)+gx_1+fy_1+c.$

Ex. Find the mid-point of the segment of the line $lx+my+n=0$ intercepted by the circle $x^2+y^2+2gx+2fy+c=0$.

43. The tangent at a given point. A tangent to a curve at a given point is the limiting position, if such exists, of a secant joining the given point to another point which approaches the given point along the curve and tends to coincide with it.

43.1. The tangent at a given point to a circle. Let $P(x_1, y_1)$ be the given point and suppose that the circle is given by the equation

$$S \equiv x^2+y^2+2gx+2fy+c=0.$$

The equation of an arbitrary st. line through $P(x_1, y_1)$ is

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r \qquad \text{......}(i)$$

The distances PU, PV of the points of intersection U and V from P are the roots of the equation

$$r^2+2r[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+S_1=0.$$

Since P lies on the circle, it coincides with U or V, say U, then $S_1=0$, and PV is given by the equation

$$r+2[(x_1+g)\cos\theta+(y_1+f)\sin\theta]=0.$$

Let V moving along the circle approach P and ultimately coincide with it : $PV \to 0$, and this limiting position of the line is the required tangent. The condition $PV \to 0$ demands

$$(x_1+g)\cos\theta+(y_1+f)\sin\theta=0 \qquad \text{......}(ii)$$

Thus line (i) is a tangent if condition (ii) is satisfied along with $S_1=0$. Hence the equation of the tangent is

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)=0, \qquad S_1=0$$

$$\left.\begin{array}{ll}\text{or} & x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0\\ \text{i.e.} & xx_1+yy_1+g(x+x_1)+f(y_1+f)+c=0\end{array}\right\} \qquad \text{......}(7)$$

44. Def. Normal The normal at a given point P of a plane curve is a st. line through P perpendicular to the tangent ot P to the curve.

To find the equation of the normal at $P(x_1, y_1)$ to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

The equation of the tangent at P to the circle being

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0,$$

the line at right angles to this through P is

$$(x-x_1)(y_1+f)=(y-y_1)(x_1+g) \qquad \text{.....}(8)$$

This is the required equation. It passes through the centre. Hence the proposition that the tangent to a circle is perpendicular to the radius through the point.

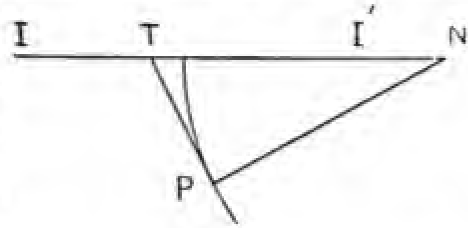
It follows that every st. line from the centre is normal to the circle.

44.1. We have shown that the circular lines through the centre are tangents to a circle at the circular points. Thus the circular lines through the centre are both tangent and normal to the circle.

It appears that a circular line is perpendicular to itself. It may also be proved as follows :—

Any pair of perpendicular st. lines are harmonically conjugate to the circular lines and conversely.

Let I and I′ be the circular points. Let the tangent at any point P on a curve meet the line at infinity at T.



Take the fourth harmonic conjugate N to T with respect to I, I′, then (TN, II′) = − 1.

Join NP. Then NP is normal to the curve at P.

If PT passes through a circular point *i.e.*, if T coincides with I, then N also coincides with I. Hence IP is both tangent and normal to the curve at P.

45. Geometrical significance of the expression

$$S \equiv x^2 + y^2 + 2gx + 2fy + c.$$

The line $\dfrac{x - x_1}{\cos\theta} = \dfrac{y - y_1}{\sin\theta}$ meets the $S = 0$ in points U and V and PU, PV are roots of the equation

$$r^2 + 2r[(x_1 + g)\cos\theta + (y_1 + f)\sin\theta] + S_1 = 0$$

$$\therefore \qquad PU.PV = S_1 \qquad \ldots\ldots(9)$$

This is called the **power** of the point P with respect to the circle S. The power depends only on the position of P and is independent of the direction of the line. If U and V coincide at T, the line PT becomes the tangent at T, and

$$PT^2 = S_1 \qquad \ldots\ldots(10)$$

Hence S_1 is the measure of the square of the tangent from P to the circle S.

If P is outside the circle, the segments PU, PV have the same sign and PU.PV is, therefore, positive. The product PU.PV is negative if P is inside the circle, since PU and PV have different signs. Hence

$$x^2 + y^2 + 2gx + 2fy + c \gtrless 0$$

is the analytic representation of outside and inside regions respectively of the circle $S = 0$.

46. Equation of tangents from a point.

Let $P(x_1, y_1)$ be a point in the plane of the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0.$$

The intersections U, V of the line

$$\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\cos\theta} = r \text{ with } S=0$$

are given by the equation

$$r^2+2r[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+S_1=0.$$

If the points U and V coincide, the line PUV becomes a tangent and therefore the quadratic in r has equal roots.

$$\therefore \qquad [(x_1+g)\cos\theta+(y_1+f)\sin\theta]^2=S_1.$$

Thus the equations of the tangents will be obtained by eliminating $\cos\theta$ and $\sin\theta$ between the equation of the line and the condition found above.
Eliminating we get

$$[(x-x_1)(x_1+g)+(y-y_1)(y_1+f)]^2=r^2S_1$$
$$=[(x-x_1)^2+(y-y_1)^2]S_1$$

or $\qquad (T-S_1)^2=[S+S_1-2T]S_1$ yes

where $T\equiv x(x_1+g)+y(y_1+f)+gx_1+fy_1+c$

i.e., $\qquad SS_1=T^2 \qquad$ ......(11)

This is of the second degree in x and y, hence two tangents can be drawn from a point to a circle.

These tangents are real, coincident or imaginary, according as the lines parallel to these through the origin are real, coincident, or imaginary. The equation of these lines is

$$(x^2+y^2)S_1-[x(x_1+g)+y(y_1+f)]^2=0$$

or
$$x^2[S_1-(x_1+g)^2]-2xy(x_1+g)(y_1+f)$$
$$+y^2[S_1-(y_1+f)^2]=0.$$

These are real, coincident or imaginary if

$$(x_1+g)^2(y_1+f)^2-[S_1-(x_1+g)^2][S_1-(y_1+f)^2] \gtreqless 0$$

or $\quad S_1[(x_1+g)^2+(y_1+f)^2-S_1] \gtreqless 0$

i.e., $\qquad (g^2+f^2-c)S_1 \gtreqless 0$

The circle being real, the square of the radius g^2+f^2-c is positive. The condition thus becomes $S_1 \gtreqless 0$. Hence :—

From a point two tangents can be drawn to a real circle, and these are real, coincident or imaginary, according as the point is outside, on, or inside the circle.

Illustrative Examples

(1) *Find the equation of the circle which circumscribes the triangle formed by the points* (x_i, y_i), $i=1, 2, 3$.

Suppose that the equation of the circle is

$$x^2+y^2+2gx+2fy+c=0.$$

Since the three points are supposed to lie on it,

$$\therefore\quad x_1^2+y_1^2+2gx_1+2fy_1+c=0$$
$$x_2^2+y_2^2+2gx_2+2fy_2+c=0$$
$$x_3^2+y_3^2+2gx_3+2fy_3+c=0.$$

Eliminating g, f, c we get the required equation, *viz.*,

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix}=0$$

The circle will be a proper circle if the co-efficient of x^2+y^2 is not zero. This requires that

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

should be different from zero, *i.e.* the three points should not be collinear.

(2) *Find the angle between the tangents from the point* (x_1, y_1) *to the circle* $x^2+y^2+2gx+2fy+c=0$. *Find also the locus of the point the tangents from which include an angle* ϕ. *What is the locus when* ϕ *is* $\frac{\pi}{2}$?

The equation of the pair of tangents from the point (x_1, y_1) to the circle $S\equiv x^2+y^2+2gx+2fy+c=0$ is

$$SS_1=T^2$$

where $T\equiv x(x_1+g)+y(y_1+f)+gx_1+fy_1+c$.

The lines parallel to these through the origin are given by the equation

$$S_1(x^2+y^2)=[x(x_1+g)+y(y_1+f)]^2$$

or $x^2[S_1-(x_1+g)^2]-2xy(x_1+g)(y_1+f)+y^2[S_1-(y_1+f)^2]=0$.

If ϕ be the angle between these lines,

$$\tan\phi=\pm\frac{2\sqrt{(x_1+g)^2(y_1+f)^2-[S_1-(x_1+g)^2][S_1-(y_1+f)^2]}}{2S_1-(x_1+g)^2-(y_1+f)^2}$$

$$=\frac{\pm 2\sqrt{S_1(g^2+f^2-c)}}{S_1-(g^2+f^2-c)}.$$

The locus of the point from which the tangents to the circle $S=0$ include an angle ϕ is therefore

$$[S-(g^2+f^2-c)]^2\tan^2\phi=4S(g^2+f^2-c).$$

If $\phi=\dfrac{\pi}{2}$, the required locus is

$$S=g^2+f^2-c.$$

which is a circle concentric with the given circle. It is called the **orthoptic** or director circle of the given circle.

(3) *Find the equation of the circle, the extremities of a diameter being* $A(x_1, y_1)$, $B(x_2, y_2)$.

Let $P(x, y)$ be a point on the circle, then

$$\angle APB=\frac{\pi}{2}.$$

The slopes of the lines PA and PB are

$$\frac{y-y_1}{x-x_1},\qquad \frac{y-y_2}{x-x_2};$$

Since $\angle APB=\dfrac{\pi}{2}$,

$$\therefore\quad \frac{y-y_1}{x-x_1}\times\frac{y-y_2}{x-x_2}=-1.$$

Hence the required equation is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \qquad \ldots\ldots(12)$$

(4) *The line* $lx+my=1$ *meets the lines* $ax^2+2hxy+by^2=0$ *in points P and Q. Show that the circle on PQ as diameter is*

$$(am^2-2hlm+bl^2)(x^2+y^2)+2x(hm-bl)+2y(hl-am)+a+b=0.$$

Deduce the condition of perpendicularity of the lines

$$ax^2+2hxy+by^2=0.$$

Eliminating y, we get the equation

$$x^2(am^2-2hlm+bl^2)-2x(bl-hm)+b=0$$

which gives the abscissae x_1, x_2 of the points of intersection (x_1, y_1), (x_2, y_2) of the line $lx+my=1$ with the given pair of lines. Hence this equation is identical with

$$(am^2-2hlm+bl^2)(x-x_1)(x-x_2)=0.$$

Similarly, eliminating x, we get the equation which gives y_1 and y_2. This equation will be found to be

$$y^2(am^2-2hlm+bl^2)-2y(am-hl)+a=0$$
$$\equiv(am^2-2hlm+bh^2)(y-y_1)(y-y_2).$$

Hence the equation of the circle is

$$(am^2-2hlm+bl^2)[(x-x_1)(x-x_2)+(y-y_1)(y-y_2)]=0$$

or

$$(am^2-2hlm+bl^2)(x^2+y^2)+2x(hm-bl)+2y(hl-am)+a+b=0.$$

If the given pair of lines be at right angles, this circle will pass through the origin,

$$\therefore \quad a+b=0.$$

(5) *Find the condition that the line $lx+my+n=0$ may touch the circle $x^2+y^2+2gx+2fy+c=0$.*

Let (x_1, y_1) be the point of contact, then the tangent

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$$

is identical with $\quad lx+my+n=0.$

$$\therefore \quad \frac{x_1+g}{l}=\frac{y_1+f}{m}=\frac{gx_1+fy_1+c}{n}=\lambda \text{ say.}$$

$$\begin{aligned} \therefore \quad x_1+g-\lambda l &=0 \\ y_1+f-\lambda m &=0 \\ gx_1+fy_1+c-\lambda n &=0 \\ \text{and} \quad lx_1+my_1+n &=0. \end{aligned}$$

Elimination of x_1, y_1, and λ gives the required condition

$$\begin{vmatrix} 1 & 0 & g & l \\ 0 & 1 & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0 \qquad \ldots\ldots(13)$$

This condition is equivalent to

$$(lg+mf-n)^2=(l^2+m^2)(g^2+f^2-c) \qquad \ldots\ldots(13\text{A})$$

which can be obtained immediately from the property of the circle that the measure of the perpendicular from the centre on the tangent equals the radius.

Omit (6) *Find the locus of the mid-points of a system of concurrent chords of a circle.*

Let the chords of the circle

$$S \equiv x^2+y^2+2gx+2fy+c=0$$

be drawn through the point (h, k). If (x_1, y_1) be the mid-point of one of the chords, its equation is

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=S_1.$$

Since it passes through (h, k),

$$\therefore \quad h(x_1+g)+k(y_1+f)+gx_1+fy_1+c=S_1.$$

Hence the locus of (x_1, y_1) is

$$h(x+g)+k(y+f)+gx+fy+c=S$$

i.e., $\quad x^2+y^2+gx+fy=h(x+g)+k(y+f)$

$$x^2+y^2+x(g-h)+y(f-k)=gh+fk.$$

The locus of the mid-points of a system of parallel chords may be deduced from this equation when the point (h, k) moves to infinity, then $h\to\infty$, $k\to\infty$ and suppose $k \div h=m$, so that

m is the slope of the chords. Writing the equation in the form

$$\frac{1}{h}(x^2+y^2)+x\left(\frac{g}{h}-1\right)+y\left(\frac{f}{h}-\frac{k}{h}\right)=g+f\,\frac{h}{h}$$

and making $h\to\infty$, $k\to\infty$, $k:h\to m$, the equation takes the limiting form

$$-(x+my)=g+fm$$

or $\qquad (x+g)+m(y+f)=0$

which can be easily obtained otherwise.

Note. The equation of the locus can also be written in the form

$$(x-h)(x+g)+(y-k)(y+f)=0$$

which represents a circle drawn on the join of (h, k), $(-g, -f)$ as diameter.

(7) *Find the equation of the circumcircle of the triangle formed by the lines* $U_i \equiv a_ix+b_iy+c_i=0$ $(i=1, 2, 3)$

Consider the equation

$$l(a_2x+b_2y+c_2)(a_3x+b_3y+c_3)+m(a_3x+b_3y+c_3)(a_1x+b_1y+c_1)$$
$$+n(a_1x+b_1y+c_1)(a_2x+b_2y+c_2)=0$$

where l, m, n are arbitrary.

The equation is satisfied by the co-ordinates of points which satisfy the equations of any two of the given lines. This equation, therefore represents some locus that circumscribes the triangle. This locus will be a circle if the co-efficient of x^2 equals the co-efficient of y^2 and the co-efficient of xy vanishes.

$$\therefore\quad l(a_2a_3-b_2b_3)+m(a_3a_1-b_3b_1)+n(a_1a_2-b_1b_2)=0$$
$$l(a_2b_3+a_3b_2)+m(a_3b_1+b_1a_3)+n(a_1b_2+a_2b_1)=0.$$

Hence by cross-multiplication we get

$$l:m:n=(a_1^2+b_1^2)(a_2b_3-a_3b_2):(a_2^2+b_2^2)(a_3b_1-a_1b_3)$$
$$:(a_3^2+b_3^2)(a_1b_2-a_2b_1)$$

consequently the equation of the circle is

$$\begin{vmatrix} \dfrac{a_1^2+b_1^2}{U_1} & \dfrac{a_2^2+b_2^2}{U_2} & \dfrac{a_3^2+b_3^2}{U_3} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=0$$

Examples XI

1. Find the equation of the diameter of the circle $x^2+y^2-6x+2y-8=0$ which passes through the origin.

2. Prove that the centres of the three circles $x^2+y^2=1$, $x^2+y^2+6x-2y=1$, $x^2+y^2-12x+4y=1$ lie on a st. line.

3. Find the equation af the circle whose centre is at the point (4, 5) and whose circumference passes through the centre of the circle $x^2+y^2+4x+6y=12$.

4. Show that the circle
$x^2+y^2-2ax\cos\alpha-2by\sin\alpha-a^2\sin^2\alpha=0$ intercepts a constant length $2a$ on the x-axis.

5. Show that the circle $x^2+y^2=a^2$ intercepts a length equal to $2\sqrt{a^2(1+m^2)-c^2} \div \sqrt{1+m^2}$ on the line $y=mx+c$.

6. *Show that the line $y=mx\pm a\sqrt{1+m^2}$ is a tangent to the circle $x^2+y^2=a^2$.*

7. Show that the line $lx+my+n=0$ is a tangent to the circle $x^2+y^2+2gx+2fy+c=0$ if $(lg+mf-n)^2=(l^2+m^2)\times(g^2+f^2-c)=0$; hence deduce that the tangents to the above circle parallel to $lx+my=1$ are given by the equations $l(x+g)+m(y+f)=\pm\sqrt{(1+m^2)(g^2+f^2-c)}$.

8. Show that the circles which pass through the points $(a, 0)$, $(b, 0)$ and touch the y-axis are

$$x^2+y^2-(a+b)x\pm 2\sqrt{ab}\,y+ab=0.$$

9. Find the equations of the circles which touch both the axes and are such that the length of the tangents drawn to each of them from the point $(2, -1)$ is $2\sqrt{5}$.

10. Show that the equation of the circle of radius ρ which touches the axis of y at a distance h from the origin (the centre of the circle being in the first quadrant) is $(x-\rho)^2+(y-h)^2=\rho^2$. Show that the equation of the other tangent to the circle through the origin is given by the equation

$$(\rho^2-h^2)x+2\rho hy=0.$$

11. Prove that the tangent to $x^2+y^2+gx+fy=0$ at the origin is $gx+fy=0$.

12. Find the equations of tangents from the point (8, 1) to the circle $x^2+y^2-2x-4y-20=0$.

13. Find the co-ordinates of the mid-point of the chord which the circle $x^2+y^2+4x-2y-3=0$ cuts off on the line $y=x+5$.

14. Show that the line $3x+4y+7=0$ touches the circle $x^2+y^2-4x-6y=12$ and find the point of contact.

15. If the points (x_1, y_1), (x_2, y_2) are the ends of the segment of a chord of a circle which subtends an angle ϕ at the circumference, show that the equation of the circle is

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=\pm\cot\phi[(x-x_1)(y-y_2)-(x-x_2)(y-y_1)].$$

16. Show that the equation of the nine point circle of the triangle whose vertices are $(m, 0)$, $(-n, 0)$ and $(0, p)$ is

$$2p(x^2+y^2)-(m-n)px-(mn+p^2)y=0.$$

17. The line $y=px$ meets a circle of radius a in points O and U (O being the origin) and the x-axis passes through the centre of the circle. Show that the equation of the circle described on OU as diameter is

$$(1+p^2)(x^2+y^2)-2a(x+py)=0.$$

18. Find the equation of the circle which passes through the point (2, 0) and which touches the st. line $x+2y-1=0$ at the point $(3, -1)$.

19. Show that the equation of the circle which touches the line $4x+3y=12$ at the point (3, 0) and passes through $(-1, -3)$ is $x^2+y^2-2x+3y-3=0$.

20. Show that the circle $x^2+y^2+2gx+2fy+c=0$ intercepts equal lengths on the lines $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ if $(gl_1+fm_1-n)^2(l_2^2+m_2^2)=(gl_2+fm_2-n_2)^2(l_1^2+m_1^2)$

47. Definitions:

Chord of contact. Let the tangents at points U and V of a circle meet in P, then the line UV is called the chord of contact of tangents through P or simply the chord of contact of P.

Pole and Polar. Let P be a point in the plane of the circle S. Any line through P meets the circle S in two points (say) U and V. Take a point Q such that

$$(PQ, UV)=-1 \quad \text{or} \quad \frac{2}{PQ}=\frac{1}{PU}+\frac{1}{PV}.$$

The locus p of Q, as the line varies, is called the polar of P, which will be proved to be a st. line, and P is called the pole of p.

47.1. To find the chord of contact of a point *w. r.* **to a circle.** Let $U(h_1, k_1)$ and $V(h_2, k_2)$ be the points of contact of the tangents through $P(x_1, y_1)$ to the circle

$$x^2+y^2+2gx+2fy+c=0.$$

The equations of the tangents at $U(h_1, k_1)$ and $V(h_2, k_2)$ are

$$x(h_1+g)+y(k_1+f)+gh_1+fk_1+c=0$$
$$x(h_2+g)+y(k_2+f)+gh_2+fk_2+c=0$$

and as these tangents pass through P

$$x_1(h_1+g)+y_1(k_1+f)+gh_1+fk_1+c=0$$
$$x_1(h_2+g)+y_1(k_2+f)+gh_2+fk_2+c=0.$$

But these are the conditions that the points U and V may lie on the line.

$$x_1(x+g)+y_1(y+f)+gx+fy+c=0,$$

which is the required equation ......(14)

47.2. To find the locus of the points of intersection of tangents to a circle drawn at the extremities of chords through a fixed point.

Let $P(x_1, y_1)$ be a fixed point in the plane of the circle

$$x^2+y^2+2gx+2fy+c=0.$$

Let a line through P meet the circle in U and V; and suppose that the tangents at U and V meet in $Q(x_2, y_2)$; then PUV is the chord of contact of Q, and its equation is, therefore

$$x(x_2+g)+y(y_2+f)+gx_2+fy_2+c=0.$$

Since it passes through P,

$$\therefore \quad x_1(x_2+g)+y_1(y_2+f)+gx_2+fy_2+c=0;$$

which shows that (x_2, y_2) lies on the line

$$x_1(x+g)+y_1(y+f)+gx+fy+c=0$$

or $\quad x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0 \quad$ ......(14A)

47·3. Polar of a point *w. r.* to a circle.

Let $\quad S\equiv x^2+y^2+2gx+2fy+c=0$

be the given circle and $P(x_1, y_1)$ the given point.

The line

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$$

meets the circle $S=0$ in U and V, where PU, PV are the roots of the equation

$$r^2+2r[(x_1+g)\cos\theta+(y_1+f)\sin\theta]+S_1=0$$

Denoting the roots by r_1, r_2,

$$\frac{1}{r_1}+\frac{1}{r_2}=-2\,[(x_1+g)\cos\theta+(y_1+f)\sin\theta]/S_1.$$

Let the co-ordinates of Q be (x, y) and $PQ=r$.

Since $\quad (PU\ QV)=-1,$

i. e., $$\frac{2}{PQ}=\frac{1}{PU}+\frac{1}{PV}$$

$$\therefore \quad \frac{2}{r}=\frac{1}{r_1}+\frac{1}{r_2}$$

or $$\frac{1}{r}=-\frac{(x_1+g)\cos\theta+(y_1+f)\sin\theta}{S_1}$$

Thus from the equation of the line, we have for the locus of Q the equation

$$(x-x_1)(x_1+g)+(y-y_1)(y_1+f)+S_1=0$$

i.e., $\quad x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0 \quad$ ......(14B)

which represents a st. line.

Ex. If the polar p of a point P with respect to a circle with centre C meets CP in Q, show that $CP \perp p$, and $CP.CQ = R^2$, R being the radius of the circle.

47.4. The slope of the line joining $P(x_1, y_1)$ with the centre $C(-g, -f)$ of the circle $S=0$ is

$$(y_1+f)/(x_1+g).$$

This is the negative reciprocal of the slope of the polar p of $P(x_1, y_1)$. *Hence the polar of* P *is perpendicular to the line that joins the centre with* P.

Let the polar p of P meet CP in Q. Then CQ being the perpendicular from C on p,

$$CQ = (g^2+f^2-c) \div [(x_1+g)^2+(y_1+f)^2]^{\frac{1}{2}};$$

and $$CP = [(x_1+g)^2+(y_1+f)^2]^{\frac{1}{2}}.$$

Consequently $CP.CQ = g^2+f^2-c = R^2$

where R is the radius of the circle.

If C be the centre of a circle of radius R, P and Q two points collinear with C and on the same side of it, such that $CP.CQ = R^2$; then P and Q are called inverse points *w.r.* to the circle.

The result of this Art. may be stated as follows:—

The polar of a point w. r. to a circle passes through the inverse of the point and is perpendicular to the line that joins the point with the centre.

47.5. The pole of a given line *w. r. to* **a given circle.** Let the equation of the given circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Suppose that $P(x_1, y_1)$ is the pole of the line

$$lx+my+n=0,$$

which is therefore identical with the polar of P, *viz.*,

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0.$$

Hence $$\frac{x_1+g}{l} = \frac{y_1+f}{m} = \frac{gx_1+fy_1+c}{n}.$$

and each one of these ratios,

$$= \frac{g^2+f^2-c}{lg+mf-n}$$

$\therefore\ x_1 : y_1 : 1 =$

$$l(f^2-c)-mfg+ng : -lfg+m(g^2-c)+nf : lg+mf-n.$$

If the line is a diameter, $lg+mf-n=0$, and the pole is at infinity in the direction $y : x = m : l$ or in homogeneous co-ordinates, its co-ordinates are $(l : m : 0)$.

If the given line be the line at infinity, $l=0$ $m=0$, the co-ordinates of the pole are, therefore

$$x_1 : y_1 : 1 = +g : +f : -1$$

and these are the co-ordinates of the centre of the circle. Thus :—*The pole of a diameter is at infinity in a direction perpendicular to the diameter and the pole of the line at infinity is the centre of the circle.*

47.6. Conjugate points. Conjugate lines.

If $Q(x_2, y_2)$ lies on the polar

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$$

of $P(x_1, y_1)$ *w. r.* to the circle $S=0$, then

$$x_2(x_1+g)+y_2(y_1+f)+gx_1+fy_1+c=0.$$

The symmetry of this relation proves that P also lies on the polar of Q. Such points are called *conjugate points.*

From this, it follows that if p contains the pole of q, then q contains the pole of p. Such lines are called *conjugate lines.*

If $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ be two conjugate lines, the pole of the first, *viz.*,

$$x_1 : y_1 : 1 = l_1(f^2-c)-m_1fg+n_1g : -l_1fg+m_1(g^2-c)+n_1f$$
$$: l_1g+m_1f-n_1$$

lies on the second, therefore

$$l_1l_2(f^2-c)+m_1m_2(g^2-c)-n_1n_2+(m_1n_2+m_2n_1)f$$
$$+(n_1l_2+n_2l_1)g-(l_1m_2+l_2m_1)fg=0.$$

If the equation of the circle be $x^2+y^2-a^2=0$, the condition reduces to

$$l_1l_2+m_1m_2=\frac{n_1n_2}{a^2}.$$

47.7. On the definition of polar.

If we are given a circle $S\equiv x^2+y^2+2gx+2fy+c=0$ and a point $P(x_1, y_1)$, the equation

$$x(x_1+g)+y(y_1+f)+gx_1+fy_1+c=0$$

defines a definite line which we have called the polar of P. The line coincides with the chord of contact of P if P lies outside the circle. The identity of the two loci is not so obvious if P be inside the circle. If, however, we assume the principle of continuity, we may say that two imaginary tangents can be drawn to the circle (Art. 46) from the point P when it lies inside the circle. The points of contact of these tangents are conjugate imaginaries *i.e.*, their co-ordinates are of the form $(a+ib, c+id)$, $(a-ib, c-id)$, and the line joining such points is a real line. It is also obvious from the fact that the algebra of Art. 47.1 takes no notice of whether the points of contact U and V are real or imaginary.

Again the identity of the locus of Q of Art. 47.2 with the polar of P may be established from the fact that the line PUV is the polar (chord of contact) of Q. Hence the points P and Q are conjugate ; consequently if any one of the points, say P, be fixed, the other point Q will trace out the polar of P. The identity of the three definitions of polar as given in Arts. 47.1, 47.2, 47.3 is thus established.

Also the *form* of the equation of the tangent is the same as that of the polar. This is because, if P is on the circle, the pair of tangents drawn from it to the two circles coincide ; the points of contact of these tangents also coincide with P. The chord of contact therefore is the line that joins two coincident points at P, *i.e.*, the tangent at P.

The difference between the tangent and the polar may however be noticed. In the case of a tangent, the point P lies on the tangent and the circle, but in the case of a polar, the point lies neither on the curve nor on the polar. If the point lies on any one of these, the polar will become the tangent. Hence the polar of a point on the curve is the tangent to the curve at the point, or the polar becomes the tangent if it contains its pole.

Illustrative Examples

(1) *Prove that if the pole of a st. line w. r. to the circle $x^2+y^2=c^2$ lies on the circle $x^2+y^2=\lambda^2c^2$, the polar touches the circle $x^2+y^2=\dfrac{c^2}{\lambda^2}$.*

Let the given line be $x\cos\alpha+y\sin\alpha=p$. If the pole of this line be (x_1, y_1), it is identical with $xx_1+yy_1-c^2=0$.

$$\therefore \quad \frac{x_1}{\cos\alpha}=\frac{y_1}{\sin\alpha}=\frac{c^2}{p}$$

since this point lies on the circle $x^2+y^2=\lambda^2c^2$

$$\therefore \quad \frac{c^4}{p^2}=\lambda^2c^2, \quad \therefore \quad c^2=\lambda^2p^2,$$

and this is the condition that the line $x\cos\alpha+y\sin\alpha=p$ may touch the circle $x^2+y^2=c^2/\lambda^2$.

(2) *Show that there exist, in general, two points which have the same polars w. r. to two given circles.*

Let the equations of the two circles be

$$x^2+y^2=a^2,\ x^2+y^2-2gx+c=0.$$

Suppose (x_1, y_1) is a point whose polars w. r. to the two circles coincide, *i.e.*, the lines

$$xx_1+yy_1=a^2 \quad x(x_1-g)+yy_1-gx_1+c=0.$$

$$\therefore \quad \frac{x_1-g}{x_1}=\frac{y_1}{y_1}=\frac{gx_1-c}{a^2} \qquad \ldots\ldots(i)$$

$\therefore$ either $g=0$ or $y_1=0$.

If $g=0$, both x_1 and y_1 are i inite or zero, otherwise (i) will not hold. The circles are then ncentric and the common centre is the only finite definite point whose polars coincide. The other point which is at infinity is indeterminate.

If $y_1=0$, then x_1 is given by the quadratic equation

$$gx^2-(a^2+c)x+a^2g=0,$$

which has two roots distinct or coincident.

If $a^2+c=\pm 2ag$, the circles touch each other and the point of contact is the only point whose polars are identical. In all other cases, there exist two points whose polars are the same for both the circles. But they are real only if the given circles do not cut each other in real points, and lie on the line of centres.

Proceeding on the lines of the preceding articles, it will be found that if the equation of the circle be

$$S\equiv x^2+y^2+2xy\cos\omega+2gx+2fy+c=0,$$

the line

$$T\equiv xx_1+yy_1+(xy_1+yx_1)\cos\omega+g(x+x_1)+f(y+y_1)+c=0$$

represents a tangent to the circle at (x_1, y_1) if the point lies on the circle, otherwise the polar of the point *w.r.* to the circle.

The equation of the pair of tangents from (x_1, y_1) is

$$SS_1=T^2$$

and the chord whose mid-point is (x_1, y_1) is represented by the equation

$$T=S_1.$$

These equations, however, will be very rarely used.

48. Conjugate Triangles. Let PQR be any triangle, and let p, q, r be the polars of P, Q, R *w.r.* to the circle S. Denote by (qr) the point of intersection of the polars q and r. Since the point (qr) lies on the polars of Q and R, the polar of (qr) must pass through the points Q and R and is therefore the line QR. Similarly the polars of (rp), (pq) are RP, PQ. Thus the triangles PQR; (qr) (rp), (pq) are such that each vertex of the one triangle is the pole of a side of the other. Such pairs of triangles are called *conjugate* or reciprocal *w.r.* to the circle.

In particular, if two conjugate triangles coincide, *i.e.*, each vertex of a triangle is the pole of the opposite side, the triangle is called *self-polar* or *self-conjugate*. The existence of such a triangle can be easily proved. Let p be the polar of a point P. On p take a point Q and draw its polar. This will pass through Q and meet p in R. Since R lies on the polars of P and Q, the polar of R is PQ. Hence PQR is the required triangle. There exist ∞^2 of such triangles.

48.1. Properties of conjugate triangles.

If $P_1P_2P_3$, $Q_1Q_2Q_3$ *are two triangles which are reciprocal for a circle, then* P_1Q_1, P_2Q_2, P_3Q_3, *meet in a point.*

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

and suppose that the co-ordinates of the points P_1, P_2, P_3 are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$. Set

$$U_i \equiv x(x_i+g)+y(y_i+f)+gx_i+fy_i+c, \quad i=1, 2, 3$$

and $$U_{ij} \equiv x_ix_j+y_iy_j+g(x_i+x_j)+f(y_i+y_j)+c$$

$$i=1, 2, 3, \quad j=1, 2, 3, \quad i \neq j$$

The equations of the lines Q_2Q_3, Q_3Q_1, Q_1Q_2, *i.e.*, the polars of P_1, P_2, P_3 are

$$U_1=0, U_2=0, U_3=0$$

The equation of an arbitrary st. line through Q_1 is

$$U_2+\lambda U_3=0.$$

This passes through P_1, if $U_{12}+\lambda U_{13}=0$. Thus the equation of P_1Q_1 is

$$U_{13}U_2-U_{12}U_3=0.$$

Similarly, the equations of P_2Q_2, P_3Q_3 are

$$U_{12}U_3-U_{23}U_1=0,$$
$$U_{23}U_1-U_{13}U_2=0.$$

These equations when added vanish. Hence etc.

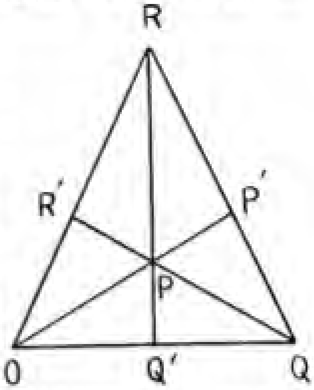
Let the points P_1, P_2, P_3 be on the circle. The polars of these points, *viz.*, the tangents at P_1, P_2, P_3 form the triangle $Q_1Q_2Q_3$ which is conjugate to the triangle $P_1P_2P_3$. We have thus the following theorem :—

The lines joining the vertices of a triangle to the points of contact of the in-circle with the opposite sides meet in a point.

48.2. *Self-conjugate triangle.* The existence of a self-polar triangle has already been proved. We proceed to prove the following theorem :—

There exists one and only one circle with respect to which a given triangle is self-conjugate, and this circle is real only if the triangle is obtuse-angled.

Let PQR be the given triangle. Then, since QR is the polar of P, the perpendicular from P on QR passes through the centre of the circle. Similarly the other two altitudes pass through the centre of the required circle. Thus the orthocentre O of the triangle is the centre of the circle. Let OP meet QR in P′, OQ meet PR in Q′, OR meet PQ in R′. Then P, P′ ; Q, Q′ ; R, R′ are pairs of inverse points *w.r.* to the circle ; hence the square of the radius must be equal to OP.OP′ or OQ.OQ′ or OR.OR′ and these are equal only if O is the ortho-centre. If ρ be the radius of the circle



$$\rho^2 = OP.OP' = OQ.OQ' = OR.OR'.$$

The given triangle then is self-conjugate *w. r.* to the circle with centre O and radius ρ, since QR is perpendicular to OP through the inverse P′ of P : so for RP and PQ.

The circle is imaginary if the triangle is acute-angled for then O is inside the triangle and hence $\rho^2 = OP.OP'$ is negative.

48 3. Another expression for the radius ρ.

$$OP' = P'Q \cot QOP' = P'Q \cot R = PQ \cot R \cos Q$$
$$= d \sin R \cot R \cos Q = d \cos R \cos Q,$$

where d is the circum-diameter.

$$PP' = PQ \sin Q = d \sin Q \sin R$$

$$\therefore \quad OP = OP' - PP' = d \cos (Q + R) = -d \cos P$$

$$\therefore \quad \rho^2 = OP.OP' = -d^2 \cos P \cos Q \cos R.$$

Obviously, ρ^2 is positive if one of the angles is obtuse. Hence the polar circle is only real if the triangle is obtuse-angled.

48.4. The self-conjugate triangle has one vertex within the circle and the other two outside.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ be the vertices of the self-conjugate triangle *w.r.* to the circle

$$S \equiv x^2 + y^2 + 2gx + 2fy + c = 0.$$

Now
$$PQ^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$$
$$= S_1 + S_2 - 2[x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c]$$
$$= S_1 + S_2$$

since P and Q are conjugate points.

Similarly $\quad QR^2 = S_2 + S_3, \; RP^2 = S_1 + S_3.$

$$\therefore \quad \cos P = \frac{PQ^2+PR^2-QR^2}{2PQ.PR}$$
$$= \frac{S_1}{PQ.PR}.$$

Similarly $\cos Q = S_2/QP.QR$, $\cos R = S_3/RP.RQ$.

Since the triangle is obtuse-angled at P, S_1 is negative and S_2, S_3 positive. Hence P is within the circle and Q, R outside.

It should be noticed that the vertex at which the angle is obtuse is within the circle and the other two outside.

Otherwise if P be within the circle, let OP cut the circle in T, T′, and QR in P′, then $(PP', TT') = -1$. P′ and hence Q, R and the line QR are outside the circle

$$S_1S_2S_3 = PQ^2.PR^2.\ QR^2 \cos P \cos Q \cos R$$
$$= PQ^2.PR^2.\ \sin^2 P.d^2 \cos P \cos Q \cos R$$
$$= -4\Delta^2\rho^2,$$

where Δ is the area of the triangle and d the circumdiameter.

But $S_1 = t_1^2$, $S_2 = t_2^2$, $S_3 = t_3^2$,

t_1, t_2, t_3 being the measures of the tangents from P, Q, R to the polar circle.

$$\therefore \quad t_1^2\, t_2^2\, t_3^3 = -4\Delta^2\rho^2.$$

This also proves that one of the t's is imaginary, therefore one vertex is inside the circle.

(1) Assuming the equation of the circle as $x^2+y^2-\rho^2=0$, obtain this result by multiplying the determinants

$$-4\Delta^2\rho^2 = \begin{vmatrix} x_1 & y_1 & \rho \\ x_2 & y_2 & \rho \\ x_3 & y_3 & \rho \end{vmatrix} \begin{vmatrix} x_1 & y_1 & -\rho \\ x_2 & y_2 & -\rho \\ x_3 & y_3 & -\rho \end{vmatrix}$$

(2) Show that $t_1^{-2}+t_2^{-3}+t_3^{-2} = -\rho^{-2}$, t_1, t_2, t_3 being the measures of the tangents to the circle from the vertices of the self-conjugate triangle.

With the notation of Arts 48.3, 48 4,

$$t_1^{-2}+t_2^{-2}+t_3^{-2} = \frac{1}{PQ.PR\cos P}+\frac{1}{QR.QP\cos Q}+\frac{1}{RP.RQ\cos R}$$
$$= \frac{1}{d^2 \sin P \sin Q \sin R}[\tan P + \tan Q + \tan R]$$
$$= \frac{\tan P \tan Q. \tan R}{d^2 \sin P. \sin Q \sin R} \quad \because \quad P+Q+R = (ii)$$
$$= \frac{1}{d^2 \cos P \cos Q\, R} = -\rho^{-2}.$$

Second method. By the identity

$$0=\begin{vmatrix} x_1 & y_1 & -\rho & 0 \\ x_2 & y_2 & -\rho & 0 \\ x_3 & y_3 & -\rho & 0 \\ 0 & 0 & -\rho & 0 \end{vmatrix}\begin{vmatrix} x_1 & y_1 & \rho & 0 \\ x_2 & y_2 & \rho & 0 \\ x_3 & y_3 & \rho & 0 \\ 0 & 0 & \rho & 0 \end{vmatrix}\equiv\begin{vmatrix} t_1^2 & 0 & 0 & \rho^2 \\ 0 & t_2^2 & 0 & \rho^2 \\ 0 & 0 & t_3^2 & \rho^2 \\ -\rho^2 & -\rho^2 & -\rho^2 & -\rho^2 \end{vmatrix}$$

the equation of the circle being assumed to be $x^2+y^2-\rho^2=0$].

(3) Find the equation of the polar circle of the triangle whose vertices are (x_i, y_i) $i=1, 2, 3$.

Let the equation of the circle be $x^2+y^2+2gx+2fy+c=0$. The polar of (x_1, y_1), *viz.*

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0,$$

passes through (x_2, y_2), (x_3, y_3); therefore

$$x_1x_2+y_1y_2+g(x_1+x_2)+f(y_1+y_2)+c=0$$
$$x_1x_3+y_1y_3+g(x_1+x_3)+f(y_1+y_3)+c=0$$

Similarly $\quad x_2x_3+y_2y_3+g(x_2+x_3)+f(y_2+y_3)+c=0.$

The elimination of g, f, c gives the required equation

$$\begin{vmatrix} x^2+y^2 & 2x & 2y & 1 \\ x_2x_3+y_2y_3 & x_2+x_3 & y_2+y_3 & 1 \\ x_3x_1+y_3y_1 & x_1+x_3 & y_1+y_3 & 1 \\ x_1x_2+y_1y_2 & x_1+x_2 & y_1+y_2 & 1 \end{vmatrix}=0.$$

Examples XII

V. Imp

1. Prove that the point $(1, -2)$ has the same polar *w. r.* to the circles whose equations are $x^2+y^2+x+7y+5=0$ and $x^2+y^2+2x+8y+5=0$ and deduce the existence of another point having the same property.

2. Show that there exist two points whose polars *w.r.* to the circle $x^2+y^2-2y+5=0$ coincide with their polars *w.r.* to the circle $x^2+y^2+2x+5=0$

Show also that these points are reflection of each other in the line $x+y=0$.

3. If P and Q are a pair of conjugate points *w.r.* to a circle, prove that the sum of their powers *w.r.* to the circle equals PQ^2.

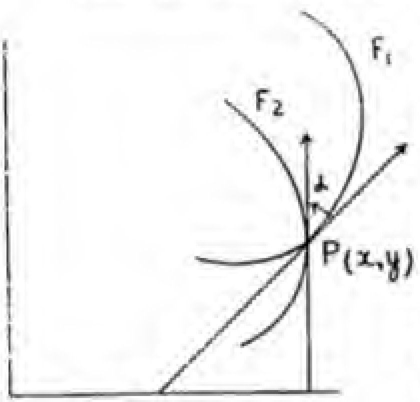
4. Show that the ratio of the distances of any two points from the polar of the other *w.r.* to a given circle is equal to the ratio of their distances from the centre of the other.

5. Prove that the equation of the circle for which the triangle formed by the points $(a, 0)$, $(-b, 0)$, $(0, c)$ is self-conjugate is $\quad c(x^2+y^2)-2aby+abc=0$.

49. Angle of intersection of two curves. In order to define the angle of intersection of two curves, it is necessary to assign a positive direction along the tangent at any point of a curve and the direction of arc s increasing along the curve. The direction of the arc s increasing along the curve can be assigned arbitrarily. We then make the following convention with regard to the positive direction along the tangent :—

The positive direction of the tangent is in the direction of a line drawn along the tangent in the direction of s increasing (Fowler).

Let two curves F_1 and F_2 intersect in a point $P(x, y)$.

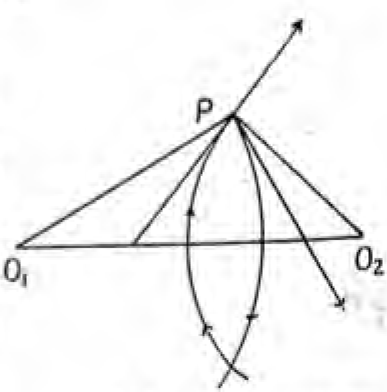


The angle of intersection of the curves F_1 and F_2 is defined to be the angle between the positive directions of the tangents to the two curves at the point (x,y). If $\alpha = \frac{\pi}{2}$, the curves are said to be orthognal or to cut each other at right angles. The curves are said to touch internally if $\alpha = 0$ and externally if $\alpha = \pi$.

49.1. The angle of intersection of two circles.

Let the equations of the circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$x^2+y^2+2g_2x+2f_2y+c_2=0$$



whose centres are O_1, O_2 and radii R_1, R_2. Suppose that the circles intersect at P. The tangents at P to the two circles being perpendicular to the radii OP_1, OP_2, the angle between the tangents is equal to the angle between the radii, *i.e.*, $\angle O_1PO_2$.

The circles intersect at one more point Q and it can easily be proved by elementary geometry that the angle of intersection at Q is equal to the angle of intersection at P. Let $\angle O_1PQ_2 = \alpha$ and $O_1O_2 = d$. Then from ΔO_1PO_2, $\quad d^2 = R_1^2 + R_2^2 - 2R_1R_2 \cos \alpha$

$$\therefore \cos \alpha = (O_1P^2 + O_2P^2 - O_1O_2^2) \div 2O_1P.O_2P$$

$$= \frac{(2g_1g_2 + 2f_1f_2 - c_1 - c_2)}{2\sqrt{(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)}}.$$

If the circles cut each other orthogonally, $\cos\alpha=0$.

$\therefore\ d^2=R_1^2+R_2^2$ i.e., $2g_1g_2+2f_1f_2-c_1-c_2=0$.

If the circles touch, $\cos\alpha=\pm1$. $d=R_1+R_2$ or $R_1\sim R_2$

i.e., $4(g_1^2+f_1^2-c_1)(g_2^2+f_2^2-c_2)=(2g_1g_2+2f_1f_2-c_1-c_2)^2$.

It should be noticed that the condition of orthogonality of two circles is linear in the co-efficients of the equations of both the circles. Conversely, if the co-efficients in the equation

$$a(x^2+y^2)+2gx+2fy+c=0,$$

of a circle, satisfy a linear relation

$$2gg'+2ff'+a'c+ac'=0,$$

the given circle is orthogonal to a fixed circle. For, comparing this with the result in Art. 49.1, we see that the given circle is orthogonal to the circle

$$a'(x^2+y^2)-2g'x-2f'y+c'=0.$$

50. Given two circles S, S′, prove that $S+S'=0$ represents the locus of the centre of a variable circle which is cut by S (or S′) orthogonally and by S′ (or S) at the extremities of a diameter.

Let O be the centre (x, y) and r the radius of the variable circle σ. Then since σ cuts S orthogonally, the tangent from O to S $= r$

$$\therefore\quad S=r^2 \qquad\qquad \ldots\ldots(i)$$

Again σ is cut by S′ at the extremities P, Q of a diameter

$\therefore$ $OP.OQ=-r^2$ (OP, OQ being of opposite signs).

But $OP.OQ=$ square of tangent from O to $S'=S'$

$$\therefore\quad S'=-r^2 \qquad\qquad \ldots\ldots(ii)$$

$\therefore$ The locus of O is given by the equation $S+S'=0$, which is a circle (called the radical circle of the coaxal system determined by S and S′.

50.1. The orthogonality of two circles may be expressed by any of the following :—

Two circles cut each other orthogonally :

(*i*) when the tangents to them at a common point are at right angles to each other ;

(*ii*) when the tangent from the centre of one to the other is equal to the radius of the former ;

(*iii*) when the sum of the squares of their radii is equal to the square of the distance between their centres ;

(*iv*) when the tangent to each at a common point passes through the centre of the other ;

(v) when the four points of intersection of any diameter with the circles form a harmonic range ;

(vi) when their radical circle passes through the centre of either circle and therefore of both ;

(vii) when the tangents at a common point and the circular lines through it form a harmonic pencil.

50.2. *Oblique axes.* Let the equations of the circles be

$$S_1 \equiv x^2+y^2+2xy\cos\omega+2g_1x+2f_1y+c_1=0$$
$$S_2 \equiv x^2+y^2+2xy\cos\omega+2g_2x+2f_2y+c_2=0,$$

then

$$2R_1R_2\cos\alpha=R_1^2+R_2^2-O_1O_2^2,$$

where α is the angle of intersection. R_1, R_2 their radii.

Let the co-ordinates of O_1 be (x, y), since the power of the centre *w.r.* to the circle is square of the radius with the sign changed

$$-R_1^2=S_1.$$

Also the power of O_1 with respect to S_2 being $O_1O_2^2-R_2^2$

$$\therefore \quad O_1O_2^2-R_2^2=S_2$$

$$\therefore \quad -2R_1R_2\cos\alpha=S_1+S_2 \qquad \text{......}(i)$$

Also the co-ordinates of the centre O_1 are given by the equations (Art. 38.2)

$$x+y\cos\omega+g_1=0 \qquad \text{......}(ii)$$
$$x\cos\omega+y+f_1=0 \qquad \text{......}(iii)$$

Subtracting from (*i*) the sum of $2x$ times (*ii*) and $2y$ times (*iii*) we have

$$2g_2x+2f_2y+c_1+c_2+2R_1R_2\cos\alpha=0.$$

Elimination of x, y gives

$$\begin{vmatrix} 1 & \cos\omega & g_1 \\ \cos\omega & 1 & f_1 \\ g_2 & f_2 & \dfrac{c_1+c_2}{2}+R_1R_2\cos\alpha \end{vmatrix}=0$$

or $\qquad R_1R_2\cos\alpha\sin^2\omega+\begin{vmatrix} 1 & \cos\omega & g_1 \\ \cos\omega & 1 & f_1 \\ g_2 & f_2 & (c_1+c_2)/2 \end{vmatrix}=0$

Substituting the values of R_1, R_2

$$\begin{vmatrix} 1 & \cos\omega & g_1 \\ \cos\omega & 1 & f_1 \\ g_1 & f_1 & f_1 \end{vmatrix}^{\frac{1}{2}} \begin{vmatrix} 1 & \cos\omega & g_2 \\ \cos\omega & 1 & f_2 \\ g_2 & f_2 & c_2 \end{vmatrix}^{\frac{1}{2}}\cos\alpha+\begin{vmatrix} 1 & \cos\omega & g_1 \\ \cos\omega & 1 & f_1 \\ g_2 & f_2 & \dfrac{c_1+c_2}{2} \end{vmatrix}=0$$

The condition of orthogonality is

$$\begin{vmatrix} 1 & \cos\omega & g_1 \\ \cos\omega & 1 & f_1 \\ g_2 & f_2 & \dfrac{c_1+c_2}{2} \end{vmatrix} = 0,$$

as $a = \dfrac{\pi}{2}$.

Illustrative Examples

(1) *If* S_1, S_2, S_3, S_4 *are the powers of a point w.r. to four mutually orthogonal circles whose radii are* r_1, r_2, r_3, r_4, *prove that*

$$\frac{S_1^2}{r_1^2}+\frac{S_2^2}{r_2^2}+\frac{S_3^2}{r_3^2}+\frac{S_4^2}{r_4^2}\equiv 0, \qquad \frac{S_1}{r_1^2}+\frac{S_2}{r_2^2}+\frac{S_3}{r_3^2}+\frac{S_4}{r_4^2}=-2.$$

Let the equations of the circles be

$$S_i \equiv x^2+y^2+2g_ix+2f_i+c_i=0, \qquad i=1, 2, 3, 4.$$

and (x, y) the co-ordinates of the point.

Multiply the two zero determinants,

$$0 \equiv \begin{vmatrix} 1 & 2g_1 & 2f_1 & c_1 & 0 \\ 1 & 2g_2 & 2f_2 & c_2 & 0 \\ 1 & 2g_3 & 2f_3 & c_3 & 0 \\ 1 & 2g_4 & 2f_4 & c_4 & 0 \\ 1 & -2x & -2y & x^2+y^2 & 0 \end{vmatrix} \begin{vmatrix} c_1 & -g_1 & -f_1 & 1 & 0 \\ c_2 & -g_2 & -f_2 & 1 & 0 \\ c_3 & -g_3 & -f_3 & 1 & 0 \\ c_4 & -g_4 & -f_4 & 1 & 0 \\ x^2+y^2 & x & y & 1 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} +2r_1^2 & 0 & 0 & 0 & S_1 \\ 0 & -2r_2^2 & 0 & 0 & S_2 \\ 0 & 0 & -2r_3^2 & 0 & S_3 \\ 0 & 0 & 0 & -2r_4^2 & S_4 \\ S_1 & S_2 & S_3 & S_4 & 0 \end{vmatrix}$$

or $\quad \Sigma \dfrac{S_1^2}{r_1^2} \equiv 0.$

Equating the co-efficient of (x^4+y^4) on both sides, we get the additional relation

$$\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}=0$$

or change x, y into g_4, f_4 in the last rows of the determinants.

If in the last row of the second determinant the elements x^2+y^2, x, y, 1, 0, be replaced by 1, 0, 0, 0, 0, the product determinant becomes

$$\begin{vmatrix} -2r_1^2 & 0 & 0 & 0 & 1 \\ 0 & -2r_2^2 & 0 & 0 & 1 \\ 0 & 0 & -2r_3^2 & 0 & 1 \\ 0 & 0 & 0 & -2r_4^2 & 1 \\ S_1 & S_2 & S_3 & S_4 & 1 \end{vmatrix}=0$$

i.e., $\dfrac{S_1}{r_1^2}+\dfrac{S_2}{r_2^2}+\dfrac{S_3}{r_3^2}+\dfrac{S_4}{r_4^2}+2=0.$

The same result is obtained by changing x, y into g_4, f_4 in the last row of the 1st determinant.

(2) *If* S_1, S_2, S_3, S_4 *be four mutually orthogonal circles, with radii* r_i $(i=1, 2, 3, 4)$ *prove that the circles*

$$\lambda_1 S_1+\lambda_2 S_2+\lambda_3 S_3+\lambda_4 S_4=0$$
$$\mu_1 S_1+\mu_2 S_2+\mu_3 S_3+\mu_4 S_4=0$$

will be orthogonal if $\sum\limits_{i=1}^{4} \lambda_i\mu_i r_i^2=0.$

Suppose the equations of the circles are written in the standard form. We may then write equations of the above two circles in the form

$$(x^2+y^2)\Sigma\lambda_1+2x\Sigma\lambda_1 g_1+2y\Sigma\lambda_1 f_1+\Sigma\lambda_1 c_1=0$$
$$(x^2+x^2)\Sigma\mu_1+2x\Sigma\mu_1 g_1+2y\Sigma\mu_1 f_1+\Sigma\mu_1 c_1=0$$

These circles cut each other orthogonally if

$$2\Sigma\lambda_1 g_1\Sigma\mu_1 g_1+2\Sigma\lambda_1 f_1\Sigma\mu_1 f_1-\Sigma\lambda_1\Sigma\mu_1 c_1-\Sigma\mu_1\Sigma\lambda_1 c_1=0$$

or $$2\Sigma\lambda_1\mu_1(g_1^2+f_1^2-c_1)+\Sigma\lambda_i\mu_j(2g_ig_j+2f_if_j-c_i-c_j)=0$$
$$i=1, 2, 3, 4. \qquad j=1, 2, 3, 4;\ i\neq j.$$

Since the circles are mutually orthogonal, the expressions $2g_ig_j+2f_if_j-c_i-c_j=0$ $\quad i\neq j$ and $g_i^2+f_i^2-c_i=r_i^2$. Hence the condition becomes

$$\Sigma\lambda_i\mu_i r_i^2=0, \qquad i=1, 2, 3, 4.$$

Examples XIII

1. Show that the circles given by the equations $(x-a)^2+p(x+a)^2+(1+p)y^2=0$, $(x-a)^2+q(x+a)^2+(1+q)y^2=0$ will cut each other orthogonally if $p+q=0$.

2. Show that the circle described on the join of two points as diameter is orthogonal to a circle *w.r.* to which the given points are conjugate.

3. Show that there are two circles which touch the line $lx+my+n=0$ and are orthogonal to circles

$$x^2+y^2+2g_1x+c=0 \quad x^2+y^2+2g_2x+c=0.$$

Show that their equations are $x^2+y^2+2kx-c=0$, where k is a root of the equation $(km-n)^2=(l^2+m^2)(k^2+c)$.

4. Show that a unique circle which passes through the point (x_1, y_1) and is cut orthogonally by the circles of Ex. 3 is

$$(x^2+y^2-c)y_1=(x_1^2+y_1^2-c)y.$$

Miscell neous Examples XIV

1. Find the centre and radius of the circle which passes through the three points $(2, 8)$, $(7, 3)$, $(-1, -1)$.

Find also the points in which the circle meets the line

$x+7y+2=0$. [Math. Trip. I. 1925]

2. Two circles touch the axis of y and intersect in the points $(1, 0)$, $(2, -1)$. Find the radii and show that they will both touch the line $y+2=0$. [Math. Trip. 1912]

3. A circle passes through the origin and touches the st. line $x=c$. Find the locus of the other extremities of the diameter through the origin.

4. Find the equation of the pair of tangents from the origin to the circle $x^2+y^2+2gx+2fy+k^2=0$ and show that their intercept on the line $y=h$ is $2hk\div(k^2-g^2)$ times the radius of the circle. [Math. Trip. 1916]

5. Show that the circum-circle of the triangle formed by the lines $y=\pm kx$ and $x\cos\alpha+y\sin\alpha-p=0$ is

$$(\cos^2\alpha-k^2\sin^2\alpha)(x^2+y^2)-p(1+k^2)(x\cos\alpha-y\sin\alpha)=0.$$

6. Show that the circumcircle of the triangle formed by the lines $ax+by+c=0$, $cx+ay+b=0$, $bx+cy+a=0$ passes through the origin if

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=abc(b+c)(c+a)(a+b).$$

7. The st. line $x\cos\alpha+y\sin\alpha-p=0$ being called the line (αp), find the equation of the circumcircle of the triangle formed by the lines (αp), (βq), (γr) and show that if it passes through the origin, then

$$qr\sin(\beta-\gamma)+rp\sin(\gamma-\alpha)+pq\sin(\alpha-\beta)=0.$$

8. A pair of parallel tangents is drawn to one of two equal circles and another pair of tangents perpendicular to the first pair is drawn to the other. Prove that each of the diagonals of the square formed by the four tangents passes through a fixed point. [Math. Trip. 1918]

9. Show that the locus of the poles of tangents to the circle $x^2+y^2=a^2$ w.r. to the circle $x^2+y^2=2bx$ is the conic

$$(a^2-b^2)x^2+a^2y^2-2a^2bx+a^2b^2=0.$$

10. Two variable st. lines are at right angles and are such that the mid. points of their intercepts on the axes are the fixed points $(h, 0)$, $(0, k)$. Show that the locus of their point of intersection is the circle $x^2+y^2-hx-ky=0$.

11. The circle $x^2+y^2+2gx-\delta^2=0$ is cut by the line $x=\lambda$ in points P, P′. Prove that the product of the perpendiculars from $(0, \pm\delta)$ on the tangents at P, P′ is equal to λ^2 for all values of g.

12. A circle passes through a fixed point and cuts two fixed perpendicular st. lines through O in points P and Q such that the line PQ always passes through a fixed point. Find the equation of the locus of the centre of the circle.

13. Find the locus of the point of intersection of two st. lines at right angles to one another each of which touches one of the two circles $(x-a)^2+y^2=b^2$, $(x+a)^2+y^2=c^2$.

14. Show that the st line $lx+my+n=0$ cuts the circle $x^2+y^2+2gx+2fy+c=0$ at an angle

$$\tan^{-1}\pm\sqrt{(g^2+f^2-c)(l^2+m^2)-(gl+fm-n)^2}\div(gl+fm-n).$$

15. Show that the equation of the polar circle of the triangle formed by the points $(\mathrm{R}\cos\alpha_i, \mathrm{R}\sin\alpha_i)$ $(i=1, 2, 3)$ is

$$(x-\Sigma\mathrm{R}\cos\alpha_i)^2+(y-\Sigma\mathrm{R}\sin\alpha_i)^2$$
$$=4\,\mathrm{R}^2\cos\frac{\alpha_1-\alpha_2}{2}\cos\frac{\alpha_2-\alpha_3}{2}\cos\frac{\alpha_3-\alpha_1}{2}.$$

CHAPTER VI

SYSTEMS OF CIRCLES

51. Radical axis. The radical axis of two circles is the locus of a point which has equal powers *w.r.* to the circles.

To find the radical axis of two given circles.

Let the equation of the circles be

$$S_1 \equiv x^2+y^2+2g_1x+2f_1y+c_1=0$$
$$S_2 \equiv x^2+y^2+2g_2x+2f\,y+c_2=0.$$

If (x, y) be the co-ordinates of a point on the locus, its powers S_1 and S_2 *w.r.* to the given circles are equal, *i.e.*,

$$S_1=S_2$$

or $$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0.$$

The required locus is therefore a st. line.

It is easy to show that *the radical axis of two given circles is a st. line perpendicular to the line of centres.*

51.1. The values of x and y which satisfy the equations of the given circles equally satisfy the equation of the radical axis. The radical axis, therefore, passes through the intersections of the given circles. The radical axis is real whether the circles meet in real or imaginary points. If the circles meet in real points, the common chord is the radical axis. When the points of intersection are imaginary, the radical axis is entirely outside the circles. But in either case, there exist real points on the radical axis which are outside both the circles and these points are tangentially equidistant from both of them. It thus follows that the radical axis bisects the segments intercepted between the points of contact of the common tangents of the given circles.

If the circles touch at a point P, the two points of intersection coincide at P and the line joining them, *viz.*, the common tangent at P is, therefore, the radical axis. The contact will be internal or external according as the centres of the circles lie on the same or opposite sides of the radical axis.

51.2. The radical axes of three given circles taken in pairs meet in a point, (provided that they are not co-axal).

Let equations of the given circles be $S_1=0$, $S_2=0$, $S_3=0$, (it being assumed that the equations are written in the

standard form). The equations of the radical axes are $S_2-S_3=0$, $S_3-S_1=0$, $S_1-S_2=0$. The sum of these equations vanishes identically, hence the three lines meet in a point.

The point of concurrence is called the **radical centre.** It is tangentially equidistant from the given circles.

It therefore follows that a circle, with the radical centre as centre and radius equal to the length of the tangent from the radical centre to any one of the given circles, will cut each of the given circles orthogonally.

The circle is called *orthotomic circle.*

The circle will be imaginary (because its radius will be imaginary) if the given circles overlap so as to enclose an area. It will be a point circle if the given circles have a common point.

Illustrative Examples

(1) *Find the radical centre of three given circles.*

Let the equations of the three circles be $S_i=0$, $i=1, 2, 3$. If (x, y) be the co-ordinates of the radical centre and λ the length of tangent from it to the given circles, then

$$x^2+y^2-\lambda^2+2g_1x+2f_1y+c_1=0$$
$$x^2+y^2-\lambda^2+2g_2x+2f_2y+c_2=0$$
$$x^2+y^2-\lambda^2+2g_3x+2f_3y+c_3=0$$

Eliminating $x^2+y^2-\lambda^2$, y and $x^2+y^2-\lambda^2$, x in turn we have

$$\begin{vmatrix} 1 & 2g_1x+c_1 & f_1 \\ 1 & 2g_2x+c_2 & f_2 \\ 1 & 2g_3x+c_3 & f_3 \end{vmatrix}=0 \qquad \begin{vmatrix} 1 & g_1 & 2f_1y+c_1 \\ 1 & g_2 & 2f_2y+c_2 \\ 1 & g_3 & 2f_3y+c_3 \end{vmatrix}=0$$

Whence

$$2x\begin{vmatrix} 1 & g_1 & f_1 \\ 1 & g_2 & f_2 \\ 1 & g_3 & f_3 \end{vmatrix} - \begin{vmatrix} 1 & f_1 & c_1 \\ 1 & f_2 & c_2 \\ 1 & f_3 & c_3 \end{vmatrix}=0$$

$$2y\begin{vmatrix} 1 & g_1 & f_1 \\ 1 & g_2 & f_2 \\ 1 & g_3 & f_3 \end{vmatrix} + \begin{vmatrix} 1 & g_1 & c_1 \\ 1 & g_2 & c_2 \\ 1 & g_3 & c_3 \end{vmatrix}=0$$

The radical centre will be a finite point if the determinant

$$\begin{vmatrix} 1 & g_1 & f_1 \\ 1 & g_2 & f_2 \\ 1 & g_3 & f_3 \end{vmatrix} \neq 0,$$

i e., the centres $(-g_i, -f_i)$, $i = 1, 2, 3$ of the three given circles are not *collinear.*

(2) *Prove that if the points in which the straight line* $l_1x+m_1y+n_1=0$ *meets the circle* $x^2+y^2+2g_1x+2f_1y+c_1=0$ *and those in which the straight line* $l_2x+m_2y+n_2=0$ *meets the circle* $x^2+y^2+2g_2x+2f_2y+c_2=0$ *lie on a circle then*

$$2(g_1-g_2)(m_1n_2-m_2n_1)+2(f_1-f_2)(n_1l_2-n_2l_1)+(c_1-c_2)(l_1m_2-l_2m_1)=0.$$

If S be the circle that passes through the four points and S_1, S_2 the given circles, the radical axis of S and S_1 is the line $l_1x+m_1y+n_1=0$ and that of S and S_2 is $l_2x+m_2y+n_2=0$. The point of intersection of these two lines lies on the radical axis $2(g_1-g_2)x+2(f_1-f_2)y+c_1-c_2=0$ of S_1 and S_2,

Hence
$$\begin{vmatrix} 2(g_1-g_2) & 2(f_1-f_2) & c_1-c_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

Thus the necessary and sufficient condition is that the given lines and the radical axis of the given circles should be concurrent.

52. Pencil of curves. Pencil of circles.

Def. A system of curves of any order, passing through a number of given points which is one less than the number required to determine a proper curve of that order, is called a pencil of curves.

The equation of the circles contains three disposable constants g, f, c, which can be determined by three independent conditions. Thus three non-collinear points are sufficient to determine a circle uniquely. *Hence all circles which pass through two given points form a pencil of circles.*

Let $S_1=0$, $S_2=0$ be two given circles. The equation

$$S_1+\lambda S_2=0 \qquad \ldots\ldots(2)$$

is satisfied by the common points of $S_1=0$, $S_2=0$, and the co-efficients of x^2 and y^2 are equal. The equation (2) therefore, represents for different values of λ a system of circles passing through two fixed points, which is consequently called a ***pencil of circles***. In particular, for $\lambda=0$, it represents

the circle $S_1=0$: for $\lambda = \infty$, the circle $S_2=0$: for $\lambda=-1$, the radical axis: and $\lambda=1$, the radical circle.

Similarly, the equation

$$S_1+\lambda U=0 \qquad \text{.....(2A)}$$

where U is linear in x and y represents a pencil of circles.

52.1. Coaxal Circles.

Let $S_1+\lambda_1 S_2=0$, $S_1+\lambda_2 S_2=0$ be two members of the pencil of circles generated by the circles $S_1=0$, $S_2=0$. The co-efficients of x^2+y^2 in the two equations being respectively $1+\lambda_1$, $1+\lambda_2$, the radical axis of these circles is

$$(1+\lambda_2)(S_1+\lambda_1 S_2)-(1+\lambda_1)(S_1+\lambda_2 S_2)=0$$

i.e., $\quad (\lambda_2-\lambda_1)(S_1-S_2)=0$

or $\quad S_1-S_2=0.$

Thus any two members of the pencil of circles have the same radical axis.

For this reason, a pencil of circles is called a *coaxal system* or coaxal family.

The same conclusion follows from the consideration of the pencil of circles given by the equation $S+\lambda U=0$. The radical axis of any two members of the family is always $U=0$.

Since the line of centres is perpendicular to the radical axis, it follows that the centres of all circles of the system lie on a st. line called the *line of centres*. The result can be obtained analytically. For if $S_i \equiv x^2+y^2-2a_i x-2b_i y+c_i=0$ $(i=1, 2)$, the centre of the circle $S_1+\lambda S_2=0$ is the point $\left(\dfrac{a_1+\lambda a_2}{1+\lambda}, \dfrac{b_1+\lambda b_2}{1+\lambda}\right)$ which is collinear with centres (a_1, b_1), (a_2, b_2) of the generating circles. Similarly the centres of all circles of the family $S+\lambda U=0$, lie on the st. line

$$\frac{x-a}{l}=\frac{y-b}{m},$$

where $U\equiv lx+my+n$ and $S\equiv x^2+y^2-2ax-2by+c$. Hence

The centres of all circles of a coaxal system lie on a st. line and any two members of the system have the same radical axis.

leave

52.2. A coaxal system may be generated by any two of its members.

Let $S_1+\lambda S_2=0$ be the pencil of circles, and suppose $\Sigma_1=0$ and $\Sigma_2=0$ are two given members of the system, where $\quad \Sigma_1\equiv S_1+\lambda_1 S_2$ and $\Sigma_2\equiv S_1+\lambda_2 S_2$.

Let $\Sigma \equiv S_1 + \lambda S_2 = 0$ be another member of the family. We then want to express Σ in terms of Σ_1, Σ_2. This is easily done by eliminating S_1 and S_2 from the equations of three circles. We then have

$$\begin{vmatrix} \Sigma_1 & 1 & \lambda_1 \\ \Sigma_2 & 1 & \lambda_2 \\ \Sigma & 1 & \lambda \end{vmatrix} = 0, \qquad i.e., \ \Sigma_1(\lambda-\lambda_2) - \Sigma_2(\lambda-\lambda_1) + \Sigma(\lambda_2-\lambda_1) = 0$$

$$\therefore \quad \Sigma \equiv \frac{\lambda-\lambda_2}{\lambda_1-\lambda_2}\Sigma_1 + \frac{\lambda_1-\lambda}{\lambda_1-\lambda_2}\Sigma_2.$$

52.3. Degenerate circles of the coaxal family.

Let $S_i \equiv x^2+y^2+2g_ix+2f_iy+c_i=0$, $i=1, 2$, be the equations of the generating circles, so that the equation of the family is

$$(1+\lambda)(x^2+y^2)+2x(g_1+\lambda g_2)+2y(f_1+\lambda f_2)+c_1+\lambda c_2=0.$$

This will degenerate if its discriminant is zero.

i.e., $(1+\lambda)[(1+\lambda)(c_1+\lambda c_2)-(f_1+\lambda f_2)^2-(g_1+\lambda g_2)^2]=0.$

Hence $\lambda=-1$ and for this value of λ, the equation $S_1+\lambda S_2=0$, *i.e.*, $S_1-S_2=0$ represents the radical axis. Two other values of λ are the roots of the equation

$$(1+\lambda)(c_1+\lambda c_2)-(f_1+\lambda f_2)^2-(g_1+\lambda g_2)^2=0, \qquad \ldots\ldots(3)$$

which expresses the property that the radius of the circle $S_1+\lambda S_2=0$ is zero. We have thus proved that *one of the circles of coaxal system is a circle of infinitely large radius (a st. line, viz., radical axis) and two are of infinitely small radius.*

It may be pointed out that a circle of zero radius can be interpreted as a pair of lines. For, any such circle is of the form

$$(x-a)^2+(y-b)^2=0$$

which can be written as

$$\{(x-a)+i(y-b)\}\{(x-a)-i(y-b)\}=0.$$

The centres of circles of zero radius of a coaxal system are called **limiting points** of the system.

The equation (3) can be written as

$$\lambda^2(g_2^2+f_2^2-c_2)+\lambda(2g_1g_2+2f_1f_2-c_1-c_2)+(g_1^2+f_1^2-c_1)=0$$

or
$$R_2^2\lambda^2+2\lambda R_1R_2\cos\alpha+R_1^2=0, \qquad (3A)$$

where R_1 and R_2 are the radii of the circles $S_1=0$, $S_2=0$ and α is their angle of intersection.

The equations of the point circles will be obtained by eliminating λ between equation 3 (or 3A) and $S_1+\lambda S_2=0$.

The required equation is

$$R_2^2S_1^2-2R_1R_2S_1S_2\cos\alpha+R_1^2S_2^2=0, \qquad \ldots\ldots(4)$$

or
$$R_2S_1=R_1S_2e^{i\alpha}, \qquad R_2S_1=R_1S_2e^{-i\alpha} \qquad \ldots\ldots(4A)$$

From equations (4A) or from the discriminant $-4R_1^2R_2\sin^2\alpha$ of equation (3A) it follows that limiting points are real if α is imaginary, coincident if $\alpha=0$ or π, and imaginary if α is real.

Hence a coaxal system has real, coincident or imaginary limiting points according as the circles S_1, S_2 do not intersect, do touch or do intersect each other.

It is easy to show that a common tangent to any two circles of the system subtends a right angle at either limiting point.

Ex. Write the equation (3A) in the form $R_2^2\lambda^2+\lambda(R_1^2+R_2^2-d^2)+R_1^2=0$ (where d is the distance between the centres) and from its discriminant discuss the nature of the limiting points.

53. The equation of a coaxal system in the simplest form.

Let the x-axis be the line of centres, then the equation of any member of the system is of the form

$$x^2+y^2-2\lambda x+c=0.$$

If the y-axis be chosen as the radical axis, the line $x=0$ meets each circle of the system in the same two points, real, coincident or imaginary. The abscissa of each of these two points of intersection is zero and their ordinates are given by the equation

$$y^2+c=0.$$

Hence if the points of intersection are the same, then c is the same for all circles. Consequently the equation of the system is

$$x^2+y^2-2\lambda x+c=0,$$

where c is constant and λ is a variable parameter.

The circles of the system intersect in real, coincident or imaginary points according as c is negative, zero or positive.

Evidently concentric circles from a coaxal system.

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53.1. Properties of coaxal system.

Let the pencil of circles be given by the equation

$$x^2+y^2-2\lambda x+c=0 \qquad \ldots\ldots(5)$$

where c is a constant and λ a variable parameter.

The parameter λ can be determined uniquely if one more point of the circle be given.

Thus one and only one circle can be drawn through a given point coaxal with the system.

A circle of the system will touch the line $lx+my+n=0$

if $\qquad (l\lambda+n)^2=(l^2+m^2)(\lambda^2-c).$

This is a quadratic in λ and therefore gives two values of λ.

Hence there are two circles of the system which touch a given line.

53.2. *An intersecting species of coaxal circles has imaginary limiting points, a non-intersecting species has real limiting points and a touching species has coincident limiting points which coincide with the point of contact.*

The equation (5) of the system of circles can be written as

$$(x-\lambda)^2+y^2=\lambda^2-c.$$

This will be a circle of zero radius if $\lambda=\pm\sqrt{c}$ and the co-ordinates of the centres of these circles are $(\pm\sqrt{c}, 0)$ which are real, coincident or imaginary according as c is positive, zero or negative, but with such values of c, the points of intersection of the radical axis $x=0$ with the circles of the system, whose ordinates are given by equation $y^2+c=0$, are imaginary, coincident or real. Hence, etc. Also each limiting point is the reflection of the other in the radical axis. No centre of a circle of the system can lie between the limiting points.

53.3. The polar of the limiting point $(\sqrt{c}, 0)$ *w.r.* to the circle (5) is given by the equation

$$\sqrt{c}x-\lambda(x+\sqrt{c})+c=0,\ c\neq 0$$

$$i.e.,\quad (x+\sqrt{c})(\sqrt{c}-\lambda)=0,$$

$$\text{or}\quad x+\sqrt{c}=0.$$

The equation of the line is independent of λ. Also the line passes through the other limiting point $(-\sqrt{c}, 0)$. *Hence the polars of a limiting point w.r. to the family coincide with a line through the other limiting point parallel to the radical axis.*

The limiting points are therefore ***inverse points*** *w.r.* to each circle of the system.

53.4. *The polars of a point* P *w.r. to the pencil of circles pass through a point* Q *and* PQ *subtends a right angle at either limiting point.*

The polars of $P(x_1, y_1)$ *w.r.* to the family are given for different values of λ by the equation

$$xx_1+yy_1-\lambda(x+x_1)+c=0,$$

which always passes through the intersection of the lines

$$xx_1+yy_1+c=0$$

$$x+x_1=0.$$

Hence the co-ordinates of Q are $\left(-x_1, \dfrac{x_1^2-c}{y_1}\right)$.

If L be the limiting point $(\sqrt{c}, 0)$ the slopes of PL and QL are respectively $y_1 \div (x_1 - \sqrt{c})$ and $-(x_1 - \sqrt{c}) \div y$ and their product is -1.

It is also easy to see that the mid.-point of PQ lies on the radical axis.

53.5. *Every circle through the limiting points of a coaxal system is orthogonal to each circle of the system.*

Let $x^2 + y^2 + 2Ax + 2By + D = 0$
be the equation of a circle, which goes through the limiting points $(\pm\sqrt{c}, 0)$ of the family

$$x^2 + y^2 - 2\lambda x + c = 0.$$

Consequently

$$c^2 + 2A\sqrt{c} + D = 0$$
$$c - 2A\sqrt{c} + D = 0$$
$$\therefore \quad A = 0, \quad D = -c.$$

Hence the equation of the circle takes the form

$$x^2 + y^2 - 2\mu y - c = 0 \qquad \ldots\ldots(6)$$

Since μ is variable this equation therefore represents a coaxal system of circles, every member of which evidently cuts each member of the given family orthogonally.

54. The angle of intersection of a given circle with a pencil of circles.

Let the circle

$$\sigma \equiv x^2 + y^2 - 2ux - 2vy + w = 0$$

with radius ρ cut three arbitrary circles S_1, S_2, S_3 of the pencil at angles α, β, γ. Let r_1, r_2, r_3 be the radii and A, B, C the centres of these circles S_1, S_2, S_3, where

$$S_i \equiv x^2 + y^2 - 2\lambda_i x + c = 0, \quad (i = 1, 2, 3).$$

Then

$$2\rho r_1 \cos\alpha = \rho^2 - (u^2 + v^2 + c) + 2u\lambda_1$$
$$2\rho r_2 \cos\beta = \rho^2 - (u^2 + v^2 + c) + 2u\lambda_2$$
$$2\rho r_3 \cos\gamma = \rho^2 - (u^2 + v^2 + c) + 2u\lambda_3$$

$$\therefore \quad \Sigma(\lambda_2 - \lambda_3) r_1 \cos\alpha = 0,$$

or

$$BC.r_1 \cos\alpha + CA.r_2 \cos\beta + AB.r_3 \cos\gamma = 0.$$

If $\alpha = \beta = \pi/2$, then $\gamma = \dfrac{\pi}{2}$ also. Hence, *if a variable circle cuts two circles of a coaxal system orthogonally, it cuts every circle of the system orthogonally.*

If α and β be constants, γ is also constant.

And $\gamma = \dfrac{\pi}{2}$ if λ_3 is determined from the equation

$(\lambda_2 - \lambda_3) r_1 \cos\alpha + (\lambda_3 - \lambda_1) r_2 \cos\beta = 0$ which gives a unique value of λ_3. Thus a variable circle, which cuts two circles of a coaxal system at constant angles, cuts every circle of the system at a constant angle, and is orthogonal to one circle of the system.

If in addition to α and β, $\gamma\left(\neq\frac{\pi}{2}\right)$ be also given, λ_3 is determined from a quadratic equation. Thus, if a variable circle σ cuts two circles of a coaxal system at constant angles, there exist two circles of the system, which are cut by σ at a given angle. In particular, σ will touch two fixed circles of the system.

Suppose that σ touches S_1 and S_2 and cuts S_3 orthogonally, then

$$BC.r_1 \pm CA.r_2 = 0.$$

Thus, the variable circle having similar contacts with two given circles cut, at right angles, the coaxal circle whose centre is the external centre of similitude; and, if the contacts are dissimilar, the coaxal circle whose centre is the internal centre of similitude.

Again, if $\alpha=\beta$ or $\pi-\beta$ and $\gamma=\frac{\pi}{2}$, $BC.r_1 \pm CA.r_2 = 0$.

Hence a variable circle which cuts two circles of a coaxal system at equal angles cuts orthogonally the fixed circle of the system having its centre at the external centre of similitude and a circle which cuts the given circles of the system at supplementary angles cuts orthogonally the fixed circle of the system whose centre is the internal centre of similitude. In particular, if σ is of infinite radius, *i e.*, the circle degenerates into a line (and the line at infinity), the theorem becomes :

The st. lines that cut two circles of the system at equal or supplementary angles pass respectively through the external or internal centres of similitude.

It is easy to show that a variable circle which cuts the three given circles at equal or suplementary angles belongs to one of four coaxal systems.

55. Conjugate system of coaxal circles.

A circle which cuts two given circles orthogonally belongs to a pencil every member of which cuts orthogonally each member of the pencil of circles generated by the given circles.

Let
$$x^2+y^2-2\lambda_1 x+k^2=0$$
$$x^2+y^2-2\lambda_2 x+k^2=0$$
be two circles of pencil $x^2+y^2-2\lambda x+k^2=0$ ... (i)
which are met orthogonally by the circle
$$x^2+y^2-2Ax-2By+C=0.$$
$\therefore$ $2A\lambda_1=C+k^2$
$2A\lambda_2=C+k^2$.
Hence $2A(\lambda_1-\lambda_2)=0$.
$\therefore$ $A=0$ and $C=-k^2$.

So the equation of the circle takes the form

$$x^2+y^2-2\mu y-k^2=0. \qquad \ldots\ldots(ii)$$

As μ varies, it will represent circles belonging to the coaxal system with the x-axis as the radical axis and the y-axis as the line of centres. Each member of the family passes through the points $(\pm k, 0)$ which are the limiting points of the system (i).

Conversely, every circle which passes through the limiting points of a pencil of circles belongs to a pencil and the two pencils of circles are mutually orthogonal.

Now the limiting points $(\pm k, 0)$ of the pencil (i) are the points of intersection of the pencil (ii). If these points are real, the first system is of a non-intersecting species and the second is of an intersecting species and has therefore imaginary limiting points, and *vice versa*.

With each coaxal system of circles is associated an orthogonal coaxal system, such that the line of centres of one is the radical axis of the other. Each system passes through the limiting points of the other and if one pencil has real limiting points, the other has imaginary limiting points. Consequently if one is an intersecting system the other is of a non-intersecting species.

Illustrative Examples

(1) *Find the co-ordinates of the limiting points of the system of coaxal circles determined by the circles*

$$x^2+y^2-6x-6y+4=0,\ x^2+y^2-2x-4y+3=0$$

and find also the equations of the circles of this coaxal system which touch the line $x+y-5=0$. [Math. Trip. 1931]

The pencil of circles generated by these two circles is

$$S\lambda \equiv x^2+y^2-2x\,\frac{3+\lambda}{1+\lambda}-2y\,\frac{3+2\lambda}{1+\lambda}+\frac{4+3\lambda}{1+\lambda}=0.$$

This will be a circle of zero radius if

$$(3+\lambda)^2+(3+2\lambda)^2-(4+3\lambda)(1+\lambda)=0$$

$$2\lambda^2+11\lambda+14=0$$

$$\therefore\quad \lambda=-2, \text{ or } \ -\frac{7}{2}.$$

For these values of λ, the centres of the circles are

$$(-1, 1),\quad \left(\frac{1}{5}, \frac{8}{5}\right).$$

These are the co-ordinates of the limiting points.

Now the circle $S\lambda$ will touch the line $x+y-5=0$ if

$$\frac{(3+\lambda)+(3+2\lambda)-5(1+\lambda)}{\sqrt{2}}=\sqrt{(3+\lambda)^2+(3+2\lambda)^2-(4+3\lambda)(1+\lambda)}$$

or $$(1-2\lambda)^2=2(2\lambda^2+11\lambda+14)$$
$$1+4\lambda^2-4\lambda=4\lambda^2+22\lambda+28$$
$$0.\lambda^2+26\lambda+27=0.$$

$\therefore\ \lambda\to\infty$ or $\dfrac{-27}{26}$. Hence the circles are

$x^2+y^2-2x-4y+3=0$ and $x^2+y^2+102x+48y-23=0$

(2) *If the circle $x^2+y^2-2ax-2by+c=0$ is a circle of a coaxal system having the origin at a limiting point, prove that the equation of the system is*
$$\lambda(x^2+y^2)-2ax-2by+c=0$$
and that the equation of the conjugate system is
$$(a+\mu b)(x^2+y^2)-c(x+\mu y)=0.$$

Since $x^2+y^2=0$, $x^2+y^2-2ax-2by+c=0$ are two members of the system, the equation of the system determined by them is
$$x^2+y^2-2ax-2by+c+k(x^2+y^2)=0$$
or $$\lambda(x^2+y^2)-2ax-2by+c=0$$
where $1+k$ is replaced by λ.

The radical axis of the system is given by the equation
$$2ax+2by=c,$$
and it meets the axes in points whose co-ordinates are $\left(\dfrac{c}{2a},\ 0\right),\left(0,\ \dfrac{c}{2b}\right)$. Consequently the co-ordinates of any point on the line can, by suitable choice of k, be represented by
$$\left(\frac{c}{2a(1+k)},\ \frac{ck}{2b(1+k)}\right).$$

The co-ordinates of the centres of circles of the orthogonal system are of this form. Hence the equation of the system is $x^2+y^2-\dfrac{c}{a(1+k)}\left(x+\dfrac{a}{b}ky\right)=0$ which takes the required form if we put $\mu=\dfrac{a}{b}k$.

Otherwise. Let the conjugate system be
$$x^2+y^2+2gx+2fy+k=0.$$

Since it passes through the limiting points of the 1st coaxal system and the limiting points are
$$(0,0)\left(\frac{ac}{a^2+b^2},\ \frac{bc}{a^2+b^2}\right)$$
$$\therefore\ k=0,\ 2ga+2fb+c=0.$$

The latter result follows also from the fact that the centre of each conjugate circle lies on the radical axis of the given coaxal system.

$\therefore$ the equation of the conjugate system takes the form

$$x^2+y^2+2gx-\frac{2ga+c}{b}y=0$$

Put $\dfrac{2ga+c}{b}=\dfrac{c\mu}{a+\mu b}$.

Thus the conjugate system is given by the equation

$$(a+\mu b)(x^2+y^2)-c(x+\mu y)=0.$$

(3) *Find the limiting points of the system of circles*

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+k)=0$$

and show that the square of the distance between them is

$$\{(c-k)^2-4f^2g^2+4f^2c+4g^2k\}\div(g^2+f^2).$$

Show also that the limiting points will subtend a right angle at the origin if $\dfrac{c}{g^2}+\dfrac{k}{f^2}=2$.

The circle $x^2+y^2+2x\dfrac{g}{1+\lambda}+2y\dfrac{\lambda f}{1+\lambda}+\dfrac{c+\lambda k}{1+\lambda}=0$ will be of zero radius if

$$g^2+\lambda^2 f^2-(1+\lambda)(c+\lambda k)=0$$

i.e. $$\lambda^2(f^2-k)-\lambda(k+c)+g^2-c=0.$$

If λ_1, λ_2 be the roots of this equation, the co-ordinates of the limiting points L_1, L_2 will be

$$\left(\frac{-g}{1+\lambda_1}, \frac{-\lambda_1 f}{1+\lambda_1}\right), \left(\frac{-g}{1+\lambda_2}, \frac{-\lambda_2 f}{1+\lambda_2}\right).$$

Now $$L_1L_2^2=g^2\left(\frac{1}{1+\lambda_1}-\frac{1}{1+\lambda_2}\right)^2+f^2\left(\frac{\lambda_1}{1+\lambda_1}-\frac{\lambda_2}{1+\lambda_2}\right)$$

$$=\frac{(\lambda_1-\lambda_2)^2}{(1+\lambda_1)^2(1+\lambda_2)^2}(g^2+f^2)$$

$$=\frac{[(\lambda_1+\lambda_2)^2-4\lambda_1\lambda_2](g^2+f^2)}{[1+(\lambda_1+\lambda_2)+\lambda_1\lambda_2]^2}$$

$$=\frac{[(k+c)^2-4(f^2-k)(g^2-c)](g^2+f^2)}{[f^2-k+(k+c)+g^2-c]^2},$$

which leads to the required result.

The segment L_1L_2 will subtend a right angle at the origin, if the circle on L_1L_2 as diameter passes through the origin. The equation of such a circle is,

$$\left(x+\frac{g}{1+\lambda_1}\right)\left(x+\frac{g}{1+\lambda_2}\right)+\left(y+\frac{\lambda_1 f}{1+\lambda_1}\right)\left(y+\frac{\lambda_2 f}{1+\lambda_2}\right)=0.$$

This passes through the origin if

$$g^2+\lambda_1\lambda_2 f^2=0$$

or $$g^2(f^2-k)+f^2(g^2-c)=0$$

or $$\frac{c}{g^2}+\frac{k}{f^2}=2.$$

Second Method. Let $x^2+y^2+2Gx+2Fy+C=0$ be the circle which meets the generating circles orthogonally, then

$$2gG=C+c,\ 2fF=C+k.$$

The equation of the orthogonal family is therefore

$$x^2+y^2+\frac{C+c}{g}x+\frac{C+k}{f}y+C=0$$

or $$x^2+y^2+\frac{c}{g}x+\frac{k}{f}Y+C\left(\frac{x}{g}+\frac{y}{f}+1\right)=0,$$

where C is the parameter of the family.

The circle

$$x^2+y^2+\frac{c}{g}x+\frac{k}{f}y=0 \qquad \ldots\ldots(i)$$

belongs to the family of which the radical axis is the line

$$\frac{x}{g}+\frac{y}{f}+1=0.$$

This is therefore the equation of the line L_1L_2. If R be the radius of the circle (i) and p the perpendicular from the centre of this circle on L_1L_2,

$$L_1L_2^2=4(R^2-p^2)$$

or $$L_1L_2^2=4\left[\tfrac{1}{4}\left(\frac{c^2}{g^2}++\frac{k^2}{f^2}\right)-\left(\frac{c}{2g^2}+\frac{k}{2f^2}-1\right)^2\frac{g^2f^2}{g^2+f^2}\right]$$

$$=\frac{c^2f^2+k^2g^2}{g^2f^2}-\frac{(cf^2+2kg^2-2g^2f^2)^2}{g^2f^2(g^2+f^2)}$$

$$=\frac{1}{g^2f^2(g^2+f^2)}\Big[c^2f^2g^2+c^2f^4+k^2g^4+k^2g^2f^2$$
$$-c^2f^4-k^2g^4-4g^4f^4-2ckf^2g^2+4cg^2f^4+4kg^4f^2\Big]$$

$$=\frac{1}{g^2+f^2}\Big[(c-k)^2-4f^2g^2+4f^2c+4g^2k\Big].$$

The circle given by equation (i) passes through the origin and the points L_1, L_2. If $\angle L_1OL_2$ is a right angle, the line L_1L_2 will be the diameter of the circle and therefore will pass through the centre $\left(\frac{-c}{2g}, \frac{-c}{2f}\right)$ of the circle.

$$\therefore\quad \frac{c}{g^2}+\frac{k}{f^2}=2.$$

(4) *Show that the condition that the two circles*

$$a_i(x^2+y^2)+g_ix+f_iy+c_i=0 \quad i=1, 2$$

may touch each other is

$$(a_1f_2-a_2f_1)(c_1f_2-c_2f_1)+(a_1g_2-a_2g_1)(c_1g_2-c_2g_1)+(a_1c_2-a_2c_1)^2$$
$$=\tfrac{1}{4}(f_1g_2-f_2g_1)^2.$$

The equation of the radical axis of the given circles is

$$(a_1g_2-a_2g_1)x+(a_1f_2-a_2f_1)y+(a_1c_2-a_2c_1)=0.$$

If the circles touch, the radical axis will be a tangent to every member of the family to which the given circles belong. A member of the family which passes through the origin is

$$(a_1c_2-a_2c_1)(x^2+y^2)+(g_1c_2-g_2c_1)x+(f_1c_2-f_2c_1)y=0.$$

The lines which join the origin with points of intersection of this circle with the radical axis are given by the equation

$$(x^2+y^2)(a_1c_2-a_2c_1)^2+[(c_1g_2-c_2g_1)x+(c_1f_2-c_2f_1)y]$$
$$\times[(a_1g_2-a_2g_1)x+(a_1f_2-a_2f_1)y]=0$$

or $x^2[(a_1c_2-a_2c_1)^2+(a_1g_2-a_2g_1)(c_1g_2-c_2g_1)]+y^2[(a_1c_2-a_2c_1)^2$
$+(a_1f_2-a_2f_1)(c_1f_2-c_2f_1)]+xy[(c_1g_2-c_2g_1)(a_1f_2-a_2f_1)$
$+(c_1f_2-c_2f_1)(a_1g_2-a_2g_1)]=0.$

These lines will coincide if

$$4[(a_1c_2-a_2c_1)^2+(a_1g_2-a_2g_1)(c_1g_2-c_2g_1)][(a_1c_2-a_2c_1)^2$$
$$+(a_1f_2-a_2f_1)(c_1f_2-c_2f_1)]=[(c_1g_2-c_2g_1)(a_1f_2-a_2f_1)$$
$$+(c_1f_2-c_2f_1)(a_1g_2-a_2g_1)]^2$$
$$4(a_1c_2-a_2c_1)^2[(a_1c_2-a_2c_1)^2+(a_1g_2-a_2g_1)(c_1g_2-c_2g_1)$$
$$+(a_1f_2-a_2f_1)(c_1f_2-c_2f_1)]$$
$$=[(c_1g_2-c_2g_1)(a_1f_2-a_2f_1)-(c_1f_2-c_2f_1)(a_1g_2-a_2g_1)]^2$$
$$=(a_1c_2-a_2c_1)^2(f_1g_2-f_2g_1)^2$$

and this leads to the desired result.

(5) Show that the generel equation of a circle which touches the two circles

$$x^2+y^2+2ax+c^2=0$$
$$x^2+y^2+2bx+c^2=0$$

may be written in the form

$$\{(c^2+\mu^2)(c^2+ab)\}^{\frac{1}{2}}\ (x^2+y^2+2\lambda x+c^2)$$
$$+c(ab-\lambda^2)^{\frac{1}{2}}\ (x^2+y^2+2\mu y-c^2)=0$$

$x^2+y^2+2\lambda x+c^2=0$ is a circle of the co-axal system determined by $x^2+y^2+2ax+c^2=0$, $r_1^2=a^2-c^2$ ......(*i*)
and $x^2+y^2+2bx+c^2=0$, $r_2^2=b^2-c^2$. ......(*ii*)
$x^2+y^2+2\mu y-c^2=0$ is a circle of the conjugate system.

Any circle through their intersections is given by the equations

$$x^2+y^2+2\lambda x+c^2+k(x^2+y^2+2\mu y-c^2)=0,$$
$$r^2(1+k)^2=\lambda^2+\mu^2k^2-c^2(1-k^2) \qquad ......(iii)$$

We have three variable parameters λ ; μ, k at our disposal.

(*iii*) cuts (*i*) and (*ii*) at ϕ_1, ϕ_2 respectively (say).

Then $c^2+c^2\,\dfrac{1-k}{1+k}+2rr_1\cos\phi_1-\dfrac{2a\lambda}{1+k}=0$

i.e., $c^2-a\lambda+rr_1\cos\phi_1\,(1+k)=0$ ......(*iv*)

and $c^2-b\lambda+rr_2\cos\phi_2\,(1+k)=0$ ......(*v*)

For $\phi_1=\phi_2=0$ or π, or $\phi_1=0$ or π, $\phi_2=\pi$ or 0,

$$\frac{c^2-a\lambda}{r_1}=\pm\frac{c^2-b\lambda}{r_2}, \quad \therefore \quad \frac{(c^2-a\lambda)^2}{a^2-c^2}=\frac{(c^2-b\lambda)^2}{b^2-c^2}$$

i.e. $(a+b)(\lambda^2+c^2)=2\lambda(c^2+ab)$ ......(*vi*)

Thus λ is equal to either root of this equation.

From (*iv*) or (*v*)

$$\frac{(c^2-a\lambda)^2}{a^2-c^2}=r^2(1+k)^2=\lambda^2-c^2+(\mu^2+c^2)k^2 \qquad \text{from } (iii)$$

$$\therefore \quad (\mu^2+c^2)k^2=\frac{c^4-2a\lambda c^2+a^2\lambda^2}{a^2-c^2}-\lambda^2+c^2$$

$$=\frac{c^2}{a^2-c^2}(\lambda^2-2a\lambda+a^2)=\frac{c^2}{a^2-c^2}\left[\lambda^2-\frac{a(a+b)(\lambda^2+c^2)}{c^2+ab}+a^2\right]$$

$$=\frac{(ab-\lambda^2)c^2}{c^2+ab}, \qquad \text{from } (vi)$$

Thus we have for the equation of the required circle

$$(x^2+y^2+2\lambda x+c^2)+\sqrt{\frac{ab-\lambda^2}{(\mu^2+c^2)(c^2+ab)}}\;c(x^2+y^2+2\mu y-c^2)=0.$$

56. Common tangents to two circles.

Let the equations of two circles be

$$S_1\equiv(x-a_1)^2+(y-b_1)^2-R_1^2=0$$
$$S_2\equiv(x-a_2)^2+(y-b_2)^2-R_2^2=0.$$

The line $(x-a_1)\cos\theta+(y-b_1)\sin\theta=R_1$ ......(*i*)
is a tangent to the circle $S_1=0$ at the point

$$(R_1\cos\theta+a_1,\ R_1\sin\theta+b_1).$$

Parallel tangents to S_2 are given by the equations

$$(x-a_2)\cos\theta+(y-b_2)\sin\theta=R_2$$
$$(x-a_2)\cos\theta+(y-b_2)\sin\theta=-R_2.$$

If any one of these tangents coincides with the tangent to S_1, the tangent will be a common tangent, and the conditions for this are

$(a_1-a_2)\cos\theta+(b_1-b_2)\sin\theta=R_2-R_1$ ......(*ii*)

or $(a_1-a_2)\cos\theta+(b_1-b_2)\sin\theta=-R_2-R_1$ ......(*iii*)

Each one of these equations gives two values of $\cos\theta$ and therefore two corresponding values of $\sin\theta$. There are, therefore, four common tangents to two given circles.

The line (i) subject to the condition (ii) will be a direct common tangent. For substituting the co-ordinates of the centres of the circles in (ii), we find that the centres are on the same side of (i). Hence the two tangents corresponding to the values of θ given by (ii) will give direct common tangents, and the other two are the transverse common tangents.

56.1. Combined equation of pair of tangents.

Taking the equations of circles as Art. 56 the line

$$(x-a_1)\cos\theta+(y-b_1)\sin\theta=R_1$$

is a direct common tangent if

$$(a_1-a_2)\cos\theta+(b_1-b_2)\sin\theta=R_2-R_1$$

$$\therefore\quad \cos\theta:\sin\theta:1=\begin{vmatrix} R_1 & y-b_1 \\ R_2-R_1 & b_1-b_2\end{vmatrix}:\begin{vmatrix} x-a_1 & R_1 \\ a_1-a_2 & R_2-R_1\end{vmatrix}:\begin{vmatrix} x-a_1 & y-b_1 \\ x-a_2 & y-b_2\end{vmatrix}.$$

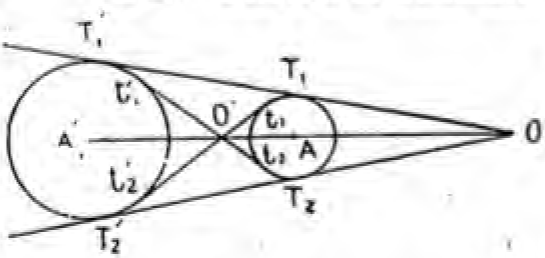
Eliminating $\cos\theta$, $\sin\theta$, we get the equation of the direct tangents in the form

$$\begin{vmatrix} x-a_1 & R_1 \\ a_1-a_2 & R_2-R_1\end{vmatrix}^2+\begin{vmatrix} y-b_1 & R_1 \\ b_1-b_2 & R_2-R_1\end{vmatrix}^2=\begin{vmatrix} x+a_1 & y-b_1 \\ x-a_2 & y-b_2\end{vmatrix}^2.$$

The equation of transverse common tangents will be obtained by replacing R_2 by $-R_2$.

56.2. Chords of contact.

Let T_1T_1', T_2T_2' be the direct common tangents and $t_1t'_2$, $t_2t'_1$ the transverse common tangents. It is required to find the equations of the lines T_1T_2, t_1t_2, $T'_1T'_2$, $t'_1t'_2$.



Let (x, y) be the point of contact of a direct common tangent with the circle S_1, then

$$x=a_1+R_1\cos\theta,\ y=b_1+R_1\sin\theta, \qquad ..(iv)$$

and $\cos\theta$, $\sin\theta$ satisfy the relation (ii) (Art. 56). Hence the equation of T_1T_2 will be obtained by the elimination of $\cos\theta$, $\sin\theta$ between (iv) and (ii), consequently the equation of T_1T_2 is

$$(x-a_1)(a_1-a_2)+(Y-b_1)(b_1-b_2)=R_1(R_2-R_1).$$

Similarly we get the following equations

$T'_1T'_2$ $\quad (x-a_2)(a_1-a_2)+(y-b_2)(b_1-b_2)=R_2(R_2-R_1)$

t_1t_2 $\quad (x-a_1)(a_1-a_2)+(y-b_1)(b_1-b_2)=-R_1(R_1+R_2)$

$t'_1t'_2$ $\quad (x-a_2)(a_1-a_2)+(y-b_2)(b_1-b_2)=-R_2(R_1+R_2)$.

57. Centres of similitude. The point of intersection of direct common tangents of two given circles is called the external centre of similitude. The internal centre of similitude is the point of intersection of transverse common tangents. The circle on the join of the two centres of similitude as diameter is called the circle of similitude.

The co-ordinates of the centres of similitude.

Let O (x_1, y_1) be the co-ordinates of the external centre of similitude. Its polar *w.r.* to S_1, *viz.*

$$(x-a_1)(x_1-a_1)+(y-b_1)(y_1-b_1)-R_1^2=0$$

is identical with the line T_1T_2 whose equation is

$$(x-a_1)(a_1-a_2)+(y-b_1)(y_1-b_2)=R_1(R_2-R_1).$$

$$\therefore \quad \frac{x_1-a_1}{a_1-a_2}=\frac{Y_1-b_1}{b_1-b_2}=\frac{R_1}{R_2-R_1}$$

$$\text{or} \quad x_1=\frac{R_1a_2-R_2a_1}{R_1-R_2}, \quad y_1=\frac{R_1b_2-R_2b_1}{R_1-R_2} \qquad \ldots\ldots(7)$$

Similarly, the co-ordinates of the internal centre of similitude are

$$\frac{R_1a_2+R_2a_1}{R_1+R_2}, \quad \frac{R_1b_2+R_2b_1}{R_1+R_2} \qquad \ldots\ldots(7A)$$

Or let the common tangent TT′ meet the line of centres AA′ in O. Therefore from the similarity of Δs ATO, A″T′O′ we have AO : A′O = AT : A′T′ = R : R′.

Hence the *centres of similitude of two given circles may be defined as the points which divide the line joining the centres of two circles internally and externally in the ratio of the radii.*

From the definition Arts 56, 56.1, 56.2 follow once.

Ex. 1. The lines PP′, QQ′ meet in one centre of similitude and the lines PQ′, P′Q in the other, PQ, P′Q′ being parallel diameters of the two circles.

Ex. 2. Any secant drawn through either centre of similitude is divided by the circles in the ratio of their radii.

57.1. Circle of similitude.

Let $S_1=0$, $S_2=0$ be the circles.

The equation of the circles on the join of the centres of similitude given by relations 7 and 7A as diameter is

$$\left(x-\frac{R_1a_2-R_2a_1}{R_1-R_2}\right)\left(x-\left(\frac{R_1a_2+R_2a_1}{R_1+R_2}\right)\right)$$
$$+\left(y-\frac{R_1b_2-R_2b_1}{R_1-R_2}\right)\left(y-\frac{R_1b_2+R_2b_1}{R_1+R_2}\right)=0$$

or $$[R_1(x-a_2)-R_2(x-a_1)][R_1(x-a_2)+R_2(x-a_1)]$$
$$+[R_1(x-b_2)-R_2(x-b_1)][R_1(x-b_2)+R_2(x-b_1)]=0$$

y

or $R_1^2[(x-a_2)^2+(y-b_2)^2-R_2^2]-R_2^2[(x-a_1)^2+(y-b_1)^2-R_1^2]=0$.

$$i.e., \quad \frac{S_1}{R_1^2}-\frac{S_2}{R_2^2}=0, \qquad \ldots\ldots(8)$$

which belongs to the co-axal system generated by $S_1=0$, $S_2=0$.

Thus the circle of similitude may be regarded as the locus of a point from which the tangents to the two given circles or whose distances from the centres are in the ratio of their radii.

Evidently the circles of similitude of three circles taken in pairs are co-axal. They, also, cut orthogonally the circle through the centres of the given circles. For if C be the mid-point of the centres of similitude O, O′ in virtue of relation $(OO', AA')=-1$, $CA.CA'=CO^2$. *i.e.*, the power of C *w.r.* to the circle through the centres of the given circles equals the square of the radius of circle of similitude.

The six centres of similitude of three circles (taken two by two) lie by threes on four lines called the axes of similitude and through each centre of similitude there pass two axes of similitude.

Let the equations of the three circles be

$$S_k\equiv(x-a_k)^2+(y-b_k)^2-R_k^2=0, \quad k=1, 2, 3$$

and suppose that $I_{i,j}$, $E_{i,j}$ are respectively the internal and external centres of similitude of circles S_i and S_j. The co-ordinates of $I_{i,j}$ and $E_{i,j}$ are respectively

$$\frac{a_iR_j+a_jR_i}{R_j+R_i}, \frac{b_iR_j+b_jR_i}{R_j+R_i} \qquad i, j=1, 2, 3 \; i\neq j$$

$$\frac{a_iR_j-a_jR_i}{R_j-R_i}, \frac{b_iR_j-a_jR_i}{R_j-R_i}.$$

Let the equation of the line that joins E_{12}, and E_{13} be

$$lx+my+n=0.$$

Substituting the co-ordinates of the point E_{12} in the equation of the line, we have

$$\frac{la_1+mb_1+n}{R_1}=+\frac{la_2+mb_2+n}{R_2}=\lambda.$$

Similarly, substituting the co-ordinates of E_{13}, it is found that

$$\frac{la_1+mb_1+n}{R_1}=\frac{la_3+mb_3+n}{R_3}$$

Consequently $\quad \dfrac{la_1+mb_1+n}{R_1}=\dfrac{la_2+mb_2+n}{R_2}=\dfrac{la_3+mb_3+n}{R_3}$

To find the equation of the line put each rotio equal to λ.

$$\therefore \quad \begin{aligned} lx+my+n &= 0 \\ la_1+mb_1+n-R_1\lambda &= 0 \\ la_2+mb_2+n-R_2\lambda &= 0 \\ la_3+mb_3+n-R_3\lambda &= 0. \end{aligned}$$

Hence the equation of the line is

$$L(R_1\ R_2, R_3) \equiv \begin{vmatrix} x & y & 1 & 0 \\ a_1 & b_1 & 1 & R_1 \\ a_2 & b_2 & 1 & R_2 \\ a_3 & b_3 & 1 & R_3 \end{vmatrix} = 0 \qquad \ldots\ldots(9)$$

It is similarly found that the six points arrange themselves on the four lines according to the following scheme. The points

E_{12}	E_{23}	E_{31}	lie on the	$L(R_1, R_2, R_3) = 0$
E_{12}	I_{23}	E_{31}	,,	$L(R_1, R_2, -R_3) = 0$
I_{12}	E_{23}	I_{31}	,,	$L(-R_1, +R_2, R_3) = 0$
I_{12}	I_{23}	E_{31}	,,	$L(R_1, -R_2, R_3) = 0.$

It is seen from the table that while each line contains three points, through each point there pass two lines. The fact is expressed as follows :—

points	lines
6	4
3	2

Menelaus's theorem yields the above result at once.

Illustrative Examples

(1) *Find the common tangents of the circles*

$$x^2+y^2+10x-6Y+25=0, \quad x^2+y^2+2x-10y+25=0.$$

The equations of the circles can be thrown in the form

$$(x+5)^2+(y-3)^2=9, \quad (x+1)^2+(y-5)^2=1.$$

The line $(x+5)\cos\theta+(y-3)\sin\theta-3=0$ is a tangent to the first circle, and it will be a tangent to the second circle also if the perpendicular from its centre is equal to its radius.

$$\therefore \quad 4\cos\theta+2\sin\theta-3=\pm 1 \qquad \ldots\ldots(i)$$

Taking first the positive sign, the condition is

$$2\cos\theta+\sin\theta=2$$

or $$4(\cos\theta-1)^2=\sin^2\theta=1-\cos^2\theta$$
$$5\cos^2\theta-8\cos\theta+3=0$$
i.e., $$(5\cos\theta-3)(\cos\theta-1)=0$$

$\therefore$ $\cos\theta=1$ or $\cos\theta=\dfrac{3}{5}$

Hence $\sin\theta=0$, or $\sin\theta=\dfrac{4}{5}$.

Consequently the equations of two of the tangents are
$$x+2=0 \qquad 3x+4y=12.$$

Now taking the negative sign in (*i*) the condition becomes
$$2\cos\theta+\sin\theta=1$$
or $$4(1-\sin^2\theta)=1-2\sin\theta+\sin^2\theta$$
or $$5\sin^2\theta-2\sin\theta-3=0$$
or $$(5\sin\theta+3)(\sin\theta-1)=0$$

$$\therefore \left.\begin{array}{l}\sin\theta=1\\ \cos\theta=0\end{array}\right\} \text{ and } \left\{\begin{array}{l}\sin\theta=-\dfrac{3}{5}\\ \cos\theta=\dfrac{4}{5}\end{array}\right\}$$

Hence the equations of two other tangents are
$$y=6 \qquad 4x-3y+14=0.$$

(2) *Find the common tangents to the circles*
$$x^2+y^2-2x-2y+1=0 \quad x^2+y^2-8x-8y+28=0.$$

The co-ordinates of the centre of the first circle are (1, 1) and its radius is unity, while the centre of the second circle is (4, 4) and its radius is 2. Consequently the centres of similitude of the two circles are
$$(2, 2), (-2, -2).$$

The equation of the pair of tangents from (2, 2) is
$$x^2+y^2-2x-2y+1=(x+y-3)^2 \qquad \text{(Chap. V)}$$
or $$x=2,\ y=2.$$

The equation of the pair of tangents that pass through $(-2, -2)$ is
$$17(x^2+y^2-2x-2y+1)=(3x+3y-5)^2$$
or $$4(x^2+y^2)-9xy-4x-4y-8=0$$
or $$8x=(9y+2)\pm(y+2)\sqrt{17}.$$

58. Net of Circles. If S_1, S_2, S_3 are three circles not belonging to a co-axal system, the equation
$$\lambda S_1+\mu S_2+v S_3=0 \qquad \text{.........(10)}$$
where λ, μ, v, are variable parameters, represents a circle which is member of the system with two degrees of freedom. This is called a *net* or *bundle* of circles.

A net of circles is a system with a common radical centre.

Let
$$\lambda_1 S_1 + \mu_1 S_2 + v_1 S_3 = 0$$
$$\lambda_2 S_1 + \mu_2 S_2 + v_2 S_3 = 0$$
be two members of the system. Then the radical axis of the two circles is
$$\frac{\lambda_1 S_1 + \mu_1 S_2 + v_1 S_3}{\lambda_1 + \mu_1 + v_1} = \frac{\lambda_2 S_1 + \mu_2 S_2 + v_2 S_3}{\lambda_2 + \mu_2 + v_2}$$
or
$$(\mu_1 v_2 - \mu_2 v_1)(S_2 - S_3) + (v_1 \lambda_2 - v_2 \lambda_1)(S_3 - S_1) + (\lambda_1 \mu_2 - \lambda_2 \mu_1)(S_1 - S_2) = 0.$$

Unless $\dfrac{\lambda_1}{\lambda_2} = \dfrac{v_1}{v_2} = \dfrac{\mu_1}{\mu_2}$, which is the condition that circles may coincide, the line always passes through the point of intersection of the radical axes $S_2 - S_3 = 0$, $S_3 - S_1 = 0$, $S_1 - S_2 = 0$ of S_1, S_2, S_3 and these lines meet at the radical centre of the three circles. Thus the radical axis of any two circles of the system passes through a fixed point. Hence any trio of circles has the same radical centre, v is the radical centre of the circles which generate the system.

It, therefore, follows from Art. 15.2 that *a net of circles has a common orthogonal circle, which is called the orthotomic circle of the system.*

58.1. Equation of the orthotomic circle of the system.

Now as all circles of the net have the same orthotomic circle, it is only necessary to find the orthotomic circle of S_1, S_2, S_3.

Let $S_i \equiv x^2 + y^2 + 2a_i x + 2b_i y + c_i = 0$, $i = 1, 2, 3$

be the equations of the fundamental circles of the system and suppose that the equation of the common orthogonal circle is
$$x^2 + y^2 + 2Ax + 2By + C = 0$$
$$\therefore \quad -c_1 + 2Aa_1 + 2Bb_1 - C = 0$$
$$-c_2 + 2Aa_2 + 2Bb_2 - C = 0$$
$$-c_3 + 2Aa_3 + 2Bb_3 - C = 0.$$

Hence the required equation is
$$\begin{vmatrix} x^2 + Y^2 & x & y & 1 \\ -c_1 & a_1 & b_1 & -1 \\ -c_2 & a_2 & b_2 & -1 \\ -c_3 & a_3 & b_3 & -1 \end{vmatrix} = 0$$

Adding the first row to the other three and in the new determinant subtracting the sum of x times the second column and y times the third from the first, we get

$$\begin{vmatrix} a_1x+b_1Y+c_1 & x+a_1 & y+b_1 \\ a_2x+b_2y+c_2 & x+a_2 & y+b_2 \\ a_3x+b_3y+c_3 & x+a_3 & y+b_3 \end{vmatrix}=0 \qquad \ldots\ldots(11)$$

If the equation of the three circles be made homogeneous by the introduction of a new variable $=z$, the equation of the orthotomic circle, after an interchange of columns, takes the form

$$\begin{vmatrix} \dfrac{\partial S_1}{\partial x} & \dfrac{\partial S_1}{\partial y} & \dfrac{\partial S_1}{\partial z} \\ \dfrac{\partial S_2}{\partial x} & \dfrac{\partial S_2}{\partial y} & \dfrac{\partial S_2}{\partial z} \\ \dfrac{\partial S_3}{\partial x} & \dfrac{\partial S_3}{\partial y} & \dfrac{\partial S_3}{\partial z} \end{vmatrix}=0.$$

The expression on the left is called the Jacobian of S_1, S_2, S_3 and is written briefly as

$$\frac{\partial(S_1, S_2, S_3)}{\partial(x, y, z)} \text{ or } J(S_1, S_2, S_3)=0.$$

58.2. Properties of the orthotomic circle of a net of circles. The orthotomic circle of a net is the locus of a point whose polars w. r. to the net are concurrent.

Let $P(x_1, y_1)$ be the point whose polars *w. r.* to all members of the net

$$\lambda S_1+\mu S_2+v S_3=0$$

meet in a point. The polar of the point (x_1, y_1) is

$$x\left(\lambda\frac{\partial S_1}{\partial x_1}+\mu\frac{\partial S_2}{\partial x_1}+v\frac{\partial S_3}{\partial x_1}\right)+y\left(\lambda\frac{\partial S_1}{\partial y_1}+\mu\frac{\partial S_2}{\partial y_2}+v\frac{\partial S_3}{\partial Y_3}\right)$$
$$+z\left(\lambda\frac{\partial S_1}{\partial z_1}+\mu\frac{\partial S_2}{\partial z_1}+v\frac{\partial S_3}{\partial z_1}\right)=0.$$

If (x_2, y_2, z_2) is the point of concurrence of the polar,

$$\lambda\left(x_2\frac{\partial S_1}{\partial x_1}+y_2\frac{\partial S_1}{\partial y_1}+z_2\frac{\partial S_1}{\partial z_1}\right)+\mu\left(x_2\frac{\partial S_2}{\partial x_1}+y_2\frac{\partial S_2}{\partial y_1}+z_2\frac{\partial S_2}{\partial z_1}\right)$$
$$+v\left(x_2\frac{\partial S_3}{\partial x_1}+y_2\frac{\partial S_3}{\partial y_1}+z_2\frac{\partial S_3}{\partial z_1}\right)=0.$$

Since this relation is satisfied for all values of λ, μ, v,

$$\therefore \quad x_2\frac{\partial S_i}{\partial x_1}+y_2\frac{\partial S_i}{\partial y_1}+z_2\frac{\partial S_i}{\partial z_1}=0, \qquad i=1, 2, 3$$

Eliminating (x_2, y_2, z_2) and changing (x_1, y_1, z_1) in (x, y, z) we get the required locus

$$\frac{\partial(S_1, S_2, S_3)}{\partial(x, y, z)}=0.$$

58.3. Let the polars of P meet at Q, then the polars of Q meet at P, therefore P and Q are both on the orthotomic circle. Let C be the centre and R the radius of the circle (C, R) of the net, and O the radical centre. Suppose CP meets the orthotomic circle again in P′. Since the circle (CR) is orthogonal to the orthotomic circle,

$$CP.CP' = R^2.$$

The point P′ is, therefore, inverse to P *w. r.* to the circle (C, R). Hence the polar of P passes through P′ and is perpendicular to CPP′. It also passes through Q. Hence $\angle PP'Q=\frac{\pi}{2}$.

Consequently PQ is a diameter of the orthotomic circle. Hence *the extremities of any diameter of the orthotomic circle are conjugate points w. r. to every member of the net.*

58.4. The orthotomic circle is the locus of the point-circles of the net.

Let (x, y) be the centre of the point circle

$$\lambda S_1+\mu S_2+v S_3=0 \qquad \ldots\ldots(i)$$

of the net. Hence

$$\lambda(x+a_1)+\mu(x+a_2)+v(x+a_3)=0 \qquad \ldots\ldots(ii)$$

$$\lambda(y+b_1)+\mu(y+b_2)+v(y+b_3)=0 \qquad \ldots\ldots(iii)$$

Also the circle being a point circle, its centre lies on the circle. Subtracting x times (ii) and y times (iii) from (i) we have

$$\lambda(a_1x+b_1y+c_1)+\mu(a_2x+b_2y+c_2)+v(a_3x+b_3y+c_3)=0.$$

Elimination of λ, μ, v gives the locus,

$$J(S_1, S_2, S_3)=0.$$

58.5. Canonical form of the equation of a net of circles.

Let $S_i\equiv x^2+y^2-2a_ix-2b_iy+c_i=0$

be the equation of three circles of the net.

Take the axes so that the radical centre is the origin.

Since the lines $S_1=S_2=S_3$ meet at the origin $c_1=c_2=c_3=c$.

Hence the equation of the net $\lambda S_1+\mu S_2+vS_3=0$ takes the form

$$x^2+y^2-2x\,\frac{\lambda a_1+\mu a_2+va_3}{\lambda+\mu+v}-2y\,\frac{\lambda b_1+\mu b_2+vb_3}{\lambda+\mu+v}+c=0$$

or

$$x^2+y^2-2ax-2by+c=0 \qquad \ldots\ldots(12)$$

where a and b are parameters of the system and c is the same for all circles of the system.

The orthotomic circle of the system is

$$x^2+y^2-c=0,$$

and obviously contains the centres of the point circles of the net.

The circle is real if $c>0$. If $c=0$, the circle then represents a point circle and all circles of the net pass through the centre of this circle. If c is negative, the radical axis of the real circle

$$x^2+y^2+c=0$$

and the circle (12) is

$$ax+by=0,$$

which is the diameter of the circle $x^2+y^2+c=0$.

Hence every circle (12) meets the circle $x^2+y^2+c=0$ at the ends of a diameter.

Illustrative Examples

(1) *If O_1, O_2, O_3, O_4 are the centres of four circles which have a common orthotomic circle and S_1, S_2, S_3, S_4, are the powers of a point O w. r. to the four circles, prove that*

$$S_1(O_2O_3O_4)+S_2(O_3O_1O_4)+S_3(O_1O_2O_4)+S_4(O_3O_2O_1)=0.$$

The four circles belong to a net, since they have a common orthotomic circle. Let the equation of the circles be

$$S_i \equiv x^2+y^2-2a_ix-2b_iy+c=0, \qquad i=1, 2, 3, 4,$$

where c is the same for all the circles. Now

$$\begin{vmatrix} S_1 & S_2 & S_3 & S_4 \\ a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} \equiv 0.$$

The result follows from the expansion of the determinant *w. r.* to the first row.

(2) *Prove that the equation of a circle which cuts each of the three given circles $x^2+y^2-2a_ix-2b_iy+c_i=0$, $i=1, 2, 3$ at the same angle ϕ can be written as*

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ c_1 & a_1 & b_1 & 1 \\ c_2 & a_2 & b_2 & 1 \\ c_3 & a_3 & b_3 & 1 \end{vmatrix} + 2r\cos\phi \begin{vmatrix} 0 & x & y & 1 \\ r_1 & a_1 & b_1 & 1 \\ r_2 & a_2 & b_2 & 1 \\ r_3 & a_3 & b_3 & 1 \end{vmatrix} = 0$$

where r_1, r_2, r_3 are the radii of the given circles and r the radius of the required circle. Hence prove that all circles which cut three given circles at equal (not specified) angles form four co-axal systems, the pairs of base-points being the points of intersection of orthotomic circle with the homothetic axes.

Let the equation of the required circle be

$$x^2+y^2-2ax-2by+c=0$$

$$\therefore \quad c_1+2rr_1\cos\phi-2aa_1-2bb_1+c=0$$
$$c_2+2rr_2\cos\phi-2aa_2-2bb_2+c=0$$
$$c_3+2rr_3\cos\phi-2aa_3-2bb_3+c=0.$$

Hence the required equation is

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ c_1+2rr_1\cos\phi & a_1 & b_1 & 1 \\ c_2+2rr_2\cos\phi & a_2 & b_2 & 1 \\ c_3+2rr_3\cos\phi & a_3 & b_3 & 1 \end{vmatrix}=0,$$

or
$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ c_1 & a_1 & b_1 & 1 \\ c_2 & a_2 & b_2 & 1 \\ c_3 & a_3 & b_3 & 1 \end{vmatrix}+2r\cos\phi\begin{vmatrix} 0 & x & y & 1 \\ r_1 & a_1 & b_1 & 1 \\ r_2 & a_2 & b_2 & 1 \\ r_3 & a_3 & b_3 & 1 \end{vmatrix}=0 \quad (13)$$

The first determinant equated to zero is the equation of the common orthotomic circle; and the second determinant equated to zero is the axis of similitude [Art 56.2 (9)]. The equation (13) therefore represents a co-axal system for the different values of ϕ, the axis of similitude being the common radical axis. Consequently the base points of the system are the intersections of the orthotomic circle with the radical axis. If the angle ϕ is not specified it can be replaced by its supplement which is equivalent to replacing $\cos\phi$ by $-\cos\phi$ or some one of r_1, r_2, r_3 by $-r_1, -r_2, -r_3$. Hence there are four systems of co-axal circles, cutting the given circles at equal angles, the radical axes of the four systems being the four axes of similitude

$$L(r_1, r_2, r_3)=0,\quad L(-r_1, r_2, r_3)=0,\quad L(r_1, -r_2, r_3)=0$$
$$L(r_1, r_2, -r_3)=0 \qquad \text{(Art. 56.2).}$$

The base points of the four systems are the four pairs of intersection of these lines with the common orthogonal circle.

The centres of the circles of the system lie on four lines which are perpendicular to the axes of similitude through the centre of the common orthotomic circle.

This result is obtained independently in the next example.

(3) *Find the locus of the centre of a circle cutting the three given circles at equal angles.*

Let $S_i \equiv (x-a_i)^2+(y-b_i)^2-R_i^2=0, \quad i=1, 2, 3,$
be the given circles and (x, y) the co-ordinates of the centre of the variable circle and R the radius. If ϕ be the angle of intersection, then

$$S_1=R^2-2RR_1\cos\phi,\ S_2=R^2-2RR_2\cos\phi,\ S_3=R^2-2RR_3\cos\phi.$$

Eliminating R^2, $2R\cos\phi$, the required locus is obtained in the form

$$\begin{vmatrix} S_1 & R_1 & 1 \\ S_2 & R_2 & 1 \\ S_3 & R_3 & 1 \end{vmatrix}=0,$$

which is a st. line through the radical centre of the three circles.

This line will be seen to be at right angles to an axis of similitude.

It is of course optional which of two supplemental angles we consider to be the angle at which two circles intersect. The formula used above assumes that the angle at which two circles intersect is measured by the angle which the distance between their centres subtends at the point of meeting, and with this convention, the locus under consideration is a perpendicular to the external axis of similitude. If this limitation be removed, the formula we have used becomes $S_1=R^2\pm 2RR_1\cos\phi$, or in other words, we may change the sign of either R_1, R_2, R_3 in the preceding formulæ and therefore the locus is a perpendicular to any of the four axes of similitude (Salmon).

Inversion

59. Let (O, R) be a fixed circle with centre O and radius R, and suppose P, P′ are points collinear with O such that

$$OP.OP'=R^2,$$

the points P and P′ are called inverse of each other. The circle (O, R) is called the *circle of inversion*, its centre O is the *centre of inversion* and R the *radius of inversion* and R^2 the *constant of inversion.*

If R^2 is positive, the circle of inversion is real and P, P′ are on the same side of O. If R^2 is a negative, the circle of inversion is imaginary with real centre and P, P′ are on the opposite

sides of O. However, inversion *w. r.* to an imaginary circle can be avoided. Let P′ be the inverse of P with R^2 as the constant of inversion and P_1 with $-R^2$ as constant of inversion, then

$$OP.OP' = R^2, \qquad OP.OP_1 = -R^2,$$

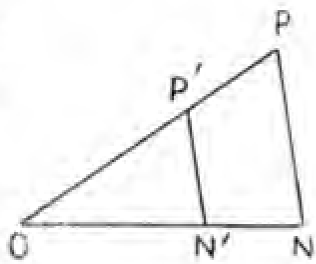
$$\therefore \qquad OP' = -OP_1.$$

Hence the inverse P_1 of P with negative constant can be obtained by first taking the inverse P′ of P with the positive constant and the same centre of inversion, and then taking the reflection of P′ in the diameter at right angles to OP.

59.1. Transformation by inversion.

Suppose P, P′ are inverse points *w. r.* to a circle $x^2+y^2=R^2$ and let the cartesian and polar co-ordinates of the points be

$$P\left.\begin{array}{l}(x, y)\\(r, \theta)\end{array}\right\} \qquad P'\begin{array}{l}(x', y')\\(r', \theta').\end{array}$$



It is evident from triangles ON′P′, ONP, that

$$\frac{x'}{x} = \frac{y'}{y} = \frac{r'}{r}$$

$$= \frac{rr'}{r^2} = \frac{R^2}{x^2+y^2} \qquad \because OP'.OP = R^2.$$

$$\therefore x' = \frac{R^2x}{x^2+y^2}, \; y' = \frac{R^2y}{x^2+y^2} \qquad \ldots\ldots(13)$$

or $\qquad x = R^2x' \div (x'^2+y'^2)\,; \quad y = R^2y' \div (x'^2+y'^2)$

and $\qquad \theta = \theta', \qquad r' = \dfrac{R^2}{r}. \qquad \ldots\ldots(14)$

59.2. Inverse of a curve. When the point P traces out a certain locus K, its inverse point P′ also traces out some locus K′ which is called the inverse of K. The equation of K′ will be obtained from that of K by the transformation (13) or (14).

59.3. The inverse of a st. line is a circle.

Let the equation of the line be $lx+my+n=0$. The equation of the inverse curve is

$$\frac{lR^2x'}{x'^2+y'^2} + \frac{mR^2y'}{x'^2+y'^2} + n = 0.$$

Multiplying and dropping the dashes, the equation of the locus is

$$n(x^2+y^2) + R^2(lx+my) = 0$$

which represents a circle through O and having the tangent at O parallel to the given line.

If $n=0$, the inverse of $lx+my=0$ is the line itself. Further, when $l=i$, $m=1$ *i. e.*, when the line is the circular line $y+ix+n=0$, its inverse is the other circular line

$$n(y-ix)+R^2=0.$$

If R be allowed to vary, the inverse curves are coaxal circles which touch at O, $lx+my=0$ being the common tangent.

59.4. *The inverse of a circle is in general a circle.*

The inverse of the circle

$$x^2+y^2+2gx+2fy+c=0 \qquad \ldots\ldots(i)$$

is the curve

$$\frac{R^4(x^2+y^2)}{(x^2+y^2)^4}+\frac{2gR^2x}{x^2+y^2}+\frac{2fR^2y}{x^2+y^2}+c=0$$

or

$$c(x^2+y^2)+2R^2(gx+fy)+R^4=0 \qquad \ldots\ldots(ii)$$

This is a circle if $c=\neq 0$. If $c=0$ *i.e.*, the circle passes through the origin, the inverse curve is a st. line. Further, if $R^2=c$, *i.e.*, if the circle of inversion is orthogonal to a given circle, the given circle inverts into itself.

Subtracting (ii) from R^2-times (i) we get

$$(R^2-c)(x^2+y^2-R^2)=0, \quad i.e., \ x^2+y^2-R^2=0.$$

Hence *a circle* S, *its inverse circle* S' *and the circle of inversion belong to a coaxal system.*

The centre of inversion is either centre of similitude of a given circle and its inverse.

Hence it follows that any two circles are inverse with respect to either centre of similitude and with respect to no other point and the two circles of inversion are orthogonal.

Miscellaneous Examples XV

1. Find the equation of the circle which has for its diameter the chord cut off on the st. line $ax+by+c=0$ by the circle $(a^2+b^2)(x^2+y^2)=2c^2$. (Peterhouse, etc.)

2. Show that the condition that the circle
$$x^2+y^2-2a_1x-2b_1y+c_1=0$$
should cut the circle
$$x^2+y^2-2a_2x-2b_2y+c_2=0$$
at the ends of a diameter is
$$2a_1a_2+2b_1b_2=c_1-c_2+2(a_1^2+b_1^2).$$

3. If S_1, S_2 are two circles which cut two given circles each at the ends of a diameter, prove that every circle of the system $S_1-\lambda S_2=0$ cuts the given circles in the same manner. (Sommerville).

4. Prove that there are in general two circles in any coaxal system which are cut diametrically. (Sommerville).

5. Prove that there is in general just one circle of a given coaxal system which cuts a given circle diametrically. (Sommerville).

6. The co-ordinates of any two points P, Q are (x_1, y_1), (x_2, y_2), the origin being at O. Circles are described on OP, OQ as diameters. Prove that the length of their common chord is

$$(x_1 y_2 \sim x_2 y_1)/PQ. \qquad \text{(Math. Tip. 1919.)}$$

7. There are two segments on the line $3x-4y+7=0$, each of length 10 units, which subtend right angles at the origin. Find the co-ordinates of their extremities.

(Math. Trip. 1924)

8. Show that the circumcircle of the triangle formed by the lines

$$ax+by+c=0,\ cx+ay+b=0,\ bx+cy+a=0$$

passes through the origin if

$$(b^2+c^2)(c^2+a^2)(a^2+b^2)=abc\ (b+c)(c+a)(a+b).$$

9. Show that the length of the least chord of the circle

$$x^2+y^2+2gx+2fy+c=0$$

which passes through an internal point (x_1, y_1) is

$$2\sqrt{[-(x_1^2+y_1^2+2gx_1+2fy_1+c)]}.$$

10. Find the common tangents to the circles

$$x^2+y^2-3x-4y=0,\ x^2+y^2-21x+90=0.$$

11. Find the common tangents of the following pairs of circles.

(*i*) $x^2+y^2-6y-2x+9=0,\ x^2+y^2+6x-2y+1=0$ (B. U.)

(*ii*) $x^2+y^2+4x+2y-4=0,\ x^2+y^2-4x-2y+4=0.$ (C. U.)

12. Two circles of radii a and b touch the axis of y on the opposite sides at the origin. The axes being rectangular, prove that the other two common tangents are given by

$$(b-a)x \pm 2\sqrt{ab}\ y-2ab=0.$$

13. Prove that the equations of the common tangents to the circle $x^2+y^2=289$ and the circle whose diameter is the chord of the first circle made by the line $x\cos\alpha+y\sin\alpha=15$ are

$$3(x\cos\alpha+y\sin\alpha)\pm4(y\cos\alpha-x\sin\alpha)=85.$$

14. Find the co-ordinates of the limiting points of the coaxal system determined by the circles

$$x^2+y^2-2x+8y+11=0,\ x^2+y^2+4x+2y+5=0. \qquad \text{(C.U.)}$$

15. Show that the circles with respect to which a fixed line $ax+by+c$ is the polar of the origin form a coaxal system, and find the line of centres, the radical axis and the limiting points.

(Peterhouse etc., 1936)

16. Prove that the equation to the common tangents to the circles

$$x^2+y^2=2ax \text{ and } x^2+y^2=2by$$

can be expressed in the form

$$2ab(x^2+y^2-2by)=(ax-by+ab)^2.$$

17. Find the equations of the radical axes of the circles $(x-a)^2+(y-b)^2=b^2$, $(x-b)^2+(y-a)^2=a^2$, $(x-a-b-c)^2+y^2=ab+c^2$, and prove that they are concurrent.

Find also the equation of the circle which cuts the three circles orthogonally.

18. If the equation of one circle and the radical axis of this circle and another are respectively

$$a(x^2+y^2)+2gx+2fy+c=0 \text{ and } lx+my+n=0,$$

find the equation of the other circle with the proper number of arbitrary constants and the co-ordinates of the limiting points.

19. Show that the limiting points of the circle $x^2+y^2=a^2$ and an equal circle with centre on the line $lx+my+n=0$ lie on the locus

$$(x^2+y^2)(lx+my+n)+a^2(lx+my)=0.$$

20. Show that the square of the diameter of the circum-circle of the triangle formed by the lines

$$ax^2+2hxy+by^2=0,\ lx+my+1=0$$

is

$$\frac{[(a-b)^2+4h^2](l^2+m^2)}{(am^2-2hlm+bl^2)^2}.$$

21. If the tangents drawn from a point to the circle

$$x^2+y^2+2ax+c^2=0$$

are conjugate lines with respect to the circle

$$x^2+y^2+2bx+c^2=0,$$

show that the locus of the point is

$$(a^2+b^2-2c^2)(x^2+c^2)+2(ab-c^2)\{y^2+(a+b)x\}=0.$$

22. Show that the necessary and sufficient condition that four points (x_r, y_r), $r=1, 2, 3, 4$ be concyclic is that the powers of the four points with respect to an arbitrary circle

$$S(x, y)\equiv x^2+y^2+2gx+2fy+c=0$$

are connected by the linear relation

$$\sum_{r=1}^{4} \lambda_r S(x_r, y_r)\equiv 0.$$

23. If A, B, C, D are four concyclic points and O any other point, prove that

$$OA^2(BCD)+OB^2(CAD)+OC^2(ABD)+OD^2(CBA)=0$$

where (BCD) represents the area with its proper sign of the triangle BCD etc.

24. Show that the locus of a point which moves so that

$$\lambda\, PA^2 + \mu\, PB^2 + \nu PC^2 = 0$$

where λ, μ, ν are constants, is a circle which cuts the circum-circle of the triangle ABC orthogonally.

25. If O_1, O_2, O_3, O_4 are the centres and r_1, r_2, r_3, r_4 the radii of the circles BCD, CDA, DAB, ABC, where A, B, C, D are any four points in a plane, then will

$$(AO_1^2 - r_1^2)^{-1} - (BO_2^2 - r_2^2)^{-1} + (CO_3^2 - r_3^2) - (DO_4^2 - r_4^2)^{-1} = 0.$$

26. If A, B, C are the centres of the co-axal circles S_1, S_2, S_3, prove that

$$BC.S_1 + CA.S_2 + AB.S_3 = 0.$$

27. If t_1, t_2, t_3 be the lengths of the tangents from any point to three given circles, show that any circle or any st. line can be represented by an equation of the form

$$at_1^2 + bt_2^2 + ct_3^2 = d.$$

What relation will hold between a, b, c for st. lines.

28. Show that the powers of a point w. r. to five given circles S_1, S_2, S_3, S_4, S_5 are connected by a fixed relation of the form

$$a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4 + a_5 S_5 = 0.$$

29. Show that the power of a point w. r. to four given circles are connected by a relation of the form

$$a_1 S_1 + a_2 S_2 + a_3 S_3 + a_4 S_4 = k.$$

30. A variable circle is one of a definite co-axal system, and a perpendicular is drawn from a fixed point to its polar with respect to the variable circle. Show that the locus of the foot of the perpendicular is a circle whose centre is on the common radical axis of the system of circles. (Math. Trip. I. 1913)

31. A variable chord PQ of a given circle subtends a right angle at a given point A. Find the locus of the pole of PQ with respect to the circle. Interpret the result when A lies on the given circle. (Downing 1931)

32. The inverse of a co-axal system is a co-axal system.

33. The inverse of a pencil of lines is a pencil of circles.

34. Concentric circles invert into co-axal circles with one limiting point at the centre of inversion.

35. A co-axal system having real limiting points is the inverse of a concentric system, and a system having imaginary limiting points the inverse of a pencil of lines.

CHAPTER VII

LOCI OF THE SECOND DEGREE : THEIR CLASSIFICATION AND COMMON PROPERTIES

60. First Definition. A conic is a plane curve which is cut by an arbitrary st. line in its plane in two and only two points. Thus, the general equation of the second degree,

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \qquad \ldots\ldots(1)$$

or $\phi(x,y,z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \ldots\ldots(1A)$

represents a conic.

Conversely, the equation of a conic is of the 2nd degree.

For, if possible, let the equation of a conic be

$$U_n + U_{n-1} + \ldots\ldots U_r + \ldots\ldots U_2 + U_1 + U_0 = 0$$

where U_r is a homogeneous expression in (x, y) of the rth degree.

The intersections with the conic of any st. line

$$\frac{x-x'}{l} = \frac{y-y'}{m} = r \qquad (l^2 + m^2 = 1)$$

are obtained by substituting $x' + lr$, $y' + mr$ for x and y respectively. The resulting equation in r is of the nth degree and it has only two roots if the co-efficients of $r^n, r^{n-1} \ldots\ldots r^3$ vanish identically, for they must be zero for all values of $l : m$.

Thus the co-efficients of the terms in U_n, $U_{n-1}, \ldots\ldots U_3$ are each zero. Hence the equation of the conic is

$$U_2 + U_1 + U_0 = 0.$$

Again, there exist no functions of the variables x and y which when expanded give only second degree terms in the variables. Hence the equation of the conic is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

61. Second Definition. A conic is also defined to be the locus of a point in a plane whose distance from a fixed point bears a constant ratio to its distance from a fixed line.

The fixed point is called a **focus**, the fixed line a **directrix** and the constant ratio an **eccentricity**.

This geometrical condition is necessary and sufficient to generate a conic.

Necessity. Let $S(x_1, y_1)$ be the fixed point and

$$x \cos \alpha + y \sin \alpha - p = 0.$$

the equation of the directrix. Suppose that the eccentricity is e. The equation of the conic is then

$$(x-x_1)^2+(y-y_1)^2=e^2(x \cos \theta + y \sin \theta - p)^2 \qquad \ldots\ldots(2)$$

which is of the second degree in x and y.

Sufficiency. Shift the origin to the point $S(x_1, y_1)$ then the equation (1) with the notation of Art. 63.1, takes the form $\quad ax^2+2hxy+by^2+2xX_1+2yY_1+\phi_1=0.$

Assume that it can be written as

$$x^2+y^2=e^2(x \cos \alpha + y \sin \alpha - p)^2.$$

Comparing we have

$$1-e^2\cos^2\alpha=\lambda a, \qquad 1-e^2\sin^2\alpha=\lambda b \qquad -e^2\cos\alpha\sin\alpha=\lambda h$$

$$e^2 p\cos\alpha=\lambda X_1, \qquad e^2 p \sin\alpha=\lambda Y_1 \qquad -e^2 p^2=\lambda\phi_1$$

$$\therefore \quad -e^2\cos 2\alpha=\lambda(a-b) \quad e^4p^2\cos 2\alpha=\lambda^2(X_1^2-Y_1^2)$$

$$-e^2\sin 2\alpha=2\lambda h. \qquad e^4p^2\sin 2\alpha=2\lambda^2X_1Y_1.$$

Thus

$$\frac{X_1^2-Y_1^2}{a-b}=\frac{-e^2p^2}{\lambda}=\phi_1$$

$$\frac{X_1Y_1}{h}=\frac{-e^2p^2}{\lambda}=\phi_1.$$

Hence (x_1, y_1) is given by the equations

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h}=\phi. \qquad \ldots\ldots(3)$$

There are four positions of S.

Further

$$x\cos\alpha+y\sin\alpha-p\equiv\frac{\lambda}{e^2p}(xX_1+yY_1+\phi_1)$$

$$\equiv\frac{xX_1+yY_1+\phi_1}{\sqrt{X_1^2+Y_1^2}} \qquad \ldots\ldots(4)$$

Consequently, the equation can be written in the form

$$x^2+y^2=e^2\left[\frac{xX_1+yY_1+\phi_1}{\sqrt{X_1^2+Y_1^2}}\right]^2.$$

The directrix, is, therefore, $xX_1+yY_1+\phi_1=0.$

To determine the eccentricity, we note that

$$2-e^2=\lambda(a+b), \qquad \lambda^2(ab-h^2)=1-e^2.$$

Thus e is given by the equation

$$\frac{(2-e^2)^2}{1-e^2}=\frac{(a+b)^2}{ab-h^2}. \qquad \ldots\ldots(5)$$

It follows that a second degree equation represents a conic.

The conic is called an ellipse, parabola or hyperbola according as $e \gtreqless 1$. Thus when $\triangle \neq 0$.

for an ellipse	$ab - h^2 > 0$.
for a parabola	$ab - h^2 = 0$.
for a hyperbola	$ab - h^2 < 0$.
for a circle	$e = 0$, $\therefore$ $4h^2 + (a-b)^2 = 0$,
i.e., $a = b$, and	$h = 0$.

Thus the foci coincide with the centre and the directrix is the line at ∞.

If the conic passes through the origin *i. e.*, if the directrix passes through the focus S, the conic will be a pair of st. lines which are real, coincident (or parallel) or imaginary according as $e \gtreqless 1$.

Remarks. The equation (2) can be written as

$$[(x - x_1) + i\,(y - y_1)]\,[(x - x_1) - i\,(y - y_1)] = e^2\,(x \cos \alpha + y \sin \alpha - p)^2.$$

Each factor on the left equated to zero is an isotropic line through the focus S (x_1, y_1) which meets the conic in coincident points that lie on the line

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Thus these isotropic lines are tangents to the conic through (x_1, y_1), the line $x \cos \alpha + y \sin \alpha - p = 0$ being the chord of contact. Hence *the foci of a conic are the four intersections, other than the circular points, of the isotropic tangents to the conic. The polars of these foci are directrices.*

61.1. The equation of the directrix with the focus as the origin is $x\mathrm{X}_1 + y\mathrm{Y}_1 + \phi_1 = 0$. Translating the axes back through the old origin, the equation becomes

$$(x - x_1)\,\mathrm{X}_1 + (y - y_1)\,\mathrm{Y}_1 + \phi_1 = 0$$

i.e., $$x\mathrm{X}_1 + y\mathrm{Y}_1 + \mathrm{Z}_1 = 0$$

Note. The equation of the conic contains five disposable constants, the ratio of any five of a, b, c, f, g, h to the remaining sixth. The equation of the conic is thus, in general, determined by five independent conditions.

62. Centre of a conic. The centre of a conic, when it exists, is a point C such that the segment made by the conic on every line through C is bisected at C.

Let $\phi = 0$ be the conic and (x', y') the co-ordinates of the centre. Translate the axes to the point (x', y'). The equation takes the form

$$ax^2 + 2hxy + by^2 + 2x\mathrm{X}' + 2y\mathrm{Y}' + \phi' = 0.$$

Now the centre being the origin, if (x, y) be a point on the conic, $(-x, -y)$ is also a point on the conic, for which the condition is

$$ax^2+2hxy+by^2-2xX'-2yY'+\phi'=0$$

$$\therefore \qquad xX'+yY'=0.$$

This being the condition for all sets of values of x and y which satisfy the equation of the conic

$$X'=0, \quad Y'=0.$$

The point (x', y') is, therefore, the intersection of the lines

$$X \equiv ax+hy+g=0$$

$$Y \equiv hx+by+f=0.$$

Thus $$x=\frac{G}{C}\cdot y=\frac{F}{C} \qquad \ldots\ldots\ldots(6)$$

The centre is a finite point if $C \neq 0$. If $C=0$ and G and F do not vanish simultaneously, the point is at infinity. If $C=0$, $G=0$, $F=0$, its position is not defined.

62.1. The line

$$lX+mY=0 \qquad \ldots\ldots(7)$$

for all values of l and m excepting $l=m=0$, passes through the centre of the conic. The line is called a **diameter.**

62.2. The equation of the conic when the centre is the origin can be readily obtained. For $X'=0$, $Y'=0$

$$\phi'=x'X'+y'Y'+Z'=Z'$$

$$\therefore \qquad ax'+hy'+g=0$$

$$hx'+by'+f=0$$

$$gx'+fy+c-\phi'=0$$

Hence $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-\phi' \end{vmatrix}=0$$

or $$\Delta-\phi' C=0$$

Consequently, the equation of the conic takes the form

$$ax^2+2hxy+by^2+\frac{\Delta}{C}=0 \qquad \ldots\ (8).$$

63. Classification of conics.

In Art 61, the conics have been classified according as $e \gtreqless 1$. The classification in the present article is based on the forms of their equations, and in later chapters the indentity of the two classifications will be established.

63.1. 1st case : When $C \neq 0$.

If $C \neq 0$, the conic has a finite centre.

Referring the equation of the conic to the centre as origin, the equation is of the form as (8) Art. 62.2. Now rotate the axes through an angle θ by the transformation

$$x=\xi \cos\theta-\eta \sin\theta,\ y=\xi \sin\theta+\eta \cos\theta.$$

The equation thus becomes

$$a(\xi \cos\theta-\eta \sin\theta)^2+2h(\xi \cos\theta-\eta \sin\theta)(\xi \sin\theta+\eta \cos\theta)$$
$$+b(\xi \sin\theta+\eta \cos\theta)^2+\frac{\Delta}{C}=0.$$

The co-efficient of $\xi\eta$ is

$$-(a-b)\sin 2\theta+2h\cos 2\theta.$$

If θ be so chosen that

$$\tan 2\theta=\frac{2h}{a-b},$$

the resulting equation in ξ, η will not contain the term $\xi\eta$. The equation will then be of the form

$$\frac{\xi^2}{\lambda_1}+\frac{\eta^2}{\lambda_2}+\frac{\Delta}{C}=0$$

$$\therefore \quad \frac{1}{\lambda_1}+\frac{1}{\lambda_2}=a+b$$

$$\frac{1}{\lambda_1\lambda_2}=ab-h^2.$$

If we set $\mu_1=\dfrac{-\lambda_1\Delta}{C}$, $\mu_2=\dfrac{-\lambda_2\Delta}{C}$, then

$$\mu_1+\mu_2=-\frac{\Delta}{C}(\lambda_1+\lambda_2)=-\frac{-(a+b)\Delta}{C^2} \qquad \ldots\ldots(i)$$

$$\mu_1\mu_2=\frac{\lambda_1\lambda_2\Delta^2}{C^2}=\frac{\Delta^2}{C^3} \qquad \ldots\ldots(ii)$$

and the equation of the conic assumes the form

$$\frac{\xi^2}{\mu_1}+\frac{\eta^2}{\mu_2}=1 \qquad \ldots\ldots(9)$$

(*i*) If $C>0$ and $(a+b)\Delta<0$, μ_1, μ_2 are both positive, and the equation (9) can be written in the form

$$\frac{\xi^2}{\lambda^2}+\frac{\eta^2}{\mu^2}=1 \qquad \ldots\ldots(10)$$

The locus represented by this equation is called an *ellipse.*

(*ii*) If $C>0$, $(a+b)\Delta>0$, μ_1 and μ_2 are both negative, the equation becomes

$$\frac{\xi^2}{\lambda^2}+\frac{\eta^2}{\mu^2}+1=0. \qquad \ldots\ldots(10A)$$

The locus contains no real point and is called an *imaginary ellipse.*

(*iii*) When $C<0$ and $\Delta\neq0$, μ_1 and μ_2 are of opposite signs, the equation of the conic takes either of the forms

$$\left.\begin{aligned}\frac{\xi^2}{\lambda^2}-\frac{\eta^2}{\mu^2}=1\\ -\frac{\xi^2}{\lambda^2}+\frac{\eta^2}{\mu^2}=1.\end{aligned}\right\} \qquad \ldots\ldots(11)$$

The loci of these equations are called *hyperbolas.*

(*iv*) If in addition to $C<0$, $a+b=0$, then $\lambda^2=-\mu^2$, the locus has for its equation

$$\xi^2-\eta^2=\pm\lambda^2 \qquad \ldots\ldots(11A)$$

and is called a *rectangular* or *equilateral hyperbola.*

(*v*) If $\mu_1=\mu_2$, $\dfrac{(a+b)^2\Delta^2}{C^4}=\dfrac{4\Delta^2}{C^3}=4\mu_1^2$

$\therefore\quad (a+b)^2-4C=0$ or $(a-b)^2+4h^2=0$.

Thus $a=b$, $h=0$. The equation of the conic is of the form

$$\xi^2+\eta^2=\lambda^2$$

which represents a circle.

(*vi*) When $C>0$, $\Delta=0$, λ_1 and λ_2 have the same sign, the equation of the conic becomes

$$\frac{\xi^2}{\lambda^2}+\frac{\eta^2}{\mu^2}=0$$

which represents a pair of *imaginary lines.*

(*vii*) When $C<0$, $\Delta=0$, λ_1 and λ_2 are of opposite signs, the equation of the conic assumes the form

$$\frac{\xi^2}{\lambda^2}-\frac{\eta^2}{\mu^2}=0$$

which represents a *pair of real distinct lines.*

63.2. 2nd case. When $C=0$. The equation of the conic becomes

$$(\alpha x+\beta y)^2+2gx+2fy+c=0$$

where $a=\alpha^2$, $b=\beta^2$, $h=\alpha\beta$. Also, then $\Delta=-(\alpha f-\beta g)^2$.

Shift the origin to a point (x', y') on the curve $\phi=0$.

The equation becomes

$$(\alpha x+\beta y)^2+2xX'+2yY'=0$$

where $X'=\alpha(\alpha x'+\beta y')+g$; $Y'=\beta(\alpha x'+\beta y')+f$

and $$\phi(x', y')=0.$$

It is possible to choose (x', y') uniquely, such that the lines $\alpha x+\beta y=0$ and $xX'+yY'=0$ are at right angles. For, this requires

$$\alpha X'+\beta Y'=0$$

or $(\alpha^2+\beta^2)(\alpha x'+\beta y')+\alpha g+\beta f=0.$

Also $0=\phi(x', y')=(\alpha x'+\beta y')^2+2gx'+2fy'+c$

$$=2gx'+2fy'+c+\frac{(\alpha g+\beta f)^2}{(\alpha^2+\beta^2)^2}.$$

Thus (x', y') is the point of intersection of the lines

$$(\alpha^2+\beta^2)(\alpha x+\beta y)+\alpha g+\beta f=0$$

$$2gx+2fy+c+\frac{(\alpha g+\beta f)^2}{(\alpha^2+\beta^2)^2}=0.$$

The point of intersection is unique and finite if $\alpha f-\beta g\neq 0$ *i.e.*, $\Delta\neq 0$. Having so chosen (x', y') take the lines

$$\alpha x+\beta y=0\ ;\ -xX'-yY'=0$$

as axes of co-ordinates. Thus

$$\eta=\frac{\alpha x+\beta y}{\sqrt{\alpha^2+\beta^2}},\qquad \xi=\frac{-xX'-yY'}{\sqrt{X'^2+Y'^2}}.$$

The equation of the conic becomes

$$\eta^2=2\frac{(X'^2+Y'^2)^{\frac{1}{2}}}{\alpha^2+\beta^2}\xi.$$

Also $X'=\alpha(\alpha x'+\beta y')+g=-\alpha\dfrac{\alpha g+\beta f}{\alpha^2+\beta^2}+g$

$$=\frac{-\beta(\alpha f-\beta g)}{\alpha^2+\beta^2}.$$

$$Y'=\beta(\alpha x'+\beta y')+f=\frac{\alpha(\alpha f-\beta g)}{\alpha^2+\beta^2}.$$

$$\therefore\quad X'^2+Y'^2=\frac{(\alpha f-\beta g)^2}{\alpha^2+\beta^2}=\frac{-\Delta}{a+b}.$$

Consequently the equation of the conic takes the form

$$\eta^2=2\sqrt{\frac{-\Delta}{(a+b)^3}}\ \xi \qquad ......(12)$$

(*ix*) The conic is called a parabola and is real if $\Delta<0$, and imaginary if $\Delta>0$.

(*x*) Let now $\Delta=0, C=0$.

$\therefore \quad \frac{g}{\alpha}=\frac{f}{\beta}=\lambda$, say.

The equation of the conic becomes

$$(\alpha x+\beta y)^2+2\lambda(\alpha x+\beta y)+C=0$$

or $$(\alpha x+\beta y+\lambda)^2=\lambda^2-c,$$

which represents a pair of parallel lines.

The lines are real if $\lambda^2>c$ and imaginary if $\lambda^2<c$, and coincident if $\lambda^2=c$. Since $A=\beta^2(c-\lambda^2)=bc-f^2$ and $B=\alpha^2(c-\lambda^2),=ac-g^2$, it follows that the equation $\phi=0$ will represent a pair of lines if $\Delta=0$, $C=0$ and these are real, coincident or imaginary, according as

$$A \text{ (or B)} \gtreqless 0.$$

(*xi*) If each of $a, b, h\to 0$, the equation represents a finite straight line and the line at infinity.

(*xii*) When each of $a, b, f, g, h\to 0$, the equation $\phi=0$ represents two coincident st. lines at infinity.

The following table summarises the results of the above articles.

Sign of C	Sign of Δ	Additional criterion	Nature of the conic.
$C>0$	$\Delta\neq 0$	$(a+b)\Delta<0$	The conic is a real ellipse.
		$(a+b)\Delta>0$	The conic is an imaginary ellipse.
		$a=b, h=0$	The conic is a circle.
	$\Delta=0$		The conic is a pair of imaginary st. lines.

Sign of C	Sign of Δ	Additional criterion	Nature of the conic.
$C<0$	$\Delta \neq 0$		The conic is a hyperbola.
		$a+b=0$	The conic is a rectangular hyperbola.
	$\Delta=0$		The conic is a pair of real st. lines.
$C=0$	$\Delta>0$		The conic is an imaginary parabola.
	$\Delta<0$		The conic is a real parabola.
	$\Delta=0$	$A<0$	The conic is a pair of real parallel st. lines.
		$A>0$	The conic is a pair of imaginary and parallel lines.
		$a, b, h \to 0$	The conic contiss of a finite line and the line at infinity.
		$a, b, f, g, h \to 0$	The conic consists of two coincident lines at infinity.
		$A=0$	Two finite coincident lines.

64. Form of the ellipse.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

1. If (x, y) be a point on the ellipse, the points $(x, -y)$ $(-x, y)$ are also points of the ellipse. The curve is, therefore, symmetrical *w. r.* to x-axis and y-axis.

The point $(-x, -y)$ also belongs to the conic, which shows that every chord of the conic through the origin is

bisected by it. The origin is thus the centre of the conic and there is a symmetry about the centre.

2. Writing the equation in the form

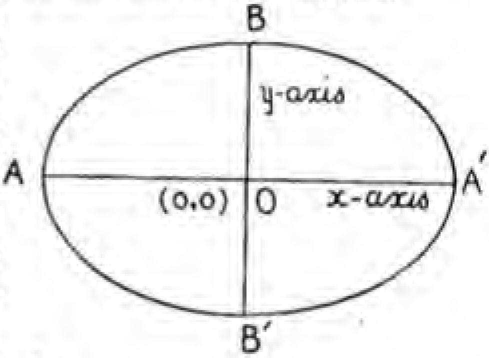
$$\frac{y}{b} = \pm \frac{\sqrt{a^2 - x^2}}{a}$$

it is seen that y is real only if $-a \leqslant x \leqslant a$. As x varies from $-a$ to 0, y increases or decreases according as $+$ or $-$ sign is taken before the radical; assuming its maximum or minimum value b when $x=0$, and it decreases or increases as x changes from 0 to a. The lines $x = \pm a$ meet the conic in concident points and are thus tangents to the ellipse.

Similarly, the lines $y = \pm b$ are tangents to the ellipse and it lies between these two lines.

The shape of the ellipse is as shown in the figure.

Def. The lines which bisect the chord of the conic drawn perpendicular to them are called **axes of symmetry**. In the present case $x=0$, $y=0$ are the axes of symmetry. If $a > b$, then $AA' = 2a$ and $BB' = 2b$ are called the major and minor axes which lie along the co-ordinate axes.



64.1. Form of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

If (x, y) be a point on the hyperbola, the points $(x, -y)$ $(-x, y)$, $(-x, -y)$ also lie on the hyperbola. Consequently, the x-axis and y-axis are axis of symmetry of the conic and origin is its centre. The axis of symmetry $(y=0)$ which meets the conic in real points is called the transverse axis and the other $(x=0)$ which meets it in imaginary points is called the conjugate axis.

For real points of the conic $x \geqslant a$ or $x \leqslant -a$ and the lines $x = \pm a$ are tangents to the hyperbola. Thus no point of the curve lies within the lines $x = \pm a$.

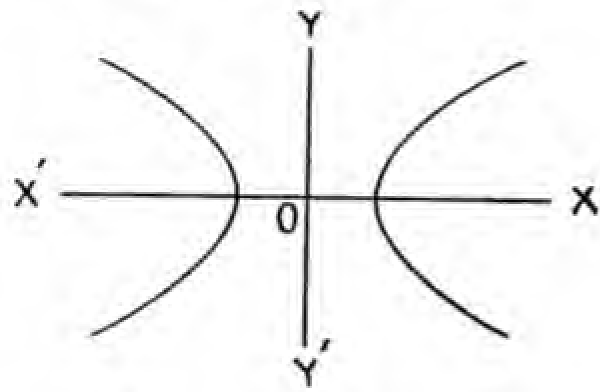
Now

$$y = \pm b\left(\frac{x^2}{a^2} - 1\right)^{\frac{1}{2}}$$

$$= \pm \frac{bx}{a}\left(1 - \frac{a^2}{x^2}\right)^{\frac{1}{2}}.$$

The first part of the equation shows that if $-a \leqslant x$ or $x \geqslant a$, $/y/$ increases as $/x/$ increases and as $/x/ \to \infty$, $y \to \pm\infty$. Hence the curve extends to infinity and does not appear to be a closed curve.

For given values of x, the ordinates of points on the curve lie between $\frac{-bx}{a}$ and $\frac{bx}{a}$ and as $x \to \infty$, $\frac{y}{x} \to \pm \frac{b}{a}$. Hence the curve lies within the lines $y = \pm \frac{bx}{a}$ and approaches these lines as $x \to \pm\infty$.



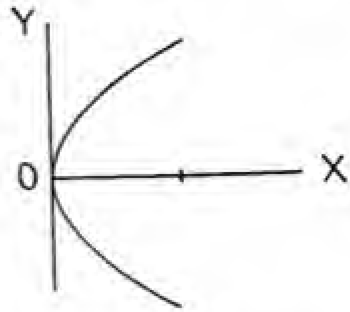
The lines $y = \frac{b}{a}x$, $y = -\frac{b}{a}x$ are tangents to the curve, the points of contact being at infinity and each may be regarded as touching this curve on either side of the origin. For, when $x \to \infty$, $y \to \infty$, or $x \to -\infty$, $y \to -\infty$, then $\frac{y}{x} \to \frac{b}{a}$; when $x \to +\infty$, $y \to -\infty$ or $x \to -\infty$, $y \to \infty$, then $\frac{y}{x} \to -\frac{b}{a}$. But the positive and negative infinities of a finite st. line coincide and a st. line cannot touch this curve at two distinct points. The hyperbola is also a closed curve.

64.2. Form of the parabola $y^2 = 4ax$.

If (x, y) be a point on the parabola, $(x, -y)$ is equally a point on the conic. Hence the x-axis is an axis of symmetry of the parabola and is called the axis of the conic.

If x is negative, y is imaginary. The curve thus lies entirely to the right of the y-axis. The line $x=0$ is a tangent to the parabola at $(0, 0)$. The point is called the vertex of the parabola.

As x increases, $|y|$ increases and tends to infinity with x. The curve is thus an open curve and its shape is as shown in the annexed figure.



This curve will also be a closed curve if we regard the positive and negative infinities of the st. line at infinity also coincident.

We will show later on that a parabola may be regarded indifferently as the limit of an ellipse or a hyperbola. The parabola so obtained is a closed curve, for $e \to 1$, and this parabola does not appear to be the same as the parabola for which $e=1$.

Illustrative Examples.

(1) *Find the equation of the conic which passes through the points (x_i, y_i) $(i=1, 2, 3, 4, 5)$ no three of which are collinear.*

Let the equation of the conic be

$$\phi \equiv ax^2+by^2+2hxy+2gx+2fy+c=0,$$

since it goes through the points (x_i, y_i) $(i=1, 2, 3, 4, 5)$.

$$\therefore \quad ax_i^2+by_i^2+2hx_iy_i+2gx_i+2fy_i+c=0, \; i=1, 2, 3, 4, 5.$$

The elimination of a, b, h, g, f, c gives the required equation.

$$\begin{vmatrix} x^2 & y^2 & xy & x & y & 1 \\ x_1^2 & y_1^2 & x_1y_1 & x_1 & y_1 & 1 \\ x_2^2 & y_2^2 & x_2y_2 & x_2 & y_2 & 1 \\ x_3^2 & y_3^2 & x_3y_3 & x_3 & y_3 & 1 \\ x_4^2 & y_4^2 & x_4y_4 & x_4 & y_4 & 1 \\ x_5^2 & y_5^2 & x_5y_5 & x_5 & y_5 & 1 \end{vmatrix}=0.$$

(2) *Find the foci and directrices of the conic*

$$\phi \equiv 3x^2+4xy-2x-6y-4=0.$$

The foci of the conic are given by the equations

$$\frac{X^2-Y^2}{3}=\frac{XY}{2}=\phi$$

i.e., $$\frac{(3x+2y-1)^2-(2x-3)^2}{3}=\frac{(3x+2y-1)(2x-3)}{2}$$
$$=3x^2+4xy-2x-6y-4.$$

The first part of the equation gives

$$(X-2Y)(2X+Y)=0$$

or $x=2y+5$ and $8x+4y-5=0$.

Putting $x=2y+5$ in the second part of the equation

$$\frac{(3x+2y-1)(2x-3)}{2}=3x^2+4xy-2x-6y-4,$$

it is found that

$$2y^2+7y+6=0$$

$\therefore\ y=-2$ or $-\frac{3}{2}$.

The corresponding values of x are 1, 2. Hence the co-ordinates of the real foci are $(1, -2)$, $(2, -\frac{3}{2})$. The equation of the directrix is $xX_1+yY_1+Z_1=0$.

The equation in the present case reduces to

$$2x+y-3=0 \text{ and } 2x+y+\tfrac{1}{2}=0.$$

(3) *A conic is given by the equation*

$$x^2+2(2\lambda-1)xy+\lambda^2y^2+2\lambda x+2\lambda^2y+\lambda^2+\tfrac{1}{4}\lambda-\tfrac{1}{4}=0,$$

where λ is a variable parameter which takes all real values. Show that the conic is always real and find the values for which the conic degenerates. [King's 1912]

In the case in hand

$$c=-3(\lambda-1)(\lambda-\tfrac{1}{3}),\ \Delta=-(\lambda-1)^2(\lambda+1)(\lambda-\tfrac{1}{4}).$$

The conic degenerates into a pair of st. lines if

$$\lambda=1, -1, \tfrac{1}{4}.$$

For $\lambda=-1$, or $\frac{1}{4}$, $c<0$, the equation represents a pair of real st. lines.

For $\lambda=1$, $c=0$, $\Delta=0$ the equation represents a pair of coincident lines.

Suppose now $\Delta\neq 0$.

If $c<0$, $(\lambda-1)(\lambda-\frac{1}{3})>0$ and λ cannot lie between $\frac{1}{3}$ and 1, the equation then represents a hyperbola.

If $\lambda=\frac{1}{3}$, $c=0$, $\Delta<0$, the locus represented by the equation is a real parabola.

The locus will be an ellipse if $\frac{1}{3}<\lambda<1$, since for such values of λ, $c>0$ Also since $(a+b)\Delta=-(1+\lambda^2)(1+\lambda)(\lambda-1)^2(\lambda-\frac{1}{4})<0$, the ellipse is a real ellipse.

(4) *Discuss the nature of the conic given by the equation*

$$y=\lambda x+\mu\pm(\alpha x^2+2\beta x+\gamma)^{\frac{1}{2}}.$$

We require the following lemmas :—

1. The sign of the expression $E\equiv\alpha x^2+2\beta x+\gamma$, for all real values of x, is that of α except in the case when $E=0$ has real and distinct roots and x lies between them.

2. If $\alpha \to 0$, the equation $E=0$ has one infinite root, and one finite root. The expression E changes sign as x passes through the finite root of $E=0$.

Case I. $\alpha<0$.

(*i*) Suppose that $\beta^2-\alpha\gamma>0$, the equation $E=0$ has real roots x_1, x_2. The expression E is positive or negative according as x lies or does not lie between x_1 and x_2. Thus y is real only if x lies between x_1 and x_2. The curve is therefore a closed real curve, *i.e.*, an ellipse.

(*ii*) If $\beta^2-\alpha\gamma<0$ the equation $E=0$ has imaginary roots and the expression E is negative, for all real values of x, since $\alpha<0$. The locus, thus contains no real point and consequently the equation represents an imaginary ellipse.

(*iii*) $\beta^2-\alpha\gamma=0$, then $E\equiv\alpha(x-x_1)^2$ which is negative since $\alpha<0$. The equation which reduces to

$$y=\lambda x+\mu\pm(x-x_1)\sqrt{\alpha}$$

represents a pair of imaginary lines.

Case II. $\alpha>0$.

(*i*) Suppose $\beta^2-\alpha\gamma$ is positive and x_1, x_2 the real roots of $E=0$. All lines $x=k$ with the exception of those which lie between the lines $x=x_1$, $x=x_2$ meet the conic in real points. The conic is therefore a hyperbola with no part of the curve between the lines $x=x_1$, $x=x_2$.

(*ii*) Let $\beta^2-\alpha\gamma<0$, then $E=0$ has complex roots and the expression E is positive for all real values of x, since $\alpha>0$. Also, y tends to infinity with x. The conic in this case also is a hyperbola.

(*ii*) If $\beta^2-\alpha\gamma=0$ and x_1 is the double root of $E=0$, the equation represents the two real lines

$$y=\lambda x+\mu\pm(x-x_1)\sqrt{\alpha}.$$

Case III. $\alpha=0$, the equation of the locus is

$$y=\lambda x+\mu\pm[2\beta(x-x_1)]^{\frac{1}{2}}$$

where $\quad 2\beta x_1+\gamma=0$.

(*i*) $\beta\neq 0$. The equation $E=0$ has one infinite root, hence every line parallel to $y=\lambda x+\mu$ meets the conic in one point at infinity and another finite point. The line $x=x_1$ is a tangent to the conic and the conic lies to the positive side of the line $x-x_1=0$ if $\beta>0$ and to the negative side of the line if $\beta<0$. The conic is, therefore, a parabola.

(*ii*) For $\beta=0$, the locus is a pair of parallel lines

$$y=\lambda x+\mu\pm\sqrt{\gamma}$$

which are real, coincident or imaginary according as

$$\gamma \gtreqless 0$$

65. Joachimsthal's ratio-equation.

Let P (x_1, y_1), Q (x, y) be two points in the plane of the conic $\phi = 0$ and suppose that the line PQ meets the conic ϕ in a point R which divides the segment PQ in the ratio $\lambda : 1$. The co-ordinates of R are

$$\left(\frac{x_1+\lambda x}{1+\lambda}, \frac{y_1+\lambda y}{1+\lambda}\right)=0 \qquad \text{......(13)}$$

The point lies on the conic ϕ, therefore

$$\phi\left(\frac{x_1+\lambda x}{1+\lambda}, \frac{y_1+\lambda y}{1+\lambda}\right)=0,$$

or

$$\lambda^2\phi + 2\lambda T + \phi_1 = 0 \qquad \text{......(14)}$$

where ϕ_1 stands for $\phi\,(x_1, y_1)$ and $T \equiv xX_1 + yY_1 + Z_1 \equiv x_1X + y_1Y + Z$, which in homogeneous co-ordinates will be written as $xX_1 + yY_1 + zZ_1$ or $x_1X + y_1Y + z_1Z$.

65.1. The equation (14) gives two values of λ to which correspond two points R and S (say) given by the formula (13). The points will be real, coincident, or imaginary according as the roots of the equation (14) are real, equal, or imaginary. *Thus the conic ϕ meets every line of its plane in two points.*

65.2 Equation of the tangent at a point.

Let the point P (x_1, y_1) be on the conic $\phi = 0$.

If the line PQ is a tangent (at P) its intersections R and S will fall on P. The equation (14), therefore has two zero roots, since the point (13) will represent the point P if $\lambda = 0$. Consequently

$$T = 0 \qquad \phi_1 = 0.$$

Thus the equation of the tangent is

$$\begin{aligned} T &\equiv xX_1 + yY_1 + Z_1 \\ &\equiv x_1X + y_1Y + Z = 0 \end{aligned} \qquad \text{......(15)}$$

The relation $\phi = 0$ expresses the condition that P lies on $\phi = 0$.

In particular, the equation of the tangent at (x_1, y_1) to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \text{......(15 A)}$$

The line

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1 \qquad \text{...... (15 B)}$$

is a tangent at (x', y') to the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$,

while the line $$yk=2a(x+h) \qquad \text{......(15 C)}$$
touches the parabola $y^2=4ax$ at (h, k).

An alternative method of finding the equation of a tangent to a conic is given below when the conic is an ellipse given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The application of the method to the other two conics is left to the student.

Let (x_1, y_1), (x_2, y_2) be two points on the ellipse. The equation

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=0$$

represents an ellipse whose axes are parallel to the axes of the original ellipse and which passes through the points (x_1, y_1), (x_2, y_2), the centre of the ellipse being the mid-point of the join of (x_1, y_1), (x_2, y_2).

Of the four points of intersections of the ellipses, two are the points (x_1, y_1), (x_2, y_2) and the other two are imaginary. The conic

$$\frac{(x-x_1)(x-x_2)}{a^2}+\frac{(y-y_1)(y-y_2)}{b^2}=\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)$$

passes through the four intersections of the two ellipses and in particular the points (x_1, y_1), (x_2, y_2). For $\lambda=1$, the conic reduces to a line which is therefore the chord of the conic that joins the points (x_1, y_1), (x_2, y_2), and the equation takes the form

$$\frac{x(x_1+x_2)}{a^2}+\frac{y(y_1+y_2)}{b^2}=\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+1.$$

Let the point (x_2, y_2) moving along the curve approach (x_1, y_1) and in the limiting position when $x_2\to x_1$, $y_2\to y_1$, the equation takes the form

$$\frac{2xx_1}{a^2}+\frac{2yy_1}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+1$$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \qquad \left[\because \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1\right]$$

which is, therefore, the equation of the tangent.

65.3. The condition of Tangency. Tangential equation.

Let the line

$$lx+my+n=0$$

be a tangent to the conic ϕ at the point (x_1, y_1). The line is therefore, identical with the tangent at P, *viz.*,

$$xX_1+yY_1+Z_1=0.$$

Hence $$\frac{X_1}{l}=\frac{Y_1}{m}=\frac{Z_1}{n}=k \text{ (say)}$$

or
$$ax_1+hy_1+g-kl=0$$
$$hx_1+by_1+f-km=0$$
$$gx_1+fy_1+c-kn=0$$
$$lx_1+my_1+n=0$$

since (x_1, y_1) lies on the line. The elimination of $x_1, y_1, -k$ gives the required condition

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}=0$$

or $Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$ ......(16)

where A, B etc., are the co-factors of the corresponding small letters in the determinant

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Def. The condition of tangency of a line with a conic is called the tangential or line equation of a conic.

66. Equation of the normal to a conic.

Let $P(x_1, y_1)$ be a point of ϕ; the equation of the tangent at P is

$$xX_1+yY_1+Z_1=0.$$

Consequently, the equation of a line through P at right angles to the tangent is

$$\frac{x-x_1}{X_1}=\frac{y-y_1}{Y_1} \qquad \text{...... (17)}$$

67. Equation of the polar of a point.

Let P (x_1, y_1) be the point whose polar *w. r.* to $\phi=0$, is required. If Q be another point in the plane of the conic, the ratios λ_1, λ_2, in which PQ is divided by the points R

and S of its intersections with ϕ are given by the equation (65. 14)

$$\lambda^2 \phi + 2\lambda \mathrm{T} + \phi_1 = 0.$$

If (PQ, RS) $= -1$, the locus of Q is the polar of P. But the condition that the segment PQ be harmonically divided by R and S is that $\lambda_1 + \lambda_2 = 0$, or $\mathrm{T} = 0$. Thus the locus of Q is the line

$$\begin{aligned} \mathrm{T} &\equiv x\mathrm{X}_1 + y\mathrm{Y}_1 + \mathrm{Z}_1 \\ &\equiv x_1 \mathrm{X} + y_1 \mathrm{Y} + \mathrm{Z} = 0 \end{aligned} \qquad \ldots\ldots(18)$$

The equation in homogeneous co-ordinates will be written as

$$\mathrm{T} \equiv x\,\mathrm{X}_1 + y\,\mathrm{Y}_1 + z\mathrm{Z}_1 \equiv x_1\,\mathrm{X} + y_1\,\mathrm{Y} + z_1\mathrm{Z} = 0.$$

The following particular cases may be noted.

The polars of the point (x_1, y_1) with respect to the conics given by the equations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad y^2 = 4ax$$

are respectively

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 \qquad \ldots\ldots(18\ \mathrm{A})$$

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1 \qquad \ldots\ldots(18\ \mathrm{B})$$

$$yy_1 = 2a\ (x + x_1) \qquad \ldots\ldots(18\ \mathrm{C})$$

Remark. 1. If the point P be at infinity, its co-ordinates will be $(x_1, y_1, 0)$ and its polar is the line $x_1\mathrm{X} + y_1\mathrm{Y} = 0$ which passes through the centre of the conic.

2. The remarks made as regards the definition of a polar *w. r.* to a circle equally apply to the present case.

67.1. Conjugate Points.

The symmetry in the co-ordinates of P and Q in the expression for T shows that if the polar of P passes through Q, the polar of Q passes through P. Such points are called **conjugate points**.

From the above statement it follows that if the pole of a line p lies on q, the pole of q lies on p.

Such lines are called **conjugate lines**.

67.2. The pole of a line. Conjugate lines.

Let (x_1, y_1) be the pole of the line $lx + my + n = 0$, therefore, it is identical with $x\mathrm{X}_1 + y\mathrm{Y}_1 + \mathrm{Z}_1 = 0$.

Thus $$\frac{\mathrm{X}_1}{l} = \frac{\mathrm{Y}_1}{m} = \frac{\mathrm{Z}_1}{n} = k, \text{ say.}$$

or
$$ax_1+hy_1-kl+g=0$$
$$hx_1+by_1-km+f=0$$
$$gx_1+fy_1-kn+c=0.$$

Hence $x_1\begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}+\begin{vmatrix} g & h & l \\ f & b & m \\ c & f & n \end{vmatrix}=0$

$$y_1\begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}+\begin{vmatrix} a & g & l \\ h & f & m \\ g & c & n \end{vmatrix}=0$$

or $x_1=\dfrac{Al+Hm+Gn}{Gl+Fm+Cn},\ y_1=\dfrac{Hl+Bm+Fn}{Gl+Fm+Cn}$ ......(19)

If the pole (x_1, y_1) of $lx+my+n=0$ lies on
$$l'x+m'y+n'=0,$$
we have
$$All'+Bmm'+Cnn'+F(mn'+m'n)$$
$$+G(nl'+n'l)+H(lm'+l'm)=0 \qquad ......(20)$$
which is the condition of conjugacy of the lines.

68. The equation of the chord in terms of the co-ordinates of its mid. point.

Let P (x_1, y_1) be the mid. point of the chord R S of the conic ϕ. If Q be a point collinear with P, R, S, the ratios $\dfrac{PR}{RQ}=\lambda_1$, $\dfrac{PS}{SQ}=\lambda_2$ are the roots of the equation
$$\lambda^2\phi(x, y)+2\lambda T+\phi(x_1, y_1)=0.$$

Since $\dfrac{PR}{RQ}=\lambda_1$, $\dfrac{PS}{SQ}=\lambda_2$,

$\therefore$ $\dfrac{PQ}{PR}=\dfrac{1+\lambda_1}{\lambda_1}$, $\dfrac{PQ}{PS}=\dfrac{1+\lambda_2}{\lambda_2}$.

But PR $=-$ PS,

hence $\dfrac{1+\lambda_1}{\lambda_1}=-\dfrac{1+\lambda_2}{\lambda_2}$

or $\lambda_1+\lambda_2+2\lambda_1\lambda_2=0$

i.e., $\quad T=\phi(x_1, y_1)$.

Thus the equation of the chord is
$$xX_1+yY_1+Z_1=\phi(x_1, y_1) \qquad .....(21)$$

68.1. Locus of the mid-points of a system of parallel chords.

Let $P(x_1, y_1)$ be the mid-point of one of the chords of slope m. The equation of the chord being

$$xX_1 + yY_1 + Z_1 = \phi(x_1, y_1).$$

$$\therefore \quad -\frac{X_1}{Y_1} = m.$$

Thus the locus of (x_1, y_1) is

$$X + mY = 0, \qquad \ldots\ldots(22)$$

which is a line through the centre. The line is called a **diameter**.

Conjugate diameters. *If two diameters are such that the first bisects chords parallel to the second, then the second bisects chords parallel to the first.*

Let $X + mY = 0$, $X + m'Y = 0$ be two diameters.

The diameter $X + mY = 0$ bisects chords of slope m. If these chords are parallel to the diameter $X + m'Y = 0$,

$$m = -\frac{a + m'h}{h + bm'}.$$

$$\therefore \qquad bmm' + h(m + m') + a = 0, \qquad \ldots\ldots(23)$$

and since the relation remains unaltered if m and m' are interchanged, the proposition is proved.

68.2. *Conjugate diameters of a conic are conjugate lines through the centre of the conic.*

Let $X + mY = 0$, $X + m'Y = 0$ be the diameters. Suppose the pole of the first is (x', y'), therefore

$$X + mY = 0 \text{ and } x'X + y'Y + Z = 0$$

are identical, consequently

$$\frac{y'}{x'} \to m \text{ and } x' \to \infty.$$

This point lies on the second diameter, hence

$$X' + mY' = 0$$

$$\text{or} \quad \left(a + h\frac{y'}{x'} + \frac{g}{x'}\right) + m'\left(h + b\frac{y'}{x'} + \frac{f}{x'}\right) = 0$$

$$\text{or} \qquad (a + hm) + m'(h + bm) = 0$$

$$\text{i.e.,} \qquad bmm' + h(m + m') + a = 0.$$

This condition is the same as (23). Hence we may also define a pair of conjugate diameters as follows :—

Def. A pair of diameters so related that each bisects chords parallel to the other are called conjugate diameters.

68.3. Axes. Perpendicular conjugate diameters are called the **axes** of the conic.

Let $X + mY = 0$, $X + m'Y = 0$ be conjugate diameters. The slopes of these diameters are m' and m. Since they are at right angles

$$mm' + 1 = 0.$$

The diameters being conjugate, the condition (23) is satisfied which in virtue of $mm' + 1 = 0$ becomes

$$m + m' = \frac{b-a}{h}.$$

The joint equation of the axes therefore is

$$0 = (X + mY)(X + m'Y) = X^2 + \frac{b-a}{h} XY - Y^2$$

or

$$\frac{X^2 - Y^2}{a-b} = \frac{XY}{h}. \tag{24}$$

The equation (3) shows that the foci lie on the axes.

Exercises XVI

1. Show that the abscissæ of the points of intersection of the line $y = mx + c$ with the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad y^2 = 4ax$$

are given by the equations

$$(a^2m^2 + b^2)x^2 + 2a^2mcx + a^2(c^2 - b^2) = 0$$
$$(a^2m^2 - b^2)x^2 + 2a^2mcx + a^2(c^2 + b^2) = 0$$
$$m^2x^2 + 2x(mc - 2a) + c^2 = 0.$$

(*i*) Show that the points of intersection are in each case real, coincident and imaginary according as

$$a^2m^2 + b^2 \gtreqless c^2, \; a^2m^2 - b^2 <, = \text{or} > 0, \; mc <, = \text{or} > a.$$

(*ii*) Show that the tangents to these conics parallel to the line $y = mx$ are

$$y = mx \pm \sqrt{a^2m^2 + b^2}, \; y = mx \pm \sqrt{a^2m^2 - b^2}, \; y = mx + \frac{a}{m}.$$

(*iii*) Show that the lengths of the chords intercepted on the line $y = mx + c$ by these conics are respectively

$$2ab\,[(1 + m^2)(a^2m^2 + b^2 - c^2)]^{\frac{1}{2}}/(a^2m^2 + b^2)$$
$$2ab\,[(1 + m^2)(c^2 - a^2m^2 - b^2)]^{\frac{1}{2}}, 4[a(1 + m^2)(a - mc)]/m^2$$

2. Show that the equations of the normals to the conics

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \; y^2 = 4cx \text{ at } (x_1, y_1)$$

are respectively

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}},\quad \frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{-\frac{y_1}{b^2}},\quad \frac{y-y_1}{-y_1}=\frac{x-x_1}{2a}.$$

3. Find the equation of the lines which join the origin with the intersections of the line $lx+my+n=0$ and the conic $\phi=0$. Deduce also the tangential equation of the conic.

4. Show that the locus of the mid-points of the chords which pass through the point (x_1, y_1) is $xX_1+yY_1+Z_1=\phi$.

69. Forms of Equations. In the present section, conics will be denoted by $\phi=0$, $\psi=0$ and lines by $L_i=0$. The tangent to the conic $\phi=0$ at the point (x_i, y_i) will be denoted by $T_i=0$.

Theorem. *Two conics intersect in four points.*

To find the points of intersection of two conics, we have to solve the equations in x and y. To do so, we write their equations as quadratics in x

$$ax^2+2x(hy+g)+by^2+2fy+c=0 \qquad \ldots\ldots(i)$$

$$a'x^2+2x(h'y+g')+b'y^2+2f'y+c'=0 \qquad \ldots\ldots(ii).$$

We then eliminate x in the usual way and get an equation in y which is of the fourth degree. The roots of this equation are the ordinates of the points of intersection of the conics ϕ and ψ. We can eliminate x^2 from the above quadratics and obtain a linear equation for x in terms of y. Thus the four corresponding values of x can then be determined. Hence we see that there are only four points of intersection.

As regards the nature of the points of intersection it may be pointed out that they can be all real, two real, and two imaginary, four imaginary occuring as conjugate pairs, two pairs of coincident points, two coincident, three coincident or four coincident points.

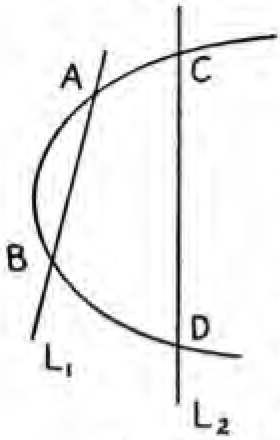
We now proceed to interpret the various forms of equations obtained by the combination of the equations of two proper or degenerate conics

70. The equation

$$\phi+\lambda\psi=0 \qquad \ldots\ldots(25)$$

is of the second degree, and is satisfied by the values of x and y which simultaneously satisfy $\phi=0$, $\psi=0$. The locus of the equation thus passes through the four points of intersections of ϕ and ψ and this number is one less than the number of points which determine a proper conic. The equation (25) therefore represents a *pencil of conics.*

70.1. Let the conic $\psi=0$ break up into right lines L_1 and L_2. The conic meets the line L_1 in A and B and L_2 in C and D.



The equation

$$\phi+\lambda\, L_1\, L_2=0 \qquad \ldots\ldots(26)$$

therefore represents a pencil of conics whose base points are A. B. C. D.

If A and C coincide, *i.e.*, if the lines $L_1=0$, $L_2=0$ intersect on the conic $\phi=0$, then the conic $\phi+\lambda L_1L_2=0$ touches $\phi=0$ at the point of intersection of $L_1=0$, $L_2=0$.

70.2. Let the line L_2 move up and ultimately coincide with L_1 so that C coincides with A and D with B. The equation

$$\phi+\lambda\, L_1^2=0 \qquad \ldots\ldots(27)$$

then represents a pencil of conics whose base points are A. A. B. B. The conics of the pencil, therefore, touch each other at A and B and are said to have a double contact.

As a particular case, if $L_1=0$, $L_2=0$, $L_3=0$ be the equations to the three st. lines, the equation $L_2\,L_3=\lambda\, L_1^2$ represents a conic to which $L_2=0$, $L_3=0$ are tangents and $L_1=0$ is the chord of contact. Interpret the equation $\phi=u^2+\lambda\, u$, where $\phi=0$ is a conic and $u=0$ a st. line.

70.3. The equation $\phi=\lambda$ L represents a conic through intersections of $\phi=0$ with the line L=0 and the line at ∞.

Thus $\phi=\lambda L$ and $\phi=0$ have their asymptotes in the same direction.

As a particular case if $\phi\equiv x^2+y^2-r^2=0$, the equation of any other circle $x^2+y^2+2gx+2fy+c=0$ is of the form $\phi=\lambda L$. Thus (any two circles intersect the line at ∞ in the same two circular) points.

The equation $y^2=\lambda x$ represents a conic which touches the line at ∞ and the line $x=0$ where they are met by the line $y=0$.

70.4. The equation $\phi=\lambda$ represents a conic which has double contact with $\phi=0$ where the line at ∞ meets it; and these two conics have therefore the same asymptotes.

If $\phi=0$ denote a circle, $\phi=\lambda$ represents a concentric circle.

Hence two concentric circles have double contact at the circular points.

70.5. Suppose that the line L_2 is the tangent T_1 to the conic ϕ at C, then

$$\phi+\lambda L_1T_1=0 \qquad \ldots\ldots(28)$$

is the equation of a pencil of conics which has A, B, C, C as base points. All members of the pencil therefore touch at C, and cut at two distinct points A and B.

If the line T_1 is a tangent to ϕ at A, the pencil of conics given by the equation

$$\phi+\lambda L_1 T_1=0 \qquad \ldots\ldots(29)$$

has three coincident base points at A, and fourth point at B. The conics of the system are said to have a *double contact*, a contact of the 2nd order, at A.

Interpret the equation $\phi=\lambda T$.

70.7. The conics of the pencil

$$\phi+\lambda T_1^2=0 \qquad \ldots\ldots(30)$$

have four coincident base points. They are said to have 'four pointic contact' or contact of the third order.

70.8. The equation

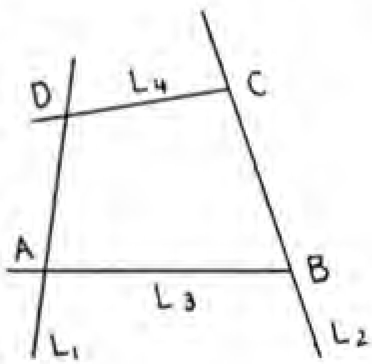
$$\phi+\lambda T_1 T_2=0 \qquad \ldots\ldots(31)$$

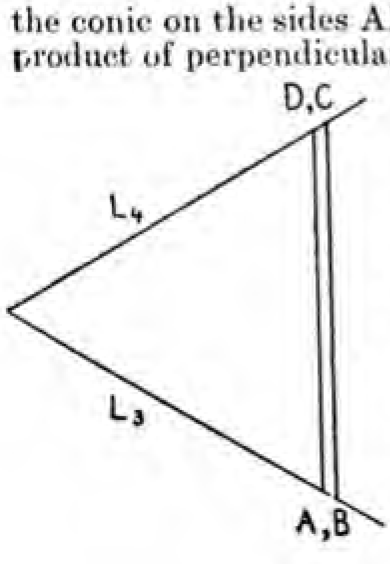
is another form of (70.2, 27) and represents a pencil of conics having double contact at the points of contact of the tangents T_1, T_2.

70.9. Let ϕ also break up into two right lines L_3, L_4. The equation

$$L_3 L_4+\lambda L_1 L_2=0 \qquad \ldots\ldots(32)$$

then represents a pencil of conics with A, B, C, D as the base points, since the equation is satisfied by the co-ordinates of points where L_3 and L_4 are met by L_1 and L_2. Hence if a conic circumscribe a quadrangle ABCD, the product of the perpendiculars from a variable point P of the conic on the sides AB, CD bears a constant ratio to the product of perpendiculars from P to the sides AD, BC.





70.10. A particular case of the above equation is when L_1 and L_2 coincide. The equation

$$L_3 L_4+\lambda L_1^2=0 \qquad \ldots\ldots(33)$$

represents a pencil of conics whose base points coincide by two at A and D.

The conics of the pencil have double contact, the lines L_3, L_2 being the common tangents and L_1 the chord of contact (*c.f.* 70.2).

Exercises XVII

1. Prove that all conics through the intersections of two rectangular hyperbolas are rectangular hyperbolas.

2. If two rectangular hyperbolas intersect in four points, each point is the orthocentre of the triangle formed by the other three.

3. If a rectangular hyperbola pass through the vertices of a triangle, it will also pass through the orthocentre.

4. Prove that two and only two parabolas can be drawn through four given points.

5. Show that only one parabola can be drawn touching a given conic in two given points.

6. Show that only one rectangular hyperbola can be drawn touching a given conic in two given points.

7. Show that the equation of a parabola touching the axes at points A, B distant a, b from the origin may be reduced to the form $\pm\sqrt{\dfrac{x}{a}} \pm \sqrt{\dfrac{y}{b}} = 1$.

8. Find the equation of a conic passing through the point $(2, -2)$ and having double contact with the conic

$$2x^2 - 3xy + 5y^2 + x + 2y - 1 = 0$$

where it meets the y-axis.

9. Show that there are three values of λ for each of which the conic $\phi + \lambda\psi = 0$ breaks up into a pair of st. lines.

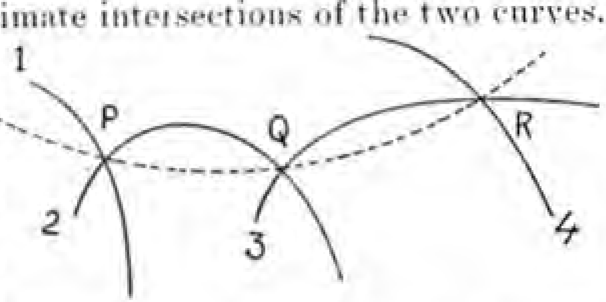
71. Envelopes. Let $P(x, y)$ be an arbitrary point of a plane which is subjected to certain geometrical conditions. The point will trace out a locus whose equation can be determined. If the number of conditions is not sufficient, the locus will not be unique and its equation will contain a number of arbitrary constants which can be determined by imposing the requisite number of conditions. Thus, for example, a st. line is uniquely determined if two points on it be given. If only one point (x', y') be given, the equation of all lines through it can be represented by the equation $y - y' = m(x - x')$. By varying m we get different lines all belonging to a pencil and for a definite value of m we get a perfectly definite st. line of the pencil. As another instance, the equation $C_1 + \lambda C_2 = 0$, where $C_1 = 0$, and $C_2 = 0$ are circles, represents a family of co-axal circles whose equation depends on λ. The quantity λ is called a variable **parameter** and the system of curves is said to depend upon one parameter. The equation of curves $\lambda C_1 + \mu C_2 + \nu C_3 = 0$ depends upon two parameters. Thus a system of circles

with a common radical centre depends upon two parameters.

Def. *The envelope of a system or family of curves is a curve (or group of lines) which touches every member of the family, and which, at each point, is touched by some member of the family.*

71.1. Let $f(x, y, \alpha)=0$ be the equation of a family of curves which depends upon a single parameter α. If we give to α a particular value α_1, we obtain perfectly a definite curve of the family. Now give to α the value $\alpha_1+\delta\alpha_1$. The corresponding curve is then represented by the equation $f(x, y, \alpha_1+\delta\alpha_1)=0$. These two curves cut each other in a certain number of points and when $\delta\alpha_1\to 0$ these points are called the ultimate intersections of the two curves.

Consider the curve (2) of the family which is met by curves (1) and (3) of the family in points P and Q respectively. Suppose that each of the curves (1) and (3) approach indefinitely close to the curve (2). The points P and Q also approach coincidence, and hence the curve PQR touches the curve (2).



Thus *the locus of the ultimate intersections of the curves of the family $f(x, y, \alpha)=0$ is, in general, the envelope of the family.* For the envelope may still exist when the neighbouring members do not intersect, and thus, there is no locus of the ultimate intersections of the members of the family.

71.2. To find the envelope of the family of curves

$$f(x, y, \alpha)=0 \qquad \ldots\ldots(34)$$

where α is a parameter.

Let $f(x, y, \alpha+\delta\alpha)=0$ be a curve adjascent to $f(x,y,\alpha)=0$. The common points of the two curves satisfy the equation

$$f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)=0$$

i.e.,
$$\frac{f(x, y, \alpha+\delta\alpha)-f(x, y, \alpha)}{\delta\alpha}=0$$

and as $\delta\alpha\to 0$, this approaches

$$f_\alpha(x, y, \alpha)=0 \qquad \ldots\ldots(35)$$

where $f_\alpha(x, y, \alpha)$ is the derivative of $f(x, y, \alpha)$ *w. r.* to α.

The envelope of the family will, therefore, be obtained by the elimination of α between the equations (34) and (35) The eliminant is called the α-**discriminant**.

Remark. The α-discriminant of a family of curves may contain factors which equated to zero will give loci which are not touched by the members of the family, and are not, therefore, envelopes.

The complete rigorous discussion of the subject is beyond the scope of the present work.

71.3. To find the envelope of the family of curves
$$f(x, y, \lambda) \equiv \lambda^2 P + 2\lambda Q + R = 0$$
where λ is a variable parameter and P, Q, R are functions of x and y.

$$\tfrac{1}{2} f_\lambda (x, y, \lambda) \equiv \lambda P + Q = 0$$

Elimination of λ gives the equation of the envelope, *viz.*,

$$Q^2 = PR \qquad \ldots\ldots(36)$$

The result may be obtained otherwise thus :—For a given set of values of x and y, the equation $\lambda^2 P + 2\lambda Q + R = 0$ gives two values of λ, λ_1 and λ_2, which are the parameters of the curves which pass through the given point. If the point is on the envelope, these curves are adjascent curves, and therefore, $\lambda_1 \to \lambda_2$. Hence $Q^2 = PR$.

71.4. To find the envelope of the family of curves given by the equation
$$f(x, y, \theta) \equiv P \cos\theta + Q \sin\theta = R$$
where θ is a variable parameter, and P, Q, R are functions of x, y

$$f_\theta(x, y, \theta) \equiv -P \sin\theta + Q \cos\theta = 0$$

Squaring and adding we get the required result
$$P^2 + Q^2 = R^2 \qquad \ldots\ldots(37)$$

Or, if we put $t = \tan\dfrac{\theta}{2}$, $\cos\theta = \dfrac{1-t^2}{1+t^2}$, $\sin\theta = \dfrac{2t}{1+t^2}$,

the equation of the family takes the form
$$t^2(P + R) - 2tQ - (P - R) = 0$$
and the envelope of this is by Art. 71.3.
$$P^2 + Q^2 = R^2.$$

71.5. To find the envelope of the family of lines
$$lx + my + n = 0$$
whose co-efficients are connected by the relation
$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0.$$

If we eliminate n between the given equations, then
$$al^2 + bm^2 + c(lx + my)^2 - 2(fm + gl)(lx + my) + 2hlm = 0$$
i.e., $l^2(a + cx^2 - 2gx) + 2lm(cxy + h - fx - gy)$
$$+ m^2(b - 2fy + cy^2) = 0.$$

For a given set of values of x and y, there are two values of $l : m$. Thus, through a point, two lines of the system can be drawn. If the point $P(x, y)$ is on the envelope, the lines approach coincidence, and therefore, the above quadratic in $l : m$ has equal roots.

Hence

$$(a-2gx+cx^2)(b-2fy+cy^2)=(cxy+h-fx-gy)^2$$

which reduces to the form

$$Ax^2+2Hxy+By^2+2Gx+2Fy+C=0$$

where A, B, etc., are co-factors of a, b, etc., in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0.$$

Solved Examples

(1) *Find the envelope of the family of conics given by the equation*

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

The equation can be written as

$$\lambda^2+\lambda(a^2+b^2-x^2-y^2)+(a^2b^2-b^2x^2-a^2y^2)=0$$

The envelope of the family is given by the equation

$$(x^2+y^2-a^2-b^2)^2+4(b^2x^2+a^2y^2-a^2b^2)=0.$$

The equation can be written as

$$(x\pm\sqrt{a^2-b^2})^2+y^2=0$$

which represents two pairs of isotropic lines through the points $(\pm\sqrt{a^2-b^2}, 0)$.

(2) *Find the envelope of the system of lines*

$$ax\cos\theta+by\sin\theta=c$$

Differentiating *w. r.* to θ,

$$ax\sin\theta-by\cos\theta=0.$$

Squaring and adding we have the envelope

$$a^2x^2+b^2y^2=c^2.$$

(3) *Find the envelope of the family of lines* $lx+my+n=0$ *when* l, m, n *are connected by the relation* $a^2l^2+b^2m^2=c^2n^2$.

The equation $a^2l^2+b^2m^2=c^2(lx+my)^2$

or $$l^2(a^2-c^2x^2)-2lmc^2xy+m^2(b^2-c^2y^2)=0$$

gives two lines of the system that pass through (x, y). If the point (x, y) lies on the envelope, the two lines will coincide

$$\therefore \quad c^4x^2y^2=(a^2-c^2x^2)(b^2-c^2y^2),$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{c^2}.$$

Exercises XVIII

1. Find the envelope of the lines $ty=x+at^2$, where t is the parameter.

2. Find the envelope of the lines $y=mx+\sqrt{a^2m^2+b^2}$ when m varies.

3. N is the foot of the perpendicular from P$(at^2, 2at)$ on the x-axis. The parallelogram ONPQ is completed. Show that the diagonal NQ touches the parabola $y^2+16ax=0$.

4. A line moves so that the sum of its intercepts on two fixed lines is constant. Show that the line always touches a parabola.

5. A line moves so that the product of the perpendiculars drawn on it from the points $(\pm a, 0)$ is the constant k. Show that it envelopes the conic $\dfrac{x^2}{\lambda+a^2}+\dfrac{y^2}{\lambda}=1$.

6. Show that the line that joins the points $(a\cos\theta,\ b\sin\theta)$, $(-a\sin\theta,\ b\cos\theta)$, θ being the variable parameter, is the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{1}{2}.$$

Exercises XIX

1. Discuss the nature of the following conics.

(*i*) $5x^2-4xy+y^2+2x-y=0$
(*ii*) $3x^2-4xy+y^2+2x-y=0$
(*iii*) $3x^2-4xy+y^2+15x-6y+7=0$
(*iv*) $2x^2-7xy+3y^2-9x+7y+4=0$
(*v*) $4x^2-12xy+9y^2+4x-5y+3=0$
(*vi*) $4x^2-12xy+9y^2+8x+12y-7=0$. (Niewen glowski)

2. Discuss the nature of the conics represented by the following equations, λ being a variable parameter which assumes every real value :—

(*i*) $x^2+2\lambda xy+\lambda y^2+2\lambda x+2y+\lambda+1=0$.
(*ii*) $x^2+2\lambda xy+y^2-2\lambda x+2y+2=0$.
(*iii*) $\lambda x^2-2xy+\lambda y^2-2(\lambda+1)x+2y+2=0$
(*iv*) $x^2-2\lambda xy+(\lambda+2)\,y^2-2x-2\lambda y-3=0$.
(*v*) $x^2+2\lambda xy+y^2-2x+2y-\lambda=0$.
(*vi*) $\lambda x^2+2\lambda xy+y^2-2x-2\lambda y+\lambda=0$.

(Anbert and Papelier)

3. Discuss the nature of the locus of the following equations :—

(*i*) $y=2x+1\pm\sqrt{(\lambda^2-1)\,x^2-2x\,(\lambda+1)+(2\lambda-1)}$,

where λ is a variable parameter.

4. Find the envelope of the lines $y+tx=2at+at^3$ where t is a variable parameter.

5. Show that the envelope of the circles $(x-c)^2+y^2=d^2$, where $c^2+d^2=k^2$, is $x^2-2y^2+2k^2=0$.

6. Find the envelope of the polars *w. r.* to the circle $x^2+y^2+2gx+2fy+c=0$ of points that lie on the parabola $y^2=4ax$.

7. Show that the polars of the points of the circle $(x-h)^2+(y-k)^2=a^2$ *w. r.* to the circle $x^2+y^2=b^2$ is the conic $a^2(x^2+y^2)=(hx+ky-b^2)^2$.

8. Show that the polars of the points of the circle $x^2+y^2=b^2$ *w. r.* to the circle $(x-h)^2+(y-k)^2=a^2$ is the conic

$$b^2[(x-h)^2+(y-k)^2]=[h(x-h)+k(y-k)+a^2]^2.$$

9. Show that the polars of points of circle $x^2+y^2=2bx$ *w. r.* to the circle $x^2+y^2=a^2$ touch the conic $b^2y^2+a^2(2bx-a^2)=0$.

10. Show that the envelope of a st. line which is such that the circles $x^2+y^2-2\lambda x+\delta^2=x^2+y^2-2\lambda' x+\delta^2=0$ make equal intercepts on it, is the parabola $y^2=2(\lambda+\lambda')x$.

11. The centroid of a triangle inscribed in the hyperbola $xy=a^2$ is at the point $(a\lambda, 0)$. Show that its sides touch the conic

$$4xy=(a+3\lambda y)^2.$$

12. Find the pole of the line $y+2=0$ *w. r.* to the conic

$$x^2+2xy-y^2-4x-6y-1=0.$$ [Math. Trip. 1932]

CHAPTER VIII

ELLIPSE

72. It has been shown in Chapter VII, that the equation of the ellipse can be reduced to the canonical form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

when the major axis and minor axis be along the x-axis and y-axis respectively. The two axes are the axes of symmetry of the ellipse and the origin is the centre.

The present chapter deals with the properties of the ellipse.

√72.1. Geometrical Method of Generating an Ellipse.

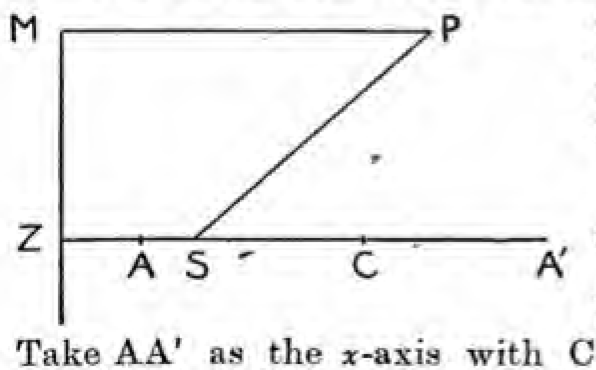
An ellipse has been defined to be the locus of a point which moves in a plane, such that its distance from a fixed point bears a constant ratio less than unity to its distance from a fixed line.

The fixed point is called focus, the fixed line the directrix and the constant ratio the eccentricity.

We proceed to obtain its equation in the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from this definition.

Let S be the focus, MZ the directrix and e the eccentricity. Draw SZ perpendicular to MZ. Divide SZ in the ratio $e:1$ at the points A and A′, so that

$$SA=e.AZ,\quad A'S=e.A'Z.$$



The points A and A′ are on the ellipse.

Let AA′ $=2a$, and C be the mid-point of AA′. Take AA′ as the x-axis with C as the origin. Let the co-ordinates of S be $(-x', 0)$ and the equation of MZ $x+k=0$. If (x, y) be the co-ordinates of an arbitrary point P on the ellipse, and PM the perpendicular on the directrix, the conditions of the problem require

$$SP=e.PM.$$

$$\therefore \qquad (x+x')^2+y^2=e^2(x+k)^2$$

i.e., $$x^2(1-e^2)+y^2+2x(x'-e^2k)=e^2k^2-x'^2$$

Since $y=0$ cuts the curve in A and A′ which are equidistant from C.

$\therefore x'=e^2k$ and $a^2=\dfrac{e^2k^2-x'^2}{1-e^2}=e^2k^2$

Thus the equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

where $$b^2=a^2(1-e^2)$$

Evidently CS $=-ae$, CZ $=-\dfrac{a}{e}$.

omit **73**. We know that the focus of a conic is the intersection of the tangents one from either circular point and the corresponding directrix is the polar of the focus *w. r.* to the conic. Now the equations of the four isotropic tangents of the ellipse in question are

$$y=ix\pm i\sqrt{a^2-b^2}$$
$$y=-ix\pm i\sqrt{a^2-b^2}.$$

The first pair intersect the other pair in four points

$$(\pm\sqrt{a^2-b^2},\ 0),\ (0,\ \pm\sqrt{b^2-a^2})$$

which are the foci of the ellipse. Since $a>b$, the first pair of foci is real and lies on the x-axis and the second is imaginary, and lies on the y-axis.

Since $\sqrt{a^2-b^2}<a$, it is convenient to represent this expression as a fraction of the semi-major axis 'a'. Accordingly we set

$$\sqrt{a^2-b^2}=ae \qquad 0<e<1$$

or $$b^2=a^2(1-e^2).$$

The co-ordinates of the foci are then $(\pm ae,\ 0)$, $(0,\ \pm iae)$.

The equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ of the ellipse can then be written in the form

$$x^2(1-e^2)+y^2=a^2(1-e^2)$$

or $$(x+ae)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2. \qquad \text{......(1)}$$

If P(x,y) be any point on the ellipse, the left-hand side of the equation is the square of the distance PS of P from

the focus $S(-ae, 0)$ and the right-hand side is e^2PM^2 where PM is the measure of the perpendicular from P on the corresponding directrix $x+\frac{a}{e}=0$.

It has thus been proved that *the ratio of the distances of any point on an ellipse from a focus and the corresponding directrix is a positive constant less than unity.*

A similar conclusion can be drawn when the equation is written in the form

$$(x-ae)^2+y^2=e^2\left(\frac{a}{e}-x\right)^2 \qquad \text{......(2)}$$

Note. The equations of the two directrices corresponding to the real foci $(-ae, 0)$, $(ae, 0)$ are

$$x+\frac{a}{e}=0, \qquad x-\frac{a}{e}=0.$$

73.1. The points $S(-ae, 0)$, $S'(ae, 0)$ being the foci and $P(x, y)$ any point on the ellipse, it follows from equations

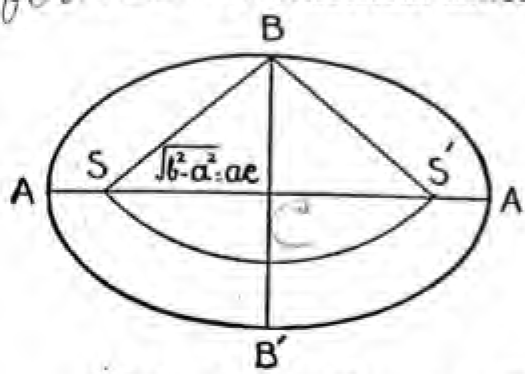
$$x+\frac{a}{e}=0, \qquad x-\frac{a}{e}=0$$

that
$$SP=a+ex \qquad S'P=a-ex$$
$$\therefore \qquad SP+S'P=2a. \qquad \text{......(3)}$$

Thus *the sum of the distances of any point of an ellipse from its real foci is constant and equal to the length of its major-axis.*

The base of a triangle is fixed and the sum of the other two sides is constant, show that the locus of the vertex is an ellipse whose foci are the extremities of the base.

73.2. Geometrical construction of the real foci.



Let AA′, BB′ ($=2a$, $2b$) be the major and minor axes of the ellipse. With B as centre and radius equal to a describe a circle cutting AA′ in S and S′. Then S, S′ are the foci.

It is obvious that

$$\cos CSB=e.$$

73.3. Latus Rectum. Def. The length of the chord drawn through the focus S or S′ at right angles to the major axis is called the *latus rectum.*

If LSL′ be the latus rectum, the co-ordinates of L are $(-ae, \text{SL})$. Since L lies on the ellipse,

$$\frac{a^2e^2}{a^2}+\frac{\text{SL}^2}{b^2}=1$$

$$\therefore \qquad \text{SL}^2=\frac{b^4}{a^2}$$

$$\therefore \qquad \text{LL}'=2\text{SL}=\frac{2b^2}{a} \qquad \text{......(4)}$$

Imp.

74. To interpret the inequation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1 \gtreqless 0$.

Let $P(x, y)$ be a point in the plane of the ellipse. Join P to the centre C to meet the ellipse in Q (x', y'). Suppose $\text{CP}=r$, $\text{CQ}=\rho$ and $\angle \text{A}'\text{CP}=\theta$, then

$$x=r\cos\theta,\ y=r\sin\theta,\ x'=\rho\cos\theta,\ y'=\rho\sin\theta.$$

Now P is outside or inside the ellipse according as $r \gtrless \rho$.

But since Q lies on the ellipse

$$\frac{\rho^2\cos^2\theta}{a^2}+\frac{\rho^2\sin^2\theta}{b^2}=1$$

or $$\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{\rho^2}$$

$$>\frac{1}{r^2} \text{ if P is outside for then } r>\rho$$

and $$<\frac{1}{r^2} \text{ if P is inside the ellipse for then } r<\rho.$$

Thus $r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}-1\right)\gtrless 1$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\gtrless 1$ according as P is outside or inside the ellipse. It is thus seen that as $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ is the analytic representation of all points on the ellipse, the inequation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1>0$$

is the analytic representation of the region of the plane that lies outside the ellipse, and the inequation

$$\frac{x}{a^4}+\frac{y^2}{b^2}-1<0$$

represents analytically the region that lies inside the ellipse.

75. Auxiliary circle. The circle whose diameter is the major axis AA′ of the ellipse is called the *major auxiliary circle*, or simply, the auxiliary circle.

If P be a point on the ellipse and MP be drawn perpendicular to AA′, the point Q where MP meets the auxiliary circle is said to correspond to P.

Similarly, the circle drawn on the minor axis as diameter is called the *minor auxiliary circle*. If P be a point on the ellipse and the perpendicular PN on the minor axis meets the minor auxiliary circle in Q′, then P and Q′ are said to correspond.

75.1. In an ellipse, the locus of the foot of the perpendicular from a focus on a tangent is the auxiliary circle.

The line $y - mx = \sqrt{a^2m^2 + b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m.

Consider the focus $(ae, 0)$.

The line through the focus and perpendicular to the tangent is,

$$my + x = ae.$$

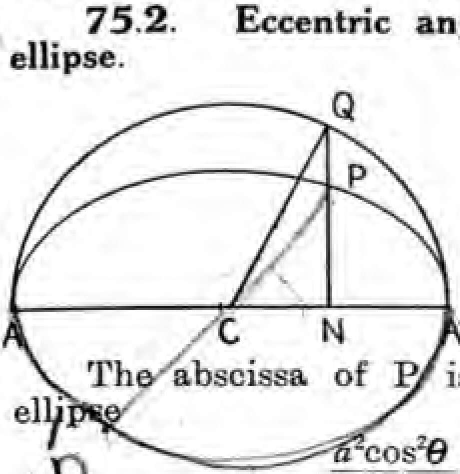
We eliminate m by squaring and adding the two equations and thus get the equation of the locus

$$x^2 + y^2 = a^2.$$

In like manner, the feet of the perpendiculars from the other real focus on the tangents lie on this circle.

But the feet of the perpendiculars from the imaginary foci on the tangents lie on the circle $x^2 + y^2 = b^2$.

75.2. Eccentric angle. Freedom equations of an ellipse.



Let P and Q be two corresponding points on the ellipse and its auxiliary circle. If the angle $A'CQ=\theta$, the freedom equations of the auxiliary circle are

$$x = a\cos\theta,\ y = a\sin\theta.$$

The abscissa of P is $a\cos\theta$. Since it lies on the ellipse

$$\frac{a^2\cos^2\theta}{a^2} + \frac{y^2}{b^2} = 1.$$

$$\therefore\quad y = b\sin\theta.$$

Hence the freedom equations of the ellipse are

$$x=a\cos\theta,\ y=b\sin\theta. \qquad \ldots\ldots(5)$$

It follows that the ordinates NP, NQ of the corresponding points on the ellipse and the auxiliary circle are in the ratio of $b : a$.

The angle θ is called the *eccentric angle* of P. The point whose eccentric angle is θ is called the θ-point.

The eccentric angle of the other extremity P′ of the diameter through P is $\theta+\pi$. For the point $\theta+\pi$ is $(-a\cos\theta, -b\sin\theta)$ and this lies on CP as well as on the ellipse.

The equation of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the parametric form may also be expressed as follows:—

Writing the equation in the form $\frac{y^2}{b^2}=1-\frac{x^2}{a^2}$

i.e., $$\frac{\frac{y}{b}}{1+\frac{x}{a}}=\frac{1-\frac{x}{a}}{\frac{y}{b}}=t \text{ (say)}$$

We have $$x=\frac{a(1-t^2)}{1+t^2},\quad y=\frac{2bt}{1+t^2}.$$

76. Equation of the chord joining the points α, α'.

The equation of the chord is

$$\frac{x-a\cos\alpha}{y-b\sin\alpha}=\frac{a(\cos\alpha-\cos\alpha')}{b(\sin\alpha-\sin\alpha')}=-\frac{a}{b}\,\frac{\sin\frac{1}{2}(\alpha+\alpha')}{\cos\frac{1}{2}(\alpha+\alpha')}$$

which reduces to

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\alpha')+\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\alpha')=\cos\tfrac{1}{2}(\alpha-\alpha').$$

76.1. Equation of the tangent at α.

Let the point α' moving along the curve approach α, the limiting position of the chord, *viz.*, the tangent at α has the equation

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1. \qquad \ldots\ldots(6)$$

If the point of contact be (x', y'), $x'=a\cos\alpha$, $y'=b\sin\alpha$ the equation of the tangent is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1 \qquad \ldots\ldots(7)$$

Cor. If $\alpha+\alpha'=2\phi$, the equation of the chord (α, α') becomes

$$\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi=\cos\tfrac{1}{2}(\alpha-\alpha')$$

which represents a system of chords parallel to the tangent at ϕ.

Hence *for a system of parallel chords, the sum of the eccentric angles of their extremities is constant, and is equal to double the eccentric angle of the point of contact of a tangent parallel to the chord.*

76.2. Tangential Equation of the ellipse.

Let $lx+my+n=0$ be a tangent to the ellipse. It is then for some α identical with

$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$$

$$\therefore \quad \frac{\cos\alpha}{al}=\frac{\sin\alpha}{bm}=-\frac{1}{n}$$

$$a^2l^2+b^2m^2=n^2 \qquad \ldots\ldots(8)$$

We give an alternative method for finding the condition that the line

$$lx+my+n=0$$

may touch the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The lines joining the origin to the points of intersection are given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{lx+my}{n}\right)^2=0$$

i.e., $$\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)x^2-2\frac{lm}{n^2}xy+\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right)y^2=0.$$

If the line touches the ellipse, these two lines coincide.

i.e., $$\frac{l^2m^2}{n^4}=\left(\frac{1}{a^2}-\frac{l^2}{n^2}\right)\left(\frac{1}{b^2}-\frac{m^2}{n^2}\right).$$

Thus the condition is

$$a^2l^2+b^2m^2=n^2.$$

76.3. The product of the perpendiculars from the real foci on any tangent of an ellipse is equal to the square on the semi-minor axis.

$(ae, 0)$ and $(-ae, 0)$ are the real foci of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Any tangent is $x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$.

Let p_1, p_2 be the perps. from the foci on it.

$$p_1 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} - ae\cos\alpha,$$

$$p_2 = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha} + ae\cos\alpha$$

$$\therefore \quad p_1p_2 = (a^2\cos^2\alpha + b^2\sin^2\alpha) - (a^2 - b^2)\cos^2\alpha = b^2.$$

In particular the line

$$y = mx + \sqrt{a^2m^2 + b^2}$$

is a tangent to the ellipse for all values of m.

76.4. The equation of the tangent at a point θ is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$$

or $\quad \dfrac{x}{a}\,\dfrac{1-t^2}{1+t^2} + \dfrac{y}{b}\,\dfrac{2t}{1+t^2} = 1$ where $t = \tan\dfrac{\theta}{2}$

i.e., $\quad bt^2(x+a) - 2aty + b(a-x) = 0.$

For a given value of x and y, there are two values of t, hence from an arbitrary point two tangents can be drawn to an ellipse which are real, coincident, or imaginary, according as

$$a^2y^2 - b^2(a^2 - x^2) \gtreqless 0$$

or

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \gtreqless 0$$

i.e., according as the point (x, y) is outside, on or inside the ellipse.

77. The equation of the pair of tangents through a point.

Let (x_1, y_1) be the point from which the tangents are drawn, then

$$bt^2(x+a) - 2aty + b(a-x) = 0$$
$$bt^2(x_1+a) - 2aty_1 + b(a-x_1) = 0$$

$$\therefore \quad \frac{t^2}{-ab[y(a-x_1) - y_1(a-x)]}$$

$$= \frac{2t}{b^2[(a+x_1)(a-x) - (a+x)(a-x)]}$$

$$= \frac{1}{-ab[y_1(x+a) - y(x_1+a)]}$$

$$\therefore \quad 4a^2[y(a-x_1) - y_1(a-x)][y_1(x+a) - y(x_1+a)]$$
$$= b^2[(a+x_1)(a-x) - (a+x)(a-x_1]$$

i. e., $\quad (xy_1 - x_1y)^2 - a^2(y-y_1)^2 = b^2(x-x_1)^2$

or $\left(\frac{x-x_1}{a}\right)^2+\left(\frac{y-y_1}{b}\right)^2=\left(\frac{xy_1-x_1y}{ab}\right)^2$

It can be reduced to the form

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2 \quad \text{......(9)}$$

Alternative method :—

The polar of the point $(x_1\ y_1)$ is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0$$

The conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=\lambda\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$ at the points where the polar cuts it.

If this conic passes through $(x_1\ y_1)$ it will then be a pair of tangents from $(x_1\ y_1)$ to the ellipse.

Thus the equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

represents the required tangents.

77.1. Orthoptic Locus or Director Circle.

Def. *The locus of the point of intersection of perpendicular tangents of a conic is called its orthoptic locus.*

In the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the tangents from (x_1, y_1) are given by the equation

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

If the two lines are at right angles, the sum of the co-efficients of x^2 and y^2 is zero.

i.e., $\frac{1}{a^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}$

or, on simplifying, $x_1^2+y_1^2=a^2+b^2$.

Hence the locus required is the circle

$$x^2+y^2=a^2+b^2,$$

which is concentric with the ellipse.

There are other useful methods of solving the foregoing problem.

We have seen that the line $y=mx\pm\sqrt{a^2m^2+b^2}$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ for all values of m.

If the tangent passes through $(x_1\ y_1)$, we have

$$y_1-mx_1=\pm\sqrt{a^2m^2+b^2}$$

or $$(y_1-mx_1)^2=a^2m^2+b^2$$

or $$m^2(a^2-x_1^2)+2mx_1y_1+b^2-y_1^2=0$$

which is a quadratic in m.

This gives the directions of the two tangents from (x_1, y_1). If the tangents are at right angles, the product of the roots $m_1, m_2=-1$.

$$\therefore \qquad \frac{b^2-y_1^2}{a^2-x_1^2}=-1.$$

Whence the locus is the circle $x^2+y^2=a^2+b^2$.

Or

The line $x\cos\alpha+y\sin\alpha=\sqrt{a^2\cos^2\alpha+b^2\sin^2\alpha}$ touches the ellipse for all values of α.

The tangent at right angles to it is therefore

$$x\cos(90+\alpha)+y\sin(90+\alpha)=\sqrt{a^2\cos^2(90+\alpha)+b^2\sin^2(90+\alpha)}$$

i.e., $$-x\sin\alpha+y\cos\alpha=\sqrt{a^2\sin^2\alpha+b^2\cos^2\alpha}.$$

Squaring each equation and adding, we get

$$x^2+y^2=a^2+b^2.$$

78. Equation of the chord with the given mid-point.

Let (x_1, y_1) be the mid-point and α_1, α_2 the eccentric angles of the extremities. The slope of the chord m is

$$m=\frac{b(\sin\alpha_1-\sin\alpha_2)}{a(\cos\alpha_1-\cos\alpha_2)}=-\frac{b}{a}\ \frac{\cos\frac{1}{2}(\alpha_1+\alpha_2)}{\sin\frac{1}{2}(\alpha_1+\alpha_2)}$$

$$=-\frac{b}{a}\ \frac{\cos\alpha_1+\cos\alpha_2}{\sin\alpha_1+\sin\alpha_2}=-\frac{b^2}{a^2}\ \frac{x_1}{y_1}$$

$$\because \left[\begin{array}{l} 2x_1=a(\cos\alpha_1+\cos\alpha_2) \\ 2y_1=b(\sin\alpha_1+\sin\alpha_2). \end{array}\right.$$

Hence the equation of the chord is

$$(y-y_1)=-\frac{b^2}{a^2}\ \frac{x_1}{y_1}\ (x-x_1)$$

or $$\frac{x_1(x-x_1)}{a^2}+\frac{y_1(y-y_1)}{b^2}=0. \qquad \text{......(10)}$$

Imp **79.** Polar of a point P $(x'\ y')$ with respect to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

The points of intersection of any line

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r.$$

are given by the quadratic

$$r^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)+2r\left(\frac{x'\cos\theta}{a^2}+\frac{y'\sin\theta}{b^2}\right)$$
$$+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0.$$

If r_1, r_2 be the roots,

$$\frac{1}{r_1}+\frac{1}{r_2}=-2\,\frac{\dfrac{x'\cos\theta}{a^2}+\dfrac{y'\sin\theta}{b^2}}{\dfrac{x'^2}{a^2}+\dfrac{y'^2}{b^2}-1}.$$

Let R be the point on the line distant ρ from P such that

$$\frac{2}{\rho}=\frac{1}{r_1}+\frac{1}{r_2}.$$

$$\therefore\quad \frac{x'\rho\cos\theta}{a^2}+\frac{y'\rho\sin\theta}{b^2}+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0.$$

Thus the locus of R is the polar of P

$$\frac{x'}{a^2}(x-x')+\frac{y'}{b^2}(y-y')+\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1=0.$$

i.e., $$\frac{xx'}{a^2}+\frac{yv'}{b^2}=1. \qquad \text{......(11)}$$

If P be the mid-point of the chord,

$$\frac{x'\cos\theta}{a^2}+\frac{y'\sin\theta}{b^2}=0$$

Thus the chord which is bisected at $(x'\ y')$ is given by the equation $\dfrac{x'(x-x')}{a^2}+\dfrac{y'(y-y')}{b^2}=0$

i.e., $$\frac{xx'}{a^2}+\frac{yy'}{b^2}=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}.$$

It is parallel to the polar of P.

If θ be given, we get the locus of the middle points of parallel chords

$$y=-\frac{b^2}{a^2m}x. \qquad \text{.....(12)}$$

It follows at once that if the polar of $P(x_1\ y_1)$ passes through $Q(x_2\ y_2)$, then the polar of Q passes through P.

P and Q are the conjugate points.

Find the pole of the line $lx+my+n=0$.

79.1. Conjugate Diameters. Two conjugate diameters are conjugate lines through the centre of the conic.

Condition for conjugate diameters.

Let $y=m_1x, y=m_2x$ be two conjugate diameters of the ellipse. The pole of the diameter $y=m_2x$, lies on $y=m_1x$. Suppose its co-ordinates are (x_1, m_1x_1). The polar of this point is

$$\frac{xx_1}{a^2}+\frac{m_1x_1y}{b^2}=1$$

or $\quad y=-\dfrac{b^2}{a^2m_1}x+\dfrac{b^2}{m_1x_1}$,

which is identical with $y=m_2x$. Thus $x_1\to\infty$ and $m_2=-\dfrac{b^2}{a^2m_1}$. Thus the condition of conjugacy of the two diameters $y=m_1x$, $y=m_2x$ is

$$m_1m_2=-b^2/a^2. \qquad \ldots\ldots(13)$$

and the pole of the diameter $y=m_2x$ is the point at ∞ on the line $y=m_1x$ and *vice versa.*

If $P(x_1\, y_1)$, $D(x_2\, y_2)$ be the extremities of the conjugate diameters CP, CD,

$$m_1=\frac{y_1}{x_1}, \quad m_2=\frac{y_2}{x_2}$$

$$\therefore \quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=0.$$

79.2. Let θ_1, θ_2 be the extremities of a pair of conjugate diameters. The equations of the diameters are

$y=\dfrac{b}{a}x\tan\theta_1$, $y=\dfrac{b}{a}x\tan\theta_2$ and they will be conjugate if

$$\tan\theta_1\tan\theta_2=-1.$$

Hence $$\theta_1\sim\theta_2=\frac{\pi}{2} \qquad \ldots\ldots(14)$$

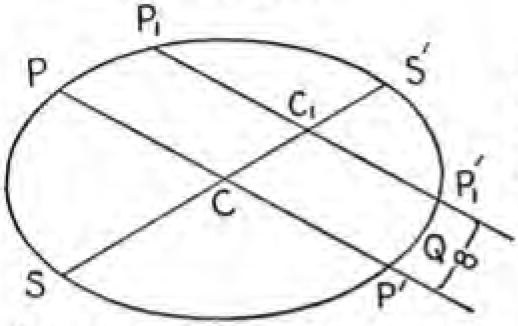
79.3. *If two diameters are conjugate, then each bisects chords parallel to the other.*

First Proof. Let $y=m_1x$, $y=m_2x$ be two conjugate diameters. An arbitrary point on the diameter $y=m_1x$ can be taken as (x_1, m_1x_1). The equation of the chord whose mid-point is (x_1, m_1x_1) is

$$\frac{x_1(x-x_1)}{a^2}+\frac{m_1x_1(y-m_1x_1)}{b^2}=0.$$

The slope of the chord is $-\dfrac{b^2}{a^2m_1}$ which equals m_2 in virtue of $m_1m_2=-\dfrac{b^2}{a^2}$.

Second Proof.



Let PCP′, SCS′ be a pair of conjugate diameters and $P_1C_1P_1'$ a chord parallel to the diameter PP′ meeting SS′ in C_1, then C_1 is the mid-point of P_1P_1'.

Since CP′, CS′ are conjugate, the pole of CS′ is the point at infinity on CP′. Let the point be $Q(\infty)$ which also lies on P_1P_1'. Since CS′ is the polar of $Q(\infty)$,

$$(P_1P_1', C_1\,Q)=-1.$$

Hence C_1 is the mid-point of $P_1\,P_1'$.

79.4. *The sum of the squares of two conjugate diameters is constant.*

Let P $(a\cos\theta, b\sin\theta)$,

$$D\ \left\{a\cos\left(\theta\pm\frac{\pi}{2}\right),\ b\sin\left(\theta\pm\frac{\pi}{2}\right)\right\}$$

be the extremities of the conjugate diameters, this

$$CP^2=a^2\cos^2\theta+b^2\sin^2\theta$$
$$CD^2=a^2\sin^2\theta+b^2\cos^2\theta$$
$$\therefore\quad CP^2+CD^2=a^2+b^2. \qquad \ldots\ldots(15)$$

79.5. Equi-conjugate diameters. Conjugate diameters which are equal are called equi-conjugate diameters.

The diameters being equal and conjugate, CP=CD hence from Art. 79.4.

$$(\cos^2\theta-\sin^2\theta)(a^2-b^2)=0$$
$$\therefore\qquad \cos 2\theta=0$$
$$\text{or}\qquad \theta=\frac{\pi}{4},\ \frac{3\pi}{4}.$$

Thus the equi-conjugate diameters are equally inclined to the axis and are given by the equations

$$y=\pm\frac{b}{a}x. \qquad \ldots\ldots(16)$$

The product of two conjugate diameters is greatest when they are equal.

79.6. *The area of a parallelogram which touches an ellipse at the extremities of conjugate diameters is constant.*

Let $\theta, \theta \pm \frac{\pi}{2}$ be the eccentric angle of the extremities of conjugate diameters, then

$$CD^2 = a^2 \sin^2\theta + b^2 \cos^2\theta.$$

The tangent at P to the ellipse is $\frac{x \cos\theta}{a} + \frac{y \sin\theta}{b} = 1.$

If p be the perpendicular from the centre on this tangent

$$p = \frac{ab}{[a^2 \sin^2\theta + b^2 \cos^2\theta]^{\frac{1}{2}}}$$

$$\therefore \quad p.CD = ab.$$

Hence the area of the parallelogram is $4ab$.

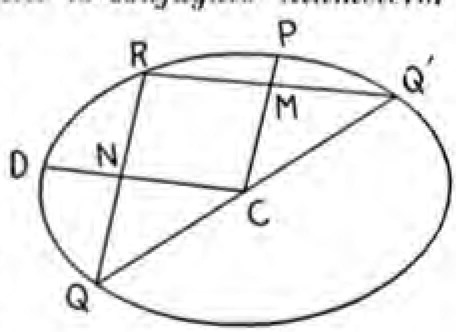
79.7. To prove that $CD^2 = SP.S'P$

Let P be $(a \cos\theta, b \sin\theta)$. D is $(-a \sin\theta, b \cos\theta)$.

$$CD^2 = a^2 \sin^2\theta + b^2 \cos^2\theta = a^2 - a^2e^2 \cos^2\theta$$
$$= (a - ae \cos\theta)(a + ae \cos\theta)$$
$$= SP.S'P.$$

80. Supplemental chords. Def. Two st. lines drawn from any point of an ellipse to the extremities of any diameter are called supplemental chords.

Supplemental chords are parallel to conjugate diameters.



Let QCQ' be a diameter, and R an arbitrary point on the ellipse, then RQ, RQ$'$ are supplemental chords.

Suppose that M, N are the mid-points of RQ$'$, RQ, the CMP, CND are respectively parallel to QR, Q$'$R and bisect respectively the chords Q$'$R, QR. Hence etc.

80.1. Equation of the supplemental chords.

The equations of two conjugate diameters CP, CD can be taken as $\lambda \frac{x}{a} + \mu \frac{y}{b} = 0$, $\frac{\mu x}{a} - \frac{\lambda y}{b} = 0$. If Q$'$ be the point $(a \cos\theta, b \sin\theta)$, the co-ordinates of Q are $(-a \cos\theta, -b \sin\theta)$. Thus the equations of QR, Q$'$R are respectively,

$$\frac{\lambda}{a}(x + a\cos\theta) + \frac{\mu}{b}(y + b \sin\theta) = 0,$$

$$\frac{\mu}{a}(x - a\cos\theta) - \frac{\lambda}{b}(y - b \sin\theta) = 0.$$

80.2 The property of supplemental chords proved in Art. 78 enables us to construct a pair of conjugate diameters of an ellipse inclined at a given angle.

Construction. On a given diameter QQ′ describe a circle capable of the given angle. If the circle meets the ellipse in R, RQ, RQ′ give the directions of the diameters.

To determine the possibility of the construction.

Let θ be the angle between the diameters CP and CD. If ϕ be the eccentric angle of P, the slopes CP and CD are

respectively $\dfrac{b}{a}\tan\phi,\ \dfrac{-b}{a}\cot\phi,$

hence
$$\tan\theta=\frac{2ab}{a^2-b^2}\operatorname{cosec}2\phi.$$

The maximum value $\dfrac{\pi}{2}$ of the smaller of the two values of θ is attained when $\phi=0$ or $\dfrac{\pi}{2}$ and the principal axes are the only pair of conjugate diameters which are at right angles. The minimum value of the two values of θ is reached when $\phi=\dfrac{\pi}{4}$, *i. e.*, when the diameters are equi-conjugate. Hence the solution is always possible if the given angle is not less than the angle $\tan^{-1}2ab/(a^2-b^2)$ between the equi-conjugate diameters.

The result may be obtained otherwise as follows :—

Let the given diameter chosen be the major axis. The equation of the segment of the circle capable of angle θ is

$$x^2+y^2\pm2ay\cot\theta-a^2=0,\ i.e.,\frac{x^2+y^2}{a^2}\pm2\frac{y}{a}\cot\theta=1.$$

This meets the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, in points whose ordinates are given by the equation

$$y\left\{\frac{a^2-b^2}{ab^2}y\pm2\cot\theta\right\}=0.$$

The value $y=0$ corresponds to the extremities of the major axis. The ordinate of the other point of intersection is

$$y=\pm\frac{2ab^2\cot\theta}{a^2-b^2},$$

But since $|y| \leqslant b$.

$$\therefore \qquad \left| \frac{2ab^2 \cot \theta}{a^2 - b^2} \right| \leqslant b.$$

$$\text{Hence} \qquad \left| \tan \theta \right| > \frac{2ab}{a^2 - b^2}.$$

81. The equation of the ellipse referred to a pair of conjugate diameters as axes.

Let the equation of the conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Since the centre is the origin, for every point (x, y) on the conic, there is also a point $(-x, -y)$ on the conic, therefore $g = 0$, $f = 0$. The equation becomes

$$ax^2 + 2hxy + by^2 + c = 0.$$

For a given y there are two equal and opposite values of x. This demands $h = 0$. If the diameters be of length $2a'$, $2b'$, the points $(\pm a', 0)$, $(0, \pm b')$ are on the ellipse,

$$\therefore \qquad \mathrm{A}a'^2 + c = 0, \qquad \mathrm{A}b'^2 + \mathrm{C} = 0.$$

Hence the equation of the ellipse reduces to

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1.$$

If the equi-conjugate diameters be taken as the axes $a' = b'$, the equation of the ellipse becomes

$$x^2 + y^2 = a'^2.$$

This equation should be distinguished from that of a circle in rectangular co-ordinates.

Illustrative Examples

(1) *Find the area of the triangle formed by the points* ϕ_1, ϕ_2, ϕ_3 *of the ellipse* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. *Hence find the triangle of maximum area inscribed in the above ellipse.*

The area of the triangle formed by the points $(a \cos \phi_i, b \sin \phi_i)$ $i = 1, 2, 3$ is given by

$$\Delta = \tfrac{1}{2} \begin{vmatrix} a \cos \phi_1 & b \sin \phi_1 & 1 \\ a \cos \phi_2 & b \sin \phi_2 & 1 \\ a \cos \phi_3 & b \sin \phi_3 & 1 \end{vmatrix}$$

$$= 2ab \sin \frac{\phi_2 - \phi_3}{2} \sin \frac{\phi_3 - \phi_1}{2} \sin \frac{\phi_1 - \phi_2}{2}.$$

The area Δ' of the triangle formed by the corresponding points $(a \cos \phi_i, a \sin \phi_i)$ on the auxiliary circle, is similarly given by the equation

$$\Delta' = 2a^2 \sin \frac{\phi_2 - \phi_3}{2} \sin \frac{\phi_3 - \phi_1}{2} \sin \frac{\phi_1 - \phi_2}{2}$$

$$\therefore \qquad \frac{\Delta}{\Delta'} = \frac{b}{a}.$$

Thus the ratio $\Delta : \Delta'$ is constant. Consequently if Δ' is maximum, so will be Δ. But the triangle of maximum area inscribed in a circle is equilateral, hence

$$\phi_2 - \phi_3 = \phi_3 - \phi_1 = \phi_1 - \phi_2 = \frac{2\pi}{3}.$$

Thus the eccentric angles of a triangle of a maximum area inscribed in an ellipse differ by $\frac{2\pi}{3}$, and its area Δ is given by

$$\Delta = \frac{3\sqrt{3}}{4} ab.$$

(2) *If the tangents to* $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ *at the extremities of a chord meet at right angles, show that the locus of the mid-point of the chord is the curve.*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2 + y^2}{a^2 + b^2}.$$

Since the tangents drawn at the extremities of the chord are perpendicular, they intersect on the director of the ellipse, *viz.* $x^2 + y^2 = a^2 + b^2$, and the chord is the polar of the point *w. r.* to the ellipse. Let $(c \cos \theta, c \sin \theta)$, $c = \sqrt{a^2 + b^2}$, be a point on the director. Its polar *w. r.* to the ellipse is

$$\frac{cx \cos \theta}{a^2} + \frac{cy \sin \theta}{b^2} = 1.$$

If (x_1, y_1) be the mid-point of the chord, it is identical with

$$\frac{x_1(x - x_1)}{a^2} + \frac{y_1(y - y_1)}{b^2} = 0$$

$$\therefore \qquad \frac{c \cos \theta}{x_1} = \frac{c \sin \theta}{y_1} = \frac{1}{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}}$$

Thus
$$\frac{c^2}{x_1^2 + y_1^2} = \frac{1}{\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right)^2}$$

Hence (x_1, y_1) lies on the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 = \frac{x^2+y^2}{a^2+b^2}.$$

(3) *One of the bisectors of the angles between the tangents from a point P on the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *passes through a given point* $(d, 0)$ *on the major axis. Prove that P lies either on the major axis or on the circle*

$$d(x^2+y^2)-x(a^2-b^2+d^2)+d(a^2-b^2)=0$$ [Math. Trip. 1922]

Let the co-ordinates of P be (x_0, y_0). The equation of the pair of tangents from (x_0, y_0) is

$$\left(\frac{x-x_0}{a}\right)^2+\left(\frac{y-y_0}{b}\right)^2=\left(\frac{xy_0-x_0y}{ab}\right)^2.$$

Transforming the axes through the point (x_0, y_0) the equation takes the form

$$\frac{\xi^2}{a^2}+\frac{\eta^2}{b^2}=\left(\frac{\xi y_0-\eta x_0}{ab}\right)^2$$

or $$\xi^2(b^2-y_0^2)+2\xi\eta\, y_0x_0+\eta^2(a^2-x_0^2)=0.$$

The equation of the bisectors of the angles between the lines is

$$\frac{\xi^2-\eta^2}{b^2-a^2+x_0^2-y_0^2}=\frac{\xi\eta}{y_0x_0}.$$

Transforming back to the old axis, the equation takes the form

$$[(x-x_0)^2-(y-y_0)^2]\,x_0y_0=(x-x_0)(y-y_0)(x_0^2-y_0^2-a^2+b^2)$$

Since $(d, 0)$ lies on the locus,

$$[(d-x_0)^2-y_0^2]\,x_0y_0=-y_0\,(d-x_0)(x_0^2-y_0^2-a^2+b^2).$$

Thus either $y_0=0$, *i. e.*, P lies on the major axis, or

$$d\,(x_0^2+y_0^2)-x_0\,(a^2-b^2+d^2)+d(a^2-b^2)=0$$

or (x_0, y_0) lies on the circle

$$d(x^2+y^2)-x(a^2-b^2+d^2)+d(a^2-b^2)=0.$$

(4) *Through the points* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ *are drawn parallel chords* P_1QR, $P_2Q'R'$ *in variable directions. Show that the radical axis of circles on* QR, Q′R′ *as diameters pass through a fixed point.*

Let $y-y_1=m(x-x_1)$ be a chord through $P(x_1, y_1)$. It meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points Q, R and their abscissæ are given by the equation

$$\frac{x^2}{a^2}+\frac{[y_1+m(x-x_1)]^2}{b^2}=1$$

or $x^2(a^2m^2+b^2)+2a^2m(y_1-mx_1)x+a^2[(y_1-mx_1)^2-b^2]=0$ (*i*)

If x', x'' be the roots of this equation, it is identical with

$$(x-x')(x-x'')(a^2m^2+b^2)=0.$$

Similarly, the ordinates y', y'' of the points Q and R are given by the equation

$$y^2(a^2m^2+b^2)-2b^2y(y_1-mx_1)+b^2[(y_1-mx_1)^2-a^2m^2]=0 \quad \ldots\ldots(ii)$$

i. e. $\quad (a^2m^2+b^2)(y-y')(y-y'')=0.$

But the equation of the circle on (x', y'). (x'', y'') as diameter is $\quad (a^2m^2+b^2)[(x-x')(x-x'')+(y-y')(y-y'')]=0.$

Hence the equation of the circle is obtained by adding (*i*) and (*ii*) and is found to be

$$(a^2m^2+b^2)(x^2+y^2)+2a^2m(y_1-mx_1)x-2b^2y(y_1-mx_1)$$
$$+(y_1-mx_1)^2(a^2+b^2)-a^2b^2(1+m^2)=0 \quad \ldots\ldots(iii)$$

Similarly the equation of the circle on Q′R′ as diameter is

$$(a^2m^2+b^2)(x^2+y^2)+2a^2m(y_2-mx_2)x-2b^2y(y_2-mx_2)$$
$$+(y_2-mx_2)^2(a^2+b^2)-a^2b^2(1+m^2)=0 \quad \ldots\ldots(iv)$$

Now the radical axis of circles (*iii*) and (*iv*) is

$$2a^2mx[(y_1-mx_1)-(y_2-mx_2)]-2b^2y[(y_1-mx_1)-(y_2-mx_2)]$$
$$+(a^2+b^2)[(y_1-mx_1)^2-(y_2-mx_2)^2]=0$$

or $\quad 2a^2mx-2b^2y+(a^2+b^2)[(y_1+y_2)-m(x_1+x_2)]=0$

i. e., $m[2a^2x-(a^2+b^2)(x_1+x_2)]-[2b^2y-(a^2+b^2)(y_1+y_2)]=0,$

and this line for all values of m passes through the intersection of the lines

$$2a^2x-(a^2+b^2)(x_1+x_2)=0, \qquad 2b^2y-(a^2+b^2)(y_1+y_2)=0$$

i. e., the point

$$\left[\frac{a^2+b^2}{2a^2}(x_1+x_2),\ \frac{a^2+b^2}{2b^2}(y_1+y_2)\right].$$

(5) *A pair of conjugate diameters of an ellipse, whose centre is* O, *are cut by a fixed st. line in points* P *and* P′. *Prove that the locus of a centre of the circle circumscribing the triangle* OPP′ *is a st. line.* (Selwyn, 1913)

Let the equation of the ellipse be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

and that of the given line be $y=mx+c$. Suppose that the equation of the circle is

$$x^2+y^2-2xx_0-2yy_0=0.$$

The equation of the lines OP, OP′ is therefore

$$x^2+y^2-2(xx_0+yy_0)\frac{(y-mx)}{c}=0$$

i. e., $\quad y^2(c-2y_0)-2xy(x_0-my_0)+x^2(c-2mx_0)=0,$

If m', m'' be the slopes of these lines

$$m'm'' = \frac{c - 2mx_0}{c - 2y_0} = -\frac{b^2}{a^2}.$$

Hence the locus of (x_0, y_0) is

$$a^2(c - 2mx) + b^2(c - 2y) = 0$$

i.e., $$2a^2mx + 2b^2y - c(a^2 + b^2) = 0.$$

Exercises XX

1. Prove that the extremities of the latera recta a ellipses having a given major axis $2a$ lie on the parabola

$$x^2 = -a(y - a) \text{ or } x^2 = a(y + a).$$

2. A series of ellipses are described with a given focus and a corresponding directrix, show that the locus of the extremities of their minor axis is a parabola.

3. If Q and Q′ are the points which correspond to P (lying on an ellipse) on the major and minor auxiliary circles respectively, show that QQ′ passes through the centre.

4. Show that if the line $lx + my = 1$ meets the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in real points, the co-ordinates of the mid-point of the intercepted portion are

$$\frac{a^2l}{a^2l^2 + b^2m^2}, \quad \frac{b^2m}{a^2l^2 + b^2m^2}.$$

5. If P and D are extremities of conjugate diameters and the tangent at P cut the major axis in T and the tangent at D cut the minor axis in T′, prove that TT′ is parallel to one of the equi-conjugates.

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6. If P and D are extremities of conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

show that the tangents at P and D meet on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2$$

and the locus of the mid-point of PD is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}.$$

7. Show that the equation of the tangents to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the points of intersection with the line $lx+my+n=0$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(a^2l^2+b^2m^2-n^2\right)=\left(lx+my+n\right)^2.$$

8. Show that the area of the triangle formed by the tangents at the points whose eccentric angles are ϕ_1, ϕ_2, ϕ_3 respectively is

$$ab \tan\frac{\phi_2-\phi_3}{2}\tan\frac{\phi_3-\phi_1}{2}\tan\frac{\phi_1-\phi_2}{2}.$$

Deduce that the area of minimum triangle circumscribed to an ellipse touches the ellipse at the vertices of maximum inscribed triangle and the area of such triangle is $3\sqrt{3}\,ab$.

9. Prove that a tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

meets the ellipse $\dfrac{x^2}{a}+\dfrac{y^2}{b}=a+b$ in points the tangents at which are at right angles.

10. Show that the maximum rectangle circumscribed about an ellipse is a square of area $2(a^2+b^2)$. [Hint. Such a rectangle is inscribed in the director circle].

11. Show that the sides of an inscribed parallelogram of an ellipse are parallel to a pair of its conjugate diameters.

12. Show that only one square can be inscribed in an ellipse, and its area is $4a^2b^2 \div (a^2+b^2)$.

13. The points M and N are the projections of a point P on the axis of an ellipse, find the locus of P if MN be a tangent to the ellipse.

14. If the st. line $y=\frac{1}{2}(x+1)$ meets the ellipse $x^2+4y^2=4$ in P and Q, show that the equation of the circle on PQ as diameter is $8(x^2+y^2)+16x-8y-15=0$.

82. Equation of the normal at ϕ to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of the tangent to the ellipse at ϕ is

$$\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1.$$

The equation of the normal at ϕ is, therefore,

$$(x-a\cos\phi)\frac{\sin\phi}{b}-(y-b\sin\phi)\frac{\cos\phi}{a}=0$$

or, $\quad ax\sin\phi-by\cos\phi=(a^2-b^2)\cos\phi\sin\phi \qquad \ldots\ldots(17)$

The equation may be written as

$$ax \sec\phi - by \operatorname{cosec}\phi = a^2 - b^2 \qquad \ldots\ldots(17\text{A})$$

If $\tan\dfrac{\phi}{2}$ be put equal to t, the equation of the normal in virtue of the relations

$$\cos\phi = \frac{1-t^2}{1+t^2}\ ,\ \sin\phi = \frac{2t}{1+t^2}$$

takes the form

$$byt^4 + 2(ax+a^2-b^2)t^3 + 2(ax-a^2+b^2)t - by = 0 \qquad \ldots\ldots(17\text{B})$$

82.1. Condition for the concurrence of three normals.

Let the equations of the three normals be

$$by(t_1^4-1) + 2(ax+a^2-b^2)t_1^3 + 2(ax-a^2+b^2)t_1 = 0$$
$$by(t_2^4-1) + 2(ax+a^2-b^2)t_2^3 + 2(ax-a^2+b^2)t_2 = 0$$
$$by(t_3^4-1) + 2(ax+a^2-b^2)t_3^3 + 2(ax-a^2+b^2)t_3 = 0.$$

The *necessary* and *sufficient* condition that these normals may meet in a point is

$$\begin{vmatrix} t_1^4-1 & t_1^3 & t_1 \\ t_2^4-1 & t_2^3 & t_2 \\ t_3^4-1 & t_3^3 & t_3 \end{vmatrix} = 0$$

The determinant can be written as

$$\begin{vmatrix} t_1^4 & t_1^3 & t_1 \\ t_2^4 & t_2^3 & t_2 \\ t_3^4 & t_3^3 & t_3 \end{vmatrix} - \begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix} = 0$$

or

$$t_1 t_2 t_3 \begin{vmatrix} t_1^3 & t_1^2 & 1 \\ t_2^3 & t_2^2 & 1 \\ t_3^3 & t_3^2 & 1 \end{vmatrix} - \begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix} = 0.$$

Each one of these two determinants vanishes for $t_1=t_2$, $t_1=t_3$, $t_2=t_3$, thus $(t_2-t_3)(t_3-t_1)(t_1-t_2)$ is a factor of each determinant. Since the first determinant is of the fifth degree, the other possible factor of the first determinant is $a\Sigma t_1^2 + b\Sigma t_2 t_3$. For similar reason, the other factor of the second determinant is $c\Sigma t_1$. A comparison of the leading terms show that $a=0$, $b=-1$ $c=-1$. Thus the result reduces to

$$-(t_1-t_2)(t_2-t_3)(t_3-t_2)[t_1t_2t_3)\Sigma t_2t_3 - \Sigma t_1] = 0$$

$$i.\,e.,\quad \Sigma t_2t_3 - \Sigma\frac{1}{t_2t_3} = 0 \qquad \text{......(18)}$$

since $t_1 - t_2 \neq 0$ $t_2 - t_3 \neq 0$, $t_3 - t_1 \neq 0$.

82.2. Burnside's condition for concurrence of three normals.

The necessary and sufficient condition that the normals at t_1, t_2, t_3 may meet in a point has been found to be

$$\Sigma\left(\frac{1}{t_2t_3} - t_2t_3\right) = 0.$$

The condition can be written as

$$\Sigma\,\frac{\cos^2\frac{\phi_2}{2}\cos^2\frac{\phi_3}{2} - \sin^2\frac{\phi_2}{2}\sin^2\frac{\phi_3}{2}}{\cos\frac{\phi_2}{2}\sin\frac{\phi_2}{2}\cos\frac{\phi_3}{2}\sin\frac{\phi_3}{2}} = 0$$

$$\text{or}\quad \Sigma\,\frac{\cos\phi_2 + \cos\phi_3}{\sin\phi_2\sin\phi_3} = 0$$

$$\text{or}\quad \Sigma\sin\phi_1(\cos\phi_2 + \cos\phi_3) = 0$$

which is equivalent to

$$\sin(\phi_2+\phi_3) + \sin(\phi_3+\phi_1) + \sin(\phi_1+\phi_2) = 0 \qquad \text{......(18A)}$$

This is Burnside's necessary and sufficient condition for the concurrence of three normals. This condition is necessary and sufficient, since (18A) and (18) are equivalent conditions and we can pass from one to the other.

82.3. Conditions of concurrence of four normals.

The necessary and sufficient conditions that the normals at t_3 and t_4 may concur with the normals at t_1 and t_2 are

$$t_3t_1 + t_3t_2 + t_1t_2 - \left(\frac{1}{t_3t_1} + \frac{1}{t_3t_2} + \frac{1}{t_1t_2}\right) = 0 \qquad \text{......}(i)$$

$$t_4t_1 + t_4t_2 + t_1t_2 - \left(\frac{1}{t_4t_1} + \frac{1}{t_4t_2} + \frac{1}{t_1t_2}\right) = 0 \qquad \text{......}(ii)$$

Thus t_1, t_2 being given, t_3, t_4 are the roots of the quadratic

$$t(t_1+t_2) - \frac{1}{t}\left(\frac{1}{t_1} + \frac{1}{t_2}\right) + \left(t_1t_2 - \frac{1}{t_1t_2}\right) = 0$$

$$\text{or}\quad t^2t_1t_2(t_1+t_2) + t(t_1^2t_2^2 - 1) - (t_1+t_2) = 0$$

$$\therefore\quad t_3t_4 = -\frac{1}{t_1t_2} \quad\text{or}\quad t_1t_2t_3t_4 = -1$$

$$\text{and}\quad (t_3+t_4) = \frac{1 - t_1^2t_2^2}{t_1t_2(t_1+t_2)}$$

$$\text{or}\qquad (t_1+t_2)(t_3+t_4) = \frac{1}{t_1t_2} - t_1t_2$$

$$= -t_3t_4 - t_1t_2 \qquad \text{......(19)}$$

The condition reduces to $\Sigma_1^4\, t_1t_2=0$. ... (19A)

Thus the conditions that the normals at t_1, t_2, t_3, t_4 may meet in a point are

$$^4\Sigma_1 t_1t_2=0, \qquad t_1t_2t_3t_4=-1.$$

These conditions are necessary and sufficient, for they at-once lead to the conditions (*i*) and (*ii*) which are necessary and sufficient. Thus :

82 4. *From a point, four normals can be drawn to an ellipse and their parameters satisfy the equations* (19).

A necessary condition for the concurrence of four normals can be deduced from conditions (19) Art. 82.3.

$$\tan\left(\frac{\phi_1}{2}+\frac{\phi_2}{2}+\frac{\phi_3}{2}+\frac{\phi_4}{2}\right)=\frac{\Sigma t_1-\Sigma t_1t_2t_3}{1-\Sigma t_1t_2+t_1t_2t_3t_4}\rightarrow\infty$$

$$\therefore \qquad \tfrac{1}{2}(\phi_1+\phi_2+\phi_3+\phi_4)=(2n+1)\frac{\pi}{2}$$

or $$\phi_1+\phi_2+\phi_3+\phi_4=(2n+1)\pi \qquad \text{......(20)}$$

82.5. The result of Art. 82.3 can be obtained directly as follows.

If the normal at a point 't' passes through the point (x, y), we have

$$byt^4+2(ax+a^2-b^2)\,t^3+2(ax-a^2+b^2)\,t^2-by=0 \quad (17\text{B})$$

This equation is of the fourth degree in t, and has therefore, four roots, and corresponding to each root, there exists a normal given by the equation (17B). Thus through *an arbitrary point four normals can be drawn to an ellipse.*

If t_1, t_2, t_3, t_4 be the roots of the equation (17 B).

$$\Sigma t_1t_2=0 \qquad t_1t_2t_3t_4=-1 \qquad \text{......(21)}$$

Thus these conditions are necessary for the concurrence of four normals. These conditions are also sufficients. For, let the normals be

$$U_1\equiv by(t_1^4-1)+2(ax+a^2-b^2)t_1^3+2(ax-a^2+b^2)t_1=0$$
$$U_2\equiv by(t_2^4-1)+2(ax+a^2-b^2)t_2^3+2(ax-a^2+b^2)t_2=0$$
$$U_3\equiv by(t_3^4-1)+2(ax+a^2-b^2)t_3^3+2(ax-a^2+b^2)t_3=0$$
$$U_4\equiv by(t_4^4-1)+2(ax+a^2-b^2)t_4^3+2(ax-a^2+b^2)t_4=0$$

U_1, U_2, U_3 will meet at a point if

$$\begin{vmatrix} t_1^4-1 & t_1^3 & t_1 \\ t_2^4-1 & t_2^3 & t_2 \\ t_3^4-1 & t_3^3 & t_3 \end{vmatrix}=0$$

i. e., if $t_1t_2t_3 \begin{vmatrix} t_1^3 & t_1^2 & 1 \\ t_2^3 & t_2^2 & 1 \\ t_3^3 & t_3^2 & 1 \end{vmatrix} = \begin{vmatrix} t_1^3 & t_1 & 1 \\ t_2^3 & t_2 & 1 \\ t_3^3 & t_3 & 1 \end{vmatrix}$

i. e., if $t_1t_2+t_2t_3+t_3t_1=\dfrac{1}{t_1t_2}+\dfrac{1}{t_2t_3}+\dfrac{1}{t_3t_1}$; $t_1 \neq t_2 \neq t_3$,

which holds by virtue of (21).

Similarly U_1, U_3, U_4, meet at a point.

Thus all the four normals meet at a point.

The condition (18) of concurrence of the three normals at t_1, t_2, t_3 can be easily deduced by the elimination of t_4 from conditions (21). Burnside's condition of concurrence can then be deduced as in Art. 82.2.

Note. 1. The sines and cosines of the eccentric angles of the feet of the four normals that pass through (x, y) are the roots of the equations

$$(a^2-b^2)^2 \sin^4\phi+2(a^2-b^2)by \sin^3\phi+[a^2x^2+b^2y^2-(a^2-b^2)^2] \sin^2\phi \\ -2(a^2-b^2)\, by \sin\phi-b^2y^2=0 \quad \ldots\ldots(22)$$

$$(a^2-b^2)^2 \cos^4\phi-2(a^2-b^2)\, ax \cos^3\phi+[a^2x^2+b^2y^2-(a^2-b^2)^2] \cos^2\phi \\ +2(a^2-b^2)\, ax \cos\phi-a^2x^2=0 \quad \ldots\ldots(23)$$

Exercise. Obtain the condition (20) from the equations (22) and (23)

Note 2. From the equations (22) and (23)

$$(a^2-b^2)^2 \Sigma \sin\phi_1 \sin\phi_2=a^2x^2+b^2y^2-(a^2-b^2)^2 \\ =(a^2-b^2)^2 \Sigma \cos\phi_1 \cos\phi_2$$

$$\therefore \quad \sum_1 \cos(\phi_1+\phi_2)=0.$$

This is a necessary condition for the concurrence of four normals.

Note 3. If the normals at ϕ_1, ϕ_2, ϕ_3, ϕ_4 meet at (x, y) we have from equations (22) and (23) that

$$x=\frac{a^2-b^2}{2a}\sum_{j=1}^{4}\cos\phi_j \quad y=-\frac{a^2-b^2}{2}\sum_{j=1}^{4}\sin\phi_j \quad \ldots\ldots(24)$$

Note 4. The necessary and sufficient conditions for the concurrence of normals at ϕ_i $(i=1, 2, 3, 4)$ are

$\Sigma\sin\phi_i+\Sigma\sin\phi_1 \sin\phi_2 \sin\phi_3=0$, $\Sigma\cos\phi_i+\Sigma\cos\phi_1 \cos\phi_2 \cos\phi_3=0$.

These follow from (22), (23).

82.6. Joint equation of the four normals that pass through a point.

Let $y-y_1=m(x-x_1)$ be a normal through (x_1, y_1), then for some values of m, this equation will be identical with

$$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2$$

$$\therefore\quad \frac{m\cos\phi}{a}=\frac{\sin\phi}{b}=\frac{mx_1-y_1}{a^2-b^2}$$

$$\therefore\quad \cos\phi=\frac{a}{m}\,\frac{mx_1-y_1}{a^2-b^2}\,,\ \sin\phi=b\,\frac{mx_1-y_1}{a^2-b^2}$$

$$\therefore\quad \left(\frac{mx_1-y_1}{a^2-b^2}\right)^2\left(\frac{a^2}{m^2}+b^2\right)=1$$

or $\quad (mx_1-y_1)^2(a^2+b^2m^2)=(a^2-b^2)^2m^2.$

Eliminating m between this equation and $y-y_1=m(x-x_1)$, the equation of the four normals is obtained in the form

$$(a^2-b^2)^2(x-x_1)^2(y-y_1)^2=(x_1y-xy_1)^2\{a^2(x-x_1)^2+b^2(y-y_1)^2\} \quad \ldots\ldots(25)$$

83. If the normal at $P(x_1\, y_1)$ on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ make an acute angle θ with the x-axis and p be the perpendicular from the centre on the tangent at that point, prove that

$$\cos\theta=\frac{px_1}{a^2}\,,\quad \sin\theta=\frac{py_1}{b^2}.$$

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}$$

and $\quad \dfrac{x-x_1}{\dfrac{x_1}{a^2}}=\dfrac{y-y_1}{\dfrac{y_1}{b^2}}$ represent the normal at P.

Comparing, we have

$$\frac{\cos\theta}{\dfrac{x_1}{a^2}}=\frac{\sin\theta}{\dfrac{y_1}{b^2}}=\frac{1}{\sqrt{\left(\dfrac{x_1^2}{a^4}+\dfrac{y_1^2}{b^4}\right)}}$$

But the perpendicular from the centre on the tangent

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1$$

is given by $\quad \dfrac{1}{p}=\pm\sqrt{\dfrac{x_1^2}{a^4}+\dfrac{y_1^2}{b^4}}$

Hence, as θ is acute,

$$\cos\theta=\frac{px_1}{a^2}\,,\quad \sin\theta=\frac{py_1}{b^2}\,.$$

83.1. If the normal at P meet the axes in G and F,

$$PG = \frac{b^2}{p}, \quad PF = \frac{a^2}{p}.$$

The co-ordinates of a point on the normal at a distance r, measured inwards, are

$$x_1 - r\frac{px_1}{a^2}, \quad y_1 - r\frac{py_1}{b^2}.$$

Since G lies on $y = 0$,

$$\therefore \quad y_1 - r\frac{py_1}{b^2} = 0, \text{ i. e., } PG = \frac{b^2}{p}.$$

Since F lies on $x = 0$

$$\therefore \quad x_1 - r\frac{px_1}{a^2} = 0, \text{ i.e., } PF = \frac{a^2}{p}.$$

83.2. If PQ, PQ′ be the lengths along the normal at P measured outwards and inwards, each equal to CD where CP and CD are conjugate semi-diameters then $CQ = a + b$, $CQ' = a - b$, and CQ and CQ′ are equally inclined to the axes.

Let Q and Q′ be (α, β) ; (α', β') ; let $CD = r$.

$$\alpha = x_1 + r\frac{px_1}{a^2}, \quad \beta = y_1 + r\frac{py_1}{b^2}$$

$$\alpha' = x_1 - r\frac{px_1}{a^2}, \quad \beta' = y_1 - r\frac{py_1}{b^2}.$$

But $\qquad pr = ab$

$$\therefore \qquad \alpha = \frac{x_1}{a}(a + b), \quad \beta = \frac{y_1}{b}(a + b)$$

$$\alpha' = \frac{x_1}{a}(a - b), \quad \beta' = -\frac{y_1}{b}(a - b).$$

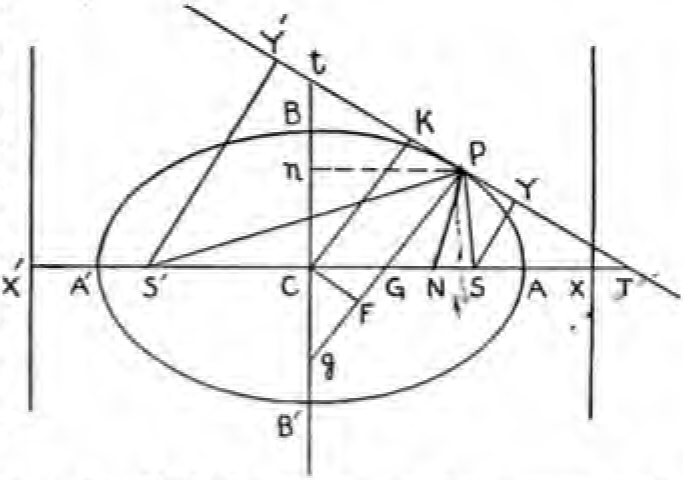
$$\therefore \qquad CQ^2 = \alpha^2 + \beta^2 = (a + b)^2\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}\right) = (a + b)^2$$

$\therefore \qquad CQ = a + b$. Similarly $CQ' = a - b$.

Since $\dfrac{\beta}{\alpha} = -\dfrac{\beta'}{\alpha'}$, CQ, CQ′ are equally inclined to the axes.

Exercises

84. The student is expected to establish the following exercises :—



1. In the annexed figure AA′ is the major axis, BB′ the minor axis of the ellipse. The major axis meets the directrices in points X and X′ ; S, S′ are the foci. The tangent at P(x', y') meets the axes in T and t. The normal PG meets the axes in G and g. The lines CK, CF are the perpendiculars from the centre C of the ellipse on the tangent and normal at P, while SY, S′Y′ are the perpendiculars on the tangent at P. PN is the ordinate of P. Prove the following

(*i*) $CS.\ CX = CA^2$ (*ii*) $AS.\ A'S = CB^2$,
(*iii*) $SX : CX = CB^2 : CA^2$ (*iv*) $CB^2 = CA^2 - CS^2$
(*v*) $CS = e^2.\ CX$ (*vi*) $CT.\ CN = CA^2$, (*vii*) $Ct.\ Cn = CB^2$

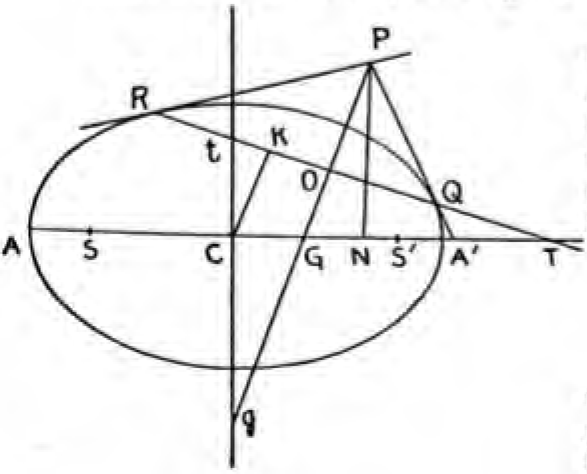
(*viii*) The subtangent $NT = \dfrac{a^2 - x'^2}{x'}$, (*ix*) $CG = e^2.\ CN$.

(*x*) The sub-normal $NG = (1 - e^2)CN$. (*xi*) $NG : NC = b^2 : a^2$.
(*xii*) Show that $SG = e.\ SP$. Deduce $SG : S'G = SP : S'P$.

The tangent and normal at P bisect the angle SPS′.

(*xiii*) Show that the points Y and Y′ lie on the auxiliary circle.

(*xiv*) $PF.\ PG = CB^2$; $PF.\ Pg = CA^2$.



2. In the annexed figure P is the pole of the chord RtOQT and the line through P at right angles to RT meets the axes in G and g. Prove the following :—

(*i*) $CN.\ CT = CA^2$
(*ii*) $NP.\ Ct = CB^2$
(*iii*) $CG = e^2CN$
(*iv*) $KC.\ PG = CB^2$.

where K is the foot of the perpendicular from C on RQ.

Illustrative Examples

(1) *Find the locus of the point of intersection of normals to the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the extremities of two variable conjugate diameters.*

Let the eccentric angles of the feet of the normals be $\phi,\ \phi+\frac{\pi}{2}$. The equations of the normals are therefore

$$ax\sin\phi - by\cos\phi-(a^2-b^2)\cos\phi\sin\phi=0$$
$$by\sin\phi+ax\cos\phi+(a^2-b^2)\cos\phi\sin\phi=0$$

$$\frac{\sin\phi}{ax-by}=\frac{\cos\phi}{-by-ax}=\frac{(a^2-b^2)\cos\phi\sin\phi}{a^2x^2+b^2y^2}$$

$$\therefore\quad (a^2-b^2)\cos\phi=(a^2x^2+b^2y^2)/(ax-by)$$
$$(a^2-b^2)\sin\phi=-(a^2x^2+b^2y^2)/(ax+by).$$

Squaring and adding we get the required locus *viz.*,

$$(a^2-b^2)^2(a^2x^2-b^2y^2)^2=2(a^2x^2+b^2y^2)^3.$$

(2) *Normals to an ellipse are drawn at the extremities of a chord parallel to one of the equi-conjugate diameters, prove that they intersect on a diameter perpendicular to the other equi-conjugate diameter.*

Let the eccentric angles of the extremities of the chords be $\frac{\pi}{4}\pm\alpha$. The equation of the chord is

$$\frac{x}{a}\cos\frac{\pi}{4}+\frac{y}{b}\sin\frac{\pi}{4}=\cos 2\alpha$$

or

$$\frac{x}{a}+\frac{y}{b}=\sqrt{2}\cos 2\alpha,$$

which is parallel to one of the equi-conjugates

$$\frac{x}{a}+\frac{y}{b}=0.$$

The equation of the normal at $\left(\frac{\pi}{4}+\alpha\right)$ is

$$ax\sin\left(\frac{\pi}{4}+\alpha\right)-by\cos\left(\frac{\pi}{\alpha}+\alpha\right)=\tfrac{1}{2}(a^2-b^2)\sin\left(\frac{\pi}{2}+2\alpha\right)$$
$$=\tfrac{1}{2}(a^2-b^2)\cos 2\alpha.$$

Also the equation of the normal at $\frac{\pi}{4}-\alpha$ is

$$ax\sin\left(\frac{\pi}{4}-\alpha\right)-by\cos\left(\frac{\pi}{4}-\alpha\right)=\tfrac{1}{2}(a^2-b^2)\cos 2\alpha.$$

The line given by the equation

$$ax\left[\sin\left(\frac{\pi}{4}+\alpha\right)-\sin\left(\frac{\pi}{4}-\alpha\right)\right]$$
$$+by\left[\cos\left(\frac{\pi}{4}-\alpha\right)-\cos\left(\frac{\pi}{4}+\alpha\right)\right]=0$$

is a diameter through the intersection of the given normals. The equation reduces to

$$ax\cos\frac{\pi}{4}\sin\alpha+by\sin\frac{\pi}{4}\sin\alpha=0$$

or $\qquad ax+by=0$

which is perpendicular to the diameter $\frac{x}{a}-\frac{y}{b}=0.$

Take the equi-conjugate diameters as axes.

The equation of the ellipse is of the form $x^2+y^2=c^2$.

Normals at the points $(c\cos\mu, c\sin\mu)$, $(c\cos\mu, -c\sin\mu)$ are given by the equations

$$y(\cos\mu-\sin\mu\cos\omega)-x(\sin\mu-\cos\mu\cos\omega)=c\cos 2\mu\cos\omega$$

and $y(\cos\mu+\sin\mu\cos\omega)+x(\sin\mu+\cos\mu\cos\omega)=c\cos 2\mu\cos\omega$

Subtracting, we get for the required locus the line

$$y\cos\omega+x=0$$

which is perpendicular to $y=0$.

(3) **Fregier's Theorem.** *A chord of an ellipse which subtends a right angle at a given point P of the ellipse passes through a fixed point F on the normal at P. Find the locus of F when P varies.*

Let a chord $lx+my=1$ subtend a right angle at P $(x'y')$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Shift the origin to (x',y') and the equations of the chord and ellipse take the form

$$lx+my=1-lx'-my'$$

and $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2xx'}{a^2}+\frac{2yy'}{b^2}=0.$$

The lines through the origin to the extremities of the chord are represented by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+2\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}\right)\left(\frac{lx+my}{-lx'-my}\right)=0.$$

They are at right angles,

$$\therefore\quad\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\left(1-lx'-my'\right)+\frac{2lx'}{a^2}+\frac{2my'}{b^2}=0$$

i. e., $(lx'-my')(a^2-b^2)=a^2+b^2.$

Thus the chord $lx+my=1$ passes through the point $\left(x'\frac{a^2-b^2}{a^2+b^2}, -y'\frac{a^2-b^2}{a^2+b^2}\right)$, which evidently lies on the normal

$$\frac{x-x'}{x'/a^2}=\frac{y-y'}{y'/b^2} \text{ at P.}$$

If P varies, let F be (x, y), then

$$x=x'\frac{a^2-b^2}{a^2+b^2}, \qquad y=-y'\frac{a^2-b^2}{a^2+b^2}$$

Thus the locus of F is a concentric homothetic ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2$$

85. Intersections of a circle and an ellipse.

Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

This meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ or $x=a\cos\theta$, $y=b\sin\theta$ in points whose eccentric angles are given by the equation

$$a^2\cos^2\theta+b^2\sin^2\theta+2ag\cos\theta+2bf\sin\theta+c=0.$$

This can be written in either of the forms

$$\rho^4\cos^4\theta+4ag\,\rho^2\cos^3\theta+2[2a^2g^2+(c+b^2)\,\rho^2+2f^2b^2]\cos^2\theta$$
$$+4ag(c+b^2)\cos\theta+(c+b^2)^2-4f^2b^2=0 \qquad \text{.....(25 A)}$$

$$\rho^2\sin^4\theta-4bf\,\rho^2\sin^3\theta+2[b^2f^2-\rho^2(c+a^2)+2a^2g^2]\sin^2\theta$$
$$+4bf(c+a^2)\sin\theta+(c+a^2)^2-4a^2g^2=0 \text{......(25B)}$$

$$t^4(a^2-2ag+c)+4bft^3+(4b^2-2a^2+2c)t^2$$
$$+4bft+(a^2+2ag+c)=0 \text{......(25 C)}$$

where $\rho^2=a^2-b^2$, $t=\tan\frac{\theta}{2}$.

85.1. Condition that four points of an ellipse may be concyclic.

If $\theta_1, \theta_2, \theta_3, \theta_4$ be the eccentric angles of the four points, $t_i\equiv\tan\frac{\theta_i}{2}$, $i=1, 2, 3, 4$ are the roots of the equation (25C),

$$\therefore\ \tan\left(\frac{\theta_1+\theta_2+\theta_3+\theta_4}{2}\right)=\frac{\Sigma t_1-\Sigma t_1t_2t_3}{1-\Sigma t_1t_2+t_1t_2t_3t_4}=0$$

$$\therefore\quad \frac{\theta_1+\theta_2+\theta_2+\theta_3}{2}=n\pi, \quad n \text{ being an integer.}$$

or $$\theta_1+\theta_2+\theta_3+\theta_4=2n\pi. \qquad \text{......(26)}$$

This is therefore a necessary condition that four points of an ellipse be concyclic. The condition is also sufficient. For let the circle through $\theta_1, \theta_2, \theta_3$ meet the ellipse again in θ'_4.

$\therefore \quad \theta_1+\theta_2+\theta_3+\theta'_4=2m\pi.$

Hence $\qquad \theta_4 \sim \theta'_4=$ an even multiple of π.

The points θ_4, and θ'_4 therefore coincide.

85.2. The co-ordinates of the centre of the circle being $(-g, -f)$ and since $\cos\theta_4=\cos(\theta_1+\theta_2+\theta_3)$ and
$\sin\theta_4=-\sin(\theta_1+\theta_2+\theta_3)$
it follows from (25A) and (25 B) that the co-ordinates of the centre of the circle through $\theta_1, \theta_2, \theta_3$ are given by the equations

$$x=\frac{a^2-b^2}{4a}\left\{\Sigma\cos\theta_1+\cos(\theta_1+\theta_2+\theta_3)\right\} \qquad \ldots\ldots(27)$$

$$y=-\frac{a^2-b^2}{4b}\left\{\Sigma\sin\theta_1-\sin(\theta_1+\theta_2+\theta_3)\right\}$$

85.3. Suppose that the normals at $\theta_1, \theta_2, \theta_3$ are concurrent and θ_4' is the foot of the fourth normal from the point of concurrence. Let θ_4 be the fourth intersection of the ellipse and the circle through $\theta_1, \theta_2, \theta_3$.

$$\therefore \quad \theta_1+\theta_2+\theta_3+\theta'_4=(2m+1)\pi$$
$$\theta_1+\theta_2+\theta_3+\theta_4=2n\pi$$

Hence $\qquad \theta_4-\theta'_4=$ odd multiple of π.

Thus the points θ_4 and θ'_4 are diametrically opposite points. Hence the theorem : *If* P, Q, R *be three co-normal points of an ellipse the circumcircle of* PQR *meets the ellipse again in a point which is diametrically opposite to the point which is co-normal with* P, Q, R. (Joachisthal)

86. Let the circle

$$x^2+y^2+2gx+2fy+c=0$$

meet the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$

in four points P, Q, R, S. The equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x^2+y^2+2gx+2fy+c)=0,$$

represents for different values of λ, an infinity of conics passing through P, Q, R, S. The condition that this may break up into two right lines gives a cubic in λ. Hence *three pairs of rt. lines belong to the system*, as is otherwise obvious from the figure, *viz.*, (PQ, RS), (PR, QS), (PS, QR)

For any one of these three values of λ, the parallel pair through the origin is given by the equation

$$x^2\left(\frac{1}{a^2}+\lambda\right)+y^2\left(\frac{1}{b^2}+\lambda\right)=0$$

which are equally inclined to the axes. Hence *three pairs of common chords of an ellipse and a circle are equally inclined to the major (minor) axis.*

Ex. Deduce the condition of Art. 85.1 from Art. 86 and conversely.

86.1. Nature of the points of intersection of acircle and an ellipse.

The equations (25 A), (25 B), (25 C) are all of the fourth degree with real coefficients, hence their four roots and the corresponding four points P, Q, R, S fall in the following classes.

1. All the points P, Q, R, S are imaginary.

2. (*i*) Two real and two imaginary.
   (*ii*) Two real coincident points and two imaginary points which cannot coincide.

3. (*i*) The points P, Q, R, S are all real and distinct.
   (*ii*) The points P, Q concide and R and S are distinct.
   (*iii*) The points coincide in pairs, *e. g.*, P=Q, R=S.
   (*iv*) The points P, Q, R coincide and S is distinct.
   (*v*) All the four points coincide.

The cases [1], [2 (*i*)], [3 (*i*)] offer no special interest. We pass on to the cases [2 (*ii*)], [3 (*ii*)].

Suppose P and Q coincide,

$$2\theta_1+\theta_3+\theta_4=2n\pi.$$

The equation gives two values of θ_1, *viz.* $\pi-\frac{1}{2}(\theta_3+\theta_4)$, $2\pi-\frac{1}{2}(\theta_3+\theta_4)$ when θ_3 and θ_4 are given. These values of θ_1 differ by π. Hence if R and S be given there exist two circles which touch the ellipse and the points of contact are at ends of a diameter.

In case 3 (*iii*), $\theta_1=\theta_2$, $\theta_3=\theta_4$, hence $\theta_1+\theta_3=n\pi$. The common chord is thus, perpendicular to one of the axes and the centre of the circle lies on one of the axes.

86.2. Let the points P, Q, R, coincide and S be distinct. The circle is then said to have a contact of second order. Since four points, in general, do not lie on a circle, a circle cannot have a contact of order higher than the second. If, however, P, Q, R, S all coincide $4\phi,=2n\pi$ and the point therefore, must be the extremity of one of the axes.

Def. The circle which has contact of the highest order with a curve at a given point is called an **osculating circle.**

Consequently the osculating circle of a conic is, in general, one that has a contact of the second order.

Suppose P, Q, R coincide, the eccentric angles of the points satisfy the relation

$$3\theta_1+\theta_4=2n\pi$$

$$\therefore \quad \theta_1=\tfrac{1}{3}(2\pi-\theta_4),\ \tfrac{1}{3}(4\pi-\theta_4),\ \tfrac{1}{3}(6\pi-\theta_4)$$

Thus given θ_4, there are three values of θ_1, hence through a given point on an ellipse, there pass three osculating circles. The three points at which the three circles through a point S, osculate are the vertices of a maximum triangle inscribed in an ellipse. Also these three points are concyclic with θ_4, since $\frac{1}{3}(2\pi-\theta_4)+\frac{1}{3}(4\pi-\theta_4)+\frac{1}{3}(6\pi-\theta_4)+\theta_4=4\pi$.

86.3 Def. The point of intersection of the normals at P, and Q when Q→P along the curve, is called the **centre of curvature** at P.

The distance of the centre of curvature from P is called the **radius of curvature**.

The **circle of curvature** is the circle whose centre is the centre of curvature and radius is the radius of curvature.

The osculating circle when it exists is identical with the circle of curvature. This will be verified for the ellipse later on. The proof of the general statement is beyond the scope of the book.

Illustrative Examples

(1) *A circle $x^2+y^2+2gx+2fy+c=0$ passes through the extremities of three semi-diameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that the circle*

$$x^2+y^2+\frac{2fb}{a}x-\frac{2ag}{b}y-(a^2+b^2+c)=0$$

passes through the extremities of three conjugate semi-diameters.

Let θ_1, θ_2, θ_3 be the eccentric angles of the points which lie on the circle

$$x^2+y^2+2gx+2fy+c=0,$$

$$\therefore \quad -g=\frac{a^2-b^2}{4a}\left\{\Sigma\cos\theta_1+\cos(\theta_1+\theta_2+\theta)_3\right\}$$

$$-f=-\frac{a^2-b^2}{4b}\left\{\Sigma\sin\theta_1-\sin(\theta_1+\theta_2+\theta_3)\right\}$$

If $(-g', -f')$ be the centre of the circle which passes through the extremities of three conjugate semi-diameters whose eccentric angles are $\left(\theta_i+\frac{\pi}{2}\right)$, $i=1, 2\ 3$.

$$-g' = \frac{a^2-b^2}{4a}\left\{-\Sigma \sin\theta_1 + \sin(\theta_1+\theta_2+\theta_3)\right\} = -\frac{bf}{a}$$

$$-f' = -\frac{a^2-b^2}{4b}\left[\Sigma \cos\theta_1 + \cos(\theta_1+\theta_2+\theta_3)\right] = \frac{ag}{f}.$$

Thus the equation of the circle is of the form

$$x^2+y^2+\frac{2bf}{a}x - \frac{2ag}{b}y + c' = 0.$$

This circle passes through $\theta_1 + \frac{\pi}{2}$ and θ_1 lies on the first circle

$$\therefore \quad a^2 \sin^2\theta_1 + b^2\cos^2\theta_1 - 2bf\sin\theta_1 - 2ag\cos\theta_1 + c' = 0$$
$$a^2\cos^2\theta_1 + b^2\sin^2\theta_1 + 2bf\cos\theta_1 + 2ag\cos\theta_1 + c = 0.$$

Thus $a^2+b^2+c+c'=0$, or $c' = -(a^2+b^2+c)$.

Hence the equation of the circle in the above form.

(2) *A circle passes through the focus $(ae, 0)$ of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ and touches the ellipse at the point whose eccentric angle is ϕ. Prove that the line joining the other two points of intersection of the two curves is*

$$\left(\frac{x}{a}-\frac{1}{e}\right)\cos\phi - \frac{y}{b}\sin\phi = \frac{1-e^2}{e^2}. \text{ [St. Catherine, 1928]}$$

As the circle meets the ellipse in two coincident points at ϕ one of the common chords is the tangent $\frac{x}{a}\cos\phi + \frac{y}{b}\sin\phi - 1 = 0$ at ϕ. Since the pairs of common chords are equally inclined to the axis, the equation of the other common chord is of the form $\frac{x}{a}\cos\phi - \frac{y}{b}\sin\phi - k = 0$, where k is to be determined. The conic given by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1\right)\left(\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi-k\right)=0$$

meets the ellipse in two coincident points at ϕ and two other points. This will be a circle if

$$\frac{1}{a^2}+\frac{\lambda}{a^2}\cos^2\phi = \frac{1}{b^2}-\frac{1}{b^2}\sin^2\phi$$

$$\therefore \quad \lambda = +\frac{a^2-b^2}{b^2\cos^2\phi + a^2\sin^2\phi} = +\frac{e^2}{1-e^2\cos^2\phi}.$$

With this value of λ, the circle will pass through $(ae, 0)$ if
$$e^2-1+\lambda(e\cos\phi-1)(e\cos\phi-k)=0$$
$$\therefore \quad k=\frac{1}{e}\cos\phi+\frac{1-e^2}{e^2}.$$
Hence the equation of the chord is
$$\left(\frac{x}{a}-\frac{1}{e}\right)\cos\phi-\frac{y}{b}\sin\phi=\frac{1-e^2}{e^2}.$$

(3) *Find the equation of the osculating circle of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *at the point* ϕ.

By Art. 85.2, the centre of the circle is
$$-g=\frac{a^2-b^2}{4a}[3\cos\phi-\cos 3\phi]=\frac{a^2-b^2}{a}\cos^3\phi$$
$$-f=-\frac{a^2-b^2}{4b}[3\sin\phi-\sin 3\phi]=-\frac{a^2-b^2}{b}\sin^3\phi$$
The equation of the circle is therefore of the form
$$x^2+y^2-2x\frac{a^2-b^2}{a}\cos^3\phi+2y\frac{a^2-b^2}{b}\sin^3\phi+k=0.$$
This passes through the point $(a\cos\phi, b\sin\phi)$,
$$a^2\cos^2\phi+b^2\sin^2\phi-2(a^2-b^2)(\cos^4\phi-\sin^4\phi)+k=0$$
or $\quad a^2\cos^2\phi+b^2\sin^2\phi-2(a^2-b^2)\cos 2\phi+k=0.$

Hence the equation of the circle is
$$x^2+y^2-2x\frac{a^2-b^2}{a}\cos^3\phi+2y\frac{a^2-b^2}{b}\sin^3\phi$$
$$=a^2\cos^2\phi+b^2\sin^2\phi-2(a^2-b^2)\cos 2\phi$$

Alternative method :—

Let the osculating circle at P $(a\cos\phi, b\sin\phi)$ cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at Q. Then the tangent at P and PQ are equally inclined to the axis of the curve. The conic
$$\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)+\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi-1\right)$$
$$\left(\frac{x}{a}\cos\phi-\frac{y}{b}\sin\phi-\cos 2\phi\right)=0$$
has contact of the 2nd order at P with the ellipse

This conic will be a circle if the co-efficient of $x^2=$ the co-efficient of y^2.

i.e., if $\dfrac{\lambda}{a^2}+\dfrac{\cos^2\phi}{a^2}=\dfrac{\lambda}{b^2}-\dfrac{\sin^2\phi}{b^2}$

i.e., if $\lambda=\dfrac{a^2\sin^2\phi+b^2\cos^2\phi}{a^2-b^2}$

Thus the osculating circle is given by the equation

$$\left(x-\frac{a^2-b^2}{a}\cos^3\phi\right)^2+\left(y+\frac{a^2-b^2}{b}\sin^3\phi\right)^3$$
$$=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^2}{a^2b^2}$$

(4) *Find the equation of the circle of curvature at the point ϕ of the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of the normal at ϕ is

$$F(\phi)\equiv ax\sin\phi-by\cos\phi-\tfrac{1}{2}a^2e^2\sin 2\phi \qquad \ldots\ldots(i)$$

The equation of a neighbouring normal at $\phi+\delta\phi$ is

$$F(\phi+\delta\phi)=ax\sin(\phi+\delta\phi)-by\cos(\phi+\delta\phi)$$
$$-\tfrac{1}{2}a^2e^2\sin(2\phi+2\delta\phi)=0.$$

The point of intersection satisfies the equation

$$F(\phi+\delta\phi)-F(\phi)=0$$

or
$$\frac{F(\phi+\delta\phi)-F(\phi)}{\delta\phi}=0.$$

If $\delta\phi$ be made to tend to zero, this approaches the line

$$F'(\phi)=0$$

or $\quad ax\cos\phi+by\sin\phi-a^2e^2\cos 2\phi=0 \qquad \ldots\ldots(ii)$

The lines (i) and (ii) are easily seen to intersect at

$$\left(\frac{a^2-b^2}{a}\cos^3\phi,\ -\frac{a^2-b^2}{b}\sin^3\phi\right)$$

The distance of this point from $(a\cos\phi,\ b\sin\phi)$ is the radius ρ of the circle of curvature.

$$\rho^2=\left(\frac{a^2-b^2}{a}\cos^3\phi-a\cos\phi\right)^2+\left(\frac{a^2-b^2}{b}\sin^3\phi+b\sin\phi\right)^2$$
$$=(a^2\sin^2\phi+b^2\cos^2\phi)^3\div a^2b^2$$

The equation of the circle is therefore

$$\left(x-\frac{a^2-b^2}{a}\cos^3\phi\right)^2+\left(y+\frac{a^2-b^2}{b}\sin^3\phi\right)^2$$
$$=\frac{(a^2\sin^2\phi+b^2\cos^2\phi)^3}{a^2b^2}$$

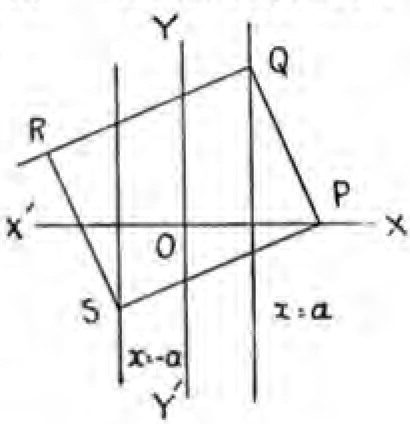
The identity of the circles of Ex. 3 and Ex. 4 can be easily verified or inferred from the fact that both the circles have the same centre and pass through the point $(a\cos\phi,\ b\sin\phi)$.

It follows that the locus of the centre of curvature is the curve (called the evolute of the ellipse)

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$$

which is evidently the envelope of the normals of the ellipse since the elimination of ϕ between $F(\phi)=0$ and $F'(\phi)=0$ gives this equation.

(5) *A variable rectangle* PQRS *has its sides parallel to fixed directions.* Q *and* S *lie respectively on the lines* $x=a$, $x=-a$ *and* P *lies on the line* $y=0$. *Prove that the locus of* R *is a st. line, and for all directions of the sides of the rectangle, this st. line always touches a fixed ellipse.* (Math. Trip. 1917)



Let the co-ordinates of P be $(x_0, 0)$ and m the slope of PQ. The equation of PQ is therefore

$$y = m(x - x_0).$$

The co-ordinates of Q are therefore $[a, m(a - x_0)]$

Hence the equation of QR is

$$y - m(a - x_0) = -\frac{1}{m}(x - a) \quad \ldots\ldots(i)$$

Again, the equation of PS is

$$y = -\frac{1}{m}(x - x_0).$$

Hence the co-ordinates of S are $\left[-a, \frac{1}{m}(a + x_0)\right]$.

The equation of SR is therefore

$$y - \frac{1}{m}(a + x_0) = m(x + a) \quad \ldots\ldots(ii)$$

Multiplying (ii) by m^2 and adding to (i) we eliminate x_0 and get the locus of R, *viz.*,

$my = x(m^2 - 1) + a(m^2 + 1)$, or $m^2(x + a) - my - (x - a) = 0$, which is a st. line. The envelope of the line as m varies is the ellipse

$$y^2 + 4(x^2 - a^2) = 0.$$

(6) *A variable tangent to the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ *meets the lines* $x + d = 0$, $x - d = 0$ *in points* P *and* Q. *The other two tangents from* P *and* Q *meet in* O. *Find the locus of* O.

Let $\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha - 1 = 0$ be a variable tangent.

The second tangent to the ellipse through P is of the form

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}-1+\lambda(x+d)=0,$$

where $(\lambda d-1)^2=a^2\left(\frac{\cos\alpha}{a}+\lambda\right)^2+\sin^2\alpha.$

One value of λ is zero which corresponds to the tangent $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-1=0$, and the other value of λ is given by

$$\lambda=\frac{2(a\cos\alpha+d)}{d^2-a^2}.$$

So the equation of the tangent is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}-1+\frac{2(a\cos\iota}{d^2-a^2}(x+d)=0\text{.....}(i)$$

Similarly the equation of the other tangent through Q can be obtained from this by replacing d by $-d$. The equation is

$$\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}-1+\frac{2(a\cos\alpha-d)}{d^2-a^2}(x-d)=0. \quad \text{......}(ii)$$

The locus of the point of intersection will be obtained by eliminating α between these equations. Subtracting, we have

$$a\cos\alpha+x=0$$

Substituting this value of $\cos\alpha$ in (i) or (ii) we have

$$\frac{-x^2}{a^2}+\frac{y\sin\alpha}{b}-1-\frac{2(x^2-d^2)}{d^2-a^2}=0$$

or $$\frac{y\sin\alpha}{b}=\frac{(x^2-a^2)(a^2+d^2)}{a^2(d^2-a^2)}.$$

Squaring and substituting for $\sin^2\alpha$

$$\frac{y^2(a^2-x^2)}{a^2b^2}=\frac{(x^2-a^2)^2(a^2+d^2)^2}{a^4(d^2-a^2)^2}$$

So the locus consists of the lines $x=\pm a$ and the conic

$$\frac{y^2}{b^2}+\frac{(x^2-a^2)(a^2+d^2)^2}{a^2(d^2-a^2)^2}=0.$$

(7) A parallelogram circumscribes the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

and two of its corners move on $Ax^2+2Hxy+By^2=1$. Show that the other two move on the conic

$$Bb^4x^2-2Ha^2b^2xy+Aa^4y^2=b^2x^2+a^2y^2-a^2b^2.$$

Let α, $\alpha+\pi$, β, $\beta+\pi$ be the eccentric angles of the points of contact, then the pairs of opposite vertices are

$$\left\{\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\right\},\ \left\{-\frac{a\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},-\frac{b\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right\}$$

$$\left\{\frac{a\cos\frac{\alpha+\beta+\pi}{2}}{\cos\frac{\alpha+\pi-\beta}{2}},\ \frac{b\sin\frac{\alpha+\beta+\pi}{2}}{\cos\frac{\alpha+\pi-\beta}{2}}\right\},$$

$$\left\{\frac{a\cos\frac{\alpha+\beta+\pi}{2}}{\cos\frac{\beta+\pi-\alpha}{2}},\ \frac{b\sin\frac{\alpha+\beta+\pi}{2}}{\cos\frac{\beta+\pi-\alpha}{2}}\right\}.$$

Suppose the first pair lies on the locus $Ax^2+2Hxy+By^2=1$.

$$\therefore\quad Aa^2\cos^2\frac{\alpha+\beta}{2}+Bb^2\sin^2\frac{\alpha+\beta}{2}+Hab\sin(\alpha+\beta)$$
$$=\cos^2\frac{\alpha-\beta}{2}$$

$$\therefore\quad Aa^2\frac{\cos^2\frac{\alpha+\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}}+Bb^2\frac{\sin^2\frac{\alpha+\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}}$$
$$+2Hab\frac{\cos\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}$$
$$=\frac{\cos^2\frac{\alpha-\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}}=\operatorname{cosec}^2\frac{\alpha-\beta}{2}-1\ldots\ldots(i)$$

Now, $x^2=\dfrac{a^2\sin^2\frac{\alpha+\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}}$, $y^2=b^2\dfrac{\cos^2\frac{\alpha+\beta}{2}}{\sin^2\frac{\alpha-\beta}{2}}$..

$$\therefore\quad\frac{x}{a^2}+\frac{y^2}{b^2}=\operatorname{cosec}^2\frac{\alpha-\beta}{2}.$$

Hence substituting in (i), we have

$$Aa^2\frac{y^2}{b^2}+Bb^2\frac{x^2}{a^2}-2Hab\frac{y}{b}\cdot\frac{x}{a}=\frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

or $\quad Bb^4x^2 - 2Ha^2b^2xy + Aa^4y^2 = b^2x^2 + a^2y^2 - a^2b^2.$

(8) *If the square of the eccentricity of an ellipse exceeds* $2(\sqrt{2}-1)$, *prove that there are eight normal chords each of which meets the ellipse at the extremities of a pair of conjugate diameters and the length* (l) *of any such chord is a root of the equation.*

$$l^4 - (a^2+b^2)l^2 + 2a^2b^2 = 0$$ [Math. Tip. I, 1923]

The equation of the normal at $P(\phi)$ is

$$\frac{x - a\cos\phi}{\dfrac{p\cos\phi}{a}} = \frac{y - b\sin\phi}{\dfrac{p\sin\phi}{b}} = r$$

The co-ordinates of an arbitrary point on the normal are

$$\frac{pr\cos\phi}{a} + a\cos\phi,\ \frac{pr}{b}\sin\phi + b\sin\phi.$$

If the normal passes through $D(-a\sin\phi, b\cos\phi)$; then

$$\frac{pr}{a}\cos\phi + a\cos\phi = -a\sin\phi,\ \frac{pr}{b}\sin\phi + b\sin\phi = b\cos\phi$$

$$\therefore \quad \frac{\cos\phi}{\sin\phi} = -\frac{a^2}{b^2}\ \frac{\cos\phi + \sin\phi}{\cos\phi - \sin\phi}$$

i.e., $\quad (a^2+b^2) + (a^2-b^2)\sin 2\phi - (a^2-b^2)\cos 2\phi = 0 \ldots\ldots(i)$

Now $\quad l^2 = a^2(\cos\phi + \sin\phi)^2 + b^2(\sin\phi - \cos\phi)^2 = (a^2+b^2)$
$\qquad + (a^2-b^2)\sin 2\phi \ \ldots\ldots(ii)$

$\therefore$ From (i) and (ii),

$$l^4 = (a^2-b^2)^2(1-\sin^2 2\phi) = (a^2-b^2)^2 - (l^2-a^2-b^2)^2$$

i.e., $\quad l^4 - (a^2+b^2)l^2 + 2a^2b^2 = 0.$

The roots of this quadratic in l^2 will be real if

$$(a^2+b^2)^2 > 8a^2b^2$$

i.e., $\quad (2-e^2)^2 > 8(1-e^2)$

i.e., $\quad e^4 + 4e^2 - 4 > 0 \quad$ or $\quad (e^2+2)^2 > 8$

i.e., $\quad e^2 > 2(\sqrt{2}-1)$

The roots in l^2 are then positive and hence all the four roots are real, two being positive and two negative. The two positive roots give the lengths of the normal chords. Let them be l_1, l_2.

From (i) $\quad \cos 2\phi = \dfrac{l_1^2}{a^2-b^2}$ or $\dfrac{l_2^2}{a^2-b^2}$

i.e., $\quad \cos 2\phi = \cos\alpha$ or $\cos\beta$.

$$\therefore \quad \phi = \pm\frac{\alpha}{2},\ \pi \pm \frac{\alpha}{2},\ \pm\frac{\beta}{2},\ \pi \pm \frac{\beta}{2}.$$

Thus there are eight normal chords, four being equal to l_1 and four equal to l_2.

Miscellaneous Exercises XXI

1. Show from definition that the st. lines $3x-y=0$ and $x+17y=0$ represent a pair of conjugate diameters of the conic
$$2x^2-xy+3y^2=0.$$

2. The points A, A′ are the ends of the major axis of a conic and PAQ, P′A′Q′ are tangents to the conic there; if PP′, QQ′ are two other tangents to the conic, prove that
$$AP.A'P=AQ.A'Q'$$
and the lines PQ′, P′Q intersect on AA′. [Math. Trip. 1913]

3. A conic $Ax^2+By^2=1$ and a point $P(h, k)$ being given, prove that the locus of a point Q, whose polar makes a constant angle with QP is a conic passing through P and through the origin. What is the nature of the locus when the constant angle is (*i*) zero, (*ii*) a right angle? [Math. Trip. 1915]

4. Two chords of an ellipse are drawn parallel to the minor axis, and equidistant from it. Any tangent to the curve meets the chords in P, Q. Prove that SP^2+SQ^2 is always proportional to PQ^2, S being either of the foci.

[Math. Trip. 1917]

5. P and Q are extremities of two conjugate diameters of an ellipse of minor axis $2b$, and S is a focus. Prove that
$$PQ^2-(SP-SQ)^2=2b^2.$$

6. If the points of intersection of the ellipses
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad \frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}=1$$
are at the extremities of conjugate diameters of the former, prove that, the point (α, β) lies on the curve
$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=2.$$

7. Prove that the length of the normal chord of the ellipse of axes $2a$, $2b$ which makes equal angles with the axes is
$$4\sqrt{2}\, a^2b^2\,(a^2+b^2)^{-\frac{3}{2}}.$$

8. SP, S′P′ are focal radii of an ellipse drawn in the same direction, and the tangents at P and P′ meet S′P′ and SP in Q′ and Q respectively. Prove that QQ′ is parallel to PP′.

9. If P, Q are the extremities of conjugate diameters of an ellipse, and PP′, QQ′ be chords parallel to an axis of the ellipse: show that PQ′ and P′Q are parallel to equi-conjugates.

10. QQ′ is any chord of an ellipse parallel to one of the equi-conjugates, and the tangents at Q, Q′ meet in T; show that the circle QTQ′ passes through the centre.

11. A tangent to an ellipse is a chord of a concentric circle whose radius is equal to the distance between the ends of the axes of the ellipse ; show that the st. lines which join the ends of the chords to the centre are conjugate diameters.

12. If θ and θ' are the eccentric angles of the ends of a focal chord of an ellipse, prove that

$$\tan\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta' = -\frac{1-e}{1+e} \text{ or } -\frac{1+e}{1-e}.$$

Distinguish the two cases.

13. If PSQ. PS′R are two focal chords of an ellipse and the eccentric angles of Q and R are θ and θ': show that the ratio $\tan\frac{1}{2}\theta : \tan\frac{1}{2}\theta'$ is constant for all positions of P.

14. If P be the point $(a\cos\theta, b\sin\theta)$, show that the equation of QR is

$$\frac{x}{a}\cos\theta + \frac{y}{b}\,\frac{1+e^2}{e^2-1}\sin\theta = 1$$

and it touches the conic

$$\frac{x^2}{a^2} + \frac{(a^2-b^2)^2y^2}{b^6} = 1.$$

15. If the tangents at Q and R of the last exercise meet in T, show that the locus of T as P moves on the curve is

$$(1+e^2)^2\frac{x^2}{a^2} + (1-e^2)^2.\frac{y^2}{b^2} = (1+e^2)^2.$$

16. Show that the locus of points, the tangents from which to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ contain a given angle α is

$$(x^2+y^2-a^2-b^2)^2\tan^2\alpha = 4(b^2x^2+a^2y^2-a^2b^2).$$

The locus is called the *isoptic locus.*

17. Given the base of a triangle and the product of the tangents of the base angles, the locus of the vertex is an ellipse.

18. Given the base and the sum of the two sides, the locus of the in-centre is an ellipse.

Prove also that the locus of the centre of the escribed circle which touches the base externally is also an ellipse.

[*Hint.* PSS′ be the triangle, prove $\tan\frac{1}{2}S\ \tan\frac{1}{2}S'$ is constant].

19. The tangents AP, A′Q at the extremities A, A′, the major axis of an ellipse, (centre C) are met by an arbitrary tangent PRQ at R in the points P and Q. If CP, CQ meet the ellipse in P′, Q′, prove that

(*i*) $AP.A'Q = b^2$.

(*ii*) CP′, CQ′ are semi conjugate diameters of the ellipse.

(*iii*) PR.RQ = square of the parallel semi-conjugate diameter.

(*iv*) The circle on PQ as diameter passes through the foci.

20. If the chord joining two points whose eccentric angles are α, β cut the major axis of an ellipse at a distance d from the centre, show that

$$\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2}=\frac{d-a}{d+a}.$$

21. If any two chords be drawn through two points on the major axis of an ellipse equidistant from the centre, show that $\tan\frac{\alpha}{2}\ \tan\frac{\beta}{2}\ \tan\frac{\gamma}{2}\ \tan\frac{\delta}{2}=1$, where α, β, γ, δ are eccentric angles of the extremities of the chords.

22. Show that the co-ordinates of the intersection of normals at the points whose eccentric angles are $\alpha+\beta$, $\alpha-\beta$, are

$$x=\frac{(a^2-b^2)\cos\alpha\cos(\alpha+\beta)\cos(\alpha-\beta)}{a\cos\beta},$$

$$y=\frac{(b^2-a^2)\sin\alpha\sin(\alpha+\beta)\sin(\alpha-\beta)}{b\cos\beta}.$$

23. The normal at a point P of an ellipse meets the major axis in G, show that the locus of the mid-point of PG is an ellipse whose eccentricity e' is connected with that of the given ellipse e, by the equation

$$(1-e^2)=(1+e^2)^2(1-e'^2).$$

24. Show that the line $lx+my+n=0$, is a normal to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, provided $\frac{a^2}{l^2}+\frac{b^2}{m^2}=\frac{(a^2-b^2)^2}{n^2}$.

25. Show that the locus of the poles of the normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve $\frac{a^6}{x^2}+\frac{b^6}{y^2}=(a^2-b^2)^2$.

26. The normals at four points on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet in the point (α, β), show that the co-ordinates of the centroid of the four points are

$$\frac{a^2\alpha}{2(a^2-b^2)},\ \frac{b^2\beta}{b^2-a^2}.$$

27. If F is a focus of an ellipse and the normal at P meets the major and minor axes in G, g, and CD is conjugate to CP prove that FG : CF = FP : AC and Fg : CF = CD : BC.

28. A triangle is inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and has its centre of gravity at the centre of the ellipse, show that the locus of the circumcentre is

$$a^2x^2+b^2y^2=\frac{1}{16}(a^2-b^2)^2.$$

29. The locus of the centroid of an equi-lateral triangle inscribed in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, is

$$\frac{x^2(a^2+3b^2)}{a^2}+\frac{y^2(b^2+3a^2)}{b^2}=(a^2-b^2)^2.$$

[*Hint.* The centroid coincides with the circum-centre.]

30. If $P(x_1, y_1)$, $Q(x_2, y_2)$ are two adjacent vertices of a parallelogram circumscribed about the ellipse

$$S(x, y)\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0,$$

show that $S(x_1, y_1)\,S(x_2, y_2)=1$.

31. If PCP′, DCD′ are conjugate diameter of an ellipse, and ϕ is the eccentric angle of P. Prove that $\frac{1}{2}\pi-3\phi$ is the eccentric angle of the point where the circle PP′D again cuts the ellipse. (Math. Trip. 1910)

32. Prove that the tangent and normal at any point of an ellipse cut the minor axis at points which subtend a right angle at either focus.

33. If the normals at the extremities of the polars of $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ meet in a point, show that

$$x_1x_2+a^2=0, \quad y_1y_2+b^2=0.$$

34. Prove that the envelope of the chord of contact of two perpendicular tangents to an ellipse is another ellipse.

35. Two lines are conjugate *w. r.* to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and always pass through the ends of the major axis of the ellipse, show that they intersect on the ellipse $\frac{x^2}{a^2}+\frac{2y^2}{b^2}=1$.

36. Prove that the length of the perpendicular from the the centre on the chord joining the ends of two conjugate diameters lies between $\frac{a}{\sqrt{2}}$ and $\frac{b}{\sqrt{2}}$, where a and b are the semi-axes. [Math. Trip. 1922]

37. Find the equations of the four st. lines other than the axes which are normal to both of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

38. T is the pole of a chord PQ of an ellipse whose centre is C and CT meets the curve in R. Prove that if eccentric angles of P and Q are $\alpha+\beta$ and $\alpha-\beta$, the eccentric angle of R is α.

39. Prove that the circle whose diameter is the chord

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2}+\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha-\beta}{2}$$

cuts the ellipse $b^2x^2+a^2y^2=a^2b^2$ in two other points whose join is the line

$$\frac{x}{a}\cos\tfrac{1}{2}(\alpha+\beta)-\frac{y}{b}\sin\tfrac{1}{2}(\alpha+\beta)-\frac{a^2+b^2}{a^2-b^2}\cos\tfrac{1}{2}(\alpha-\beta)=0.$$

40. The tangent at one end P of a diameter PP′ of an ellipse and any chord P′Q through the other end meet in R, prove that the tangent at Q bisects PR. [Math. Trip 1921]

41. CP, CP′, CQ, CQ′ are pairs of conjugate diameters of an ellipse; show that P′Q′ is either parallel to PQ or to the diameter which bisects PQ.

42. Prove that if $c^2a^2=a^4-b^4$, the circle $x^2+y^2+2cx+a^2=0$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are so related that the polar *w. r.* to the circle of any point on the conic touches the conic and the polar *w. r.* to the conic of any point on the circle touches the circle. [Downing 1929]

CHAPTER IX

HYPERBOLA

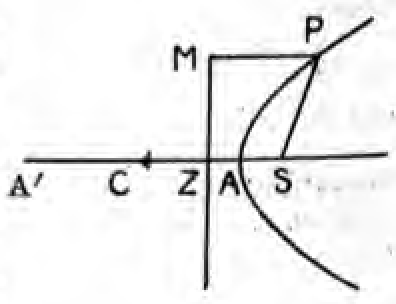
87. The equation of the hyperbola whose transverse axis is the x-axis and conjugate axis is the y-axis has been reduced to the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1. \qquad \ldots\ldots(1)$$

This equation can be directly obtained from the following definition of a hyperbola :—

A hyperbola is a curve traced out by a point moving ni a plane such that its distance from a fixed point (called the focus) bears a constant ratio greater than unity to its distance from a fixed line (called the directrix).

The constant ratio is called the **eccentricity** and is denoted by e.



Let S be the focus and MZ the directrix. Draw SZ perpendicular to MZ, and divide SZ in the ratio $e:1$ such that

$$\left.\begin{array}{l}SA=e.AZ\\ A'S=e.A'Z\end{array}\right\} \quad e>1$$

Thus A and A′ are on the curve.

Suppose that the mid-point C of AA′ $(=2a)$ is the origin and AA′ is the x-axis. The co-ordinates of S can be supposed to be $(p, 0)$ and the equation of the directrix MZ can be taken as $x-k=0$. The condition $SP=e.PM$ gives the equation

$$(x-p)^2+y^2=e^2(x-k)^2$$

i.e., $$x^2(e^2-1)-y^2-2x(e^2k-p)+(e^2k^2-p^2)=0$$

The line $y=0$ meets the conic in points A $(a, 0)$ and A′$(-a, 0)$. This requires

$$p=e^2k,\ e^2k^2-p^2=-a^2(e^2-1)$$

whence $k=\dfrac{a}{e}$, $p=ae$, $k=\dfrac{-a}{e}$, $p=-ae$.

Thus the equation of the conic takes the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \qquad b^2=a^2(e^2-1).$$

Its real foci are $S(ae, 0)$, $S'(-ae, 0)$ and the corresponding real directrices are $x-\dfrac{a}{e}=0$, $x+\dfrac{a}{e}=0$

87.1. The lines

$$y=ix \pm i\sqrt{a^2+b^2}$$
$$y=-ix \pm i\sqrt{a^2+b^2}$$

are the isotropic tangents of the hyperbola.

These lines intersect in six points two of which are the circular points at infinity and the other four are

$$(\pm\sqrt{a^2+b^2}, 0), (0, \pm i\sqrt{a^2+b^2})$$

The first pair of points is real and the second imaginary.

Since $\sqrt{a^2+b^2} > a$, we may put $\sqrt{a^2+b^2}=ae$ where $e > 1$, then $b^2=a^2(e^2-1)$. The equation of the hyperbola can then be written as

$$\frac{x^2}{a^2}-\frac{y^2}{a^2(e^2-1)}=1$$

or
$$(x-ae)^2+y^2=e^2\left(x-\frac{a}{e}\right)^2 \qquad \text{......}(i)$$

Thus the distance of a point $P(x, y)$ of the curve from the fixed point S (ae, 0 bears a constant) ratio greater than unity to its distance from the fixed line $x-\dfrac{a}{e}=0$. A similar inference can be drawn when the equation is written in the form

$$(x+ae)^2+y^2=e^2\left(x+\frac{a}{e}\right)^2 \qquad \text{......}(ii)$$

87.2. From the equation $x \pm \dfrac{a}{e}=0$ of the directrices, it follows that

$$SP=e.PM=ex-a$$
$$S'P=e.PM'=ex+a,$$

since $$x > a$$

$\therefore$ $$S'P-SP=2a.$$

Thus the difference between the focal distances of a point P on the hyperbola is constant.

87.3. The length of the latus rectum which is a focal chord of a hyperbola at right angles to the transverse axis is easily seen to be $2\dfrac{b^2}{a}=2a(e^2-1)$.

88. To interpret $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 \gtrless 0.$

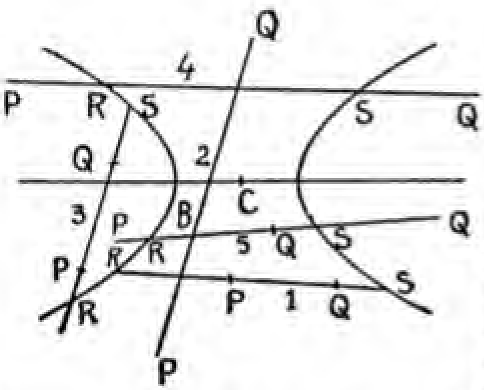
To interpret it, we may use an argument similar to that for an ellipse, but it will be instructive to use a different method.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points in the plane of the hyperbola. The Jaochimsthal's ratio equation is

$$\lambda^2 S_2 + 2\lambda T + S_1 = 0,$$

where $S_2 \equiv \frac{x_2^2}{a^2} - \frac{y_2^2}{b^2} - 1, S_1 \equiv \frac{x_1^2}{a^2} \quad \frac{y_1^2}{b^2} - 1.$

$$T \equiv \frac{x_1 x_2}{a^2} - \frac{y_1 y_2}{b^2} - 1.$$



If the line PQ meets the hyperbola in points R and S and λ_1, λ_2 be the corresponding values of λ for R and S,

$$\lambda_1 \lambda_2 = S_1 / S_2.$$

It will be seen that if the points P and Q lie in the region in which the centre C lies, the points R and S are either both real and external to the segment PQ (line 1) or both conjugate imaginary (line 2). In either case $\lambda_1\lambda_2$ is positive. If P and Q both lie in the region in which the centre C does not lie, the points R and S are either both internal or both external to the segment PQ (lines 4, 3). In either case $\lambda_1\lambda_2$ is positive. If one of the points P and Q is in the centre region and the other in the non-centre region, one of the points R and S is internal and the other external to the segment PQ. The product $\lambda_1\lambda_2$ is therefore negative. *Thus if the points* P *and* Q *lie both in the centre region or non-centre region, the expressions* S_1 *and* S_2 *are of the same sign, and the expression* S_1 *and* S_2 *are of different signs if* P *and* Q *lie in different regions.*

Now for the centre C (0,0) the expression $S = -1$. Hence $\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1$ is negative for all points on the centre side and positive for all points on the non-centre side. *Thus the inequation* $S<0$ *represents the region in which the centre lies and* $S>0$ *is the analytic expression of the region in which the centre does not lie.*

89. Asymptotes. *A line which meets an algebraic curve at two points at infinity but is not wholly at infinity is called an asymptote.*

Since there is only one point at infinity on a line, the points of intersection of the asymptote with the curve coincide. Thus a *rectilinear asymptote is the limiting position of a tangent when its point of contact moves along the curve to infinity.*

89.1. Let $\dfrac{x - x_0}{\cos\theta} = \dfrac{y - y_0}{\sin\theta} \; (=r)$

be the equation of a line. It meets the hyperbola in points whose distances from (x_0, y_0) are given by the equation

$$r^2\left(\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2}\right) + 2r\left(\frac{x_0\cos\theta}{a^2} - \frac{y_0\sin\theta}{b^2}\right)$$
$$+\left(\frac{x_0^2}{a^2} - \frac{y_0^2}{b^2} - 1\right) = 0.$$

If both the points of intersection are at infinity,

$$\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = 0, \qquad \frac{x_0\cos\theta}{a^2} - \frac{y_0\sin\theta}{b^2} = 0.$$

$$\therefore \qquad \tan\theta = \pm\frac{b}{a} \qquad \ldots\ldots(i)$$

$$\text{and} \qquad \frac{y_0}{x_0} = \frac{b^2}{a^2}\cot\theta \qquad \ldots\ldots(ii)$$

$$\therefore \qquad \frac{y_0}{x_0} = \pm\frac{b}{a}.$$

Thus the equations of the two asymptotes are

$$\frac{x}{a} \pm \frac{y}{b} = 0. \qquad \ldots\ldots(2)$$

Cor. It is thus seen that the pair of asymptotes are tangents to the hyperbola through the centre.

Exercise. Find the asymptotes of the ellipse

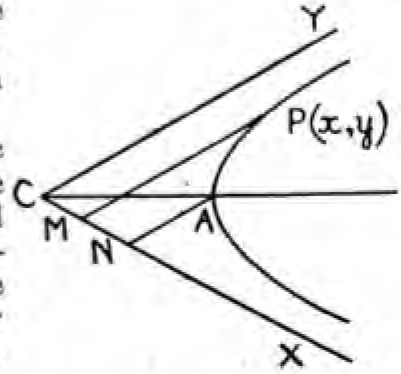
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

90. Equation of a hyperbola referred to asymptotes as axes.

(*a*) The lines $x=0$, $y=0$ are by supposition tangents to the conic, its equation is therefore of the form

$$xy = (lx + my + n)^2.$$

The line $y=0$ meets the conic in points whose abscissæ are the roots of the equation $(lx+n)^2 = 0$, and since the point of contact is at infinity, this equation has 1 infinite roots. Thus $l=0$. For a similar reason, $m=0$. The equation of the



conic is therefore of the form

$$xy = n^2.$$

Let AN be parallel to CY. The co-ordinates of A are (CN, NA). From the isosceles triangle CNA, in which

$CA = a$, $\angle NCA = \alpha = \tan^{-1}\dfrac{b}{a}$

$$a^2 = 2\,CN^2 + 2\,CN^2 \cos 2\alpha$$

$$= 4\,CN^2 \cos^2\alpha = 4.\,CN^2.\,\frac{a^2}{a^2+b^2},$$

$$\therefore \qquad CN = NA = \frac{\sqrt{a^2+b^2}}{2}$$

Since A lies on the hyperbola,

$$n^2 = \frac{a^2+b^2}{4}.$$

Thus the equation of the hyperbola is

$$xy = \frac{a^2+b^2}{4}. \qquad \ldots\ldots(3)$$

90.1. *Second method.* Let $P(x, y)$ be the co-ordinates of a point on the hyperbola when the axes are the transverse and conjugate axes and (ξ, η) the co-ordinates of the same point when the asymptotes are taken as the axes. Suppose that p_1, p_2 are the measures of the perpendiculars from P on the asymptotes, then

$$\eta = p_1 \operatorname{cosec} 2\alpha = \frac{\dfrac{x}{a} - \dfrac{y}{b}}{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)^{\frac{1}{2}}} \operatorname{cosec} 2\alpha$$

$$\xi = p_2 \operatorname{cosec} 2\alpha = \frac{\dfrac{x}{a} + \dfrac{y}{b}}{\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)^{\frac{1}{2}}} \operatorname{cosec} 2\alpha$$

$$\therefore \qquad \xi\eta = \left(\frac{x^2}{a^2} - \frac{y^2}{b^2}\right) \cdot \frac{a^2b^2}{4(a^2+b^2)\sin^2\alpha \cos^2\alpha}$$

$$\text{or} \qquad \xi\eta = \frac{a^2+b^2}{4}. \qquad \because \tan\alpha = \frac{b}{a}.$$

It should be noticed that the axes are oblique.

90.2. Def. If the asymptotes of a hyperbola be at right angles, the conic is called a **rectangular** or **equilateral hyperbola.**

Since the angle between the asymptotes is $2\tan^{-1}\frac{b}{a}$, it will be a right angle if $a=b$. The equation of the curve is then $x^2-y^2=a^2$ or $xy=c^2$, according as the axes or the asymptotes are the axes of co-ordinates.

90.3. The similarity of the equation of the hyperbola with that of the ellipse suggests that several results proved for the ellipse will give the corresponding results for the hyperbola by a change of b^2 to $-b^2$. In the following articles, some of these results will be obtained by a different method which is also applicable to an ellipse.

91. Equation of a tangent. Let P(x', y'), Q(x'', y'') be two points on a hyperbola, therefore

$$\frac{x'^2}{a^2}-\frac{y'^2}{b^2}=1,\quad \frac{x''^2}{a^2}-\frac{y''^2}{b^2}=1.$$

Thus $$\frac{(x'-x'')(x'+x'')}{a^2}=\frac{(y'-y'')(y'+y'')}{b^2}. \qquad \ldots\ldots(4)$$

Also the equation of the line PQ is

$$\frac{x-x'}{x'-x''}=\frac{y-y'}{y'-y''}$$

which in virtue of relation (4) becomes

$$\frac{(x-x')(x'+x'')}{a^2}-\frac{(y-y')(y'+y'')}{b^2}=0. \qquad \ldots\ldots(5)$$

Let the point (x'', y'') moving on the curve approach P indefinitely. In the limiting position the equation of the line becomes

$$\frac{x'(x-x')}{a^2}-\frac{y'(y-y')}{b^2}=0$$

$$\text{or}\qquad \frac{xx'}{a^2}-\frac{yy'}{b^2}=1, \qquad \ldots\ldots(6)$$

which is therefore the equation of the tangent to the hyperbola at (x', y').

91.1. The equation of the normal. It can be easily proved that the equation of the normal at P (x', y') is

$$\frac{x-x'}{\dfrac{x'}{a^2}}=\frac{y-y'}{-\dfrac{y'}{b^2}} \qquad \ldots\ldots(7)$$

91.2. The following results are left to the reader as exercise.

The line $lx+my+n=0$ is a tangent to the hyperbola if

$$a^2l^2-b^2m^2=n^2. \qquad \ldots\ldots(8)$$

In particular the line

$$y=mx+\sqrt{a^2m^2-b^2} \qquad \ldots\ldots(9)$$

is a tangent to the hyperbola for all values of m.

The equation of the pair of tangents from (x', y') is

$$\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right) \ldots\ldots(10)$$

and the equation of the director circle is

$$x^2+y^2=a^2-b^2 \qquad \ldots\ldots(11)$$

The polar of the point $(x'y')$ is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1. \qquad \ldots\ldots(12)$$

If the pole of a line p lies on the line q, the pole of q lies on p. ......(13)

If the polar of P passes through Q, the polar of Q passes through P. ......(14)

Exercises XXII

1. The tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes and the portion of it intercepted between the asymptotes is bisected at that point.

2. Any st. line cuts a hyperbola in Q, Q′ and its asymptotes in R, R′. QQ′ and RR′ have the same middle point.

3. The mean centre of the intersections of a rectangular hyperbola and a circle is the mid-pt. of the line joining the centres of the curves.

4. If a rectangular hyperbola $xy=c^2$ passes through the vertices of a triangle, it passes through the ortho-centre and its centre lies on the nine-points circle of the triangle.

5. If a conic circumscribing a triangle pass through the ortho-centre, it must be a rectangular hyperbola.

6. In a rectangular hyperbola, the angle between any chord PQ and the tangent at P is equal to the angle subtended by PQ at the other extremity of the diameter through P.

7. Any chord of a rectangular hyperbola subtends at the ends of any diameter equal or supplementary angles.

92. The equation of the chord with a given mid-point.

Let (x_0, y_0) be the mid-point of the chord whose end-points are (x', y'), (x'', y'')

$$2x_0 = x' + x'',\ 2y_0 = y' + y''.$$

The equation of the chord is

$$\frac{x - x_0}{\cos\theta} = \frac{y - y_0}{\sin\theta},$$

where in virtue of equation (4),

$$\tan\theta = \frac{y' - y''}{x' - x''} = \frac{b^2(x' + x'')}{a^2(y' + y'')} = \frac{b^2 x_0}{a^2 y_0}.$$

The equation of the chord thus becomes

$$\frac{x_0(x - x_0)}{a^2} - \frac{y_0(y - y_0)}{b^2} = 0 \qquad \text{......(15)}$$

It is parallel to the polar of $(x_0\ y_0)$.

93. Locus of the mid-points of a system of parallel chords. Conjugate diameters.

As in the case of an ellipse, we obtain, that two diameters $y = mx$, $y = m'x$ are conjugate *w. r.* to the hyperbola if

$$mm' = \frac{b^2}{a^2}. \qquad \text{......(16)}$$

The result may be obtained as follows :—

If m be the slope of the system of parallel chords and (x_0, y_0) the mid-point of one of the chords whose extremities are (x', y'), (x'', y''), we have in virtue of equation (15)

$$\frac{(x' - x'')x_0}{a^2} = \frac{(y' - y'')y_0}{b^2}.$$

Thus (x_0, y_0) lies on the diameter $y = \dfrac{b^2}{a^2 m}x$

since $m = \dfrac{y' - y''}{x' - x''}$.

If we call this diameter $y = m'x$, we have $mm' = \dfrac{b^2}{a^2}$.

It is, convenient to take the equations of two conjugate diameters in the form

$$y = \frac{b\lambda}{a}x, \quad y = \frac{b}{a\lambda}x. \qquad \text{......(17)}$$

The slope of the diameter which is conjugate to itself, is given by the equation

$$m^2 = \frac{b^2}{a^2}$$

since $m = m'$,

$$\therefore \; m = \pm \frac{b}{a}.$$

Thus *each asymptote is a diameter which is conjugate to itself.*

93.1. Let $P(x_1, y_1)$ be an extremity of the diameter $y = \frac{b}{a}\lambda x$ and $D(x_2, y_2)$ that of the diameter $y = \frac{b}{a\lambda}x$.

$$x_1^2 = \frac{a^2}{1-\lambda^2}, \quad x_2^2 = \frac{a^2\lambda^2}{\lambda^2-1}.$$

If C be the centre of the hyperbola,

$$\begin{aligned} CP^2 + CD^2 &= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) \\ &= x_1^2\left(1 + \frac{b^2\lambda^2}{a^2}\right) + x_2^2\left(1 + \frac{b^2}{a^2\lambda^2}\right) \\ &= \frac{a^2}{1-\lambda^2} \cdot \frac{a^2 + \lambda^2 b^2}{a^2} + \frac{a^2\lambda^2}{\lambda^2-1} \cdot \frac{a^2\lambda^2 + b^2}{a^2\lambda^2} \\ &= a^2 - b^2. \end{aligned} \qquad \ldots\ldots(18)$$

Thus the sum of the squares of two conjugate diameters of a hyperbola is constant.

93.2. The values of x_1 and x_2 in Art 93.1 show that if one diameter meets the hyperbola in real points, the other meets it in imaginary points. If x_1 be real, $|\lambda| < 1$, the diameter $y = \frac{b\lambda}{a}x$ for which $\left|\frac{b\lambda}{a}\right| < \frac{b}{a}$ meets the curve in real points and the other in imaginary points. If, however, $|\lambda| > 1$, the second diameter $y = \frac{b}{a\lambda}x$ for which $\left|\frac{b}{a\lambda}\right| < \frac{b}{a}$ meets the curve in real and the other in imaginary points.

If α be the acute angle at which the asymptote $y = \frac{b}{a}x$ is inclined to the x-axis, then every diameter which lies within the angles $-\alpha$ and α meets the hyperbola in real points and its conjugate meets the hyperbola in imaginary points.

94. Conjugate Hyperbola. The equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \text{ or } -\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots\ldots(19)$$

represents a hyperbola whose transverse axis is along the

y-axis and the conjugate axis is along the x-axis. This hyperbola is called conjugate to the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

and the relation is mutual. The two hyperbolas have common asymptotes. Since the ratio $b^2 : a^2$ is the same for both, a pair of diameters which is conjugate for one is conjugate for the other also.

95. Let a pair of conjugate diameters CP, CQ of the hyperbolas be given by the equations

$$y=\frac{b}{a}\lambda x \quad y=\frac{b}{a\lambda}x.$$

Suppose that the first diameter meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ in real points $P(x_1,y_1)$, $P'(x'_1, y'_1)$, so that $\lambda<1$, and

$$x_1^2=\frac{a^2}{1-\lambda^2} \qquad \text{by (Art. 93.1)}$$

$$\text{Thus } CP^2=x_1^2+y_1^2=x_1^2\left(1+\frac{b^2\lambda^2}{a^2}\right)$$

$$=\frac{a^2+\lambda^2b^2}{1-\lambda^2}.$$

The second diameter meets the conjugate hyperbola in $Q(x_2, y_2)$, where

$$x_2^2=\frac{a^2\lambda^2}{1-\lambda^2}$$

and

$$CQ^2=\frac{a^2\lambda^2+b^2}{1-\lambda^2}.$$

The point Q is real if P is real and conversely. If $\lambda\to1$ the points move along the curve to infinity and the conjugate diameters approach the position of the asymptotes. *Thus the conjugate diameters of a hyperbola and its conjugate are separated by their common asymptotes into two sets, one of them meets one hyperbola in real points and the other meets the second hyperbola in real points.*

Also $$CP^2-CQ^2=a^2-b^2. \qquad \text{......(20)}$$

Hence the differene of the squares of two conjugate diameters of a hyperbola and the conjugate hyperbola is constant.

If the diameter CQ meets the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

in an imaginary point D.

$$CD^2 = \frac{a^2\lambda^2+b^2}{\lambda^2-1} \qquad (c.\ f.\ (93.1)$$

$$\therefore \qquad CQ^2 = -CD^2$$

Thus if CP and CD be conjugate diameters of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$CP^2+CD^2 = a^2 - b^2$$

Exercises XXIII

1. In a rectangular hyperbola, conjugate diameters are equal and are equally inclined to either asymptote.

2. Any diameter of a rectangular hyperbola is equal to the diameter perpendicular to it of the conjugate hyperbola.

3. If a hyperbola has two perpendicular diameters equal to one another, the one belonging to the hyperbola itself and the other to its conjugate, the hyperbola must be a rectangular one.

4. The lines joining the extremities of conjugate diameters of a rectangular hyperbola are perpendicular to the asymptotes.

5. The circles described on parallel chords of a rectangular hyperbola are co-axal.

6. The base of a triangle and the difference of its base-angles being given, the locus of its vertex is a rectangular hyperbola.

96. The equation of the tangent at $P(x_1, y_1)$ or $P\left(\frac{a}{\sqrt{1-\lambda^2}}, \frac{b\lambda}{\sqrt{1-\lambda^2}}\right)$ to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$\frac{x}{a}-\frac{\lambda y}{b}=\sqrt{1-\lambda^2},$$

and the equation of the tangent to the second hyperpola at $Q\left(\frac{a\lambda}{\sqrt{1-\lambda^2}}, \frac{b}{\sqrt{1-\lambda^2}}\right)$ is

$$\frac{\lambda x}{a}-\frac{y}{b}=-\sqrt{1-\lambda^2}.$$

These tangents are parallel to a pair of conjugate diameters and intersect on the asymptote. Hence, *the tangents at the real points where the two conjugate diameters cut the two conjugate hyperbolas form a parallelogram whose vertices lie on the asymptotes.*

The area of this parallelogram is constant. To prove this we notice that

$$CP^2=\frac{a^2+\lambda^2b^2}{1-\lambda^2}.$$

If CK be the measure of the perpendicular from the centre on the tangent at Q

$$CK = \frac{ab\sqrt{1-\lambda^2}}{\sqrt{b^2+a^2\lambda^2}}.$$

$\therefore$ Area $=4$. CP.CK $=4\,ab$. ......(21)

The mid-point of the segment PQ is

$$x = a\sqrt{\frac{1+\lambda}{1-\lambda}},\ y = b\sqrt{\frac{1+\lambda}{1-\lambda}}$$

and it lies on an asymptote.

96.1. The polars of $P(x_1, y_1)$ *w. r.* to the two hyperbolas are given by the equations

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,\quad \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = -1$$

and these are parallel and equi-distant from the origin.

Let P be a point of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{{}^2} = 1.$$

The polar of this *w. r.* to the conjugate hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = -1$$

or $$\frac{x(-x_1)}{a^2} - \frac{y(-y_1)}{b^4} = 1,$$

and this is a tangent at $(-x_1, -y_1)$ to the original hyperbola. Hence, *the polar of any point of a hyperbola w. r. to the conjugate hyperbola touches the original hyperbola at the other end of the diameter through the given point.*

97. Freedom Equations. It can easily be seen that the point $(a \cosh\theta, b \sinh\theta)$ lies on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

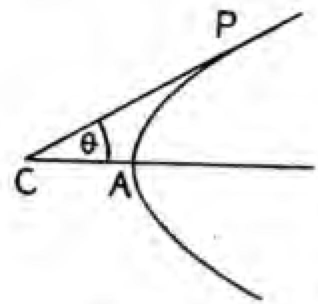
Thus the freedom equations of the hyperbola are

$$x = a\cosh\theta,\ y = b\sinh\theta. \qquad ......(22)$$

As θ varies from $-\infty$ to ∞, the point describes the whole of one branch of the curve, the other branch is expressed by the equations

$$x = -a\cosh\theta,\ y = -b\sinh\theta. \qquad ......(22A)$$

Since $\cosh(\theta+i\pi) = -\cosh\theta$, $\sinh(\theta+i\pi) = -\sinh\theta$, the parameters of the extremities of a diameter of a hyperbola differ by $i\pi$.



To interpret θ, let ACP be a sector, the point P being $(a \cosh \theta, b \sinh \theta)$

The area of the sector ACP is

$$\tfrac{1}{2}\int (xdy - ydx)$$

$$= \tfrac{1}{2}\, ab \int (\cosh^2 \theta - \sin^2 h^2\theta)d\theta$$

$$= \tfrac{1}{2}\, ab \int d\theta = \tfrac{1}{2}\, ab\theta.$$

Thus θ is proportional to the area of the sector ACP.

Exercises XXIV

1. If $P(a \cosh \phi, b \sinh \phi)$ be a point of hyperbola (1) show that $Q(a \sinh \phi, b \cosh \phi)$ is a point on the conjugate hyperbola and if C be the common centre, then CP, CQ constitute a pair of conjugate diameters.

2. Prove that $CP^2 - CQ^2 = a^2 - b^2$.

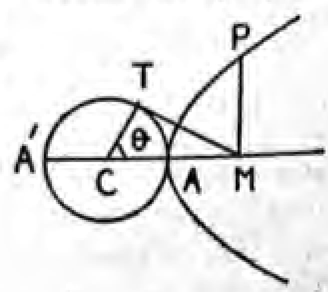
3. Show that the mid-point of PQ lies on an asymptote.

97.1. The points on the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ can be expressed by means of circular functions of an angle by the equations

$$x = a \sec \theta,\ y = b \tan \theta \qquad \text{......(23)}$$

Let P be a point on the hyperbola. From its projection M on the x-axis draw a tangent MT to its auxiliary circle. Join the centre C to T and $\angle ACT = \theta$. Obviously

$$x = CM = a \sec \theta.$$



Since P lies on the hyperbola,

$$\frac{a^2 \sec^2\theta}{a^2} - \frac{y^2}{b^2} = 1$$

$\therefore\ y = b \tan \theta.$

As θ varies from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$, one branch of the curve is described and as θ varies from $\dfrac{\pi}{2}$ to $\dfrac{3\pi}{2}$, the other branch is described.

Exercises XXV

1. *Show that the equation of the chord joining the points θ_1 and θ_2 is*

$$\frac{x}{a} \cos \frac{1}{2}(\theta_1 - \theta_2) = \frac{y}{b} \sin \frac{1}{2}(\theta_1 + \theta_2) + \cos \frac{1}{2}(\theta_1 + \theta_2)$$

Deduce that the equations of the tangent and normal at θ are respectively

$$\frac{x}{a}=\cos\ \theta+\frac{y}{b}\sin\theta$$

$$ax+\frac{by}{\sin\theta}=\frac{a^2+b^2}{\cos\theta}.$$

2. *Show that if the normals at $\theta_1, \theta_2, \theta_3$ are concurrent* $\sin(\theta_2+\theta_3)+\sin(\theta_3+\theta_1)+\sin(\theta_1+\theta_2)=0$, *and conversely.*

The equations of the normals at θ_i $(i=1, 2, 3)$ can be written in the form

$+ax$

$$by(t_i^4-1)+2t_i(a^2+b^2-ax)+2t_i^3(a^2+b^2)=0, \quad i=1,\ 2,\ 3$$

where $t_i=\tan\dfrac{\theta_i}{2}$. The *necessary* and *sufficient* condition that these normals may meet in a point is

$$\begin{vmatrix} t_1^4-1 & t_1 & t_1^3 \\ t_2^4-1 & t_2 & t_2^3 \\ t_3^4-1 & t_3 & t_3^3 \end{vmatrix}=0.$$

This reduces, after discarding the non-zero factor $(t_1-t_2)(t_2-t_3)(t_3-t_1)$, to

$$\sum_1^3\left(t_2t_3-\frac{1}{t_2t_3}\right)=0,$$

which can easily be put in the desired form.

3. *Show that from an arbitrary point four normals can be drawn to a hyperbola. If t_i $(i=1, 2, 3, 4)$ be the parameters of the feet of four concurrent normals, show that $\Sigma t_1t_2=0$ $t_1t_2t_3t_4=-1$ and conversely.*

4. *Show that a necessary condition that the normals at $\theta_1, \theta_2, \theta_3, \theta_4$ may be concurrent is $\Sigma\theta_1=(2n+1)\pi$.*

5. *Show that the necessary and sufficient condition that the points $\theta_1, \theta_2, \theta_3, \theta_4$ may be concyclic is $\Sigma\theta_1=2n\pi$.*

97.2. A third parametric representation of the hyperbola, which is most workable is

$$x=\frac{a}{2}\left(t+\frac{1}{t}\right),\ y=\frac{b}{2}\left(t-\frac{1}{t}\right) \qquad \text{......(24)}$$

This can be deduced by writing the equation of the hyperbola in the form

$$\frac{x}{a}+\frac{y}{b}=\frac{1}{\dfrac{x}{a}-\dfrac{y}{b}}=t \quad \text{(say)}$$

$$\therefore \qquad \frac{x}{a}+\frac{y}{b}=t$$

$$\frac{x}{a}-\frac{y}{b}=\frac{1}{t},$$

from which the required freedom equations follow.

The equation of the chord joining t_1, t_2 is

$$\frac{x}{a}(1+t_1 t_2)+\frac{y}{b}(1-t_1 t_2)=t_1+t_2.$$

The equation of the tangent at t is

$$\frac{x}{a}(1+t^2)+\frac{y}{b}(1-t^2)=2t$$

and that of the normal is

$$2at(1-t^2)x-2bt(1+t^2)y=(a^2+b^2)(1-t^4)$$

Exercises XXVI

1. *Show that the necessary and sufficient conditions that the normals at t_1, t_2, t_3, t_4 may be concurrent are*

$$\Sigma t_1 t_2=0, \quad t_1 t_2 t_3 t_4=-1.$$

2. *The necessary and sufficient condition for the concurrence of the normals at t_1, t_2, t_3 is*

$$\Sigma t_2 t_3=\Sigma\frac{1}{t_2 t_3}.$$

3. *The necessary and sufficient condition that the points t_1, t_2, t_3, t_4 may be concyclic is $t_1 t_2 t_3 t_4=1$.*

4. If we write the equation of the hyperbola in the form

$$\frac{\dfrac{x}{a}+1}{\dfrac{y}{b}}=\frac{\dfrac{y}{b}}{\dfrac{x}{a}-1}=t,$$

show that $x=a\dfrac{t^2+1}{t^2-1}$, $y=\dfrac{2bt}{t^2-1}$,

are the freedom equations of the hyperbola.

97.3. The freedom equations of the hyperbola $xy=c^2$ are

$$\frac{x}{c}=\frac{c}{y}=t$$

i.e., $\qquad x = ct,\ y = \dfrac{c}{t}$ ......(25)

The equation of the line that joins the points t_1 and t_2 is

$$x + t_1 t_2 y - c(t_1 + t_2) = 0$$

from which can be easily deduced the equation of the tangent *at* 't'

$$x + t^2 y = 2ct.$$

Exercises XXVII

1. Show that the polar of (x_1, y_1) *w. r.* to $xy = c^2$ is

$$xy_1 + x_1 y = 2c^2.$$

2. Show that the normal at 't' to the rectangular hyperbola $xy = c^2$ is given by the equation

$$xt^3 \quad ty - ct^4 + c = 0.$$

Deduce the necessary and sufficient conditions

$$\Sigma t_1 t_2 = 0\ ;\ t_1 t_2 t_3 t_4 = -1$$

for the co-normality of the four points t_1, t_2, t_3, t_4.

Illustrative Examples

(1) *Show that the common tangents of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

and the rectangular hyperbola $xy = c^2$ *are the lines*

$$\{4c^2x^2 + a^2(2xy - 4c^2)\}\{4c^2y^2 + b^2(2xy - 4c^2)\} + (b^2x^2 - a^2y^2)^2 = 0.$$

The line

$$x + t^2 y - 2ct = 0,$$

is for all values of t, a tangent to the hyperbola $xy = c^2$.

This will be a tangent to the given ellipse if

$$(a^2l^2 + b^2m^2 = n^2)$$
$$a^2 + b^2t^4 = 4c^2t^2.$$

The equation of the tangents can therefore be obtained by the elimination of t between these two equations which can be written in the form

$$t^4y^2 + t^2(2xy - 4c^2) + x^2 = 0$$
$$t^4b^2 - 4c^2t^2 + a^2 = 0$$

$$\therefore \quad \frac{t^4}{4c^2x^2 + a^2(2xy - 4c^2)} = \frac{t^2}{b^2x^2 - a^2y^2}$$

$$= \frac{1}{-\{4c^2y^2 + b^2(2xy - 4c^2)\}}$$

Thus the equation of the tangents is

$$\{4c^2x^2 + a^2(2xy - 4c^2)\}\{4c^2y^2 + b^2(2xy - 4c^2)\} + (b^2x^2 - a^2y^2) = 0.$$

(2) *Find the locus of the pole of a chord of a hyperbola which subtends a right angle at a fixed point.*

Let the equation of the hyperbola be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The polar of the point (x', y') *w. r.* to the hyperbola is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=1.$$

Shift the origin to the point (h, k). The equations of the hyperbola and the polar take the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2hx}{a^2}-\frac{2ky}{b^2}+\lambda=0$$

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}+\mu=0$$

where $\lambda=\frac{h^2}{a^2}-\frac{k^2}{b^2}-1,\qquad \mu=\frac{hx'}{a^2}-\frac{ky'}{b^2}-1.$

The equation of the lines that join the new origin with the intersections of the polar and the hyperbola is

$$\mu^2\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)-2\mu\left(\frac{hx}{a^2}-\frac{ky}{b^2}\right)\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}\right)+\lambda\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}\right)^2=0$$

These lines will be at right angles if

$$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\mu^2-2\mu\left(\frac{hx'}{a^4}+\frac{ky'}{b^4}\right)+\lambda\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)=0,\ i.e.,$$

$$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)\left(\frac{hx'}{a^2}-\frac{ky'}{b^2}-1\right)^2-2\left(\frac{hx'}{a^2}-\frac{ky'}{b^2}-1\right)\left(\frac{hx'}{a^4}+\frac{ky'}{b^4}\right)$$

$$+\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}-1\right)\left(\frac{x'^2}{a^4}+\frac{y'^2}{b^4}\right)=0$$

Hence the locus of (x', y') is

$$\frac{x^2}{a^2}(h^2+k^2+b^2)-\frac{y^2}{b^2}(h^2+k^2-a^2)-2(hx+ky)+(a^2-b^2)=0.$$

Remarks. The equation can be written in the form

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)(h^2+k^2-a^2+b^2)+(x-h)^2+(y-k)^2=0.$$

If the point (h, k) lies on the director circle of the hyperbola, the locus breaks up into circular lines through (h, k). If, however, the point (h, k) lies on the hyperbola, let $h=a\sec\theta$, $k=b\tan\theta$, the equation of the locus becomes

$$\frac{a^2+b^2}{a^2}x^2\sec^2\theta-\frac{a^2+b^2}{b^2}y^2\tan^2\theta-2a\sec\theta\,x-2b\tan\theta\,y$$

$$+a^2-b^2=0.$$

or $\dfrac{x^2}{a^2}\sec^2\theta - \dfrac{y^2}{b^2}\tan^2\theta - \dfrac{2ax\sec\theta}{a^2+b^2} - \dfrac{2by}{a^2+b^2}\tan\theta$

$$+\frac{a^2-b^2}{a^2+b^2}=0$$

i. e. $\left(\dfrac{x}{a}\sec\theta - \dfrac{a^2}{a^2+b^2}\right)^2 - \left(\dfrac{y}{b}\tan\theta + \dfrac{b^2}{a^2+b^2}\right)^2=0$

which is also a pair of lines. One of these lines is the tangent at the point $a\sec\theta$, $b\tan\theta$. It is only in these two cases when the locus breaks up into a pair of lines as can readily be seen by writing the discriminant of the locus, *viz.*,

not
$$\left(\frac{h^2}{a^2}-\frac{k^2}{b^2}-1\right)\left(h^2+k^2-a^2+b^2\right).$$

(3) *A triangle is circumscribed about the circle* $x^2+y^2=r^2$ *and two of its angular points lie on the hyperbola* $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ *prove that the third angular point lies on the ellipse*

$$\frac{x^2}{(a^2r^2+b^2r^2-a^2b^2)^2}+\frac{y^2}{(a^2r^2+b^2r^2+a^2b^2)^2}=\frac{r^2}{(a^2r^2-b^2r^2+a^2b^2)^2}.$$

Let the equations of the three sides be

$x\cos\alpha+y\sin\alpha-r=0$, $x\cos\beta+y\sin\beta-r=0$,

$$x\cos\gamma+y\sin\gamma-r=0.$$

The co-ordinates of the three vertices are

$$\left\{\frac{r\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},\ \frac{r\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}}\right\}\ \left\{\frac{r\cos\frac{(\alpha+\gamma)}{2}}{\cos\frac{\alpha-\gamma}{2}},\ \frac{r\sin\frac{\alpha+\gamma}{2}}{\cos\frac{\alpha-\gamma}{2}}\right\}$$

$$\left\{\frac{r\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}},\ \frac{r\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}}\right\}$$

Suppose that the first vertex lies on the hyperbola,

$$\therefore\quad b^2r^2\cos^2\frac{\alpha+\beta}{2}-a^2r^2\sin^2\frac{\alpha+\beta}{2}=a^2b^2\cos^2\frac{\alpha-\beta}{2}$$

or $(b^2+a^2)r^2\cos(\alpha+\beta)-a^2b^2\cos(\alpha-\beta)=a^2r^2-b^2r^2+a^2b^2$

i. e. $\lambda\cos\alpha\cos\beta-\mu\sin\alpha\sin\beta-v=0$ ......(i)

where $\lambda=a^2r^2+b^2r^2-a^2b^2$, $\mu=a^2r^2+b^2r^2+a^2b^2$,

$$v=a^2r^2-b^2r^2+a^2b^2.$$

Similarly the second vertex lies on the same hyperbola if

$\lambda\cos\alpha\cos\gamma-\mu\sin\alpha\sin\gamma-v=0$ ......(ii)

From equations (i) and (ii)

$$\frac{\lambda \cos \alpha}{\sin \beta - \sin \gamma} = \frac{\mu \sin \alpha}{\cos \beta - \cos \gamma} = \frac{v}{\sin(\beta - \gamma)}$$

or $$\frac{\lambda \cos \alpha}{\cos \frac{\beta+\gamma}{2}} = \frac{\mu \sin \alpha}{-\sin \frac{\beta+\gamma}{2}} = \frac{v}{\cos \frac{\beta-\gamma}{2}}$$

$$\therefore \frac{1}{\lambda^2} \cos^2 \frac{\beta+\gamma}{2} + \frac{1}{\mu^2} \sin^2 \frac{\beta+\gamma}{2} = \frac{1}{v^2} \cos^2 \frac{\beta-\gamma}{2}$$

Thus the locus of the third vertex is the ellipse

$$\frac{x^2}{\lambda^2} + \frac{y^2}{\mu^2} = \frac{r^2}{v^2}.$$

(4) *From a variable point on the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, *two lines are drawn touching the rectangular hyperbola* $4xy = c^2$ *and meeting the ellipse again in* Q *and* R. *Prove that the envelope of* QR *is the conic*

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)(c^4 + a^2b^2) - 4c^2xy = a^2b^2.$$

Let $(a \cos \phi, b \sin \phi)$ be the co-ordinates of P. The equation of the pair of tangents from P to $4xy = c^2$ is

$$(4xy - c^2)(4ab \cos \phi \sin \phi - c^2) = [2(bx \sin \phi + ay \cos \phi) - c^2]^2$$

Any conic through the points P, P, Q, R is

$$[2(bx \sin \phi + ay \cos \phi) - c^2]^2 - (4xy - c^2)(4ab \cos \phi \sin \phi - c^2)$$
$$+ \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) = 0.$$

This will break for some values of λ into pairs of st. lines, one such pair will consist of the tangent at P and the line QR, the equation of which we assume to be $lx + my + n = 0$. Thus the conic for some value of λ is identical with

$$\left(\frac{x \cos \phi}{a} + \frac{y \sin \phi}{b} - 1\right)(lx + my + n) = 0.$$

Comparing the co-efficients of x, y, xy, we have

$$\rho\left(-l + \frac{n \cos \phi}{a}\right) = -4c^2 b \sin \phi, \quad \rho\left(-m + \frac{n \sin \phi}{b}\right)$$
$$= -4c^2 a \cos \phi,$$

$$\rho\left(\frac{l \sin \phi}{b} + \frac{m \cos \phi}{a}\right) = 4c^2 - 8ab \sin \phi \cos \phi.$$

$$\therefore \rho l = 4b(c^2 \sin \phi - ab \cos \phi), \quad \rho m = 4a(c^2 \cos \phi - ab \sin \phi)$$
$$\rho n = -4a^2b^2.$$

Thus the equation of QR takes the form

$$F(\phi) \equiv a \cos \phi \, (b^2x - c^2y) + b \sin \phi \, (a^2y - c^2x) + a^2b^2) = 0.$$
$$\therefore F'(\phi) = -a \sin \phi \, (b^2x - c^2y) + b \cos \phi \, (a^2y - c^2x) = 0.$$

Squaring and adding we get the required envelope

$$a^2(b^2x-c^2y)^2+b^2(a^2y-c^2x)^2=a^4b^4$$

which can be put in the required form.

Remarks. 1. The equation of the envelope can be obtained thus :—Write $t=\tan\dfrac{\phi}{2}$. The equation of the line takes the form

$$t^2[a^2b^2-a(b^2x-c^2y)]+2bt(a^2y-c^2x)+[a^2b^2+a(b^2x-c^2y)]=0$$

from which the equation of the envelope follows, by equating to zero the discriminant of the equation in 't'.

2. The equation of the envelope can also be written in the form

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)(c^4+a^2b^2)-c^2(4xy-c^2)=0$$

which shows that the conic passes through the intersections of the given conics.

Miscellaneous Exercises XXVIII

1. Show that the eccentricity of a rectangular hyperbola is $\sqrt{2}$.

2. From first principles, find the diameter conjugate to the diameter $y=x$ with respect to the hyperbola $x^2-4y^2=4$.

3. From first principles, find the diameter conjugate to $x=2y$ *w. r.* to the hyperbolas $\dfrac{x^2}{16}-\dfrac{y^2}{9}=1$, $\dfrac{-x^2}{16}+\dfrac{y^2}{9}=1$.

4. Find *ab initio* the asymptotes of the conic

$$25x^2-16y^2=400.$$

5. Find the foci and directrices of the rectangular hyperbola $2xy=a^2$.

6. From definition only, show that the lines $2x-3y=6$, $4x-3y=6$ are conjugate lines *w. r.* to the hyperbola

$$\frac{x^2}{9}-\frac{y^2}{4}=1.$$

7. If the two hyperbolas

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,\quad \frac{x^2}{a_1^2}-\frac{y^2}{b_1^2}=1$$

have the same eccentricity, prove that $a:a_1=b:b_1$.

8. A st. line cuts an hyperbola at P and p, and the asymptotes at Q and q ; show that PQ $=pq$.

9. Show that the line $y=mx+2c\sqrt{-m}$ always touches the hyperbola $xy=c^2$ at the point $\left(\dfrac{c}{\sqrt{-m}}, c\sqrt{-m}\right)$.

$y = mx + d$

10. Show that, if perpendiculars be drawn from each point of the line $x=k$ to its polar with respect to the conic $ax^2+by^2=1$, the locus of the feet of the perpendiculars is a circle, which passes through the pole of the line and the intersections of the line with the conic.

11. Show that the locus of the poles of tangents to the conic $a_1x^2+b_1y^2=1$ *w. r.* to the conic $ax^2+by^2=1$, is the conic

$$\frac{a^2x^2}{a_1}+\frac{b^2y^2}{b_1}=1.$$

12. Show that the polar of any point on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *w. r.* to $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ will touch $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

13. Prove that the locus of the pole *w. r.* to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ of any tangent to the circle, whose diameter is the line joining the foci, is the ellipse $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$.

14. Show that the polars of any point *w. r.* to all conics $ax^2+\lambda y^2=1$ where λ is a variable parameter pass through a fixed point.

15. P is any point on the line $lx+my=1$ and the polars of P *w. r.* to the circle $x^2+y^2=R^2$ and the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ meet at Q. Prove that the locus of Q is the conic

$$\left(\frac{x}{R^2}-l\right)\left(\frac{y}{b^2}+m\right)+\left(\frac{x}{a^2}-l\right)\left(\frac{y}{R^2}-m\right)=0.$$

16. From points on the circle $x^2+y^2=a^2$, tangents are drawn to the hyperbola $x^2-y^2=a^2$; prove that the locus of the mid-points of the chords of contact is the curve

$$(x^2-y^2)^2=a^2(x^2+y^2).$$

17. A straight line has its extremities on two fixed st. lines, and passes through a fixed point, show that the locus of the mid-point of the line is a hyperbola.

18. A st. line has its extremities on two fixed st. lines and cuts off from them a triangle of constant area, show that the locus of the mid-point of the line is a hyperbola.

19. The lines $x-\alpha=0$, $y-\beta=0$ are conjugate *w. r.* to the hyperbola $xy=c^2$. Prove that the locus of (α, β) is the hyperbola $xy=2c^2$.

20. If two vertices of a self-conjugate triangle $w.\ r.$ to the conic $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ lie on the lines $y=\pm x$, show that the third vertex lies on the conic

$$\frac{x^2}{a^4}-\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

21. Show that the poles of the line $lx+my+n=0$ $w.\ r.$ to the conics $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\lambda$ where λ is a variable parameter lie on the diameter conjugate to $lx+my=0$.

22. Show that the points of intersection of the polars of all points on the line $\frac{x}{l}=\frac{y}{m}$ $w.\ r.$ to the hyperbolas

$x^2-y^2=a^2$, $2xy=a^2$ lie on the line $l(x-y)=m(x+y)$.

23. Show that the points of intersection of the polars of all points on the circle $x^2+y^2=r^2$, $w.\ r.$ to the hyperbolas $x^2-y^2=a^2$, $2xy=a^2$ lie on the circle

$$x^2+y^2=\frac{2a^4}{r^2}.$$

24. Show that the inclination θ to the axis of x of the tangents drawn from the point $(p,\ q)$ to the conic $ax^2+by^2=1$ are determined by the equation

$$(ap^2+bq^2-1)(a+b\tan^2\theta)=(ap+bq\tan\theta)^2.$$

25. Four points are taken on a rectangular hyperbola $xy=c^2$. Find the condition that the chord joining two of the points shall be perpendicular to that joining the other two. Prove that this holds good for all the three pairs of chords if it is true for one. (Math. Trip I 1918)

98. Apollonius Hyperbola. The normal at the point $(x_1,\ y_1)$ to the conic

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

is

$$L\equiv\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2$$

This passes through the point $O(x_0,\ y_0)$ if

$$\frac{a^2x_0}{x_1}-\frac{b^2y_0}{y_1}=a^2-b^2$$

which shows that the foot $(x_1,\ y_1)$ of the normal L lies on the hyperbola

$$H'\equiv\frac{a^2x_0}{x}-\frac{b^2y_0}{y}=a^2-b^2 \qquad \ldots\ldots(26)$$

or $\quad H \equiv (a^2-b^2)xy+b^2y_0x-a^2x_0y=0 \qquad \ldots\ldots(27)$

This is a rectangular hyperbola of Apollonius which contains the feet of the normals that meet at $O(x_0, y_0)$, since the feet also lie on S, and the two conics S and H meet in general in four points, it follows that there exist four normals to the conic S which pass through a point O.

The hyperbola H passes through the centre of S and also through the point O.

98.1. Suppose the normals at the intersections of the lines

$$U_i \equiv l_i x + m_i y + n_i = 0 \qquad i=1, 2$$

with the conic S meet at the point $O(x_0, y_0)$. Then the conic

$$b^2x^2+a^2y^2-a^2b^2+\lambda(l_1x+m_1y+n_1)(l_2x+m_2y+n_2)=0,$$

which passes through the feet of the four normals through O, will, for some value of λ, be identical with the conic H

$$\therefore \quad b^2+\lambda l_1l_2=0, \quad a^2+\lambda m_1m_2=0, \quad -a^2b^2+\lambda n_1n_2=0.$$

Thus

$$a^2l_1l_2=b^2m_1m_2=-n_1n_2 \qquad \ldots\ldots(28)$$

are the necessary and sufficient conditions for the concurrence of the normals at the extremities of the chords U_1 and U_2.

98.2. Let (x_1, y_1), (x_2, y_2) be the poles of the lines U_1 and U_2. The lines U_1 and U_2 are therefore identical with

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \qquad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}=1.$$

Thus

$$\frac{x_1}{a^2l_1}=\frac{y_1}{b^2m_1}=-\frac{1}{n_1}; \quad \frac{x_2}{a^2l_2}=\frac{y_2}{b^2m_2}=-\frac{1}{n_2}.$$

The conditions (28) become

$$x_1x_2+a^2=0, \; y_1y_2+b^2=0. \qquad \ldots\ldots(29)$$

98.3. Suppose that the normals drawn at the intersections of the polars of (x_1, y_1), (x_2, y_2) with the conic S meet at $O(x_0, y_0)$, then with conditions (29), the feet of the normals lie on the rectangular hyperbola

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}-1\right)=0.$$

or $\quad xy(x_1y_2+x_2y_1)-b^2(x_1+x_2)x-a^2(y_1+y_2)y=0.$

This hyperbola is, therefore, identical with the conic H.

$$\therefore \quad \frac{x_1y_2+x_2y_1}{a^2-b^2}=-\frac{x_1+x_2}{y_0}=\frac{y_1+y_2}{x_0} \qquad \ldots\ldots(30)$$

Thus

$$x_0 = -\frac{(a^2-b^2)x_1(y_1^2-b^2)}{a^2y_1^2+b^2x_1^2},\ y_0 = \frac{(a^2-b^2)y_1(x_1^2-a^2)}{a^2y_1^2+b^2x_1^2} \quad \ldots (31)$$

The equations (30) and (31) connect the *tangential poles* with the *normal poles.*

98.4. Joachimsthal's Theorem. *The normals at the points* A, B, C, D *meet in* $O(x_0, y_0)$. *Show that the circle through* A, B, C *meets the conic again in a point diametrically opposite to* D.

Find also the equation of the circle ABC if the co-ordinates of D are (x_1, y_1).

The theorem has already been proved. For an alternative proof, we notice if D and D′ are the extremities of a diameter and B a point on the conic, the chords BD, BD′ are supplemental and therefore parallel to a pair of conjugate diameters and conversely.

If the equation of AB be $lx+my+1=0$, the equation of CD is $\frac{x}{a^2l}+\frac{y}{b^2m}-1=0$ (equations 28). Since ABCD′ are concyclic, CD′ is parallel to $lx-my=0$, for AB and CD′ are equally inclined to the axes.

But $lx-my=0$ and $\frac{x}{a^2l}+\frac{y}{b^2m}=0$ are conjugate diameters of the conic S.

Thus CD and CD′ are supplemental chords and consequently D, D′ are the extremities of a diameter.

To find the equation of the circle ABC, we notice that it passes through $D(-x_1, -y_1)$. Thus if the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0,$$

then $$x_1^2+y_1^2-2gx_1-2fy_1+c=0$$

Thus the equation of the circle becomes

$$(x+x_1)(x-x_1+2g)+(y+y_1)(y-y_1+2f)=0 \quad \ldots\ldots (32)$$

The equation of the conic being

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \text{ with } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1=0$$

$$\therefore \quad \frac{(x-x_1)(x+x_1)}{a^2}+\frac{(y-y_1)(y+y_1)}{b^2}=0$$

Thus a conic through ABC is given by the equation

$$a^2(y-y_1)(x-x_1+2g)=b^2(x-x_1)(y-y_1+2f).$$

This conic is a rectangular hyperbola and passes through the point D. It is thus identical with the hyperbola H.

$\therefore \quad x_1(a^2-b^2)-2a^2g=a^2x_0$

$y_1(a^2-b^2)-2b^2f=-b^2y_0$

and $\quad (a^2-b^2)x_1y_1+2b^2fx_1-2a^2g\,y_1=0$

Hence $\quad 2g-x_1=-x_0-\dfrac{b^2}{a^2}x_1$

$$2f-y_1=-y_0-\frac{a^2}{b^2}y_1$$

The equation (32) of the circle ABC becomes

$$\left(x+x_1\right)\left(x-x_0-\frac{b^2}{a^2}x_1\right)+\left(y+y_1\right)\left(y-y_0-\frac{a^2}{b^2}y_1\right)=0 \quad\ldots\ldots(33)$$

98.5. Second Proof of Joachimsthal's Theorem.

From Art. 98 (27).

$$\frac{a^2x_0}{x}+b^2=\frac{b^2y_0}{y}+a^2,\ \frac{a^2x_0}{x_1}+b^2=\frac{b^2y_0}{y_1}+a^2=h \text{ say}$$

we get $\quad \dfrac{a^2x_0(x-x_1)}{xx_1}=\dfrac{b^2y_0(y-y_1)}{yy_1}$

The equation of the conic can be written in the form

$$\frac{(x-x_1)(x+x_1)}{a^2}=-\frac{(y-y_1)(y+y_1)}{b^2}$$

since $D(x_1, y_1)$ is a point on the conic. Division by the corresponding members gives the equation

$$\frac{xx_1(x+x_1)}{a^4x_0}+\frac{yy_1(y+y_1)}{b^4y_0}=0$$

of a conic through ABC but not D. Thus the general equation of the conic through ABC is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left[\frac{xx_1(x+x_1)}{a^4x_0}+\frac{yy_1(y+y_1)}{b^4y_0}\right]$$
$$+\mu[(a^2-b^2)xy+b^2y_0x-a^2x_0y]=0.$$

This will be a circle if $\mu=0$ and

$$\frac{1}{a^2}+\frac{\lambda x_1}{a^4x_0}=\frac{1}{b^2}+\frac{\lambda y_1}{b^4y_0}$$

or $\quad \dfrac{1}{a^2}+\dfrac{\lambda}{a^2}\ \dfrac{1}{h-b^2}=\dfrac{1}{b^2}+\dfrac{\lambda}{b^2(h-a^2)}$

$\therefore \quad (h-a^2)(h-b^2)=-\lambda h.$

With this value of λ, the equation of the circle becomes

$$h\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)-\left(h-a^2\right)\left(h-b^2\right)\left[\frac{x(x+x_1)}{a^2(h-b^2)}+\frac{y(y+y_1)}{b^2(h-a^2)}\right]=0,$$

which assumes the form

$$x^2+y^2+xx_1+yy_1-h\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+1\right)=0 \qquad \ldots\ldots(34)$$

The circle obviously passes through $D'(-x_1, -y_1)$.

If the normals at A, B, C meet at D on the conic,

$$x_1=x_o,\ y_1=y_0,\ h=a^2+b^2,$$

the equation of the circle becomes

$$x^2+y^2-\frac{b^2}{a^2}xx_o-\frac{a^2}{b^2}yy_0-a^2-b^2=0. \qquad \ldots\ldots(35)$$

Illustrative Examples

(1) *From any point on the hyperbola* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, *three normals other than the one at the point are drawn, show that the centroid of the triangle formed by the feet of the three normals lies on the hyperbola*

$$9\left[\frac{x^2}{a^2}-\frac{y^2}{b^2}\right]=\left[\frac{a^2-b^2}{a^2+b^2}\right]^2.$$

Let the normal at a point 't'

$$\left[\frac{a}{2}\left(t+\frac{1}{t}\right),\ \frac{b}{2}\left(t-\frac{1}{t}\right)\right]$$

meet the hyperbola at a point (x_0, y_0) *i. e.* $x_0=\frac{a}{2}\left(t_0+\frac{1}{t_0}\right)$

$y_0=\frac{b}{2}\left(t_0-\frac{1}{t_0}\right)$. Hence the parameters of the feet of the normals are the roots of the equation

$$(a^2+b^2)t^4-2t^3(ax_0+by_0)+2t(ax_0-by_0)-(a^2+b^2)=0.$$

If t_1, t_2, t_3, t_0 are the roots of the equation

$$t_1+t_2+t_3+t_0=\frac{2(ax_0+by_0)}{a^2+b^2}$$

$$\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}+\frac{1}{t_0}=\frac{2(ax_0-by_0)}{a^2+b^2}$$

$$\therefore\ \left(t_1+t_2+t_3+\frac{1}{t_1}+\frac{1}{t_2}+\frac{1}{t_3}\right)=\frac{4ax_0}{a^2+b^2}-\left(t_0+\frac{1}{t_0}\right)$$

$$=\frac{2a^2}{a^2+b^2}\left(t_0+\frac{1}{t_0}\right)-\left(t_0+\frac{1}{t_0}\right)$$

$$=\frac{a^2-b^2}{a^2+b^2}\left(t_0+\frac{1}{t_0}\right).$$

Similarly $\left(t_1+t_2+t_3-\dfrac{1}{t_1}-\dfrac{1}{t_2}-\dfrac{1}{t_3}\right)$

$$=\frac{a^2-b^2}{a^2+b^2}\left(t_0-\frac{1}{t_0}\right)$$

If (x, y) be the co-ordinates of the centroid,

$$3x=\frac{a^2-b^2}{a^2+b^2}\,\frac{a}{2}\left(t_0+\frac{1}{t_0}\right),\quad 3y=\frac{a^2-b^2}{a^2+b^2}\,\frac{b}{2}\left(t_0-\frac{1}{t_0}\right)$$

Hence the locus of (x, y) is the hyperbola

$$9\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)=\left(\frac{a^2-b^2}{a^2+b^2}\right)^2.$$

(2) *Find the equation of the chords through (h, k), the normals at the intersections of which with the conic $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$ are concurrent.*

Let the joint equation of the chords be

$$l(x-h)^2+2m(x-h)(y-k)+n^2(y-k)^2=0.$$

The conic given by the equation

$$b^2x^2+a^2y^2-a^2b^2+\lambda\,[l(x-h)^2+2m(x-h)(y-k)+n(y-k)^2]=0$$

passes through the feet of the concurrent normals, and is thus identical with the Apollonius hyperbola, hence

$$b^2+\lambda l=0,\ a^2+\lambda n=0,\ -a^2b^2+\lambda(lh^2+2mhk+nk^2)=0,$$

$\therefore$ $\lambda l=-b^2,\ \lambda n=-a^2$

and $a^2b^2=-b^2h^2-a^2k^2+2mkh\lambda.$

Thus the equation of the chords is

$$b^2(x-h)^2\,hk-(a^2b^2+b^2h^2+a^2k^2)(x-h)(y-k)$$
$$+a^2(y-k)^2\,hk=0.$$

The result can be put in a simpler form

$$b^2h(x-h)(kx-hy)-a^2b^2(x-h)(y-k)-a^2k(y-k)(kx-hy)=0$$

i. e. $\dfrac{h}{a^2(y-k)}-\dfrac{k}{b^2(x-h)}-\dfrac{1}{kx-hy}=0.$

(3) *From a point P are drawn four normals to the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The tangents at the feet of the normals form a quadrilateral. Find the locus of P if a circle can be inscribed in the quadrilateral. Find also the locus of the centre of the circle.

Let P be (α, β), and suppose (ξ, η) is a foot of a normal, then the tangent at (ξ, η) is

$$\frac{x\xi}{a^2}+\frac{y\eta}{b^2}-1=0.$$

This will be a tangent to a circle with centre (x_0, y_0) and radius R if

$$\left(\frac{\xi x_0}{a^2}+\frac{\eta y_0}{b^2}-1\right)^2-R^2\left(\frac{\xi^2}{a^4}+\frac{\eta^2}{b^4}\right)=0.$$

Thus the feet of the normals as (ξ, η) lie on the conic

$$\left(\frac{xx_0}{a^2}+\frac{yy_0}{b^2}-1\right)^2-R^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)=0.$$

Hence for a certain value of λ, the conic

$$\left(\frac{xx_0}{a^4}+\frac{yy_0}{b^4}-1\right)^2-R^2\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)-\lambda\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)=0$$

will be the Apollonius hyperbola $c^2xy+b^2\beta x-a^2\alpha y=0$.

$$c^2=a^2-b^2.$$

$$\therefore \qquad \frac{x_0^2}{a^4}-\frac{R^2}{a^4}-\frac{\lambda}{a^2}=0$$

$$\frac{y_0^2}{b^4}-\frac{R^2}{b^4}-\frac{\lambda}{b^2}=0$$

$$1 \qquad +\lambda=0$$

$$\therefore \begin{vmatrix} x_0^2 & 1 & a^2 \\ y_0^2 & 1 & b^2 \\ 1 & 0 & -1 \end{vmatrix}=0$$

or $\quad (x_0^2+a^2)-(y_0^2+b^2)=0.$

Hence the locus of the centre of the circle is

$$x^2-y^2+a^2-b^2=0.$$

Also comparing with the Apollonius hyperbola

$$\frac{x_0y_0}{c^2}=\frac{-x_0}{\beta}=\frac{y_0}{\alpha},$$

$$\therefore \qquad y_0=\frac{-c^2}{\beta}, \quad x_0=\frac{c^2}{\alpha},$$

$$\therefore \qquad \frac{1}{\alpha^2}-\frac{1}{\beta^2}+\frac{1}{c^2}=0.$$

So the locus of (α, β) is the curve

$$\frac{1}{x^2}-\frac{1}{y^2}+\frac{1}{c^2}=0.$$

(4) *From a point on the ellipse* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1=0$, *three normals, other than the normal at the point are drawn, and*

their feet are A, B, C. Show that the sides of the triangle ABC touch the ellipse

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2-b^2)^2}.$$

Suppose $D(x_0, y_0)$ is the point from which the normals are drawn. The equations of AB and CD can be taken in the form

$$blx+amy+ab=0,\ \frac{bx}{l}+\frac{ay}{m}-ab=0.$$

The curve

$$\left(blx+amy+ab\right)\left(\frac{bx}{l}+\frac{ay}{m}-ab\right)-b^2x^2-a^2y^2+a^2b^2=0$$

passes through the feet of the four normals and is an equilateral hyperbola. It is, therefore, identical with the rectangular hyperbola H. Comparing

$$ab\left(\frac{l}{m}+\frac{m}{l}\right)=\lambda\,(a^2-b^2)=\lambda c^2 \quad \text{where } c^2=a^2-b^2$$

$$a\left(l-\frac{1}{l}\right)=-\lambda y_0,\ b\left(m-\frac{1}{m}\right)=\lambda x_0,$$

and $$\frac{bx_0}{l}+\frac{ay_0}{m}-ab=0,$$

since D is supposed to lie on the line CD $\frac{bx}{l}+\frac{ay}{m}=ab.$

Eliminating (x_0, y_0) from the last three equations, we have

$$\frac{b^2}{l}\left(m-\frac{1}{m}\right)+\frac{a^2}{m}\left(\frac{1}{l}-l\right)=\lambda ab$$

$$=\frac{a^2b^2}{c^2}\left(\frac{l}{m}+\frac{m}{l}\right)$$

or $$\frac{a^4}{c^2}l^2+\frac{b^4}{c^2}m^2=c^2.$$

This is then the condition under which the envelope of AB is to be determined. Making this condition homogeneous in l, m with the equation of AB, we have

$$\frac{a^4l^2}{c^2}+\frac{b^4m^2}{c^2}=c^2\left(\frac{lx}{a}+\frac{my}{b}\right)^2$$

or $$l^2\left(\frac{a^4}{c^2}-\frac{c^2}{a^2}x^2\right)-\frac{2c^2}{ab}lm\,xy+m^2\left(\frac{b^4}{c^2}-\frac{c^2}{b^2}y^2\right)=0.$$

Thus the required envelope is

$$\frac{c^4}{a^2b^2}\ x^2y^2=\left(\frac{a^4}{c^2}-\frac{c^2}{a^2}\ x^2\right)\left(\frac{b^4}{c^2}-\frac{c^2}{b^2}y^2\right)$$

i.e., $$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{c^4}\ .$$

Alternative method.

Let the normal at ϕ pass through $O(\phi')$

$$\therefore\quad a^2\frac{\cos\phi'}{\cos\phi}-b^2\ \frac{\sin\phi'}{\sin\phi}=a^2-b^2$$

i.e., $$a^2\frac{\cos\phi'-\cos\phi}{\cos\phi}+b^2\ \frac{\sin\phi-\sin\phi'}{\sin\phi}=0$$

i.e., $$a^2\ \frac{\sin\frac{\phi+\phi'}{2}}{\cos\phi}+b^2\ \frac{\cos\frac{\phi+\phi'}{2}}{\sin\phi}=0$$

i. e., $$2a^2t(t+t')+b^2(1-t^2)(1-tt')=0$$

where $t=\tan\dfrac{\phi}{2}$, $t'=\tan\ \dfrac{\phi'}{2}$

rejecting $\sin\dfrac{\phi-\phi'}{2}=0$ as it gives O.

Thus the feet A, B, C of the normals from O are given by

$$b^2t't^3+(2a^2-b^2)t^2+(2a^2-b^2)t't+b^2=0.$$

If A, B, C correspond to t_1, t_2, t_3,

$$t_1+t_2+t_3=-\frac{2a^2-b^2}{b^2t'}=\frac{2a^2-b^2}{b^2}\ t_1t_2t_3=k\ t_1t_2t_3$$

$$t_1t_2+t_1t_3+t_2t_3=\frac{2a^2-b^2}{b^2}=k$$

Eliminating t_1, $(t_2+t_3)^2+k(t_2t_3)^2-(1+k^2)t_2t_3+k=0$......$(i)$

The equation to BC is

$$\frac{x}{a}\ \cos\ \frac{\phi_2+\phi_3}{2}\ +\frac{y}{b}\sin\frac{\phi_2+\phi_3}{2}=\cos\frac{\phi_2-\phi_3}{2}$$

i.e., $$\frac{x}{a}\left(1-t_2t_3\right)+\frac{y}{b}\left(t_2+t_3\right)=1+t_2t_3 \qquad ......(ii)$$

We have to find the envelope of (ii) with condition (i)

Eliminate t_2+t_3 and we get

$$\left\{\left(\frac{x}{a}-1\right)-t_2t_3\left(\frac{x}{a}+1\right)\right\}^2+$$

$$\frac{y^2}{b^2}\left\{\ k\left(\ t_2t_3\ \right)^2-\left(1+k^2\right)t_2t_3+k\ \right\}=0$$

$$i.e., \left[\left(\frac{x}{a}+1\right)^2+\frac{ky^2}{b^2}\right]\left(t_2t_3\right)^2-\left\{2\left(\frac{x^2}{a^2}-1\right)\right.$$
$$\left.+\left(1+k^2\right)\frac{y^2}{b^2}\right\}t_2t_3+\left(\frac{x}{a}-1\right)^2+\frac{ky^2}{b^2}=0.$$

It has equal roots in t_2t_3 if

$$\left\{2\left(\frac{x^2}{a^2}-1\right)+\left(1+k^2\right)\frac{y^2}{b^2}\right\}^2=$$
$$4\left\{\frac{ky^2}{b^2}+\left(\frac{x}{a}-1\right)^2\right\}\left\{\frac{ky^2}{b^2}+\left(\frac{x}{a}+1\right)^2\right\}=0.$$

Thus the envelope is given by the equation

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{1}{(a^2-b^2)^2}.$$

Miscellaneous Exercises XXIX

1. If e, e' be the eccentricities of an hyperbola and of the conjugate hyperbola, then will $\frac{1}{e^2}+\frac{1}{e'^2}=1$.

2. If A, A′ are the vertices of a rectangular hyperbola, and P is any point on the curve, show that the bisectors of the angle APA′ are parallel to the asymptotes.

3. From a point of one hyperbola, tangents are drawn to another which has the same asymptotes, show that the chord of contact cuts off a constant area from the asymptotes.

4. One circle lies completely outside another. If a variable circle move so that it touches both circles, externally, prove that the locus of the centre of the variable circle is a hyperbola, having for foci the centres of the fixed circles. Discuss the case where the contact is one internal and the other external.

5. Through a given point P, outside a fixed circle, centre C, is described any circle of the same radius as the fixed circle. If the line joining the centre of the variable circle to P meets the common chord of the two circles in Q, prove that the locus of Q is a hyperbola whose foci are C and P.

6. A variable circle touches two fixed st. lines on which A and B are fixed points. The second tangents drawn from A, B to the circle meet in P. Prove that the locus of P is a hyperbola whose foci are A and B.

7. P is any point on a fixed line $y=mx$, A and B are the fixed points $(c, 0)$, $(-c, 0)$. The line PQ subtends a right angle at each of the points A and B. Prove that the locus of Q is a hyperbola, one of whose asymptotes is the y-axis and the other the perpendicular through the origin to the locus of P.

8. The area of the triangle formed with the asymptotes by the normal of the hyperbola $x^2-y^2=a^2$, at the point (x', y'), is

$$\frac{(x'^2-y'^2)^2}{a^2}.$$

9. Show that the normal to the rectangular hyperbola $xy=c^2$ at the point 't' meets the curve again at a point t' such that

$$t^3t'+1=0.$$

10. If x_1, x_2, x_3 be the abscissae of three points on the rectangular hyperbola $xy=c^2$, show that the area of the triangle formed by these points is

$$\frac{c^2}{2}\,\frac{(x_2-x_3)(x_3-x_1)(x_1-x_2)}{x_1x_2x_3},$$

and the tangents at these points form a triangle whose area is

$$2c^2\frac{(x_2-x_3)(x_3-x_1)(x_1-x_2)}{(x_2+x_3)(x_3+x_1)(x_1+x_2)}.$$

11. Three points (x', y'), (x'', y''), (x''', y''') lie on the rectangular hyperbola $xy=a^2$. Prove that the orthocentre (x, y) of the triangle formed by the points is given by

$$xx'x''x'''=yy'y''y'''=-a^4$$

and lies on the hyperbola. (Math Trip. I 1920).

12. Find the locus of the centroid of an equilateral triangle inscribed in a rectangular hyperbola.

13. Show that the circle through three given points of a rectangular hyperbola meets the conic again in a point which is diametrically opposed to the orthocentre of the triangle formed by the given points.

14. The rectangular hyperbola $xy=a^2$ is cut by a circle passing through its centre C in four points P_1, P_2, P_3, P_4. Prove that if p_1, p_2 be the perpendiculars from C on the chords P_1P_2, P_3P_4, then

$$p_1p_2=a^2.$$

15. Show that the rectangular hyperbola which cuts the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

at an angle α and has the principal axes of the ellipse for asymptotes, is

$$xy=a^2b^2\cos\alpha\div[(a^2+b^2)^2\sin^2\alpha+4a^2b^2\cos^2\alpha]^{\frac{1}{2}}.$$

16. The centroid of a triangle inscribed in the hyperbola $xy=a^2$ is at the point $(ka, 0)$. Show that its sides touch the conic

$$4xy=(a+3\,ky)^2.$$

17. If P, P' be a diameter of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

prove that the locus of the intersection of the normal at P with the ordinate at P' is

$$\frac{x^2}{a^2} - \frac{b^2y^2}{(2a^2+b^2)^2} = 1.$$

18. PQ is a chord of an ellipse at right angles to the major axis AA'; PA and QA' meet in R; show that the locus of R is a hyperbola having the same axis as the ellipse.

19. If a circle be described passing through any point P of a given hyperbola and the extremities of the transverse axis, and the ordinate MP be produced to meet the circle in Q, show that the locus of Q is a hyperbola.

20. A st. line is drawn parallel to the axis of y meeting the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and its conjugate at points P, Q; show that the normals at P and Q intersect on the x-axis.

Show also that the tangents at P and Q intersect on the curve whose equation is

$$y^4(a^2y^2 - b^2x^2) = 4\,b^6x^2.$$

21. P and Q are the extremities of the conjugate diameters of a hyperbola and its conjugate, P being on one and Q on the other, show that the locus of the orthocentre of the triangle CPQ is the line $ax - by = 0$.

22. Show that the tangents to the rectangular hyperbola $x^2 - y^2 = a^2$ at the extremities of its latera recta pass through the vertices of the conjugate hyperbola

$$x^2 - y^2 = -a^2.$$

23. The tangents at the ends of a chord PQ of a hyperbola meet in T, and TM, TN are drawn parallel to the asymptotes to meet them in M, N. Prove that MN is parallel to PQ.

24. If from any point on the hyperbola $x^2 - y^2 = a^2 + b^2$, a pair of tangents be drawn to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

prove that the four points where they cut the axes are concyclic.

[*Hint.* If the pair of tangents

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right)=\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2$$

meet the axes in P, P′; Q, Q′ and C be the centre, then

CP. CP′ = CQ. CQ′].

25. Prove that the lines joining a variable point on a hyperbola to two fixed points on the hyperbola intercept a segment of constant length on either asymptote.

26. The locus of the centre of a circle which circumscribes the triangle formed by the asymptotes and any tangent to a given hyperbola is another hyperbola whose asymptotes are perpendicular to those of the given hyperbola.

27. P and Q are points on the rectangular hyperbola $xy=k^2$, such that the asculating circle at P passes through Q.

Show that the locus of the pole of PQ is the curve

$$(x^2+y^2)^2=4\,k^2xy.$$

28. Prove that the normals to the rectangular hyperbola $xy=c^2$ at the extremities of the chord $px+qy=1$ intersect in the point

$$\frac{1}{p}-c^2\left(q-\frac{p^2}{q}\right),\ \frac{1}{q}-c^2\left(p-\frac{q^2}{p}\right).$$ (Radford)

CHAPTER X

PARABOLA

99. The general equation of the second degree in x and y, when its discriminant is different from zero, and the second degree terms form a perfect square, can by proper choice of axes be reduced to the form

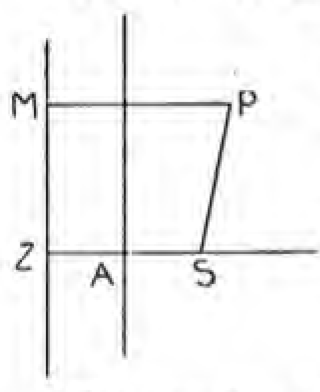
$$y^2=4ax \qquad \text{......(1)}$$

The locus of this equation we have named a **parabola**.

99.1. *A parabola may be generated by the motion of a point in a plane such that its distance from a fixed point remains equal to its distance from a fixed line.*

The fixed point is called the **focus** and the fixed line the **directrix**.

Let S be the focus and MZ the directrix. From S draw SZ perpendicular to MZ. Take SZ as the axis of x and a line through the mid-point A of ZS as the y-axis. Let $AS=a$. The co-ordinates of S are $(a, 0)$ and the equation of ZM is $x+a=0$. If $P(x, y)$ be the moving point, the condition $SP=PM$ gives the equation



$$(x-a)^2+y^2=(x+a)^2$$

i. e., $$y^2=4ax.$$

The equation shows that the curve touches the line $z=0$, *i.e.*, the line at ∞.

99.2. To prove the converse, we notice that the lines

$$y=i(x-a),\ y=-i(x-a)$$

are the isotropic tangents to the parabola $y^2=4ax$ and these lines meet at an accessible point $(a, 0)$. The focus-directrix property can now be obtained by throwing the equation in the form

$$(x-a)^2+y^2=(x+a)^2$$

which expresses that the locus is traced by a point whose distance from the line $x+a=0$ remains equal to its distance from the point $(a, 0)$. The point $(a, 0)$ is thus the focus and the line $x+a=0$ is the corresponding directrix and is the polar of $(a, 0)$. The eccentricity is unity.

100. To interpret the inequation $y^2 - 4ax \gtreqless 0$.

Let $P(x, y)$ be any point in the plane of the parabola. Join P to the focus $S(a, 0)$. The line PS will meet the parabola in points Q and R. Now the co-ordinates of an arbitrary point on PS are

$$\frac{\lambda a + x}{1+\lambda}, \quad \frac{y}{1+\lambda}.$$

This will be the point Q or R if

$$\frac{y^2}{(1+\lambda)^2} - \frac{4a(\lambda a + x)}{1+\lambda} = 0$$

i.e., $\quad 4a^2\lambda^2 + 4a\lambda(x+a) - (y^2 - 4ax) = 0.$

The roots of this equation are

$$\lambda_1 = \frac{PQ}{QS}, \quad \lambda_2 = \frac{PR}{RS}.$$

The parabola $y^2 = 4ax$ divides the plane in two regions, one of which contains the focus and is called the interior region and the other exterior. If P is in the exterior region or outside the parabola, one of the points Q and R is internal and the other external to the segment PQ. Thus $\lambda_1\lambda_2$ is negative and consequently $y^2 - 4ax > 0$. If P lies inside the parabola, the points Q and R are external to the segment PS, $\lambda_1\lambda_2$ is therefore positive, hence $y^2 - 4ax < 0$.

Thus $y^2 - 4ax > 0$ and $y^2 - 4ax < 0$ are respectively the analytic representations of the external and the internal regions of the parabola $y^2 = 4ax$.

101. The freedom equations of the parabola $y^2 - 4ax = 0$.

Writing the equation of the parabola in the form

$$\frac{y}{2a} = \frac{2x}{y} = t$$

we find $\quad x = at^2 \quad y = 2at \quad \ldots\ldots(2)$

as the freedom equations of the parabola $y^2 = 4ax$.

Replacing t by $\frac{1}{m}$, we get $\quad x = \frac{a}{m^2}, \; y = \frac{2a}{m} \quad \ldots\ldots(2A)$

102. The equation of a line joining two points on the parabola.

Let $P(x_1, y_1)$, $Q(x_2, y_2)$ be two points on the parabola $y^2 = 4ax$. The equation

$$\lambda(y - y_1)(y - y_2) = y^2 - 4ax$$

represents a family of parabolas through the intersections of the lines $y - y_1 = 0$, $y - y_2 = 0$ with the parabola $y^2 = 4ax$.

For $\lambda=1$, it reduces to the line at infinity and the line PQ given by the equation

$$y(y_1+y_2)=4\,ax+y_1y_2. \qquad \ldots\ldots(3)$$

102.1. If the points P and Q be given as $(at_1^2, 2at_1)$ $(at_2^2, 2at_2)$ the equation of the chord PQ may be obtained as follows:—

Suppose that the equation of the chord is

$$lx+my+n=0.$$

This meets the parabola in points whose parameters are given by the equation

$$G_1(t)\equiv alt^2+2atm+n=0.$$

Therefore

$$l(t_1+t_2)+2m=0,$$

$$alt_1t_2=n,$$

consequently the chord is given by the equation

$$(t_1+t_2)y=2(x+at_1t_2). \qquad \ldots\ldots(3A)$$

102.2. Tangential equation of the parabola.

If the line $lx+my+n=0$ be a tangent to the parabola, the quadratic $G(t)=0$ which gives the parameters of the points of intersection of the line with the parabola has equal roots, therefore,

$$am^2=ln. \qquad \ldots\ldots(4)$$

In particular, $y=mx+c$ will be a tangent to the parabola if $c=\dfrac{a}{m}$ and thus for all values of m,

$$y=mx+\frac{a}{m}$$

is a tangent to the parabola at the point $\left(\dfrac{a}{m^2},\ \dfrac{2a}{m}\right)$.

(4) is satisfied when $l=0$, $m=0$. Thus the line at ∞ touches the parabola.

103. *Tangent at a point of the parabola.*

Let the point Q (Art. 102) approach P, then $x_2\to x_1$, $y_2\to y_1$, the equation of the limiting position of the chord PQ *viz.* the tangent at P becomes

$$yy_1=2a(x+x_1), \qquad \ldots\ldots(5)$$

for $y_1^2=4ax_1$. The equation (3) becomes

$$ty=x+at^2. \qquad \ldots\ldots(5A)$$

The form of the equation shows that for given values of x and y, there are two values of t. Hence from an arbi-

trary point (x, y), two tangents can be drawn to a parabola and these are real, coincident, or imaginary according as

$$y^2-4ax \gtreqless 0.$$

103.1. Tangents from a point. Let (x', y') be a point from which the tangents are drawn and suppose that the tangent

$$y=mx+\frac{a}{m}$$

passes through (x', y').

Then $$y'=mx'+\frac{a}{m}.$$

Eliminating m, we see that the pair is given by the equation

$$a(x-x')^2=(y-y')(xy'-x'y) \qquad \text{......(6)}$$

which can be reduced to the form

$$(y^2-4ax)(y'^2-4ax')=\{\,yy'-2a(x+x')\,\}^2 \qquad \text{...(6A)}$$

This equation may be obtaind from the next Art. (7).

104. Pole and Polar. The line through $O(x', y')$ which makes an angle θ with the x-axis is given parametrically by the equations

$$x=x'+r\cos\theta$$
$$y=y'+r\sin\theta$$

where r is the variable parameter and denotes the distance of a variable point (x, y) of the line from the point O. The line meets the parabola in points P and Q whose distances from O are the roots of the equation

$$(y'+r\sin\theta)^2-4a(x'+r\cos\theta)=0$$

i.e., $r^2\sin^2\theta+2r(y'\sin\theta-2a\cos\theta)+(y'^2-4ax')=0$...(7)

On the line take a point R such that (OR, PQ) = −1, and this implies that

$$\frac{1}{OP}+\frac{1}{OQ}=\frac{2}{OR}.$$

$$\therefore \qquad -\frac{y'\sin\theta-2a\cos\theta}{y'^2-4ax'}=\frac{1}{OR}.$$

If (x, y) be the co-ordinates of R and OR$=r$, the locus of R is the polar of O and is given by the equation

$$y'(y-y')-2a(x-x')+y'^2-4ax'=0,$$

i.e., $$yy'-2a(x+x')=0. \qquad \text{......(8)}$$

104.1. Pole of a line. Let $O(x_0, y_0)$ be the pole of the line $lx+my+n=0$. The equation is therefore identical with $yy_0-2a(x+x_0)=0$, therefore

$$\frac{2a}{l}=-\frac{y_0}{m}=\frac{2ax_0}{n}.$$

Thus $$x_0=\frac{n}{l},\ y_0=-\frac{2am}{l}. \qquad \text{......(9)}$$

104.2. Conjugate lines. If the line $l_1x+m_1y+n_1=0$ is conjugate to $l_2x+m_2y+n_2=0$, the pole of either lies on the other. The pole of the first line is $\left(\frac{n_1}{l_1}, \ -\frac{2am_1}{l_1}\right)$ and it lies on the second line if

$$l_1n_2+l_2n_1=2am_1m_2. \qquad \text{......(10)}$$

which is, therefore, the condition of conjugacy of two lines.

Exercises

1. Show that conjugate line through the focus of a parabola are perpendicular to each other.

2. Prove the same property for the ellipse and hyperbola.

3. If a line passes through the focus of a conic, its perpendicular conjugate line also passes through the same focus.

105. Chord with a given mid-point. Let $O(x_0, y_0)$ be the given point, the line

$$y-y_0=m(x-x_0)$$

passes through (x_0, y_0) for all values of m. The line meets the parabola in points whose parameters are given by the equation

$$2at-y_0=m(at^2-x_0)$$

i.e., $$amt^2-2at+y_0-mx_0=0.$$

If t_1, t_2 are the roots of this equation,

$$y_0=a(t_1+t_2)=\frac{2a}{m} \qquad \text{or } m=\frac{2a}{y_0}$$

Thus the equation of the line is

$$y_0(y-y_0)=2a(x-x_0) \qquad \text{......(11)}$$

105.1. Locus of the mid-points of a system of parallel chords. Let (x_0, y_0) be the mid-point of one of the chords of slope m, then proceeding as in Art. 105, it is found that

$$y_0=\frac{2a}{m}.$$

Thus the locus of (x_0, y_0) is the line

$$y = \frac{2a}{m}, \qquad \ldots\ldots(12)$$

which is parallel to the axis of the parabola and is called a **diameter** of the parabola. The diameter meets the parabola in only one accessible point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ which is called its **vertex**.

105.2. In a parabola, there is no diameter which is conjugate to a given diameter. But for a given system of parallel chords there exists a diameter which bisects them and for every diameter there exists a system of chords bisected by the diameter. The directions of the parallel chords and the corresponding diameter are called **conjugate directions.** The direction conjugate to the diameter $y = \frac{2a}{m}$ is m and if we call $\frac{2a}{m} = m'$, we have

$$mm' = 2a,$$

which is the condition of conjugacy of the directions.

The tangent at the vertex $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ of the diameter $y = \frac{2a}{m}$ is $y = mx + \frac{a}{m}$ which has a slope m. Thus the direction which is conjugate to a diameter is that of the tangent at its vertex. We have incidentally proved that the tangent at the vertex of a diameter is parallel to the system of chords bisected by the diameter.

Further it will be proved that a parabola can be regarded as a limiting case of an ellipse or a hyperbola whose centre is at infinity, and this explains why the diameters of a parabola are parallel.

Chap **106. Equation of a parabola referred to a diameter and the tangent at its vertex as axes.**

Since the origin lies on the conic, its equation can be taken as

$$ax^2 + 2hxy + by^2 + 2gx + 2fy = 0.$$

Since the line $x = 0$ is tangent at the origin, the equation

$$by^2 + 2fy = 0,$$

which gives the ordinates of the points of intersection of the conic with $x = 0$, has two zero roots, therefore $f = 0$. Again, the line $y = 0$ being a diameter meets the parabola

in one finite point (the vertex) and one point at infinity. The equation

$$ax^2+2gx=0$$

has therefore one zero root and one infinite root. This requires $a=0$. The equation takes the form

$$by^2+2hxy+2gx=0.$$

Now, every chord parallel to the y-axis is bisected by the x-axis, thus for every x, there are two equal and opposite values of y, this demands $h=0$. Hence the equation of the parabola is of the form

$$y^2=4px. \qquad \text{......(13)}$$

If we write the equation in the form

$$y^2=(\epsilon x+\epsilon' y+4p)x \qquad \epsilon\to 0 \quad \epsilon'\to 0$$

it shows that the line $x=0$ and the line at infinity are tangents to the parabola, the chord of contact being the diameter through the point of contact of the accesible tangent. Thus *the pole of every diameter is at infinity in the conjugate direction.* The same inference can be drawn if the equation of the parabola be written in homogeneous co-ordinates, *viz.* $y^2=4pxz$.

106.1. The relation between the constants 'p' of $y^2=4px$ and 'a' of $y^2-4ax=0$ may be determined. For, writing the equations as $\frac{1}{4p}y^2-x=0$, $\frac{1}{4a}\ y^2-x=0$, by Chapter IV,

$$\frac{1}{4p\sin^2\theta}=\frac{1}{4a}$$

i.e.,

$$p=a\operatorname{cosec}^2\theta,$$

where θ is the angle between the axes in the first case. It is easy to see that if O be the origin when the equation is $y^2-4px=0$, $p=\text{OS}$, S being the focus. Also, the length of the focal chord bisected by the diameter is $4p$ and is called the **parameter** of the diameter.

Illustrative Examples.

(1) *Show that, if parabolas ($y^2=4ax$) are drawn corresponding to different values of 'a', the feet of the perpendiculars from a fixed point on its polar lines all lie on a circle passing through the point.* [Math. Trip. 1916]

Let $P(x_0, y_0)$ be the fixed point. Its polar $w.\ r.$ to $y^2=4ax$ is

$$yy_0=2a(x+x_0).$$

The line perpendicular to it through (x_0, y_0) is

$$2a(y-y_0)+(x-x_0)y_0=0.$$

The locus of the point of intersection of these two lines is obtained by the elimination of the arbitrary constant 'a'. The locus in question is the curve

$$\frac{y}{x-x_0} = -\frac{x+x_0}{y-y_0}$$

i.e., $x^2+y^2-yy_0-x_0^2=0$,

which is a circle through (x_0, y_0).

(2) *Through a point of the st. line $y=mx+c$ is drawn a chord of the parabola $y^2=4ax$ which is bisected at the point. Show that the chords touch the parabola*

$$\left(y+\frac{2a}{m}\right)^2=8a\left(x+\frac{c}{m}\right).$$

Let (x_1, mx_1+c) be a point on the st. line. The equation of the chord whose mid-point is (x_1, mx_1+c) is

$$(mx_1+c)[y-(mx_1+c)]=2a(x-x_1)$$

i.e., $$\lambda^2-\lambda\left(y+\frac{2a}{m}\right)+2a\left(x+\frac{c}{m}\right)=0.$$

where $\lambda=mx_1+c$. The envelope of the line as λ varies is

$$\left(y+\frac{2a}{m}\right)^2=8a\left(x+\frac{c}{m}\right).$$

(3) *A chord through the fixed point (x_0, y_0) meets the parabola $y^2=4ax$ in Q and R. If P be the pole of QR, show that the centroid of the triangle PQR lies on the parabola*

$$2y^2-yy_0-6ax+2ax_0=0.$$

Let the line $y-y_0=m(x-x_0)$ meet the parabola in Q and R. The parameters of these two points are the roots of the equation

$$mat^2-2at+(y_0-mx_0)=0.$$

If t_1, t_2 be the roots of this equation, the co-ordinates of P are $(at_1t_2, a(t_1+t_2))$. If (x, y) be the co-ordinates of the centroid

$$3x=a(t_1^2+t_2^2+t_1t_2)=a(t_1+t_2)^2-at_1t_2$$

$$=\frac{4a}{m^2}-\frac{y_0-mx_0}{m},$$

$$y=a(t_1+t_2)=\frac{2a}{m}.$$

Elimination of m gives the required locus

$$2y^2-yy_0-6ax+2ax_0=0.$$

(4) *The tangent at any point P of the parabola $y^2=4ax$ is met in Q by a line through the vertex A at right angles to AP and Z is the foot of the perpendicular from A on the tangent at P. Show that there are three positions of the point P on the*

parabola, for which Z lies on the st. line $lx+my+na=0$, and the corresponding point Q lies on the line
$(2l-n)x+4my+2na=0$. [Pembroke etc., 1910, St. Catherine, 1928, Downing, 1931]

Let the co-ordinates of P be $(at^2, 2at)$. The equation of the tangent at the point is

$$x-ty+at^2=0.$$

The equation of AZ is $tx+y=0$.

These two lines meet on the line

$$lx+my+na=0,$$

therefore

$$\begin{vmatrix} 1 & -t & t^2 \\ t & 1 & 0 \\ l & m & n \end{vmatrix} =0$$

which reduces to

$$mt^3+t^2(n-l)+n=0. \qquad \text{......}(i)$$

The equation, being a cubic in t, determines three possible positions of P.

The equation of AP being $ty-2x=0$, the perpendicular to it through the vertex is

$$tx+2y=0. \qquad \text{......}(ii)$$

The co-ordinates of Q which lies on the line (ii) and the tangent at P are

$$x=\frac{-2at^2}{2+t^2},\ y=\frac{at^3}{2+t^2}.$$

Therefore from (i)

$$m(2+t^2)y+at^2(n-l)+an=0,$$

i.e.,
$$my-\frac{x}{2}(n-l)+\frac{2a+x}{4}.n=0$$

or
$$x(2l-n)+4my+2an=0.$$

Exercises XXX

1. *Show that the subtangent at any point of a parabola is bisected at the vertex.*

2. The tangent at any point of a parabola is equally inclined to the focal distance of the point and the diameter through the point.

3. Prove that the normal at a point is equally inclined to the focal distance of the point and the diameter through that point.

Note.—This property is the principle of parabolic reflector. Suppose a source of light be placed at the focus F. Now the law

of reflection of light from a surface says that the incident and reflected rays are equally inclined with the normal to the surface. So the light from a source of light at F will be projected at a great distance by the rays parallel to the axis of the parabolic mirror. This property is used in the mirror of a search light, of a motor headlight, etc.

Conversely, parallel rays of light are brought to a focus at F (focus of the parabola) after reflection.

4. *Show that the sub-normal at a point is constant.*

5. *The foot of the perpendicular from the focus on the tangent at any point lies on the tangent at the vertex.*

6. *Show that the perpendicular tangents to a parabola intersect on the directrix and the chord of contact passes through the focus.*

[*Hint.*—If t_1, t_2 be the parameters of the points of contact, then

$$t_1t_2+1=0.]$$

7. Show that the co-ordinates of the mid-point of the chord of the parabola along the line $y=\mu(x-h)$ are

$$x=h+\frac{2a}{\mu^2},\quad y=\frac{2a}{\mu}.$$

Hence show that the locus of the mid-points of all chords of the parabola, which pass through $(h, 0)$ is the parabola

$$y^2=2a(x-h).$$

8. Show that a chord of the parabola $y^2=4a(x+a)$ touches $y^2=4ax$ at its mid-point.

9. A tangent of the parabola $y^2=4a(x+a)$ is at right angles to a tangent of the parabola $y^2=4a'(x+a')$. Show that the point of intersection of the tangents lies on the line

$$x+a+a'=0.$$

10. Show that chords of a parabola, which subtend a right angle at the vertex, pass through a fixed point.

11. Tangents are drawn to a parabola at points whose abscissæ are in the ratio $p:1$, prove that they intersect on the curve

$$y^2=(p^{\frac{1}{4}}+p^{-\frac{1}{4}})^2\,ax.$$

12 Show that the locus of the mid. points of the chords of the parabola $y^2=4ax$ which subtend a right angle at the vertex is the parabola

$$y^2=2a(x-4a).$$

[Let (h, k) be the mid-point of a chord. Its equation is

$$k(y-k)=2a(x-h), \text{ then } y^2=4a\frac{2ax-ky}{2ah-k^2}x$$

represents the lines which join A to the intersection of the parabola with the chord. Since they are at right angles,

$$\therefore \quad k^2 - 2ah + 8a^2 = 0, \text{ etc.}].$$

13. Prove that the locus of the poles, of tangents to the parabola $y^2 = 4ax$, *w. r.* to the circle $x^2 + y^2 = 2ax$, is the circle $x^2 + y^2 = ax$.

14. Show that the locus of the poles of tangents to the parabola $y^2 = 4ax$ *w.r.* to the parabola $y^2 = 4bx$ is the parabola

$$y^2 = \frac{4b^2}{a} x.$$

15. The tangents to the parabola $y^2 = 4ax$ which intersect at an angle α have their point of intersection on the curve

$$y^2 - 4ax = (a + x)^2 \tan^2 \alpha.$$

16. Prove that the parabolas $y^2 = 4ax$, $x^2 = 4by$ intersect at an angle

$$\tan^{-1} \frac{3a^{\frac{1}{3}} b^{\frac{1}{3}}}{2(a^{\frac{2}{3}} + b^{\frac{2}{3}})} .$$

17. O is the vertex of the parabola $y^2 = 4ax$ and P, Q are tho points in which it meets the line $lx + my = 4na$. Prove that the internal and external bisectors of the angle POQ are given by the equation $m(x^2 - y^2) = 2(l + n)xy$. (Selwyn, 1928)

18. Through the vertex A of a parabola two chords AP, AQ are drawn at right angles to one another. The chord PQ meets the axis in G. Prove that AG is equal to the latus rectum of the parabola, and conversely.

19. Show that $y = 2x + \dfrac{a}{2}$ and $y = -2x - \dfrac{a}{2}$ are the real common tangents to the parabola $y^2 = 4ax$ and the circle

$$20(x^2 + y^2) = a^2.$$

20. Show that the real common tangents of the curves $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$ are $y = \pm x \pm 2a$.

21. Show that the real common tangent to the parabolas $y^2 = 4ax$ and $x^2 = 4by$ is $a^{\frac{1}{3}} x + b^{\frac{1}{3}} y + a^{\frac{2}{3}} b^{\frac{2}{3}} = 0$.

22. If a tangent to the parabola $y^2 = 4ax$ meets the axis in T and the tangent at the vertex A in Y and the rectangle TAYQ is completed, show that the locus of Q is the parabola $y^2 + ax = 0$.

23. From H, a fixed point on a parabola, chords HP, HQ are drawn perpendicular to each other. Show that the locus of the intersection of the tangents at P and Q is a st. line.

(King's etc. 1927]

24. The orthocentre of a triangle formed by three tangents to a parabola lies on its directrix and the circumcircle passes through the focus.

25. TP, TQ are tangents to a parabola, prove that Δs SPT, SQT are similar, S being the focus.

26. Parabolas touch a given line and have a common focus; show that the vertices lie on a circle.

27. If SY be the perpendicular from the focus S on the tangent at P, $SY^2 = AS.SP$. *i.e.*, $p^2 = ar$.

107. Normal at a point.

The equation of the tangent at (x', y') to the parabola $y^2 = 4ax$ is $yy' = 2a(x + x')$. The normal at (x', y') is therefore given by the equation

$$y'(x - x') + 2a(y - y') = 0. \qquad \ldots\ldots(14)$$

The equations of the normals at the points $-t$ and $+t$ are respectively given by the equations

$$y = tx - 2at - at^3, \qquad \ldots\ldots(14A)$$

$$y + tx = 2at + at^3. \qquad (14B)$$

107.1. Co-normal points. Let the normal at t pass through the point $O(x_0, y_0)$, then the values of t are given by the equation

$$at^3 + t(2a - x_0) - y_0 = 0. \qquad \ldots\ldots(15)$$

The equation being a cubic has three roots. Thus, unlike an ellipse or hyperbola, *through a point three normals with accessible feet can be drawn to a parabola.*

If the roots of equation (15) be t_1, t_2, t_3, which are the parameters of the feet of the co-normal points, then

$$t_1 + t_2 + t_3 = 0 \qquad \ldots\ldots(16)$$

i.e., the sum of the gradients of the three normals that pass through a point is zero.

The condition (16) can be written as

$$(2at_1) + (2at_2) + (2at_3) = 0$$

i.e., the sum of the ordinates of three co-normal points is zero and thus *the centre of gravity of three co-normal points lies on the axis of the parabola.*

The condition is also sufficient. For, let the normals be

$$y + t_1(x - 2a) - at_1^3 = 0,$$
$$y + t_2(x - 2a) - at_2^3 = 0,$$
$$y + t_3(x - 2a) - at_3^3 = 0.$$

Multiplying these by t_2-t_3, t_3-t_1, t_1-t_2 in turn and adding, we get

$$\Sigma t_1^3(t_2-t_3)=-(t_2-t_3)(t_3-t_1)(t_1-t_2)(t_1+t_2+t_3)=0$$

for $t_1+t_2+t_3=0$. Hence the three lines meet in a point.

107.2. The result of the preceding article can be otherwise obtained as follows :—

The *necessary* and *sufficient* condition that the three normals

$$y+t_i(x-2a)-at_i^3=0, \qquad i=1, 2, 3,$$

may meet at a point can be obtained by the elimination of $x-2a$, and y between the three equations. The eliminant is

$$\begin{vmatrix} 1 & t_1 & t_1^3 \\ 1 & t_2 & t_2^3 \\ 1 & t_3 & t_3^3 \end{vmatrix}=0$$

i.e., $\qquad (t_2-t_3)(t_3-t_1)(t_1-t_2)(t_1+t_2+t_3)=0.$

As $\qquad t_1 \neq t_2 \neq t_3$

$$t_1+t_2+t_3=0. \qquad \text{......(16)}$$

107.3. Let y_1, y_2 be the ordinates of the extremities of a chord which belongs to a parallel system. Since the mid-points of a system of parallel chords lie on a st. line parallel to the axis, $y_1+y_2=$ constant. If y_3 be the ordinate of the third co-normal point,

$$y_3=-(y_1+y_2)=\text{constant}.$$

Thus the third normal is fixed. Hence *the normals at the extremities of a system of parallel chords intersect on a fixed normal of the parabola.*

107.4. Suppose that the normal at 't' meets the parabola again at λ. therefore

$\therefore \qquad 2a\lambda+at\lambda^2=2at+at^3$

or $\qquad t^2+\lambda t+2=0$

if the factor $t-\lambda$ be removed. Hence

$$\lambda=-t-\frac{2}{t}.$$

Thus the normal at t meets the parabola again in a point whose parameter is $-t-\dfrac{2}{t}$.

If λ be given, the above quadratic in t gives the parameters of the feet of the two normals that pass through λ other than the normal at λ.

If t_1, t_2 be the roots of this equation,

$$t_1 t_2 = 2. \qquad \ldots\ldots(17)$$

This is the necessary and sufficient condition that the normals at t_1 and t_2 may meet on the parabola.

The normals that pass through λ are real, coincident or imaginary, according as

$$\lambda^2 \gtreqless 8, \ i.\ e., \ a\lambda^2 \gtreqless 8a\ ;$$

i. e., according as the abscissa of the point is greater than, equal to, or less than $8a$.

107.5. The relation between the tangential and normal poles of a line.

Let the tangents at the extremities of the line meet at (x_1, y_1) and the normals at (ξ_1, η_1). The equation of the chord is therefore

$$yy_1 = 2a\ (x + x_1).$$

The parameters of the extremities of the chord are the roots of the equation

$$at^2 - ty_1 + x_1 = 0.$$

Let t_1, t_2 be the roots of this equation, then $t_1, t_2, -(t_1+t_2)$ are the feet of the normals from (ξ_1, η_1) and are therefore the roots of the equation

$$at^3 + t\ (2a - \xi_1) - \eta_1 = 0$$

whence $\quad t_1 t_2 - (t_1 + t_2)^2 = (2a - \xi_1)/a$

$$-t_1 t_2\ (t_1 + t_2) = \eta_1 / a$$

and therefore $\quad x_1 - \dfrac{y_1^2}{a} = 2a - \xi_1$

and $\quad -x_1 y_1 = a\eta_1.$

Thus the relations that connect the co-ordinates of the tangential pole (x, y) with the normal pole (ξ, η) are

$$\left.\begin{aligned} ax - y^2 &= a(2a - \xi) \\ xy &= -a\eta. \end{aligned}\right\} \qquad \ldots\ldots(18)$$

108. Radius of curvature. Circle of curvature. Evolute.

The normal at the point t is

$$\phi(t) \equiv at^3 + t(2a - x) - y = 0.$$

A neighbouring normal meets this normal on the line

$$\phi'(t) \equiv 3at^2 + 2a - x = 0.$$

The point of intersection of these two lines is the centre of curvature. The point of intersection is given by the equations

$$x = 2a + 3at^2, \quad y = -2at^3.$$

The locus of the centre of curvature is the curve

$$4(x-2a)^3=27ay^2$$

and is called the **evolute** of the parabola.

The equation of the circle of curvature is of the form

$$x^2+y^2-2(2a+3at^2)x+4at^3y+c=0.$$

This passes through $(at^2, 2at)$, therefore $c=-3a^2t^4$. Thus the equation of the circle of curvature is

$$x^2+y^2-2(2a+3at^2)x+4at^3y-3a^2t^4=0.$$

The radius ρ of this circle is given by the equation

$$\rho^2=(2a+3at^2)^2+4a^2t^6+3a^2t^4$$
$$=4a^2(1+t^2)^3.$$

109. Concyclic Points.

The circle

$$C\equiv x^2+y^2+2gx+2fy+c=0.$$

meets the parabola $x=at^2$, $y=2at$ in points whose parameters are roots of the equation

$$a^2t^4+2a(2a+g)t^2+4aft+c=0. \qquad \ldots\ldots(19)$$

If t_1, t_2, t_3, t_4 be the roots of this equation

$$t_1+t_2+t_3+t_4=0 \qquad \ldots\ldots(20)$$

$$\text{or} \quad y_1+y_2+y_3+y_4=0 \qquad \ldots\ldots(20A)$$

where $y_i=2at_i$. Conversely, if either of these relations holds, the four points of the parabola lie on a circle. For, let the circle through t_1, t_2, t_3 meet the parabola in t', therefore

$$t_1+t_2+t_3+t'=0.$$

Consequently $\qquad t'=t_4.$

Thus *the necessary and sufficient condition that four points of a parabola may lie on a circle is that their centre of gravity lies on the axis of the parabola.*

109.1 The four concyclic points P_i $(i=1, 2, 3, 4)$ of the parabola $y^2=4ax$ can be joined in six ways giving three pairs of lines, *viz.*, (P_1P_2, P_3P_4), (P_1P_3, P_2P_4), (P_1P_4, P_2P_3).

The equations of the lines of the first pair are

$$(t_1+t_2)y=2(x+at_1t_2), \quad (t_3+t_4)y=2(x+at_3t_4),$$

and these are equally inclined to the axis of the parabola in virtue of the relation

$$t_1+t_1+t_3+t_4=0.$$

The same is true for other pairs. Hence *the common chords of a circle and a parabola are in pairs equally inclined to the axis of the parobola.*

109.2. If the point $P_4 \to P_3$, the chord P_3P_4 approaches the tangent at P_3. The circle and the parabola touch at P_3 and cut at P_1, P_2. The chord P_1P_2 and the tangent at P_3 are therefore equally inclined to the axis of the parabola.

109.3. Suppose that $P_3 \to P_1$ and $P_4 \to P_2$, the circle and parabola have double contact and the relation that connects their parameters is $t_1 + t_2 = 0$. Thus the common chord P_1P_2 is perpendicular to the axis and the tangents at P_1 and P_2 are equally inclined to the axis and the points P_1, P_2 are reflections of each other in the axis of the parabola.

109.4. Osculating circle. Circle of curvature. Let $P_2 \to P_1 \cdot P_3 \to P_1$ independently of each other, then the circle is said to **osculate** the parabola and the circle is called the **osculating circle** and is identical with the **circle of curvature** defined as the locus of the point of intersection of two coincident normals of a parabola. (*c.f.* Art. 108).

The relation that connects the parameters of the points is $3t_1 + t_4 = 0$. Thus the osculating circle at t meets the parabola again at the point $(9at_1^2, -6at_1)$.

The biquadratic (19) has three equal roots and a root $-3t_1$. Thus

$$\begin{aligned} a^2t^4 + 2a(2a+g)t^2 + 4aft + c &\equiv a^2(t-t_1)^3(t+3t_1) \\ &\equiv a^2(t^4 - 6t^2t_1^2 + 8tt_1^3 - 3t_1^4). \end{aligned}$$

Hence

$$\left.\begin{aligned} 2a+g &= -3at_1^2 \\ f &= 2at_1^3 \\ c &= -3a^2t_1^4 \end{aligned}\right\} \qquad \ldots\ldots(21)$$

Thus the centre of the osculating circle at (t) is given by the equations

$$\begin{aligned} x &= 2a + 3at^2, \\ y &= -2at^3. \end{aligned} \qquad \ldots\ldots(22)$$

The locus of the centre of the circle is the cubic

$$4(x-2a)^3 = 27ay^2. \qquad \ldots\ldots(23)$$

The radius ρ of the circle is given by the equation

$$\begin{aligned} \rho^2 &= (2a+3at_1^2)^2 + 4a^2t_1^6 + 3a^2t_1^4 \\ &= 4a^2(1+t_1^2)^3. \end{aligned}$$

$$\therefore \qquad \rho = 2a(1+t_1^2)^{\frac{3}{2}}. \qquad \ldots\ldots(24)$$

The value of ρ shows that the minimum value of ρ is $2a$ when $t_1 = 0$. *The circle that osculates a parabola at the vertex has minimum radius.*

If $\rho > 2a$, there are two equal but opposite values t_1. of Consequently there exist two circles of curvature which are symmetrically placed *w. r.* to the axis of the parabola and have a given diameter greater than the latus rectum of the parabola.

The equation of the osculating circle is

$$x^2+y^2-2(3at^2+2a)+4at^3y-3a^2t^4=0. \qquad \text{......(25)}$$

The equation of the circle may also be obtained as follows :—

The tangent at t is $ty=x+at^2$ and the equation of $P_1 P_4$ is

$$ty+x-3at^2=0.$$

The equation of the system of conics passing through P_4 and three coincident points at P_1 is

$$(x+ty-3at^2)(x-ty+at^2)+\lambda(y^2-4ax)=0.$$

This will be a circle if $1=\lambda-t^2$. Thus the equation of the osculating circle is

$$(x+ty-3at^2)(x-ty+at^2)+(1+t^2)(y^2-4ax)=0 \text{......(25A)}$$

110. Harvey's Theorem. *The circle which passes through co-normal points of a parabola passes also through the vertex of the parabola.*

If the normals at t_1, t_2, t_3 meet in a point, $t_1+t_2+t_3=0$. The circle through t_1, t_2, t_3 meets the parabola again in t_4 such that $t_1+t_2+t_3+t_4=0$, hence $t_4=0$ and this is the parameter of the vertex.

110.1. The equation of the circle can be determined, when the point of concurrence of the normals is known. Suppose that the normals at t_1, t_2, t_3 meet at (x_0, y_0), then t_1, t_2, t_3 are the roots of the cubic

$$at^3+t(2a-x_0)-y_0=0.$$

If the equation of the circle be supposed to be

$$x^2+y^2+2gx+2fy=0,$$

(since it passes through the origin), then t_1, t_2, t_3 are also the roots of the equation

$$at^3+2(2a+g)t+4f=0$$

whence identifying the two cubics, we have

$$2(2a+g)=2a-x_0, \qquad 4f=-y_0.$$

Hence the equation of the circle is

$$x^2+y^2-(2a+x_0)x-\frac{y_0}{2}y=0 \qquad \text{......(26)}$$

110.2. The results of Arts. 110, 110.1 can otherwise be deduced as follows :—

The equation of the normal at (x', y') is

$$y'(x-x')+2a(y-y')=0.$$

Thus the feet of the normals through (x_0, y_0) lie on the rectangular hyperbola

$$H(x, y)\equiv xy+y(2a-x_0)-2ay_0=0 \qquad \text{......(27)}$$

The asymptotes of the hyperbola are $y=0$ and $x+2a-x_0=0$. The hyperbola and the parabola $y^2=4ax$ have therefore one

common point at infinity on the line $y=0$; hence the foot of the 4th normal is always at ∞. We deduce the equation of a conic which passes through the given co-normal points by deleting the point at infinity. Writing the equations as

$$y^2=4ax, \qquad y\ [x+(2a-x_0)]=2ay_0,$$

the curve $\dfrac{x+(2a-x_0)}{y}=\dfrac{y_0}{2x}$

$$i.\ e.,\quad 2x^2+2x(2a-x_0)-yy_0=0$$

satisfies the given conditions. Now the conic

$$2x^2+2x(2a-x_0)-yy_0+2(y^2-4ax)$$

is a circle and passes through the three co-normal points. It evidently passes through the vertex $(0, 0)$.

Illustrative Examples.

(1) *If the normals at the points* P, Q, R *on the parabola* $y^2=4ax$ *meet in the point* (x_0, y_0), *the orthocentre of the triangle* PQR *will be* $\left(x_0-6a, -\frac{1}{2}y_0\right)$. *Prove also that the centroid of* PQR *is* $[\frac{2}{3}(x_0-2a), 0]$.

The parameters of the feet of the normals from (x_0, y_0) are the roots of the equation

$$at^3+t(2a-x_0)-y_0=0.$$

Let the roots of this equation be t_1, t_2, t_3 which are the parameters of the points P, Q, R respectively. The equation of QR is

$$(t_2+t_3)\ y=2(x+at_2t_3)$$

or $$t_1y+2x+2at_2t_3 \qquad \because\ t_1+t_2+t_3=0$$

The equation of the altitude through P is

$$2(y-2at_1)-t_1(x-at_1^2)=0$$

i. e., $$2y-t_1x-4at_1+at_1^3=0.$$

Similarly the equation of the altitude from Q is

$$2y-t_2x-4at_2+at_2^3=0.$$

Subtraction gives

$$(t_2-t_1)x=4a(t_1-t_2)-a(t_1^3-t_2^3)$$

i. e., $$\begin{aligned} x&=-4a+a(t_1^2+t_1t_2+t_2^2)\\ &=-4a-a(t_1t_2+t_2t_3+t_3t_1)\\ &=x_0-6a. \end{aligned}$$

Again, $2(t_2-t_1)y+at_1t_2(t_1^2-t_2^2)=0$

or $y=\frac{1}{2}at_1t_2(t_1+t_2)=-\frac{1}{2}at_1t_2t_3=-\frac{1}{2}y_0$.

Since $\Sigma t_1^2 = (\Sigma t_1)^2 - 2\Sigma t_2 t_3 = -2\,\dfrac{2a - x_0}{a}$, the co-ordinates of the centroid

$$\left(\frac{a}{3}\,\Sigma t_1^2,\ \frac{2a}{3}\,\Sigma t_1 \right)$$

can be written as

$$\left[\frac{2}{3}(x_0 - 2a),\ 0 \right].$$

(2) *If tangents are drawn to the parabola $y^2 = 4ax$ from a point P and the corresponding normals meet in a point Q such that PQ cuts the axis at a fixed point O within the curve at a distance K from the vertex, show that P lies on the circle*

$$x^2 + y^2 - x(a+k) + a(2a-k).$$

Let the co-ordinates of P be (x_1, y_1), then the co-ordinates (ξ, η) of Q are given by the equations

$$ax_1 - y_1^2 = a(2a - \xi),$$
$$x_1 y_1 = -a\eta_1.$$

The equation of PQ is

$$\frac{x - x_1}{x_1 - \xi} = \frac{y - y_1}{y_1 - \eta}.$$

This passes through $(k, 0)$

$$ky_1 - \eta(k - x_1) - \xi y_1 = 0$$

i. e., $$ky_1 + \frac{x_1 y_1}{a}(k - x_1) + \frac{ax_1 - y_1^2 - 2a^2}{a}\, y_1 = 0.$$

Therefore the locus of (x_1, y_1) is the circle

$$x^2 + y^2 - x(a+k) - ak + 2a^2 = 0.$$

(3) *Show that three circles can be drawn to touch a parabola and also to touch at the focus a given st. line through the focus. Prove also that the tangents at the points of contact parabola form an equilateral triangle.*

Let the circle touch the parabola at 't.' One of the chords of contact is the tangent at t and the other chord is equally inclined to the axes. The equation of the circle is of the form

$$(x - ty + at^2)(x + ty + ac) + \lambda(y^2 - 4ax) = 0.$$

where $\quad 1 = -t^2 + \lambda \quad$ *i.e.*, $\quad \lambda = 1 + t^2$.

The circle also passes through $(a, 0)$,

$$(1 + t^2)(1 + c) - 4\lambda = 0$$
$$\therefore\ c = 3.$$

Thus the equation of the circle is

$$(x - ty + at^2)(x + ty + 3a) + (1 + t^2)(y^2 - 4ax) = 0$$

i.e., $$x^2 + y^2 - ax(3t^2 + 1) - ay(3t - t^3) + 3a^2t^2 = 0.$$

The equation of the tangent at the point $(a, 0)$ is

$$(1-3t^2)(x-a)-y(3t-t^3)=0$$

which is identical with the fixed line $y=m(x-a)$.

$$\therefore \quad \frac{1-3t^2}{3t-t^3}=m.$$

It is obvious that for a given m, there are three values of 't'. Hence there exist three circles which satisfy the conditions of the problem.

Suppose that θ is the angle that the tangent at 't' makes with the axis, then $t=\cot\theta$,

$$\therefore \quad \frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=m$$

$$\text{or} \quad \tan 3\theta=m$$

$$\therefore \quad 3\theta=\tan^{-1}m,\ \tan^{-1}m+\pi,\ \tan^{-1}m+2\pi.$$

$$\text{Hence} \quad \theta=\tfrac{1}{3}\tan^{-1}m,\ \tfrac{1}{3}\tan^{-1}m+\frac{\pi}{3},\ \tfrac{1}{3}\tan^{-1}m+\frac{2\pi}{3}.$$

Remarks. 1. If (x, y) be the co-ordinates of the centre of the circle,

$$2x=a(3t^2+1), \quad 2y=a(3t-t^3).$$

Eliminating 't' we get the locus of the centre of the circle

$$27ay^2=(x-5a)^2(2x-a).$$

2. The radical axis of the circles at P't_1', Q't_2' is

$$3x(t_1+t_2)+y[3-(t_1^2+t_1t_2+t_2^2)]=3a(t_1+t_2).$$

If the pole of PQ be T, the centre of gravity of TPQ is

$$3x=a(t_1^2+t_2^2+t_1t_2),\ y=a(t_1+t_2)$$

and this evidently lies on the radical axis.

Exercises XXXI

1. *Prove that the normal at a point is equally inclined to the focal distance of the point and the diameter through that point.*

2. *Show that the sub-normal at a point of a parabola is constant and equal to the semi-latus rectum.*

3. Show that the line $lx+my+n=0$ will be a normal to the parabola $y^2=4ax$ if

$$al^3+2alm^2+m^2n=0.$$

4. If the sum of the inclinations to the axis of normals from P to the parabola $y^2=4ax$ is ω, show that the locus of P is the st. line $y\cos\omega+(x-a)\sin\omega=0$.

5. The normal at a point P of the parabola $y^2=4ax$ meets the axis in G, if S be the focus and if the triangle SPG be equilateral, show that the co-ordinates of P are $(3a, 2\sqrt{3}a)$.

6. If the normals at P, Q of the parabola $y^2=4ax$ meet at a point R on the curve, show that the locus of the pole of PQ is $x=2a$.

7. If the normals to the parabola $y^2=4ax$ which meet at the point (x_0, y_0) make angles α, β, γ with the axis, prove that

$$\alpha+\beta+\gamma=\tan^{-1}\frac{y_0}{a-x_0}.$$

8. Find the co-ordinates of the feet of the normals from the point $\left(\frac{27a}{4}, \frac{-15a}{4}\right)$ to the parabola $y^2=4ax$.

9. From a point $P(at^2, 2at)$ on the parabola $y^2=4ax$ two chords PQ, PR are drawn normal to the curve at Q, R. Prove that the equation of QR is $ty+2(x+2a)=0$. [Math. Trip. 1918.]

10. If tangents be drawn from points on the line $x=c$ to the parabola $y^2=4ax$, show that the locus of intersection of the corresponding normals is the parabola

$$ay^2=c^2(x+c-2a).$$

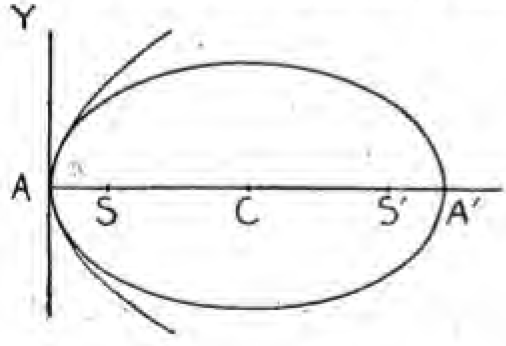
11. If the normal at P to a parabola meet the curve again in Q and the normals at P, Q make angles α, β with the axis, show that

$$3\cos\beta+\cos(2\alpha-\beta)=0.$$

V.V.9-b

111. Parabola as a limiting case of an ellipse or hyperbola.

(i) Let the equation of the ellipse be



i. e. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Shift the origin to the left-hand vertex $A(-a, 0)$ and let S be the corresponding focus. The equation of the ellipse becomes

$$\frac{y^2}{b^2}+\frac{x^2}{a^2}-\frac{2x}{a}=0$$

i. e., $y^2=\frac{2b^2}{a}x-\frac{b^2}{a^2}x^2.$

Put $p=AS=a-ae=a(1-e)$,
then $b^2=2ap-p^2$,
the equation of the ellipse becomes

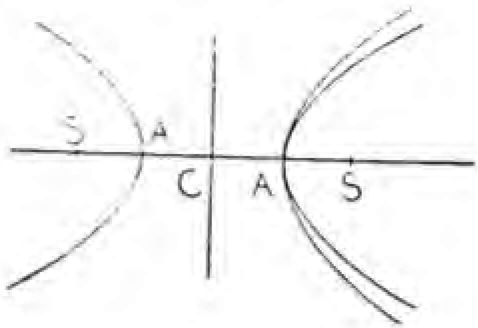
$$y^2=4px-\frac{2p^2}{a}x-x^2\left(\frac{2p}{a}-\frac{p^2}{a^2}\right)$$

Keep p *i.e.*, S fixed and let $a \to \infty$, then C, S$'$, A$'$ all move to infinity and the equation takes the form

$$y^2 = 4px.$$

(*ii*) Suppose that the equation of the hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$



Shift the origin to the positive vertex A$(a, 0)$,

$$\frac{(x+a)^2}{a^2} - \frac{y^2}{b^2} = 1$$

or $y^2 = \dfrac{2b^2}{a}\, x + \dfrac{b^2}{a^2} x^2$.

Set $p = \text{AS} = ae - a$,

then $b^2 = 2ap + p^2$.

The equation of the hyperbola becomes

$$y^2 = 4px + \frac{2p^2 x}{a} + \left(\frac{2p}{a} + \frac{p^2}{a^2} \right) x^2.$$

Keeping S, *i.e.*, p fixed, let 'a' approach infinity, then C,S$'$,A$'$ all move to infinity and the equation takes the form

$$y^2 = 4px.$$

Thus a parabola may be regarded indifferently as the limit of an ellipse or hyperbola.

111.1. It has been seen that like a hyperbola, a parabola has infinite branches. But the infinite branches of the hyperbola tend ultimately to coincide with two divergent right lines, *viz.*, the asymptotes. This is not the case with the parabola. If we find the intersections of the parabola $y = 4ax$ and the line $lx + my = 1$, the ordinates of the points of intersection are the roots of the equation

$$ly^2 + 4amy - 4a = 0,$$

and these roots can never be infinite, unless $l \to 0$, $m \to 0$, *i.e.*, when the line tends to coincide with the line at infinity.

Also, no finite right line meets the parabola at two coincident points at infinity. For, the above quadratic will have equal roots if $am^2 + l = 0$ when the quadratic becomes

$$m^2 y^2 - 4my + 4 = 0,$$

and this has infinite roots if $m \to 0$, then $l \to 0$ and the line tends to coincide with the line at infinity.

111.2. A circle as a limiting case of an ellipse.

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ will tend to coincide with the circle

$$x^2+y^2=a^2 \text{ when } b^2\to a^2, \textit{ i.e.,}$$
$$a^2(1-e^2)\to a^2.$$

This demands that $e\to 0$. Thus a circle may be regarded as a conic whose eccentricity approaches zero. The real foci $(\pm ae, 0)$ tend to coincide with the centre $(0, 0)$. The real directrices $x\pm\frac{a}{e}=0$ tend to coincide with the line at infinity. The focus-directrix property can be expressed by the equation

$$a=e.\,\text{PM},$$

when $e\to 0$, $\text{PM}\to\infty$.

112. Foci of a conic. We have already seen that if t_1, t_2 be two tangents to a conic Σ from a circular point I and t_1', t_2' the corresponding conjugate isotropic tangents from the circular point J, the four tangents intersect in four vertices of a quadrilateral and are the foci of the conic Σ. The foci obtained as the intersections of conjugate tangents (t_1, t_1'), (t_2, t_2') are real and the intersections $(t_1\ t_2')$, $(t_2 t_1')$ are imaginary. In case of a parabola, we have obtained only one accessible focus. For, the line at infinity I J being a tangent to the parabola, the tangents t_2, t_2' from I and J coincide with the line I J and the quadrilateral of tangents reduces to a triangle of tangents. The point S, the intersection of t_1, t' is the only accessible real focus that remains. The point of contact S$'$ of the line at infinity with the parabola, which also lies on the axis, is the second focus. The two complex foci are obtained as the intersections of t_1, t_1' with the line at infinity and are the points I and J.

If a curve passes through I and J, the tangents at I and J to a conic meet in points which are not usually included among the 'ordinary' foci and are called *singular foci*. Now in case of a circle with centre O, OI, OJ are the tangents at I and J to the circle. These tangents meet at O. Thus the centre of a circle is a singular focus and the circle has no ordinary foci.

113. Normals to a conic. The normals through $\text{O}(x_0, y_0)$ to a conic Σ are the lines which join O with the intersections of Σ with its Apollonius hyperbola. Four normals were obtained in case of a central conic. In case of the parabola $y^2=4ax$, the feet of the normals are the intersections $y^2=4ax$ with its Apollonius hyperbola

$$\text{H}(x, y)\equiv xy+y(2a-x_0)2ay_0=0.$$

The asymptotes of this hyperbola are $y=0$, $x+2a-x_0=0$. Thus the axis of the parabola is a tangent to the hyperbola at

infinity. The point at infinity, on the parabola also lies on the $y=0$. Thus one of the four points of intersection is at infinity and the normal to the parabola at this point is, therefore, the line through O parallel to the axis of the parabola.

Through a point O *four normals can be drawn to a parabola and one of these normals is a diameter through* O.

We may now consider the case of a circle. The Apollonius hyperbola of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$(a^2-b^2)xy-a^2x_0yz+b^2y_0xz=0.$$

When $b^2\to a^2$, the ellipse tends to a circle with the same centre and radius equal to 'a'. The Apollonius hyperbola breaks up into the line at infinity $z=0$ and the line

$$x_0y-y_0x=0.$$

This line joins (x_0, y_0) with the centre of the circle, and is therefore a diameter. The lines that join O with the intersections of this line and the circle are the normals from O to the circle. Thus the diameter through O is a double normal. But OI, OJ are not normals through O, as either is regarded as perpendicular to itself and neither touches the circle when O is other than the centre. When O is at the centre, for an ellipse, the hyperbola breaks up into the axes $xy=0$. Thus the major and minor axes are double normals. When an ellipse tends to a circle, any pair of perpendicular diameters may be taken as the axes of symmetry for the circle or let the normal at $(x'y')$ to the ellipse meet the major axis at G, $CG=e^2x'$. Thus for a circle, G coincides with C and every diameter is a double normal.

Illustrative Examples.

(1) *Show that if a chord of the parabola $y^2=4ax$, touches the parabola $y^2=4bx$, the tangents at its extremities meet on the parabola $by^2=4a^2x$, and the normals on the curve*

$$(4a-b)^3y^2=4b^2(x-2a)^3.$$

Let (x', y') be the meet of the tangents; the equation of the chord is

$$yy'=2a(x+x').$$

This touches $\quad y^2=4bx$ if

$$\frac{2ax'}{y'}=\frac{b.y'}{2a}, \qquad \left[c=\frac{b}{m}\right]$$

Hence (x', y') lies on the curve

$$by^2=4a^2x. \qquad \text{......(1)}$$

To find the locus of the corresponding normal pole, we use

relations $a(2a-\xi)=ax-y^2$, $a\eta=-xy$. We have to eliminate x, y from these relations and relation (1) above. Thus

$$a(2a-\xi)=\frac{by^2}{4a}-y^2=\frac{(b-4a)}{4a}y^2$$

$$a\eta=-xy=-\frac{by^3}{4a^2}$$

$$\therefore \qquad \left[\frac{4a^2}{b-4a}(2a-\xi)\right]^3=\left[\frac{4a^3}{b}\eta\right]^2$$

or
$$(b-4a)^3\eta^2=4b^2(2a-\xi)^3.$$

Hence the required locus is the curve

$$(4a-b)^3y^2=4b^2(x-2a)^3.$$

(2) *Tangents q_1, q_2, q_3 are drawn at three points P_1, P_2, P_3 on the parabola $y^2=4ax$ and Q_1, Q_2, Q_3 are the vertices of the triangle formed by q_1, q_2, q_3 (Q_1 being opposite q_1 etc.) Through Q_1 are drawn lines parallel to q_1 and to P_2P_3, and similarly for other vertices. Prove that the six lines thus obtained all touch the parabola $(y-2as_1)^2+8a(x-as_2)=0$ where $s_1=t_1+t_2+t_3$, $s_2=t_2t_3+t_3t_1+t_1t_2$, t_1, t_2, t_3 being the parameters of P_1, P_2, P_3 in the parameteric representation $x=at^2$, $y=2at$.*

(King's College etc. 1931)

The equation of q_1 is $\qquad t_1y=x+at_1^2$,
and that of P_2P_3 is $\qquad (t_2+t_3)y=2(x+at_2t_3)$.

The lines q_2, q_3, *i. e.*

$$t_2y=x+at_2^2$$
$$t_3y=x+at_3^2$$

intersect at Q_1 [at_2t_3, $a(t_2+t_3)$].

The line through Q_1 parallel to q_1 is

$$t_1[y-a(t_2+t_3)]=x-at_2t_3$$
$$t_1y-at_1(s_1-t_1)=x-a[s_2-t_1(s_1-t_1)]$$

or $\quad 2at_1^2+t_1(y-2as_1)-(x-as_2)=0$,

and the envelope of this line is the parabola

$$(y-2as_1)^2+8a(x-as_2)=0.$$

Also, the equation of the line through Q, parallel to P_2P_3 is

$$(t_2+t_3)[y-a(t_2+t_3)]=2[x-at_2t_3]$$

or
$$(s_1-t_1)[y-a(s_1-t_1)]=2x-2a[s_2-t_1(s_1-t_1)]$$

i. e., $\quad at_1^2-t_1y+(s_1y-2x-as_1^2+2as_2)=0$.

The envelope of this is the parabola

$$y^2-4a(s_1y-2x-as_1^2+2as_2)=0$$

i. e.,
$$(y-2as_1)^2+8a(x-as_2)=0.$$

Similarly, the other four lines can be proved to touch this conic.

(3) *Find the locus of the centre of an equilateral triangle whose sides are normals to the parabala* $y^2=4ax$.

Let the equations of the three normals be

$$y=t_r x-2at_r-at_r^3.$$

If the normals make angles

$$\theta_1=\theta,\ \theta_2=\theta+\frac{2\pi}{3},\ \theta_3=\theta+\frac{4\pi}{3}$$

with the axis,

$$\tan 3\theta_r=\tan 3\theta=\lambda \text{ (say)}$$

or $$\frac{3\tan\theta_r-\tan^3\theta_r}{1-3\tan^2\theta_r}=\lambda.$$

So t_r are the roots of the equation

$$\frac{3t-t^3}{1-3t^2}=\lambda$$

or $$t^3-3\lambda t^2-3t+\lambda=0.$$

The vertices of the triangle are

$$[2a+a(t_2^2+t_2t_3+t_3^2),\ at_2t_3(t_2+t_3)] \text{ etc.}$$

The abscissæ of the centre of gravity of the triangle is

$$\begin{aligned}3x&=6a+a\Sigma(t_2^2+t_2t_3+t_3^2)=6a+a\Sigma t_2t_3+2a\Sigma t_1^2\\&=32a+2a[(\Sigma t_1)^2-2\Sigma t_2t_3]\\&=15a+18a\lambda^2\end{aligned}$$

$$\therefore\quad x=5a+6a\lambda^2. \qquad \ldots\ldots(i)$$

The ordinate y of the centre of gravity is given by

$$\begin{aligned}3y&=a\Sigma t_2t_3(t_2+t_3)=a\Sigma t_2t_3(3\lambda-t_1)\\&=3\lambda a\Sigma t_2t_3-3at_1t_2t_3=-6\lambda a\end{aligned}$$

$$\therefore\quad y=-2\lambda a. \qquad \ldots\ldots(ii)$$

Thus the locus of (x, y) is the parabola

$$3y^2=2a(x-5a).$$

(4) *A variable triangle is inscribed in the parabola* $y^2=4ax$ *and two of its sides touch the parabola* $y^2-4bx=0$. *prove that the third side touches the parabola* $y^2-4cx=0$, *where*

$$(2a-b)^2c-ab^2=0. \qquad \text{[Math. Trip. I, 1931].}$$

If t_1, t_2, t_3 be the parameters of the vertices Q_1, Q_2, Q_3, the equations of the sides are

$$\begin{array}{ll}Q_1Q_2 & (t_1+t_2)y=2(x+at_1t_2).\\ Q_1Q_3 & (t_1+t_3)y=2(x+at_1t_3).\\ Q_2Q_3 & (t_2+t_3)y=2(x+at_2t_3).\end{array}$$

The first two lines will touch $y^2=4bx$ if

$$b(t_1+t_2)^2=4at_1t_2$$
$$b(t_1+t_3)^2=4at_1t_3,$$

which show that t_2, t_3 are the roots of the equation

$$b(t_1+t)^2=4at_1t.$$

or $bt^2+2tt_1(b-2a)+bt_1^2=0$

$$\therefore \quad t_2+t_3=\frac{2(2a-b)t_1}{b},\ t_2t_3=t_1^2.$$

The equation of Q_2Q_3 then reduces to

$$\frac{2(2a-b)t_1}{b}y=2x+2at_1^2$$

i.e., $$2abt_1^2-2(2a-b)t_1y+2bx=0.$$

The envelope of this is the parabola

$$(2a-b)^2y^2=4ab^2x$$

$$\text{or} \quad y^2=4cx.$$

(5) *A, B, C are the feet of the normals to the parabola $y^2=4ax$. The points of contact of tangents parallel to BC, CA, AB are A_1, B_1, C_1. Show that the normals at A_1,B_1,C_1 meet in a point P_1.*

If P_2 be deduced from P_1 as P_1 is deduced from P and so on show that PP_1, P_1P_2 touch a cubic curve at P. P_1,.. ..

Let t_1, t_2, t_3 be the parameters of the points A, B, C. The equation of BC is

$$(t_2+t_3)y=2(x+at_2t_3)$$

$$\text{or} \quad -t_1y=2x+2at_2t_3.$$

The parallel tangent is

$$-t_1y=2x+\frac{at_1^2}{2}$$

and its point of contact is $\left(\frac{at_1^2}{4},-at_1\right)$ *i.e.*, the point $-\frac{1}{2}t_1$.

Similarly the points of contact of tangents parallel to CA, AB are the points with parameters $-\frac{1}{2}t_2$, $-\frac{1}{2}t_3$. But since

$$\left(\frac{-t_1}{2}\right)+\left(\frac{-t_2}{2}\right)+\left(\frac{-t_3}{2}\right)=0,$$

the points are co-normal.

If the normals at A, B, C meet at $P(\alpha,\beta)$ and the normals at A_1, B_1, C_1 meet at $P_1(\alpha_1,\beta_1)$, then

$$at^3+t(2a-\alpha)-\beta=0$$

$$at_1^3+4(2a-\alpha_1)+8\beta_1=0.$$

The roots of these equations are t_1, t_2, t_3.

$$\alpha_1-2a=\frac{1}{4}(\alpha-2a).\ \beta_1=-\frac{1}{8}\beta.$$

Similarly $\alpha_2 - 2a = \frac{1}{4}(\alpha_1 - 2a) = \frac{1}{4^2}(\alpha - 2a)$,

and in general

$$\alpha_n - 2a = \frac{1}{4^n}(\alpha - 2a),\ \beta_n = (-1)^n \frac{\beta}{8^n}.$$

The equation of the line P_nP_{n+1} is

$$\frac{y - \beta_n}{\beta_n - \beta_{n+1}} = \frac{x - \alpha_n}{\alpha_n - \alpha_{n+1}}$$

or

$$\frac{2(y - \beta_n)}{3\beta_n} = \frac{x - \alpha_n}{\alpha_n - 2a}.$$

Put $\left(\frac{-1}{2}\right)^n = \lambda$, then $\alpha_n - 2a = \lambda^2(\alpha - 2a)$, $\beta_n = \lambda^3\beta$,

so the equation of the line becomes

$$\frac{2(y - \lambda^3\beta)}{3\lambda\beta} = \frac{(x - 2a) - \lambda^2(\alpha - 2a)}{\alpha - 2a}$$

or $F(\lambda) \equiv \lambda^3\beta(\alpha - 2a) - 3\lambda\beta(x - 2a) + 2y(\alpha - 2a) = 0$ ......(*i*)

$F'(\lambda) = 3\beta[\lambda^2(\alpha - 2a) - (x - 2a)] = 0$ ......(*ii*)

In order to obtain the envelope, we have to eliminate λ between (*i*) and (*ii*).

$\therefore$ $\lambda\beta(x - 2a) = y(\alpha - 2a)$

$\therefore$ $y^2 = \left(\frac{x - 2a}{\alpha - 2a}\right)^3\beta^2$ represents the envelope.

It can also be seen that the points P_n all lie on this curve, and the equation (*i*) is the tangent at the point P_n.

(6) *Show that there exists an infinity of triangles ABC which are inscribed in the cubic $x^3 = ay^2$ and whose sides are normals to the parabola $y^2 = 4a(x + 2a)$. Show also that the three normals to the parabola which are at right angles to the sides of the triangle ABC meet at a point.*

It can easily be seen that an arbitrary normal to the parabola $y^2 = 4a(x + 2a)$ can be written in the form

$$y + mx - am^3 = 0 \qquad \text{......}(i)$$

and this is the normal at $(am^2 - 2a, 2am)$. The freedom equation of the cubic can be written in the form

$$x = at^2,\ y = at^3.$$

If t_1, t_2, t_3 are the parameters of A, B, C, the equation of AB is

$$\frac{y - at_1^3}{a(t_1^3 - t_2^3)} = \frac{x - at_1^2}{a(t_1^2 - t_2^2)}$$

$$y(t_1 + t_2) - x(t_1^2 + t_1t_2 + t_2^2) + at_1^2t_2^2 = 0.$$

This will be a normal to the parabola if, for some value of m,

$$\frac{1}{t_1+t_2}=\frac{m}{-(t_1^2+t_1t_2+t_2^2)}=\frac{-m^3}{t_1^2t_2^2}$$

$$\therefore \quad (t_1^2+t_1t_2+t_2^2)^3=t_1^2t_2^2(t_1+t_2)^2. \qquad \ldots\ldots(ii)$$

Similarly the line joining t_1, t_3 is a normal if

$$(t_1^2+t_1t_3+t_3^2)^3=t_1^2t_3^2(t_1+t_3)^2. \qquad \ldots\ldots(iii)$$

Subtracting (iii) from (ii) we get after reducing

$$t_1+t_2+t_3=0. \qquad \ldots\ldots(iv)$$

$$\therefore \quad t_1^2+t_1t_2+t_2^2=t_1(t_1+t_2)+t_2^2=(t_2+t_3)t_3+t_2^2.$$

So relation (ii) becomes

$$(t_2^2+t_2t_3+t_3^2)^3=t_2^2t_3^2(t_2+t_3)^2,$$

which shows that the line joining t_2 and t_3 is a normal to the parabola.

The slope of the normal which is perpendicular to BC is

$$-\frac{t_2+t_3}{t_2^2+t_3^2+t_2t_3}=\frac{t_1}{t_2^2+t_3^2+\dfrac{t_1^2-t_2^2-t_3^2}{2}}$$

$$(t_2+t_3)^2=t_1^2 \text{ etc.}$$

$$=\frac{2t_1}{t_1^2+t_2^2+t_3^2}.$$

Similarly the slopes of ths normals perpendicular to AC, AB are

$$\frac{2t_2}{t_1^2+t_2^2+t_3^2}, \quad \frac{2t_3}{t_1^2+t_2^2+t_3^2}.$$

Since the sum of the slopes is zero, the three normals are concurrent.

114. Equation of a parabola referred to a pair of tangents as axis.

Let the tangents OA, OB touch the parabola at points $A\left(\frac{1}{a}, 0\right)$, $B\left(0, \frac{1}{b}\right)$ and suppose that the equation of the parabola is

$$(\alpha x+\beta y)^2+2gx+2fy+1=0.$$

The line $y=0$ meets the parabola in points whose abscissæ are the roots of the equation

$$\alpha^2x^2+2gx+1=0$$

$$\therefore \quad \frac{-g}{\alpha^2}=\frac{1}{a}, \quad a^2=\alpha^2.$$

Similarly, since the y-axis touches the parabola at the point $\left(0, \frac{1}{b}\right)$.

$$\therefore \quad \frac{-f}{\beta^2} = \frac{1}{b}, \ \beta^2 = b^2$$

Hence $\alpha = \pm a, g = -a, \beta = \pm b, f = -b.$

Thus the equation of the parabola becomes

$$(\pm ax \pm by)^2 - 2(ax + by) + 1 = 0.$$

If similar signs be taken with a and b, the expression on the left becomes a perfect square and the equation represents a pair of coincident lines. Leaving this case aside, the equation of the parabola is

$$(ax - by)^2 - 2(ax + by) + 1 = 0. \qquad \text{......(28A)}$$

or $$(ax + by - 1)^2 = 4abxy. \qquad \text{......(28)}$$

114.1. The result may be obtained otherwise thus :—

Since $x=0$, $y=0$ are tangents to the curve, the line $ax + by - 1 = 0$ being the chord of contact, its equation is of the form

$$(ax + by - 1)^2 = 2\lambda xy.$$

This will be a parabola if $(ax + by)^2 - 2\lambda xy$ is a perfect square. This requires $\lambda = 2ab$, which gives the desired form

$$(ax + by - 1)^2 = 4abxy.$$

114.2. The equation of the parabola can be put in an irrational form also. For, the equation can be written in the form

$$ax + by - 1 = \pm 2\sqrt{abxy},$$

i.e., $$(\sqrt{ax} \pm \sqrt{by})^2 = 1$$

or $$\sqrt{ax} \pm \sqrt{by} = \pm 1.$$

Thus the equation can be written in either of the forms

$$\left.\begin{array}{rl} \sqrt{ax} + \sqrt{by} = 1 & (i) \\ \sqrt{ax} - \sqrt{by} = 1 & (ii) \\ -\sqrt{ax} + \sqrt{by} = 1 & (iii) \end{array}\right\} \qquad \text{......(29)}$$

The equation (29 i) is valid for points which lie on the portion of the curve between the points of contact A, B, as for such points $\sqrt{ax} < 1$, $\sqrt{by} < 1$. If the point lies on the portion of the curve A∞, $\sqrt{ax} > 1$ and $\sqrt{by} < \sqrt{ax}$. This portion is, therefore, represented by the equation (29 ii). For points on the portion B ∞, $by > 1$, and $ax < by$ and thus its equation is (29 iii).

There is no part of the parabola which is represented by the equation $\sqrt{ax}+\sqrt{by}=-1$.

114.3. Parametric Equations of a parabola.

The equation of the parabola can be written in the form (Cf. 28 A):

$$(ax-by)^2-1=2(ax+by-1).$$

Writing it as

$$\frac{ax-by+1}{2}=\frac{ax+by-1}{ax-by-1}=t,$$

we have

$$ax-by=2t-1$$
$$ax(1-t)+by(1+t)=1-t$$

whence $\quad ax=t^2,\ by=(t-1)^2 \qquad \text{......(30)}$

which are the required equations.

Remark. Every line parallel to $ax-by=0$ meets the parabola at one point at infinity. Thus the line $ax-by=c$ for all values of c is a diameter of the parabola.

115. Equation of the chord joining two points on a parabola.

Let (x_1, y_1), (x_2, y_2) be two points on the parabola

$$\sqrt{ax}+\sqrt{by}=1.$$

The equation of the chord joining these two points is

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2},$$

with the conditions

$$\sqrt{ax_1}+\sqrt{by_1}=1.$$
$$\sqrt{ax_2}+\sqrt{by_2}=1.$$

Thus $\sqrt{a}\,(\sqrt{x_1}-\sqrt{x_2})+\sqrt{b}\,(\sqrt{y_1}-\sqrt{y_2})=0.$

Hence the equation of the chord takes the form

$$(x-x_1)\sqrt{\frac{a}{x_1+x_2}}+(y-y_1)\sqrt{\frac{b}{y_1+y_2}}=0. \qquad \text{......(31)}$$

115.1. Let t_1, t_2 be the parameters of the points and suppose that the equation of the line that join these points is

$$lax+mby=1\ ;$$

then t_1, t_2 are the roots of the quadratic

$$lt^2+m(t-1)^2+1=0$$

i.e., $\quad t^2(l+m)-2mt+(m+1)=0$;

$$\therefore (t_1+t_2)=\frac{2m}{l+m}, \qquad t_1t_2=\frac{m+1}{l+m};$$

whence $\quad l=\dfrac{2-t_1-t_2}{2t_1t_2-t_1-t_2}; \quad m=\dfrac{t_1+t_2}{2t_1t_2-t_1-t_2}.$

Thus the equation of the chord is

$$ax(t_1+t_2-2)-(t_1+t_2)by=2t_1t_2-t_1-t_2. \qquad \text{......(32)}$$

115.2. Tangent at a point. Let the point (x_2, y_2) move along the curve towards (x_1, y_1). In the ultimate position, when the two points approach coincidence, the equation (31) of the chord becomes

$$(x-x_1)\sqrt{\frac{a}{x_1}}+(y-y_1)\sqrt{\frac{b}{y_1}}=0,$$

or

$$x\sqrt{\frac{a}{x_1}}+y\sqrt{\frac{b}{y_1}}=1, \qquad \text{......(33)}$$

which is the equation of the tangent at (x_1, y_1), for

$$\sqrt{ax_1}+\sqrt{by_1}=1.$$

The equation of the tangent at t_1 can similarly be obtained from equation (32) by making $t_2 \to t_1$. Thus the equation of the tangent at t is

$$ax(t-1)-tby=t^2-t. \qquad \text{......(34)}$$

115.3. Condition of Tangency. Let $lx+my+n=0$ be a tangent to the parabola. This equation is, therefore, identical with equation (34) for some value of t. Thus

$$\frac{a(t-1)}{l}=\frac{-bt}{m}=\frac{-t(t-1)}{n}.$$

whence $\quad \dfrac{a}{l}=\dfrac{-t}{n}, \quad \dfrac{+b}{m}=\dfrac{t-1}{n}.$

Hence the condition for tangency is

$$\frac{a}{l}+\frac{b}{m}+\frac{1}{n}=0. \qquad \text{......(35)}$$

116. Tangent at the vertex. The line $ax-by=0$ is a diameter of the parabola (Cf. 114.3). The line at right angles to it is

$$\frac{x}{a+b\cos\omega}+\frac{y}{b+a\cos\omega}=k,$$

and this will be tangent if (35)

$$a(a+b\cos\omega)+b(b+a\cos\omega)-\frac{1}{k}=0;$$

$$\therefore \quad k=\frac{1}{a^2+b^2+2ab\cos\omega}.$$

Thus the equation of the tangent at the vertex is

$$\frac{x}{a+b\cos\omega}+\frac{y}{b+a\cos\omega}=\frac{1}{a^2+b^2+2ab\cos\omega} \qquad \ldots\ldots(36)$$

116.1. Vertex. The vertex is the point of contact of the tangent given by the equation (36). Comparing it with (33) we see that

$$\frac{\sqrt{x_1}}{(a+b\cos\omega)\sqrt{a}}=\frac{\sqrt{y_1}}{(b+a\cos\omega)\sqrt{b}}=\frac{1}{a^2+b^2+2ab\cos\omega}.$$

Hence the vertex is given by the equations

$$\therefore\quad x=\frac{a(a+b\cos\omega)^2}{(a^2+b^2+2ab\cos\omega)^2},\quad y=\frac{b(b+a\cos\omega)^2}{(a^2+b^2+2ab\cos\omega)^2} \ldots\ldots(37)$$

116.2 Axis. The axis of the parabola is parallel to the diameter $ax-by=0$ and passes through the vertex. Its equation is, therefore

$$ax-by=\frac{a^2(a+b\cos\omega)^2-b^2(b+a\cos\omega)^2}{(a^2+b^2+2ab\cos\omega)^2}$$

$$i.e.,\ ax-by=\frac{a^2-b^2}{a^2+b^2+2ab\cos\omega}. \qquad \ldots\ldots(38)$$

The equation $(ax-by)^2=2(ax+by)-1$ of the parabola can be written as

$$(ax-by+\lambda)^2=2ax(1+\lambda)+2by(1-\lambda)+\lambda^2-1.$$

The lines $ax-by=0$ and $ax(1+\lambda)+by(1-\lambda)=0$ are at right angles if

$$a^2-b^2+\lambda(a^2+b^2+2ab\cos\omega)=0.$$

Hence the equation of the axis is

$$ax-by=\frac{a^2-b^2}{a^2+b^2+2ab\cos\omega};$$

and the tangent at vertex is given by the equation

$$\frac{x}{a+b\cos\omega}+\frac{y}{b+a\cos\omega}=\frac{1}{a^2+b^2+2ab\cos\omega}.$$

116.3 Directrix. The directrix of a parabola is the locus of the points of intersection of perpendicular tangents.

Let (x', y') be a point on the directrix. The parameters of the points of contact of tangents drawn from (x', y') are the roots of the equation

$$t^2-t(ax'-by'+1)+ax'=0.$$

If the roots of this equation be t_1 and t_2

$$t_1+t_2=ax'-by'+1,\quad t_1t_2=ax'.$$

The tangents at t_1, t_2 have respectively the equations

$$ax(t_1-1)-bt_1y=t_1^2-t_1,$$
$$ax(t_2-1)-bt_2y=t_2^2-t_2.$$

These are at right angles if

$$a^2(t_1-1)(t_2-1)+b^2t_1t_2+ab\,[t_2(t_1-1)+t_1(t_2-1)]\cos\omega=0$$

i.e., $a^2[t_1t_2-(t_1+t_2)+1]+b^2t_1t_2+ab[2t_1t_2-(t_1+t_2)]\cos\omega=0$

or $ay'+bx'+(ax'+by'-1)\cos\omega=0.$

Hence the locus of (x', y') is the line

$$x(b+a\cos\omega)+y(a+b\cos\omega)=\cos\omega, \qquad \ldots\ldots(39)$$

which is, therefore, the equation of the directrix.

116.4. Second method. Let h, k be the intercepts on the axes made by the directrix. One of the tangents from $(h, 0)$ is $y=0$. The line at right angles to it through $(h, 0)$ is $x+y\cos\omega=h$. This will be a tangent to the parabola if (35.)

$$a+\frac{b}{\cos\omega}=\frac{1}{h}. \qquad \therefore \quad h=\frac{\cos\omega}{a\cos\omega+b}.$$

Similarly, one tangent from $(0,k)$ is $x=0$, the second tangent from it is perpendicular to $x=0$ and is therefore of the form $x\cos\omega+y=k$. This will be a tangent if

$$\frac{a}{\cos\omega}+b=\frac{1}{k},$$

$$\therefore \qquad k=\frac{\cos\omega}{a+b\cos\omega}.$$

Substituting the values of h and k in the equation

$$\frac{x}{h}+\frac{y}{k}=1$$

we get the equation (39) of the directrix.

117. Focus. Let $S(\xi, \eta)$ be the focus. The line

$$y-\eta+(x-\xi)\operatorname{cis}\omega=0$$

i.e., $\qquad x\operatorname{cis}\omega+y-(\eta+\xi\operatorname{cis}\omega)=0,$

where $\operatorname{cis}\omega\equiv\cos\omega+i\sin\omega$, is an isotropic line through (ξ, η).

This will be a tangent to the parabola if

$$\frac{a}{\operatorname{cis}\omega}+b-\frac{1}{\eta+\xi\operatorname{cis}\omega}=0,$$

i.e., $[a\cos\omega+b)-i\,a\sin\omega][(\eta+\xi\cos\omega)+i\,\xi\sin\omega]=1.$

Separating the real and imaginary parts, we have

$$\xi(a+b\cos\omega)+\eta(b+a\cos\omega)=1,$$
$$a\eta-b\xi=0.$$

$$\therefore \quad \frac{\xi}{a}=\frac{\eta}{b}=\frac{1}{a^2+b^2+2ab\cos\omega}.$$

117.1. Second Method. The feet of the perpendiculars from the focus on the tangents to a parabola lie on the tangent at the vertex.

Let (ξ, η) be the focus. The feet of the perpendiculars from (ξ, η) on the tangents $y=0$, $x=0$ are respectively

$$(\xi+\eta \cos \omega,\ 0),\ (0,\ \eta+\xi \cos \omega).$$

The equation of the line joining these points is

$$\frac{x}{\xi+\eta \cos \omega}+\frac{y}{\eta+\xi \cos \omega}=1,$$

which must be identical with equation (36).

$$\therefore \quad \frac{\xi+\eta \cos \omega}{a+b \cos \omega}=\frac{\eta+\xi \cos \omega}{b+a \cos \omega}=\frac{1}{a^2+b^2+2ab \cos \omega}$$

whence immediately

$$\frac{\xi}{a}=\frac{\eta}{b}=\frac{1}{a^2+b^2+2ab \cos \omega}.$$

117.2. Latus Rectum. Latus rectum is twice the perpendicular from the focus on the directrix. The length will be found to be

$$\frac{4ab \sin^2\omega}{(a^2+b^2+2ab \cos \omega)^{\frac{3}{2}}}.$$

Illustrative Examples

(1) *If a parabola whose latus rectum is $4p$ slides between two rectangular axes, prove that the locus of its vertex is the curve*

$$x^{\frac{2}{3}}\, y^{\frac{2}{3}}(x^{\frac{2}{3}}+ y^{\frac{2}{3}})=p^2.$$

First Method. If the equation of the parabola be

$$(ax+by-1)^2=4abxy$$

its latus rectum (Art. 117.2) is

$$\frac{4ab}{(a^2+b^2)^{\frac{3}{2}}}=4p$$

and the vertex is given by the equations (Art. 116.1)

$$x=\frac{a^3}{(a^2+b^2)^3},\ y=\frac{b^3}{(a^2+b^2)^3};$$

whence $\quad x^{\frac{2}{3}}+ y^{\frac{2}{3}}=\dfrac{1}{(a^2+b^2)^{\frac{1}{3}}},\quad x^{\frac{2}{3}}\, y^{\frac{2}{3}}=\dfrac{a^2b^2}{(a^2+b^2)^{\frac{8}{3}}}.$

$$\therefore \quad x^{\frac{2}{3}}\, y^{\frac{2}{3}}\, (x^{\frac{2}{3}}+ y^{\frac{2}{3}})=p^2.$$

Second Method. Let (x', y') be the vertex. Since the axes are orthogonal tangents to the parabola, the origin lies on

the directrix. Consequently, the directrix and the tangent at the vertex are given by equations of the form

$$lx+my=0. \quad l(x-x')+m(y-y')=0.$$

The perpendicular from the vertex on the directrix equals p,

i.e., $$\frac{lx'+my'}{\sqrt{l^2+m^2}}=p.$$

The intercepts made by the tangent at the vertex on the axes of x and y are $\dfrac{lx'+my'}{l}$, $\dfrac{lx'+my'}{m}$. These are therefore the co-ordinates of the focus. The line that joins the focus and the vertex is the axis of the parabola which is perpendicular to the directrix. This gives the condition $\dfrac{l^3}{m^3}=\dfrac{y'}{x'}$, whence

$$\frac{m^2}{l^2+m^2}=\frac{x'^{\frac{2}{3}}}{x'^{\frac{2}{3}}+y'^{\frac{2}{3}}}, \quad \frac{l^2}{\sqrt{l^2+m^2}}=\frac{y'^{\frac{2}{3}}}{x'^{\frac{2}{3}}+y'^{\frac{2}{3}}}$$

Thus $$p=\frac{y'^{\frac{1}{3}}x'+x'^{\frac{1}{3}}y'}{(x'^{\frac{2}{3}}+y'^{\frac{2}{3}})^{\frac{1}{2}}}=x'^{\frac{1}{3}}y'^{\frac{1}{3}}(x'^{\frac{2}{3}}+y'^{\frac{2}{3}})^{\frac{1}{2}}.$$

Hence the locus of (x', y') is the curve

$$x^{\frac{2}{3}}\,y^{\frac{2}{3}}\,(x^{\frac{2}{3}}+y^{\frac{2}{3}})=p^2.$$

(2) *A parabola of latus rectum 4c slides between two rectangular lines. Prove that the locus of the focus referred to those lines as axes is the curve* $x^2y^2=c^2(x^2+y^2)$.

Let the equation of the parabola be

$$(ax+by-1)^2=4abxy.$$

The latus rectum of the parabola is (Art. 117.2)

$$\frac{4ab}{(a^2+b^2)^{\frac{3}{2}}}=4c. \qquad \ldots\ldots(i)$$

and the focus is given by the equations

$$\frac{x}{a}=\frac{y}{b}=\frac{1}{a^2+b^2}.$$

whence $$x^2+y^2=\frac{1}{a^2+b^2}, \quad ab=xy(a^2+b^2)^2,$$

and from (i) $$c=xy(a^2+b^2)^{\frac{1}{2}}=\frac{xy}{(x^2+y^2)^{\frac{1}{2}}},$$

i.e., $$x^2y^2=c^2(x^2+y^2).$$

Second method. Let (ξ, η) be the focus, then the feet of the perpendiculars $(\xi, 0)$, $(0, \eta)$ on the axes which are orthogonal tangents to the parabola lie on the tangent at the vertex. The equation of the tangent is, therefore,

$$\frac{x}{\xi}+\frac{y}{\eta}-1=0.$$

The perpendicular from the focus on this tangent is c,

i.e., $$\frac{1}{\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right)^{\frac{1}{2}}}=c.$$

Hence the locus of (ξ, η) is the curve

$$x^2y^2=c^2(x^2+y^2).$$

(3) *Parabolas are drawn to touch the axes, which are inclined at an angle* ω, *and their directrices all pass through a fixed point* (h, k). *Prove that all the parabolas touch the line*

$$\frac{x}{h+k\sec\omega}+\frac{y}{k+h\sec\omega}=1.$$

Let the equation of the parabolas be

$$(ax+by-1)^2=4abxy,$$

where a and b are variable. The equation of the directrix of the parabola is

$$x(b+a\cos\omega)+y(a+b\cos\omega)=\cos\omega.$$

This passes through (h, k)

$\therefore$ $\quad h(b+a\cos\omega)+k(a+b\cos\omega)=\cos\omega$

or $\quad a(h\cos\omega+k)+b(h+k\cos\omega)=\cos w$

i.e., $\quad a(h+k\sec\omega)+b(k+h\sec\omega)=1$

and this is the condition that the line

$$\frac{x}{h+k\sec\omega}+\frac{y}{k+h\sec\omega}=1$$

may touch the parabola.

Exercises XXXII

1. Parabolas are drawn to touch two given rectangular axes and their foci are all at a constant distance c from the origin. Prove that the locus of the vertices of these parabolas is the curve

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}.$$

2. A tangent to the parabola $(ax+by-1)^2=4abxy$ meets the co-ordinate axes at the points A and B. Show that the locus of the mid-point of AB is the st. line $ax+by=\frac{1}{2}$.

3. Parabolas are drawn to touch two given st. lines which are inclined at an angle ω; if the chords of contact all pass through a fixed point, prove that

(*i*) their directrices all pass through another fixed point,

(*ii*) their foci all lie on a circle which passes through the intersection of the two given lines.

4. A parabola touches three given lines, prove that each of the lines joining the points of contact passes through a fixed point.

5. A parabola touches the axes of co-ordinates. If its axis passes through the point (h, k), prove that its focus lies on the conic $x^2 - y^2 - hx + ky = 0$.

6. A tangent to the parabola $\sqrt{ax} + \sqrt{by} = 1$ meets the axes of x and y in P, Q, and perpendiculars are drawn from P, Q to the y and x axes : prove that the locus of their points of intersection is the line

$$b(x + y\cos\omega) + a(y + x\cos\omega) = \cos\omega.$$

7. Show that the parallels through the origin to the tangents from (x', y') to $\sqrt{x} + \sqrt{y} = \sqrt{a}$ are the lines

$$axy + (x - y)(xy' - x'y) = 0.$$

8. Show that the pair of tangents from (x', y') to the parabola $\sqrt{ax} + \sqrt{by} = 1$ are given by the equation

$$\frac{a}{y - y_0} - \frac{b}{x - x_0} + \frac{1}{xy_0 - x_0y} = 0.$$

Miscellaneous Examples XXXIII

1. PNP′ is a double ordinate of the parabola $y^2 = 4ax$. If the tangent at P meets the diameter through P′ in Q, prove that the locus of Q is the parabola $3y^2 + 4ax = 0$.

2. Tangents are drawn to the circle $x^2 + y^2 = a^2$ from two points on the axis of x equidistant from the point $(c, 0)$; show that the locus of their intersection is the parabola $y^2 = 4ax$.

3. If P moves on the line $2ax + by + 4a^2 = 0$, show that a line through P perpendicular to its polar with regard to the parabola $y^2 = 4ax$ touches the parabola $x^2 = 4by$.

4. Through the vertex A of the parabola $y^2 = 4ax$, two chords AP, AQ are drawn and the circles on AP, AQ as diameters intersect in R. Prove that if θ_1, θ_2 and ϕ be the angles made with the axis by the tangents at P and Q and by AR, then

$$\cot\theta_1 + \cot\theta_2 + 2\tan\phi = 0.$$

5. Through the vertex A of the parabola $y^2 = 4ax$ is drawn a chord AP meeting the parabola at P. The perpendicular at A to AP meets the tangent and normal at P in T and N. Show that the loci of T and N are respectively the curves

$$2y^2(x + 2a) + x^3 = 0, \qquad 8ay^2 = x^2(x - 4a).$$

If the perpendicular to PF at F (F being the focus) meets the tangent and normal at P in T′, N′ show that the loci of T′, N′ are respectively the curves

$$x+a=0, \qquad 8y^2=x(x-2a)^2.$$

6. A point P moves on the parabola $y^2=4ax$. The tangent at P meets the tangent at the vertex in Y. Through Y draw a parallel to AP and through A draw a parallel to PY. Let these lines meet in Y′. Show that

(*i*) the envelope of Y′Y is $y^2=8ax$,

(*ii*) the locus of Y′ is $y^2+ax=0$,

(*iii*) the envelope of PY′ is $y^2-3ax=0$,

(*iv*) the locus of the centre of the circle APY is

$$2y^2-x+a^2=0.$$

7. Show that the locus of the mid-points of chords of constant length of the parabola $y^2=4ax$ is the curve

$$(4ax-y^2)(y^2+4a^2)=a^2c^2.$$

8. The tangents at P and Q of the parabola $y^2=4ax$ meet at (x_1, y_1); show that $PQ^2=\dfrac{(y_1^2-4ax_1)(y_1^2+4a^2)}{a^2}$.

9. Show that the distance between a tangent to a parabola and the parallel normal is a cosec θ $\sec^2\theta$, where θ is the angle that either makes with the axis of the parabola and $4a$ is the latus rectum.

10. Prove that the length of the normal chord at P of the parabola $y^2=4ax$ is $\dfrac{4ax}{\sin\theta\cos^2\theta}$, where θ is the inclination of the tangent at P with the axis.

11. Prove that the point of intersection of the normals to the parabola $y^2=4ax$ at the points of intersection with the line $lx+my+n=0$ are given by

$$l^2x=2al^2+4am^2-nl, \quad l^2y=2mn.$$

12. The tangents at Q and R of the parabola $y^2=4ax$ meet in P. Show that the locus of the mid-point of QR when P lies on $lx+my+n=0$ is the parabola $l(y^2-4ax)+2a(lx+my+n)=0$.

13. If the perpendicular AY drawn from the vertex A to the tangent at any point P to the parabola $y^2=4ax$ meets the curve again at Z, prove that AY.AZ$=4a^2$.

14. MP is an ordinate of a point P on a parabola. A st. line is drawn parallel to the axis bisecting MP and meeting the curve at Q, MQ cuts the tangent at the vertex A at T; show that AT$=\frac{2}{3}$MP.

15. Two tangents are drawn to a parabola, so that they intercept a constant length c on the tangent at the vertex. Show that the locus of their intersection is a parabola.

16. If a chord of the parabola $y^2=4ax$ be a tangent to the parabola $y^2=8a(x-c)$, show that the mid-point of the chord lies on the line $x=c$.

17. PQ is a double ordinate of a parabola and the line joining P to the foot of the directrix cuts the curve in P′. Show that P′Q passes through the focus.

18. If $\dfrac{b}{a-c}>2$, the parabolas $y^2=4c(x-b)$ and $y^2=4ax$, have a pair of common normals equally inclined to the common axis and the distance d between the curves measured along one of these common normals is given by

$$d^2=4(c-a)(a-b-c).$$

19. If the normal at $P(at^2, 2at)$ to the parabola $y^2=4ax$ meets the curve again at Q and if A be the vertex of the parabola, the area of the triangle APQ is $2a^2(1+t^2)(2+t^2)/t$.

20. Prove that perpendicular normal chords of a parabola divide one another in the ratio 3 : 1

21. If normals PO, QO to a parabola intersect at right angles in O, the third normal RO through O cuts the axis of the parabola in G, such that 3OG = OR. If the equation of the parabola be $y^2=4ax$ prove that the locus of O is the parabola

$$y^2+a(3a-x)=0.$$

22. If the normals at two points P and Q on the parabola $y^2=4ax$ intersect on the fixed diameter $y=\eta$, prove that the pole of PQ lies on the hyperbola $xy+a\eta=0$.

23. P is any point on a parabola whose vertex is A, and Q, R are the feet of the normals from P to the curve. Show that QR passes through a fixed point and that AP, AQ meet on a fixed line.

24. The normals at P, Q of the parabola $y^2=4ax$ meet at a point R on the curve. If T be the pole of PQ, prove that the centre of the circum-circle of PTQ for all positions of R on the parabola lies on the curve $2y^2=a(x-a)$.

25. Show that the mid-points of the sides of a triangle formed by the tangents at P, Q, R to the parabola $y^2=4ax$ lie on the parabola $2y^2+ax=0$ if the normals at P, Q, R meet in a point.

26. If the chord PQ of the parabola $y^2=4ax$, passes through $(-2a, 0)$, the normals at P, Q meet on the curve and contain an angle equal to PAQ.

27. If T and N are the tangential and normal poles of the chord PQ of a parabola whose focus is S and if M is the mid-point of TN, show that $\angle TSM=\dfrac{\pi}{2}$.

28. If P, Q, R is a triad of co-normal points on the parabola $y^2=4ax$ and if P and Q approach coincidence at (ξ, η) show that the equation of PR is

$$\frac{x}{\xi}+\frac{y}{\eta}=2.$$

29. A chord of the parabola $y^2=4ax$ passes through the point $(\lambda a, 0)$. Prove that the normals at its extremities intersect on the curve $y^2=\lambda^2 a(x-\lambda a-2a)$.

30. The normals at P, Q, the ends of a focal chord, meet the curve again at P′, Q′. Show that P′Q′ is parallel and equal to 3PQ.

31. AQ, AR are chords of the parabola $y^2=4ax$ drawn at right angles to each other from the vertex A. The rectangle AQPR is completed on AQ, AR. Prove that the locus of P is the parabola

$$y^2=4a(x-8a).$$

32. Circles are drawn on any two focal chords of a parabola as diameters. Prove that their common chord passes through the vertex of the parabola.

33. Show that all circles on focal chords of a parabola as diameters touch the directrix, and that all circles on focal radii as diameters touch the tangent at the vertex.

34. A circle is described on a focal chord as diameter; if m be the tangent of the inclination of the chord to the axis, prove that the equation of the circle is

$$x^2+y^2-2ax\left(1+\frac{2}{m^2}\right)-\frac{4ay}{m}-3a^2=0.$$

35. Show that the locus of the poles of normal chords of the parabola $y^2=4ax$ is the curve $(x+2a)y^2+4a^3=0$.

36. Show that the locus of the poles of the chords which subtend a constant angle α at the vertex is the curve

$$(x+4a)^2=4(y^2-4ax)\cot^2\alpha.$$

37. A tangent to $y^2+4bx=0$ meets $y^2=4ax$ at P and Q. Prove that the locus of mid-point of PQ is

$$y^2(2a+b)=4a^2x.$$

38. A tangent to the parabola $y^2=4ax$ at $(at^2, 2at)$ meets the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles are θ and ϕ, prove that

$$\tan\frac{\theta+\phi}{2}=-\frac{bt}{2}.$$

39. At the point of intersection of the rectangular hyperbola $xy=k^2$ and the parabola $y^2=4ax$, the tangents to the hyperbola

and parabola make angles θ and ϕ respectively with the axis of x. Prove that $\tan\theta = -2\tan\phi$.

40. If the line $lx+my+na=0$ meets $y^2=4ax$ in P and Q, and if the lines joining P, Q to the focus F meet the parabola in T, U, show that the equation of TU is $nx-my+la=0$.

41. Show that the area of the triangle formed by three normals $y=m_r x+c_r$ to the parabola $y^2=4ax$ is

$$\frac{a^2}{2}\cdot(m_1\sim m_2)(m_2\sim m_3)(m_3\sim m_1)(m_1+m_2+m_3)^2.$$

42. The triangle formed by the points (x_r, y_r), $r=1, 2, 3$ is self-conjugate *w. r.* to the parabola $y^2=4ax$; show that its area is $\frac{1}{4}(y_2\sim y_3)(y_3\sim y_1)(y_1\sim y_2)/a$.

43. A st. line is drawn perpendicular to the axis of a parabola. With any point P on it as centre, and any radius, a circle is drawn cutting the parabola in four points. Show that the sum of the focal distances of the four points is the same for all points P on the line.

44. Two lines are drawn at right angles, one being a tangent to $y^2=4ax$, and the other to $x^2=4by$. Show that the locus of their point of intersection is the curve

$$(x^2+y^2)(ax+by)+(bx-ay)^2=0.$$

45. Show that the tangents to the parabola $y^2=4ax$ at the intersections of the line $px+qy+a=0$ include an angle

$$\tan^{-1}\frac{2\sqrt{q^2-p}}{1+p}.$$

46. A circle through the vertex of the parabola $y^2=4ax$ meets it again in points the tangents at which form a triangle PQR. Show that the mid-points of the sides of the triangle lie on the parabola

$$2y^2+ax=0.$$

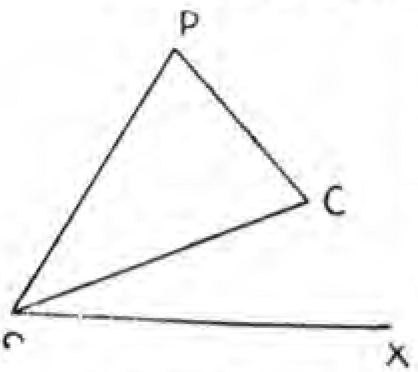
47. The normals at points P, Q, R of the parabola $y^2=4ax$ meet at a point (x_0, y_0), prove that the centre of the nine-point circle of the triangle PQR is

$$\left(\frac{3x_0-10a}{4}, \frac{-y_0}{8}\right).$$

CHAPTER XI

POLAR EQUATION OF A CONIC

118. Polar Equation of a Circle.



Let OX be the initial line with O as pole. Suppose C (ρ, α) is the centre of the circle of radius 'a' and P (r, θ) any point on it. From the triangle OCP

$$a^2 = r^2 + \rho^2 - 2r\rho \cos(\theta - \alpha) \qquad \text{......(1)}$$

Particular cases.

(*i*) If the origin is on the circle $a = \rho$, the equation of the circle takes the form

$$r = 2a \cos(\theta - \alpha). \qquad \text{......}(i)$$

(*ii*) If the origin lies on the circle and the initial line passes through the centre, $a = \rho$, $\alpha = 0$, the equation of the circle then becomes

$$r = 2a \cos\theta. \qquad \text{......}(ii)$$

(*iii*) If the origin is the centre of the circle, $\alpha = 0$, $\rho = 0$, the equation reduces to

$$r = a. \qquad \text{......}(iii)$$

Illustrative Examples

1. *Find the equation of the chord joining the points whose vectorial angles are θ_1, θ_2 on the circle $r = 2a\cos\theta$ and deduce the equation of the tangent at the point θ_1.*

Let the equation of the chord be

$$p = r\cos(\theta - \alpha)$$

which joins the points $(2a\cos\theta_1, \theta_1)$, $(2a\cos\theta_2, \theta_2)$,

$$\therefore \quad p = 2a\cos\theta_1 \cos(\theta_1 - \alpha),$$
$$p = 2a\cos\theta_2 \cos(\theta_2 - \alpha),$$

whence $\cos\theta_1 \cos(\theta_1 - \alpha) = \cos\theta_2 \cos(\theta_2 - \alpha)$

i. e. $\quad \cos(2\theta_1 - \alpha) = \cos(2\theta_2 - \alpha)$

$\therefore \quad 2\theta_1 - \alpha = -(2\theta_2 - \alpha)$

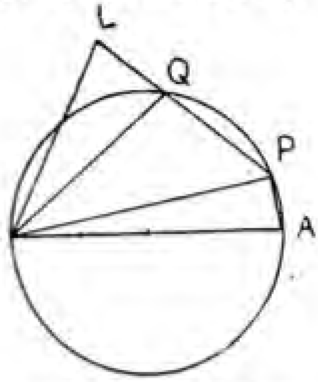
Hence $\quad \alpha = \theta_1 + \theta_2$

and therefore $\quad p = 2a\cos\theta_1 \cos\theta_2.$

Consequently the equation of the chord is

$$2a\cos\theta_1 \cos\theta_2 = r\cos(\theta - \theta_1 - \theta_2). \qquad \text{......(2)}$$

Second method. Let $P(2a\cos\theta_1, \theta_1)$, $Q(2a\cos\theta_2, \theta_2)$ be the two points, and OL the perpendicular from O on PQ.

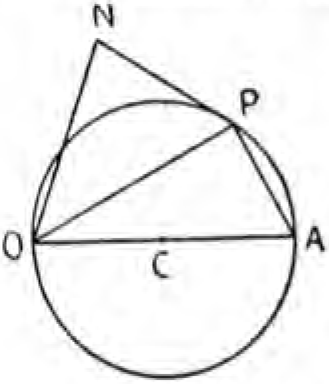


$$\angle QOL = \frac{\pi}{2} - \angle OQL = \frac{\pi}{2} - \angle PAO$$
$$= \theta_1$$
$$\therefore \quad \angle AOL = \theta_1 + \theta_2,$$
and $\quad OL = OQ\cos\theta_1 = 2a\cos\theta_1\cos\theta_2$

Hence the equation of the chord is
$$2a\cos\theta_1\cos\theta_2 = r\cos(\theta - \theta_1 - \theta_2).$$

To obtain the equation of the tangent we make $\theta_2 \to \theta_1$ along the curve. The required equation is
$$2a\cos^2\theta_1 = r\cos(\theta - 2\theta_1). \qquad \ldots\ldots(3)$$

The following is an alternative method.



If PN be the tangent at P, (r_1, θ_1).
$$\angle OPN = \angle OAP$$
$$\therefore \quad \angle AOP = \angle PON$$
$$ON = r_1\cos\theta_1 = 2a\cos^2\theta_1.$$

Also ON makes an angle $2\theta_1$ with the initial line.

Thus the equation to the tangent at P is
$$2a\cos^2\theta_1 = r\cos(\theta - 2\theta_1).$$

(2) **Show that the Polar of (r_1, θ_1) w. r. to the circle $r = 2a\cos\theta$ is given by the equation**
$$rr_1\cos(\theta - \theta_1) = a(r\cos\theta + r_1\cos\theta_1).$$

Let $P(r_1, \theta_1)$ be the point whose polar w. r. to the circle with centre C is the line MN. Draw OM perpendicular to OM from the pole O,
$$\therefore \quad CP \parallel OM$$

Let M be (p, α) and $CP = \rho$,
$$r_1\cos\theta_1 - a = \rho\cos\alpha, \qquad \ldots\ldots(i)$$
$$r_1\sin\theta_1 = \rho\sin\alpha. \qquad \ldots\ldots(ii)$$

Also $$CN = \frac{a^2}{\rho}.$$

$$\therefore \quad p = OM = CN + OC\cos\alpha = \frac{a^2}{\rho} + a\cos\alpha$$
$$= \frac{a\,r_1\cos\theta_1}{\rho}, \qquad \text{from } (i).$$

Thus the polar of P is given by the equation

$$ar_1 \cos\theta_1 = r\rho\cos(\theta-\alpha)$$
$$= r\cos\theta\,(r_1\cos\theta_1 - a) + r\sin\theta\, r_1\sin\theta_1$$
$$= rr_1\cos(\theta-\theta_1) - ar\cos\theta$$

i. e., $rr_1\cos(\theta-\theta_1) = ar\cos\theta + ar_1\cos\theta_1$

or
$$\frac{\cos(\theta-\theta_1)}{a} = \frac{\cos\theta}{r_1} + \frac{\cos\theta_1}{r}.$$

Second method. Let QN be the polar of P, cutting OP in Q (ρ, θ_1) and let OP meet the circle again in R. Then $(\mathrm{OR}, \mathrm{QP}) = -1$.

$$\therefore \qquad \frac{2}{\mathrm{OR}} = \frac{1}{\mathrm{OQ}} + \frac{1}{\mathrm{OP}}$$

i. e.,
$$\frac{1}{a\cos\theta_1} = \frac{1}{\rho} + \frac{1}{r_1}.$$

The equation CP, where C is $(a, 0)$, $\mathrm{P}(r_1, \theta_1)$ is

$$\frac{\sin\theta_1}{r} + \frac{\sin(\theta-\theta_1)}{a} = \frac{\sin\theta}{r_1}.$$

The polar of P is perpendicular to CP and will have an equation of the form

$$\frac{k}{r} + \frac{\sin\left(\theta + \frac{\pi}{2} - \theta_1\right)}{a} = \frac{\sin\left(\theta + \frac{\pi}{2}\right)}{r_1}$$

i. e.,
$$\frac{k}{r} + \frac{\cos(\theta-\theta_1)}{a} = \frac{\cos\theta}{r_1},$$

where k is constant.

This passes through Q.

$$\therefore \qquad k\left(\frac{1}{a\cos\theta_1} - \frac{1}{r_1}\right) + \frac{1}{a} = \frac{\cos\theta_1}{r_1}$$

i.e.,
$$k = -\cos\theta_1.$$

Thus the equation of the required polar is

$$\frac{\cos(\theta-\theta_1)}{a} = \frac{\cos\theta}{r_1} + \frac{\cos\theta_1}{r}.$$

Third method. Let a line through P (r_1, θ_1) cut the circle in Q (p, α) and R (q, β). Then the locus of the intersection of the tangents at Q and R is the polar of P.

The equation to QR is

$$2a\cos\alpha\cos\beta = r\cos(\theta-\alpha-\beta).$$

Since QR passes through P, $2a\cos\alpha\cos\beta = r_1\cos(\theta_1-\alpha-\beta)$

i.e., $\dfrac{2a}{r_1} = \cos\theta_1(1-\tan\alpha\tan\beta) + \sin\theta_1(\tan\alpha+\tan\beta)$ ......(*i*)

The tangents at Q and R are given by the equations
$2a\cos^2\alpha = r\cos(\theta - 2\alpha)$; $2a\cos^2\beta = r\cos(\theta - 2\beta)$.
Hence $\tan\alpha$ and $\tan\beta$ are the roots of the equation in t, *i.e.*

$$\cos\theta . t^2 - 2\sin\theta . t + \frac{2a}{r} - \cos\theta = 0,$$

$$\therefore \qquad \tan\alpha + \tan\beta = \frac{2\sin\theta}{\cos\theta},$$

$$\tan\alpha\tan\beta = \left(\frac{2a}{r} - \cos\theta\right) \Big/ \cos\theta. \qquad \ldots\ldots (ii)$$

Eliminating $\tan\alpha$, $\tan\beta$ from relations (i) and (ii)

$$\frac{2a\cos\theta}{r_1} = \cos\theta_1\left(2\cos\theta - \frac{2a}{r}\right) + 2\sin\theta\sin\theta_1$$

i. e., $$\frac{\cos(\theta - \theta_1)}{a} = \frac{\cos\theta}{r_1} + \frac{\cos\theta_1}{r}.$$

This method is general.

(3) *Show that the equation* $r^2 - 2(b - \lambda) r\cos\theta + 2\lambda d = 0$, *for different values of* λ, *represents a system of co-axal circles. Find the radical axis and the limiting points of the system.*

The equation can be written in the form
$$r^2 - 2br\cos\theta + 2\lambda(r\cos\theta + d) = 0$$
which is a circle and passes through the common points of the loci
$$r = 2b\cos\theta, \qquad r\cos\theta + d = 0.$$

The first equation represents a circle and the second a st. line. Thus the equation represents a co-axal system with $r\cos\theta + d = 0$ as the radical axis. This is at right angles to the initial line, consequently the line of centres is the initial line. A comparison of the equation of the system with the equation (i) Art. 118 shows

$$a = 0 \quad b - \lambda = \rho, \quad 2\lambda d = \rho^2 - a^2.$$

For point-circles of the system, the radius $a = 0$,

$$\therefore \qquad \rho^2 = 2d(b - \rho)$$

$$\therefore \qquad \rho = d \pm \sqrt{d^2 + 2bd}.$$

Hence the co-ordinates of the limiting points are

$$(-d \pm \sqrt{d^2 + 2bd}, 0).$$

Exercises XXXIV

1. If st. lines through a point O meet a circle in pairs of points P_n, Q_n $(n = 1, 2\ldots\ldots)$, show that $OP_n . OQ_n$ is constant.

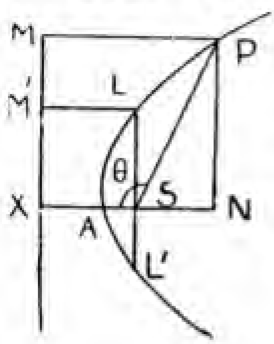
2. Show that the st. line $\frac{1}{r} = a\cos\theta + b\sin\theta$ will touch the circle $r = 2c\cos\theta$ if $b^2c^2 + 2ac = 1$.

3. The polar equation of the circle on the join of the points $A(a, \alpha)$, $B(b, \beta)$ as diameter is

$$r^2 - r\{a \cos(\theta - \alpha) + b \cos(\theta - \beta)\} + ab \cos(\alpha - \beta) = 0.$$

[*Hint.* $PA^2 + PB^2 = AB^2$.]

119. Equation of a conic, the focus being the pole.



Let S be the focus, A the vertex, MX the directrix and $LSL' = 2l$, the latus rectum of the conic of eccentricity e. Suppose S is the pole and SX the initial line. Let $P(r, \theta)$ be an arbitrary point on the conic. Draw PM, PN perpendiculars on XM, SX respectively. Also let LM′ be parallel to SX. Now

$$r = SP = ePM = eNX = eSX + eSN$$
$$= l - er \cos\theta,$$

$$\therefore \qquad \frac{l}{r} = 1 + e \cos\theta, \qquad \ldots\ldots(4)$$

which is the required equation.

119.1. If the axis of the conic makes an angle γ with the initial line, the equation of the conic takes the form

$$\frac{l}{r} = 1 + e \cos(\theta - \gamma). \qquad \ldots\ldots(4\text{ A})$$

119.2. Polar Equation of the Directrix.

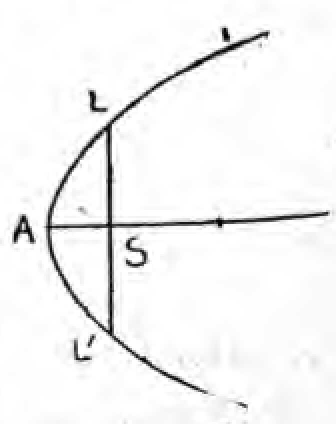
Let $M(r, \theta)$ be a point on the directrix.

$$SX = r \cos\theta$$

i.e., $$l/e = r \cos\theta$$

or $$\frac{l}{r} = e \cos\theta. \qquad \ldots\ldots(5)$$

120. To trace the conic $\dfrac{l}{r} = 1 + e \cos\theta$.



The discussion will be divided into three parts according as $e \gtreqless 1$.

(*a*) Let $e = 1$, the conic is a parabola and its equation becomes

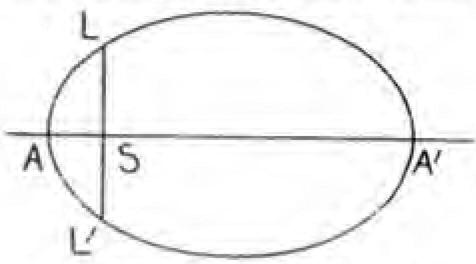
$$\frac{l}{r} = 1 + \cos\theta.$$

When $\theta = 0$, $r = \dfrac{l}{2}$, these values of θ and r correspond to the point A. As θ increases from zero to $\dfrac{\pi}{2}$, $\cos\theta$ decreases from 1 to 0, consequently r increases from

$\frac{l}{2}$ to l. As θ increases from $\frac{\pi}{2}$ to π, $\cos\theta$ decreases from 0 to -1 and therefore r increases from l to ∞. As θ increases from π to $3\pi/2$, $1+\cos\theta$ increases from 0 to 1, and r decreases from ∞ to l. Between the values $\frac{3\pi}{2}$ and 2π of θ, r decreases from l to $\frac{l}{2}$.

(*b*) Suppose $e<1$, the curve is then an ellipse.

As θ increases from 0 to π, $1+e\cos\theta$ decreases from $1+e$ to $1-e$ and therefore r increases from $\frac{l}{1+e}$ to $\frac{l}{1-e}$.

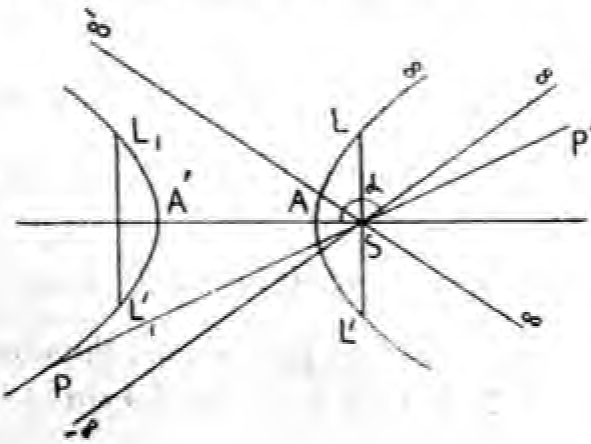


These two values of r are the radii vectors of the points A and A′ respectively. When θ changes continuously from π to 2π, $\cos\theta$ increases from -1 to 1 and r decreases from $\frac{l}{1-e}$ to $\frac{l}{1+e}$. The portion A′L′A is then described.

(*c*) Let $e>1$, the conic represented by the equation

$$\frac{l}{r}=1+e\cos\theta$$

is a hyperbola. Suppose α is the least angle which satisfies the equation $1+e\cos\theta=0$ and obviously $\frac{\pi}{2}<\alpha<\pi$. When $\theta=0$, $r=\frac{l}{1+e}$, these co-ordinates belong to the point A. As θ increases from 0 to α, r increases from $\frac{l}{1+e}$ to ∞, assuming the value l when $\theta=\frac{\pi}{2}$. The branch AL∞ of the curve is described.



When $\theta = \alpha + \epsilon$, where ϵ is an arbitrarily small positive angle, $\cos(\alpha+\epsilon) < -\frac{1}{e}$ and $1+e\cos(\alpha+\epsilon)$ is small and negative and thus the corresponding value of r approaches $(-\infty)$. Hence when θ increases from α to π, r increases from $-\infty$ to $\frac{l}{1-e}$ (<0), $\frac{l}{1-e}$ being the radius vector of A′. The branch $(-\infty)$ L_1' A′ is described.

As θ increases from π to $2\pi - \alpha - \epsilon$, r decreases from $\frac{l}{1-e}$ to $-\infty$, the branch $A'L_1$ $(-\infty)$ of the conic is described. When $\theta = 2\pi - \alpha + \epsilon$, $\cos\theta = \cos(\alpha - \epsilon) > \cos\alpha = -\frac{1}{e}$, thus $1+e\cos\theta$ is small and positive, and hence r is large and positive, therefore as θ increases from $2\pi - \alpha$ to 2π, r decreases from ∞ to $\frac{l}{1+e}$. The branch ∞ L′A is described. The curve is thus described as

$$AL\infty(-\infty)\,L_1'A'L_1\,(-\infty)\,\infty\,L'A.$$

Remarks. It must be borne in mind that the equation $\frac{l}{r} = 1+e\cos\theta$ $(e>1)$ will represent the further branch of the hyperbola, if negative radii vectors are introduced. For instance, if the point P is on the further branch, the vectorial angle of P is not ASP (which is the vectorial angle of Q) but ASP′, the angle that PS makes with SA *i.e.*, $\angle$ASP′, where P′ is on PS produced. The discussion of Art. 120(*c*) shows that if $\alpha < \theta < 2\pi - \alpha$, the corresponding radii vectors are negative for the whole branch $(-\infty)PL_1'AL_1(-\infty)$ if the equation is to be satisfied. Thus only if negative radii vectors are introduced, the equation will represent both the branches of the hyperbola.

121. Equation of the line joining two points of the conic.

Let $\alpha \pm \beta$ be the vectorial angles of the two points on the conic and

$$\frac{l}{r} = a\cos\theta + b\sin\theta$$

the equation of the line that joins them ; therefore

$$1+e\cos(\alpha+\beta) = a\cos(\alpha+\beta) + b\sin(\alpha+\beta),$$

$$1+e\cos(\alpha-\beta)=a\cos(\alpha-\beta)+b\sin(\alpha-\beta),$$

or $(a-e)\cos(\alpha+\beta)+b\sin(\alpha+\beta)-1=0$

$$(a-e)\cos(\alpha-\beta)+b\sin(\alpha-\beta)-1=0.$$

Solving the equations, we get

$$\frac{a-e}{\sin(\alpha-\beta)-\sin(\alpha+\beta)}=\frac{b}{\cos(\alpha+\beta)-\cos(\alpha-\beta)}=\frac{1}{-\sin 2\beta}$$

i.e., $\dfrac{a-e}{\cos\alpha}=\dfrac{b}{\sin\alpha}=\dfrac{1}{\cos\beta}$.

Thus the equation of the chord becomes

$$\frac{l}{r}=(e+\cos\alpha\sec\beta)\cos\theta+\sin\theta\sin\alpha\sec\beta$$

i.e., $\dfrac{l}{r}=e\cos\theta+\cos(\theta-\alpha)\sec\beta.$ ......(6)

The equation of the chord of the conic

$\dfrac{l}{r}=1+e\cos(\theta-\gamma)$ is found to be

$$\frac{l}{r}=e\cos(\theta-\gamma)+\cos(\theta-\alpha)\sec\beta. \qquad \text{......(6A)}$$

Another method. Let the extremities of the chord be $A(\rho\alpha)$ and $B(\rho'\beta)$ and $P(r\theta)$ any point on it.

The area of $\triangle$ APB=0.

But $\triangle$ APB=$\triangle$ SAP+$\triangle$ SPB+$\triangle$ SBA.

$\therefore\quad \rho r\sin(\alpha-\theta)+r\rho'\sin(\theta-\beta)+\rho\rho'\sin(\beta-\alpha)=0,$

$$\therefore\quad \frac{l}{r}\sin(\alpha-\beta)=\frac{l}{\rho'}\sin(\alpha-\theta)+\frac{l}{\rho}\sin(\theta-\beta)$$

$$=(1+e\cos\beta)\sin(\alpha-\theta)+(1+e\cos\alpha)\sin(\theta-\beta)$$

$$=2\sin\left(\frac{\alpha-\beta}{2}\right)\cos\left(\theta-\frac{\alpha+\beta}{2}\right)+e.\sin(\alpha-\beta)\cos\theta.$$

Thus the chord is represented by the equation

$$\frac{l}{r}=e.\cos\theta+\sec\frac{\alpha-\beta}{2}\cos\left(\theta-\frac{\alpha+\beta}{2}\right).$$

121.1. Equation of the tangent. The equation of the chord which joins points whose vectorial angles are $(\alpha\pm\beta)$, is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha)\sec\beta.$$

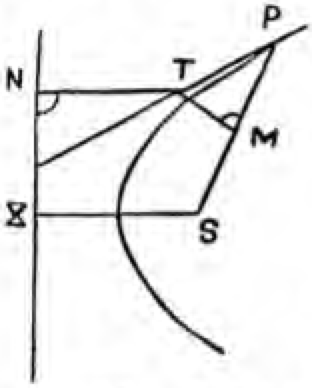
When $\beta\to 0$, the points $(\alpha\pm\beta)$ approach coincidence, and in the limiting position when chord becomes a tangent, its equation takes the form

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha). \qquad \text{......(7)}$$

The equation of the tangent of the conic $\frac{l}{r}=1+e\cos(\theta-\gamma)$ is

$$\frac{l}{r}=e\cos(\theta-\gamma)+\cos(\theta-\alpha). \qquad \ldots\ldots(7A)$$

121.2. Geometrical interpretation of the tangent equation.



P is the point α. Let T be any point (r, θ) on the tangent at P

Draw TN perp. to the directrix and TM perp. to SP.

$\therefore \angle TSP=\alpha-\theta.$

By the equation of the tangent,

$l-er\cos\theta=r\cos(\theta-\alpha)$

$=ST\cos TSP=SM$

and $l-er\cos\theta=e.SX-er\cos\theta=e.TN.$

$\therefore$ SM$=e.$ TN,

whichis Adam's property of the conic.

122. Polar equation of the normal.

Let $\left(\frac{l}{1+e\cos\alpha}, \alpha\right)$ be a point on the curve

$$\frac{l}{r}=1+e\cos\theta.$$

The equation of the tangent at α is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha).$$

Any line at right angles to this is given by the equation

$$\frac{c}{r}=e\cos\left(\theta+\frac{\pi}{2}\right)+\cos\left(\frac{\pi}{2}+\theta-\alpha\right)$$

$$=-e\sin\theta-\sin(\theta-\alpha).$$

This passes through

$$\frac{c(1+e\cos\alpha)}{l}=-e\sin\alpha$$

$$\therefore \qquad c=\frac{-le\sin\alpha}{1+e\cos\alpha}.$$

Hence the required equation of the normal is

$$\frac{le\sin\alpha}{1+e\cos\alpha}\cdot\frac{1}{r}=e\sin\theta+\sin(\theta-\alpha). \qquad \ldots\ldots(8)$$

123. Equation of the polar.

Let (r_1, θ_1) be the point the tangents from which touch

the conic at the points whose vectorial angles are $\alpha \pm \beta$. The equation of the chord joining these points is

$$\frac{l}{r}=e \cos \theta+\cos (\theta-\alpha) \sec \beta,$$

and the equations of the tangents at these points are

$$\frac{l}{r}=e \cos \theta+\cos (\theta-\alpha-\beta)$$

and $$\frac{l}{r}=e \cos \theta+\cos (\theta-\alpha+\beta).$$

They pass through (r_1, θ_1).

$$\therefore \quad \frac{l}{r_1}=e \cos \theta_1+\cos (\theta_1-\alpha-\beta),$$

and $$\frac{l}{r_1}=e \cos \theta_1+\cos (\theta_1-\alpha+\beta).$$

$\therefore$ $\cos (\theta_1-\alpha-\beta)-\cos (\theta_1-\alpha+\beta)=0$

i.e., $\sin (\theta_1-\alpha) \sin \beta=0$

Hence $\theta_1=n\pi+\alpha$, for $\beta \neq n\pi$.

$$\therefore \quad \frac{l}{r_1}-e \cos \theta_1=\cos (n\pi \pm \beta)=(-1)^n \cos \beta.$$

Hence the polar is represented by the equation

$$\left(\frac{l}{r}-e \cos \theta\right)\left(\frac{l}{r_1}-e \cos \theta_1\right)=\cos (\theta-\theta_1). \qquad \ldots (9)$$

2nd Method.

Let a st. line through P $(r_1\theta_1)$ cut the conic at Q and R whose vectorial angles are α, β.

The equation to QR is

$$\frac{l}{r}=e \cos \theta+\sec \frac{\beta-\alpha}{2} \cos \left(\theta-\frac{\alpha+\beta}{2}\right).$$

It passes through (r_1, θ_1),

$$\therefore \quad \frac{l}{r_1}=e \cos \theta_1+\sec \frac{\beta-\alpha}{2} \cos \left(\theta_1-\frac{\alpha+\beta}{2}\right). \qquad \ldots\ldots (i)$$

The tangents at Q and R are given by the equations

$$\frac{l}{r}=e \cos \theta+\cos (\theta-\alpha) \qquad \ldots\ldots (ii)$$

$$\frac{l}{r}=e \cos \theta+\cos (\theta-\beta). \qquad \ldots\ldots (iii)$$

The locus of their intersection is the polar of P.

Now at the point of intersection,

$$\cos (\theta-\alpha)=\cos (\theta-\beta).$$

$$\therefore \theta = \frac{\alpha+\beta}{2}.$$

From (*i*), (*ii*), the polar is given by the equation

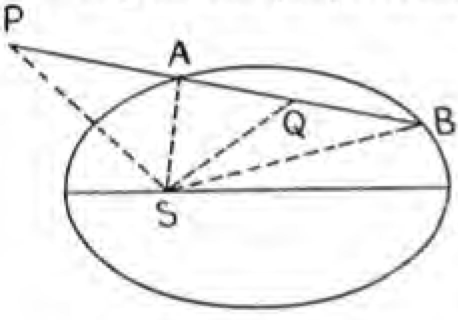
$$\left(\frac{l}{r} - e\cos\theta\right)\left(\frac{l}{r_1} - e\cos\theta_1\right) = \cos(\theta - \theta_1).$$

3rd Method.

Let a variable line through $P(r_1, \theta_1)$ cut the conic $\frac{l}{r} = 1 + e\cos\theta$ at points A (ρ, α) and B (ρ', β).

Take a point $Q(r, \theta)$ on this line such that $(PAQB) = -1$. Then the locus of Q is the polar of P.

If p be the perp. from the pole S on the line PQ,



$p. PA = r_1\rho \sin(\alpha - \theta_1)$, etc.

Since $(PAQB) = -1$,

$$PA.QB + PB.QA = 0.$$

$$\therefore \quad r_1\rho \sin(\alpha - \theta_1).\, r\rho' \sin(\beta - \theta) + r_1\rho' \sin(\beta - \theta_1).\, r\rho \sin(\alpha - \theta) = 0.$$

i.e., $\cos(\alpha - \beta + \theta - \theta_1) - \cos(\alpha + \beta - \theta - \theta_1) + \cos(\beta - \alpha + \theta - \theta_1) - \cos(\alpha + \beta - \theta - \theta_1) = 0.$

i.e., $\cos(\theta - \theta_1)\cos(\alpha - \beta) = \cos(\theta + \theta_1 - \alpha - \beta)$. ......(*i*)

Now the equation to AB is

$$\frac{l}{r} - \cos\theta = \cos\left(\theta - \frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2}.$$

It passes through P and Q.

$$\therefore \quad \frac{l}{r_1} - e\cos\theta_1 = \cos\left(\theta_1 - \frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2} \qquad \text{......}(ii)$$

$$\frac{l}{r} - e\cos\theta = \cos\left(\theta - \frac{\alpha+\beta}{2}\right)\sec\frac{\alpha-\beta}{2}. \qquad \text{......}(iii)$$

Eliminate α, β from (*i*), (*ii*), (*iii*) and we get the locus of Q.

Thus the locus of Q is given by the equation

$$\left(\frac{l}{r} - e\cos\right)\left(\frac{l}{r_1} - e\cos\theta_1\right)$$

$$= \sec^2\frac{\alpha-\beta}{2}\cos\left(\theta - \frac{\alpha+\beta}{2}\right)\cos\left(\theta_1 - \frac{\alpha+\beta}{2}\right)$$

$$= \frac{1}{2}\sec^2\frac{\alpha-\beta}{2}\left[\cos\left(\theta + \theta_1 - \alpha - \beta\right) + \cos\left(\theta - \theta_1\right)\right]$$

$$=\frac{1}{2}\sec^2\frac{\alpha-\beta}{2}\left[1+\cos\left(\alpha-\beta\right)\right]\cos\left(\theta-\theta_1\right)$$
$$=\cos(\theta-\theta_1).$$

Thus the polar of P is given by the equation

$$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos\left(\theta-\theta_1\right).$$

Remarks. If T be the point (r_1, θ_1) and P, Q have vectorial angles $\alpha\pm\beta$, then it is seen $\theta_1=n\pi+\frac{1}{2}\angle PSQ$.

Thus ST bisects the angle SPQ. If, however, the conic be a hyperbola and the points be on the different branches of the curve, ST will bisect the exterior angle PSQ, for the vectorial angle of P (if P be on the further branch) is not the angle which SP makes with SX, but the angle that PS produced makes with SX.

124. Equation of the pair of tangents. Let (r_1, θ_1) be the point from which tangents are drawn, and suppose $\alpha\pm\beta$ are the vectorial angles of the points of contact, then the tangents are

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha-\beta),\quad \frac{l}{r}=e\cos\theta+\cos(\theta-\alpha+\beta),$$

where $\theta_1=n\pi+\alpha$, $\frac{l}{r_1}-e\cos\theta_1=(-1)^n\cos\beta$.

The combined equation of the tangents is

$$\left\{\frac{l}{r}-e\cos\theta-\cos(\theta-\alpha-\beta)\right\}\left\{\frac{l}{r}-e\cos\theta-\cos(\theta-\alpha+\beta)\right\}=0$$

i.e., $$\left(\frac{l}{r}-e\cos\theta\right)^2-2\cos\beta\cos\left(\theta-\alpha\right)\left(\frac{l}{r}-e\cos\theta\right)$$
$$+\frac{1}{2}\left[\cos 2\left(\theta-\alpha\right)+\cos 2\beta\right]=0,\ i.e.,$$

$$\left(\frac{l}{r}-e\cos\theta\right)^2-2\left(\frac{l}{r_1}-e\cos\theta_1\right)\left(\frac{l}{r}-e\cos\theta\right)\cos(\theta-\theta_1)$$
$$+\left(\frac{l}{r_1}-e\cos\theta_1\right)^2+\cos^2(\theta-\theta_1)=1. \qquad (10)$$

Or we may combine the equations thus: The tangents are

$$\cos^{-1}\left(\frac{l}{r}-e\cos\theta\right)=\theta-\alpha\pm\beta=\theta-\theta_1+n\pi\pm\beta$$

i.e., $$\cos^{-1}\left(\frac{l}{r}-e\cos\theta\right)-\cos^{-1}\left(\frac{l}{r_1}-e\cos\theta_1\right)=\theta-\theta_1$$

i.e., $$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)$$
$$+\sqrt{\left\{1-\left(\frac{l}{r}-e\cos\theta\right)^2\right\}\left\{1-\left(\frac{l}{r_1}-e\cos\theta_1\right)^2\right\}}$$
$$=\cos(\theta-\theta_1)$$

or $$\left[\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)\right]^2$$
$$=\left\{1-\left(\frac{l}{r}-e\cos\theta\right)^2\right\}\left\{1-\left(\frac{l}{r_1}-e\cos\theta_1\right)^2\right\} \quad \ldots\ldots(11)$$

Alternative Method. The chord of contact of (r_1, θ_1) with respect to the conic

$$\frac{l}{r}=1+e\cos\theta \qquad \ldots\ldots(i)$$

is $$\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)=\cos(\theta-\theta_1) \quad \ldots\ldots(ii)$$

The conic

$$\left(\frac{l}{r}-e\cos\theta\right)^2-1=\lambda\left\{\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)\right\}^2$$

touches the given conic (i) at the intersections of (i) and (ii). If it passes through (r_1, θ_1), then it represents two tangents from (r_1, θ_1) to (i). Thus the tangents are given by the equation

$$\left\{\left(\frac{l}{r}-e\cos\theta\right)^2-1\right\}\left\{\left(\frac{l}{r_1}-e\cos\theta_1\right)^2-1\right\}$$
$$=\left\{\left(\frac{l}{r}-e\cos\theta\right)\left(\frac{l}{r_1}-e\cos\theta_1\right)-\cos(\theta-\theta_1)\right\}^2.$$

124.1. Asymptotes. The pair of tangents from the centre are the asymptotes of the conic.

The centre is the point $\left(\frac{le}{e^2-1}, 0\right)$

Putting $r_1=\frac{le}{e^2-1}$, $\theta_1=0$, in Art. 124, we see that the asymptotes are represented by the equation

$$\left\{\left(\frac{l}{r}-e\cos\theta\right)^2-1\right\}\left(\frac{1}{e^2}-1\right)$$
$$=\left[\left(\frac{l}{r}-e\cos\theta\right)\left(-\frac{1}{e}\right)-\cos\theta\right]^2$$
$$=\frac{l^2}{e^2r^2}$$

i.e., $$\frac{l^2}{r^2}-\frac{2l(e^2-1)}{er}\cos\theta+(e^2-1)\cos^2\theta=\frac{e^2-1}{e^2}$$

i.e., $$\left(\frac{l}{r}-\frac{e^2-1}{e}\cos\theta\right)^2=\frac{e^2-1}{e^2}\sin^2\theta,$$

whence $$\frac{l}{r}-\left(e-\frac{1}{e}\right)\cos\theta=\pm\frac{\sqrt{e^2-1}}{e}\sin\theta.$$

Second Method.

Let α be a point of contact, $1+e\cos\alpha=0$, whence $\sin\alpha=\pm\dfrac{\sqrt{e^2-1}}{e}$. The equation of the tangent at α is

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\alpha)$$

or $$\frac{l}{r}-e\cos\theta=\cos\theta\cos\alpha+\sin\theta\sin\alpha.$$

Hence the equations of the asymptotes are

$$\frac{l}{r}-e\cos\theta+\frac{1}{e}\cos\theta=\pm\frac{\sqrt{e^2-1}}{e}\sin\theta. \quad\ldots\ldots(12)$$

125. Auxiliary Circle. The locus of the foot of the perpendicular from either focus on the tangent to a conic is a circle, which we have already defined as the auxiliary circle.

The equation of the tangent at α to the conic

$$\frac{l}{r}=1+e\cos\theta$$

is $$\frac{l}{r}=e\cos\theta+\mathrm{ces}\,(\theta-\alpha).$$

The equation of the line through the focus perpendicular to this tangent is

$$e\sin\theta+\sin(\theta-\alpha)=0.$$

Thus the required locus is obtained by the elimination of α between these two equations. Thus the auxiliary circle is

$$\left(\frac{l}{r}-e\cos\theta\right)^2+e^2\sin^2\theta=1$$

i.e., $$\frac{l^2}{r^2}-\frac{2l}{r}e\cos\theta+e^2=1$$

or $$(1-e^2)r^2+2lre\cos\theta-l^2=0. \quad\ldots\ldots(13)$$

126. Director Circle. Let the tangents from (r_1, θ_1) be perpendicular to each other, and suppose that $\alpha \pm \beta$ are the vectorial angles of the points of contact, then

$$\theta_1 = n\pi + \alpha, \quad \frac{l}{r_1} - e \cos \theta_1 = \cos (n\pi \pm \beta)$$

$$= (-1)^n \cos \beta.$$

The equation of the tangent at $\alpha + \beta$ is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha - \beta)$$

$$= [e + \cos (\alpha + \beta)] \cos \theta + \sin (\alpha + \beta) \sin \theta$$

$$= \rho \cos (\theta - \phi),$$

where $\rho \cos \phi = e + \cos (\alpha + \beta), \ \rho \sin \phi = \sin (\alpha + \beta)$.

Similarly the tangent at $(\alpha - \beta)$ is

$$\frac{l}{r} = \rho' \cos (\theta - \phi'),$$

where $\rho' \cos \phi' = e + \cos (\alpha - \beta), \ \rho' \sin \phi' = \sin (\alpha - \beta)$.

These tangents are at right angles if

$$\phi \sim \phi' = \frac{\pi}{2}.$$

i.e., $\cos \phi \cos \phi' + \sin \phi \sin \phi' = 0$;

whence by substitution

$$[e + \cos (\alpha + \beta)] [e + \cos (\alpha - \beta)] + \sin (\alpha + \beta) \sin (\alpha - \beta) = 0$$

i.e., $\cos 2\beta + 2 \cos \alpha \cos \beta + e^2 = 0$.

Elimination of α and β and change of r_1, θ_1 into r, θ give the required locus

$$2\left(\frac{l}{r} - e \cos \theta\right)^2 - 1 + 2e \cos \theta \left(\frac{l}{r} - e \cos \theta\right) + e^2 = 0$$

i.e., $(1 - e^2) r^2 + 2ler \cos \theta - 2l^2 = 0$. ......(14)

Illustrative Examples

Imp

(1) *If a chord of a conic subtends a constant angle at a focus, the tangents at the ends of the chord will meet on a fixed conic and the chord will touch another fixed conic.*

Let 2β be the angle subtended at the focus by the chord and $\alpha \pm \beta$ the vectorial angles of its extremities. The tangents at these extremities meet at (r_1, θ_1) where

$$\theta_1 = \alpha, \ \frac{l}{r_1} = e \cos \alpha + \cos \beta.$$

Hence $$\frac{l}{r_1} = e \cos \theta_1 + \cos \beta.$$

Thus the locus of (r_1, θ_1) is the conic

$$\frac{l \sec\beta}{r} = e \sec\beta \cos\theta + 1,$$

which has the same focus as the original conic and whose latus rectum is $2l \sec\beta$ and eccentricity is $e \sec\beta$.

The equation of the chord is

$$\frac{l}{r} = e \cos\theta + \sec\beta \cos(\theta - a)$$

i.e., $$\frac{l \cos\beta}{r} = e \cos\beta \cos\theta + \cos(\theta - a),$$

and this is a tangent to the conic

$$\frac{l \cos\beta}{r} = 1 + e \cos\beta \cos\theta.$$

(2) *The semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.*

Let PSQ be the focal chord and suppose θ is the vectorial angle of P, then $\pi + \theta$ is the vectorial angle of Q. Hence

$$\frac{l}{SP} = 1 + e \cos\theta, \quad \frac{l}{SQ} = 1 - e \cos\theta$$

$$\therefore \quad \frac{1}{SP} + \frac{1}{SQ} = \frac{2}{l}.$$

Remark. If the curve is a hyperbola and P is on the nearest branch while Q is on the further branch of the conic then the vectorial angle of P and Q are still θ and $\pi + \theta$, but the radius vector of Q is negative and numerically equal to SQ. Thus we have

$$\frac{l}{SP} = 1 + e \cos\theta, \; -\frac{l}{SQ} = 1 - e \cos\theta$$

whence $$\frac{1}{SP} - \frac{1}{SQ} = \frac{2}{l}.$$

(3) *If the ellipses whose latera recta are l_1, l_2 and eccentricities e_1, e_2 have a common focus, and touch one another, show that their axes are inclined to each other at an angle*

$$\cos^{-1}\left\{\frac{e_1^2 l_2^2 + e_2^2 l_1^2 - (l_2 - l_1)^2}{2 l_1 l_2 e_1 e_2}\right\}.$$

Take the common focus as the pole and the axis of one of the ellipses as the initial line. The equations of the ellipses can be written as

$$\frac{l_1}{r} = 1 + e_1 \cos\theta, \quad \frac{l_2}{r} = 1 + e_2 \cos(\theta - \gamma).$$

Let α be the vectorial angle of the point of contact of the conics.

The equations of the tangents to the conics at α are

$$\frac{l_1}{r}=e_1 \cos \theta+\cos (\theta-\alpha), \quad \frac{l_2}{r}=e_2 \cos (\theta-\gamma)+\cos (\theta-\alpha).$$

As the tangents are identical

$$\frac{l_1}{l_2}=\frac{e_1+\cos \alpha}{e_2 \cos \gamma+\cos \alpha}=\frac{\sin \alpha}{e_2 \sin \gamma+\sin \alpha}$$

$$\therefore \qquad (l_2-l_1) \cos \alpha=l_1 e_2 \cos \gamma-e_1 l_2$$

$$(l_2-l_1) \sin \alpha=e_2 l_1 \sin \gamma.$$

Squaring and adding,

$$(l_2-l_1)^2=l_1^2 e_2^2+l_2^2 e_1^2-2e_1 e_2\, l_1 l_2 \cos \gamma,$$

which gives γ.

The relation found above is the condition of tangency of the two conics.

(4) *Show that the equation to the circle, which passes through the focus and touches the curve* $\frac{l}{r}=1+e \cos \theta$ *at the point* $\theta=\alpha$, is

$$\frac{r}{l}(1+e \cos \alpha)^2=\cos (\theta-\alpha)+e \cos (\theta-2\alpha).$$

Let the equation of the circle be

$$r=2a \cos (\theta-\alpha+\beta).$$

The circle passes through $\left(\frac{l}{1+\cos \alpha}, \alpha\right)$,

$$\therefore \qquad \frac{l}{1+e \cos \alpha}=2a \cos \beta.$$

The centre of the circle $(a, \alpha-\beta)$ must lie on the normal at α to the conic,

$$\therefore \qquad \frac{l\, e \sin \alpha}{a\,(1+e \cos \alpha)}=e \sin (\alpha-\beta)-\sin \beta$$

i.e.,

$$\frac{l\, e \sin \alpha}{1+e \cos \alpha}=ae \sin \alpha \cos \beta-a(1+e \cos \alpha) \sin \beta$$

$$\therefore \qquad a \sin \beta=-\frac{le \sin \alpha}{2(1+e \cos \alpha)^2}.$$

Hence the equation of the circle, *viz.*,

$$r=2a[\cos (\theta-\alpha) \cos \beta-\sin (\theta-\alpha) \sin \beta]$$

becomes

$$r=\frac{l \cos (\theta-\alpha)}{1+e \cos \alpha}+\frac{le \sin \alpha \sin (\theta-\alpha)}{(1+e \cos \alpha)^2}$$

i.e., $\frac{r}{l}(1+e\cos\alpha)^2=\cos(\theta-\alpha)(1+e\cos\alpha)+e\sin\alpha\sin(\theta-\alpha)$

or $\frac{r}{l}(1+e\cos\alpha)^2=\cos(\theta-\alpha)+e\cos(\theta-2\alpha).$

(5) *Find the equation of the circle circumscribing the triangle formed by three tangents to a parabola.*

Let α, β, γ be three points A, B, C on the parabola

$$\frac{l}{r}=1+\cos\theta.$$

The equations of the three tangents are

$$\frac{l}{r}=\cos\theta+\cos(\theta-\alpha),\ \frac{l}{r}=\cos\theta+\cos(\theta-\beta),$$

$$\frac{l}{r}=\cos\theta+\cos(\theta-\gamma).$$

These tangents meet at the points A′, B′, C′, whose coordinates are

$$\left[\frac{l}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2},\ \tfrac{1}{2}(\beta+\gamma)\right],$$

$$\left[\frac{l}{2}\sec\frac{\gamma}{2}\sec\frac{\alpha}{2},\ \tfrac{1}{2}(\gamma+\alpha)\right],$$

and
$$\left[\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2},\ \tfrac{1}{2}(\alpha+\beta)\right].$$

Let the equation of the circumcircle be

$$r^2-2r\rho\cos\left(\theta-\frac{\alpha+\beta+\gamma}{2}+\phi\right)+\rho^2-a^2=0.$$

Hence

$$\frac{l^2}{4}-\rho l\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\cos\left(\phi-\frac{\alpha}{2}\right)$$
$$+(\rho^2-a^2)\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}=0 \quad \ldots\ldots(i)$$

$$\frac{l^2}{4}-\rho l\cos\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\left(\phi-\frac{\beta}{2}\right)$$
$$+(\rho^2-a^2)\cos^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}=0 \quad \ldots\ldots(ii)$$

$$\frac{l^2}{4}-\rho l\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\left(\phi-\frac{\gamma}{2}\right)$$
$$+(\rho^2-a^2)\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}=0 \quad \ldots\ldots(iii)$$

From the first two after subtraction, we get

$$\rho l\left[\cos\frac{\beta}{2}\cos\left(\phi-\frac{\alpha}{2}\right)-\cos\frac{\alpha}{2}\cos\left(\phi-\frac{\beta}{2}\right)\right]$$
$$=(\rho^2-a^2)\cos\frac{\gamma}{2}\left(\cos^2\frac{\beta}{2}-\cos^2\frac{\alpha}{2}\right),$$

$$\rho l\left[\cos\left(\phi+\frac{\beta}{2}-\frac{\alpha}{2}\right)-\cos\left(\phi+\frac{\alpha}{2}-\frac{\beta}{2}\right)\right]$$
$$=2(\rho^2-a^2)\cos\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$$

or $$\rho l\sin\phi=(\rho^2-a^2)\cos\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}.$$

Similarly, from (*ii*) and (*iii*),

$$\rho l\sin\phi=(\rho^2-a^2)\cos\frac{\alpha}{2}\sin\frac{\beta+\gamma}{2}.$$

Thus $$(\rho^2-a^2)\left[\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma}{2}-\sin\frac{\beta+\gamma}{2}\cos\frac{\alpha}{2}\right]=0$$

or $$(\rho^2-a^2)\cos\frac{\beta}{2}\sin\frac{\alpha-\gamma}{2}=0.$$

Since β is arbitrary, $\cos\dfrac{\beta}{2}\neq 0$ and $\sin\dfrac{\alpha-\gamma}{2}\neq 0$ for $\alpha\neq\gamma$.

Hence $\rho^2-a^2=0$, and therefore $\phi=0$, and from any one of the relations (*i*), (*ii*), (*iii*)

$$2\rho=\frac{l}{2\cos\dfrac{\alpha}{2}\cos\dfrac{\beta}{2}\cos\dfrac{\gamma}{2}}.$$

Thus the equation of the circle is

$$r=\frac{l}{2}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\cos\left(\theta-\frac{\alpha+\beta+\gamma}{2}\right),$$

which passes through the focus.

If this fact be assumed, the equation of the circle can be obtained more easily. For, the equation of the circle is of the form

$$r=2\rho\cos\left(\theta-\frac{\alpha+\beta+\gamma}{2}+\phi\right).$$

The points A′, B′, C′ lie on it,

$$l=4\rho\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\cos\left(\phi-\frac{\alpha}{2}\right)$$

and two similar relations ; and these are obviously satisfied when

$$\phi=0,\qquad \rho=\frac{l}{4}\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}.$$

Examples XXXV

1. Prove that the equations $\dfrac{l}{r} = e\cos\theta \pm 1$ represent the same curve.

2. The tangents drawn from any point to a conic subtend equal angles at the focus.

3. The portion of a tangent to a conic between the point of contact and the directrix subtends a right angle at the focus.

4. The general equation of all conics having the same focus and directrix is $\dfrac{1}{r} = a + p\cos\theta$, where p is the same for all conics of the system.

5. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

6. Prove that the sum of reciprocals of two perpendicular focal chords of a conic is constant.

7. If PSP$'$, QSQ$'$ be any two perpendicular focal chords of a conic, show that $\dfrac{1}{PS.SP'} + \dfrac{1}{QS.SQ'}$ is constant.

8. If A B, C be the three points on a parabola and the tangents at these points meet at A$'$, B$'$, C$'$, show that

$$SA.SB.SC = SA'.SB'.SC',$$

S being the focus.

9. Two conics have the same focus and directrix. If any tangent to one cut the other in P and Q, prove that $\cos\frac{1}{2}$ PSQ $= e/e'$, e, e' being the eccentricities.

10. A system of conics have the same focus and latus rectum. Prove that the tangents at all points on a fixed line through the focus cut the latus rectum produced at the same distance from the focus.

11. If a st. line drawn through the focus S of a hyperbola parallel to an asymptote, meet the curve in P, prove that SP is one-quarter of the latus rectum.

12. Two conics have a common focus, prove that two of their common chords pass through the intersection of their directrices.

13. If a focal chord of an ellipse makes an angle α with the axis, the angle between the tangents at its extremities is

$$\tan^{-1}\frac{2e\sin\alpha}{1-e^2}.$$

14. If a normal is drawn at the extremity of the latus rectum, prove that the distance from the focus of the other point in which it meets the curve is $\dfrac{1+3e^2+e^4}{1+e^2-e^4}\,l$.

15. From the focus S of an ellipse whose eccentricity is e, radii SP, SQ are drawn at right angles to one another, and the tangents at P and Q meet at T. Show that the locus of T is an hyperbola, parabola, or ellipse according as $e \gtreqless \dfrac{1}{\sqrt{2}}$.

16. If T is the pole of a chord PQ of the conic $\dfrac{l}{r} = 1 + e\cos\theta$ which subtends an angle 2β at the focus, $\dfrac{1}{SP} + \dfrac{1}{SQ} - \dfrac{2\cos\beta}{ST}$ is constant.

17. Show that the conic $\dfrac{l}{r} = 1 + e\cos\theta$ intercepts on its normal at $\theta = \alpha$ a length which subtends at the focus an angle

$$2\tan^{-1}\frac{1+e^2+2e\cos\alpha}{e\sin\alpha}.$$

18. Two parabolas have a common focus and axes inclined at an angle 2α. Prove that the locus of the intersection of two perpendicular tangents, one to each of the parabolas, is a conic.

19. A conic is described having the same focus and eccentricity as the conic $\dfrac{l}{r} = 1 + e\cos\theta$, and the two conics touch at the point $\theta = \alpha$, prove that the length of the latus rectum will be

$$\frac{2l(1-e^2)}{1+2e\cos\alpha+e^2}.$$

20. The conic $\dfrac{l}{r} = 1 + e\cos\theta$ is cut by a circle which passes through the pole in the points (r_i, θ_i), $i = 1, 2, 3, 4$, prove that $\sum_1^4 \dfrac{1}{r_i} = \dfrac{2}{l}$, $(1+e)\sum_{i=1}^4 \tan\dfrac{\theta_i}{2}$

$$= (1-e)\,\Sigma\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}\tan\frac{\theta_4}{2},$$

and $r_1r_2r_3r_4 = d^2l^2/e^2$, where d is the diameter of the circle.

21. If the normals at α, β, γ on $\dfrac{l}{r} = 1 + \cos\theta$ meet in the point (ρ, ϕ), then will $2\phi = \alpha + \beta + \gamma$.

22. Show that if the normals at the points, whose vectorial angles are $\theta_1, \theta_2, \theta_3, \theta_4$, on $\frac{l}{r} = 1 + e\cos\theta$ meet in the point (ρ, ϕ), then will $\theta_1+\theta_2+\theta_3+\theta_4-2\phi=(2n+1)\pi$.

23. If the normals at the points $\theta_1, \theta_2, \theta_3$ are concurrent,

prove that $$\frac{\Sigma \tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}}{\Sigma \cot\frac{\theta_2}{2}\cot\frac{\theta_3}{2}}=\left(\frac{1+e}{1-e}\right)^2.$$

24. Find the condition that the line $\frac{l}{r}=a\cos\theta+b\sin\theta$ may be a tangent to the conic $\frac{l}{r}=1+e\cos(\theta-\alpha)$.

25. Prove that the two equal conics which have a common focus and whose axes are inclined at an angle 2α intersect at an angle

$$\tan^{-1}\left[\frac{e^2\sin 2\alpha+2e\sin\alpha}{e^2\cos^2\alpha+2e\cos\alpha+1}\right].$$

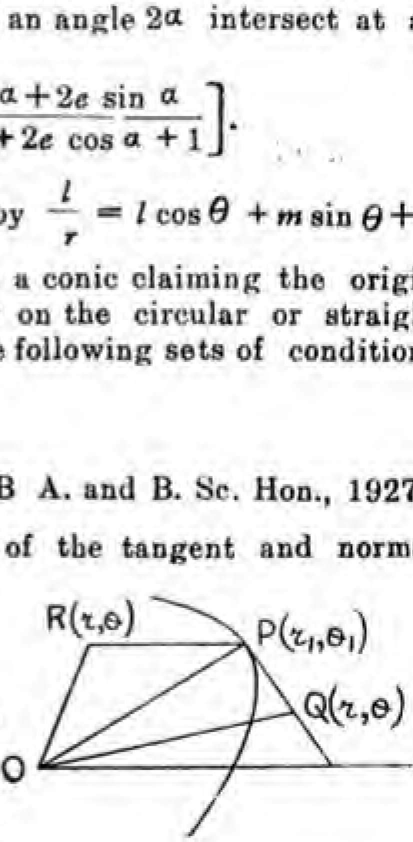
26. Prove that the curve Σ given by $\frac{l}{r}=l\cos\theta+m\sin\theta+n$ will for non-zero values of l, m, n be a conic claiming the origin as a focus. Also prove that Σ takes on the circular or straight form according as one or other of the following sets of conditions is satisfied,

(a) $l=m=0,\ n\neq 0$

(b) $n=0,\ l, m\neq 0$. [C. U., B A. and B. Sc. Hon., 1927.]

27. Find the polar equations of the tangent and normal at a point (r_1, θ_1) to the conic

$$\frac{l}{r}=1+e\cos\theta.$$



(i) Take any point $Q(r,\theta)$ on the tangent at P and let $\angle OPQ=\phi$.

From $\triangle OPQ$, $\frac{\sin OPQ}{OQ}=\frac{\sin OQP}{OP}$

i.e., $\frac{\sin\phi}{r}=\frac{\sin(\phi+\theta_1-\theta)}{r_1}$

i.e., $\frac{r_1}{r}=\cos(\theta_1-\theta)+\cot\phi\sin(\theta_1-\theta)$

$$=\cos(\theta_1-\theta)+\frac{1}{r_1}\,\frac{dr_1}{d\theta_1}\sin(\theta_1-\theta).$$

Since $\dfrac{l}{r_1}=1+e\cos\theta_1$ $\quad\therefore\ \dfrac{1}{r_1}\,\dfrac{dr_1}{d\theta_1}=\dfrac{e\sin\theta_1}{1+e\cos\theta_1}.$

Thus the tangent is given by the equation

$$\frac{l}{r}=e\cos\theta+\cos(\theta-\theta_1).$$

(*ii*) Take any point R (r,θ) on the normal at P.

From ΔOPR, $\dfrac{\sin \text{OPR}}{\text{OR}}=\dfrac{\sin \text{ORP}}{\text{OP}}$

i.e., $$\frac{r_1}{r}=\frac{\sin\left(\frac{\pi}{2}-\phi+\theta-\theta_1\right)}{\sin\left(\frac{\pi}{2}-\phi\right)}=\frac{\cos(\phi+\theta_1-\theta)}{\cos\phi}$$

$$=\cos(\theta_1-\theta)-\tan\phi\sin(\theta_1-\theta)$$

$$=\cos(\theta-\theta_1)+\frac{1+e\cos\theta_1}{e\sin\theta_1}\sin(\theta-\theta_1).$$

Thus the equation of the normal is

$$\frac{e\sin\theta_1}{1+e\sin\theta_1}\cdot\frac{l}{r}=e\sin\theta+\sin(\theta-\theta_1).$$

CHAPTER XII.

THE GENERAL CONIC.

127. A line is drawn through a given point in a given direction: to find the distances from the given point of the points in which it is met by a given conic.

Let O(x', y') be a given point and let the line in the direction θ be given by the equation

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r. \qquad \ldots\ldots(1)$$

Then the co-ordinates of any point on the line distant r from O are

$$x=x'+r\cos\theta,\ y=y'+r\sin\theta.$$

If this point be on the conic

$$\phi(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

we must have

$$\phi(x'+r\cos\theta,\ y'+r\sin\theta)=0.$$

i.e., $r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)$
$+2r\{(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta\}$
$+ax'^2+2hx'y'+by'^2+2gx'+2fy'+c=0$

i.e., $r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)$
$+2r(X'\cos\theta+Y'\sin\theta)+\phi(x', y')=0.$ ......(2)
where $X'=ax'+hy'+g$, $Y'=hx'+by'+f$.

This quadratic in r will be called *the r-quadratic.*

The two roots of the r-quadratic are the required distances from O of the intersections of the line (1) with the conic $\phi(x, y)=0$.

It follows that *a st. line cannot cut a conic in more than two points.*

If P, Q be the intersections of the line with the conic, OP, OQ are the roots of the r-quadratic.

By the theory of quadratics we have

(*i*) $$\text{OP}+\text{OQ}=-\frac{2(X'\cos\theta+Y'\sin\theta)}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta};$$

(*ii*) $$\text{OP}.\ \text{OQ}=\frac{\phi(x', y')}{a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta};$$

(*iii*) $$\frac{1}{\text{OP}}+\frac{1}{\text{OQ}}=-\frac{2(X'\cos\theta+Y'\sin\theta)}{\phi(x', y')}.$$

This proof assumes that the axes are rectangular.

We now proceed to make deductions from the r-quadratic.

127.1. Polar of a point. If on the line OPQ, we take a point R such that $(\mathrm{OPRQ})=-1$, then

$$\frac{1}{\mathrm{OP}}+\frac{1}{\mathrm{OQ}}=\frac{2}{\mathrm{OR}}$$

$\therefore$ from Art. 127 (*iii*), we have

$$\frac{1}{\mathrm{OR}}+\frac{\mathrm{X}'\cos\theta+\mathrm{Y}'\sin\theta}{\phi(x',y')}=0.$$

Since $\dfrac{x-x'}{\cos\theta}=\dfrac{y-y'}{\sin\theta}=\mathrm{OR}$

$\therefore$ the locus of R is the st. line.

$$\mathrm{X}'(x-x')+\mathrm{Y}'(y-y')+\phi(x',y')=0,$$

i.e., $x\mathrm{X}'+y\mathrm{Y}'+gx'+fy'+c=0.$ ......(3)

This st. line is the polar of O (x', y').

127.2. To find the condition that the two st. lines.

$$l_1x+m_1y+n_1=0$$

and

$$l_2x+m_2y+n_2=0$$

may be conjugate for the conic $\phi(x, y)=0$.

Let the pole of the 1st. line be (x', y'). Then it is the same as (3), and therefore

$$\begin{aligned} ax'+hy'+g-\lambda l_1&=0\\ hx'+by'+f-\lambda m_1&=0\\ gx'+fy'+c-\lambda n_1&=0\end{aligned}$$

Also $\qquad l_2x'+m_2y'+n_2=0$

Eliminating x', y', λ we get the required condition.

$$\begin{vmatrix} a & h & g & l_1\\ h & b & f & m_1\\ g & f & c & n_1\\ l_2 & m_2 & n_2 & 0\end{vmatrix}=0.$$

Ex. If on the secant OPQ of the conic $\phi(x, y)$ a point R is taken such that $\mathrm{OP}+\mathrm{OQ}=2\,\mathrm{OR}$ or $\mathrm{OP.OQ}=\mathrm{OR}^2$, then the loci of R are conics σ_1, σ_2 passing through the intersections of the polar of O with the conic ϕ and having their asymptotes parallel to those of the conic ϕ. σ_1 passes through O and σ_2 has its centre at O.

128. If through fixed point O, two chords are drawn in fixed directions to meet a conic in P, Q and P′, Q′ respectively, the ratio of the rectangles OP.OQ and OP′.OQ′ is independent of the position of O.

Let the conic be $\phi(x, y)=0$ and $O(x', y')$, and suppose that the chords are drawn in the directions θ, θ', then it follows from Art. 127 (*ii*) that

$$\frac{OP.OQ}{OP'.OQ'}=\frac{a\cos^2\theta'+2h\cos\theta'\sin\theta'+b\sin^2\theta'}{a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta} \quad \ldots(5)$$

The ratio is independent of the position of O and only depends upon the directions of the lines. Thus we get Newton's Theorem.

If O be a variable point in the plane of a conic and PQ, RS be chords in fixed directions through O, then $\dfrac{OP.OQ}{OR.OS}$ *is constant.*

Take two positions of O, (say) O, O′. Let chords P′Q′, R′S′ be drawn through O′ parallel to PQ, RS respectively. Then evidently

$$\frac{OP.OQ}{OR.OS}=\frac{O'P'.O'Q'}{O'R'.O'S'} \quad (6)$$

and hence the theorem.

128.1. If we take O′ at the centre C of the conic, the parallel chords are bisected at the centre, and the right-hand side of (6) becomes $\dfrac{CP'^2}{CR'^2}$. Hence *the ratio of the rectangles OP.OQ and OR.OS is equal to the ratio of the squares of the parallel semi-diameters.*

128.2. In particular, if P and Q coincide in T and P′, Q′ in T′, then the lines OPQ, OP′Q′ are tangents from O, and we infer that *the tangents from a point to a central conic are in the same ratio as the parallel semi-diameters.*

128.3. Let a circle meet a conic in four points P,Q,R,S and let PQ, RS meet in O. If r_1, r_2 be the semi-diameters parallel respectively to PQ and RS,

$$\frac{r_1^2}{r_2^2}=\frac{OP.OQ}{OR.OS}=1,$$

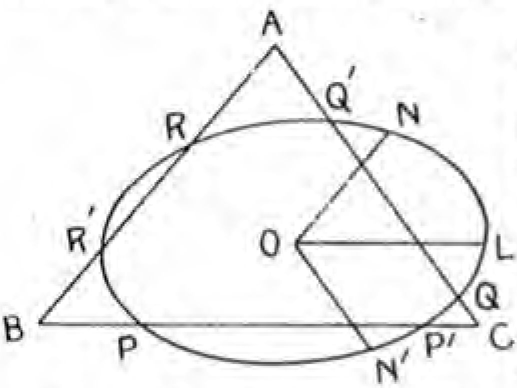
since P, Q, R, S are concyclic. Thus $r_1=r_2$. Also, in a conic, equal radii from the centre are equally inclined to the axis of the conic. Hence r_1, r_2 and therefore PQ, RS are equally inclined to the axis of the conic. Thus we have the theorem :—

If a circle cut a conic in four points, the chords joining their points of intersection in pairs are equally inclined to the axes of the conic.

128.4. If parallel chords OPQ, O′P′Q′ through fixed points O and O′ be drawn to meet the conic in P, Q and P′,Q′, the ratio of the rectangles OP.OQ, O′P′.O′Q′ is independent of the direction of the chords. Let O,O′ be (x_0, y_0), (x_0', y_0'). From Art. 127 (*ii*).

$$\frac{OP.OQ}{O'P'.O'Q'} = \frac{\phi(x_0, y_0)}{\phi(x_0', y_0')}.$$

The ratio depends, only on the positions of the points through which the chords pass.



128.5. Carnot's Theorem. If the sides BC, CA, AB of a triangle ABC meet a conic at P, P′; Q,Q′; R,R′ respectively then

$$\frac{BP.BP'.CQ.CQ'.AR.AR'}{CP.CP'.AQ.AQ'.BR.BR'} = 1 \qquad \ldots\ldots(7)$$

Let $\theta_1, \theta_2, \theta_3$ be the angles that the lines BC, CA, AB make with the x-axis. If the equation of the conic is

$$\phi(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

$$\frac{BP.BP'}{BR.BR'} = \frac{a\cos^2\theta_1 + 2h\cos\theta_1\sin\theta_1 + b\sin^2\theta_1}{a\cos^2\theta_3 + 2h\cos_3\sin\theta_3 + b\sin^2\theta_3}$$

$$\frac{CQ.CQ'}{CP.CP'} = \frac{a\cos^2\theta_2 + 2h\cos\theta_2\sin\theta_2 + b\sin^2\theta_2}{a\cos^2\theta_1 + 2h\cos\theta_1\sin\theta_1 + b\sin^2\theta_1}$$

$$\frac{AR.AR'}{AQ.AQ'} = \frac{a\cos^2\theta_3 + 2h\cos\theta_3\sin\theta_3 + b\sin^2\theta_3}{a\cos^2\theta_2 + 2h\cos\theta_2\sin\theta_2 + b\sin^2\theta_2}$$

Multiply and we get the desired result.

If the conic is central and OL, OM, ON the semi-diameters are parallel to BC, CA, AB, the above relations can be written as

$$\frac{BP.BP'}{BR.BR'} = \frac{OL^2}{ON^2}, \quad \frac{CQ.CQ'}{CP.CP'} = \frac{OM^2}{OL^2}, \quad \frac{AR.AR'}{AQ.AQ'} = \frac{ON^2}{OM^2}.$$

The result then follows by multiplication.

The converse of this is also true. For, if a conic through P, P′, Q, Q′, R meets AB again in R″

$$\frac{BP.BP'}{CP.CP'}, \frac{CQ.CQ'}{AQ.AQ'} . \frac{AR.AR''}{BR.BR''} = 1$$

$$=\frac{BP.BP'}{CP.CP'}\cdot\frac{CQ.CQ'}{AQ.AQ'}\cdot\frac{AR.AR'}{BR.BR'}$$

$$\therefore \qquad \frac{AR''}{BR''}=\frac{AR'}{BR'}.$$

Thus R'' coincides with R'.

The present theorem gives a necessary and sufficient condition for six points to lie on a conic.

Exercise XXXVI

1. A st. line is drawn through a point O(1, 2) making an angle of 45° with the x-axis and it meets the conic

$$x^2+xy+y^2+x+y+1=0$$

in P, Q. Obtain the r-quardratic and show that

$$OP+OQ=-\frac{11\sqrt{2}}{3},\ OP.OQ=\frac{22}{3}.$$

2. Show that the lines $6x+9y+4=0$ and $2x+y-3=0$ are conjugate for the conic $x^2+2xy+3y^2+2x+y+\frac{1}{2}=0$.

3. Prove that the polars of a point (ξ, η) with respect to conics $x^2+y^2+2\lambda xy=1$ where λ varies, all intersect in a point.

If (ξ, η) moves on a fixed line, prove that the intersection of the polars moves on a fixed hyperbola.

4. If a conic be inscribed in a triangle, show that the three lines from the angular points of the triangle to the points of contact of the opposite sides meet in a point.

5. If OPQ, ORS be two perpendicular chords of a conic through a fixed point O, show that $\dfrac{1}{OP.OQ}+\dfrac{1}{OR.OS}$ is constant.

6. If a circle touches a conic at one point and cuts it at two other points, prove that the tangent at the point of contact and the chord through the other intersections make equal angles with the axes of the conic.

7. If two chords of a conic PP′, QQ′ intersect in O, prove that the ratio OP.OP′ : OQ.OQ′ is equal to that of the lengths of the focal chords parallel to PP′ and QQ′. Show further that PQ and P′Q′ meet on the polar of O.

8. By means of Newton's Theorem, prove that if PN be the ordinate of a point P on a parabola whose vertex is A, PN^2 : AN is independent of the position of P on the curve.

129. Two zero roots. Tangent at a point.

The r-quadratic has one zero root if $\phi(x', y')=0$, which is just the condition that the point $O(x', y')$ should be on the curve, *i.e.* O should coincide with one intersection.

The equation has two zero roots if $O(x', y')$ is on the curve and in addition, $X' \cos\theta + Y' \sin\theta = 0$. The line (1) then meets the curve in two points coinciding with $O(x', y')$, and, therefore, the above equation gives the direction of the tangent at (x', y') *viz.* $\tan\theta = -\frac{X'}{Y'}$.

Hence *the equation of the tangent at the point* (x', y') *is*

$$y - y' = -\frac{X'}{Y'}(x - x')$$

i.e., $(x - x')(ax' + hy' + g) + (y - y')(hx' + by' + f) = 0$

i.e., $x(ax' + hy' + g) + y(hx' + by' + f) + gx' + fy' + c = 0$,

since $\phi(x', y') = 0$. ......(8)

129.1 To find the condition that a given st. line may be a tangent to the conic $\phi(x, y) = 0$.

Let the st. line be given by the equation

$$lx + my + n = 0.$$

The tangent at (x', y'), *viz.*, (8) coincides with the given line if

$$\begin{aligned} ax' + hy' + g - \lambda l &= 0 \\ hx' + by' + f - \lambda m &= 0 \\ gx' + fy' + c - \lambda n &= 0 \end{aligned}$$

Also $$lx' + my' + n = 0$$

Eliminating x', y', λ, we get the condition.

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0. \qquad \text{......(9)}$$

which is equivalent to

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

where A, B, C etc., are the minors of a, b, c etc., in the determinant $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Putting $l = 0$, $m = 0$, we get $C = 0$ as the condition of tangency.

Thus we see that *the necessary and sufficient condition that the equation* $\phi(x, y)=0$ *may represent a parabola is* $C=ab-h^2=0$, *i.e., the 2nd degree terms should form a perfect square.*

130. Equal and opposite roots.

The necessary and sufficient condition that the r-quadratic may have equal and opposite roots is that the co-efficient of r should be zero, viz.

$$(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta=0. \qquad \text{......(10)}$$

The point $O(x', y')$ is therefore the middle point of the chord PQ.

Thus *the equation of the chord which is bisected at the point* (x', y') *is*

$$(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')=0$$

i.e., $xX'+yY'+gx'+fy'+c=\phi(x', y')$, ...(11)

which is parallel to the polar of (x', y').

130.1. If θ be given and $\tan\theta=m$, by dropping the dashes in (10) we get the theorem :

The locus of the middle points of parallel chords is the st. line $(ax+hy+g)+m(hx+by+f)=0$

or $\quad (a+mh)x+(h+mb)y+g+mf=0.$ ......(12)

If this line be parallel to the line $y=m'x$, then

$$m'=-\frac{a+mh}{h+mb}$$

i.e., $\quad a+h(m+m')+bmm'=0.$ ......(13)

This is the condition that the lines $y=mx$, $y=m'x$ may be parallel to the conjugate diameters of the conic $\phi(x, y)=0$.

130.2. If (10) holds for all values of θ, then

$$ax'+hy'+g=0,$$

and

$$hx'+by'+f=0.$$

Every chord which passes through (x', y') is therefore bisected at the point.

Thus the point $x'=\dfrac{G}{C}$, $y'=\dfrac{F}{C}$ is the centre of the conic.

It follows that the locus of the middle points of parallel chords of a central conic is a diameter of the conic, for the lines

$$ax+hy+g=0, \quad hx+by+f=0$$

intersect in the centre.

If the conic be a parabola, these lines are parallel, so that, for varying values of m, the loci are parallel st. lines.

130.3. When (10) holds, the radii vectors from $O(x', y')$ are given by the equation

$$r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)+\phi(x', y')=0.$$

But if (x', y') be the centre of the conic, the semi-diameters are given by

$$r^2(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)+\frac{\Delta}{C}=0$$

$$i.e., \left(b+\frac{\Delta}{C}\frac{1}{r^2}\right)\tan^2\theta+2h\tan\theta+\left(a+\frac{\Delta}{C}\frac{1}{r^2}\right)=0. \quad \ldots(14)$$

$\therefore$ For a given r, there are two directions in which r has equal values. If r be either semi-axis, these two directions coincide and therefore the condition for the equality of the roots in $\tan\theta$ gives the axes in magnitude.

$$\text{Thus} \quad \left(a+\frac{\Delta}{C}\cdot\frac{1}{r^2}\right)\left(b+\frac{\Delta}{C}\cdot\frac{1}{r^2}\right)=h^2, \quad \ldots\ldots(15)$$

gives the axes in magnitude.

Multiplying (14) by $a+\frac{\Delta}{C}\cdot\frac{1}{r^2}$, we get by (15)

$$h\tan\theta+\left(a+\frac{\Delta}{C}\frac{1}{r^2}\right)=0$$

which gives the directions of the axes.

If $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$ be the roots of (15), we have the *equations of the axes*:

$$\left.\begin{aligned} hy+\left(a+\frac{\Delta}{C}\frac{1}{r_1^2}\right)x=0 \\ hy+\left(a+\frac{\Delta}{C}\frac{1}{r_2^2}\right)x=0. \end{aligned}\right\} \quad \ldots\ldots(16)$$

If e be the eccentricity of the conic, then

$$e^2=\frac{r_1^2-r_2^2}{r_1^2}=1-\frac{r_2^2}{r_1^2}.$$

$$\therefore \quad \frac{(2-e^2)^2}{1-e^2}=\frac{(a+b)^2}{ab-h^2}. \quad \ldots\ldots(17)$$

We see that the axes of the conic $ax^2+2hxy+by^2=1$ are given in magnitude and position by the equations

$$\frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0, \quad \ldots\ldots(18)$$

$$hy+\left(a-\frac{1}{r_1^2}\right)x=0,\ hy+\left(a-\frac{1}{r_2^2}\right)x=0, \quad \ldots(19)$$

where r_1^2, r_2^2 are the roots of (18).

✓ **130.4.** We indicate an alternative method for finding the equations and lengths of the axes of a central conic.

Consider a central conic $\alpha x^2+\beta y^2=1$, and a concentric circle $x^2+y^2=\rho^2$.

The equation $(\alpha x^2+\beta y^2-1)-\lambda(x^2+y^2-\rho^2)=0$ represents a conic through the intersections of the central conic and the circle.

This conic breaks up into a pair of st. lines if

$$(\alpha-\lambda)(\beta-\lambda)(1-\lambda\rho^2)=0.$$

For $\lambda=\frac{1}{\rho^2}$, we see that the lines through the common points of the curves are the diameters of the curves given by the equation

$$\left(\alpha-\frac{1}{\rho^2}\right)x^2+\left(\beta-\frac{1}{\rho^2}\right)y^2=0.$$

and this equation plainly represents a pair of lines equally inclined to the axes of the conic.

The two lines coincide only when ρ is equal to a semi-axis of the conic, and then they coincide in the corresponding axis.

Since the general equation of a central conic $\phi(x, y)=0$ is reducible to the form

$$ax^2+2hxy+by^2+\frac{\Delta}{C}=0,$$

it is, therefore, enough to find the axes in magnitude and position of the conic

$$ax^2+2hxy+by^2=1.$$

Note that the right-hand side is unity.

Now through the common points of such a conic and the concentric circle $x^2+y^2=r^2$, we have a conic

$$ax^2+2hxy+by^2-1+\lambda(x^2+y^2-r^2)=0,$$

which breaks up into a pair of st. lines if

$$-(a+\lambda)(b+\lambda)(1+\lambda r^2)+(1+\lambda r^2)h^2=0.$$

One pair given by $1+\lambda r^2=0$ is a pair of common diameters of the curves :

$$ax^2+2hxy+by^2-\frac{1}{r^2}(x^2+y^2)=0,$$

i.e., $$\left(a-\frac{1}{r^2}\right)x^2+2hxy+\left(b-\frac{1}{r^2}\right)y^2=0.$$

These diameters coincide in one of the axes of the conic if r is the length of the semi-axis. Now this pair of lines will be a coincident pair if

$$h^2=\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right).$$

Thus the semi-axes of the conic are given by the equation

$$\frac{1}{r^4}-(a+b)\frac{1}{r^2}+ab-h^2=0.$$

Let $\frac{1}{r_1^2}$ and $\frac{1}{r_2^2}$ be the roots of the equation, then r_1, r_2 are the semi-axes.

and $$\left(a-\frac{1}{r_1^2}\right)x^2+2hxy+\left(b-\frac{1}{r_1^2}\right)y^2=0$$

and $$\left(a-\frac{1}{r_2^2}\right)x^2+2hxy+\left(b-\frac{1}{r_2^2}\right)y^2=0.$$

are the squares of the equations of the axes.

Thus the equations of the axes are

$$\left(a-\frac{1}{r_1^2}\right)x+hy=0,$$

and $$\left(a-\frac{1}{r_2^2}\right)x+hy=0.$$

Omit

130.5. If the axes are oblique, the equation of the concentric circle considered will be

$$x^2+y^2+2xy\cos\omega=r^2$$

and the equation of the pair of lines through the centre and the intersections of the circle and the conic is

$$\left(a-\frac{1}{r^2}\right)x^2+2\left(h-\frac{\cos\omega}{r^2}\right)xy+\left(b-\frac{1}{r^2}\right)y^2=0.$$

The equation which gives the length of the semi-axes will be

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=\left(h-\frac{\cos\omega}{r^2}\right)^2.$$

If r_1^2, r_2^2 be the roots of this equation, the equations of the corresponding axes will be

$$\left(a-\frac{1}{r_1^2}\right)x+\left(h-\frac{\cos\omega}{r_1^2}\right)y=0;$$

and $$\left(a-\frac{1}{r_2^2}\right)x+\left(h-\frac{\cos\omega}{r_2^2}\right)y=0. \qquad \dots\dots(20)$$

130.6. We will now proceed to determine the combined equation of the axes of the conic $\phi(x, y)=0$.

Let (x', y') be a point on an axis, its equation is then

$$\frac{Y}{X}=\frac{Y'}{X'} \quad \text{or} \quad XY'-YX'=0,$$

i. e., $x(aY'-hX')+y(hY'-bX')+gY'-fX'=0$. ...(i)

where $X\equiv ax+hy+g$, $Y=hx+by+f$.

The polar of (x', y') is

$$xX'+yY'+gx'+fy'+c=0. \qquad \text{.....}(ii)$$

Since (x', y') is on an axis, the lines (i) and (ii) are at right angles

$$\therefore \quad X'(aY'-hX')+Y'(hY'-bX')=0,$$

which shows that (x', y') lies on the lines

$$h(X^2-Y^2)=(a-b)XY \qquad \text{.....}(21)$$

which is, therefore, the equation of the axes.

131. Two infinite roots. Asymptotes.

The r-quadratic has one root infinite if the co-efficient of r^2 be zero ; i. e., if

$$a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta=0.$$

or $$b\tan^2\theta+2h\tan\theta+a=0.$$

We infer that there are two values of θ for which the line

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r,$$

cuts the conic $\phi(x, y)=0$ at infinity. Thus, through any point two lines can be drawn each to meet the conic in one point at infinity and they are evidently the lines

$$a(x-x')^2+2h(x-x')(y-y')+b(y-y')^2=0.$$

These lines are real and distinct, coincident or imaginary, according as

$$h^2-ab \gtreqless 0,$$ Remember

i. e., according as the curve is a hyperbola, parabola or ellipse.

In order that the r-quadratic may have both roots infinite, the coefficient of r^2, r must both be zero, i. e., we must have

$$a\cos^2\theta+2h\cos\theta\sin\theta+b\sin^2\theta=0, \qquad \text{......}(i)$$

and $$(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta=0. \qquad \text{......}(ii)$$

Thus eliminating θ, we say that the point (x', y') cannot be chosen arbitrarily, but must satisfy the relation

$$bX'^2 - 2hX'Y' + aY'^2 = 0. \qquad \ldots\ldots(iii)$$

Now there can be two infinite roots only when the line drawn is an asymptote. Hence (x', y') lies on an asymptote when (iii) is satisfied.

Thus the two asymptotes are given by the equation

$$bX^2 - 2hXY + aY^2 = 0. \qquad \ldots\ldots(22)$$

But if (x', y') be the centre, (ii) is identically satisfied and the asymptotes are given by the equation

$$a(x-x')^2 + 2h(x-x')(y-y') + b(y-y')^2 = 0, \qquad \ldots\ldots(23)$$

where $$x' = \frac{G}{C},\ y' = \frac{F}{C}.$$

131.1. We now explain another method by which the equation of the asymptotes can be found.

The asymptotes of a conic $\phi(x, y) = 0$ belong to a system of conics which have a double contact at infinity with the conic. The equation of such a system can be written as

$$\phi(x, y) = \lambda.$$

This equation will represent the asymtotes if it denotes two right lines. The condition for this is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

$\therefore \quad \Delta - \lambda(ab - h^2) = 0.$

Thus the equation of the asymptotes is

$$\phi(x, y) = \frac{\Delta}{b - h^2}. \qquad \ldots\ldots(24$$

This equation can also be obtained from the condition that the conic $\phi(x, y) = \lambda$ passes through the centre of the conic

$$\phi(x, y) = 0.$$

131.2. We have pointed out that if the conic $\phi(x, y) = 0$ be a parabola, then only one st. line can be drawn through a point (x', y') to meet the curve in one point at infinity.

Since $h^2=ab$, the equation giving the directions of such lines, *viz.*, $a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta=0$ can be written as

$$(\sqrt{a}\cos\theta+\sqrt{b}\sin\theta)^2=0.$$

Thus $\sqrt{a}\cos\theta+\sqrt{b}\sin\theta=0$.

The line meets the curve in two points at ∞ if also

$$(ax'+hy'+g)\cos\theta+(hx'+by'+f)\sin\theta=0.$$

Eliminating θ, we get $g\sqrt{b}-f\sqrt{a}=0$.

Thus the locus of $(x'y')$ is the st. line at ∞. Though the line at ∞ touches the parabola, yet it is not an asymptote, because it is not within reach.

If the r-quadratic has one zero root and one infinite root, then $\phi(x', y')\equiv(\sqrt{a}\,x'+\sqrt{b}y')^2+2gx'+2fy'+c=0$, ...(*i*)

and $\sqrt{a}\cos\theta+\sqrt{b}\sin\theta=0$. ......(*ii*)

The st. line $\dfrac{x-x'}{\cos\theta}=\dfrac{y-y'}{\sin\theta}$ is then a diameter of the parabola.

The equation of the diameter takes the form by (*ii*)

$$\sqrt{a}x+\sqrt{b}y=\sqrt{a}x'+\sqrt{b}y'. \qquad ......(iii)$$

The tangent at (x',y') is $x\mathrm{X}'+y\mathrm{Y}'+gx'+fy'+c=0$...(*iv*)

The diameter will be the axis of the parabola and the tangent at (x', y') will become the tangent at the vertex if these two lines (*iii*), (*iv*) are at right angles, the condition for which is $\sqrt{a}\,\mathrm{X}'+\sqrt{b}\,\mathrm{Y}'=0$,

i. e., $\sqrt{a}x'+\sqrt{b}y'+\dfrac{g\sqrt{a}+f\sqrt{b}}{a+b}=0$.

Thus the equation of the axis is

$$\sqrt{a}x+\sqrt{b}\,y+\frac{g\sqrt{a}+f\sqrt{b}}{a+b}=0. \qquad ......(25)$$

By (*i*), the tangent at the vertex is

$$\sqrt{b}\,x-\sqrt{a}\,y-\frac{(g\sqrt{a}+f\sqrt{b})^2-c(a+b)^2}{2(a+b)(g\sqrt{b}-f\sqrt{a})}=0. \qquad ......(26)$$

If P (x, y) be any point on the curve, PM the perpendicular on the tangent at the vertex and PN the perpendicular on the axis, we have

$$\mathrm{PN}^2=2l.\mathrm{PM}$$

where l is the semi-latus rectum.

$$\therefore \qquad l=\frac{g\sqrt{b}-f\sqrt{a}}{(a+b)^{3/2}},$$

since (x, y) is on the curve

$$(\sqrt{a}x+\sqrt{b}y+\lambda)^2=2(\lambda\sqrt{a}-g)x+2(\lambda\sqrt{b}-f)y+\lambda^2-c.$$

131.3. We give another method for finding the axis and latus rectum of the parabola. This method depends upon the fact that the square of the perpendicular from a point of a parabola on its axis bears a constant ratio to the perpendicular on the tangent at the vertex. The constant ratio is the latus rectum. Hence to find the axis and tangent at the vertex, we must find two lines, bearing such a relation to the curve represented by the given equation and remember that the two lines have to be at right angles.

Let the equation of the parabola be

$$ax^2+2hxy+by^2+2gx+2fy+c=0,$$

which can be written in the form

$$(\alpha x+\beta y)^2+2gx+2fy+c=0.$$

The equation shows that the square of the perpendicular from a point of the curve on the line $\alpha x+\beta y=0$ varies as the perpendicular on the line $2gx+2fy+c=0$. If these two lines be not at right angles, we write the equation in the form

$$(\alpha x+\beta y+\lambda)^2=2x(\lambda\alpha-g)+2y(\lambda\beta-f)+\lambda^2-c.$$

Whatever λ may be, the lines

$$\alpha x+\beta y+\lambda=0,\ 2x(\lambda\alpha-g)+2y(\lambda\beta-f)+\lambda^2-c=0,$$

bear the above relation to the curve.

Choose λ so that these lines are at right angles.

$$\therefore\quad \alpha(\lambda\alpha-g)+\beta(\lambda\beta-f)=0 \text{ or } \lambda=\frac{\alpha g+\beta f}{\alpha^2+\beta^2}.$$

When λ has this value, the former line is the axis of the parabola and the latter the tangent at the vertex. The equation may further be written as

$$\left(\frac{\alpha x+\beta y+\lambda}{\sqrt{\alpha^2+\beta^2}}\right)=2l\left[\frac{2x(\lambda\alpha-g)+2y(\lambda\beta-f)+\lambda^2-c}{\sqrt{[4(\lambda\alpha-g)^2+4(\lambda\beta-f)^2]}}\right],$$

where $\quad l=\sqrt{[(\lambda\alpha-g)^2+(\lambda\beta-f)^2]}/(\alpha^2+\beta^2),$

which is therefore the semi-latus rectum. The latus rectum is equal to $\quad \dfrac{2(f\sqrt{a}-g\sqrt{b})}{(a+b)^{3/2}}.$

Exercise XXXVII

1. Find the equation of the asymptotes by considering them as a pair of tangents from the centre.

2. The equation of the asymptotes can also be deduced as follows. Let the equation of the asymptotes be

$$ax^2+2hxy+by^2+2gx+2fy+c+\lambda=0.$$

Shift the origin to a point (x', y') on an asymptote. The equation becomes

$$ax^2+2hxy+by^2+2x\mathrm{X}'+2y\mathrm{Y}'=0$$

which must be of the form

$$(lx+my)(l'x+m'y+n')=0.$$

i. e., $\quad (lx+my)(l'x+m'y)+n'(lx+my)=0.$

Thus $x\mathrm{X}'+y\mathrm{Y}'$ must be a factor of $ax^2+2hxy+by^2$.

$$\therefore \quad a\mathrm{Y}'^2-2h\mathrm{X}'\mathrm{Y}'+b\mathrm{X}'^2=0.$$

Hence (x', y') lies on the locus

$$a\mathrm{Y}^2-2h\mathrm{X}\mathrm{Y}+b\mathrm{X}^2=0.$$

3. Find the asymptotes of the following hyperbolas :—

(*i*) $10x^2+6y^2+19xy+41x+23y+27=0$,

(*ii*) $7x^2+6xy-y^2+9x+y+6=0$,

(*iii*) $2x^2-7xy+3y^2-9x+7y+8=0$,

(*iv*) $36(x^2-y^2)+48x-36y+14=0$.

4. Find the hyperbolas conjugate to the above hyperbolas.

5. Two lines are drawn through the point $(-1, 1)$ each meeting the curve

$$7x^2+6xy-y^2+9x+y+6=0$$

in one point at infinity ; find their directions and the finite points in which they meet the curve.

6. Show that only one st. line can be drawn through the point $(1, 2)$ to meet the curve

$$x^2+2xy+y^2-12x+4y+4=0$$

in one point at infinity. Why is this ? Find the finite intersection.

7. Find the inclination to the axis and the length of that chord of the conic

$$2x^2+4xy+3y^2+5x-64y+127=0$$

which is bisected at the point $(1, 3)$.

8. Find the points on the curve $x^2+xy+y^2=3$ at which the tangents are parallel to the line $y=x$.

9. Find the equation and lengths of the axes of the conic

$$2x^2-2xy+2y^2-2x-2y-3=0$$

and the equation of the curve referred to its axes.

10. Show that if

$$ax^2+2hxy+by^2=1 \text{ and } a'x^2+2h'xy+b'y^2=1$$

represent the same conic referred to two different sets of rectangular axes, then

$$a+b=a'+b',\ ab-h^2=a'b'-h'^2.$$

11. Show that the line at infinity touches the curve

$$9x^2+24xy+16y^2-98x+11y-94=0,$$

and find its axis and latus rectum.

12. Find the equations of the two ellipses that have the lines

$$2x+y-3=0,\ x-2y+2=0$$

for their principal axes, their major and minor axes being of lengths 8 and 6.

13. Show that the asymptotes of the conic

$$x^2+4xy+2y^2=1$$

are conjugate diameters of the conic

$$2x^2+4xy+4y^2=1.$$

14. Find the locus of the middle points of all chords of the conic

$$2x^2+4xy+3y^2-6x-4y+3=0$$

which pass through the origin.

15. Find the equation of the asymptotes of the hyperbola

$$x^2+2xy-y^2+2x+4y=0$$

and deduce the equation of the conjugate hyperbola.

(P. U. 1935)

16. Find the equation of the conic whose asymptotes are the lines

$$2x+3y+5=0 \text{ and } 5x+3y=8$$

and which passes through the point $(1,-1)$. (P. U. 1932)

17. Show that any two concentric conics have in general one and only one pair of common conjugate diameters.

18. Show that if

$$ax^2+2hxy+by^2=1,\quad a'x^2+2h'xy+b'y^2=1$$

represent the same conic or equal conics, the axes being rectangular, then

$$(a-b)^2+4h^2=(a'-b')^2+4h'^2.$$

Illustrative Examples.

(1) *Trace the conic*

$$21x^2-6xy+29y^2+6x-58y-151=0.$$

The conic is an ellipse, since $ab-h^2=21\times 29-3^2>0$, and $\Delta \neq 0$

The centre-giving equations are

$$21x-3y+3=0$$
$$-3x+29y-29=0$$
$$\therefore \quad x=0,\quad y=1.$$

Shifting the origin to (0, 1) the equation of the conic reduces to

$$21x^2-6xy+29y^2=180.$$

The semi-diameters of length r lie along the lines

$$x^2\left(21-\frac{180}{r^2}\right)-6xy+y^2\left(29-\frac{180}{r^2}\right)=0.$$

The semi-axes are, therefore, the roots of the equation

$$9=\left(21-\frac{180}{r^2}\right)\left(29-\frac{180}{r^2}\right),$$

which reduces to

$$r^4-15r^2+54=0.$$

$$\therefore \quad r^2=9 \quad \text{or} \quad r^2=6.$$

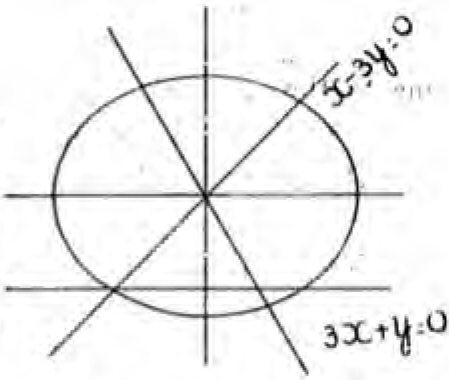
The equation of the axes whose length is 6 is

$$x-3y=0,$$

and the equation of the axis whose length is $2\sqrt{6}$ is

$$3x+y=0.$$

The ellipse is met in real points by both the original axes.



(2) *Find the species, the eccentricity, and the position of the axes of the conic*

$$x^2-11y^2-16xy+10x+10y-7=0.$$

and sketch the curve.

[Math. Trip II. 1911].

The equation of the curve is

$$x^2-16xy-11y^2+10x+10y-7=0 \qquad \ldots\ldots(i)$$

Since $h^2-ab=64+11=75>0$, and $\Delta\neq 0$, the conic is a hyperbola.

The centre is given by the equations

$$x-8y+5=0,$$
$$-8x-11y+5=0.$$

$$\therefore \quad x=-\frac{1}{5}, \quad y=\frac{3}{5}.$$

Shifting the origin to the point $\left(-\frac{1}{5}, \frac{3}{5}\right)$ the equation of the conic assumes the form

$$x^2-16xy-11y^2=5.$$

The diameters of length $2r$ lie along the lines

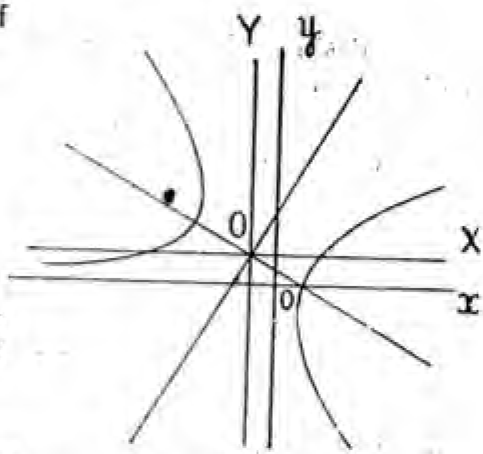
$$x^2\left(1-\frac{5}{r^2}\right)-16xy-y^2\left(11+\frac{5}{r^2}\right)=0. \qquad \ldots\ldots(ii)$$

These lines will be the axes if

$$64+\left(1-\frac{5}{r^2}\right)\left(11+\frac{5}{r^2}\right)=0$$

or $$3r^4-2r^2-1=0$$

or $$r^2=1,\ -\tfrac{1}{3}.$$



The axis of length 2 lies along the line $x+2y=0$, the equation of the other axis is

$$2x-y=0.$$

The new y-axis does not meet the curve. while the x-axis meets the curve in real points.

The shape of the curve is as shown in the figure.

Shifting the origin back, the equations of the axes become

$$x+2y=1,\quad 2x-y+1=0.$$

The eccentricity e is given by the equation

$$\frac{1}{3}=e^2-1 \quad \therefore \quad e=\frac{2}{\sqrt{3}}.$$

(3) *Trace the curve*

$$y^2-4xy-5x^2+6y+42x-63=0.$$

(Peterhouse, 1900)

Since $h^2-ab=4+5=9>0$, and $\triangle\neq 0$, the curve is a hyperbola,

The centre is the intersection of the lines

$$-5x-2y+21=0,$$
$$-2x+y+3=0.$$

$\therefore\quad x=3,\ y=3.$

Transfering the origin to the point (3, 3), the equation takes the form

$$5x^2+4xy-y^2=9.$$

The diameters of length $2r$ lie along the lines

$$\left(5-\frac{9}{r^2}\right)x^2+4xy-\left(1+\frac{9}{r^2}\right)=0.$$

These lines will coincide with either of the axes if

$$4+\left(5-\frac{9}{r^2}\right)\left(1+\frac{9}{r^2}\right)=0,$$

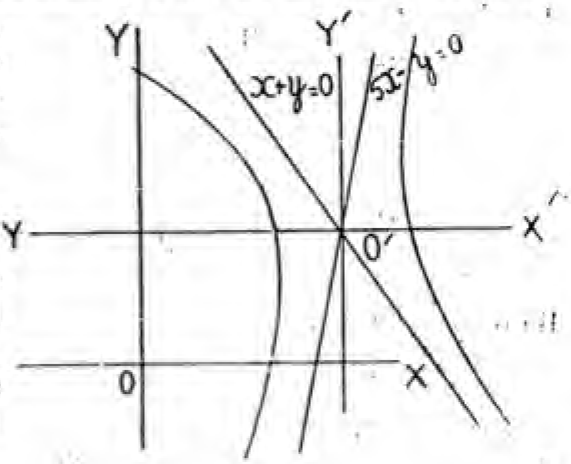
i.e., $$r^4+4r^2-9=0$$

$$\therefore\quad r^2=-2\pm\sqrt{13}.$$

One value of r^2 is approximately 1·6, and the other value is negative and so r is imaginary. The equations of the axes are also found to be in an inconvenient form. The asymptotes of the conic are

$$5x-y=0, \quad x+y=0.$$

The new y-axis meets the conic in imaginary points, while the x axis meets it in real points.



(4) *Draw the curve $9x^2+6xy+y^2+2x+3y+4=0$ and find its latus rectum.*

(Queens, 1901).

The conic is a parabola, for $h^2=ab$ and $\Delta \neq 0$.

The equation can be written in the form

$$(3x+y)^2=-2x-3y-4$$

or $\quad (3x+y+\lambda)^2=2x(3\lambda-1)+y(2\lambda-3)+(\lambda^2-4) \qquad \ldots\ldots (i)$

Choose λ so that the lines

$$3x+y+\lambda=0, \quad 2x(3\lambda-1)+y(2\lambda-3)+\lambda^2-4=0$$

may be at right angles. So

$$6(3\lambda-1)+2\lambda-3=0,$$

$$\therefore \quad 20\lambda=9 \quad i.e. \quad \lambda=\frac{9}{20}.$$

The equation (i) takes the form

$$\left(3x+y+\frac{9}{20}\right)^2=\frac{7}{10}\left(x-3y-\frac{217}{40}\right) \qquad \ldots\ldots (ii)$$

Put $\quad \mathrm{Y}=\dfrac{3x+y+\dfrac{9}{20}}{\sqrt{10}}$

$$\mathrm{X}=\left(x-3y-\frac{217}{40}\right)\Big/\sqrt{10}$$

The equation (ii) then takes the form

$$10\mathrm{Y}^2=\frac{7\mathrm{X}}{10}\cdot\sqrt{10}$$

$$\therefore \quad \mathrm{Y}^2=\frac{7\sqrt{10}}{100}\mathrm{X}.$$

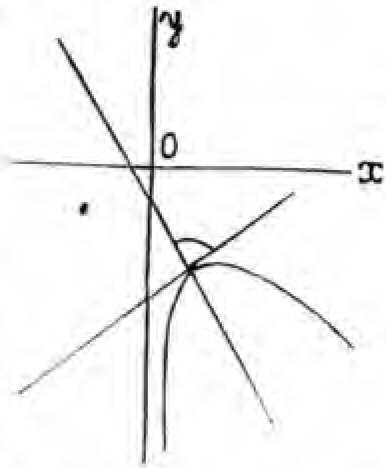
Hence the latus rectum is $\dfrac{7\sqrt{10}}{100}$.

The vertex of the parabola is the intersection of the lines

$$3x+y+\frac{9}{20}=0$$

$$x-3y-\frac{217}{40}=0$$

i.e., the point $\left(\frac{163}{400}, \frac{-669}{400}\right)$.



Now every point of the curve must lie on the positive side of the line

$$x-3y-\frac{217}{40}=0,$$

and the origin of the old axes lies on the negative side of the line. The line $x=0$ meets the conic in imaginary points and so does the y-axis.

Exercise XXXVIII

√1. Find in magnitude the axes of the following conics and sketch them :—

(*i*) $3x^2+4xy+14x+4y+5=0.$
(*ii*) $x^2+y^2+10xy-22x-14y+19=0.$
(*iii*) $11x^2+4xy+14y^2+18x-24y+15=0.$
(*iv*) $36x^2+24xy+29y^2+172x+164y-176=0.$
(*v*) $11x^2+6xy+19y^2+2x-14y+3=0.$

2. Find the latera recta of the following parabolas, and sketch them :—

(*i*) $x^2+2xy+y^2-12x+4y+4=0.$
(*ii*) $25x^2-120xy+144y^2-146x-89y=25.$

3. show that $\frac{2}{\sqrt{7}}$ is the product of the semi-axes of the ellipse

$$x^2-xy+2y^2-2x-6y+7=0,$$

and that the equation of the axes is

$$x^2-2xy-y^2+8y-8=0.$$

√4. Trace the curves

(*i*) $11x^2+4xy+14y^2-26x-32y+23=0.$
(*ii*) $3x^2-8xy-3y^2-4x+22y-12=0.$
(*iii*) $4x^2+4xy+y^2+4x+2y+1=0.$ √
(*iv*) $x^2-2xy+3y^2-2x-2y+4=0.$
(*v*) $(x+2y-2)^2+4(2x-y+1)^2=45.$
(*vi*) $2x^2-3xy-2y^2+5y+2=0.$
(*vii*) $x^2+3xy+4y^2-28x-56y+196=0.$
(*viii*) $7x^2-60xy+32y^2-106x+68y-37=0.$

132. Two equal roots. Tangents from a point.

The r-quadratic has equal roots when

$$(a\cos^2\theta+2h\sin\theta\cos\theta+b\sin^2\theta)\,\phi(x',y')=(X'\cos\theta+Y'\sin\theta)^2.$$

When it has equal roots, $OP=OQ$, and the line

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}$$

touches the conic.

Eliminating θ, we get the equation to the pair of tangents from $O(x',y')$.

Thus the tangents from O are given by the equation

$$\{a(x-x')^2+2h(x-x')(y-y')+b(y-y')^2\}\,\phi(x',y')=\{(ax'+hy'+g)(x-x')+(hx'+by'+f)(y-y')\}^2,$$

which may be written as

$$(\phi+\phi'-2T)\,\phi'=(T-\phi')^2.$$

i. e., $$\phi\phi'=T^2, \qquad \ldots\ldots(27)$$

where $T=0$ is the polar of (x',y') with respect to $\phi=0$.

132.1. The equation to the tangents from (x',y') may also be found as follows :—

The conic $\phi+\lambda T^2=0$ has double contact with the conic $\phi=0$ at the extremities of the chord $T=0$.

If it passes through (x',y') it will represent the tangents from (x',y').

Since $\phi'+\lambda T'^2=\phi'+\lambda\phi'^2=0$,

$\therefore\ \phi\phi'=T^2$ represents the required tangents.

132.2. The equation of the tangents to a conic at the extremities of a given chord.

Let $lx+my+n=0$ be the equation of the chord of the conic $\phi(x,y)=0$, then any conic which touches $\phi(x,y)=0$ at the extremities of the chord is given by the equation

$$\phi(x,y)-\lambda(lx+my+n)^2=0.$$

This equation will represent the pair of tangents, if its discriminant vanishes. Hence

$$\begin{vmatrix} a-\lambda l^2 & h-\lambda lm & g-\lambda ln \\ h-\lambda lm & b-\lambda m^2 & f-\lambda mn \\ g-\lambda ln & f-\lambda mn & c-\lambda n^2 \end{vmatrix}=0,$$

or
$$0=\begin{vmatrix} a-\lambda l^2 & h-\lambda lm & g-\lambda ln & l \\ h-\lambda lm & b-\lambda m^2 & f-\lambda mn & m \\ g-\lambda ln & f-\lambda mn & c-\lambda n^2 & n \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$=\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ \lambda l & \lambda m & \lambda n & 1 \end{vmatrix}=\lambda\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & o \end{vmatrix}+\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$\therefore \Delta-\lambda(Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm)=0$$

or $\quad \Delta-\lambda\Sigma=0$, where Σ is the co-efficient of $-\lambda$.

Hence the equation of the tangents is
$$\phi\Sigma-\Delta(lx+my+n)^2=0.$$

132.3. Director Circle. The locus of the point of intersection of perpendicular tangents to a conic is a circle concentric with the conic.

Let (x', y') be a point from which the tangents
$$\phi(x, y).\ \phi(x', y')=(xX'+yY'+gx'+fy'+c)^2$$
or $\quad (a\phi'-X'^2)x^2+(b\phi'-Y'^2)y^2+\text{etc.}=0$

to the conic
$$\phi(x, y)=0$$
are at right angles. Hence
$$(a\phi'-X'^2)+(b\phi'-Y'^2)=0$$

Thus the required locus is the circle
$$(a+b)\ \phi(x, y)=X^2+Y^2$$

or $\quad C(x^2+y^2)-2Gx-2Fy+A+B=0 \qquad \ldots\ldots(28)$

which is evidently concentric with the conic.

The circle belongs to the system (25).

If the conic is a parabola, this locus reduces to the line at infinity and the straight line
$$2Gx+2Fy-A-B=0. \qquad \ldots\ldots(29)$$

This is then the equation of the directrix.

132.4. To find the foci of a conic.

If (x', y') be a focus of the conic $\phi(x, y)=0$, the tangents from (x', y') to the conic are the circular lines through (x', y').

Putting $\tan\theta = i$ and $-i$ in the condition for the equality of roots of the r-quadratic, *viz.*,

$$(a+2h\tan\theta+b\tan^2\theta)\,\phi(x',y')=(X'+Y'\tan\theta)^2$$

we get $$(a+2hi-b)\,\phi(x',y')=(X'+iY')^2$$

and $$(a-2hi-b)\,\phi(x',y')=(X'-iY')^2.$$

Adding and subtracting, we have

$$(a-b)\,\phi(x',y')=X'^2-Y'^2,$$
$$h\,\phi(x',y')=X'Y'.$$

Thus the foci of the conic $\phi(x,y)=0$ are the intersections of the conics

$$\frac{X^2-Y^2}{a-b}=\frac{XY}{h}=\phi(x,y). \qquad \ldots\ldots(30)$$

It follows that a conic has four foci.

Since the conic $\dfrac{X^2-Y^2}{a-b}=\dfrac{XY}{h}$ passes through the foci and the centre of the conic $\phi=0$,

$\therefore$ it must be the axes of the conic $\phi=0$.

Note.—To find the foci of a conic $\phi=0$, we have to find the condition of tangency of either line

$$y-y'=\pm i(x-x'),$$

(x',y') being a focus.

Now the line $lx+my+n=0$ touches the conic if

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Put $l=\pm i$, $m=-1$, $n=\mp ix'+y'$, then

$$-A+B+C(y'\mp ix')^2-2F(y'\mp ix')\pm 2Gi(y'\mp ix')\mp 2Hi=0$$

or $$-A+B+C(y'^2-x'^2)-2Fy'+2Gx'$$
$$\mp 2i(Cx'y'-Fx'-Gy'+H)=0.$$

Adding and subtracting, we get

$$C(x'^2-y'^2)-2Gx'+2Fy'+A-B=0,$$
$$Cx'y'-Fx'-Gy'+H=0.$$

Hence, the foci are the intersections of two rectangular hyperbolas

$$C(x^2-y^2)-2Gx+2Fy+A-B=0,$$

and $$Cxy-Fx-Gy+H=0, \qquad \ldots\ldots(31)$$

which are concentric with the conic.

To solve the equations, it is convenient to proceed as follows, Shift the origin to the common centre $\left(\dfrac{G}{C},\dfrac{F}{C}\right)$, the equations of the two hyperbolas assume the forms

$$C^2(\xi^2-\eta^2)=(a-b)\,\Delta, \qquad \ldots\ldots(32)$$

and $$C^2\xi\eta=h\,\Delta. \qquad \ldots\ldots(33)$$

where $x - \dfrac{G}{C} = \xi,\ y - \dfrac{F}{C} = \eta.$

We get $C^4(\xi^2+\eta^2)^2 = C^4(\xi^2-\eta^2)^2 + 4C^4\xi^2\eta^2 = \{(a-b)^2+4h^2\}\,\Delta^2,$

hence $C^2(\xi^2+\eta^2) = \pm\{(a-b)^2+4h^2\}^{\frac{1}{2}}\,\Delta$

Thus ξ^2, η^2, and so ξ and η, are determined. The values of ξ and η are paired so as to satisfy (33). The values of x and y are then easily determined. We thus get four foci.

If the conic is a parabola, then $C=0$ and the two hyperbolas reduce to st. lines

$$\left.\begin{array}{l} 2Gx - 2Fy = A - B \\ Fx + Gy = H \end{array}\right\} \qquad \ldots\ldots(34)$$

which give the finite focus.

In the case of a parabola, first find the directrix and then its pole which will be the focus.

132.5. In the present and the next articles, we proceed to indicate methods of finding the foci which are convenient for numerical equations.

If (α, β) be a focus and $lx+my+n=0$ the corresponding directrix, the equation of the conic is of the form

$$\lambda\{(x-\alpha)^2+(y-\beta)^2\} + (lx+my+n)^2 = 0$$

which is identical with $\phi(x, y)=0$. Hence

$$\phi(x, y) - \lambda\{(x-\alpha)^2+(y-\beta)^2\} \equiv (lx+my+n)^2.$$

So λ must be so chosen that the expression on the left is a perfect square.

Ex. Find the foci and directrices of the conic whose equation is

$$7x^2 - 48xy - 7y^2 + 60x + 80y - 50 = 0.$$

If (α, β) is the focus and $lx+my+n=0$ the directrix, then

$$7x^2 - 48xy - 7y^2 + 60x + 80y - 50 - \lambda[(x-\alpha)^2+(y-\beta)^2] \equiv (lx+my+n)^2$$

i.e., $$x^2(7-\lambda) - 48xy - y^2(7+\lambda) + 2x(30+\lambda\alpha) + 2y(40+\lambda\beta) - (50+\lambda\alpha^2+\lambda\beta^2) \qquad \ldots\ldots(i)$$

must be a perfect square. This requires

$$24^2 + (49-\lambda^2) = 0,$$

$$\therefore \quad \lambda = \pm 25.$$

Taking $\lambda=25$, the left-hand side becomes

$$2[9x^2+24xy+16y^2-5x(6+5\alpha)] - 10y(8+5\beta) + 25(2+\alpha^2+\beta^2)$$

or $$2[(3x+4y)^2-5x(6+5\alpha)] - 10y(8+5\beta) + 25(2+\alpha^2+\beta^2).$$

Since this is a perfect square it must be identical with

$$2\left[3x+4y-\frac{5}{6}(6+5\alpha)\right]^2$$

i.e., $18x^2+48xy+32y^2-10(6+5\alpha)x-\dfrac{40}{3}(6+5\alpha)y$

$$+\frac{25}{18}(6+5\alpha)^2$$

$\therefore \qquad 10(8+5\beta)=\dfrac{40}{3}(6+5\alpha)$

i.e., $\qquad 4\alpha=3\beta, \qquad \ldots\ldots(ii)$

and $\qquad \dfrac{25}{18}(6+5\alpha)^2=25(2+\alpha^2+\beta^2)$

i.e., $\qquad 7\alpha^2+60\alpha-18\beta^2=0. \qquad \ldots\ldots(iii)$

Substituting from (*ii*), we get

$$25\alpha^2-60\alpha=0$$

$\therefore \qquad \alpha=0 \quad \text{or} \quad \alpha=\dfrac{12}{5}.$

The corresponding values of β are $0, \dfrac{16}{5}$.

Hence two of the foci are

$$(0, 0), \left(\frac{12}{5}, \frac{16}{5}\right).$$

The corresponding directrices are

$$3x+4y-5=0, \qquad 3x+4y-15=0.$$

Now taking $\lambda=-25$, the expression (*i*) becomes

$$32x^2-48xy+18y^2+10(6-5\alpha)x+10(8-5\beta)-25(2-\alpha^2-\beta^2).$$

This, being a perfect square, must be identical with

$$2\left[4x-3y+\frac{5}{8}(6-5\alpha)\right]^2$$

or $\qquad 32x^2-48xy+18y^2+10(6-5\alpha)x-\dfrac{15}{2}(6-5\alpha)y$

$$+\frac{25}{32}(6-5\alpha)^2$$

$\therefore \qquad 10(8-5\beta)=\dfrac{-15}{2}(6-5\alpha)$

or $\qquad 3\alpha+4\beta=10 \qquad \ldots\ldots(iv)$

and $\qquad \dfrac{25}{32}(6-5\alpha)^2=-25(2-\alpha^2-\beta^2)$

or $7\alpha^2+32\beta^2+60\alpha-100=0.$ ......(*v*)

Substituting from (*iv*), we have

$$7\alpha^2+2(10-3\alpha)^2+60\alpha-100=0$$
$$5\alpha^2-12\alpha+20=0$$
$$\therefore \quad \alpha=\frac{6\pm 8i}{5}$$

and $\therefore$ $$\beta=\frac{8\mp 6i}{20}.$$

These are the co-ordinates of the imaginary foci.

132.6. The method of the present article has the advantage that the co-ordinates of the foci and the lengths of the axes are simultaneously obtained.

Transfer the origin to the centre of the conic, the directions of the axes remaining the same, and the equation

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

reduces to

$$ax^2+2hxy+by^2+c'=0 \qquad \text{......}(i)$$

where $c'=\dfrac{\Delta}{C}$.

Let (α, β) be a focus of this conic. By central symmetry $(-\alpha, -\beta)$ is the complementary focus and the product of the perpendiculars from these foci on a tangent is equal to a constant, say λ, the square of the semi-axis of the conic.

Let $\xi=ax_1+hy_1$, $\eta=hx_1+by_1$,

so that $x_1=\dfrac{b\xi-h\eta}{ab-h^2}$, $y_1=\dfrac{h\xi-a\eta}{h^2-ab}$.

If (x_1, y_1) be a point on the conic (*i*), ξ, η satisfy the equation

$$a(b\xi-h\eta)^2-2h(b\xi-h\eta)(h\xi-a\eta)+b(h\xi-a\eta)^2+c'(ab-h^2)=0$$

i.e., $$b\xi^2-2h\xi\eta+a\eta^2=c'(h^2-ab) \qquad \text{.....}(ii)$$

Now the tangent at (x_1, y_1) to (*i*) is

$$x\xi+y\eta+c'=0,$$

and the perpendiculars from (α, β), $(-\alpha, -\beta)$ on it are

$$\frac{c'+\alpha\xi+\beta\eta}{\sqrt{\xi^2+\eta^2}}, \quad \frac{c'-\alpha\xi-\beta\eta}{\sqrt{\xi^2+\eta^2}}.$$

The product of the perpendiculars is

$$\frac{c'^2-(\alpha\xi+\beta\eta)^2}{\xi^2+\eta^2}=\lambda,$$

so that $\xi^2(\alpha^2+\lambda)+2\xi\eta\alpha\beta+\eta^2(\beta^2+\lambda)=c'^2$. ......(*iii*)

Comparing (*ii*) and (*iii*), we have

$$\frac{\alpha^2+\lambda}{b}=\frac{\beta^2+\lambda}{a}=\frac{\alpha\beta}{-h}=\frac{c'}{h^2-ab} \qquad \text{......}(iv)$$

and each is equal to $\dfrac{\alpha^2-\beta^2}{b-a}$.

Thus the foci are given by the equations

$$\alpha^2-\beta^2=\frac{c'(b-a)}{h^2-ab}, \quad \alpha\beta=\frac{-c'h}{h^2-ab} \qquad \text{......}(35)$$

and the value of λ is obtained from the relation

$$\lambda=\frac{bc'}{h^2-ab}-\alpha^2=-\frac{b\Delta}{c^2}-\alpha^2. \qquad \text{......}(36)$$

Thus when α is known, λ can be determined.

Note.—If α, β be eliminated from (*iv*), we get the equation which gives the squares of the semi-axes

$$\left\{a+\frac{c'}{\lambda}\right\}\left\{b+\frac{c'}{\lambda}\right\}=h^2$$

i.e.,
$$\left\{a+\frac{\Delta}{C\lambda}\right\}\left\{b+\frac{\Delta}{C\lambda}\right\}=h^2. \qquad \text{......}(37)$$

Ex. We consider the previous example

$$7x^2-48xy-7y^2+60x+80y-50=0.$$

The centre is given by the equations

$$7x-24y+30=0,$$
$$-24x-7y+40=0.$$

$$\therefore \qquad x=\frac{6}{5},\ y=\frac{8}{5}.$$

Transfering the origin to $\left(\frac{6}{5},\ \frac{8}{5}\right)$, the equation of the conic reduces to the form

$$7x^2-48xy-7y^2+50=0.$$

Hence $\quad \alpha^2-\beta^2=-\dfrac{28}{25}, \ \alpha\beta=\dfrac{48}{25}$

$\therefore \qquad \alpha^2+\beta^2=4.$

Consequently

$$\alpha=\pm\frac{6}{5},\ \beta=\pm\frac{8}{5}.$$

Consequently the co-ordinates of the real foci referred to the new axes are

$$\left(\frac{6}{5},\ \frac{8}{5}\right),\ \left(\frac{-6}{5},\ \frac{-8}{5}\right).$$

Hence the co-ordinates of the foci *w. r.* to the old axes are

$$\left(\frac{12}{5},\ \frac{16}{5}\right),\ (0,\ 0).$$

Taking $\alpha^2+\beta^2=-4$, we get

$$\alpha=\pm\frac{8i}{5},\ \beta=\pm\frac{6i}{5}.$$

So the co-ordinates of the imaginary foci *w. r.* to the new axes are

$$\left(\frac{8i}{5},\ \frac{-6i}{5}\right),\ \left(\frac{-8i}{5},\ \frac{6i}{5}\right)$$

and hence *w. r.* to the old axes, the co-ordinates are

$$\left(\frac{6+8i}{5},\ \frac{8-6i}{5}\right),\ \left(\frac{6-8i}{5},\ \frac{8+6i}{5}\right).$$

The squares of the semi-axes are ± 2 by (36).

Exercise XXXIX

1. Find the real foci and director circles of the following conics :—

(*i*) $x^2+8xy-5y^2+8x+4y+2=0$.

(*ii*) $3x^2+4xy+4x-2y-3=0$.

(*iii*) $5x^2-12xy+10y^2-6x+10y+6=0$.

(*iv*) $x^2+12xy-4y^2+4x-16y+4=0$.

(*v*) $55x^2-30xy+39y^2-70x+54y-449=0$.

(*vi*) $4x^2-4xy+y^2+10x-20y+25=0$.

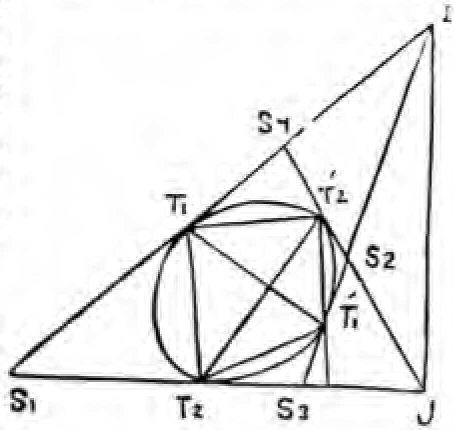
2. When is the director circle of a hyperbola imaginary ? Interpret your result.

Find the director circle of the conic $x^2+xy+y^2+x+y=0$.

3. Prove that the director circles of all conics which touch two given lines at given points are co-axel.

4. A pair of tangents to the conic $\alpha x^2+\beta y^2=1$, intercept a constant length $2k$ on the x-axis : find the locus of their point of intersection.

133. Directrices. Let the tangents from the circular points I and J at infinity to a conic meet in the four foci S_1, S_2, S_3, S_4. Suppose that T_1, T_2, T_1', T_2' are the points of contact of the tangents, then T_1T_2, $T_1'T_2'$, T_1T_2', $T_1'T_2$ are the directrices, and T_1T_1', T_2T_2' are the polars of I and J w.r. to the conic.



Consider the degenerate conic T_1T_2, $T_1'T_2'$. This passes through the intersections of the conic and the degenerate conic T_1T_1', $T_2T'_2$ which are the polars of the circular points at infinity. Thus

the directrices of a conic are the line pairs that belong to a pencil of conics generated by the given conic and the polars w.r. to the conic of the circular points at infinity.

The isotropic lines $\frac{y}{x} = \pm i$, contain the circular points at infinity, their co-ordinates can be assumed to be

$$\left(\frac{1}{\epsilon}, \frac{i}{\epsilon}\right) \left(\frac{1}{\epsilon'}, \frac{-i}{\epsilon'}\right) \text{ where } \epsilon \to 0 \quad \epsilon' \to 0.$$

The polars of these points are

$$X + iY + Z\epsilon = 0 \qquad X - iY + Z\epsilon' = 0,$$

i. e.,
$$X + iY = 0 \qquad X - iY = 0.$$

(This could be easily obtained by taking the homogeneous co-ordinates $(1, \pm i, 0)$ of the circular points at infinity). The joint equation of the polars is therefore

$$X^2 + Y^2 = 0,$$

and the system of conics that they generate with the given conic is

$$X^2 + Y^2 + \lambda\phi(x, y) = 0,$$

i. e., $(ax + hy + g)^2 + (hx + by + f)^2 + \lambda\phi(x, y) = 0.$

This will represent a pair of lines if

$$\begin{vmatrix} a^2 + h^2 + \lambda a & h(a + b + \lambda) & ag + hf + \lambda g \\ h(a + b + \lambda) & h^2 + b^2 + \lambda b & hg + bf + \lambda f \\ ag + hf + \lambda g & hg + bf + \lambda f & g^2 + f^2 + \lambda c \end{vmatrix} = 0,$$

or $$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \begin{vmatrix} a+\lambda & h & 0 \\ h & b+\lambda & 0 \\ g & f & \lambda \end{vmatrix} = 0.$$

Hence $\lambda[\lambda^2+\lambda(a+b)+ab-h^2]=0$.

For $\lambda=0$ the conic of the system is the line pair T_1T_1', $T_2T'_2$. For two other values of λ, *viz.* λ_1, λ_2 which are the roots of

$$\lambda^2+\lambda(a+b)+ab-h^2=0, \qquad \ldots\ldots(38)$$

we get the line pairs $(T_1T_2, T'_1T'_2)(T_1T'_2, T'_1T_2)$, which are the directrices.

133.1. Let the tangent at T to a conic pass through a circular point. Then the line through T perpendicular to the circular line through T is the circular line itself which touches the conic.

Thus T may be regarded as the intersection of two perpendicular tangents and does therefore lie on the director circle of the conic.

Hence *the director circle of a conic passes through the intersections of the directrices with the conic.*

Now the director circle of the conic $\phi(x, y)=0$ is

$$C(x^2+y^2)-2Gx-2Fy+A+B=0.$$

The equation

$$\lambda\phi(x, y)+C(x^2+y^2)-2Gx-2Fy+A+B=0$$

represents a conic through the intersections of the conic and its director circle.

It represents a pair of lines if

$$\begin{vmatrix} \lambda a+C & \lambda h & \lambda g-G \\ \lambda h & \lambda b+C & \lambda f-F \\ \lambda g-G & \lambda f-F & \lambda c+A+B \end{vmatrix} = 0.$$

It is a cubic in λ, thus giving three pairs of lines, one pair being the polars of the circular points and the other two being the directrices.

The line joining any two foci of a conic is normal to the conic. For a circle, all the four foci coincide in the centre and the line joining two foci becomes inderterminate. Thus every st. line from the centre of a circle is normal to it.

133.2. The lines parallel to the directrices through the origin are given by the equations

$$(a^2+h^2+\lambda a)x^2+2(ah+bg+\lambda h)xy+(h^2+b^2+\lambda b)y^2=0.$$

and the equation (38) is just the condition that these lines may coincide. Thus *the directrices are parallel by pairs.*

Miscellaneous Exercise XL.

1. Trace the curve $x^2-4xy-2y^2=1$. (P. U. 1931)
2. Trace the curve $3x^2+4xy-6=0$. (P. U. 1936)
3. Reduce the equation
$$4x^2+12xy+9y^2+2x-10y+3=0$$
to its simplest form, and give a sketch of the curve which it represents. (P. U. 1928)
4. Trace the curve $9x^2+24xy+16y^2-4y-x+7=0$. (P. U. 1935)
5. Trace and find all the elements of the equation
$$4x^2+12xy-y^2-40x-20y+24=0.$$ (P. U. 1934)
6. Show by turning the co-ordinate axes by $\frac{\pi}{4}$ that the equation $13x^2-10xy+13y^2=72$ represents an ellipse whose axes are 6 and 4. Sketch the curve. (P. U. 1932)
7. Find the focus and directrix of the parabola
$$9x^2+30xy+25y^2-206x+246y+393=0.$$ (Radford)
8. Find the equation of the directrices of the conic
$$7x^2+7y^2-1+2y-2x+2xy=0.$$ (Radford)
9. Find the equation of the directrix and the co-ordinates of the focus of the parabola $x^2+2xy+y^2-3x+6y-4=0$. (Magdalene, 1910)
10. Find the co-ordinates of the focus and the vertex of the parabola $x^2-4xy+4y^2+10x-8y+13=0$. (King's, 1912)
11. Trace the conic $34x^2+24xy+41y^2+48x+14y-108=0$ and find its eccentricity. (Corpus etc., 1913)
12. Find the centre, the directions of the axes and the eccentricity of the conic $9x^2+4xy+6y^2-10x+20y+5=0$. (P. U., B. A. 1909)
13. Show that the equation of the principal axes of the conic
$$6x^2+4xy+9y^2+12x-6y+6=0 \text{ are } x+2y=0,\ 2x-y+3=0$$
and find the eccentricity.

Find the equation of a rectangular hyperbola which has the same axes and passes through the origin. (C. U., B. A., and B. Sc., Hon. 1928)

14. Find the equation of the principal axes of the conic
$$13x^2+37y^2-32xy-14x-34y-35=0.$$ (St., Catherine, 1937)

15. Trace the conic $14x^2 - 4xy + 11y^2 + 20x - 20y - 4 = 0$, and find the co-ordinates of the foci. (Corpus, 1912)

16. Prove that the conic $30x^2 + 35y^2 = 12xy + 24x + 16y + 16$ has one focus at the origin. Find the equation of the corresponding directrix, the eccentricity, and the co-ordinates of the second focus. (Pembroke, 1909)

17. A hyperbola touches the axis of y at the origin and the line $y = 7x - 5$ at the point $(1, 2)$. One of the asymptotes is parallel to axis of x. Find the equation of the curve. (Trinity, 1909)

18. Show that the lines $y = mx$, $y = m'x$ are equal diameters of the conic

$$ax^2 + 2hxy + by^2 = 1 \text{ if } (a - b)(m + m') - 2h(1 + mm') = 0.$$

Hence find the equation of equi-conjugate diameters. (Pembroke, 1911)

19. Prove that the six points $(2, 3)$, $(3, 2)$, $(3, 1)$, $(1, 3)$, $(1, 2)$, $(2, 1)$ are on a conic whose equation referred to the axes is $3x^2 + y^2 - 2 = 0$ (Trinity, 1909)

20. Prove that the six points (a, b), (b, a), (b, c), (c, b), (c, a), (a, c) lie on a conic whose centre is the centre of mean position of the six points and the axes are parallel to the lines $x \pm y = 0$. Reduce the equation to the simplest form and find the eccentricity of the conic.

21. Find the equation of the two conics which touch the co-ordinate axes (supposed rectangular), have a focus at the point $(1, 1)$ and pass through the point $(\frac{1}{2}, 1)$. (Pembroke, 1912)

22. Prove that the conic $9x^2 - 24xy + 41y^2 = 15x + 5y$ has one extremity of the major axis at the origin and one extremity of its minor axis on the axis of x. Find the co-ordinates of its centre and foci. (Pembroke, 1910)

23. The conic $ax^2 + 2hxy + by^2 + 2x + 2y = 0$ is such that its real foci lie one on each of the co-ordinate axes; show that $(ab - h^2) = 2(h - a)(h - b)$, and that the lengths of the semi-axes are $\dfrac{-1}{2h}$ and $\dfrac{1}{\sqrt{ab - h^2}}$. (Selwyn, 1907)

24. If the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola show that its equation referred to its asymptotes as axes is

$$2(h^2 - ab)^{\frac{3}{2}}\, xy - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

25. Find the equation of the orthoptic circle of the conic $(ax + by - 1)^2 = 2\lambda xy$ and prove that for different values of λ the orthoptic circles are coaxal. (Magdalene, 1907)

26. A circle is inscribed and a rectangular hyperbola is circumscribed to an equilateral triangle. Show that each curve passes through the centre of the other. (Peterhouse, 1913)

27. Prove that the family of conics touching the four straight lines $x = \pm a$, $y = \pm b$ is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 + \frac{2\lambda xy}{ab} + \lambda^2 = 0.$$

Prove that the locus of the foci is $x^2 - y^2 = a^2 - b^2$ and if two of the conics intersect on this latter locus, they do so orthogonally. (St. Catherine, 1929)

28. Show that the co-ordinates of the foci of the general conic are given by the equations

$$\mathrm{C}x^2 - 2\mathrm{G}x + \mathrm{A} = \lambda\,\triangle$$
$$\mathrm{C}y^2 - 2\mathrm{F}y + \mathrm{B} = \lambda\,\triangle$$

where λ is either root of the quadratic

$$\mathrm{C}\lambda^2 - (a+b)\lambda + 1 = 0.$$

(King's, 1913)

29. Show that one focus of the conic

$$x^2 + y^2 + 2hxy + 2g(x+y) + \frac{g^2}{h} = 0$$ is the origin and

that the other is the point

$$x = y = -\frac{2g}{1+h}.$$

30. Show that the envelope of the chord of the conic

$$ax^2 + 2hxy + by^2 + c = 0$$

the tangents at whose extremities cut at right angles, is the conic

$$(a^2+h^2)x^2 + (b^2+h^2)y^2 + 2(a+b)hxy = \frac{(h^2-ab)c}{a+b}.$$

31. A parabola passes through the point $(4, -3)$ and touches the line $y = x$ at the origin. If its axis be parallel to the axis of x, find the equation to the parabola and the co-ordinates of its focus. [Radford]

32. Find the ellipse for which the lines $x=0$, $y-x=0$ are two conjugate diameters, and the product of the lengths of the semi-axis is 1, and which passes through the point $(\frac{3}{5}, \frac{7}{8})$.

[Math. Trip., 1932]

33. Show that one of the conics which passes throug $(1, -2)$ cuts the lines $x=1$, $y=1$ at right angles and has its centre at the origin, is given by the equation $x^2 + xy + y^2 = 3$.

34. Prove that the general equation of a conic whose centre is the origin and which cuts the lines $x=a$, $y=b$ at right-angles is

$$\frac{x^2}{a^2}+\frac{2xy}{\lambda}+\frac{y^2}{b^2}=\frac{\lambda^2}{a^2b^2}-1$$

35. Prove that the equation to the circle of curvature at the point $(1,-1)$ of the conic

$$3x^2+2xy+3y^2-8x-8y-4=0$$

is

$$3x^2+3y^2-11x-9y-4=0. \qquad \text{(Radford)}$$

36. Obtain the equation of the two parabolas which pass through the points $(0, 0)$, $(7, 0)$, $(0, 5)$ and $(3, -1)$ and the equation of their axes. (Peterhouse etc., 1931)

37. Prove that the equation $ax^2+2hxy-by^2=0$ represents a pair of conjugate diameters of the ellipse $ax^2+by^2=1$.

Prove also that there is one conic with these lines as asymptotes that cuts $ax^2+by^2=1$ orthognally at their four points of intersection, and find its equation. [Downing, 1932]

38. Prove that if the conics $px^2+2qxy+ry^2=1$ and $ax^2+by^2=1$ intersect at right angles, then

$$\frac{1}{a}-\frac{1}{p}=\frac{2}{a+b}=\frac{1}{b}-\frac{1}{r}.$$

39. Find the equation of the ellipse which passes through the origin, which has the point $(0, 4)$ as one focus and such that the minor axis lies along the line whose equation is $x+2y=3$.

Show that the equations of the equi-conjugate diameters of the ellipse are $8x+y+6=0$ and $4x-7y+18=0$.

[St. Catherine, etc., 1932].

40. Show that the locus of the point

$$x=\frac{pt^2+2qt+r}{at^2+2bt+c},\ y=\frac{p't^2+2q't+r'}{at^2+2bt+c}$$

is, in general, a conic, which is an ellipse, parabola or hyperbola according as $ac \gtrless b^2$.

[*Hint.*—The parameters of the points of intersection of the locus and the line $lx+my+n=0$ are given by the quadratic

$$t^2(lp+mp'+na)+\text{etc.}=0. \qquad \text{Hence, etc.,}$$

The parameters of the points at infinity on the conic are the roots of the equation $at^2+2bt+c=0$].

41. Show that the freedom equations of a conic in Cartesian co-ordinates can be reduced to one of the forms:

$$x=\frac{t^2+2pt+q}{r(t^2-1)},\ y=\frac{t^2+2p't+q'}{r(t^2-1)} \qquad \text{for a hyperbola}$$

$$x=\frac{t^2+2pt+q}{r(t^2+1)},\ y=\frac{t^2+2p't+q'}{r(t^2+1)} \qquad \text{for an ellipse,}$$

$$x=t^2+2pt+q,\quad y=t^2+2p't+q' \qquad \text{for a parabola.}$$

What are the parameters of the points at infinity on each curve.

42. Show that the point

$$x=\frac{t^2+1}{t^2-3t+2},\ y=\frac{t+1}{t^2-3t+2}$$

lies on a hyperbola, and find its asymptotes.

[The parameters of the points at infinity are $t=1,\ 2$.

The parameters of the intersections of the conic and the line

$$lx+my+n=0$$

are given by

$$t^2(l+n)+t(m-3n)+l+m+2n=0.$$

If the line is an asymptote touching at $t=1$, the roots of this equation are 1, 1,

$$\therefore\ l=n$$
$$m=-n.$$

Thus one asymptote is $x-y+1=0$. The other asymptote will be found to be $3x-5y-7=0$].

43. Find the asymptotes of the hyperbola given by the equations

$$x=\frac{t^2-2t-3}{t^2-1},\ y=\frac{t^2+2t-1}{t^2-1}.$$

44. Show that the equation of the chord that joins the points t, t' on the conic given by the equations

$$x=\frac{a_2t^2+2a_1t+a_0}{c_2t^2+2c_1t+c_0},\ y=\frac{b_2t^2+2b_1t+b_0}{c_2t^2+2c_1t+c_0}$$

can be written as

$$lx+my+n=0$$

where $l=A_0tt'-A_1(t+t')+A_2,\ m=B_0tt'-B_1(t+t')+B_2,$
$n=C_0tt'-C_1(t+t')+C_2$

and capital letters denote the co-factors of the corresponding small letters in the determinant.

$$\begin{vmatrix} a_0 & 2a_1 & a_2 \\ b_0 & 2b_1 & b_2 \\ c_0 & 2c_1 & c_2 \end{vmatrix}$$

Deduce the equation of the tangent at t.

45. Find the point of intersection of the tangents at t_1, t_2 of the conic in Ex. 44.

[Suppose (x', y') is the point of intersection of the tangents at t_1, t_2, then t_1, t_2 are the roots of the equation

$$(A_0t^2 - 2A_1t + A_2)x' + (B_0t^2 - 2B_1t + B_2)y' + C_0t^2 - 2C_1t + C_2 = 0.$$

$$\therefore \quad \frac{\Sigma A_0x'}{1} = \frac{2\Sigma A_1x'}{t_1+t_2} = \frac{\Sigma A_2x'}{t_1t_2}.$$

Whence $x' : y' : 1 = a_2t_1t_2 + a_1(t_1+t_2) + a_0 : b_2t_1t_2 + b_1(t_1+t_2) + b_0 : c_2t_1t_2 + c_1(t_1+t_2) + c_0$]

46. Find the centre of the conic of Ex. 44.

[The parameters of the points at infinity t_1, t_2 of the conic are the roots of $c_2t^2 + 2c_1t + c_0 = 0$ and the centre is the pole of the line joining t_1, t_2.

$$x : y : 1 = a_2c_0 - 2a_1c_1 + a_0c_2 : b_2c_0 - 2b_1c_1 + b_0c_2 : 2(c_1^2 - c_0c_2)]$$

47. General relation of pole and polar.

Let (x', y') be the pole of the line $lx + my + n = 0$, then the parameters of the points of intersection of the conic and the line $lx + my + n = 0$ are the roots of the quadratics

$$(A_0x' + B_0y' + c_0)t^2 - 2t(A_1x' + B_1y' + c_1) + A_2x' + B_2y' + C_2 = 0$$

and

$$(a_2l + b_2m + c_2n)t^2 + 2t(a_1l + b_1m + c_1n) + a_0l + b_0m + c_0 = 0,$$

whence, comparing

$$\frac{\Sigma A_0x'}{\Sigma a_2l} = \frac{-\Sigma A_1x'}{\Sigma a_1l} = \frac{\Sigma A_2x'}{\Sigma a_0l}.$$

48. Show that if $x = at^2 + bt$ and $y = ct + d$, where t is a variable parameter, the locus of the point (x, y) is a parabola whose latus rectum is $\frac{c^2}{a}$, and find the co-ordinates of the focus.

[Selwyn, 1928]

[Let (x', y') be the focus. The lines $y - y' = \pm i(x - x')$ are tangents to the parabola. Thus the equations

$$\pm i(at^2 + bt - x') - ct - d + y' = 0$$

will have equal roots].

49. Show that $x = at^2 + 2bt$, $y = a't^2 + 2b't$, where a, a', b, b' are constants and t a variable parameter, represents a parabola, and find the tangent at the point t.

Show that the directrix is $ax + a'y + b^2 + b'^2 = 0$ and find the co-ordinates of the focus. [Pembroke, 1929]

50. Show that the locus of the point.

$$x = at^2 + 2bt + c, \quad y = a't^2 + 2b't + c'$$

is a parabola whose latus rectum is

$$4(ab' - a'b)^2/(a^2 + a'^2)^{\frac{3}{2}}$$

and the directrix is

$$ax + a'y = ac + a'c' - b^2 - b'^2.$$

CHAPTER XIII.

TRILINEAR CO-ORDINATES.

134. (*a*) We have defined the cross-ratio of a range of four points A, B, C, D as $(A\ B\ C\ D) = \dfrac{AB.CD}{AD.CB} = \rho$.

If A coincides with B or C with D, $\rho = 0$.

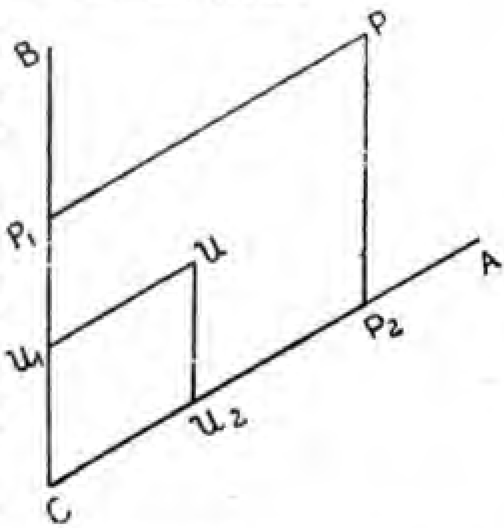
If A coincides with C or B with D, $\rho = 1$.

If A coincides with D or B with C, $\rho = \infty$.

(*b*) We have represented a point in a plane in the Cartesian system of co-ordinates as follows:—

Let CA, CB be taken as the two axes of reference.

Take any arbitrary point U as the unit point.



Let AU, BU meet CB, CA at U_1, U_2, where A, B are situated on CA, CB at infinite distances from C, *i.e.*, draw through U, the lines UU_2, UU_1 parallel to CB, CA meeting them at U_2, U_1.

Then CU_2 measures the unit length along the x-axis CA, while CU_1 is the unit length along the y-axis CB, (these unit lengths may not necessarily be equal).

Let P be any variable point in the plane of the axes.

Draw $PP_2 \| CB$, and $PP_1 \| CA$, meeting CA, CB at P_2, P_1.

Abscissa of $P = x = CP_2/CU_2$.

Ordinate of $P = y = CP_1/CU_1$.

(*c*) Since A and B are at ∞, the side AB of $\triangle$ ABC is at ∞, we can, therefore, write the co-ordinates of P as follows:—

$$x = \frac{CP_2.AU_2}{CU_2.AP_2} = (CP_2AU_2),$$

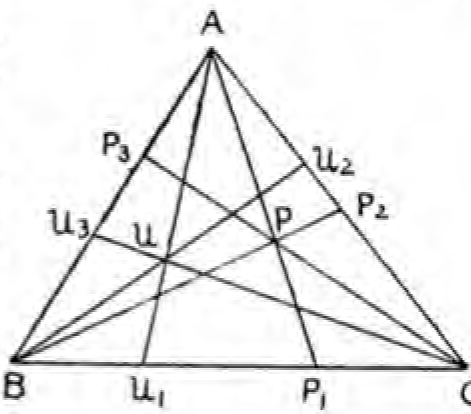
$$y = \frac{CP_1.BU_1}{CU_1.BP_1} = (CP_1BU_1),$$

i.e., the cross-ratios (CP_2AU_2) and (CP_1BU_1) define the abscissa and the ordinate of P with respect to U an arbitrary point which is taken to represent the unit point in the plane.

135. *Systems of point co-ordinates with reference to a triangle.*

If A and B in the last figure be not at ∞ and C may be any angle, we get the $\triangle$ ABC, which we may call as the *triangle of reference.*

Let U an arbitrary point in the plane of the triangle be taken as the unit point.



Draw AU, BU, CU meeting the sides of the triangle in U_1, U_2, U_3.

Let P be any variable point and let AP, BP, CP meet the sides of the triangle in P_1, P_2, P_3 respectively.

Suppose we define the co-ordinates of P as $x : y : z$.

Then $\dfrac{x}{z} = (CP_2\,AU_2)$, $\dfrac{y}{z} = (CP_1\,BU_1)$. ......(1)

If P be at U, the cross-ratios for $\dfrac{x}{z}$, $\dfrac{y}{z}$ become each equal to unity, and for the point U, we get

$$x : y : z = 1 : 1 : 1. \qquad ......(2)$$

as would otherwise be obvious, since U has been taken to be the unit point.

Let the position of U, the unit point, be supposed to be fixed. Then U_1, U_2, U_3 are also fixed, A, B, C being already fixed.

We know from Ceva's theorem that

$$\frac{BU_1.CU_2.AU_3}{U_1C.U_2A.U_3B} = 1.$$

Hence we can put

$$\frac{BU_1}{U_1C} = \frac{q}{r}, \quad \frac{CU_2}{U_2A} = \frac{r}{p}, \quad \frac{AU_3}{U_3B} = \frac{p}{q}. \qquad ......(3)$$

$p : q : r$ being thus known.

We now proceed to investigate $\frac{x}{z}$, $\frac{y}{z}$.

$$\frac{x}{z}=(CP_2AU_2)=\frac{CP_2.AU_2}{CU_2.AP_2}=\frac{CP_2}{P_2A}\cdot\frac{p}{r}$$
$$=\frac{p}{r}\cdot\frac{CP_2}{P_2A}\cdot\frac{a}{c}\cdot\frac{\sin C}{\sin A}$$

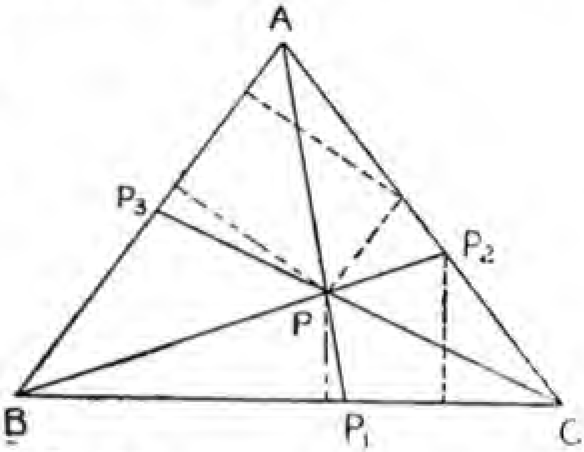
where a, b, c are the sides of the triangle ABC.

If α,β,γ be the perpendiculars from P on the sides BC, CA, AB, we have

$$CP_2 \sin C : \alpha$$
$$=BP_2:BP=P_2A\sin A:\gamma$$

$$\therefore \frac{CP_2\sin C}{P_2A\sin A}=\frac{\alpha}{\gamma}.$$

Hence $\frac{x}{z}=\frac{pa}{rc}\cdot\frac{\alpha}{\gamma}$



Thus $x:y:z=ap\alpha:bq\beta:cr\gamma$ ......(4)

where α,β,γ are subject to the condition

$$a\alpha+b\beta+c\gamma=2\Delta \qquad ......(5)$$

(4) is now the fundamental relation which is obtained by defining $\frac{x}{z}$, $\frac{y}{z}$ in terms of cross-ratios.

(x, y, z) are called the homogeneous co-ordinates, in general, of the point P.

We may choose our unit point from amongst the numerous points whose geometry in connection with the triangle of reference is known and get the various systems of co-ordinates. All that we have to know is $p:q:r$ in the fundamental relation (4).

Note.—(*a*) An *ordered triad* of numbers (x, y, z), where x, y, z are not all equal to zero, is called a *point.*

The triad (kx, ky, kz) where $k\neq 0$ represents the same point as (x, y, z).

(*b*) An *ordered triad* $[l, m, n]$, provided l, m, n are not all zero is called a *line.*

The triad $[kl, km, kn]$, $k\neq 0$ represents the same line as $[l, m, n]$.

A point (x, y, z) will be said to be on the line $[l, m, n]$ if and only if

$$lx+my+nz=0 \qquad \ldots\ldots(6)$$

With these definitions, the propositions of incidence can be easily verified.

It can be shown that the co-ordinates of the point P so determined are, but for a constant multiple, unique, *i.e.*, the same set of numbers (x, y, z) will be obtained by projecting P on any other pair of sides of triangle of reference. The student is referred to Veblen and Young Projective Geometry Vol. I.

135.1. There arise three important cases which we proceed to consider:—

(*a*) When the point U is the incentre of $\triangle$ ABC,

$$p : q : r = \frac{1}{a} : \frac{1}{b} : \frac{1}{c},$$

since the angles A, B, C are bisected

$$\therefore \quad x : y : z = k\alpha : k\beta : k\gamma$$
$$= \alpha : \beta : \gamma \quad \text{if we take } k=1.$$

$\therefore$ x, y, z are proportional to the perpendicular distances from the sides of the $\triangle$ of reference.

This system of co-ordinates is known as *the Trilinear System.*

If $x=\alpha$, $y=\beta$, $z=\gamma$, then (x, y, z) are called the actual Trilinear Co-ordinates of the point P.

(*b*) When the point U is the centroid of the $\triangle$ ABC, $p=q=r$, since the sides are bisected.

$$\therefore \quad x : y : z = a\alpha : b\beta : c\gamma$$
$$= 2 \triangle \text{PBC} : 2 \triangle \text{PCA} : 2 \triangle \text{PAB}$$

This system is known as *the Areal System.*

If the factor of proportionality be taken as $\dfrac{1}{2 \triangle \text{ABC}}$,

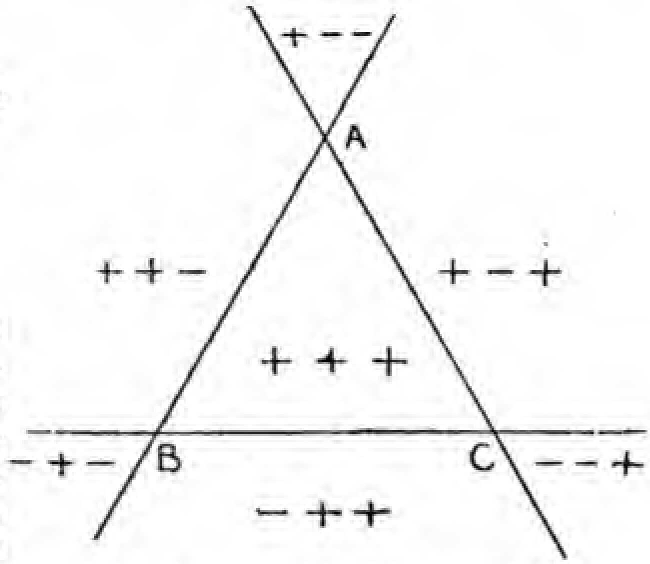
$$x = \frac{\triangle \text{PBC}}{\triangle \text{ABC}}, \; y = \frac{\triangle \text{PCA}}{\triangle \text{BCA}}, \; z = \frac{\triangle \text{PAB}}{\triangle \text{CAB}}$$

are the actual areal co-ordinates of the point P.

(*c*) When U is the orthocentre of $\triangle$ ABC,

$$\frac{q}{r} = \frac{c \cos B}{b \cos C} = \frac{\cot B}{\cot C}.$$

$\therefore p : q : r =$ $\cot A : \cot B : \cot C$, and the co-ordinates take the form $x : y : z = a\alpha \cot A : b\beta \cot B : c\gamma \cot C = \alpha \cos A : \beta \cos B : \gamma \cos C$.



In the annexed figure are shown the signs of the various co-ordinates areals or trilinears of a point in the different seven compartments in which the plane is divided by the sides of the triangle.

135.2. We now discuss the cases when P lies on the sides of the triangle of reference.

(*i*) Let P coincide with P_1. Then P_2 coincides with C,

and $\quad \dfrac{x}{z} = (CP_2AU_2) = 0.$

$\therefore$ at P_1, $x=0$.

Hence $x=0$ is the equation to the side BC of the triangle.

(*ii*) Let P coincide with P_2, then P_1 coincides with C,

and $\quad \dfrac{y}{z} = (CP_1\,BU_1) = 0.$

$\therefore$ at P_2, $\quad y=0$

i.e., the equation to the side CA is $y=0$.

(*iii*) Let P coincide with P_3, then P_1 and P_2 coincide with B and A respectively, and

$$\frac{x}{z} = (CP_2AU_2) = \infty, \quad \frac{y}{z} = (CP_1AU_1) = \infty.$$

$\therefore$ at P_3, $x = \infty = y$,

which we may express for the sake of uniformity as
$z=0$.

$\therefore$ $z=0$ is the equation to the side AB.

(*iv*) The co-ordinates of the vertices A, B, C are evidently

$$x=0=y, \; y=0=z, \; z=0=x$$

respectively.

135.3. Identical relation satisfied by the co-ordinates of a point.

(*a*) Let (x, y, z) be the actual trilinear co-ordinates of a point, *i.e.*, measures of the perpendiculars from the point on the sides of the triangle of reference, then it is obvious from a figure that

$$ax+by+cz=2\Delta \qquad \text{......(7)}$$

where Δ is the area of the triangle of reference.

(*b*) If x, y, z are the actual areal-co-ordinates, it will be found that

$$x+y+z=1 \qquad \text{.....(8)}$$

Such a relation could be expected as a point has two degrees of freedom and its position is fixed when its co-ordinates referred to two lines are known. Consequently the co-ordinates of a point referred to three or more lines must be connected by as many identical relations as the number of co-ordinates exceeds two.

Such systems are called *super abundant.*

If (ξ, η, ζ) be the proportional trilinear co-ordinates of a point whose actual trilinear co-ordinates are (α, β, γ) then

$$\frac{\xi}{\alpha}=\frac{\eta}{\beta}=\frac{\zeta}{\gamma}=\frac{a\xi+b\eta+c\zeta}{2\Delta}. \qquad \text{......(9)}$$

Similarly, if (ξ, η, ζ) be the proportional areal co-ordinates and (x, y, z) the actual areal co-ordinates,

$$\frac{\xi}{x}=\frac{\eta}{y}=\frac{\zeta}{z}=\frac{\xi+\eta+\zeta}{1}. \qquad \text{......(10)}$$

135.4. Relation between the Areal and Trilinear co-ordinates.

Let (x, y, z), (α, β, γ) be the Areal and Trilinear co-ordinates of a point respectively, then

$$\frac{x}{\Delta\,\mathrm{PBC}}=\frac{y}{\Delta\,\mathrm{PCA}}=\frac{z}{\Delta\,\mathrm{PAB}}$$

or

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma}. \qquad \text{......(11)}$$

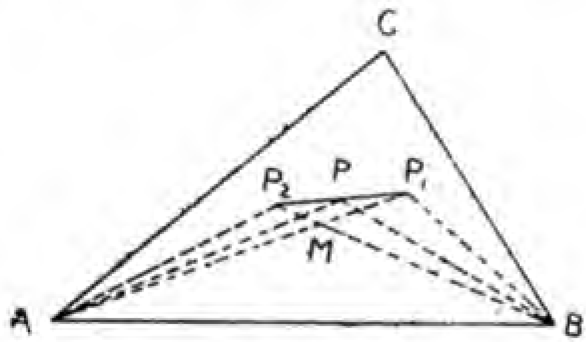
136. Metrical Theorems. Homogeneous systems of co-ordinates are, as a rule, unwieldy for metrical theorems. The most convenient system of co-ordinates to deal with metrical theorems is the Cartesian rectangular system. *In the following sections, actual co-ordinates of the points will be taken.*

136.1. Section Formula. Let $P_1(\alpha_1, \beta_1, \gamma_1)$, $P_2(\alpha_2, \beta_2, \gamma_2)$ be the trilinear co-ordinates of two points. Then α_1 and α_2 are the ordinates of P_1 and P_2 referred to BC as x-axis, hence if $P(\alpha, \beta, \gamma)$ divides P_1P_2 in the ratio $\lambda : \mu$, we have for the co-ordinates of P.

$$\alpha = \frac{\mu\alpha_1 + \lambda\alpha_2}{\lambda + \mu}.$$

Similarly $\beta = \dfrac{\mu\beta_1 + \lambda\beta_2}{\lambda + \mu}$, $\gamma = \dfrac{\mu\gamma_1 + \lambda\gamma_2}{\lambda + \mu}$.

136.2. Let (x_1, y_1, z_1), (x_2, y_2, z_2) be the areal co-ordinates of two points P_1, P_2. Let (x, y, z) be a point P which divides the line P_1P_2 in the ratio $\lambda : \mu$.



$$z_1 - z_2 = \frac{\Delta\, BP_1M - \Delta\, AMP_2}{\Delta\, ABC}$$

$$= \frac{\Delta\, BP_1P_2 - \Delta\, AP_1P_2}{\Delta\, ABC}.$$

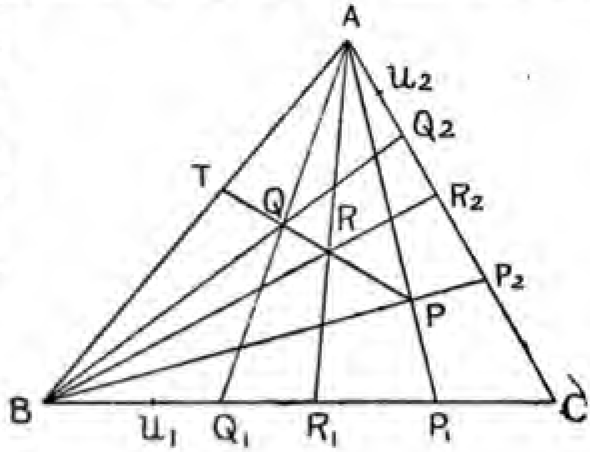
and $$z_1 - z = \frac{\Delta\, BP_1P - \Delta\, AP_1P}{\Delta\, ABC}.$$

But $\dfrac{\Delta\, BP_1P}{\Delta\, BP_1P_2} = \dfrac{\lambda}{\lambda + \mu}$, $\dfrac{\Delta\, AP_1P}{\Delta\, AP_1P_2} = \dfrac{\lambda}{\lambda + \mu}$.

Thus $\dfrac{z_1 - z}{z_1 - z_2} = \dfrac{\lambda}{\lambda + \mu}$. $\therefore$ $z = \dfrac{\mu z_1 + \lambda z_2}{\lambda + \mu}$.

Similarly $x = \dfrac{\mu x_1 + \lambda x_2}{\lambda + \mu}$, $y = \dfrac{\mu y_1 + \lambda y_2}{\lambda + \mu}$.

136.3. Let $x_p : y_p : z_p$ and $x_q : y_q : z_q$ be the homogeneous co-ordinates of the points P, Q. Let $x_r : y_r : z_r$ be the point R collinear with P, Q.



Let U be the unit point and AU, BU meet BC, CA at U_1, U_2.

We have the fundamental relations

$$\frac{x_p}{z_p} = (CP_2AU_2),$$

$$\frac{y_p}{z_p} = (CP_1BU_1)$$

and two similar relations for $\dfrac{x_q}{z_q}$ and $\dfrac{y_q}{z_q}$.

Now $(PQRT) = (P_1Q_1BR_1) = (P_2Q_2AR_2) = \lambda$ (say).

But the numbers attached to P_2, Q_2, R_2, A are $\dfrac{x_p}{z_p}$, $\dfrac{x_q}{z_q}$, $\dfrac{x_r}{z_r}$, ∞ respectively.

$$\therefore\ \lambda = \left(\frac{x_p}{z_p}\ \frac{x_q}{z_q}\ \infty\ \frac{x_r}{z_r}\right) = \frac{\dfrac{x_p}{z_p} - \dfrac{x_q}{z_q}}{\dfrac{x_p}{z_p} - \dfrac{x_r}{z_r}}$$

$$\therefore\ \frac{x_r}{z_r} = \left[\ (\lambda - 1)\ \frac{x_p}{z_p} + \frac{x_q}{z_q}\right] \div \lambda$$

whence putting $\lambda - 1 = \mu = \dfrac{z_p}{z_q}\, v$, we get

$$\frac{x_r}{z_r} = \frac{v\,x_p + x_q}{v\,z_p + z_q}.$$

$$\therefore\ x_r : y_r : z_r = vx_p + x_q : vy_p + y_q : vz_p + z_q.$$

136.4. Every equation of the first degree in homogeneous co-ordinates represents a st. line.

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be any two points on the locus

$$lx + my + nz = 0$$

and (x, y, z) any third point on the locus.

It follows that $\begin{vmatrix} x & y & z \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = 0.$

This is the condition that constants λ, μ, v (not all zero) can be found such that

$$\lambda x+\mu x_1+v x_2=0,\ \lambda y+\mu y_1+v y_2=0.\ \lambda z+\mu z_1+v z_2=0.$$

$\therefore$ the co-ordinates of the point (x, y, z) are of the form

$$k_1x_1+k_2x_2,\ k_1y_1+k_2y_2,\ k_1z_1+k_2z_2.$$

i. e., the point (x, y, z) lies on the st. line joining the points (x_1, y_1, z_1), (x_2, y_2, z_2).

Hence the locus is a st. line.

136.5. It follows that the condition that three points (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) should be collinear is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}=0.$$

It appears that, when the points are not collinear, the area of the triangle formed by them must contain this determinant as a factor.

Assuming that $\triangle\ PQR=k\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

where k is to be determined.

If P, Q, R be at the vertices of $\triangle$ ABC,

(Trilinears) $\triangle\ ABC=k\begin{vmatrix} \frac{2\triangle}{a} & 0 & 0 \\ 0 & \frac{2\triangle}{b} & 0 \\ 0 & 0 & \frac{2\triangle}{c} \end{vmatrix}$

$$\therefore\quad k=\frac{abc}{8\triangle^2}$$

$$\therefore\quad \triangle\ PQR=\frac{R}{2\triangle}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

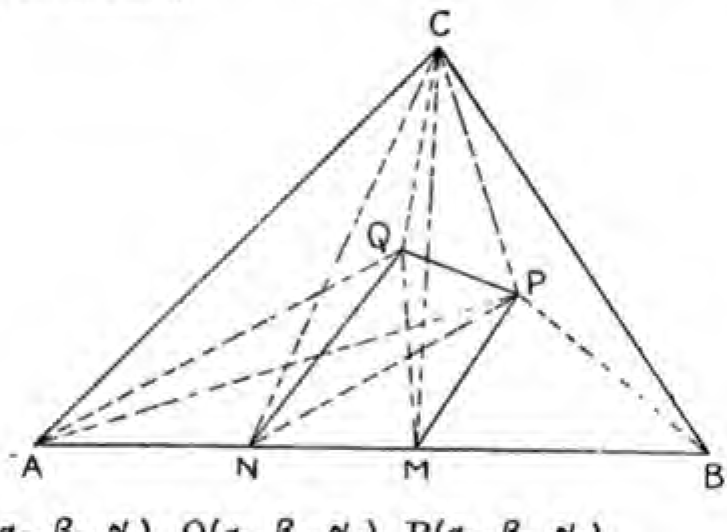
where R is the circum-radius of $\triangle$ ABC.

(Areals) $\triangle\ ABC = k \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = k$

$\therefore\ \triangle\ PQR = \triangle\ ABC \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$

This proof for the area of a triangle is incomplete.

136.6. To find the area of a triangle whose vertices are (in Trilinears).



$P(\alpha_1, \beta_1, \gamma_1),\ Q(\alpha_2, \beta_2, \gamma_2),\ R(\alpha_3, \beta_3, \gamma_3).$

Through P, Q draw parallels to AC meeting AB in M, N. Join AP, AQ, PN, QM.

$\because\ \triangle\ MPQ = \triangle\ MPN,$

$\therefore\ \triangle\ APQ = \text{quad. } AMPQ - \triangle\ AMP$

$= \triangle\ AMQ - \triangle\ ANP$

$= \frac{1}{2}(AM.\ \gamma_2 - AN.\ \gamma_1)$

$= \frac{1}{2}(\beta_1\gamma_2 - \beta_2\gamma_1) \text{ cosec A.}$

$\therefore\ \triangle\ PQR = \triangle\ APQ + \triangle\ AQR + \triangle\ ARP$

$= \frac{1}{2}(\beta_1\gamma_2 - \beta_2\gamma_1 + \beta_2\gamma_3 - \beta_3\gamma_2 + \beta_3\gamma_1 - \beta_1\gamma_3) \text{ cosec A}$

$= \frac{1}{2}\begin{vmatrix} 1 & \beta_1 & \gamma_1 \\ 1 & \beta_2 & \gamma_2 \\ 1 & \beta_3 & \gamma_3 \end{vmatrix} \text{cosec A} = \frac{R}{2\triangle}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \because a\alpha + b\beta + c\gamma = 2\triangle$

where R is the circum-radius of $\triangle$ ABC.

Alternative Method.

If $P_1P_3P_4$, $P_2P_3P_4$ be two triangles on the same base P_3P_4 and P_3P_4 meets P_1P_2 in O, then it is easy to show that

$$\frac{\triangle\, P_1P_3P_4}{\triangle\, P_2P_3P_4}=\frac{P_1O}{P_2O},\ \text{where } P_i \text{ is } (\alpha_i, \beta_i, \gamma_i).$$

Since O lies on the join of P_1 and P_2, we can take the co-ordinates of O in the form $(\alpha_1+\lambda\alpha_2, \beta_1+\lambda\beta_2, \gamma_1+\lambda\gamma_2)$.

But O, P_3, P_4 are collinear,

$$\therefore \begin{vmatrix} \alpha_1+\lambda\alpha_2 & \beta_1+\lambda\beta_2 & \gamma_1+\lambda\gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \end{vmatrix}=0 \qquad \text{where } \lambda=\frac{P_1O}{P_2O}.$$

$$\therefore \quad \frac{\triangle\, P_1P_3P_4}{\triangle\, P_2P_3P_4}=\lambda=-\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \end{vmatrix} \div \begin{vmatrix} \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \\ \alpha_4 & \beta_4 & \gamma_4 \end{vmatrix} \ldots\ldots(1)$$

Now we can find the area of a $\triangle$ PQR where P, Q, R are $(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$, $(\alpha_3, \beta_3, \gamma_3)$.

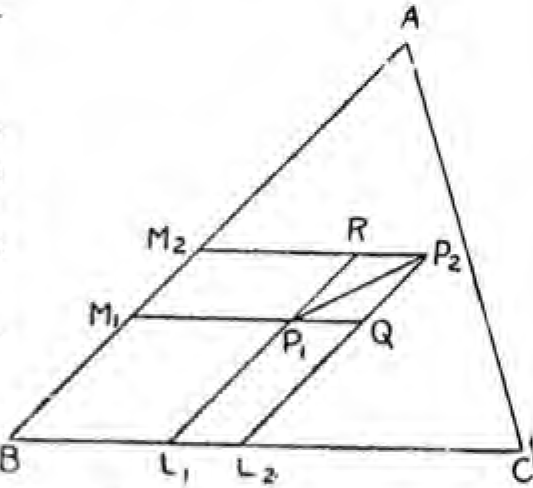
$$\triangle PQR=\frac{\triangle\, PQR}{\triangle\, AQR}\cdot\frac{\triangle\, AQR}{\triangle\, ABR}\cdot\frac{\triangle\, ABR}{\triangle\, ABC}\cdot\ \triangle\, ABC$$

$$=\triangle\, ABC \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix} \div \begin{vmatrix} \frac{2\triangle}{a} & 0 & 0 \\ 0 & \frac{2\triangle}{b} & 0 \\ 0 & 0 & \frac{2\triangle}{c} \end{vmatrix} \quad \text{from (1)}$$

$$=\frac{R}{2\triangle}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

Cor. Every equation of the 1st. degree in α, β, γ represents a st. line.

136.7. Distance between two points.



Let P_1, P_2 be two points whose actual trilinear coordinates are

$(\alpha_1, \beta_1, \gamma_1)$, $(\alpha_2, \beta_2, \gamma_2)$.

Through P_1, P_2 draw parallels to BC, AB, then $P_1L_1 = \alpha_1$ cosec B, $P_1M_1 = \gamma_1$ cosec B, $P_2L_2 = \alpha_2$ cosec B, $P_2M_2 = \gamma_2$ cosec B.

$\therefore QP_2 = (\alpha_2 - \alpha_1)$cosec B, $P_1Q = (\gamma_2 - \gamma_1)$ cosec B.

$\therefore P_1P_2^2 = P_1Q^2 + QP_2^2 - 2P_1Q.QP_2 \cos P_1QP_2$

$$=[(\alpha_2 - \alpha_1)^2 + (\gamma_2 - \gamma_1)^2 + 2(\alpha_2 - \alpha_1)(\gamma_2 - \gamma_1)\cos B] \operatorname{cosec}^2 B \qquad \ldots\ldots(1)$$

Put $\alpha_2 - \alpha_1 = \lambda$, $\beta_2 - \beta_1 = \mu$, $\gamma_2 - \gamma_1 = v$,

hence $a\lambda + b\mu + cv = 0$.

$$\therefore \quad a^2\lambda^2 + c^2v^2 + 2ac\lambda v = b^2\mu^2.$$

Substituting in (1) we get

$$P_1P_2^2 = \left[\lambda^2 + v^2 + \frac{b^2\mu^2 - a^2\lambda^2 - c^2v^2}{ac}\cos B\right]\operatorname{cosec}^2 B$$

$$= \left[a\lambda^2 \cos A + b\mu^2 \cos B + c\,v^2 \cos C\right]\frac{b}{ac \sin^2 B}$$

$$= \frac{abc}{4\Delta^2}\Big[a \cos A\,(\alpha_1 - \alpha_2)^2 + b \cos B(\beta_1 - \beta_2)^2 + c \cos C\,(\gamma_1 - \gamma_2)^2\Big] \qquad \ldots\ldots(i)$$

Again, $a\lambda^2 = -b\lambda\mu - c\lambda v$, $cv^2 = -a\lambda v - b\mu v$.

Substituting in (1) we get

$$P_1P_2^2 = -\operatorname{cosec}^2 B\left[\frac{b\lambda\mu + c\lambda v}{a} + \frac{a\lambda v + b\mu v}{c} - 2\lambda v \cos B\right]$$

$$= -\frac{b}{ac \sin^2 B}\left[a\mu v + bv\lambda + c\lambda\mu\right]$$

$$= -\frac{abc}{4\Delta^2}\Big[a(\beta_1 - \beta_2)(\gamma_1 - \gamma_2) + b(\gamma_1 - \gamma_2)(\alpha_1 - \alpha_2) + c(\alpha_1 - \alpha_2)(\beta_1 - \beta_2)\Big] \qquad \ldots\ldots(ii)$$

The expression for the distance $P_1 P_2$ may be written as

$$P_1P_2^2=\frac{R^2}{\Delta^2}(L^2+M^2+N^2-2MN\cos A-2NL\cos B-2LM\cos C) \quad \ldots\ldots(iii)$$

where $L=\beta_1\gamma_2-\beta_2\gamma_1$, $M=\gamma_1\alpha_2-\gamma_2\alpha_1$, $N=\alpha_1\beta_2-\alpha_2\beta_1$.

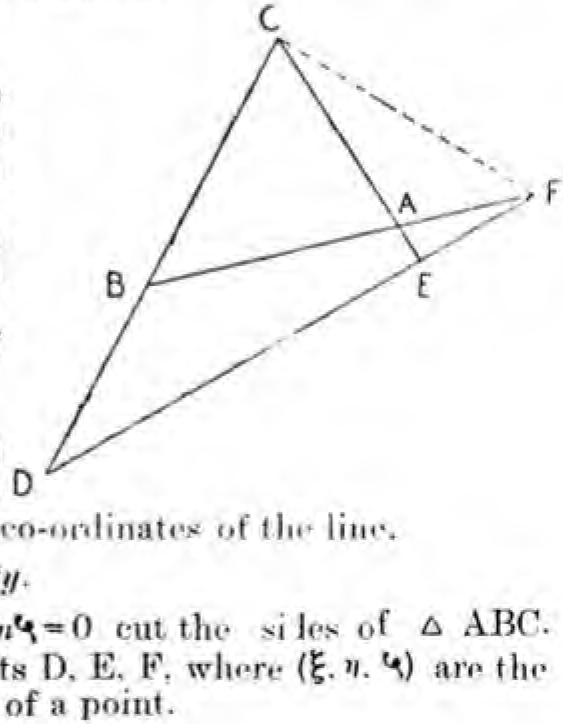
Cor. If $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ be any two points on a given line $l\alpha+m\beta+n\gamma=0$, and p the perpendicular from a given point O (f, g, h) on the line, then

$$\frac{1}{2}p.\,PQ=\triangle\, OPQ=\frac{R}{2\Delta}\begin{vmatrix} f & g & h \\ \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \end{vmatrix}$$

$$\therefore\ p=\frac{lf+mg+nh}{\sqrt{(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)}}$$

from (iii)

137. The equation of any st. line is of the form
$l\alpha+m\beta+n\gamma=0$.



Suppose DF is any line. Join CF.

Now since CF passes through the intersection of $\alpha=0$, $\beta=0$, its equation is of the form $l\alpha+m\beta=0$.

Since FD passes through the intersection of $\gamma=0$, $l\alpha+m\beta=0$,

$\therefore$ its equation is of the form

$l\alpha+m\beta+n\gamma=0$.

The numbers (l, m, n) which define the position of a st. line are called the co-ordinates of the line.

137.1. *Line at infinity*.

Let a st. line $l\xi+m\eta+n\zeta=0$ cut the sides of $\triangle$ ABC, each externally at the points D, E, F, where (ξ, η, ζ) are the homogeneous co-ordinates of a point.

It is easy to show that

$$\frac{BD}{CD}=\frac{mc}{nb},\quad \frac{CE}{AE}=\frac{na}{lc},\quad \frac{AF}{BF}=\frac{lb}{ma} \qquad \text{(Trilinears)}$$

$$\frac{BD}{CD}=\frac{m}{n},\quad \frac{CE}{AE}=\frac{n}{l},\quad \frac{AF}{BF}=\frac{l}{m} \qquad \text{(Areals).}$$

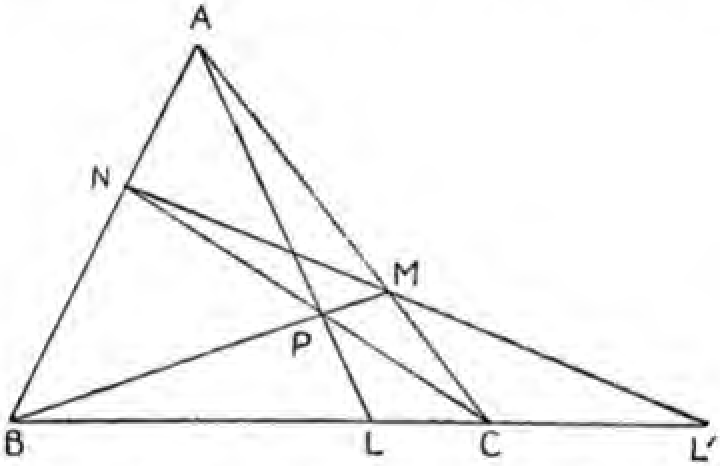
Now as the line recedes in any direction, each of the ratios tends to unity. Thus its equation tends to the limiting form :

$$(\text{Tr.})\ a\alpha + b\beta + c\gamma = 0, \quad (\text{Ar.})\ x + y + z = 0.$$

Alternative Method.

Take any point $P(\xi', \eta', \zeta')$ in the plane of the triangle of reference ABC. Let AP, BP, CP meet the sides BC, CA, AB at L, M, N ; and let MN, NL, LM meet BC, CA, AB at L′, M′, N′ respectively.

It follows from the theorems of Ceva and Menelaus that



(*i*) L, L′ are harmonic conjugates with respect to B, C; M, M′ with respect to C, A ; N, N′ with respect to A, B.

(*ii*) L′, M′, N′ lie on a st. line which is known as the polar of P *w. r.* to $\triangle$ ABC. The point P is called the pole of this line.

If the point P be taken at the centroid of $\triangle$ ABC, L, M, N will be the middle points of BC, CA, AB respectively.

$\therefore$ L′, M′, N′ the harmonic conjugates of L, M, N will be situated at an infinite distance.

Hence the polar of G *w. r.* to $\triangle$ ABC is the line at ∞.

To find the polar of a point $P(\xi', \eta', \zeta')$ *w. r.* to $\triangle$ ABC.

The equation to AP is $\dfrac{\eta}{\eta'} = \dfrac{\zeta}{\zeta'}$.

$\therefore$ the equation to AL′ where L′ is the harmonic conjugate of L *w. r.* to B and C, is $\dfrac{\eta}{\eta'} = -\dfrac{\zeta}{\zeta'}$.

The line L′M′N′ passes through L′ the intersection of the lines

$$\xi=0, \quad \frac{\eta}{\eta'}+\frac{\zeta}{\zeta'}=0.$$

Its equation is, therefore, of the form

$$\lambda\xi+\frac{\eta}{\eta'}+\frac{\zeta}{\zeta'}=0 \qquad \text{......(1)}$$

Now M′ is the intersection of the lines

$$\eta=0, \frac{\xi}{\xi'}+\frac{\zeta}{\zeta'}=0.$$

Since M′ lies on (1), $\therefore \quad \lambda=\frac{1}{\xi'}$.

Hence the polar of P is given by the equation

$$\frac{\xi}{\xi'}+\frac{\eta}{\eta'}+\frac{\zeta}{\zeta'}=0.$$

If $(\alpha', \beta', \gamma')$ be the Trilinear co-ordinates of P, we know that

$$\xi' : \eta' : \zeta'=ap\alpha' : bq\beta' : cr\gamma'.$$

If P be at G the centroid of $\triangle$ ABC, the equation to the line at ∞ is

$$\frac{\xi}{p}+\frac{\eta}{q}+\frac{\zeta}{r}=0.$$

When the centroid is the unit point, $p=q=r$, and the line at ∞ becomes (in the Areal System)

$$x+y+z=0.$$

When the unit point is at the incentre,

$$p : q : r=\frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

Hence the line at ∞ becomes (in the Trilinear System),

$$a\alpha+b\beta+c\gamma=0.$$

When the unit point is the orthocentre, we have

$$p : q : r=\cot A : \cot B : \cot C,$$

and the line at ∞ takes the form

$$\xi \tan A+\eta \tan B+\zeta \tan C=0.$$

137.2. Condition for parallelism.

The two lines

$$l_1\alpha+m_1\beta+n_1\gamma=0,$$
$$l_2\alpha+m_2\beta+n_2\gamma=0$$

will be parallel, if they concur with the line at ∞

$$a\alpha+b\beta+c\gamma=0.$$

The condition that these lines may be concurrent is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ a & b & c \end{vmatrix} = 0.$$

137.3. To find the equation of the st. line through $(\alpha', \beta', \gamma')$ parallel to the line $u \equiv l\alpha + m\beta + n\gamma = 0$.

A line parallel to u will pass through the intersection of u and the line at ∞.

$\therefore$ any line $\parallel$ u is of the form

$$l\alpha + m\beta + n\gamma - \lambda(a\alpha + b\beta + c\gamma) = 0$$

If $(\alpha', \beta', \gamma')$ lies on it, we have

$$l\alpha' + m\beta' + n\gamma' - \lambda(a\alpha' + b\beta' + c\gamma') = 0.$$

$\therefore$ Eliminating λ, we get

$$\frac{\Sigma\, l\alpha}{\Sigma\, l\alpha'} = \frac{\Sigma\, a\alpha}{\Sigma\, a\alpha'},$$

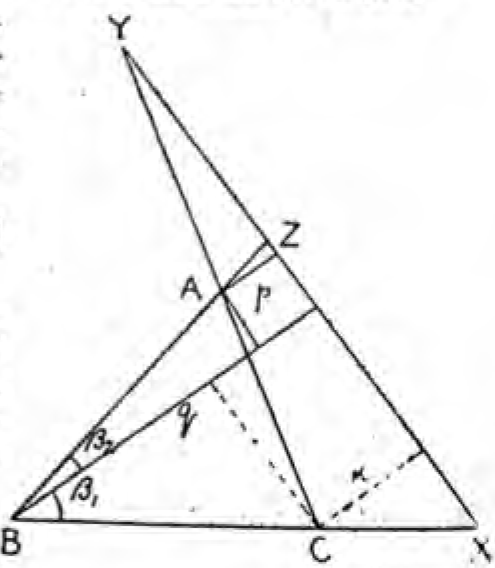
as the equation of the line required.

138. Metric Properties. Metric line-co-ordinates.

Let $[l, m, n]$ be the co-ordinates of a line and p, q, r the perpendiculars from the vertices A, B, C of the triangle of reference. The relations between

$$[l, m, n], [p, q, r]$$

will now be found.



Let the line $[l, m, n]$ meet the sides of the triangle of reference in points X, Y, Z. Suppose that the actual trilinear co-ordinates of X are $(0, \beta, \gamma)$, so that

$$m\beta + n\gamma = 0 \quad b\beta + c\gamma = 2\Delta$$

$$\therefore \quad \frac{m}{n} = -\frac{\gamma}{\beta} = -\frac{b}{c} \cdot \frac{c\gamma}{b\beta} = +\frac{b}{c} \cdot \frac{\Delta\ \text{BXA}}{\Delta\ \text{CXA}}$$

$$= +\frac{b}{c} \cdot \frac{\text{BX}}{\text{CX}} = \frac{bq}{cr},$$

$$\therefore \quad \frac{m}{bq} = \frac{n}{cr}.$$

Similarly $\dfrac{l}{ap}=\dfrac{m}{bq}=\dfrac{n}{cr}$. ......(*i*)

(*b*) If the system of co-ordinates be areal, we have the relation $\dfrac{l}{p}=\dfrac{m}{q}=\dfrac{n}{r}$. ......(*ii*)

138.1. Identical relation in line-co-ordinates.

Let p, q, r be the perpendiculars from the vertices of the triangle on a given line, and suppose that B_1, B_2 are the angles which the perpendicular q from B makes with BC, BA, then obviously

$$\frac{q-r}{a}=\cos B_1, \frac{q-p}{c}=\cos B_2. \quad ......(iii)$$

Now $\cos B=\cos(B_1+B_2)=\cos B_1 \cos B_2-\sin B_1 \sin B_2$

or $(\cos B-\cos B_1 \cos B_2)^2=\sin^2 B_1 \sin^2 B_2$
$$=(1-\cos^2 B_1)(1-\cos^2 B_2)$$

$\therefore \cos^2 B_1+\cos^2 B_2-2\cos B \cos B_1 \cos B_2=\sin^2 B.$

Substituting from (*iii*), we have

$$c^2(q-r)^2+a^2(p-q)^2+2ac(q-r)(p-q)\cos B=a^2c^2\sin^2 B$$
$$=4\Delta^2 ...(iv)$$

where Δ is the area of the triangle of reference.

Removing the brackets, the result when simplified takes the form

$$a^2p^2+b^2q^2+c^2r^2-2bc\cos A.qr-2ca\cos B.rp-2ab\cos C.pq$$
$$=4\Delta^2 ......(v)$$

Put $\lambda=q-r, \mu=r-p, v=p-q$, then $\lambda+\mu+v=0$.

$\therefore \quad \lambda+v=-\mu \quad$ or $2\lambda v=\mu^2-\lambda^2-v^2$.

Substituting in (*iv*)

$$4\Delta^2=c^2\lambda^2+a^2v^2+ac(\mu^2-\lambda^2-v^2)\cos B$$

or $\quad bc\cos A(q-r)^2+ca\cos B(r-p)^2+ab\cos C(p-q)^2$
$$=4\Delta^2 ......(vi)$$

Again $\lambda^2=-\lambda(\mu+v)=-\lambda\mu-\lambda v$, $v^2=-\lambda v-\mu v$.

From (*iv*), $\quad -c^2(\lambda\mu+\lambda v)-a^2(\lambda v+\mu v)+2ac\lambda v\cos B=4\Delta^2$

or $\quad a^2(r-p)(p-q)+b^2(q-r)(p-q)+c^2(q-r)(r-p)$
$$=-4\Delta^2 ......(vii)$$

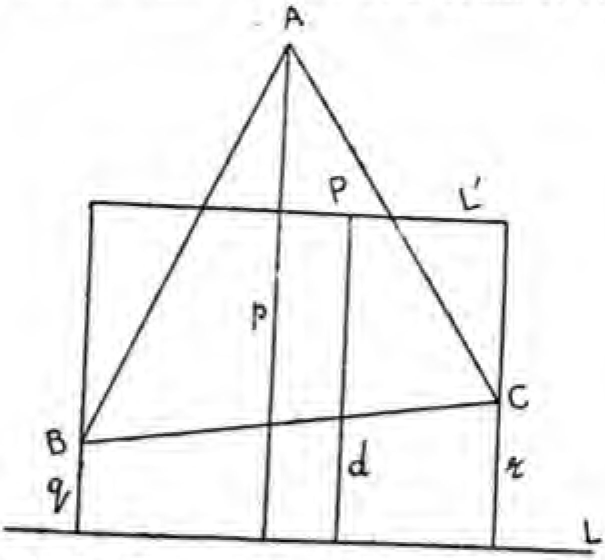
Ex. Prove that $\Sigma(q-r)^2\cot A=2\Delta$.

138.2. Distance of a point from a line.

Trilinear System. Let $(\alpha', \beta', \gamma')$ be trilinear co-ordinates of a point P and $l\alpha+m\beta+n\gamma=0$ be the equation of a given line L, then with the usual notation

$$\frac{l}{ap}=\frac{m}{bq}=\frac{n}{cr}=\frac{[\Sigma l^2-2\Sigma mn\cos A]^{\frac{1}{2}}}{2\Delta}.$$

Let d denote the distance of P from L, then perpendi-



culars from A, B, C on L' the line through P parallel to L are
$$d-p,\ d-q,\ d-r.$$
Hence the equation of L' is
$$a(d-p)\alpha+b(d-q)\beta+c(d-r)\gamma=0.$$
This passes through $(\alpha', \beta', \gamma')$,

$$\therefore \quad d=\frac{ap\alpha'+bq\beta'+cr\gamma'}{a\alpha'+b\beta'+c\gamma'}$$

$$=\frac{2\Delta}{\mathrm{K}}\cdot\frac{l\alpha'+m\beta'+n\gamma'}{a\alpha'+b\beta'+c\gamma'} \qquad \text{......}(viii)$$

where $\mathrm{K}^2=l^2+m^2+n^2-2mn\cos \mathrm{A}-2nl\cos \mathrm{B}-2lm\cos \mathrm{C}$.

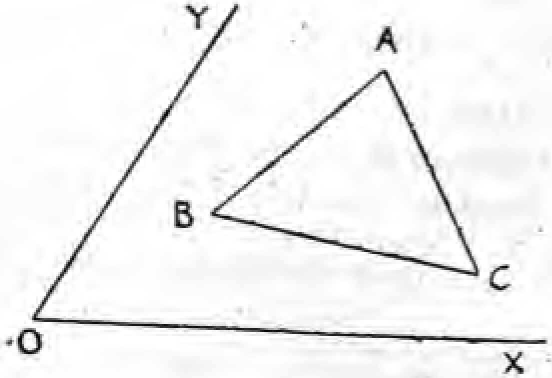
The formula in this form holds even if $(\alpha', \beta', \gamma')$ are proportional trilinear co-ordinates.

In Areals, the corresponding formula is

$$d=\frac{lx'+my'+nz'}{\sqrt{(\Sigma a^2l^2-2\Sigma bcmn\cos \mathrm{A})}}. \qquad \text{......}(ix)$$

138.3. Angle between two lines.

(a) Trilinear System.



Let $\quad l_1\alpha+m_1\beta+n_1\gamma=0,$
$\quad l_2\alpha+m_2\beta+n_2\gamma=0$

be the equations of two lines OX, OY and θ the angle between them, then with the usual notation, we have

$$\frac{l_1}{ap_1}=\frac{m_1}{bq_1}=\frac{n_1}{cr_1}=\frac{[\Sigma l_1^2-2\Sigma m_1n_1\cos A]^{\frac{1}{2}}}{2\Delta},$$

$$\frac{l_2}{ap_2}=\frac{m_2}{bq_2}=\frac{n_2}{cr_2}=\frac{[\Sigma l_2^2-2\Sigma m_2n_2\cos A]^{\frac{1}{2}}}{2\Delta}.$$

Regarding OX, OY as axes of co-ordinates, the co-ordinates of A, B, C can be taken as
$(p_1\operatorname{cosec}\theta, p_2\operatorname{cosec}\theta)$, $(q_1\operatorname{cosec}\theta, q_2\operatorname{cosec}\theta)$,
$(r_1\operatorname{cosec}\theta, r_2\operatorname{cosec}\theta)$.

The area of the triangle ABC is

$$\tfrac{1}{2}\sin\theta\begin{vmatrix} p_1 & p_2 & 1\\ q_1 & q_2 & 1\\ r_1 & r_2 & 1\end{vmatrix}\operatorname{cosec}^2\theta=\Delta$$

$$\therefore\quad \sin\theta=\frac{(q_1r_2-q_2r_1)+(r_1p_2-r_2p_1)+(p_1q_2-p_2q_1)}{2\Delta}$$

$$=2\Delta\frac{a(m_1n_2-m_2n_1)+b(n_1l_2-n_2l_1)+c(l_1m_2-l_2m_1)}{abc\ K_1^{\frac{1}{2}}\ K_2^{\frac{1}{2}}}$$

where $K_1\equiv\Sigma l_1^2-2\Sigma m_1n_1\cos A$,
and $\quad K_2\equiv\Sigma l_2^2-2\Sigma m_2n_2\cos A$.

$$\text{or }\sin\theta=\frac{(m_1n_2-m_2n_1)\sin A+(n_1l_2-n_2l_2)\sin B+(l_1m_2-l_2m_1)\sin C}{K_1^{\frac{1}{2}}\ K_2^{\frac{1}{2}}}\qquad\ldots\ldots(x)$$

Otherwise :—

The angle between any two st. lines is the same as the angle between the lines parallel to them.

ABC is the triangle of reference. Draw
BP $\parallel$ $l_1\alpha+m_1\beta+n_1\gamma=0$ and CP $\parallel$ $l_2\alpha+m_2\beta+n_2\gamma=0$, then
BP$\equiv(bl_1-am_1)\alpha+(bn_1-cm_1)\gamma=0$,
CP$\equiv(cl_2-an_2)\alpha+(cm_2-bn_2)\beta=0$.
P the point of intersection is

$$\frac{2\Delta}{bc\lambda}(cm_1-bn_1)(cm_2-bn_2),\ \frac{2\Delta}{bc\lambda}(bn_1-cm_1)(cl_2-an_2),$$
$$\frac{2\Delta}{bc\lambda}(bl_1-am_1)(cm_2-bn_2),$$

where $\lambda=a(m_1n_2-m_2n_1)+b(n_1l_2-n_2l_1)+c(l_1m_2-l_2m_1)$.

Now $PB^2 = \frac{a^2}{\lambda^2}(cm_2 - bn_2)^2 K_1$

and $PC^2 = \frac{a^2}{\lambda^2}(cm_1 - bn_1)^2 K_2$.

If θ be the angle between the lines,

$$PB.PC \sin\theta = a\alpha.$$

$$\therefore \quad \sin\theta = \frac{2\Delta}{abc}\,\frac{\lambda}{\sqrt{K_1K_2}}$$

where $K_r = l_r^2 + m_r^2 + n_r^2 - 2m_rn_r \cos A - 2n_rl_r \cos B - 2l_rm_r \cos C$.

(b) We proceed to determine $\cos\theta$.

Lemma : *If $\Sigma ax^2 + 2\Sigma fyz = 0$ be the equation of a conic whose tangential equation is $\Sigma Al^2 + 2\Sigma Fmn = 0$ (with the usual notation) and $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ two points in the plane, then*

$$(\Sigma ax_1^2 + 2\Sigma fy_1z_1)(\Sigma ax_2^2 + 2\Sigma fy_2z_2) - [\Sigma ax_1x_2 + \Sigma f(y_1z_2 + y_2z_1)]^2$$
$$= [\Sigma A(y_1z_2 - y_2z_1)^2 + 2\Sigma F(z_1x_2 - z_2x_1)(x_1y_2 - x_2y_1)].$$

Proof. The left hand side equated to zero expresses the condition that the point Q may lie on either of the tangents from P *i.e.* the condition that PQ may be a tangent to the conic and the right-hand side expresses the same condition. The comparison of the co-efficients on both sides proves this identity.

$$\cos^2\theta = \frac{K_1K_2 - [\Sigma(m_1n_2 - m_2n_1)\sin A]^2}{K_1K_2}$$

$$= \frac{[\Sigma l_1l_2 - \Sigma(m_1n_2 + m_2n_1)\cos A]^2}{K_1K_2}$$

$$\therefore \quad \cos\theta = \frac{\Sigma l_1l_2 - \Sigma(m_1n_2 + m_2n_1)\cos A}{\sqrt{K_1K_2}} \qquad \ldots\ldots(xi)$$

I. **Parallelism.** Two lines (l_1, m_1, n_1), (l_2, m_2, n_2) are parallel, if $\lambda = 0$ *i.e.* if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ a & b & c \end{vmatrix} = 0.$$

II. **Perpendicularity.** Two lines (l_1, m_1, n_1), (l_2, m_2, n_2) are perpendicular if $\cos\theta = 0$ or $\sin\theta = 1$

i.e. $K_1K_2 = \frac{4\Delta^2}{a^2b^2c^2}\lambda^2$

viz. $l_1 l_2 + m_1 m_2 + n_1 n_2 - (m_1 n_2 + m_2 n_1)\cos A - (n_1 l_2 + n_2 l_1)\cos B - (l_1 m_2 + l_2 m_1)\cos C = 0.$

III. The equation $u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$ represents a pair of perpendicular st. lines if

$$uvw + 2u'v'w' - uu'^2 - vv'^2 - ww'^2 = 0$$

and $$u + v + w - 2u'\cos A - 2v'\cos B - 2w'\cos C = 0.$$

IV. The line $l\alpha + m\beta + n\gamma = 0$ meets $\alpha = 0$ in P, where P is given by $\dfrac{\alpha_1}{o} = \dfrac{\beta_1}{n} = \dfrac{\gamma_1}{-m} = \dfrac{2\Delta}{bn - cm}$.

The line through (f, g, h) parallel to the line (l, m, n) meets $\alpha = 0$ in Q. $\dfrac{\alpha_2}{o} = \dfrac{\beta_2}{(an - cl)f + (bn - cm)g}$

$$= \frac{\gamma_2}{(bl - cm)f + (bn - cm)h}$$

$$= \frac{2\Delta}{(bn - cm)(af + bg + ch)}.$$

Now $PQ^2 = -\dfrac{abc}{4\Delta^2}\, a(\beta_1 - \beta_2)(\gamma_1 - \gamma_2)$.

If θ be the angle between the lines $\alpha = 0$ and (l, m, n),

$$\sin\theta = \frac{2\Delta}{abc}\,\frac{bn - cm}{\sqrt{K}}.$$

Hence the perpendicular from a given point (f, g, h) on the line $(l, m, n) = PQ \sin\theta$

$$= \frac{lf + mg + nh}{\sqrt{(l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2\,lm\cos C)}}.$$

V. The line passing through $Q(\alpha_2, \beta_2, \gamma_2)$, $R(\alpha_3, \beta_3, \gamma_3)$ is

$$L\alpha + M\beta + N\gamma = 0,$$

where $L = \beta_2\gamma_3 - \beta_3\gamma_2$, $M = \alpha_3\gamma_2 - \alpha_2\gamma_3$, $N = \alpha_2\beta_3 - \alpha_3\beta_2$.

Perpendicular from $P(\alpha_1, \beta_1, \gamma_1)$ on QR

$$= \frac{L\alpha_1 + M\beta_1 + N\gamma_1}{\sqrt{(L^2 + M^2 + N^2 - 2MN\cos A - 2NL\cos B - 2LM\cos C)}}.$$

Hence the area of the $\triangle$ $PQR = \frac{1}{2}\,p.\,QR$

$$= \frac{R}{2\Delta}(L\alpha_1 + M\beta_1 + N\gamma_1)$$

$$= \frac{R}{2\Delta}\begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Conversely.

$$QR^2=\frac{R^2}{\Delta^2}(L^2+M^2+N^2-2MN\cos A-2NL\cos B-2LM\cos C).$$

139. Circular points at infinity.

The equation

$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0$ breaks up into two linear factors. It does, therefore, represent two points. The corresponding point equation, *i.e.*, the envelope of the line $lx+my+nz=0$ which moves subject to the above condition is found to be the line at infinity taken twice. Thus the points represented by the equation lie at infinity.

The result may be obtained otherwise thus :—

If $[l, m, n]$ be the co-ordinates of a line and p, q, r the measures of the perpendiculars from A, B, C on the line, then

$$\frac{ap}{l}=\frac{bq}{m}=\frac{cr}{n}=k \text{ say.}$$

From Art. 138.1 (v) then

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=\frac{4\Delta^2}{k^2}.$$

If $k\to\infty$, p, q, r also tend to infinity, thus

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0 \quad \text{......}(i)$$

is the envelope of the line at infinity which is seen to reduce to two points. The coordinates $[a, b, c]$ of the line at infinity are easily seen to satisfy this equation.

The points are called the *circular points at infinity.*

(*b*) The corresponding equation of the circular points at infinity in *areals* is

$$a^2l^2+b^2m^2+c^2n^2-2mnbc\cos A-2nlac\cos B-2lmab\cos C=0 \quad \text{......}(ii)$$

If I (x_0, y_0, z_0) and J(x_0', y_0', z_0') be the co-ordinates of the circular points at infinity,

$$\frac{x_0x_0'}{1}=\frac{y_0y_0'}{1}=\frac{z_0z_0'}{1}=\frac{y_0z_0'+y_0'z_0}{-2\cos A}=\frac{z_0x_0'+z_0'x_0}{-2\cos B}$$
$$=\frac{x_0y_0'+x_0'y_0}{-2\cos C} \text{ (in trilinears)} \quad \text{...}(iii)$$

$$\frac{x_0x_0'}{a^2}=\frac{y_0y_0'}{b^2}=\frac{z_0z_0'}{c^2}=\frac{y_0z_0'+y_0'z_0}{-2bc\cos A}=\frac{z_0x_0'+z_0'x_0}{-2ac\cos B}$$
$$=\frac{x_0y_0'+x_0'y_0}{-2ab\cos C} \text{ (in areals)} \quad \text{......}(iv)$$

139.1. Co-ordinates of the Circular points at Infinity.

Writing the equation (i) as a quadratic in l, we have

$$l^2-2l\,(n\cos B+m\cos C)+(m^2+n^2-2lm\cos C)=0.$$

$$\therefore\ l=(n\cos B+m\cos C)^2\pm\sqrt{(n\cos B+m\cos C)^2-(m^2+n^2-2mn\cos A)}$$

$$=(n\cos B+m\cos C)\pm i\,(n\sin B-m\sin C)$$

or $\quad(-l+m\operatorname{cis}-C+n\operatorname{cis}B)\,(-l+m\operatorname{cis}C+n\operatorname{cis}-B)=0.$

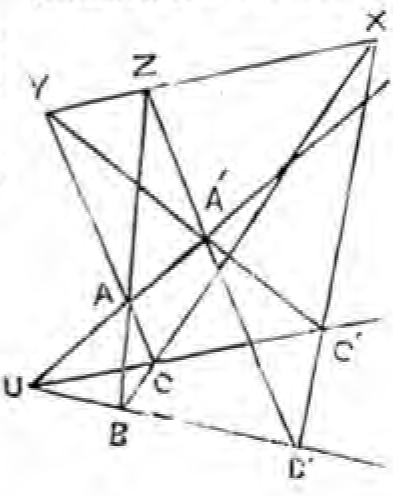
Thus the co-ordinates of the circular points I, J at infinity are

$$I\left(-1,\ e^{iC},\ e^{-iB}\right),\ J\left(-1,\ e^{-iC},\ e^{iB}\right) \qquad \text{......}(v)$$

The coordinates involve the elements of the triangle of reference only, therefore for a fixed triangle of reference the points are fixed.

Illustrative Examples.

(1) *If* ABC, A′B′C′ *are two triangles such that* AA′, BB′, CC′ *are concurrent, show that* BC *and* B′C′, CA *and* C′A′, AB *and* A′B′ *meet on a line, and conversely.* (Desargue's Theorem)



Let ABC be the triangle of reference and U the unit point. Since U is (1,1,1) and A (1, 0, 0), the co-ordinates of A′ are of the form, $(1, 1, 1+\lambda)$. Similarly, the co-ordinates of B′, C′ are of the form

$B'(1, 1+\mu, 1)$, $C'(1, 1, 1+v)$.

The equation of B′C′ is

$$\begin{vmatrix} x & y & z \\ 1 & 1+\mu & 1 \\ 1 & 1 & 1+v \end{vmatrix}=0$$

or $\quad x(\mu+v+\mu v)-yv-z\mu=0.$

Similarly the equations of C′A′, A′B′ are

$$-xv+y(\lambda+v+\lambda v)-z\lambda=0,$$
$$-\mu x-\lambda y+z(\mu+\lambda+\mu\lambda)=0.$$

The lines B′C′, C′A′, A′B′ meet the lines BC($x=0$), CA($y=0$), AB($z=0$) in points X$(0, \mu, -v)$ Y$(-\lambda, 0, v)$, Z$(\lambda, -\mu, 0)$

and all these points lie on the line

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{v}=0.$$

Otherwise. The equations of B′C′, C′A′, A′B′ can be written as

$$x\left(\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{v}+1\right)-\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{v}\right)=0$$

$$y\left(\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{v}+1\right)-\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{v}\right)=0$$

$$z\left(\frac{1}{\lambda}+\frac{1}{\mu}+\frac{1}{v}+1\right)-\left(\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{v}\right)=0$$

which shows that B′C′, BC ; C′A′, CA ; A′B′, AB meet on the line

$$\frac{x}{\lambda}+\frac{y}{\mu}+\frac{z}{v}=0.$$

To prove the converse, take ABC as the triangle of reference, and XYZ the unit line $x+y+z=0$. The equations of the sides of $\triangle$ A′B′C′ are of the form

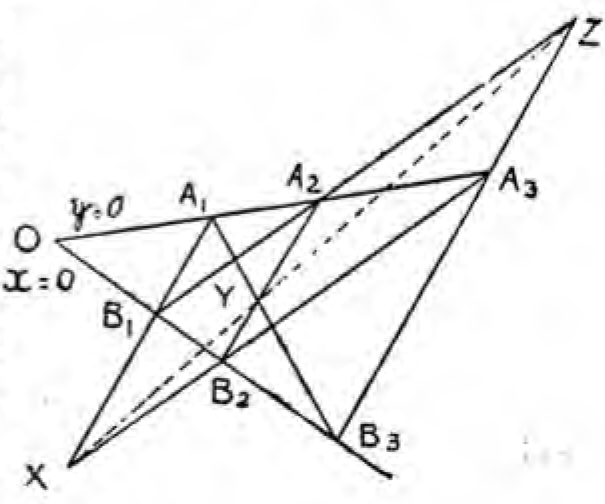
B′C′ $\quad (1+\lambda)x+y+z=0$
C′A′ $\quad x+(1+\mu)y+z=0$
A′B′ $\quad x+y+(1+v)z=0.$

The equations of the lines AA′, BB′, CC′ are clearly

$\mu y-vz=0 \quad -\lambda x+vz=0, \quad \lambda x-\mu y=0;$

and these lines obviously meet in the point $\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{v}\right)$

(2) If A_1, A_2, A_3 *be a triad of collinear points, and* B_1, B_2, B_3 *another triad of collinear points, then the three points of intersection of three pairs of lines* A_1B_1 *and* B_2A_3, A_2B_1 *and* A_3B_3, A_2B_2 *and* A_1B_3 *lie on a st. line.*



Let the lines $A_1A_2A_3$, $B_1B_2B_3$ meet in O. Take the triangle OB_1A_1 as the triangle of reference, so that the co-ordinates of A_1, B_1, O are respectively (1, 0, 0), (0, 1, 0) (0, 0, 1). Suppose that A_1B_1, B_2A_3 meet in X, A_2B_2, A_1B_3 in Y, A_3B_3, B_1A_2 in Z. Let the line which joins X, Y be taken as the unit line

$x+y+z=0.$

The equation of A_3B_2 is of the form

$x+y+zv=0$. This meets $x=0$ in B_2 which lies on $y+vz=0$.

Thus the co-ordinates of B_2 can be taken as $(0, v, -1)$, and A_3 as $(+v, 0, -1)$. Let the co-ordinates of B_3 be (o, p, q).

The equation of $A_1 B_3$ is therefore $qy - pz = 0$. Thus the equation of A_2B_2 is of the form $k(x+y+z)+qy-pz=0$.

This passes through $B_2(0, v, -1)$, hence $k(v-1)+qv+p=0$.

The equation of A_2B_2 is therefore

$$(p+qv)x+y(p+q)+z(qv+pv)=0.$$

The line $[(p+qv)x+y(p+q)+z(p+q)v]-(p+q)y=0$ passes through the intersection of A_1A_2, A_2B_2 and passes through B_1. Thus the equation of B_1A_2 is

$$(p+qv)x+z(p+q)v=0.$$

The equation of A_3B_3 is $px-vqy+zvp=0$. The identity

$$[(p+qv)x+z(p+q)v]-[px-vqy+zvp]\equiv qv(x+y+z)$$

shows that the lines B_1A_2, A_3B_3 meet on XY.

(3) The co-ordinates of some important points and lines connected with the fundamental triangle are given below for reference. Verify them.

	Trilinears.	Areals.
Incentre	$(1, 1, 1,)$	(a, b, c)
Ex-centre opposite to A	$(-1, 1, 1)$	$(-a, b, c)$
Circum-centre	$(\cos A, \cos B, \cos C)$	$(\sin 2A, \sin 2B, \sin 2C)$
Centroid	$\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$	$(1, 1, 1)$
Orthocentre	$\left(\frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}\right)$	$(\tan A, \tan B, \tan C)$
Nine-point centre	$\left(\cos\overline{B-C}, \cos\overline{C-A}, \cos\overline{A-B}\right)$	$\left(a\cos\overline{B-C}, b\cos\overline{C-A}, c\cos\overline{A-B}\right)$
Midpoint of BC	$\left(0, \frac{1}{b}, \frac{1}{c}\right)$	$(0, 1, 1)$
Median AD	$[0, b, -c]$	$[0, 1, -1]$
Interior bisector of A	$[0, 1, -1]$	$\left[0, \frac{1}{b}, \frac{1}{-c}\right]$
Exterior bisector of A	$[0, 1, 1]$	$\left[0, \frac{1}{b}, \frac{1}{c}\right]$
Altitude AL	$[0, \cos B, -\cos C]$,	$[0, \cot B, -\cot C]$

Exercises XLI

Using Trilinears, prove that the following triads of st. lines concur at a point:—

1. The medians of a triangle.
2. The altitudes of a triangle.
3. The right bisectors of the sides of a triangle.
4. The internal bisectors of the angles of a triangle.
5. Prove that the circumcentre, centroid, nine point centre and orthocentre of a triangle are collinear.
6. Prove that the middle points of the diagonals of a complete quadrilateral are collinear.
7. Given its centre and radius, find the Trilinear equation of a circle.

140. Defining a conic from the focus-directrix property or defining it as a curve which is cut by an arbitrary st. line in two and only two points, we see that the equation of a conic is of the 2nd degree in α, β, γ and conversely *i.e.*, the general equation of the 2nd degree in α, β, γ represents a conic.

Let $\phi(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$ be a conic and $P(\alpha_1, \beta_1, \gamma_1)$, $Q(\alpha_2, \beta_2, \gamma_2)$ two points in its plane. Any point on the join of these two points can be represented by $(\alpha_1+\lambda\alpha_2, \beta_1+\lambda\beta_2, \gamma_1+\lambda\gamma_2)$, where λ is a parameter. If this third point lies on the conic ϕ, the points U and V where PQ cuts ϕ are given by the equation

$$\lambda^2\phi_1 + \lambda\left(\alpha_2\frac{\partial\phi}{\partial\alpha_1} + \beta_2\frac{\partial\phi}{\partial\beta_1} + \gamma_2\frac{\partial\phi}{\partial\gamma_1}\right) + \phi_2 = 0.$$

which is a quadratic in λ

where $\qquad \phi_1 \equiv \phi(\alpha_1, \beta_1, \gamma_1). \qquad \phi_2 \equiv \phi(\alpha_2, \beta_2, \gamma_2).$

Let λ_1, λ_2 be its roots. The polar of P cuts the line PQ in a point Q which is the fourth harmonic of P *w. r.* to U, V. $\therefore$ $(0\lambda_1 \ \infty\lambda_2) = -1$, *i. e.*, $\lambda_1 + \lambda_2 = 0$. The condition for this is $\alpha_2\frac{d\phi}{d\alpha_1} + \beta_2\frac{d\phi}{d\beta_1} + \gamma_2\frac{d\phi}{d\gamma_1} = 0$. Thus

(1) the line $\alpha_1\frac{\partial\phi}{\partial\alpha} + \beta_1\frac{\partial\phi}{\partial\beta} + \gamma_1\frac{\partial\phi}{\partial\gamma} = 0$ or

$$\alpha\phi_{\alpha_1} + \beta\phi_{\beta_1} + \gamma\phi_{\gamma_1} = 0$$

is a tangent to the conic ϕ at P if $P(\alpha_1, \beta_1, \gamma_1)$ is on the conic, otherwise the polar of P *w. r.* to ϕ.

Notice that in the solution of problems, it is often useful to assign 0, ∞ to the fundamental points.

(2) The equation of the pair of tangents from $(\alpha_1, \beta_1, \gamma_1)$ is

$$\phi(\alpha, \beta, \gamma).\ \phi(\alpha_1, \beta_1, \gamma_1) = \tfrac{1}{4}\left(\alpha_1 \frac{\partial\phi}{\partial\alpha} + \beta_1 \frac{\partial\phi}{\partial\beta} + \gamma_1 \frac{\partial\phi}{\partial\gamma}\right)^2.$$

i. e., $\phi\phi_1 = P^2$ where $P=0$ is the polar of P.

140.1. Pole of a given line.

Let the line be $l\alpha + m\beta + n\gamma = 0$, and suppose that $(\alpha_1, \beta_1, \gamma_1)$ is its pole *w. r.* to ϕ.

The line $l\alpha + m\beta + n\gamma =$ is therefore identical with

$$\alpha \frac{\partial\phi}{\partial\alpha_1} + \beta \frac{\partial\phi}{\partial\beta_1} + \gamma \frac{\partial\phi}{\partial\gamma_1} = 0,$$

whence
$$\frac{\dfrac{\partial\phi}{\partial\alpha_1}}{l} = \frac{\dfrac{\partial\phi}{\partial\beta_1}}{m} = \frac{\dfrac{\partial\phi}{\partial\gamma_1}}{n} = 2\lambda \text{ say.}$$

These two equations give $\alpha_1 : \beta_1 : \gamma_1$

$$\therefore \quad \begin{aligned} u\alpha_1 + w'\beta_1 + v'\gamma_1 &= \lambda l \\ w'\alpha_1 + v\beta_1 + u'\gamma_1 &= \lambda m \\ v'\alpha_1 + u'\beta_1 + w\gamma_1 &= \lambda n. \end{aligned}$$

Hence $\dfrac{\alpha_1}{lU + mW' + nV'} = \dfrac{\beta_1}{lW' + mV + nU'} = \dfrac{\gamma_1}{lV' + mU' + nW}$

when U, V etc. are the co-factors of u, v etc. in the determinant.

$$\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}.$$

140.2. Centre of a conic.

Def. The pole of the line at infinity *w. r.* to a conic is called it centre.

The line at ∞ is $a\alpha + b\beta + c\gamma = 0$.

If $(\alpha_1, \beta_1, \gamma_1)$ be its pole *w.r.* to ϕ, this must be the same as

$$\alpha\phi_{\alpha_1} + \beta\phi_{\beta_1} + \gamma\phi_{\gamma_1} = 0$$

$$\therefore \quad \frac{\partial\phi/\partial\alpha_1}{a} = \frac{\partial\phi/\partial\beta_1}{b} = \frac{\partial\phi/\partial\gamma_1}{c}$$

Thus the coordinates of the centre are determined by

$$\frac{\partial\phi/\partial\alpha}{a} = \frac{\partial\phi/\partial\beta}{b} = \frac{\partial\phi/\partial\gamma}{c};$$

and they are $[aU + bW' + cV',\ aW' + bV + cU',\ aV' + bU' + cW)$.

140.3. The line

$$l\frac{\partial\phi}{\partial\alpha}+m\frac{\partial\phi}{\partial\beta}+n\frac{\partial\phi}{\partial\gamma}=0$$

with the condition $la+mb+nc=0$

always represents a diameter *i.e.* a line which passes through the centre of the conic ϕ.

Exercises XLII

1. Show that the line

$$\frac{m-n}{a}\frac{\partial\phi}{\partial\alpha}+\frac{n-l}{b}\frac{\partial\phi}{\partial\beta}+\frac{l-m}{c}\frac{\partial\phi}{\partial\gamma}=0,$$

represents in trilinears a diameter of the conic $\phi(\alpha,\beta,\gamma)=0$.

2. Show that the line

$$(m-n)\frac{\partial\phi}{\partial x}+(n-l)\frac{\partial\phi}{\partial y}+(l-m)\frac{\partial\phi}{\partial z}=0,$$

represents in Areals, a diameter of the conic $\phi(x,y,z)=0$.

140.4. The equation of the chord whose mid-point is given.

Let $(\alpha_1,\beta_1,\gamma_1)$ be the given point and $(\alpha_2,\beta_2,\gamma_2)$ the point at infinity on the line whose mid-point is $(\alpha_1,\beta_1,\gamma_1)$. The equation of the line is

$$\alpha_2(\beta\gamma_1-\beta_1\gamma)+\beta_2(\gamma\alpha_1-\gamma_1\alpha)+\gamma_2(\alpha\beta_1-\alpha_1\beta)=0.$$

The point $(\alpha_2,\beta_2,\gamma_2)$ lies on the polar of $(\alpha_1,\beta_1,\gamma_1)$ and the line at infinity.

$$\alpha_2\frac{\partial\phi}{\partial\alpha_1}+\beta_2\frac{\partial\phi}{\partial\beta_1}+\gamma_2\frac{\partial\phi}{\partial\gamma_1}=0.$$

$$a\alpha_2+b\beta_2+c\gamma_2=0.$$

Thus the required equation is

$$\begin{vmatrix}\alpha & \beta & \gamma\\ \alpha_1 & \beta_1 & \gamma_1\\ c\phi_{\beta_1}-b\phi_{\gamma_1} & a\phi_{\gamma_1}-c\phi_{\alpha_1} & b\phi_{\alpha_1}-a\phi_{\beta_1}\end{vmatrix}=0$$

141. Tangential Equations.

The tangential equation of a curve $\phi(\alpha,\beta,\gamma)=0$ is the same as the condition that the line $l\alpha+m\beta+n\gamma=0$ should be a tangent to ϕ.

$l:m:n$ are known as tangential coordinates.

The condition that the line $l\alpha+m\beta+n\gamma=0$ should be a tangent to the conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

is $\psi(l,m,n) \equiv Ul^2 + Vm^2 + Wn^2 + 2U'mn + 2V'nl + 2W'lm = 0$ ...(i)
where U, V etc. are the minors of u, v etc. in the determinant

$$\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix}.$$

$\therefore$ (i) represents the tangential equation of the conic ϕ.

The class of all lines which satisfy the equation. $\psi(l, m, n) = 0$ touch a conic whose point equation is $\phi(\alpha, \beta, \gamma) = 0$.

We now proceed to consider the conic as generated by a line (considered as an element).

141.1. Let $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ be two lines in the plane of the ψ-conic. The coordinates of any line through the intersection of the above lines are

$$[l_1 + \lambda l_2,\ m_1 + \lambda m_2,\ n_1 + \lambda n_2].$$

This line will touch (i) if

$$\lambda^2 \psi(l_2, m_2, n_2) + \lambda\left(l_1 \frac{\partial \psi}{\partial l_2} + m_1 \frac{\partial \psi}{\partial m_2} + n_1 \frac{\partial \psi}{\partial n_2}\right) + \psi(l_1, m_1, n_1) = 0 \quad \text{.....(ii)}$$

This is a quadratic in λ, therefore there are two lines through the intersection of $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ which touch the curve (i), *i.e.* through an arbitrary point two tangents can be drawn to (i), which is expressed by saying that the curve is of *second class* and is therefore a conic.

141.2. Point of contact of a tangent. Let $[l_1, m_1, n_1]$ be a tangent to ϕ, then $\psi(l_1, m_1, n_1) = 0$. From a point on $[l_1, m_1, n_1]$ one more tangent can be drawn and its parameter is given by the equation

$$\lambda \psi(l_2, m_2, n_2) + \Sigma l_1 \frac{\partial \psi}{\partial l_2} = 0.$$

If the point is the point of contact of $[l, m_1, n_1]$, the second value of λ will also be zero, hence $\Sigma l_1 \frac{\partial \psi}{\partial l_2} = 0$. Thus $[l_2, m_2, n_2]$ is on the point

$$l_1 \frac{\partial \psi}{\partial l} + m_1 \frac{\partial \psi}{\partial m} + n_1 \frac{\partial \psi}{\partial n} \equiv l \frac{\partial \psi}{\partial l_1} + m \frac{\partial \psi}{\partial m_1} + n \frac{\partial \psi}{\partial n_1} = 0 \quad \text{...(iii)}$$

141.3. The pole of the line $[l_1, m_1, n_1]$. Let the line $[l_1, m_1, n_1]$ meet the conic in points P, Q, and suppose that the tangents at P, Q are respectively $[l', m', n']$, $[l'', m'', n'']$, then the equations of P and Q are

$$l' \frac{\partial \psi}{\partial l} + m' \frac{\partial \psi}{\partial m} + n' \frac{\partial \psi}{\partial n} = 0, \quad l'' \frac{\partial \psi}{\partial l} + m'' \frac{\partial \psi}{\partial m} + n'' \frac{\partial \psi}{\partial n} = 0.$$

The line $[l_1, m_1, n_1]$ goes through these points.

$$l' \frac{\partial \psi}{\partial l_1} + m' \frac{\partial \psi}{\partial m_1} + n' \frac{\partial \psi}{\partial n_1} = 0, \quad l'' \frac{\partial \psi}{\partial l_1} + m'' \frac{\partial \psi}{\partial m_1} + n'' \frac{\partial \psi}{\partial n_1} = 0,$$

which shows that the tangents $[l', m', n']$, $[l'', m'', n'']$ pass through the point

$$l \frac{\partial \psi}{\partial l_1} + m \frac{\partial \psi}{\partial m_1} + n \frac{\partial \psi}{\partial n_1} \equiv l_1 \frac{\partial \psi}{\partial l} + m_1 \frac{\partial \psi}{\partial m} + n_1 \frac{\partial \psi}{\partial n} = 0.$$

Cor. 1. If the lines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ are conjugate, we have

$$l_1 \frac{\partial \psi}{\partial l_2} + m_1 \frac{\partial \psi}{\partial m_2} + n_1 \frac{\partial \psi}{\partial n_2} = 0.$$

Cor. 2. If the lines $[l_1, m_1, n_1]$, $[l_2, m_2, n_2]$ are conjugate equation (*ii*) gives two equal and opposite values of λ, *hence the conjugate lines w. r. to a conic through a point are harmonically separated by the tangents through the point.*

Cor. 3. Two lines are perpendicular if they are conjugate *w. r.* to the degenerate conic I, J.

141.4. To find the points of intersection of a line and a conic.

Let $[l_1, m_1, n_1]$ be the line, and $[l_2, m_2, n_2]$ another line in the plane. The parameter of the tangents through their intersection is given by the equation (*ii*). If these tangents coincide, the point lies on the conic and $[l_1, m_1, n_1]$ and is therefore an either point of intersection. The condition of equality of roots is

$$\psi(l_1, m_1, n_1)\ \psi(l_2, m_2, n_2) = \tfrac{1}{4}\left(l_1 \frac{\partial \psi}{\partial l_2} + m_1 \frac{\partial \psi}{\partial m_2} + n_1 \frac{\partial \psi}{\partial n_2} \right)^2.$$

Thus $[l_2, m_2, n_2]$ goes through

$$\psi(l, m, n)\ \psi(l_1, m_1, n_1) = \tfrac{1}{4}\left(l_1 \frac{\partial \psi}{\partial l} + m_1 \frac{\partial \psi}{\partial m} + n_1 \frac{\partial \psi}{\partial n} \right)^2.$$

142. Conics connected with the triangle of reference.

In the equation of a conic, there are five independent constants. So in order to fix a conic uniquely, we require five conditions. We shall suppose three conditions to be given and investigate into the form the equation takes

when the conic is (*a*) the circumconic of the triangle of reference (*b*) the inconic of the $\triangle$ of reference (*c*) the conic for which the triangle of reference is self-polar.

142.1. Circum-conic.

(*a*) Let the equation of the conic be

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0.$$

This passes through the points

$$\left(\frac{2\triangle}{a},0,0\right),\ \left(0,\frac{2\triangle}{b},0\right),\ \left(0,0,\frac{2\triangle}{c}\right),$$

viz., through the intersection of $\beta=0,\ \gamma=0$; $\gamma=0,\ \alpha=0$; $\alpha=0,\ \beta=0$.

$$\therefore \qquad u=v=w=0.$$

Thus the equation of the circumconic is of the form

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta=0. \qquad (i)$$

(*b*) **Tangential equation.** The equation of the tangent at (x', y', z') is

$$\frac{fx}{x'^2}+\frac{gy}{y'^2}+\frac{hz}{z'^2}=0$$

which is idential with $lx+my+nz=0$

$$\therefore \qquad \frac{lx'^2}{f}=\frac{my'^2}{g}=\frac{nz'^2}{h}=\lambda \text{ say}.$$

Hence the required tangentical equation is

$$\sqrt{fl}\pm\sqrt{gm}\pm\sqrt{hn}=0. \qquad (ii)$$

The rationalized form of the equation is

$$f^2l^2+g^2m^2+h^2n^2-2ghmn-2hfnl-2fglm=0. \qquad (iii)$$

142.2. Conics touching the sides of the triangle of reference.

(*a*) Let the conic be $\Sigma u\alpha^2+2\Sigma u'\beta\gamma=0$. This touches the lines $\alpha=0,\ \beta=0,\ z=0$.

$\therefore u'^2=vw,\ v'^2=uw,\ w'^2=uv.$

If therefore we replace $u,\ v,\ w$, by $u^2,\ v^2,\ w^2$, we get $u'=\pm vw,\ v'=\pm uw,\ w'=\pm uv$. Thus the equation of the conic can be written as

$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2\pm2vw\beta\gamma\pm2wu\gamma\alpha\pm2uv\alpha\beta=0.$$

For all positive or one positive and two negative signs, the expression on the left is a perfect square, therefore such combinations of signs have to be rejected. We thus have the four conics which touch the sides of the triangle of reference. Their equations are

$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2-2vw\beta\gamma-2wu\alpha\gamma-2uv\alpha\beta=0 \qquad \ldots\ldots(iv)$$
$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2-2vw\beta\gamma+2wu\alpha\gamma+2uv\alpha\beta=0 \qquad \ldots\ldots(v)$$
$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2+2vw\beta\gamma-2wu\alpha\gamma+2uv\alpha\beta=0 \qquad \ldots\ldots(vi)$$
$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2+2vw\beta\gamma+2wu\alpha\gamma-2uv\alpha\beta=0 \qquad \ldots\ldots(vii)$$

where u, v, w are positive. We may distinguish between the four conics.

The lines which join the vertices with the points of contact of conic (*iv*) with the opposite sides are respectively.

$$v\beta-w\gamma=0,\ w\gamma-u\alpha=0,\ u\alpha-v\beta=0.$$

The vertices B and C are on the opposite sides of the first line, C and A of the second line, A and B of the third line. Thus (*iv*) represents the in-conic of the triangle of reference. The equations (*v*), (*vi*), (*vii*) represent the e-conics opposite to A, B, C respectively. We may also point out that if u be replaced by $-u$, it becomes (*v*). Thus equation (*iv*) represents for different values of u, v, w all the conics which touch the sides of the triangle of reference. The signs of u, v, w are given below for each type of conic.

	u	v	w
In-conic	+	+	+
E-conic opposite to A	−	+	+
E-conic opposite to B	+	−	+
E-conic opposite to C	+	+	−

The equation (*iv*) can be written in an irrational form. We write (*iv*) as

$$(u\alpha+v\beta-w\gamma)^2=4uv\alpha\beta$$

$$\therefore \qquad u\alpha+v\beta\pm2\sqrt{uv\alpha\beta}=w\gamma$$

$$\text{or} \qquad (\sqrt{u\alpha}\pm\sqrt{v\beta})^2=w\gamma.$$

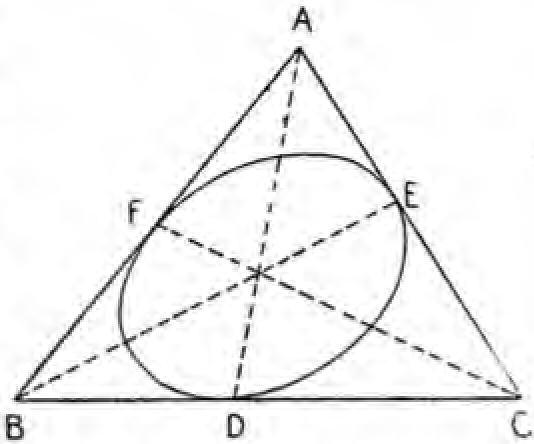
Thus the equation may be written as

$$\sqrt{u\alpha}\pm\sqrt{v\beta}\pm\sqrt{w\gamma}=0.$$

The equation (*iv*) is therefore equivalent to four equations

$$\sqrt{u\alpha}+\sqrt{v\beta}+\sqrt{w\gamma}=0 \qquad \ldots\ldots(1)$$
$$-\sqrt{u\alpha}+\sqrt{v\beta}+\sqrt{w\gamma}=0 \qquad \ldots\ldots(2)$$
$$\sqrt{u\alpha}-\sqrt{v\beta}+\sqrt{w\gamma}=0 \qquad \ldots\ldots(3)$$
$$\sqrt{u\alpha}+\sqrt{v\beta}-\sqrt{w\gamma}=0 \qquad \ldots\ldots(4)$$

Suppose that the conic is an in-conic so that u, v, w are all positive, then there is no real point on the conic which satisfies (1). For (2) $\sqrt{u\alpha}>\sqrt{v\beta}$, $\sqrt{u\alpha}>\sqrt{w\gamma}$ or $u\alpha-v\beta>0$, $u\alpha-w\gamma=0$. Now $u\alpha-v\beta=0$ is the equation of CF, $u\alpha-v\beta>0$ is the totality of all points which lie on the positive side of CF, and A is on the positive side of CF. Similarly A is on the positive side of BE. Thus



equation (2) represents those points on the conic which lie on the positive sides of CF, BE, *i. e.*, those points of the conic in which A lies. Thus part EF is represented by (2). Similarly (3) represents DF and (4) DE.

We will take (1) as the standard irrational equation of any one of the four conics.

(*b*) **Tangential equation.** The tangent at $(\alpha', \beta', \gamma')$ to (1) is

$$\alpha\sqrt{\frac{u}{\alpha'}}+\beta\sqrt{\frac{v}{\beta'}}+\gamma\sqrt{\frac{w}{\gamma'}}=0$$

which is identical with $l\alpha+m\beta+n\gamma=0$

$$\therefore \sqrt{\alpha'}=\frac{\lambda\sqrt{u}}{l},\ \sqrt{\beta'}=\frac{\lambda\sqrt{v}}{m},\ \sqrt{\gamma'}=\frac{\lambda\sqrt{w}}{n}.$$

Thus the required equation is

$$\frac{u}{l}+\frac{v}{m}+\frac{w}{n}=0. \qquad \text{......}(viii)$$

142.3. Conic with respect to which the triangle of reference is self-conjugate.

(*a*) The polar of $\left(\frac{2\Delta}{a}, 0, 0\right)$ is $u\alpha+w'\beta+v'\gamma=0$ which is $\alpha=0$, thus $w'=0$, $v'=0$. Similarly $u'=0$. Thus the equation of the conic is

$$u\alpha^2+v\beta^2+w\gamma^2=0. \qquad \text{......}(ix)$$

(*b*) **Tangential equation.** The tangent to (*ix*) at $(\alpha', \beta', \gamma')$ is $u\alpha\alpha'+v\beta\beta'+w\gamma\gamma'=0$, which is identical with $l\alpha+m\beta+n\gamma=0$,

$$\therefore \qquad \alpha'=\frac{\lambda l}{u},\ \beta'=\frac{\lambda m}{v},\ \gamma'=\frac{\lambda n}{w}.$$

Since $(\alpha', \beta', \gamma')$ lies on the conic, therefore the tangential equation is

$$\frac{l^2}{u}+\frac{m^2}{v}+\frac{n^2}{w}=0. \qquad \ldots\ldots(x)$$

Illustrative Examples.

(1) *Show that the locus of the centre of the conic (trilinears)*

$$fyz+gzx+hxy=0$$

which passes through (x', y', z') is a conic whose centre is

$$\left[\frac{1}{a}(2ax'+by'+cz'),\ \frac{1}{b}(ax'+2by'+cz'),\ \frac{1}{c}(ax'+by'+2cz')\right].$$

Let (x_0, y_0, z_0) be the co-ordinates of the centre. Its polar

$$x(hy_0+gz_0)+y(hx_0+fz_0)+z(gx_0+fy_0)=0$$

is identical with $ax+by+cz=0$.

$$\therefore \quad \begin{aligned} gz_0+hy_0-\lambda a&=0\\ fz_0+hx_0-\lambda b&=0\\ fy_0+gx_0-\lambda c&=0\\ \frac{f}{x'}+\frac{g}{y'}+\frac{h}{z'}&=0.\end{aligned}$$

Thus

$$\begin{vmatrix} 0 & z_0 & y_0 & a\\ z_0 & 0 & x_0 & b\\ y_0 & x_0 & 0 & c\\ \frac{1}{x'} & \frac{1}{y'} & \frac{1}{z'} & 0 \end{vmatrix}=0.$$

Hence the equation of the locus of (x_0, y_0, z_0) is

$$\frac{ax^2}{x'}+\frac{by^2}{y'}+\frac{cz^2}{z'}-yz\left(\frac{b}{z'}+\frac{c}{y'}\right)-zx\left(\frac{a}{z'}+\frac{c}{x'}\right)-yx\left(\frac{a}{y'}+\frac{b}{x'}\right)=0.$$

Suppose the centre of this conic is (x_1, y_1, z_1), then

$$-\frac{2ax_1}{x'}+y_1\left(\frac{a}{y'}+\frac{b}{x'}\right)+z_1\left(\frac{a}{z'}+\frac{c}{x'}\right)+\lambda a=0$$

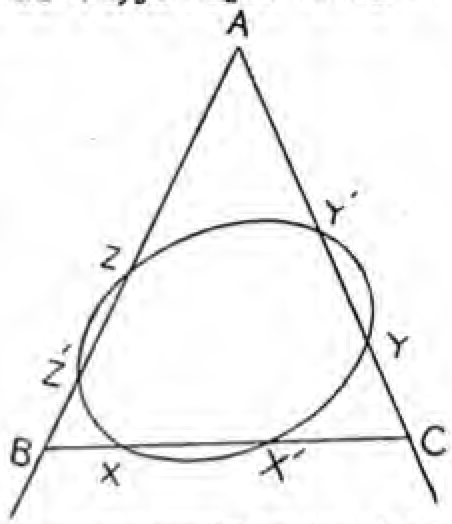
$$\left(\frac{a}{y'}+\frac{b}{x'}\right)x_1-\frac{2by_1}{y'}+z_1\left(\frac{b}{z'}+\frac{c}{y'}\right)+\lambda b=0$$

$$\left(\frac{a}{z'}+\frac{c}{x'}\right)x_1+\left(\frac{b}{z'}+\frac{c}{y'}\right)y_1-\frac{2cz_1}{z'}+\lambda c=0.$$

whence

$$x_1 : y_1 : z_1 = \frac{1}{a}(\pi' + ax') : \frac{1}{b}(\pi' + by') : \frac{1}{c}(\pi' + cz')$$

(2) *If the conic* $ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0$ *cuts the sides* BC, CA, AB *of a triangle of reference in points* X, X′; Y, Y′; Z, Z′ *respectively then will*



$$\frac{BX.BX'}{CX.CX'} \cdot \frac{CY.CY'}{AY.AY'} \cdot \frac{AZ.AZ'}{BZ.BZ'} = 1$$

(Carnot's Theorem)

Prove further, that if the lines joining the vertices of the triangle to three of the points in which the conic meets the opposite sides are concurrent, the same is true of the other three points, and the condition for this is

$$uvw-2fgh-uf^2-vg^2-wh^2=0.$$

We suppose the co-ordinates trilinear and the equation of the conic to be

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy=0.$$

If the co-ordinates of X, X′ are $(0, y_1, z_1)$, $(0, y_1', z_1')$, they satisfy the equation

$$vy^2+wz^2+2fyz=0$$

$$\text{or} \qquad w\frac{z^2}{y^2}+2f\frac{z}{y}+v=0.$$

$$\text{Now} \qquad \frac{BX.BX'}{CX.CX'}=\frac{z_1z_1'}{y_1y_1'} \cdot \frac{c^2}{b^2}=\frac{v}{w} \cdot \frac{c^2}{b^2}$$

$$\text{Similarly} \qquad \frac{CY.CY'}{AY.AY'}=\frac{w}{u} \cdot \frac{a^2}{c^2}, \quad \frac{AZ.AZ'}{BZ.BZ'}=\frac{u}{v} \cdot \frac{b^2}{a^2},$$

whence the result follows by multiplication.

Let AX, BY, CZ meet in $O(x_1, y_1, z_1)$. The equations of AX, BY, CZ respectively are

$$z_1y-y_1z=0 \quad zx_1-z_1x=0 \quad xy_1-x_1y=0,$$

and suppose AX′, BY′ meet in (x_2, y_2, z_2), the equations of AX′, BY′ are $z_2y-y_2z=0$ $zx_2-z_2x=0$.

$\therefore$ $(z_1y-y_1z)(z_2y-y_2z)$ and $vy^2+2fyz+wz^2$ are identical,

$$\text{hence} \qquad \frac{z_1z_2}{v}=\frac{y_1y_2}{w}=\frac{z_1y_2+z_2y_1}{-2f}. \qquad \ldots\ldots(i)$$

$$\text{Similarly} \qquad \frac{x_1x_2}{w}=\frac{z_1z_2}{u}=\frac{x_1z_2+x_2z_1}{-2g}. \qquad \ldots\ldots(ii)$$

If the equation of CZ′ be $xy_1'-x_1'y=0$,

$$\frac{x_1x_1'}{v}=\frac{y_1y_1'}{u};$$

$$\therefore\quad \frac{y_1'}{x_1'}=\frac{u}{v}\cdot\frac{x_1}{y_1}=\frac{w.z_1z_2}{x_1x_2}\cdot\frac{y_1y_2}{wz_1z_2}\cdot\frac{x_1}{y_1}$$

(substituting the values of u and v)

whence, $\dfrac{y_1'}{x_1'}=\dfrac{y_2}{x_2}$. Thus the equation of CZ′ is

$$xy_2-x_2y=0,$$

and this passes through (x_2, y_2, z_2). Thus AX′, BY′, CZ′ are also concurrent, and

$$\frac{u}{y_1y_2}=\frac{v}{x_1x_2}=\frac{-2h}{x_1y_2+x_2y_1}.$$

Again $$0=\begin{vmatrix} x_1 & x_2 & 0\\ y_1 & y_2 & 0\\ z_1 & z_2 & 0\end{vmatrix}\times\begin{vmatrix} x_2 & x_1 & 0\\ y_2 & y_1 & 0\\ z_2 & z_1 & 0\end{vmatrix}$$

$$=\begin{vmatrix} 2x_1x_2 & x_1y_2+x_2y_1 & x_1z_2+x_2z_1\\ x_1y_2+x_2y_1 & 2y_1y_2 & y_1z_2+y_2z_1\\ x_1z_2+x_2z_1 & y_1z_2+y_2z_1 & 2z_1z_2\end{vmatrix}$$

or $$\begin{vmatrix} 2 & \dfrac{-2h}{v} & \dfrac{-2g}{w}\\ \dfrac{-2h}{u} & 2 & \dfrac{-2f}{w}\\ \dfrac{-2g}{u} & \dfrac{-2f}{v} & 2\end{vmatrix}=0$$

or $$\begin{vmatrix} -u & h & g\\ h & -v & f\\ g & f & -w\end{vmatrix}=0$$

i.e., $$uvw-2fgh-uf^2-vg^2-wh^2=0.$$

143. Circles connected with the triangle of reference. In the foregoing three cases, we are left with two constants in the equations of the conic. Hence if two conditions more be given, our conic will be completely determined. For instance, if the centre of any one of them be given, the conic would be fixed uniquely, for the prescription of the centre means that two conditions are given.

Now the only conic that circumscribes the triangle of reference and has its centre at the circumcentre of the Δ is a circle.

Hence in order to find the equation of the circumcircle, we identify the centre of the circumconic with the circumcentre of the Δ. Similarly the centres of the incircle or the polar circle are at the incentre and the orthocentre of the Δ respectively.

143.1. Equation of the circumcircle.

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta=0$$

is the equation of the circumconic of the triangle of reference ABC.

The centre of this conic is given by the equations

$$\frac{h\beta+g\gamma}{a}=\frac{f\gamma+h\alpha}{b}=\frac{f\beta+g\alpha}{c}$$

or $$\frac{\frac{h}{\gamma}+\frac{g}{\beta}}{a\alpha}=\frac{\frac{f}{\alpha}+\frac{h}{\gamma}}{b\beta}=\frac{\frac{f}{\alpha}+\frac{g}{\beta}}{c\gamma}$$

whence $f:g:h=\alpha(-a\alpha+b\beta+c\gamma):\beta(a\alpha-b\beta+c\gamma):\gamma(a\alpha+b\beta-c\gamma)$

If the conic is to be a circumcircle. this must coincide with the circumcentre of the triangle of reference, which is given by

$$\cos A:\cos B:\cos C.$$

$\therefore\ f:\cos A(-\sin 2A+\sin 2B+\sin 2C)=$etc.

i.e., $f:\sin A\cos A\cos B\cos C=$etc.

i.e., $f:g:h=\sin A:\sin B:\sin C=a:b:c.$

and the equation of the circumcircle becomes

$$a\beta\gamma+b\gamma\alpha+a\gamma\beta=0.$$

Alternative Method.

Def. A circle is a conic which passes through the circular points at infinity. These points will be denoted by I and J and their co-ordinates by $(\alpha_0,\beta_0,\gamma_0)$, $(\alpha_0',\beta_0',\gamma')$.

The equation of the circum-conic is

$$f\beta\gamma+g\gamma\alpha+h\alpha\beta=0.$$

This passes through I and J,

$$f\beta_0\gamma_0+g\gamma_0\alpha_0+h\alpha_0\beta_0=0,$$
$$f\beta_0'\gamma_0'+g\gamma_0'\alpha_0'+h\alpha_0'\beta_0'=0,$$

$$\therefore\ \frac{f}{\alpha_0\alpha_0'(\beta_0\gamma_0'-\beta_0'\gamma_0)}=\frac{g}{\beta_0\beta_0'(\gamma_0\alpha_0'-\gamma_0'\alpha_0)}$$
$$=\frac{h}{\gamma_0\gamma_0'(\alpha_0\beta_0'-\alpha_0'\beta_0)}$$

or $$\frac{f}{\sqrt{(\beta_0\gamma_0'+\beta_0'\gamma_0)^2-4\beta_0\beta_0'\gamma_0\gamma_0'}}=\frac{g}{\sqrt{(\gamma_0\alpha_0'+\gamma_0'\alpha_0)^2-4\gamma_0\gamma_0'\alpha_0\alpha_0'}}$$
$$=\frac{h}{\sqrt{(\alpha_0\beta_0'+\alpha_0'\beta_0)^2-4\alpha_0\alpha_0'\beta_0\beta_0'}}$$

$\because \quad \alpha_0\alpha_0' = \beta_0\beta_0 = \gamma_0\gamma_0'$

$$\frac{f}{\sin A} = \frac{g}{\sin B} = \frac{h}{\sin C}.$$

Thus the equation of the circum-circle is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0$$

The equation in areals is

$$a^2yz + b^2zx + c^2xy = 0.$$

From Art. 142.1 it follows that the tangential equation in trilinears is

$$\sqrt{al} + \sqrt{bm} + \sqrt{cn} = 0$$

i.e., $$\Sigma a^2l^2 - 2\Sigma bcmn = 0.$$

143.2. Equation of the Polar Circle. The conic *w.r.* to which the triangle of reference is self-conjugate has for its equation

$$u\alpha^2 + v\beta^2 + c\gamma^2 = 0.$$

This passes through I and J,

$$u\alpha_0^2 + v\beta_0^2 + w\gamma_0^2 = 0$$
$$u\alpha_0'^2 + v\beta_0'^2 + w\gamma_0'^2 = 0$$

$$\therefore \frac{u}{(\beta_0\gamma_0' + \beta_0'\gamma_0)(\beta_0\gamma_0' - \beta_0'\gamma_0)} = \frac{v}{(\gamma_0\alpha_0' + \gamma_0'\alpha_0)(\gamma_0\alpha_0' - \gamma_0'\alpha_0)}$$
$$= \frac{w}{(\alpha_0\beta_0' + \alpha_0'\beta_0)(\alpha_0\beta_0' - \alpha_0'\beta_0)}$$

which by Art. 139 becomes

$$\frac{u}{a \cos A} = \frac{v}{b \cos B} = \frac{w}{c \cos C}.$$

Thus the equation of the polar circle is

$$a \cos A.\alpha^2 + b \cos B.\beta^2 + c \cos C.\gamma^2 = 0.$$

The equation can be written as

$$(a\alpha + b\beta + c\beta)(\alpha \cos A + \beta \cos B + \gamma \cos C) = a\beta\gamma + b\gamma\alpha + c\alpha\beta.$$

The last form of the equation shows that the polar circle meets the circum-circle in four points two of which lie on the line at infinity $a\alpha + b\beta + c\gamma = 0$ and these points are I and J, while the other two lie on

$$\alpha \cos A + \beta \cos B + \gamma \cos C = 0$$

which is therefore the radical axis of the two circles.

Alternative method.

The centre of the conic $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$ for which $\triangle$ ABC is self-polar is given by the equations

$$\frac{u\alpha}{a} = \frac{v\beta}{b} = \frac{w\gamma}{c}$$

or $$u : v : w = \frac{a}{\alpha} : \frac{b}{\beta} : \frac{c}{\gamma}.$$

But the co-ordinates of the orthocentre are

$$\sec A : \sec B : \sec C.$$

$\therefore\quad u : v : w = \sin 2A : \sin 2B : \sin 2C.$

$\therefore$ the equation to the polar circle is

$$\sin 2A.\alpha^2 + \sin 2B.\beta^2 + \sin 2C.\gamma^2 = 0.$$

(*b*) In areals the equation is

$$a^2yz + b^2zx + c^2xy = (x+y+z)(bc\cos A.x + ca\cos B.y + ab\cos C.z)$$

or $\quad x^2\cot A + y^2\cot B + z^2\cot C = 0.$

Ex. Find the tangential equation of the polar circle (*i*) in trilinears, (*ii*) in areals.

143.3. The equation of the incircle.

The equation of the in-conic is

$$\sqrt{u\alpha} + \sqrt{v\beta} + \sqrt{w\gamma} = 0.$$

This passes through I and J if

$$\sqrt{u\alpha_0} + \sqrt{v\beta_0} + \sqrt{w\gamma_0} = 0$$
$$\sqrt{u\alpha_0'} + \sqrt{v\beta_0'} + \sqrt{w\gamma_0'} = 0$$

$$\frac{\sqrt{u}}{\sqrt{\beta_0\gamma_0'} - \sqrt{\beta_0'\gamma_0}} = \frac{\sqrt{v}}{\sqrt{\gamma_0\alpha_0'} - \sqrt{\gamma_0'\alpha_0}} = \frac{\sqrt{w}}{\sqrt{\alpha_0\beta_0'} - \sqrt{\alpha_0'\beta_0}}$$

or
$$\frac{\sqrt{u}}{\sqrt{[\beta_0\gamma_0' + \beta_0'\gamma_0 - 2\sqrt{\beta_0\beta_0'\gamma_0\gamma_0'}]}}$$
$$= \frac{\sqrt{v}}{\sqrt{[\gamma_0\alpha_0' + \gamma_0'\alpha_0 - 2\sqrt{\alpha_0\alpha_0'\gamma_0\gamma_0'}]}}$$
$$= \frac{\sqrt{w}}{\sqrt{[\alpha_0\beta_0' + \alpha_0'\beta_0 - 2\sqrt{\alpha_0\alpha_0'\beta_0\beta_0'}]}}$$

or
$$\frac{\sqrt{u}}{\cos\frac{A}{2}} = \frac{\sqrt{v}}{\cos\frac{B}{2}} = \frac{\sqrt{w}}{\cos\frac{C}{2}}.$$

Thus the equation of the in-circle is

$$\sqrt{\alpha\cos^2\frac{A}{2}} + \sqrt{\beta\cos^2\frac{B}{2}} + \sqrt{\gamma\cos^2\frac{C}{2}} = 0$$

or $\quad \sqrt{a(s-a)\alpha} + \sqrt{b(s-b)\beta} + \sqrt{c(s-c)\gamma} = 0$

or $\quad \Sigma a^2(s-a)^2\alpha^2 - 2\Sigma bc(s-b)(s-c)\beta\gamma = 0$

or $\quad (a\alpha + b\beta + c\gamma)\left[\frac{(s-a)^2}{bc}\alpha + \frac{(s-b)^2}{ca}\beta + \frac{(s-c)^2}{ab}\gamma\right]$
$$= a\beta\gamma + b\gamma\alpha + c\alpha\beta.$$

The tangential equation is

$$mn\cos^2\frac{A}{2} + nl\cos^2\frac{B}{2} + lm\cos^2\frac{C}{2} = 0.$$

Alternative method.

The equation $\sqrt{u\alpha}+\sqrt{v\beta}+\sqrt{w\gamma}=0$ represents a conic touched by the three sides of $\triangle$ ABC.

Writing the conic in the proper form

$$u^2\alpha^2+v^2\beta^2+w^2\gamma^2-2vw\beta\gamma-2wu\gamma\alpha-2uv\alpha\beta=0,$$

we see that the centre is given by the equations

$$\frac{-u\alpha+v\beta+w\gamma}{a/u}=\frac{u\alpha-v\beta+w\gamma}{b/v}=\frac{u\alpha+v\beta-w\gamma}{c/w},$$

which give $u\alpha:v\beta:w\gamma=\dfrac{b}{v}+\dfrac{c}{w}:\dfrac{c}{w}+\dfrac{a}{u}:\dfrac{a}{u}+\dfrac{b}{v}$

$$\therefore\quad \frac{a\alpha}{a(bw+cv)}=\frac{b\beta}{b(cu+aw)}=\frac{c\gamma}{c(av+bu)}$$

i.e. $u:v:w=a(-a\alpha+b\beta+c\gamma):b(a\alpha-b\beta+c\gamma):c(a\alpha+b\beta-c\gamma)$.

But the in-centre is given by $1:1:1$.

$\therefore\quad u:v:w=a(s-a):b(s-b):c(s-c)$

where $2s=a+b+c$.

Thus the equation of the in-circle is of the form

$$\sqrt{a(s-a)\alpha}+\sqrt{b(s-b)\beta}+\sqrt{c(s-c)\gamma}=0.$$

143.4. E-circles. Proceeding as above, the equation of the *e*-circle opposite to A can be written in either of the forms

$$\sqrt{-\alpha\cos^2\frac{A}{2}}+\sqrt{\beta\sin^2\frac{B}{2}}+\sqrt{\gamma\sin^2\frac{C}{2}}=0$$

$$\sqrt{-as\alpha}+\sqrt{b(s-b)\beta}+\sqrt{c(s-c)\gamma}=0,$$

or $\quad a^2s^2\alpha^2+b^2(s-b)^2\beta^2+c^2(s-c)^2\gamma^2-2bc(s-b)(s-c)\beta\gamma$
$$+2cas(s-b)\gamma\alpha+2abs(s-c)\alpha\beta=0$$

or $\quad (a\alpha+b\beta+c\gamma)\left[\dfrac{s^2}{bc}\,\alpha+\dfrac{(s-c)^2}{ca}\,\beta+\dfrac{(s-b)^2}{ab}\,\gamma\right]$
$$=a\beta\gamma+b\gamma\alpha+c\alpha\beta.$$

Its tangential equation is

$$-mn\cos^2\frac{A}{2}+nl\cos^2\frac{B}{2}+lm\cos^2\frac{C}{2}=0.$$

144. Tangential equation of the circular points w, w'. w, w' are given by the intersection of any circle (say) $a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$ with the line at ∞, $a\alpha+b\beta+c\gamma=0$.

And the tangential equation of w, w' will be the condition that w, w' lie on $l\alpha+m\beta+n\gamma=0$.

$\therefore$ Solving $l\alpha+m\beta+n\gamma=0$, $a\alpha+b\beta+c\gamma=0$, we get

$$\alpha:\beta:\gamma=mc-nb:na-lc:lb-ma.$$

$\therefore$ Substituting in $\Sigma a\beta\gamma=0$, we see that the tangential equation required is

$$a(na-lc)(lb-ma)+b(lb-ma)(mc-nb)+c(mc-nb)(na-lc)=0.$$

We therefore, deduce that the tangential equation of w, w' is known if we substitute for α, β, γ,

$$mc-nb,\ na-lc,\ lb-ma,$$

respectively in the equation of any known circle in the plane. Making these substitutions in the circum-circle, in-circle and the polar circle we get these forms for the tangential equations of w, w' :—

$\Sigma a(na-lc)(lb-ma)=0$ in the circum-circle.

$\Sigma\sqrt{a(s-a)(mc-nb)}=0$ in the in-circle.

$\Sigma a[\cos A(mc-nb)^2=0$ in the polar circle.

These three forms are the same except for some constant factors, and any one of them can be written in the form

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0 \quad \ldots\ldots (i)$$

If we get $\alpha_1, \beta_1, \gamma_1$; $\alpha_2, \beta_2, \gamma_2$ as the co-ordinates of w, w', we must have $\Sigma(l^2-2mn\cos A)=\Sigma l\alpha_1\ \Sigma l\alpha_2$.

$\therefore$ equating co-efficients of l^2 etc. on both sides, we get

$$\frac{\alpha_1\alpha_2}{1}=\frac{\beta_1\beta_2}{1}=\frac{\gamma_1\gamma_2}{1}=\frac{\beta_1\gamma_2+\gamma_1\beta_2}{-2\cos A}=\frac{\gamma_1\alpha_2+\alpha_1\gamma_2}{-2\cos B}=\frac{\alpha_1\beta_2+\beta_1\alpha_2}{-2\cos C}.$$

These six equations determine w, w', but (i) is a condensed form for the co-ordinates of w, w'.

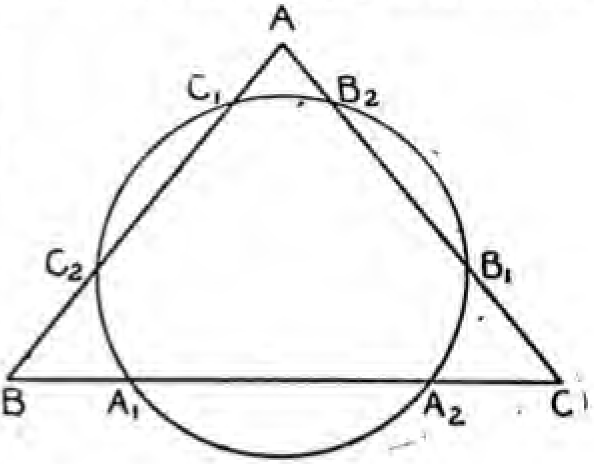
144.1. (*a*) **General equation of a circle.** A circle has been defined to be a conic which passes through the circular points at infinity. Now, the circular points at infinity, are the intersections of the line at infinity and the circum-circle. Thus the line at infinity is one of the common chords of the given circle and the circum-circle. If the other two points of intersection lie on $u\alpha+v\beta+w\gamma=0$, the required equation of the circle is

$$\phi(\alpha,\beta,\gamma)\equiv(u\alpha+v\beta+w\gamma)(a\alpha+b\beta+c\gamma)-(a\beta\gamma+b\gamma\alpha+c\alpha\beta)=0.$$

If $u\alpha+v\beta+w\gamma=0$ be fixed, $\Sigma u\alpha\ \Sigma a\alpha-\lambda\Sigma a\beta\gamma=0$ represents a system of coaxal circles for different values of λ.

(b) Geometrical significance of u, v, w.

Let the circle meet the sides of the triangle of reference in points, A_1, A_2; B_1, B_2; C_1, C_2 then the power of A *w. r.* to the circle $= AB_1.AB_2 = AC_1.AC_2 = t_1^2$, where t_1 is the length of the tangent from A to the circle.



Let the actual trilinear co-ordinates of C_1 and C_2 be $(\alpha_1, \beta_1, 0)$, $(\alpha_2, \beta_2, 0)$, then

$AC_1 = \beta_1 \operatorname{cosec} A.$

$AC_2 = \beta_2 \operatorname{cosec} A.$

$\therefore AC_1.AC_2 = \beta_1\beta_2 \operatorname{cosec}^2 A.$

Also C_1, C_2 lie on the circle, hence

$$(u\alpha + v\beta)(a\alpha + b\beta) - c\alpha\beta = 0 \text{ and } a\alpha + b\beta = 2\Delta$$

or $$\left[\frac{u}{a}(2\Delta - b\beta) + v\beta\right] 2\Delta - c\,\frac{2\Delta - b\beta}{a}\,\beta = 0$$

or $$bc\beta^2 + (\ldots\ldots\ldots)\beta + 4\Delta^2 u = 0$$

$$\therefore \qquad \beta_1\beta_2 = \frac{4\Delta^2 u}{bc}.$$

Thus $$t_1^2 = \frac{4\Delta^2 u}{bc \sin^2 A} = \frac{ub^2c^2 \sin^2 A}{bc \sin^2 A}.$$

$$\therefore \qquad u = \frac{t_1^2}{bc}.$$

Similarly if t_2, t_3 are the lengths of the tangents from B and C,

$$v = \frac{t_2^2}{ca}, \quad w = \frac{t_3^2}{ab}.$$

Thus the equation of a circle can be written as

$$\left(\frac{t_1^2}{bc}\alpha + \frac{t_2^2}{ca}\beta + \frac{t_3^2}{ab}\gamma\right)(a\alpha + b\beta + c\gamma)$$
$$- (a\beta\gamma + b\gamma\alpha + c\alpha\beta) = 0.$$

Ex. 1. Obtain the equations of the polar circle, in-circle and e-circles with the help of the present Art.

Ex. 2. *Nine-point circle.* Let D be the mid-point of BC and L the projection of A on BC, then the power of B *w. r.* to the nine-point circle is given by

$$t_2^2 = \frac{ac}{2} \cos B.$$

Similarly $t_1^2 = \frac{bc}{2}\cos A,\quad t_3^2 = \frac{ab}{2}\cos C.$

Thus the equation of the circle is

$$\tfrac{1}{2}(\alpha\cos A + \beta\cos B + \gamma\cos C)(a\alpha + b\beta + c\gamma) = a\beta\gamma + b\gamma\alpha + c\alpha\beta.$$

Ex. 3. Assuming that the feet of the perpendiculars from a point on the circumcircle of the triangle of reference to its sides are collinear, show that the trilinear equation of the circumcircle is

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta = 0.$$

Ex. 4. Find the point of contact of the in-conic of the triangle of reference and deduce that the equation of the incircle is

$$\sqrt{a(s-a)\alpha} + \sqrt{b(s-b)\beta} + \sqrt{c(s-c)\gamma} = 0.$$

144.2. Perpendicularity of two lines.

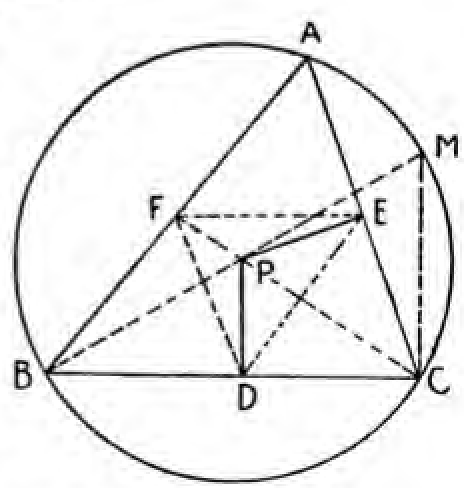
Let two lines VP, VQ meet the line at infinity in P and Q. *The lines* VP, VQ *are said to be perpendicular if*

$$(IJ, PQ) = -1.$$

Let C be any circle, then since $(IJ, PQ) = -1$, the points P and Q are conjugate *w. r.* to every circle. Thus *two lines are perpendicular if their points at infinity are conjugate w. r. to any circle* (therefore every circle).

Let $lx + my + nz = 0$, $l'x + m'y + n'z = 0$ be two lines. Their points at infinity are $(cm - bn,\ an - cl,\ bl - am)$, $(cm' - bn',\ an' - cl',\ bl' - am')$. If the lines are perpendicular the points are conjugate *w. r.* to every circle (say *e. g.*, the polar circle). The condition of perpendicularity is therefore $\Sigma(cm - bn)(cm' - bn')a\cos A = 0$.

Illustrative Examples



(1). *Find the power of* $P(x', y', z')$ *w. r. to the circle* $ayz + bzx + cxy = 0$.

From P draw the perpendiculars PD, PE, PF on the sides of the triangle of reference. Let BP meet the circum-circle in M and suppose $\angle FDE = \alpha$. We may note that, since D, C, E, P lie on a circle whose diameter is CP, $DE = CP\sin C$. Similarly $DF = BP\sin B$.

If Δ_1 be the area of the triangle DEF,

$$2\Delta_1 = DE.DF\sin\alpha = BP.CP\sin B\sin C\sin\alpha.$$

Also $2\Delta = 4R^2\sin A\sin B\sin C$.

$$\therefore \quad \frac{\Delta_1}{\Delta} = \frac{\text{BP.CP. sin } \alpha}{4\text{R}^2 \sin \text{A}}.$$

Now $\angle$ACM $= \angle$ABM $= \angle$FDP and $\angle$PCE $= \angle$PDE, hence $\angle$PCM $= \angle$FDE $= \alpha$. Thus from the triangle PCM, since $\angle$M $= \angle$A

$$\frac{\text{PC}}{\sin \text{A}} = \frac{\text{PM}}{\sin \alpha}.$$

$$\therefore \quad \frac{\Delta_1}{\Delta} = \frac{\text{BP.PM}}{4\text{R}^2}.$$

Thus the power of $\text{P} = t^2 = -\text{BP.PM} = \dfrac{-4\text{R}^2}{\Delta} \Delta_1$

$$= \frac{-2\text{R}^2}{\Delta}(y'z' \sin \text{A} + z'x' \sin \text{B} + x'y' \sin \text{C})$$

$$= \frac{-\text{R}}{\Delta}(ay'z' + bz'x' + cx'y') = \frac{-abc}{4\Delta^2} \Sigma ay'z' \quad \ldots\ldots(1)$$

where R is the circum-radius.

(2) *Find the power of* L(x', y', z') *w. r. to the circle*

$$\phi(x, y, z) \equiv \Sigma ax \Sigma \frac{t_1^2}{bc} x - \Sigma ayz = 0.$$

Let P, Q be the powers of (x', y', z') *w. r.* to the circle ϕ and the circum-circle, then by the well-known theorem of geometry

$$\text{P} - \text{Q} = 2kp,$$

where k is the distance between the centres of the two circles and p the perpendicular from (x', y', z') on the radical axis.

Now replace x', y', z' by the vertex $\text{A}\left(\dfrac{2\Delta}{a}, 0, 0\right)$, then $t_1^2 = 2kp'$ where p' is the perpendicular from A on the radical axis. Thus

$$\frac{\text{P} - \text{Q}}{t_1^2} = \frac{p}{p'}$$

Now, the radical axis being $\Sigma \dfrac{t_1^2}{bc} x = 0$, the ratio

$$\frac{p}{p'} = \frac{\Sigma \dfrac{t_1^2 x'}{bc}}{\dfrac{2\Delta}{abc} t_1^2},$$

for if LA meets the radical axis in R, $\dfrac{p}{p'} = -\dfrac{\text{LR}}{\text{RA}} = -\lambda$

Hence

$$\text{P} - \text{Q} = \frac{abc}{2\Delta} \Sigma \frac{t_1^2 x'}{bc}.$$

$$\text{Thus}\quad P = \frac{abc}{2\Delta}\Sigma\frac{t_1^2x'}{bc} + Q = \frac{abc}{2\Delta}\Sigma\frac{t_1^2x'}{bc} - \frac{abc}{4\Delta^2}\Sigma ay'z'$$

$$= \frac{abc}{4\Delta^2}\left[\Sigma ax'\Sigma\frac{t_1^2x'}{bc} - \Sigma ay'z'\right]$$

$$= \frac{abc}{4\Delta^2}\phi(x', y', z') \qquad \text{......(2)}$$

(3) *Prove Feuerbach's Theorem that the nine-point circle of a triangle touches all the four circles which touch the sides of the triangle.*

The equations of the inscribed circle and the nine-point circle respectively are

$$(\Sigma ax)\left(\Sigma\frac{(s-a)^2}{bc}x\right) = \Sigma ayz,\ (\Sigma ax).\ (\tfrac{1}{2}\Sigma x\cos A) = \Sigma ayz.$$

The radical axis of the two circles is

$$\Sigma\left(\frac{(s-a)^2}{bc} - \frac{1}{2}\cos A\right)x = 0$$

$$\text{or}\quad \Sigma[2(s-a)^2 - bc\cos A]ax = 0$$

$$\text{or}\quad \Sigma[(b+c-a)^2 - (b^2+c^2-a^2)]ax = 0$$

which reduces to $\Sigma\dfrac{ax}{b-c} = 0.$

Thus the co-ordinates of the radical axis are

$$\left[\frac{a}{b-c}, \frac{b}{c-a}, \frac{c}{a-b}\right]$$

and this satisfies the equation of the incircle

$$\Sigma mn\cos^2\frac{A}{2} = 0$$

$$\text{or}\quad \Sigma\frac{1}{l}\cos^2\frac{A}{2} = 0,\quad \text{for } \Sigma\frac{(b-c)s(s-a)}{abc} = 0$$

Similarly, it touches each of the e-circles.

145. In what follows, the system of co-ordinates will be trilinear, unless, otherwise stated and the equation of the conic will be taken as

$$\phi(x, y, z) \equiv ux^2 + vy^2 + wz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Set $X = \frac{1}{2}\dfrac{\partial\phi}{\partial x}$, $Y = \frac{1}{2}\dfrac{\partial\phi}{\partial y}$, $Z = \frac{1}{2}\dfrac{\partial\phi}{\partial z}$. Also put

$$\xi \equiv cY - bZ,\ \eta \equiv aZ - cX,\ \zeta \equiv bX - aY$$

$$\therefore\quad a\xi + b\eta + c\zeta = 0.$$

$$p=vc^2+wb^2-2bcf,$$
$$q=wa^2+uc^2-2acg$$
$$r=ub^2+va^2-2abh$$
$$-a^2p+b^2q+c^2r=2bc(ubc+a^2f-abg-ach)=2bc\lambda$$
$$a^2p-b^2q+c^2r=2ac(vac+b^2g-abf-bch)=2ac\mu$$
$$a^2p+b^2q-c^2r=2ab(wab+c^2h-acf-bcg)=2abv$$

whence $ap=c\mu+bv$

$$bq=c\lambda+av,\ cr=a\mu+b\lambda$$

and
$$\begin{vmatrix} -p & v & \mu \\ v & -q & \lambda \\ \mu & \lambda & -r \end{vmatrix}=0$$

Set
$$\theta\equiv\begin{vmatrix} u & h & g & a \\ h & v & f & b \\ g & f & w & c \\ a & b & c & 0 \end{vmatrix}$$

We also denote the discriminant of ϕ by D and the co-factors of small letters in D by the corresponding capital letters.

145.1. Discrimination of a conic. We have already classified the conics from their equations. We now classify them according to their relation with the line at infinity. A circle has already been defined as a conic which passes through I and J. We call a conic, an ellipse, parabola or hyperbola according as it meets the line at infinity in imaginary, coincident, or real points. If the real points at infinity on a hyperbola form with I J a harmonic range, the conic is a rectangular hyperbola.

The equation of the lines which join C with the intersections of the line at infinity π and the conic ϕ is obtained by eliminating z between ϕ and π, we get

$$x^2(uc^2+wa^2-2acg)+2xy(abw+c^2h-acf-bcg)$$
$$+y^2(c^2v+b^2w-2bcf)=0 \qquad \text{......}(i)$$

The conic will be an ellipse, parabola or hyperbola according as these lines are imaginary, coincident or real *i.e.*, according as

$$(abw+c^2h-acf-bcg)^2-(uc^2+wa^2-2acg)(c^2v+b^2w-2bcf)\lesseqgtr 0$$

$$\text{or}\quad a^2U+b^2V+c^2W+2bcF+2acG+2abH\gtreqless 0 \qquad \text{......}(ii)$$

The equation (*i*) may be written as

$$qx^2+2vxy+py^2=0$$

or $\quad a(c\lambda+av)x^2+2abvxy+b(c\mu+bv)y^2=0. \qquad \ldots\ldots(iii)$

The conic will be an ellipse, parabola or hyperbola according as

$$abv^2-(c\lambda+av)(c\mu+bv)\lesseqgtr 0$$

i.e., $$a\mu v+bv\lambda+c\lambda\mu \gtreqless 0 \qquad \ldots\ldots(iv)$$

In particular, if the conic is a parabola,

$$\frac{a}{\lambda}+\frac{b}{\mu}+\frac{c}{v}=0,$$

hence $\left(\frac{1}{\lambda},\ \frac{1}{\mu},\ \frac{1}{v}\right)$ is the point at infinity on the conic.

145.2. If $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be the points at infinity on ϕ, the equation (*iii*) is identical with

$$(y_1x-x_1y)(y_2x-x_2y)=0$$

hence

$$\frac{x_1x_2}{p}=\frac{y_1y_2}{q}=\frac{x_1y_2+x_2y_1}{-2v}$$

From two more equations similar to (*iii*) obtained by eliminating x and y in turn, we get

$$\frac{x_1x_2}{p}=\frac{y_1y_2}{q}=\frac{z_1z_2}{r}=\frac{y_1z_2+y_2z_1}{-2\lambda}=\frac{z_1x_2+z_2x_1}{-2\mu}=\frac{x_1y_2+x_2y_1}{-2v}$$

$\ldots\ldots(v)$

152.3. (*a*) **Condition for a circle.** Writing the equation ϕ as

$$\left(\frac{ux}{a}+\frac{vy}{b}+\frac{wz}{c}\right)\left(ax+by+cz\right)=\frac{p}{bc}yz+\frac{q}{ca}zx+\frac{r}{ab}xy$$

The left hand side must be identical with

$$ayz+bzx+cxy$$

if ϕ is to be a circle. Hence the required condition is

$$p=q=r \qquad \ldots\ldots(vi)$$

The condition can be written as $\lambda=\mu=v$.

(*b*) **Condition for a rectangular hyperbola.** The conic will be a rectangular hyperbola if besides the condition (*iv*) its points at infinity $P(x_1, y_1, z)$, $Q(x_2, y_2, z_2)$ are conjugate *w. r.* to any circle. Choosing for example the polar circle, the condition is

$$\Sigma a\cos A.\, x_1x_2=0 \qquad \ldots\ldots(vii)$$

or $\quad pa\cos A + q\,b\cos B + r\,c\cos C = 0$

If the circum-circle be chosen, the condition that P, Q may be conjugate is

$$a(y_1z_2+y_2z_1)+b(z_1x_2+z_2x_1)+c(x_1y_2+x_2y_1)=0$$

or $\quad a\lambda+b\mu+c\nu=0 \qquad \ldots\ldots(viii)$

The student can easily establish the identity of (vii) and $(viii)$.

Otherwise :—

A rectangular hyperbola is a conic for which the circular points w, w' are a pair of conjugate points.

The two points (x_1, y_1, z_1), (x_2, y_2, z_2) will be conjugate with respect to the conic

$$\phi\equiv\Sigma(ux^2+2u'yz)=0$$

if $\quad \Sigma[ux_1x_2+u'(y_1z_2+y_2z_1)]=0.$

But since the points (x_1, y_1, z_1), (x_2, y_2, z_2) are the points w, w' in this case, we have

$$(\Sigma lx_1)(\Sigma lx_2)=\Sigma(l^2-2mn\cos A)$$

$\therefore \quad x_1x_2=1$ etc.

$\therefore \quad$ the condition becomes

$$u+v+w-2u'\cos A-2v'\cos B-2w'\cos C=0$$

Ex. Find the condition that the two lines

$$l_1x+m_1y+n_1z=0$$
$$l_2x+m_2y+n_2z=0$$

may include a right angle.

146. *The polars w. r. to $\phi(x, y, z)$ of the points at infinity on $\psi(x, y, z)=0$ are given by the equation $\psi(\xi, \eta, \zeta)=0$.*

Let (x', y', z') be a point at infinity on $\psi(x, y, z)=0$. Its polar $w.\ r.$ to ϕ is

$$x'X+y'Y+z'Z=0$$
$$x'a+y'b+z'c=0$$

$$\therefore \qquad \frac{x'}{\xi}=\frac{y'}{\eta}=\frac{z'}{\zeta}.$$

Since (x', y', z') lies on $\psi(x, y, z)=0$, hence $\psi(\xi, \eta, \zeta)=0$. The lines pass through the centre, as at the centre $\xi=0$, $\eta=0$, $\zeta=0$. Hence $\psi(\xi, \eta, \zeta)=0$.

147. Asymptotes. The pair of asymptotes of ϕ is a degenerate conic which has double contact with ϕ at points where it is met by the line at infinity and which (i) passes through the centre or (ii) breaks down into two st. lines. Conics having double contact with $\phi=0$ where it is cut by the line at ∞ are of the form :—

$$ux^2+vy^2+wz^2+2fyz+2gzx+2hxy-\lambda(ax+by+cz)^2=0.$$

This will represent a pair of st. lines if

$$\begin{vmatrix} u-\lambda a^2 & h-\lambda ab & g-\lambda ac \\ h-\lambda ab & v-\lambda b^2 & f-\lambda bc \\ g-\lambda ac & f-\lambda bc & w-\lambda c^2 \end{vmatrix}=0$$

To simplify it, we write it as

$$\begin{vmatrix} u-\lambda a^2 & h-\lambda ab & g-\lambda ac & a \\ h-\lambda ab & v-\lambda b^2 & f-\lambda bc & b \\ g-\lambda ac & f-\lambda bc & w-\lambda c^2 & c \\ 0 & 0 & 0 & 1 \end{vmatrix}=0.$$

To first, second and third columns, add λa, λb, λc times the fourth, we have

$$\begin{vmatrix} u & h & g & a \\ h & v & f & b \\ g & f & w & c \\ \lambda a & \lambda b & \lambda c & 1 \end{vmatrix}=0, \text{ or } \begin{vmatrix} u & h & g & a \\ h & v & f & b \\ g & f & w & c \\ \lambda a & \lambda b & \lambda c & 0 \end{vmatrix}+\begin{vmatrix} u & h & g & 0 \\ h & v & f & 0 \\ g & f & w & 0 \\ \lambda a & \lambda b & \lambda c & 1 \end{vmatrix}=0.$$

i. e., $\lambda\theta+D=0$.

Substituting the value of λ, we get the equation

$$\phi(x, y, z)\theta+D\pi^2=0. \qquad \ldots\ldots(i)$$

147.1. The asymptotes may also be defined as the locus of a point whose polar meets the curve in one point on the line at ∞.

Let $R(x', y', z')$ be a point on an asymptote, which touches ϕ at $P(x_1, y_1, z_1)$. Since R lies on the polar of P, P lies on the polar of R.

$\therefore$ $\qquad x_1X'+y_1Y'+z_1Z'=0$

and $\qquad ax_1+by_1+cz_1=0$,

since P lies at infinity. Hence

$$\frac{x_1}{cY'-bZ'}=\frac{y_1}{aZ'-cX'}=\frac{z_1}{bX'-aY'}.$$

i. e., $$\frac{x_1}{\xi'}=\frac{y_1}{\eta'}=\frac{z_1}{\zeta'}$$

As P lies on ϕ, $\phi(\xi', \eta', \zeta')=0$. Thus the locus of the point (x', y', z') which are the asymptotes is given by the equation

$$\phi(\xi, \eta, \zeta)=0. \qquad \ldots\ldots(ii)$$

As $\qquad a\xi+b\eta+c\zeta=0$

$$2bc\eta\zeta=a^2\xi^2-b^2\eta^2-c^2\zeta^2,\ 2ac\xi\zeta=b^2\eta^2-a^2\xi^2-c^2\zeta^2$$
$$2ab\xi\eta=c^2\zeta^2-a^2\xi^2-b^2\eta^2.$$

Hence $abc\,\phi(\xi, \eta, \zeta)=abc(u\xi^2+v\eta^2+w\zeta^2)$

$$+af(a^2\xi^2-b^2\eta^2-c^2\zeta^2)+bg(b^2\eta^2-a^2\xi^2-c^2\zeta^2)$$
$$+ch(c^2\zeta^2-a^2\xi^2-b^2\eta^2)=0.$$

Thus the equation of the asypmtotes can be written as

$$a\lambda\xi^2+b\mu\eta^2+cv\zeta^2=0. \qquad \ldots\ldots(iii)$$

The equation can also be written as

$$ap\eta\zeta+bq\zeta\xi+cr\xi\eta=0. \qquad \ldots\ldots(iv)$$

Cor. The equations (*ii*), (*iii*), (*iv*) represent the polars *w. r.* to ϕ of the points at infinity on ϕ. $a\lambda x^2+b\mu y^2+cvz^2=0$ and $apyz+bqzx+crxy=0$ and as (*ii*), (*iii*), (*iv*) are the same. these three conics have common points at infinity.

Ex. Defining the asymptotes of a conic ϕ as the pair of tangents from the centre, obtain the joint equation of the asymptotes.

148. Locus of the mid-points of a system of parallel chords.

Let $lx+my+nz=0$ be one of the chords whose mid-point is P and the point at infinity is Q. If PQ meets the conic in U and V, then $(PQ, UV)=-1$. Thus the locus of P is the polar of Q. Now the co-ordinates of Q are $(mc-nb,\ na-lc,\ lb-ma)$. Its polar is

$$(mc-nb)X+(na-lc)Y+(lb-ma)Z=0$$

or $$l\xi+m\eta+n\zeta=0.$$

148.1. Equation of a pair of diameters.

Let P, Q be the points at infinity on the diameters, and suppose P′, Q′ are respectively conjugate to P and Q. Then a unique conic K can be drawn through A, B, C, and P′, Q′, whose equation is of the form $k_1yz+k_2zx+k_3xy=0$. The polars of P′, Q′ will be given by the equation

$$k_1\eta\zeta+k_2\zeta\xi+k_3\xi\eta=0. \qquad \ldots\ldots(i)$$

Similarly a unique conic can be drawn through P′, Q′ for which $\triangle$ABC is self-conjugate. Its equation is of the form $t_1x^2+t_2y^2+t_3z^2=0$, and the polars of P′. Q′ *w. r.* to ϕ are given by the equation

$$t_1\xi^2+t_2\eta^2+t_3\zeta^2=0. \qquad \ldots\ldots(ii)$$

148.2. Equation of a pair of conjugate diameters.

Let R and S be the points where a pair of conjugate diameters meet the line at infinity. The polar of R with regard to ϕ passes through S and the polar of S passes through R. Thus if P, Q be the points at infinity on ϕ (PQ, RS) = −1. Hence every conic through R and S has P, Q as a pair of conjugate points.

Suppose that the conic through R and S and having $\triangle$ ABC as self-conjugate $\triangle$ is

$$t_1x^2+t_2y^2+t_3z^2=0. \qquad \ldots\ldots(iii)$$

If (x', y', z'), (x'', y'', z'') be the co-ordinates of P and Q, then

$$t_1x'x''+t_2y'y''+t_3z'z''=0$$

or

$$pt_1+qt_2+rt_3=0, \qquad \ldots\ldots(iv)$$

and the equation of the polars of R and S is

$$t_1\xi^2+t_2\eta^2+t_3\zeta^2=0. \qquad \ldots\ldots(v)$$

Thus the equation (v) represents a pair of conjugate diameters provided condition (iv) is satisfied. Similarly the equations

$$\left.\begin{aligned} k_1\eta\zeta+k_2\zeta\xi+k_3\xi\eta=0 \\ \lambda k_1+\mu k_2+\nu k_3=0 \end{aligned}\right\} \qquad \ldots\ldots(vi)$$

represents a pair of conjugate diameters.

Ex. Show that the equation of a pair of conjugate diameters can be written in either of the following forms

$$\frac{m-n}{p}\xi^2+\frac{n-l}{q}\eta^2+\frac{l-m}{r}\zeta^2=0 \qquad \ldots\ldots(vii)$$

$$\frac{m-n}{\lambda}\eta\zeta+\frac{n-l}{\mu}\zeta\xi+\frac{l-m}{\nu}\xi\eta=0. \qquad \ldots\ldots(viii)$$

149. Axes of a central conic.

The axes of a central conic are a pair of perpendicular conjugate diameters.

The equations

$$t_1\xi^2+t_2\eta^2+t_3\zeta^2=0 \qquad \ldots\ldots(i)$$

$$t_1p+t_2q+t_3r\quad=0 \qquad \ldots\ldots(ii)$$

represent a pair of conjugate diameters. These lines will be the axes if their points at infinity R and S form with I, J a harmonic range *i.e.*, if I and J are conjugate *w.r.* to a conic through R and S. Taking $t_1x^2+t_2y^2+t_3z^2=0$ as the conic through R and S, and (x_0, y_0, z_0), (x_0', y_0', z_0') the co-ordinates of I and J, the condition of conjugacy is

$$t_1x_0x_0'+t_2y_0y_0'+t_3z_0z_0'=0$$

or

$$t_1+t_2+t_3=0 \qquad \ldots\ldots(iii)$$

Thus the equation of the axes is

$$\begin{vmatrix} \xi^2 & \eta^2 & \zeta^2 \\ p & q & r \\ 1 & 1 & 1 \end{vmatrix} = 0 \qquad \text{......}(iv)$$

Similarly the equation of the axes can be written in the form

$$\begin{vmatrix} \eta\zeta & \zeta\xi & \xi\eta \\ \lambda & \mu & v \\ \cos A & \cos B & \cos C \end{vmatrix} = 0 \qquad \text{......}(v)$$

149.1. Axis of a parabola.

Let the conic ϕ be a parabola, then its point at infinity is $\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{v}\right)$. Now all diameters of a parabola are parallel and meet at infinity on the parabola. Thus $P\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{v}\right)$ is a point on the axis. Suppose $Q(x', y', z')$ is the point where chords perpendicular to the axis meet the line at infinity. Thus the axis of the parabola will be the polar of (x', y', z'). Also P and Q are on perpendicular lines, therefore P, Q are conjugate *w.r.* to a circle.

Thus the line

$$x'X + y'Y + z'Z = 0$$

will be the axis if

$$ax' + by' + cz' = 0$$

and $$a\cos A.\frac{x'}{\lambda} + b\cos B\,\frac{y'}{\mu} + c\cos C\,\frac{z'}{v} = 0,$$

these being the conditions that (x', y', z') lies on the line at infinity and (x', y', z'), $\left(\frac{1}{\lambda}, \frac{1}{\mu}, \frac{1}{v}\right)$ are conjugate *w.r.* to the polar circle. Hence the equation of the axis is

$$\begin{vmatrix} X & Y & Z \\ a & b & c \\ \frac{a}{\lambda}\cos A & \frac{b}{\mu}\cos B & \frac{c}{v}\cos C \end{vmatrix} = 0 \qquad \text{...}(vi)$$

or $$\frac{a}{\lambda}\xi\cos A+\frac{b}{\mu}\eta\cos B+\frac{c}{\nu}\zeta\cos C=0 \quad \ldots(vi_1)$$

150. Director Circle.

The pair of tangents from a point (x_1, y_1, z_1) to the conic ϕ is given by the equation $\phi\phi_1=[\Sigma x\xi_1]^2$

or $\Sigma[x^2(u\phi_1-\xi_1^2)+2yz(u'\phi_1-\eta_1\zeta_1)]=0.$

These would include a right angle if

$$\Sigma[u\phi_1-\xi_1^2)-2\cos A\,(u'\phi_1-\eta_1\zeta_1)]=0.$$

$\therefore$ The locus of the point (x_1, y_1, z_1) is the director circle.

$$\phi\Sigma(u-2u'\cos A)=\Sigma(\xi^2-2\eta\zeta\cos A) \quad \ldots\ldots(i)$$

Now the tangential equation of two points (x_1, y_1, z_1), (x_2, y_2, z_2) is $\Sigma lx_1.\ \Sigma lx_2=0$ or $\Sigma[l^2x_1x_2+mn(y_1z_2+y_2z_1)]=0.$

The joint equation of the polars of these points is

$$\Sigma x_1\xi.\ \Sigma x_2\xi=0 \text{ or } \Sigma[\xi^2x_1x_2+\eta\zeta(y_1z_2+y_2z_1)]=0.$$

Thus the joint equation of the polars of the circular points is

$$\Sigma(\xi^2-2\eta\zeta\cos A)=0.$$

Hence if the conic be a rectangular hyperbola, the director circle is $\Sigma(\xi^2-2\eta\zeta\cos A)=0$, which represents a pair of imaginary st. lines through the centre of the conic.

Thus the director circle of a rectangular hyperbola may be regarded as a point circle represented by the centre of the hyperbola.

(i) can be put in the form

$$\Sigma ax.\ \Sigma lx=\frac{\psi}{abc}\Sigma ayz$$

where $\Sigma lx=0$ is the radical axis of the director circle and the circumcircle $\Sigma ayz=0$, and $\psi=0$ is the condition for a parabola.

If $\psi=0$, the conic is a parabola, and the director circle reduces to the st. line $\Sigma lx=0$ and the line at ∞, $\Sigma ax=0$.

Thus $\Sigma lx=0$ is the directrix of the parabola.

151. Foci of a conic.

The foci of a conic are the points of intersection of the tangents drawn to the conic from the circular points.

Let us find the foci of the conic $\phi\equiv\Sigma(ux^2+2u'yz)=0$.

Suppose that (x', y', z') are the co-ordinates of a focus

The pair of tangents from (x', y', z') to the conic is given by

$$\phi\phi' = (x\xi' + y\eta' + z\zeta')^2$$

or $\quad \Sigma\,[x^2(u\phi' - \xi'^2) + 2yz(u'\phi' - \eta'\zeta')] = 0.$

This must satisfy the conditions for a circle.

$\therefore$ the foci are given by the equations

$$4(b^2w + c^2v - 2bcu')\phi - \left(b\frac{d\phi}{dz} - c\frac{d\phi}{dy}\right)^2$$

$$= 4(c^2u + a^2w - 2cav')\phi - \left(c\frac{d\phi}{dx} - a\frac{d\phi}{dz}\right)$$

$$= 4(a^2v + b^2u - 2abw')\phi - \left(a\frac{d\phi}{dy} - b\frac{d\phi}{dx}\right)^2$$

The elimination of ϕ will give the axes of the conic.

151.1. The equations giving the foci represent two conics, both satisfying the condition for a rectangular hyperbola.

Now the conic through the intersection of two rectangular hyperbolas is itself a rectangular hyperbola or a pair of perpendicular st. lines.

Thus it is inferred that the line joining any two foci is perpendicular to the line joining the other two.

Hence the four foci of a conic appear to form an orthocentric set, *i.e.* the vertices of a triangle and its orthocentre.

151.2. If $S=0$ be a conic, and $l=0$, $l'=0$ two lines all given in point co-ordinates, then we know that

$$S = \lambda ll'$$

represents a conic passing through the four points of intersection of the two lines with the conic.

On the other hand, if $\Sigma=0$ be a conic and l, l' two points, all given in tangential co-ordinates, then

$$\Sigma = \lambda ll'$$

will represent a conic touching the four tangents to Σ through l, l'.

For in the system of line co-ordinates, a conic is to be considered as the envelope of a line, while in the system of point co-ordinates, it is the locus of a point.

Now $K \equiv l^2 + m^2 + n^2 - 2mn\cos A - 2nl\cos B - 2lm\cos C = 0$ is the tangential equation of the circular points w, w'.

Hence $\quad \Sigma + \lambda K = 0$

is a general conic touching the four tangents to Σ from w, w'.

But the tangents through w, w' to Σ intersect in the four foci of the conic Σ.

$\therefore$ the equation $\Sigma+\lambda K=0$ represents a system of conics confocal with the conic Σ.

151.3. Now for some proper choice of λ, the equation $\Sigma+\lambda K=0$ can be made to break up into two linear factors. And this can be done in three ways so as to represent the three pairs of points (*i*) two real foci (*ii*) two imaginary foci (*iii*) two circular points. Let the real foci be (x_1, y_1, z_1), (x_2, y_2, z_2). Then for some value of λ

$$\Sigma+\lambda K=(lx_1+my_1+nz_1)(lx_2+my_2+nz_2).$$

$\therefore$ $\Sigma=0$ the tangential equation of the conic is identical with

$$(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)-\lambda K=0$$

which represents the geometrical fact that the product of the perpendiculars from the two real foci upon any tangent to the conic is constant.

We also infer that the tangential equation of a conic is always capable of representation as $\Sigma lx_1\ \Sigma lx_2+\lambda K=0$ where (x_1, y_1, z_1), (x_2, y_2, z_2) are the two real foci of the conic.

151.4. The tangential equation of a conic inscribed in the triangle of reference is of the form

$$\lambda mn+\mu nl+\nu lm=0.$$

If (x_1, y_1, z_1), (x_2, y_2, z_2) be the two foci of the in-conic, we have

$$\Sigma\lambda mn=\Sigma lx_1.\Sigma lx_2+\theta K.$$

$\therefore$ Comparing the co-efficients, we get

$$x_1\,x_2+\theta=0,\ y_1\,y_2+\theta=0,\ z_1\,z_2+\theta=0.$$

$\therefore$ $x_1\,x_2=y_1\,y_2=z_1\,z_2$.

If the conic be a parabola, one of its real foci (say) (x_1, y_1, z_1) is at ∞, $\therefore$ $ax_1+by_1+cz_1=0$.

$$\therefore\quad \frac{a}{x_2}+\frac{b}{y_2}+\frac{c}{z_2}=0,$$

which shows that the finite focus lies on the circumcircle

$$ayz+bzx+cxy=0$$

In general, if the locus of one of the foci be known, we can find the locus of the other focus as well.

Illustrative Examples

(1) *The st. line $lx+my+nz=0$ meets the sides BC, CA, AB of the triangle of reference in points L, M, N respectively. Show that the circles drawn on AL, BM, CN as diameters are co-axal.*

The line which joins A to L is given by the equation $my+nz=0$ obtained by putting $x=0$. Thus the co-ordinates of L can be taken as $(0, -n, +m)$ and the co-ordinates of A are $(1, 0, 0)$.

Let $P(x', y', z')$ be a point on the circumference. The co-ordinates of PL, PA are

$$[my'+nz', -x'm, -nx'], [0, z', -y']$$

respectively. These lines are at right angles,

$$\therefore\ -mx'z'+nx'y'-(mx'y'-nx'z')\cos A+y'(my'+nz')\cos B$$
$$-z'(my'+nz')\cos C=0.$$

Thus the locus of (x', y', z') which is the circle on AL as diameter is given by the equation

$$my^2\cos B-nz^2\cos C+yz(n\cos B-m\cos C)+zx(n\cos A-m)$$
$$+xy(n-m\cos A)=0$$

$$\text{or}\quad (ax+by+cz)\left(\frac{m\cos B}{b}y-\frac{n\cos C}{c}z\right)=$$
$$\frac{cm-bn}{bc}(ayz+bzx+cxy) \qquad \ldots\ldots(i)$$

Similarly the equation of the circles on BM, CN as diameters are

$$\left(ax+by+cz\right)\left(\frac{n\cos C}{c}z-\frac{l\cos A}{a}x\right)=$$
$$\frac{an-cl}{ac}(ayz+bzx+cxy) \qquad \ldots\ldots(ii)$$

$$\left(ax+by+cz\right)\left(\frac{l\cos A}{a}x-\frac{m\cos B}{b}y\right)=$$
$$\frac{lb-am}{ab}(ayz+bzx+cxy) \qquad \ldots\ldots(iii)$$

The radical axis of circles (i) and (ii) is

$$\frac{bc}{cm-bn}\left(\frac{m\cos B}{b}y-\frac{n\cos C}{c}z\right)=$$
$$\frac{ac}{an-cl}\left(\frac{n\cos C}{c}y-\frac{l\cos A}{a}x\right)$$

$$(an-cl)(mc\cos B.y-nb\cos C.z)=(cm-bn)(na\cos C.z-lc\cos A.x)$$
$$\text{or}\quad (cm-bn)lc\cos A.x+(an-cl)mc\cos B.y$$
$$+(bl-am)nc\cos C.z=0$$
$$\text{or}\quad (cm-bn)\,l\cos A.x+(an-cl)m\cos B.y+$$
$$(bl-am)\,n\cos C.z=0$$

The radical axis of (*ii*) and (*iii*) will be found to be the same.

Thus the three circles are co-axal.

Otherwise :—If AX, BY, CZ be any three st. lines drawn from the vertices of $\triangle$ ABC to the opposite sides, the circles on AX, BY, CZ as diameters have their radical centre at the orthocentre.

Now AL, BM, CN are three lines from the vertices of each of four triangles ABC, AMN, BLN, CLM to its opposite sides. Thus the circles on AL, BM, CN as diameters have the orthocentres of these triangles for their radical centres *i.e.*, these circles have more than one radical centre. Hence they must be co-axal.

It follows that the orthocentres of four $\triangle$s formed by any four lines are collinear.

Further it follows that since AL, BM, CN are the diagonals of a complete quadrilateral formed by the sides of $\triangle$ ABC and a given line, the mid-points of the diagonals of a complete quadrilateral are collinear.

(2) *Find the condition that the conic $ux^2+vy^2+wz^2=0$ be a parabola ; and find the equation of its directrix, the co-ordinate system being trilinears. Prove also that the centre of the circum-circle of a triangle self-conjugate for a parabola lies on the directrix.*

The lines which join C to the points of intersections of the conic with the line at infinity are given by the equation

$$ux^2+vy^2+\left(\frac{-ax-by}{c}\right)^2 w=0$$

or $\quad x^2(uc^2+wa^2)+2ab\,xy\,w+y^2(vc^2+wb^2)=0$

The conic will be a parabola if these lines coincide,

i.e., if $a^2b^2w^2=(uc^2+wa^2)(vc^2+wb^2)$

or $\quad k\equiv \dfrac{a^2}{u}+\dfrac{b^2}{v}+\dfrac{c^2}{w}=0$

Let (x_1, y_1, z_1) be a point on the directrix, the equation of the pair of tangents from it to the conic is

$$(ux_1^2+vy_1^2+wz_1^2)(ux^2+vy^2+wz^2)=(uxx_1+vyy_1+wzz_1)^2$$

$$u(vy_1^2+wz_1^2)x^2+v(ux_1^2+wz_1^2)y^2+w(ux_1^2+vy_1^2)z^2$$
$$-2vwy_1z_1yz-2wux_1z_1xz-2uvx_1y_1xy=0.$$

These tangents are at right angles if the points I (x_0, y_0, z_0), J (x_0', y_0', z_0') are conjugate *w. r.* to these lines regarded as a degenerate conic. The condition for perpendicularity is

$$\Sigma u^2(vy_1^2+wz_1^2)x_0x_0'-\Sigma vwy_1z_1(y_0z_0'+y_0'z_0)=0$$

or $\quad \Sigma u^2(vy_1^2+wz_1^2)+2\Sigma vw\,y_1z_1\cos A=0$

Thus the locus of (x_1, y_1, z_1) is the conic

$$x^2(uv+uw)+y^2(uv+wv)+z^2(uw+wv)+2vwyz\cos A$$
$$+2wuzx\cos B+2uv\,xy\cos C=0.$$

If the conic is not a parabola, it gives the equation of the director circle of the given conic. If it is a parabola, we write the equation as

$$(ax+by+cz)\left[\frac{u(v+w)}{a}x+\frac{v(u+w)}{b}y+\frac{w(u+v)}{c}z\right]$$
$$=\frac{uvw}{abc}k(ayz+bzx+cxy).$$

Since the conic is a parabola, $k=0$ and the director circle degenerates into the line at infinity and the directrix

$$\frac{u}{a}(v+w)x+\frac{v}{b}(w+u)y+\frac{w}{c}(u+v)z=0.$$

The circum-circle of the triangle is $ayz+bzx+cxy=0$ and the co-ordinates of its centre are $(\cos A, \cos B, \cos C)$. The equation of the directrix can be written as

$$\frac{1}{u}\left(\frac{y}{b}+\frac{z}{c}\right)+\frac{1}{v}\left(\frac{z}{c}+\frac{x}{a}\right)+\frac{1}{w}\left(\frac{x}{a}+\frac{y}{b}\right)=0.$$

The point $(\cos A, \cos B, \cos C)$ will lie on it, if

$$\frac{1}{u}\,\frac{c\cos B+b\cos C}{bc}+\frac{1}{v}\,\frac{a\cos C+c\cos A}{ac}+\frac{1}{w}\,\frac{b\cos A+a\cos B}{ab}$$

vanishes,

or $\quad \dfrac{a^2}{u}+\dfrac{b^2}{v}+\dfrac{c^2}{w}\quad$ vanishes.

But $\quad \dfrac{a^2}{u}+\dfrac{b^2}{v}+\dfrac{c^2}{w}=0$

since the conic is a parabola. Hence the result.

(3) *An in-conic touches the sides BC, CA, AB of the triangle of reference ABC at points A', B', C'. If AA', BB', CC' meet in $P(x_1, y_1, z_1)$ prove that the normals at A', B', C' will meet in a point provided that*

$$\begin{vmatrix} x_1^2\sin^2 A & y_1^2\sin^2 B & z_1^2\sin^2 C \\ x_1\cos A & y_1\cos B & z_1\cos C \\ 1 & 1 & 1 \end{vmatrix}=0$$

the system of co-ordinates being trilinears.

Let the equation of the conic be

$$u^2x^2+v^2y^2+w^2z^2-2vwyz-2wuzx-2uvxy=0.$$

The equations of the lines AA', BB', CC' are

$$vy-wz=0,\quad wz-ux=0,\quad ux-vy=0.$$

These lines obviously meet in a point. If the point is taken to be (x_1, y_1, z_1), then $u : v : w = \dfrac{1}{x_1} : \dfrac{1}{y_1} : \dfrac{1}{z_1}$.

The co-ordinates of A′, B′, C′ are $(0, y_1, z_1)$, $(x_1, 0, z_1)$, $(x_1, y_1, 0)$.

Let the normal at A′ be

$$\begin{aligned} lx + my + nz &= 0 \\ my_1 + nz_1 &= 0 \\ -l + m\cos C + n\cos B &= 0 \end{aligned}$$

since $[l, m, n]$, $[1, 0, 0]$ are at right angles. Thus the equation of the normal at A′ is

$$x(y_1 \cos B - z_1 \cos C) - yz_1 + zy_1 = 0.$$

Similarly the normals at B′, C′ are given by the equations

$$xz_1 + y(z_1 \cos C - x_1 \cos A) - zx_1 = 0$$
$$-xy_1 + yx_1 + z(x_1 \cos A - y_1 \cos B) = 0.$$

These lines meet in a point if

$$\begin{vmatrix} y_1 \cos B - z_1 \cos C & -z_1 & y_1 \\ z_1 & z_1 \cos C - x_1 \cos A & -x_1 \\ -y_1 & x_1 & x_1 \cos A - y_1 \cos B \end{vmatrix} = 0$$

which reduces to the given form.

(4) *Show that the trilinear equation to the ellipse through B and C which has one focus at the angular point A of the triangle of reference ABC and the other focus in BC is*

$$x^2 \sin^2 \frac{A}{2} + yz + zx + xy = 0.$$

The equation of A is $l = 0$ and of a point in BC is $my' + nz' = 0$. The tangential equation of the conics which touch the lines joining these points with the points I, J is

$$2\lambda l(my' + nz') = l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C \quad \text{......}(i)$$

Also, the equation of the conic which passes through B and C is of the form

$$ux^2 + 2fyz + 2gzx + 2hxy = 0.$$

Its tangential equation is

$$-f^2l^2 - g^2m^2 - h^2n^2 + 2(gh - uf)mn + 2hfnl + 2fglm = 0.$$

Now the curve

$$\begin{aligned} -f^2l^2 - g^2m^2 - h^2n^2 + 2(gh - uf)mn + 2hfnl + 2fglm \\ + \lambda[l^2 + m^2 + n^2 - 2mn \cos A - 2nl \cos B - 2lm \cos C] = 0 \end{aligned}$$

touches the tangents to the given conic from I and J *i.e.* it is confocal with the given conic. This will be of the form

$$l(my'+nz')$$

if $f^2=g^2=h^2=\lambda$. $\quad gh-uf-\lambda\cos A=0$

or $f^2-f^2\cos A=uf$ $\quad$ or $u=2f\sin^2\dfrac{A}{2}$.

and this gives the desired result.

(5) *Show that the radius of the circle*

$$ayz+bzx+cxy=kx(ax+by+cz)$$

is $R\sqrt{1-2k\cos A+k^2}$, *where* R *is the radius of the circumcircle of the triangle of reference :*

We note that the power of the centre *w. r.* to the circle is equal to the square of the radius with the sign changed.

Suppose (x_0, y_0, z_0) is the centre of the circle

$$2\phi(x, y, z)\equiv 2akx^2-2ayz+2zx(ck-b)+2xy(bk-c)=0.$$

The polar of (x_0, y_0, z_0) viz.,

$$x[2ak\,x_0+y_0(bk-c)+z_0(ck-b)]+y[(bk-c)x_0-az_0]$$
$$+z[(ck-b)x_0-ay_0]=0$$

is identical with $\quad ax+by+cz=0$

$$2akx_0+y_0(bk-c)+z_0(ck-b)-\lambda a=0 \qquad \ldots\ldots(i)$$
$$(bk-c)x_0 \qquad -az_0 \qquad -\lambda b=0 \qquad \ldots\ldots(ii)$$
$$(ck-b)x_0 \quad -ay_0 \qquad -\lambda c=0 \qquad \ldots\ldots(iii)$$

and $\quad ax_0 \quad +by_0 \quad +cz_0 \qquad -2\Delta=0 \qquad \ldots\ldots(iv)$

Now $2\phi(x_0, y_0, z_0)=x_0[2akx_0+(bk-c)y_0+(ck-b)z_0]$

$$+y_0[(bk-c)x_0-az_0]+z_0[(ck-b)x_0-ay_0]$$
$$=\lambda(ax_0+by_0+cz_0)=2\Delta\lambda\ldots\ldots(v)$$

Also from (i), (ii), (iii), (iv)

$$\begin{vmatrix} 2ak & bk-c & ck-b & \lambda a \\ bk-c & 0 & -a & \lambda b \\ ck-b & -a & 0 & \lambda c \\ a & b & c & 2\Delta \end{vmatrix}=0$$

or $$\begin{vmatrix} 2ak & bk-c & ck-b & a \\ bk-c & 0 & -a & b \\ ck-b & -a & 0 & c \\ a & b & c & 0 \end{vmatrix}\lambda+2\Delta\begin{vmatrix} 2ak & bk-c & ck-b \\ bk-c & 0 & -a \\ ck-b & -a & 0 \end{vmatrix}=0$$

or $\quad -2abc(a\cos A+b\cos B+c\cos C)\lambda$

$$+2\Delta(k^2-2k\cos A+1)(-2abc)=0$$

$\therefore\ \lambda=-2\Delta(k^2-2k\cos A+1)/\Sigma a\cos A=-R(k^2-2k\cos A+1)$.

Hence $-\rho^2=\dfrac{abc}{4\Delta^2}\phi(x_0, y_0, z_0)$

or $\quad \rho^2=R^2(1-2k\cos A+k^2)$.

(6) *Find the equation of the director circle of the conic $ux^2+vy^2+wz^2=0$, the system of co-ordinates being trilinears.*

The equation of pair of tangents from (x_1, y_1, z_1) is

$$(ux^2+vy^2+wz^2)(ux_1^2+vy_1^2+wz_1^2)=(uxx_1+vyy_1+wzz_1)^2$$

or $$x^2u(vy_1^2+wz_1^2)+y^2v(ux_1^2+wz_1^2)+z^2w(ux_1^2+vy_1^2)$$
$$-2vwy_1z_1yz-2wuz_1x_1zx-2uvx_1y_1xy=0. \quad ......(i)$$

These tangents will be perpendicular, if their points at infinity are conjugate *w.r.* to I and J. If the co-ordinates of I and J be (x_0, y_0, z_0), (x_0', y_0', z_0')
$x_0x_0'=y_0y_0'=z_0z_0'=1$. $y_0z_0'+y_0'z_0=-2\cos A$ etc.

Thus the points I and J will be conjugate *w.r.* to the tangents if

$$u(vy_1^2+wz_1^2)+v(ux_1^2+wz_1^2)+w(ux_1^2+vy_1^2)$$
$$+2vwy_1z_1\cos A+2wuz_1x_1\cos B+2uvx_1y_1\cos C=0.$$

Thus the locus of (x_1, y_1, z_1) is the conic

$$u(v+w)x^2+v(w+u)y^2+w(u+v)z^2$$
$$+2vwyz\cos A+2wuzx\cos B+2uvxy\cos C=0.$$

(7) *Find the foci and directrices of the conic in Ex. 6.*

Suppose that the point (x_1, y_1, z_1) of Ex. 6 is a focus, the pair of tangents (i) satisfy the condition for a circle, hence

$$c^2(ux_1^2+wz_1^2)v+b^2(ux_1^2+vy_1^2)w+2bcvwy_1z_1$$
$$=a^2(ux_1^2+vy_1^2)w+c^2(vy_1^2+wz_1^2)u+2acwux_1z_1$$
$$=b^2(vy_1^2+wz_1^2)u+a^2(ux_1^2+wz_1^2)v+2abuvx_1y_1.$$

Thus, the foci are the intersections of the conics

$$ux^2(c^2v+b^2w)+vw(by+cz)^2=vy^2(a^2w+c^2u)+uw(ax+cz)^2$$
$$=wz^2(b^2u+a^2v)+uv(by+ax)^2.$$

Directrices. The pairs of directrices are the degenerate conics of the system generated by the given conic and the polar of I and J *w. r.* to the conic.

Now the polars of $I(x_0, y_0, z_0)$, $J(x_0', y_0', z_0')$ are respectively

$$(uxx_0+vyy_0+wzz_0)(uxx_0'+vyy_0'+wzz_0')$$
$$=u^2x^2+v^2y^2+w^2z^2-2vwyz\cos A-2wuzx\cos B$$
$$-2uvxy\cos C=0.$$

Thus the directrices are degenerate conics of the system

$$\Sigma u^2x^2-2\Sigma vwyz\cos A+\lambda(ux^2+vy^2+wz^2)=0. \quad ......(1)$$

The conic will degenerate if

$$\begin{vmatrix} u^2+\lambda u & -uv\cos C & -uw\cos B \\ -uv\cos C & v^2+\lambda v & -vw\cos A \\ -uw\cos B & -vw\cos A & w^2+\lambda w \end{vmatrix}=0$$

or if $$\begin{vmatrix} u+\lambda & -u\cos C & -u\cos B \\ -v\cos C & v+\lambda & -v\cos A \\ -w\cos B & -w\cos A & w+\lambda \end{vmatrix} =$$

or $\lambda^2+\lambda(u+v+w)+(vw\sin^2 A+wu\sin^2 B+uv\sin^2 C)=0.$

If λ be chosen to be a root of this equation, the equation (1) will represent a pair of lines. We thus get two pairs of directrices.

(8) *Find the equation of the circle with radius* ρ *and centre* $(\alpha_0, \beta_0, \gamma_0)$ *and prove thah the equation*

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(l\alpha+m\beta+n\gamma)(a\alpha+b\beta+c\gamma)=0$$

always represents a circle and find its centre and radius. Show further that the equation $\Sigma(u\alpha^2+2u'\beta\gamma)=0$ *will represent a circle if* $\quad b^2w+c^2v-2bcu'=c^2u+a^2w-2cav'=a^2v+b^2u-2abw'.$

(*i*) Obviously the equation is

$$a(\beta-\beta_0)(\gamma-\gamma_0)+b(\gamma-\gamma_0)(\alpha-\alpha_0)+c(\alpha-\alpha_0)(\beta-\beta_0) + \frac{4\Delta^2}{abc}\rho \;=0.$$

Making it homogeneous by means of the relation

$$a\alpha+b\beta+c\gamma=2\Delta,$$

we find that the equation can be put in the form

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta+(l\alpha+m\beta+n\gamma)(a\alpha+b\beta+c\gamma)=0.$$

(*ii*) If $f(\alpha,\beta,\gamma)\equiv u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$ and $(\alpha_0, \beta_0, \gamma_0)$ be the co-ordinates of any point O, then

$$f(\alpha,\beta,\gamma)=f(\overline{\alpha-\alpha_0}+\alpha_0, \overline{\beta-\beta_0}+\beta_0, \overline{\gamma-\gamma_0}+\gamma_0)$$

$$=f(\alpha-\alpha_0, \beta-\beta_0, \gamma-\gamma_0)+\left[(\alpha-\alpha_0)\frac{\partial f}{\partial\alpha_0} +(\beta-\beta_0)\frac{\partial f}{\partial\beta_0}+(\gamma-\gamma_0)\frac{\partial f}{\partial\gamma_0}\right]+f(\alpha_0, \beta_0, \gamma_0).$$

Applying this to the case under consideration

$$f(\alpha,\beta,\gamma)\equiv a\beta\gamma+b\gamma\alpha+c\alpha\beta+(l\alpha+m\beta+n\gamma)(a\alpha+b\beta+c\gamma)$$

and putting $\alpha_0=\mathrm{R}(l-m\cos C-n\cos B+\cos A)$

$$\beta_0=\mathrm{R}(-l\cos C+m-n\cos A+\cos B)$$

$$\gamma_0=\mathrm{R}(-l\cos B-m\cos A+n+\cos C)$$

we get

$$f(\alpha,\beta,\gamma)=a(\beta-\beta_0)(\gamma-\gamma_0)+b(\gamma-\gamma_0)(\alpha-\alpha_0)+c(\alpha-\alpha_0)(\beta-\beta_0) +f(\alpha_0, \beta_0, \gamma_0).$$

If $P(\alpha, \beta, \gamma)$ be on the locus, $f(\alpha, \beta, \gamma)=0$ and we have

$$OP^2 = \frac{abc}{4\Delta^2} f(\alpha_0, \beta_0, \gamma_0).$$

$\therefore$ OP is constant, *i. e.*, the locus is a circle and its radius is $\frac{1}{2\Delta}\sqrt{abcf(\alpha_0, \beta_0, \gamma_0)}$ and centre $(\alpha_0, \beta_0, \gamma_0)$.

If P does not lie on the circle whose radius is ρ, we have

$$f(\alpha, \beta, \gamma) = \frac{4\Delta^2}{abc}(\rho^2 - OP^2) = -\frac{4\Delta^2}{abc}PT^2$$

where PT is the tangent from P to the circle.

In particular, the square of the tangent from A to the circle is $-\frac{abc}{4\Delta^2} f\left(\frac{2\Delta}{a}, 0, 0\right)$, *viz.*, $-bcl$.

Similarly for tangents from B, C.

If t_1, t_2, t_3 be the tangents from A, B, C to the circle, its equation takes the form

$$abc(a\beta\gamma + b\gamma\alpha + c\alpha\beta) = (a\alpha + b\beta + c\gamma)(t_1^2 a\alpha + t_2^2 b\beta + t_3^2 c\gamma).$$

We now find $(\alpha_0, \beta_0, \gamma_0)$.

Any line is $\frac{\alpha - \alpha_0}{\lambda} = \frac{\beta - \beta_0}{\mu} = \frac{\gamma - \gamma_0}{v} = r$

where r is the distance between the points $O(\alpha_0, \beta_0, \gamma_0)$ and $P(\alpha, \beta, \gamma)$ with the conditions

$$a\lambda + b\mu + cv = 0, \ \frac{abc}{4\Delta^2}(a\mu v + bv\lambda + c\lambda\mu) = -1.$$

The line cuts the locus $f(\alpha, \beta, \gamma)=0$ in points whose distances from O are given by

$$f(\lambda, \mu, v)r^2 + \left[\lambda\frac{\partial f}{\partial \alpha_0} + \mu\frac{\partial f}{\partial \beta_0} + v\frac{\partial f}{\partial \gamma_0}\right]r + f(\alpha_0, \beta_0, \gamma_0) = 0,$$

$\therefore$ it cuts the locus in two points and the distance between them is bisected at $(\alpha_0, \beta_0, \gamma_0)$ for all values of λ, μ, v, if

$$\frac{u\alpha_0 + w'\beta_0 + v'\gamma_0}{a} = \frac{w'\alpha_0 + v\beta_0 + u'\gamma_0}{b} = \frac{v'\alpha_0 + u'\beta_0 + w\gamma_0}{c}$$

which determine $(\alpha_0, \beta_0, \gamma_0)$.

But if $\frac{\partial f}{\partial \alpha_0} = 0, \ \frac{\partial f}{\partial \beta_0} = 0, \ \frac{\partial f}{\partial \gamma_0} = 0$

then $\begin{vmatrix} u & w' & v' \\ w' & v & u' \\ v' & u' & w \end{vmatrix} = 0$ and the locus will be a pair of st. lines.

(*iii*) $f(\alpha, \beta, \gamma) \equiv u\alpha^2 + v\beta^2 + w\gamma^2 + 2u'\beta\gamma + 2v'\gamma\alpha + 2w'\alpha\beta = 0$ can be written as

$$f(\alpha, \beta, \gamma) \equiv \left(\frac{u}{a}\alpha + \frac{v}{b}\beta + \frac{w}{c}\gamma \right)(a\alpha + b\beta + c\gamma)$$
$$+ \frac{\beta\gamma}{bc}\left(2bcu' - vc^2 - wb^2\right) + \frac{\gamma\alpha}{ca}(2cav' - wa^2 - uc^2)$$
$$+ \frac{\alpha\beta}{ab}(2abw' - va^2 - ub^2).$$

Thus the necessary and sufficient conditions that $f(\alpha, \beta, \gamma) = 0$ should be a circle are

$$2bcu' - vc^2 - wb^2 = 2cav' - wa^2 - uc^2 = 2abw' - va^2 - ub^2.$$

Exercises XLIII

1. If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be a trial of non-collinear points, show that the co-ordinates of the centre of gravity are $\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3} \right)$, the co-ordinates of the points being actual.

2. On the sides BC, CA, AB of a triangle ABC, points D, E, F are taken such that BD : DC = CE : EA = AF : FB.

Prove that the centroid of the triangle formed by AD, BE, CF coincides with that of the triangle ABC.

3. If $O(x', y', z')$ be the point of intersection of AD, BE, CF, where D, E, F lie on the sides BC, CA, AB of the triangle of reference, prove that the equations to EF, FD, DE respectively are

$$\frac{-x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0, \quad \frac{x}{x'} - \frac{y}{y'} + \frac{z}{z'} = 0, \quad \frac{x}{x'} + \frac{y}{y'} - \frac{z}{z'} = 0.$$

4. In Ex. 3, prove that the lines BC and EF, CA and FD, AB and DE meet in three collinear points which lie on the line

$$\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0.$$

5. Prove that the straight lines $a^2x + b^2y + c^2z = 0$ and $x \cos A + y \cos B + z \cos C = 0$ are parallel (trilinear system).

6. If the system of co-ordinates is trilinear, show that the line $x(\cos B - \cos C) + y(\cos C - \cos A) + z(\cos A - \cos B) = 0$ joins the incentre and the circum-centre of the triangle of reference. Prove also, that it is perpendicular to the unit line $x + y + z = 0$.

7. Show that the point at infinity on the straight line $lx + my + nz = 0$ is $(mc - nb, na - lc, lb - ma)$ in trilinears and $(m - n, n - l, l - m)$ in areals.

8. Show that the line parallel to $lx+my+nz=0$ and passing through (x', y', z') is given by the equation

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ mc-nb & na-lc & lb-ma \end{vmatrix}=0 \text{ (trilinears)}, \quad \begin{vmatrix} x & y & z \\ x' & y' & z' \\ m-n & n-l & l-m \end{vmatrix}=0 \text{ (areals)}.$$

9. The vertices of a triangle move along three fixed concurrent lines and two of its sides pass each through a fixed point. Show that the third side passes through a fixed point collinear with the two given points.

10. Show that if θ is variable, the straight line $x\sin(\alpha+\theta)+y\sin(\beta+\theta)+z\sin(\gamma+\theta)=0$ passes through a fixed point.

11. Show that if θ is variable, the line

$$x\sin(A-\theta)+y\sin(B-\theta)+z\sin(C-\theta)=0$$

in trilinear co-ordinates is parallel to a fixed direction.

12. The three external bisectors of the angles of a triangle meet the opposite sides in three points on a st. line, which is perpendicular to the line through the in-centre and the circumcentre (see Ex. 6.)

13. The equation of the line through the mid-points of the three diagonals of the quadrilateral formed by the lines

$$lx\pm my\pm nz=0 \text{ is } \frac{l^2x}{a}+\frac{m^2y}{b}+\frac{n^2z}{c}=0 \text{ (Trilinears).}$$

14. The lines joining the vertices A, B, C of an equilateral triangle to a point P meet the sides opposite A, B, C in A′, B′, C′ respectively; prove that if BA′+CB′+AC′=A′C+B′A+C′B, then P lies on one of the medians of the triangle.

15. Points D, E, F are taken on the sides of a triangle ABC such that AD, BE and CF are concurrent and L, M, N are the middle points of EF, FD and DE respectively, show that AL, BM and CN meet in a point.

16. If parallels to $x=0$, $y=0$ be drawn through the intersection of $lx+my+nz=0$ and $l'x+m'y+n'z=0$, show that the condition in trilinears that the four lines will form a harmonic pencil is

$$c^2(lm'+l'm)+2abnn'=bc(nl'+n'l)+ca(mn'+m'n).$$

17. Show that if $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

$$\equiv(lx+my+nz)(l'x+m'y+n'z)=0$$

then $(mn'-m'n)(gh-af)=(nl'-n'l)(hf-bg)=(lm'-l'm)(fg-ch)$.

18. Show that the equation in trilinears of the circular points at infinity can be written in either of the forms

$$(cm-bn)^2 \sin 2A + (an-cl)^2 \sin 2B + (bl-am)^2 \sin 2C = 0\ ;$$
$$a(an-cl)(bl-am)+b(cm-bn)(bl-am)+c(cm-bn)(an-cl)=0.$$

19. Prove that in the trilinear system, the lines
$$(a+\lambda)x+(b+\lambda)y+cz=0,\ (a+\lambda)x+(b-\lambda)y+cz=0$$
are perpendicular for all values of λ.

20. If (x', y', z'), (x'', y'', z'') are the points at infinity on two orthogonal lines in the trilinear system, then

(*i*) $x'x'' \sin 2A + y'y'' \sin 2B + z'z'' \sin 2C = 0.$
(*ii*) $a(y'z''+y''z')+b(z'x''+z''x')+c(x'y''+x''y')=0.$

21. Show that the conditions that the conics $ux^2+vy^2+wz^2=0$ and $wz^2+2hxy=0$ may be parabolas are respectively
$$\frac{a^2}{u}+\frac{b^2}{v}+\frac{c^2}{w}=0 \text{ and } hc^2+2wab=0.$$

22. Prove that the co-ordinates of the centre of the conic
$$\sqrt{ux}+\sqrt{vy}+\sqrt{wz}=0$$
are given by
$$\frac{x}{bw+cv}=\frac{y}{cu+aw}=\frac{z}{av+bu},$$
the co-ordinates being trilinears.

23. Show that $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$
$$\text{and}\quad \frac{1}{-lx+my+nz}+\frac{1}{lx-my+nz}+\frac{1}{lx+my-nz}=0$$
represent the same conic. Explain the fact geometrically.

24. Show that $\dfrac{l}{x}+\dfrac{m}{y}+\dfrac{n}{z}=0$ and
$$\sqrt{\frac{y}{m}+\frac{z}{n}}+\sqrt{\frac{z}{n}+\frac{x}{l}}+\sqrt{\frac{x}{l}+\frac{y}{m}}=0,$$ represent the same conic. Explain the fact geometrically.

25. A variable conic touches the sides of the triangle of reference and also the line $lx+my+nz=0$. Prove that if the co-ordinate system is trilinear, the locus of the centre is the line
$$ax\left(-\frac{a}{l}+\frac{b}{m}+\frac{c}{n}\right)+by\left(\frac{a}{l}-\frac{b}{m}+\frac{c}{n}\right)$$
$$+cz\left(\frac{a}{l}+\frac{b}{m}-\frac{c}{n}\right)=0.$$

26. The conic $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ meets the sides of the triangle of reference in three point-pairs, and these on being joined to the opposite angular points of the triangle determine three line-pairs; prove that the six lines thus constructed touch the conic
$$bcl^2+cam^2+abn^2-2afmn-2bgnl-2chlm=0.$$

27. Prove that the centres of the circles which touch the sides of a triangle lie on any rectangular hyperbola *w. r.* to which the given triangle is self conjugate.

28. If $\phi(l, m, n)=0$ is the tangential equation of a conic, show that the equation

$$\phi(yz_0-y_0z,\ zx_0-z_0x,\ xy_0-x_0y)=0.$$

represents the pair of tangents from (x_0, y_0, z_0) to the conic ϕ.

29. A parabola circumscribes a triangle ABC and has its focus at the ortho-centre, prove that

$$\frac{\cos\frac{A}{2}}{\sqrt{\cos A}}+\frac{\cos\frac{B}{2}}{\sqrt{\cos B}}+\frac{\cos\frac{C}{2}}{\sqrt{\cos C}}=0.$$

30. Prove that, in the trilinear system, the normals to the conic $lyz+mzx+nxy=0$ at the three points of reference will meet in a point if

$$\frac{l}{a}(m^2-n^2)+\frac{m}{b}(n^2-l^2)+\frac{n}{c}(l^2-m^2)=0.$$

31. If the equation $\sqrt{lx}+\sqrt{my}+\sqrt{nz}=0$ in trilinears represents a parabola, the equation of its axis is

$$\frac{a^2x}{l}\left(\frac{b^4}{m^2}-\frac{c^4}{n^2}\right)+\frac{b^2y}{m}\left(\frac{c^4}{n^2}-\frac{a^4}{l^2}\right)+\frac{c^2z}{n}\left(\frac{a^4}{l^2}-\frac{b^4}{m^2}\right)=0$$

32. The directrix of the parabola which touches the sides of the triangle of reference and also the line $lx+my+nz=0$ is

$$x\cos A\left(\frac{b}{m}-\frac{c}{n}\right)+y\cos B\left(\frac{c}{n}-\frac{a}{l}\right)+z\cos C\left(\frac{a}{l}-\frac{b}{m}\right)=0;$$

the co-ordinate system being trilinear.

33. Prove that the director circle, in trilinears, of the conic $x^2+4yz\cos A=0$ is given by the equation

$$x^2+y^2+z^2-yz(\sec A-2\cos A)-a\sec A(y\cos B+z\cos C)=0.$$

34. Prove that the conics

$$a\sqrt{x\tan\theta}+b\sqrt{y\tan\phi}+c\sqrt{z\tan\psi}=0,$$
$$a\sqrt{x\cot\theta}+b\sqrt{y\cot\phi}+c\sqrt{z\cot\psi}=0,$$

are parabolas if

$$a\tan\theta+b\tan\phi+c\tan\psi=0$$
$$a\cot\theta+b\cot\phi+c\cot\psi=0$$

the co-ordinates being trilinears. Show also that a circle can be described through their six points of contact with the sides of the triangle of reference.

35. Show that the locus of the centres of rectangular hyperbolas which circumscribe the triangle of reference is the nine-point circle.

36. Show that the directrix of the parabola which touches the four lines $[l, \pm m, \pm n]$ has for its equation in trilinears

$$\Sigma \frac{x}{a}\left[l^2(b^2-c^2)-a^2(m^2-n^2)\right]=0.$$

37. In the trilinear system of co-ordinates, prove that the co-ordinates of the foci of the ellipse

$\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=0$ are respectively proportional to $\left(\frac{c}{b}, \frac{a}{c}, \frac{b}{a}\right), \left(\frac{b}{c}, \frac{c}{a}, \frac{a}{b}\right)$.

38. Prove that the foci of parabolas inscribed in the triangle of reference lie on the circum-circle of the triangle.

39. Prove that the parabola whose focus is the vertex C of the triangle of reference and whose directrix is the side AB has for its trilinear equation $x^2+y^2+2xy \cos C=z^2 \sin^2 C$.

40. Show that the equation $bcxy-abyz+(c^2-a^2)zx=0$ represents a rectangular hyperbola, and show that the equations of its asymptotes are

$$c(ax-by-cz)\pm a(ax+by-cz)=0.$$

41. If the triangle of reference is equilateral, show that $y+z+3x=0$ is a directrix of the conic $y^2+z^2-3x^2=0$ (trilinears).

[*Hint.*—Show that the pole $(-1, 1, 1)$ of the line $y+z+3x=0$ is a focus.]

CHAPTER XIV

SYSTEMS OF CONICS

152. Pencil of conics.

If $\phi \equiv ax^2+by^2+2hxy+2gx+2fy+c=0$,
$\psi \equiv a'x^2+b'y^2+2h'xy+2g'x+2f'y+c'=0$
be two conics, we have proved that the equation

$$\phi+\lambda\psi=0 \qquad \ldots\ldots(1)$$

represents for different values of λ a single infinity of conics which go through the four points of intersection of ϕ and ψ. This system of conics we call a *pencil of conics*, of which ϕ and ψ are called the base conics. Through any fifth point there will pass one and only one conic of the system.

If we put down the condition that the line

$$lx+my+nz=0$$

may be a tangent to the conic (1), we obtain a quadratic in λ, showing that *there exist two conics of the system which touch an arbitrary line of the plane.*

The conic (1) will degenerate into a pair of lines if

$$\begin{vmatrix} a+\lambda a' & h+\lambda h' & g+\lambda g' \\ h+\lambda h' & b+\lambda b' & f+\lambda f' \\ g+\lambda g' & f+\lambda f' & c+\lambda c' \end{vmatrix}=0$$

This is a cubic in λ, showing that three line-pairs belong to the system.

152.1. *A pencil of conics determines on an arbitrary line a range of points belonging to an involution, the double points of which are the points where the line touches the two members of the system.*

Take the given line as the x-axis. The abscissae of the points of intersection of (1) with $y=0$ are given by the equation

$$(ax^2+2gx+c)+\lambda(a'x^2+2g'x+c')=0 \qquad \ldots\ldots(2)$$

and this defines an involution. The line $y=0$ will touch (1) if λ is a root of the equation

$$(g+\lambda g')^2=(a+\lambda a')(c+\lambda c').$$

If λ be a root of this equation, (2) will be a perfect square, which proves the theorem.

The double points *i.e.*, the points of contact are conjugate with respect to every conic of the system.

152.2 Let $S_1 = \phi + \lambda_1\psi = 0$, $S_2 = \phi + \lambda_2\psi = 0$ be two conics of the system and $S \equiv \phi + \lambda\psi = 0$ an arbitrary conic of the system. Elimination of ϕ and ψ gives

$$\begin{vmatrix} S_1 & \lambda_1 & 1 \\ S_2 & \lambda_2 & 1 \\ S & \lambda & 1 \end{vmatrix} = 0$$

or $S = \dfrac{\lambda - \lambda_2}{\lambda_1 - \lambda_2} S_1 + \dfrac{\lambda_1 - \lambda}{\lambda_1 - \lambda_2} S_2$

This expresses S in terms of S_1 and S_2. Thus *two arbitrary conics of the system can be taken as the base conics.*

152.3. The conic (1) will be a circle if

$$a + \lambda a' = b + \lambda b' \quad \text{and} \quad h + \lambda h' = 0$$

$$\therefore \quad \frac{a-b}{h} = \frac{a'-b'}{h'} \qquad \ldots\ldots(3)$$

Now the asymptotes of ϕ are parallel to

$$ax^2 + 2hxy + by^2 = 0,$$

and its axes are therefore parallel to

$$\frac{x^2 - y^2}{a-b} = \frac{xy}{h}$$

Similarly the axes of ψ are parallel to

$$\frac{x^2 - y^2}{a'-b'} = \frac{xy}{h'}$$

Thus (3) is the condition that the axes of ϕ and ψ may be parallel.

Thus *a circle is, in general, not a member of the system, and if it is a member of the system, the conics have their axes parallel.*

152.4. *One and only one rectangular hyperbola belongs to the system, unless every conic of the system is a rectangular hyperbola.*

The conic (1) is a rectangular hyperbola if

$$(a+b) + \lambda(a'+b') = 0$$

This relation is identically satisfied if

$$a + b = 0, \; a' + b' = 0.$$

i.e., if ϕ and ψ are rectangular hyperbolas, then every conic of (1) is a rectangular hyperbola.

If $a+b$, $a'+b'$ do not vanish simultaneously, there is only one value of λ. Hence, etc.

152.5. Let the asymptotes of the rectangular hyperbola of the system be taken as the axes, so that

$$\psi \equiv xy - k^2 = 0,$$

the equation (1) then becomes

$$ax^2+2hxy+by^2+2gx+2fy+c+2\lambda(xy-k^2)=0.$$

If $\lambda=-h$, the equation becomes

$$ax^2+by^2+2gx+2fy+c+2hk^2=0$$

and this conic has its axes parallel to the co-ordinate axes *i.e.*, the asymptotes of the rectangular hyperbola of the system.

Thus there is one conic of the system which has its axes parallel to asymptotes of the rectangular hyperbola of the system.

Consequently, if we take

$$\phi\equiv ax^2+by^2+2gx+2fy+k, \quad \psi\equiv xy-c^2,$$

the equation of the system is

$$ax^2+2\lambda xy+by^2+2gx+2fy+k-2\lambda c^2=0 \tag{4}$$

This is the simplest equation of the system in *rectangular* co-ordinates.

152.6. The equation (4) will represent a parabola if

$$\lambda^2=ab.$$

Thus there are two parabolas of the system, *viz*

$$(x\sqrt{a}\pm y\sqrt{b})^2+2gx+2fy+k\mp 2c^2\sqrt{ab}=0.$$

The axes of these parabolas are parallel to $x\sqrt{a}+y\sqrt{b}=0$ and $x\sqrt{a}-y\sqrt{b}=0$ respectively *i.e.*, $ax^2-by^2=0$. These lines are harmonically separated by the lines

$$ax^2+2\lambda xy+by^2=0$$

which are parallel to the asymptotes of (4). Thus the axes of the parabolas belonging to the system are parallel to a pair of conjugate diameters of (4).

153. Centre locus of the Pencil of conics.

The equation of the system of conics has been reduced to the form

$$ax^2+by^2+2gx+2fy+2\lambda(xy-c^2)=0.$$

The co-ordinates of the centre of the conic are given by the equations

$$ax+g+\lambda y=0 \quad \lambda x+by+f=0.$$

The locus of the centre is therefore

$$\frac{ax+g}{by+f}=\frac{y}{x}$$

or $$ax^2-by^2+gx-fy=0. \qquad \ldots\ldots(5)$$

The centre locus is therefore a conic. It is an ellipse or a hyperbola according as the base conic

$$ax^2+by^2+2gx+2fy+k=0$$

is a hyperbola or an ellipse. The asymptotes of the conic are parallel to the axes of the two parabolas of the system, and to a pair of conjugate diameters of (4).

Or thus. If $\phi=0$, $\psi=0$ are two conics of the system, the equation of the pencil is

$$\phi+\lambda\psi=0.$$

The co-ordinates of the centre are given by the equations

$$\frac{\partial\phi}{\partial x}+\lambda\frac{\partial\psi}{\partial x}=0,\quad \frac{\partial\phi}{\partial y}+\lambda\frac{\partial\psi}{\partial y}=0.$$

Hence the locus of the centre is the conic

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial y}-\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial x}=0.$$

153.1. The centre of the centre-locus is the point $\left(\frac{-g}{2a},\ \frac{-f}{2b}\right)$ and this is the mid-point of the segment of the line that joins the centre$(0, 0)$ of the rectangular hyperbola of the system and the centre of the unique conic whose axes are parallel to the asymptotes of the rectangular hyperbola.

153.2. The base points *i. e.*, the points common to all conics of the system are the intersections of

$$ax^2+by^2+2gx+2fy+k=0$$

and $\qquad xy=c^2 \quad$ or $x=ct,\ y=\frac{c}{t}$,

thus the parameters of the base-points are the roots of the equation

$$ac^2t^4+2gct^3+kt^2+2fct+bc^2=0. \qquad \text{......(6)}$$

Suppose that the roots of this equation are t_1, t_2, t_3, t_4.

Now $\qquad \frac{1}{4}\Sigma ct_1=\frac{-g}{2a},\ \frac{1}{4}\Sigma\frac{c}{t_1}=\frac{-f}{2b}$.

Thus the centre of the centre-locus is the centre of mean position of the four base-points.

153.3. The co-ordinates of the mid-point of the line joining (t_1) and (t_2) are $\left[\frac{c}{2}(t_1+t_2),\ \frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\right]$. Substituting in the left-hand side of (5) we see that

$$\frac{ac}{2}(t_1+t_2)\left[c\frac{t_1+t_2}{2}+\frac{g}{a}\right]-bc\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left[\frac{c}{2}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)+\frac{f}{b}\right]$$

$$=-\frac{ac^2}{4}(t_1+t_2)(t_3+t_4)+\frac{bc^2}{4}\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t_3}+\frac{1}{t_4}\right)=0$$

$$\left[\text{ for } \Sigma ct_1 = \frac{-2g}{a},\ \ \Sigma\frac{c}{t_1} = \frac{-2f}{b} \text{ and } t_1t_2t_3t_4 = \frac{b}{a}\right].$$

Thus the centre locus passes through the six-mid points of the lines which join the four base-points.

153.4. The equations of the lines that join t_1, t_2 and t_3, t_4 are

$$\left.\begin{aligned} x + t_1t_2y &= c(t_1+t_2) \\ x + t_3t_4y &= c(t_3+t_4) \end{aligned}\right\} \qquad \ldots\ldots(7)$$

The line $2x + y(t_1t_2+t_3t_4) = c(t_1+t_2+t_3+t_4)$ which assumes the form

$$2\left(x + \frac{g}{a}\right) + y(t_1t_2+t_3t_4) = 0 \qquad \ldots\ldots(8)$$

passes through the intersection of the lines (7).

Similarly the line $x(t_1t_2+t_3t_4) + 2yt_1t_2t_3t_4 - c\Sigma t_1t_2t_3 = 0$ which reduces to

$$x(t_1t_2+t_3t_4) + \frac{2}{a}(by+f) = 0 \qquad \ldots\ldots(9)$$

also passes through the intersection of the lines (7). But the lines (8) and (9) intersect on the conic

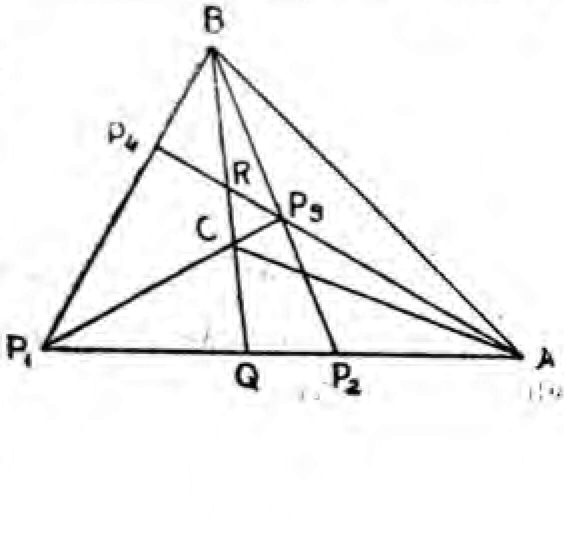
$$\frac{ax+g}{by+f} = \frac{y}{x}$$

which is the equation of the centre-locus. Thus the three line-pairs of the system intersect on the centre locus.

The result also follows from the consideration, that the line-pairs are conics of the system whose centres are their intersections.

The centre locus, therefore, passes through the six mid-points of the lines that join the four base-points, and the three intersections of the three line pairs of the system and the two points on the line at ∞ conjugate *w. r.* to every conic of the system. The centre-locus is therefore called the **eleven-point-conic**.

153.5. *The points of intersection of the line-pairs of the system form a triangle, which is self-conjugate w. r. to every conic of the system.*



1st method.

Let P_1, P_2, P_3, P_4 be the base-points. Take AP_2P_1 as the x-axis and AP_3P_4 as the y-axis, so that the co-ordinates of P_1, P_2, P_3, P_4 may be

taken as $\left(\frac{1}{a},0\right)$, $\left(\frac{1}{a'},0\right)$, $\left(0,\frac{1}{b}\right)$, $\left(0,\frac{1}{b'}\right)$. The equations of P_2P_3, P_1P_4 are respectively

$$a'x+by-1=0, \qquad ax+b'y-1=0.$$

The equation of the system of conics through P_1, P_2, P_3, P_4 is

$$(ax+b'y-1)(a'x+by-1)+\lambda xy=0.$$

The polar of A *w. r.* to the system is

$$(ax+b'y-1)+(a'x+by-1)=0$$

which obviously passes through B, the intersection of P_1P_4, and P_2P_3. Writing the equation of the polar as

$$(ax+by-1)+(a'x+b'y-1)=0,$$

it is also seen that it passes through C. Thus the polar is the line BC.

2nd method.

Let BC cut P_1P_2, P_3P_4, in Q, R. Then

$$(P_1P_2, QA)=-1, \ (P_3P_4, RA)=-1.$$

$\therefore$ the polar of A goes through Q, R *i. e.*, BC is the polar of A.

Similarly the polars of B and C are CA and AB.

Hence $\triangle$ ABC is self-conjugate *w. r.* to each conic of the system.

3rd method.

Let $P_r\left(ct_r, \frac{c}{t_r}\right)$ be the base-points. The equations of P_1P_2, P_3P_4 are

$$x+t_1t_2y=c(t_1+t_2), \qquad x+t_3t_4y=c(t_3+t_4).$$

The polar *w. r.* to (4) of the point $A(x',y')$ the intersection of P_1P_2, P_3P_4 is

$$x'(ax+\lambda y+g)+y'(\lambda x+by+f)+gx+fy-2\lambda c^2+k=0$$

with the conditions

$$x'+t_1t_2y'-c(t_1+t_2)=0$$
$$x'+t_3t_4y'-c(t_3+t_4)=0.$$

Thus the equation of the polar is

$$\begin{vmatrix} ax+g+\lambda y & 1 & 1 \\ by+f+\lambda x & t_1t_2 & t_3t_4 \\ gx+fy+k-2\lambda c^2 & -c(t_1+t_2) & -c(t_3+t_4) \end{vmatrix}=0$$

which reduces to

$$\begin{vmatrix} x & t_1t_2 & t_3t_4 \\ y & 1 & 1 \\ 2c & t_1+t_2 & t_3+t_4 \end{vmatrix}=0$$

remembering that t_1, t_2, t_3, t_4 are the roots of (6).

Writing this equation in the forms

$$(t_3-t_1)[x+yt_2t_4-c(t_2+t_4)]+(t_4-t_2)[x+yt_1t_3-c(t_1+t_3)]=0$$
$$(t_4-t_1)[x+yt_2t_3-c(t_2+t_3)]+(t_3-t_2)[x+yt_1t_4-c(t_1+t_4)]=0$$

we see that the polar passes through B and C and is therefore the line BC.

153.6. *By a proper choice of the triangle of reference, the co-ordinates of four points in a plane can be represented by* $(\pm 1, \pm 1, \pm 1)$.

Let the diagonal triangle ABC of the quadrangle $P_1P_2P_3P_4$ (See fig. art. 153.5) be taken as the triangle of reference, and suppose P_1 is the unit point (1, 1, 1). The co-ordinates of P_2, P_3, P_4 can therefore be assumed to be

$$(\lambda, 1, 1), (1, \mu, 1), (1, 1, v)$$

Now A, P_3, P_4 are collinear.

$$\therefore \quad \begin{vmatrix} 1 & 0 & 0 \\ 1 & \mu & 1 \\ 1 & 1 & v \end{vmatrix}=0 \quad \text{or} \quad \mu v=1.$$

Similarly $v\lambda=1$. $\lambda\mu=1$. Hence $\lambda^2\mu^2v^2=1$ or $\lambda\mu v=-1$, the positive sign is rejected. as it gives $\lambda=\mu=v=1$ *i.e.*, the four points coincide. Hence

$$\lambda=-1, \mu=-1, v=-1.$$

Hence the co-ordinates of the points are $(\pm 1, \pm 1, \pm 1)$.

Exercises XLIV.

1. Show that the equation of a pencil of conics can be written as

$$ux^2+vy^2+wz^2=0 \quad u+v+w=0.$$

2. If $px+qy+rz=0$ be the equation of the line at infinity, when the diagonal triangle is taken as the triangle of reference, show that the equation of the centre locus of the system of conics in Ex. 1, is

$$\frac{p}{x}+\frac{q}{y}+\frac{r}{z}=0.$$

3. Show that the equation of a conic through four given points is $u\alpha^2+v\beta^2+w\gamma^2=0$, (trilinears) with the condition

$$uf^2+vg^2+wh^2=0.$$

The co-ordinates of four points can be taken as $(\pm f, \pm g, \pm h)$.

154. Range of conics.

The tangential equation of a conic contains five disposable constants. The equation can, therefore, be made to satisfy five given conditions, in particular, one and only one conic can be made to touch five given lines. If however, only four lines are given, four of the constants can be expressed in terms of the fifth. Thus there will be a single infinity of conics which touch four given lines. The system of conics so determined is called a **Tangential Pencil** or a **Range of conics.**

154.1. *To find the equation of a range of conics.*

Let two of the lines be chosen as the co-ordinate axes, and suppose that the equations of the other two lines are

$$lx+my-1=0,\ l'x+m'y-1=0.$$

The equation of the conic which touches the co-ordinate axes is of the form

$$(ax+by-1)^2-2\lambda xy=0.$$

The lines which join the origin with the points of intersection of this conic and the line $lx+my-1=0$ are given by the equation

$$(ax+by-lx-my)^2-2\lambda xy=0$$

or $\quad (a-l)^2x^2+2xy[(a-l)(b-m)-\lambda]+(b-m)^2y^2=0.$

If the line be a tangent, these two lines will coincide:

$\therefore \quad [(a-l)(b-m)-\lambda]^2=(a-l)^2(b-m)^2$

or $\quad \lambda=2(a-l)(b-m),$

for $\lambda=0$ gives the coincident lines

$$ax+by-1=0.$$

Similarly the line $l'x+m'y-1=0$ will touch the conic if

$$\lambda=2(a-l')(b-m').$$

Thus the equation is

$$\left.\begin{aligned} &(ax+by-1)^2-2\lambda xy=0\\ \text{provided}\quad &\lambda=2(a-l)(b-m)=2(a-l')(b-m')\end{aligned}\right\} \quad (10)$$

The equation may be written so as to contain one parameter. For writing the condition in (10) as

$$\frac{a-l}{a-l'}=\frac{b-m'}{b-m}=t,$$

we get $\quad a=\dfrac{l-l't}{1-t},\ b=\dfrac{m'-mt}{1-t}$

and $\quad \lambda=2(a-l)(b-m)=-2\dfrac{(l-l')(m-m')t}{(1-t)^2}.$

Thus the equation becomes

$$[(lx+m'y-1)-t(l'x+my-1)]^2+4(l-l')(m-m')xyt=0 \quad \ldots (11)$$

Given (x, y), there are two values of t, showing that through a point there pass two conics of the system.

We may write the equation as

$$[(lx+m'y-1)-t(l'x+my-1)]^2+t[(l-l')x+(m-m')y]^2$$
$$-t[(l-l')x-(m-m')y]^2=0.$$

Put $lx+m'y-1=\mathrm{L}$, $l'x+my-1=\mathrm{M}$, $(l-l')x+(m-m')y=\mathrm{N}$, the equation can therefore be written as

$$(\mathrm{L}-\mathrm{M}t)^2+t\mathrm{N}^2-(\mathrm{L}-\mathrm{M})^2t=0.$$

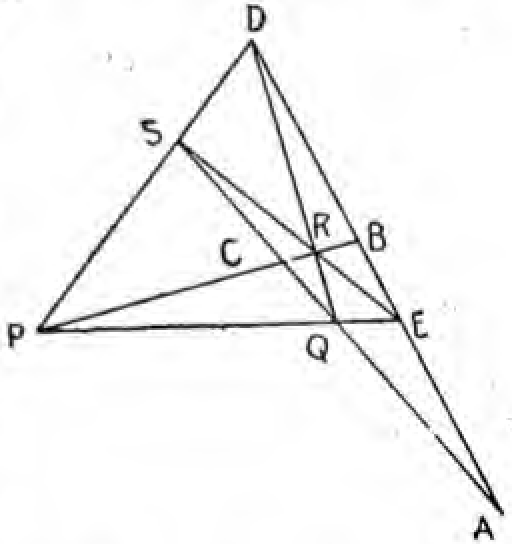
or $\quad \mathrm{L}^2(1-t)+\mathrm{M}^2t(t-1)+t\mathrm{N}^2=0.$

or $\quad \lambda_1\mathrm{L}^2+\lambda_2\mathrm{M}^2+\lambda_3\mathrm{N}^2=0.$

where $\quad \dfrac{1}{\lambda_1}+\dfrac{1}{\lambda_2}+\dfrac{1}{\lambda_3}=0.$

Ex. 1. Show that the centre locus of the range of conics is a st. line.

Ex. 2. Show that the tangents from a point to a range of conics belong to an involution and the double lines of the involution are the tangents to the two conics which pass through the point.



154.2. *The equations of four arbitrary straight lines can be represented by the equations.*

$$\mathrm{U} \pm \mathrm{V} \pm \mathrm{W}=0$$

where U, V, W *are linear functions of* x, y *and* z.

(z may be put equal to 1).

Let PQ, QR, RS, SP be four st. lines whose diagonal triangle is ABC. Suppose that the equations of BC, CA, AB are respectively

$$\mathrm{U}=0,\ \mathrm{V}=0,\ \mathrm{W}=0.$$

Since the equations of four straight lines are linearly connected, we can assume the equation of PS to be

$$\mathrm{U}+\mathrm{V}+\mathrm{W}=0.$$

Thus QR passes through the intersection of PS and AB(W=0), its equation is therefore of the form

$$\mathrm{U}+\mathrm{V}+v\mathrm{W}=0.$$

Similarly the equations of PQ, RS are of the forms

$$\lambda U+V+W=0$$

$$U+\mu V+W=0.$$

Now QR, SR, BC are concurrent.

$$\therefore \quad \begin{vmatrix} 1 & \mu & 1 \\ 1 & 1 & v \\ 1 & 0 & 0 \end{vmatrix}=0$$

i. e. $\mu v=1.$

Similarly $v\lambda=1, \quad \lambda\mu=1.$

$$\therefore \quad \lambda^2\mu^2v^2=1 \quad \text{or} \quad \lambda\mu v=\pm 1.$$

If the positive sign be taken, we get $\lambda=\mu=v=1$, *i. e.* all the lines coincide. Hence $\lambda=-1, \mu=-1, v=-1$. Hence the equations of the lines can be written in the form

$$U\pm V\pm W=0.$$

154.3. *To determine the equation of conics which touch the four lines* $U\pm V\pm W=0$.

The equation of the conic which touches $U+V\pm W=0$. can be written in the form

$$(aU+bV+cW)^2-\lambda[(U+V)^2-W^2]=0.$$

The lines which join C with the points of intersection of the conic and $U-V-W=0$ is

$$[(a+c)U+(b-c)V]^2-\lambda[(U+V)^2-(U-V)^2]=0.$$

$$\text{or } (a+c)^2U^2+2UV[(a+c)(b-c)-2\lambda]+(b-c)^2V^2=0.$$

If the line $U-V-W=0$ is a tangent

$$\lambda=(a+c)(b-c).$$

Similarly $U-V+W=0$ will be a tangent if

$$\lambda=(a-c)(b+c)$$

whence $2c(a-b)=0.$

If $a=b$, the conic reduces to two coincident st. lines, hence $c=0$, $\lambda=ab$. Thus the equation of the conic becomes

$$(aU+bV)^2-ab(U^2+V^2+2UV-W^2)=0,$$

or $$a(a-b)U^2+b(b-a)V^2+abW^2=0,$$

or $$\left(\frac{a}{b}-1\right)U^2+\left(\frac{b}{a}-1\right)V^2+W^2=0,$$

or $$(k-1)U^2+\left(\frac{1}{k}-1\right)V^2+W^2=0, \; k=\frac{a}{b}; \quad \text{......(12)}$$

or $$\lambda_1U^2+\lambda_2V^2+\lambda_3W^2=0,$$

where $$\lambda_2\lambda_3+\lambda_3\lambda_1+\lambda_1\lambda_2=0. \quad \text{......(13)}$$

Note. In order that the conic may contain real points, one of the terms must be negative.

154.4. If we write the equation of the conic

$$\lambda^2U^2+\mu^2V^2-v^2W^2=0$$

in the form

$$(\mu V+vW)(\mu V-vW)+\lambda^2U^2=0$$

we see that the line $U=0$ is the polar of A. Similarly $V=0$ is the polar of B and $W=0$ is the polar of C. Thus ABC is a self-polar triangle of every conic inscribed in the quadrilateral.

We may deduce the result from Art 153.5. or from equation (11). If PQ, PS be the axes, and the equations of QR, RS are respectively $lx+my-1=0$, $l'x+m'y-1=0$, the co-ordinates of Q and E are respectively $\left(\frac{1}{l}, 0\right)$, $\left(\frac{1}{l'}, 0\right)$ and those of S and D are $\left(0, \frac{1}{m'}\right)$, $\left(0, \frac{1}{m}\right)$. The equations of AB and AC are respectively

$$L\equiv l'x+my-1=0\ ;\ M\equiv lx+m'y-1=0.$$

Let these intersect in $A(x', y')$. The polar of A *w. r.* to (11) is

$$x[(L'-tM')(l-tl')+2(l-l')(m-m')y't]$$
$$+y[(L'-tM')(m'-tm)+2(l-l')(m-m')y't]$$
$$+[(L'-tM')(t-1)]=0$$

where L', M' are the values of L and M for $x=x'$, $y=y'$ and therefore vanish. Thus the equation reduces to

$$xy'+yx'=0$$

or $\qquad (l-l')x+(m-m')y=0.$

This obviously passes through $P(0, 0)$ and through R as it can be written in the form

$$(lx+my-1)-(l'x+m'y-1)=0.$$

Thus *the diagonal triangle of a quadrilateral is self-conjugate w. r. to every conic inscribed in the quadrilateral.*

154.5. Take the triangle formed by the diagonals of the quadrilateral as the triangle of reference. The equations of the four lines will be of the form $l\alpha\pm m\beta\pm n\gamma=0$. (trilinears). The conic

$$u\alpha^2+v\beta^2+w\gamma^2+2u'\beta\gamma+2v'\gamma\alpha+2w'\alpha\beta=0$$

will touch the line $[l, m, n]$ if

$$Ul^2+Vm^2+Wn^2+2U'mn+2V'nl+2W'lm=0.$$

If the conic touches all the four lines, then

$$U'=V'=W'=0$$

$$\therefore \qquad u'=v'=w'=0.$$

Thus the equation of the range is $u\alpha^2+v\beta^2+w\gamma^2=0$ with the condition $l^2/u+m^2/v+n^2/w=0$.

155. Centre Locus of a Range of conics..

Let the sides of the diagonal triangle be

$U\equiv a_1x+b_1y+c_1=0$, $V\equiv a_2x+b_2y+c_2=0$, $W\equiv a_3x+b_3y+c_3=0$, then the equation of the range of conics which touches $U\pm V\pm W=0$ is

$$\lambda_1U^2+\lambda_2V^2+\lambda_3W^2=0 \qquad \ldots\ldots(14)$$

where $$\frac{1}{\lambda_1}+\frac{1}{\lambda_2}+\frac{1}{\lambda_3}=0.$$

The centre of the conic is given by the equations

$$\left.\begin{aligned} a_1\lambda_1U+a_2\lambda_2V+a_3\lambda_3W=0,\\ b_1\lambda_1U+b_2\lambda_2V+b_3\lambda_3W=0.\end{aligned}\right\}\ldots\ldots(15)$$

Denote the co-factors of the small letter by the corresponding capital letters in the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}.$$

From (15), $$\frac{\lambda_1}{C_1VW}=\frac{\lambda_2}{C_2UW}=\frac{\lambda_3}{C_3UV}\cdot$$

substituting in $\Sigma\frac{1}{\lambda_1}=0$, we get the equation of the locus,

$$\frac{U}{C_1}+\frac{V}{C_2}+\frac{W}{C_3}=0. \qquad \ldots\ldots(16)$$

which is a st. line

Otherwise

Taking the equation (in trilinears) of the range in the form $u\alpha^2+v\beta^2+w\gamma^2=0$ with the condition

$$l^2/u+m^2/v+n^2/w=0,$$

we know the centre is given by $\frac{u\alpha}{a}=\frac{v\beta}{b}=\frac{w\gamma}{c}$. Hence the centre locus is the st. line $\frac{l^2\alpha}{a}+\frac{m^2\beta}{b}+\frac{n^2\gamma}{c}=0.$

The centre locus passes through the mid-points of the diagonals of the quadrilateral formed by the four tangents, since each diagonal can be regarded as a limiting form of an ellipse touching the sides of the quadrilateral whose minor axis tends to zero, and the centre of this ellipse is ultimately the mid-point of the diagonal. Thus the mid-points of the diagonals of a quadrilateral are collinear.

155.1. The following proof is taken from Serret's Geometrie de Direction.

We may observe that the line $x\cos\alpha + y\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$ is a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where α is the angle which the normal at the point makes with the axis of the conic; and the distance of the centre from the tangent is

$$\sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}.$$

Let the equations of the four lines be

$$a_r x + b_r y - p_r = 0, \qquad a_r^2 + b_r^2 = 1, \qquad r = 1, 2, 3, 4$$

and suppose that the axes of length $2a$, $2b$ of an ellipse are parallel to

$$\alpha x + \beta y = 0, \qquad \alpha^2 + \beta^2 = 1.$$
$$\alpha' x + \beta' y = 0, \qquad \alpha'^2 + \beta'^2 = 1,$$

and its centre is (x, y), then the distance of the centre from the tangent $a_1x + b_1y - p_1 = 0$ is given by either of the expressions

$$a_1x + b_1y - p_1, \text{ or } \sqrt{a^2(a_1\alpha + b_1\beta)^2 + b^2(a_1\alpha' + b_1\beta')^2}.$$

Hence

$$U_1^2 \equiv (a_1x + b_1y - p_1)^2 = a^2(a_1\alpha + b_1\beta)^2 + b^2(a_1\alpha' + b_1\beta')^2$$

and three similar relations. To find the locus, it is necessary to eliminate the six quantities

$$a^2\alpha^2,\ b^2\beta^2,\ a^2\alpha'^2,\ b^2\beta'^2,\ a^2\alpha\beta,\ b^2\alpha'\beta'.$$

Write the relations as

$$a_1^2(a^2\alpha^2 + b^2\alpha'^2) + 2a_1b_1\alpha\beta(a^2+b^2) + b_1^2(a^2\beta^2 + b^2\beta'^2) - U_1^2 = 0$$
$$a_2^2(a^2\alpha^2 + b^2\alpha'^2) + 2a_2b_2\alpha\beta(a^2+b^2) + b_2^2(a^2\beta^2 + b^2\beta'^2) - U_2^2 = 0$$

with two more similar relations. The elimination of $a^2\alpha^2 + b^2\alpha'^2$, $\alpha\beta(a^2+b^2)$, $a^2\beta^2 + b^2\beta'^2$ gives the locus

$$\begin{vmatrix} U_1^2 & U_2^2 & U_3^2 & U_4^2 \\ a_1^2 & a_2^2 & a_3^2 & a_4^2 \\ b_1^2 & b_2^2 & b_3^2 & b_4^2 \\ a_1b_1 & a_2b_2 & a_3b_3 & a_4b_4 \end{vmatrix} = 0. \qquad \ldots\ldots(17)$$

The locus which appears to be of the second degree is really a st. line, for it is easy to see that the co-efficients of x^2, xy, y^2 vanish.

Hence the locus of the centres of conics inscribed in the quadrilateral $U_1U_2U_3U_4 = 0$ is a st. line represented by the

equation

$$\lambda_1 U_1^2 + \lambda_2 U_2^2 + \lambda_3 U_3^2 + \lambda_4 U_4^2 = 0 \qquad \ldots\ldots(18)$$

rendered linear in x and y by suitable choice of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

Note. It will be noticed that the polar of any vertex of the quadrilateral *w.r.* to the system of conics $\Sigma\lambda_1 U_1^2 = 0$ passes through the opposite vertex. Thus the opposite vertices are conjugate points *w.r.* to all conics of the system $\Sigma\lambda_1 U_1^2 = 0$.

Illustrative Examples.

(1) *Find the locus of the poles of a fixed line w. r. to rectangular hyperbolas to which a given triangle is self-conjugate.*

Let $U \equiv a_1x + b_1y + c_1 = 0$, $V \equiv a_2x + b_2y + c_2 = 0$, $W \equiv a_3x + b_3y + c_3 = 0$ be the sides of the triangle and $lx + my + n = 0$ be the equation of the line. The equation of the conics *w.r.* to which $UVW = 0$ form a self-conjugate triangle is

$$\lambda_1 U^2 + \lambda_2 V^2 + \lambda_3 W^2 = 0.$$

This will be a rectangular hyperbola if

$$\lambda_1(a_1^2 + b_1^2) + \lambda_2(a_2^2 + b_2^2) + \lambda_3(a_3^2 + b_3^2) = 0.$$

If (x', y') be the pole of $lx + my + n = 0$, then the polar of (x', y') is

$$x(\lambda_1 a_1 U' + \lambda_2 a_2 V' + \lambda_3 a_3 W') + y(\lambda_1 b_1 U' + \lambda_2 b_2 V' + \lambda_3 b_3 W'$$
$$+ \lambda_1 c_1 U' + \lambda_2 c_2 V' + \lambda_3 c_3 W' = 0$$

where U', V', W' are the values of U, V, W for $x = x'$, $y = y'$.

This is identical with $lx + my + n = 0$.

$$\therefore \quad \frac{\lambda_1 a_1 U' + \lambda_2 a_2 V' + \lambda_3 a_3 W'}{l} = \frac{\lambda_1 b_1 U' + \lambda_2 b_2 V' + \lambda_3 a_3 W'}{m}$$

$$= \frac{\lambda_1 c_1 U' + \lambda_2 c_2 V' + \lambda_3 c_3 W'}{n} = \lambda.$$

Hence

$$\lambda_1 a_1 U' + \lambda_2 a_2 V' + \lambda_3 a_3 W' - \lambda l = 0$$
$$\lambda_1 b_1 U' + \lambda_2 b_2 V' + \lambda_3 b_3 W' - \lambda m = 0$$
$$\lambda_1 c_1 U' + \lambda_2 c_2 V' + \lambda_3 c_3 W' - \lambda n = 0$$
$$\lambda_1(a_1^2 + b_1^2) + \lambda_2(a_2^2 + b_2^2) + \lambda_3(a_3^2 + b_3^2) = 0.$$

Thus the locus of (x', y') is

$$\begin{vmatrix} a_1 & a_2 & a_3 & l \\ b_1 & b_2 & b_3 & m \\ c_1 & c_2 & c_3 & n \\ \dfrac{a_1^2 + b_1^2}{U} & \dfrac{a_2^2 + b_2^2}{V} & \dfrac{a_3^2 + b_3^2}{W} & 0 \end{vmatrix} = 0$$

which is a circum conic of the triangle $UVW = 0$.

Second Proof. Let the co-ordinates be trilinears. The equation of a conic to which the triangle of reference is self-conjugate is

$$ux^2+vy^2+wz^2=0, \qquad \ldots\ldots(i)$$

This will be a rectangular hyperbola if the circular points I (x_0, y_0, z_0), J(x_0', y_0', z_0') are conjugate to it, *i.e.*

$$ux_0x_0'+vy_0y_0'+wz_0z_0'=0$$

or $$u+v+w=0. \qquad \ldots\ldots(ii)$$

Let the given line be $lx+my+nz=0$ and (x', y', z') its pole, then $uxx'+vyy'+wzz'=0$ and $lx+my+nz=0$ are identical, hence

$$\frac{ux'}{l}=\frac{vy'}{m}=\frac{wz'}{n}=\lambda.$$

Substituting in (ii) for u, v, w, we have

$$\frac{l}{x'}+\frac{m}{y'}+\frac{n}{z'}=0.$$

Thus the locus of (x', y', z') is the circum-conic

$$\frac{l}{x}+\frac{m}{y}+\frac{n}{z}=0.$$

Cor. *The locus of the centres of these rectangular hyperbolas is the circum-circle.*

(2) *Prove that the equation of the family of conics inscribed in the rectangle formed by the lines $x=\pm a$, $y\pm b=0$ is*

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-1+\dfrac{2\lambda xy}{ab}+\lambda^2=0.$ *Prove that the locus of the foci is $x^2-y^2=a^2-b^2$, and that if two of the conics intersect on this latter locus, they do so at right angles.*

(Pembroke 1899, G. U., M.A., 1901, St. Catherine, 1929).

Let the tangential equation of the conic be

$$pl^2+qm^2+rn^2+2fmn+2gnl+2hlm=0.$$

As the lines $x\pm a=0$, $y\pm b=0$ for which, the values of l, m, n for the four lines are $(1, 0, a)$, $(1, 0, -a)$, $(0, 1, b)$, $(0, 1, -b)$ touch the conic,

$$p+ra^2+2ga=0, \qquad p+ra^2-2ga=0,$$
$$q+rb^2+2fb=0. \qquad q+rb^2-2fb=0.$$

$\therefore \quad g=0, f=0, p=-ra^2, q=-rb^2.$

Thus the tangential equation of the conic is

$$a^2l^2+b^2m^2-n^2-2klm\equiv 0 \text{ where } k=\frac{h}{r}.$$

Thus the conic is the envelope of $lx+my+n=0$ with the condition $a^2l^2+b^2m^2-n^2-2klm=0$. To find the equation,

we have

$$a^2l^2+b^2m^2-(lx+my)^2-2klm=0$$

or $$l^2(a^2-x^2)+2lm(xy+k)+m^2(b^2-y^2)=0.$$

Thus the envelope is the curve

$$(xy+k)^2=(a^2-x^2)(b^2-y^2)$$

or $$x^2y^2+k^2+2kxy=a^2b^2-a^2y^2-b^2x^2+x^2y^2$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2k}{a^2b^2}\ xy-1+\frac{k^2}{a^2b^2}=0$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{2\lambda xy}{ab}-1+\lambda^2=0,\ \lambda ab=k.$$

Let (α, β) be a focus, then $y-\beta\pm i(x-\alpha)=0$ are tangents, and therefore $l=\pm i,\ m=1,\ n=-\beta\mp i\alpha$ satisfy the equation

$$a^2l^2+b^2m^2-n^2-2klm=0$$

$\therefore$ $$-a^2+b^2-2ki-(\beta+i\alpha)^2=0,$$

and $$-a^2+b^2+2ki-(\beta-i\alpha)^2=0,$$

or $$-a^2+b^2-2ik-\beta^2-2i\alpha\beta+\alpha^2=0,$$

and $$-a^2+b^2+2ik-\beta^2+2i\alpha\beta+\alpha^2=0,$$

whence $$\alpha^2-\beta^2=a^2-b^2,\ \alpha\beta+k=0.$$

Thus the locus of the foci is given by $x^2-y^2=a^2-b^2$.

Let two members of the system which correspond to $\lambda=\lambda_1$, $\lambda=\lambda_2$ intersect at (x_1, y_1) on $x^2-y^2=a^2-b^2$, then

$$x_1^2-y_1^2=a^2-b^2$$

and λ_1, λ_2 are the roots of

$$\lambda^2+\frac{2\lambda x_1y_1}{ab}+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1\right)=0$$

$$\therefore\quad \lambda_1\lambda_2=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1,\ \lambda_1+\lambda_2=\frac{-2x_1y_1}{ab}.$$

The tangents at (x_1, y_1) to the two conics are

$$\frac{x}{a}\left(\frac{x_1}{a}+\frac{\lambda_1y_1}{b}\right)+\frac{y}{b}\left(\frac{\lambda_1x_1}{a}+\frac{y_1}{b}\right)+\ldots\ldots=0.$$

$$\frac{x}{a}\left(\frac{x_1}{a}+\frac{\lambda_2y_1}{b}\right)+\frac{y}{b}\left(\frac{\lambda_2x_1}{a}+\frac{y_1}{b}\right)+\ldots\ldots=0.$$

These tangents intersect at right angles if

$$\frac{1}{a^2}\left(\frac{x_1}{a}+\frac{\lambda_1y_1}{b}\right)\left(\frac{x_1}{a}+\frac{\lambda_2y_1}{b}\right)+\frac{1}{b^2}\left(\frac{\lambda_1x_1}{a}+\frac{y_1}{b}\right)\left(\frac{\lambda_2x_1}{a}+\frac{y_1}{b}\right)=0,$$

and this condition will be found to be satisfied in virtue of the values of $\lambda_1+\lambda_2$, and $\lambda_1\lambda_2$.

The last result may be proved geometrically.

Let P be a point on $x^2-y^2=a^2-b^2$.

The lines PI, PJ (I and J being the circular points at infinity) are tangents to a member of the family. The tangents

from P to the conics of the system form an involution, of which the double lines are the tangents PT, PT′ to Σ' and Σ' which pass through P. The lines PT, PT′, PI, PJ form a harmonic pencil, therefore PT, PT′ are at right angles.

Exercises XLV.

1. If ABC be taken as the triangle of reference, show that the equations of the lines PQ, QR, RS, SP can be put in the form
$$lx \pm my \pm nz = 0.$$

2. Show that the equation of the conic which touches the four lines of Ex. 1, can be written in the form
$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0, \quad al^2 + bm^2 + cn^2 = 0.$$

3. Show that the co-ordinates of the four lines of Ex. 1 can also be put in the form $(1, \pm 1, \pm 1)$, and the equation of the conic touched by these lines can be written as
$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0, \quad a + b + c = 0.$$

4. If (p, q, r) be the co-ordinates of the line at infinity in the system of co-ordinates chosen in Ex. 3, show that the equation of the centre locus is
$$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 0.$$

5. Show that the locus of the poles of a given st. line with respect to range of conics is a st. line.

6. Show that if the conic $u\alpha^2 + v\beta^2 + w\gamma^2 = 0$ be a parabola, it will touch the four lines $a\alpha \pm b\beta \pm c\gamma = 0$.

7. Show that two and only two rectangular hyperbolas can be drawn to touch four given lines.

8. Show that the director circles of all conics inscribed in the same quadrilateral are coaxal.

156. Confocal conics. Let I and J be the circular points at infinity and Σ a conic in the plane. The tangents from I and J intersect in two pairs of points F, F,′ S, S′ which are called the foci of Σ. The system of conics which touch the four tangents from I and J to Σ have the same set of foci F, F′, S, S′ and are said to form a confocal system of which Σ is a member. It follows that there is one conic of the system which touches a given line.

156.1. *Equation of a system of confocals.*

Let the equation of Σ be
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0.$$

Since all conics have common foci, they have common axes and common centre. Thus the equation of the system is of the form

$$\frac{x^2}{p^2}+\frac{y^2}{q^2}=1, \quad p^2>q^2.$$

The real foci are $(\pm\sqrt{p^2-q^2}, 0)$ which must be same as $(\pm\sqrt{a^2-b^2}, 0)$.

$\therefore \quad p^2-q^2=a^2-b^2$,

i.e., $p^2-a^2=q^2-b^2=\lambda$ (say).

Thus the equation of the confocals is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1. \qquad \ldots\ldots(19)$$

If $a>b$, then the conic $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$

(*i*) is a circle of infinite radius,
(*ii*) is an ellipse for all values of $\lambda>0$ and $\lambda>-b^2$,
(*iii*) is the line $y=0$ when $\lambda=-b^2$,
(*iv*) is an hyperbola when $\lambda<-b^2$ and $>-a^2$,
(*v*) is the line $x=0$ when $\lambda=-a^2$,
(*vi*) is an imaginary ellipse when $\lambda<-a^2$.

156.2. *The necessary and sufficient condition that a pair of conjugate lines w. r. to a conic be also conjugate w. r. to a confocal conic is that it should be at right angles.*

Let $l_1x+m_1y+n_1=0$, $l_2x+m_2y+n_2=0$ be conjugate *w. r.* to

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$\therefore \quad a^2l_1l_2+b^2m_1m_2=n_1n_2$.

If this pair of lines be also conjugate *w. r.* to the confocal (19)

$$a^2l_1l_2+b^2m_1m_2-n_1n_2-\lambda(l_1l_2+m_1m_2)=0$$

$$\therefore \quad l_1l_2+m_1m_2=0$$

which is the condition that the lines should be at right angles.

156.3. *Through a point there pass two conics of the system, one ellipse and one hyperbola and they intersect at right angles.*

Let $P(x_1, y_1)$ be the point which lies on the conic (19).

$\therefore \quad f(\lambda)\equiv\lambda^2+\lambda(x_1^2+y_1^2-a^2-b^2)+(a^2b^2-a^2x_1^2-b^2y_1^2)=0.$

Now $f(-\infty)=+\infty$, $f(b^2)=x_1^2(b^2-a^2)$,
$f(a^2)=y_1^2(a^2-b^2)$, $f(\infty)=+\infty$, $a^2>b^2$.

Thus there is one root of $f(\lambda)$ which is less than b^2, for which the conic (19) is an ellipse, and one root of $f(\lambda)=0$ lies between b^2 and a^2 for which the conic (19) is a hyperbola.

Let Σ, Σ' be the conics which pass through P, and suppose that PT, PT' are the tangents at P to Σ and Σ'. Now PS, PS' are equally inclined to PT, PT'. Thus PT, PT' are bisectors of $\angle$SPS' and are therefore at right angles.

Otherwise. PT, PT' are both conjugate to Σ and Σ', and are therefore at right angles.

156.4. Tangential equation of the confocal system.

The tangential equation of the confocal system (19) is easily seen to

$$a^2l^2+b^2m^2-n^2-\lambda(l^2+m^2)=0.$$

It follows that $l^2+m^2=0$ which represents the points I and J belonging to the system. For $\lambda=b^2, a^2$, the conic degenerates into the pairs of foci. Thus the pairs of points I, J : S, S'; F, F', are degenerate conics of the system.

Ex. 1. Show that the equation

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=\lambda(l^2+m^2)$$

represents a confocal system. Find the point equation.

Ex. 2. Show that

$$(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)=\lambda(l^2+m^2)$$

represents a system of conics with (x_1, y_1, z_1), (x_2, y_2, z_2) as foci.

156.5. The locus of the pole of a given line *w. r.* to the conics of the confocal system is a st. line, which is a normal to a conic of the system.

Let (x', y') be a pole of $lx+my+n=0$ *w. r.* to

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1.$$

The polar of (x', y') is

$$\frac{xx'}{a^2-\lambda}+\frac{yy'}{b^2-\lambda}=1.$$

$$\therefore \quad \frac{x'}{(a^2-\lambda)l}=\frac{y'}{(b^2-\lambda)m}=-\frac{1}{n},$$

or $$\frac{x'}{l}-\frac{y'}{m}=-\frac{1}{n}(a^2-b^2).$$

Thus the locus of (x', y') is the line

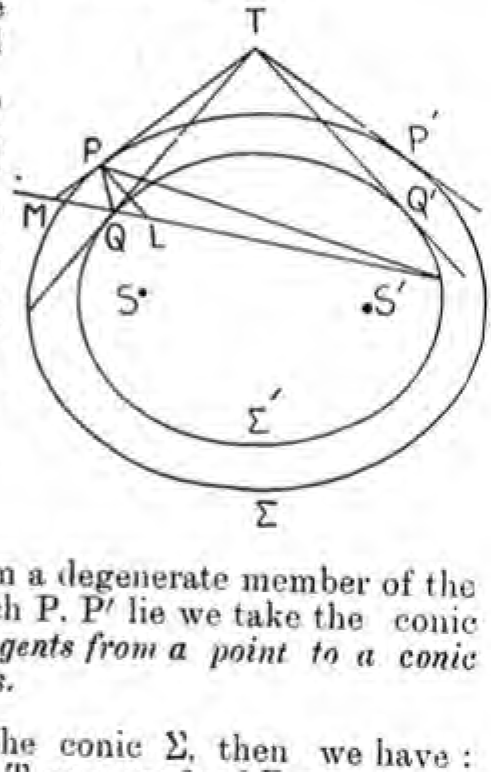
$$\frac{x}{l}-\frac{y}{m}+\frac{1}{n}(a^2-b^2)=0. \qquad \text{......(20)}$$

which is perpendicular to $lx+my+n=0$. Now there is one conic Σ of the system which touches $lx+my+n=0$. The pole of this line *w. r.* to Σ is the point of contact which lies on the line (20). Thus (20) is a normal to Σ. If we call the line (20) $l'x+m'y+n'=0$, then

$$ll' = -mm' = nn'(a^2-b^2)$$

which is symmetrical *w. r.* to l, l'; m, m'; n, n'. Hence *the tangents at P to the confocals Σ, Σ' that pass through P have the property that each is the locus of the poles of the other w. r. to the confocals of the system.*

156.6. *From any point* T *tangents* TP, TP′ ; TQ, TQ′ *are drawn to two confocal conics Σ, Σ'. Show that the lines* PQ, PQ′ *are equally inclined to the tangent at* P.



Let the line QQ′ meet the tangent PT and the normal PL at P in points M and L.

The poles of the line PT *w. r.* to the confocal conics lie on PL. The pole of QQ′ *w. r.* to Σ' is the point T which lies on PT. Thus the pole of PT *w. r.* to Σ' lies on PL and QQ′ and is therefore the point L.

Thus (ML,QQ′) $= -1$ and since $\angle MPL = \frac{\pi}{2}$, PM, PL bisect the angle QPQ′.

Cor. 1. Since (S, S′) form a degenerate member of the system, for the conic on which P. P′ lie we take the conic (S, S′). We thus get : *the tangents from a point to a conic subtend equal angles at a focus.*

Cor. 2. Let T lie on the conic Σ, then we have : *the tangents to Σ' from a point* T *on a confocal Σ are equally inclined to the tangent at* T *to Σ.*

Cor. 3. There is one member of the system which touches PQ. This will also touch PQ′ (Cor. 2). Similarly it will touch P′Q, P′Q′.

Cor. 4. Let the conic on which Q, Q′ lie degenerate into the line ellipse (SS′). The theorem becomes. *The lines joining the foci of a conic to any point* P *on the curve make equal angles with the tangent at* P.

156.7. *Two ellipses or two hyperbolas of a confocal system do not intersect in real points.*

Let
$$\frac{x^2}{a^2-\lambda_1}+\frac{y^2}{b^2-\lambda_1}=1,$$
$$\frac{x^2}{a^2-\lambda_2}+\frac{y^2}{b^2-\lambda_2}=1$$
$\lambda_1<b^2$, $\lambda_2<b^2$ be two ellipses of the system.

At their points of intersection we have
$$\frac{x^2}{(a^2-\lambda_1)(a^2-\lambda_2)}+\frac{y^2}{(b^2-\lambda_1)(b^2-\lambda_2)}=0$$
and no real values of x and y can satisfy this equation. The theorem for two hyperbolas can be proved similarly.

156.8. Definition. Two points $P(x, y)$, $P_1(x_1, y_1)$ on two confocals
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$$
are said to correspond if
$$\frac{x}{a}=\frac{x_1}{a_1},\quad \frac{y}{b}=\frac{y_1}{b_1}$$
Let $(a\cos\theta, b\sin\theta)$ be a point on the ellipse
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$
then the corresponding point on
$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$$
is $(a_1\cos\theta, b_1\sin\theta)$. Thus the *corresponding points have equal eccentric angles.*

A very convenient method of writing the equations of confocal ellipses and hyperbolas is
$$\frac{x^2}{\cosh^2\phi}+\frac{y^2}{\sinh^2\phi}=c^2 \text{ ellipses} \quad \text{......(21)}$$
$$\frac{x^2}{\cos^2\psi}-\frac{y^2}{\sin^2\psi}=c^2 \text{ hyperbolas} \quad \text{......(22)}$$

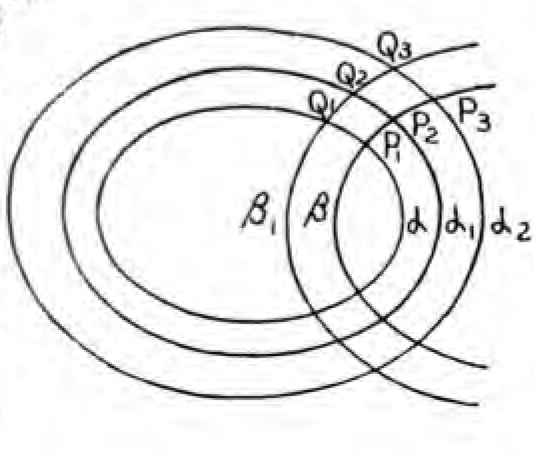
156.9. *A hyperbola intersects its confocal ellipses in series of corresponding points.*

The hyperbola (22) intersects the ellipses of the system in points whose co-ordinates are

$x=c\cosh\phi\cos\psi$,

$y=c\sinh\phi\sin\psi$ (23)

Let ψ be kept constant and ϕ variable, so that the point given by (23) lies on a unique hyperbola β of the system. This cuts the ellipses α, α_1, α_2.........in



points P_1, P_2......for the variable values of ϕ. Since

$$\frac{x}{c\cosh\phi}=\cos\psi,\ \frac{y}{c\sinh\phi}=\sin\psi$$

and ψ is fixed, it follows that $P_1P_2P_3$......are corresponding points. Similarly Q_1, Q_2......are a series of corresponding points.

If now ϕ be kept constant, and ψ variable, we get the intersections *e.g.*, P_1, Q_1......of the ellipse α with the hyperbolas β, β_1, etc., and as

$$\frac{x}{c\cos\psi}=\cosh\phi=\text{constant},$$

$$\frac{y}{c\sin\psi}=\sinh\phi=\text{constant},$$

the series of points P_1, Q_1......are corresponding points.

Similarly the series of points P_2, Q_2......; P_3, Q_3...... consist of corresponding points.

157. Confocal Parabolas. Two parabolas are confocal when they have the same focus and the same axis. The equation of the parabola whose focus is the origin and axis the x-axis is

$$x^2+y^2=(x+2\lambda)^2$$

or $$y^2=4\lambda(x+\lambda) \qquad ......(24)$$

or $$y^2=4\lambda(x+\lambda z)z$$

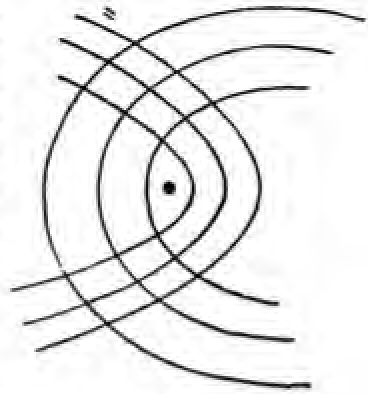
As λ varies, the equation represents a system of parabolas all having a common focus at the origin and common axis, *viz* the x-axis. The tangential equation of the system is

$$\lambda(l^2+m^2)=nl \qquad ......(25)$$

When $\lambda\to\infty$, the conic reduces to the circular points (or from (24) the line at infinity joining I and J taken twice). If $\lambda\to 0$, the equation (25) represents the finite focus (0, 0, 1) and the infinite focus (1, 0, 0), while equation (24) represents the line $y=0$ which joins them taken twice.

If $\lambda>0$ the vertex of the parabola is to the left of the focus and if $\lambda<0$, the vertex is to the right of the focus. Thus *there are two systems of confocal parabolas, one having vertices to the left and the other to the right of the focus.*

157.1. *Through a point there pass two confocal parabolas, belonging to different systems i.e., having their concavities in opposite directions and they intersect at right angles.*



If the parabola given by (24) passes through $P(x_1, y_1)$ we have

$$4\lambda^2 - 4\lambda x_1 - y_1^2 = 0. \qquad \ldots\ldots(26)$$

This equation gives two values of λ of contrary signs. If λ_1, λ_2 be the roots of this equation, the tangents to the two parabolas which correspond to $\lambda=\lambda_1$, $\lambda=\lambda_2$ have the equations

$$yy_1 = 2\lambda_1(x+x_1) + 4\lambda_1^2,$$
$$yy_1 = 2\lambda_2(x+x_1) + 4\lambda_2^2.$$

These lines will be at right angles if

$$y_1^2 + 4\lambda_1\lambda_2 = 0$$

and this relation holds in virtue of (26).

157.2. The two points (x, y), (x', y') on the two parabolas $y^2=4px$, $y^2=4p'x$ (the axes of reference not necessarily being the same for the two curves) are called corresponding points if

$$\frac{x}{p} = \frac{x'}{p'}, \quad \frac{y}{p} = \frac{y'}{p'}.$$

The corresponding points have the same parameter when the co-ordinates of a point on $y^2=4px$ are expressed in the form $(pt^2, 2pt)$ or $(t^2-p^2, 2pt)$ when the equation of the parabola is expressed in the form $y^2=4p^2(x+p^2)$.

Illustrative Examples.

(1) *Perpendicular tangents are drawn from a point, one to each conic of a confocal system, show that the point lies on a circle.*

The lines

$$x\cos\alpha + y\sin\alpha = \{(a^2-\lambda)\cos^2\alpha + (b^2-\lambda)\sin^2\alpha\}^{\frac{1}{2}}$$

$$x\sin\alpha - y\cos\alpha = \{(a^2-\mu)\sin^2\alpha + (b^2-\mu)\cos^2\alpha\}^{\frac{1}{2}}$$

are perpendicular tangents to the confocals

$$\frac{x^2}{a^2-\lambda} + \frac{y^2}{b^2-\lambda} = 1, \quad \frac{x^2}{a^2-\mu} + \frac{y^2}{b^2-\mu} = 1.$$

To find the locus of their point of intersection, we square and add. The required locus is

$$x^2 + y^2 = a^2 + b^2 - \lambda - \mu,$$

which is a circle.

(2) ***The distance between two points, one on each of the two confocal conics, is equal to the distance between their corresponding points.***

Let a, b ; a', b' be the semi-axes of the conics Σ, Σ' of the confocal system, so that $a^2-b^2 = a'^2 - b'^2$. Suppose that the eccentric angles of P and Q are α and β.

Then $PQ^2=(a\cos\alpha-a'\cos\beta)^2+(a\sin\alpha-a'\sin\beta)^2$

and $\quad P_1Q_1^2=(a'\cos\alpha-a\cos\beta)^2+(a'\sin\alpha-a\sin\beta)^2$

$\therefore PQ^2-P_1Q_1^2=(a^2-a'^2)(\cos^2\alpha-\cos^2\beta+\sin^2\alpha-\sin^2\beta)=0.$

Hence $$PQ=P_1Q_1.$$

(3) *Show that the polars of a point* $P(x_1, y_1)$ *w. r. to a system of confocal conics is a parabola which touches the axes. Find the focus and directrix of the parabola.*

The polar of (x_1, y_1) *w. r.* to the system of conics

$$\frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1$$

is
$$\frac{xx_1}{a^2-\lambda}+\frac{yy_1}{b^2-\lambda}=1$$

or
$$\lambda^2+\lambda(xx_1+yy_1-a^2-b^2)+(a^2b^2-b^2xx_1-a^2yy_1)=0.$$

The envelope of this is the conic

$$(xx_1+yy_1-a^2-b^2)^2=4(a^2b^2-b^2xx_1-a^2yy_1).$$

This can be written in the form

$$(xx_1-yy_1-a^2+b^2)^2+4x_1x_2xy=0$$

which clearly shows that the conic is a parabola which touches the lines $xy=0$. The equation of the parabola can be written as

$$\sqrt{xx_1}+\sqrt{-yy_1}+\sqrt{a^2-b^2}=0.$$

As $xy=0$ are tangents to the parabola, and perpendicular tangents of a parabola intersect on the directrix, the origin lies on the directrix. Again, the polars of P *w. r.* to Σ, Σ' which pass through P are the tangents to Σ, Σ' at P, and since these tangents are at right angles, P also lies on the directrix. Thus the equation of the directrix is

$$xy_1-x_1y=0.$$

The tangential equation of the parabola is

$$\frac{x_1}{l}-\frac{y_1}{m}+\frac{a^2-b^2}{n}=0.$$

If (ξ, η) be a focus, the isotropic lines $\pm i(x-\xi)+(y-\eta)=0$ will touch the parabola.

$$\therefore \qquad y_1+\frac{(a^2-b^2)\eta}{\xi^2+\eta^2}=0, \quad x_1-\frac{(a^2-b^2)\xi}{\xi^2+\eta^2}=0,$$

whence
$$\xi=\frac{x_1(a^2-b^2)}{x_1^2+y_1^2}, \quad \eta=\frac{-y_1(a^2-b^2)}{x_1^2+y_1^2}.$$

Note. The equation of conics which pass through the four foci is $x^2-y^2-(a^2-b^2)+2hxy=0$ where h is a parameter. It will be seen that the points (x_1, y_1), (ξ, η) are conjugate points with regard to this system of conics.

Exercises XLVI.

1. From the property $SP+S'P=$constant, show that only one ellipse having given foci can be drawn through a given point.

2. Show that the locus of the pole of the line $lx+my=1$ *w. r.* to the confocal parabolas $y^2=4\lambda(x+\lambda)$ is the st. line

$$mx-ly+m^2l=0.$$

3. If, of two lines, the first is the locus of the pole of the second, then the second is the locus of the pole of the first.

4. The difference between the squares of the central perpendiculars on two parallel tangents to two confocal conics is constant.

5. TP, TQ are tangents one to each of the two confocals whose centre is C : show that if the tangents are at right angles to one another, CT bisects PQ.

6. If the two confocal ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad \frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

be cut by the st. line $x\cos\alpha+y\sin\alpha=p$, and if T and T′ be poles of this line *w. r.* to the ellipses, prove that $TT'=\lambda/p$.

7. If OT, OT′ are the tangents from the fixed point $O(x',y')$ to one of the confocals $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, having the foci S, S′, show that (*i*) the conic OTT′SS′ passes through a fourth fixed point, (*ii*) the circles such as OTT′ are coaxal, (*iii*) the envelope of PP′ is the parabola.

$$\sqrt{xx'}+\sqrt{yy'}=\sqrt{x'^2+y'^2}$$

P, P′ being the points where the circle OTT′ cuts the corresponding confocal again.

Similar Figures

158. Let C be a curve which is generated by a point, and suppose O is a point in the plane of the curve. On the radius vector OP take a point O such that $OP=k.OQ$ (where k is constant), then as P traces out the curve C, Q will trace a curve C′ which is said to be similar and similarly situated to C or homothetic to C.

The ratio k is called the ratio of similitude. A pair of points as P, Q are called corresponding points, and the lines PP′, QQ′ joining two points of one figure, and two corresponding points of the second figure are called corresponding lines.

It may be pointed out that the line joining two points in the figure C is parallel to the line joining the corresponding points in the figure C′ which is homothetic with it, and these lines are in a constant ratio.

If now the two figures C, C′ be homothetic with O as the homothetic centre, and C′ be rotated about O through any angle, we get a new figure C_1 which is *directly similar* to C, though not homothetic to C. Such figures have the property that if P, P′ are two points of C and Q, Q′ the corresponding points of C_1, then $\angle POQ = P'OQ'$, and

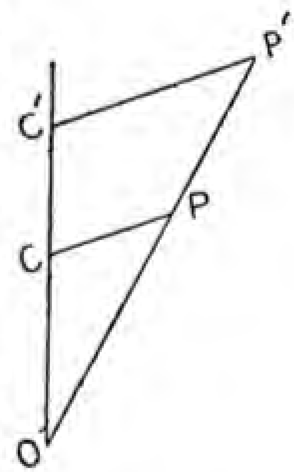
$$\frac{OP}{OQ}=\frac{OP'}{OQ'}.$$

Let C, C′ be two curves generated by the points P and Q respectively, and suppose that there exists a point O, such that $\angle POQ$ has always the same bisector and OP : OQ is constant, such curves are called *inversely similar figures.*

158.1. *The necessary and sufficient condition that two figures F and F′ be homothetic is that corresponding to a point C in the plane of the first figure, it is possible to find a point C′ in the plane of the second figure such that to a point P of F, there corresponds a point P′ of F′ with the condition that* $C'P' = kCP$, $CP \parallel C'P'$ *where k is constant.*

Let O be the homothetic centre, and suppose that C and C′ are two fixed points of the figures F and F′. If P and P′ are two other corresponding points of the two figures we have

$$OC' = k.OC,\ OP' = k.OP.$$



The triangles OCP, OC′P′ are evidently similar,

$$\therefore\quad CP \parallel C'P',\ \frac{C'P'}{CP}=\frac{OC'}{OC}=k$$

i.e., $C'P' = k.CP$.

To prove the converse, let CC′, PP′ meet in O, then from the similarity of triangles OC′P′, OCP,

$$OC' = kOC,\ OP' = k.OP.$$

158.2. Condition that two conics may be homothetic.

Let $F \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ ......(26)

and $F' \equiv a'\xi^2 + b'\eta^2 + 2h'\xi\eta + 2g'\xi + 2f'\eta + c' = 0$ ......(27)

be two homothetic conics with O(0, 0) as the homothetic centre, then clearly

$$\xi = kx,\ \eta = ky. \qquad \text{......(28)}$$

where (ξ, η), (x, y) are pairs of corresponding points. This transformation will carry the figure F′ to F.

The transform of F′ is

$$a'x^2+2h'xy+b'y^2+\frac{2}{k}(g'x+f'y)+\frac{c'}{k^2}=0. \qquad \ldots\ldots(28)$$

We have incidently proved that the transform of a conic is a conic. If the conics (26) and (27) are homothetic, the equations (26) and (28) are identical.

$$\therefore \quad a'=\lambda a,\ h'=\lambda h,\ b'=\lambda b,\ g'=k\lambda g,\ f'=k\lambda f,\ c'=k^2\lambda c.$$

Thus the two conics will be homothetic if

$$\frac{a}{a'}=\frac{h}{h'}=\frac{b}{b'}.$$

The relation shows that if two conics are homothetic, their asymptotes are parallel. Conversely, if two conics have their asymptotes parallel, they are homothetic, for the last three equations are sufficient to determine k and λ. Thus *two conics are homothetic, if they meet the line at infinity in common points.*

Now the centre of the conic (27) is given by the equations

$$a'\xi+h'\eta+g'=0$$

or

$$h'\xi+b'\eta+f'=0.$$

$$\therefore \quad \frac{\xi}{h'f'-b'g'}=\frac{\eta}{g'h'-a'f'k}=\frac{1}{a'b'-h'^2}$$

or

$$\frac{\xi}{\lambda^2(hf-bg)k}=\frac{\eta}{\lambda^2(gh-af)k}=\frac{1}{(ab-h^2)\lambda^2}$$

$$\therefore \quad \xi=\frac{hf-bg}{ab-h^2}k=kx,$$

similarly $\eta=ky$,

where (x, y) are the co-ordinates of the centre of the conic (26). Thus *in two homothetic conics, their centres are corresponding points.*

Since homothetic conics have their asymptotes parallel, and the axes of a conic bisect the angle between the asymptotes, it follows that *the necessary and sufficient condition that two conics may be homothetic is that their axes should be parallel.*

158.3. Condition of similarity of two conics.

Let the conics

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \qquad \ldots\ldots(29)$$

$$a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0 \qquad \ldots\ldots(30)$$

be similar. Then a rotation of conic (30) will make it homothetic with (29). Let the equation of the conic in the new position be

$$a_1x^2+2h_1xy+b_1y^2+2g_1x+2f_1y+c_1=0 \qquad \text{......(31)}$$

$$\therefore\ a'b'-h'^2=a_1b_1-h_1^2,\ a'+b'=a_1+b_1,$$

and $$\frac{a}{a_1}=\frac{h}{h_1}=\frac{b}{b_1}=\lambda \text{ say.}$$

$$\therefore\ ab-h^2=\lambda^2(a_1b_1-h_1^2)=\lambda^2(a'b'-h'^2)$$

$$a+b=\lambda(a_1+b_1)=\lambda(a'+b')$$

$$\therefore\ \frac{a+b}{\sqrt{ab-h^2}}=\frac{a'+b'}{\sqrt{a'b'-h'^2}}$$

i.e., the asympotes include equal angles. The condition is also sufficient, for then the asymptotes include equal angles, thus a rotation of one of the conics will make their asymptotes parallel, *i.e.* the two conics will be homothetic

Since the eccentricity of a conic depends on the ratio $(ab-h^2)/(a+b)^2$, the condition may be expressed as follows :—

The necessary and sufficient condition that two conics may be similar is that the two eccentricities of one conic should be equal to the two eccentricities of the other.

Since $\frac{b^2}{a^2}=1-e^2$, it follows that if two conics with axes $2a, 2b$; $2a', 2b'$ are similar, then $\frac{a}{b}=\frac{a'}{b'}$. Thus *the necessary and sufficient condition that two conics S, S', may be similar is that the ratio of the major axis of S to that of S' be equal to the ratio of the minor axis of S to that of S'.*

It follows that the centres of two similar conics are corresponding points.

Cor. 1. All parabolas are similar conics.

2. All rectangular hyperbolas are similar conics.

Illustrative Examples.

(1) *Find the equation of the conic which is concentric and homothetic with $f(x, y)=0$ with k as the ratio of similitude.*

Let $f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$ be the given conic. Shift the origin to the centre, the equation with the usual notation becomes

$$ax^2+2hxy+by^2+\frac{\Delta}{C}=0.$$

The equation of the conic concentric with $f(x, y)=0$ with k as the ratio of similitude is

$$ax^2+2hxy+by^2+\frac{\triangle}{k^2C}=0.$$

Transforming back to the old origin, we get the equation

$$f(x, y)+\frac{1-k^2}{k^2}\frac{\triangle}{C}=0.$$

(2) *If two similar concentric ellipses touch one another, show that the angle between their major axes is*

$$\tan^{-1}\frac{b(a'^2-a^2)}{a\sqrt{(a'^2-b^2)(a^2-b'^2)}}$$

where a, b and a', b' are the semi-axes of the conics.

Let θ be the angle which the major axis of the ellipse with axes a', b' makes with the major axis of the other ellipse. If we rotate this ellipse through θ, so that the axes of the two ellipses coincide (major axis with the major axis), the equations of the two ellipses can be taken as

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \frac{x^2}{a'^2}+\frac{y^2}{b'^2}=1, \qquad \frac{a}{a'}=\frac{b}{b'}=k.$$

Now rotate the second ellipse through θ, the equation becomes

$$\frac{(x\cos\theta-y\sin\theta)^2}{a'^2}+\frac{(x\sin\theta+y\cos\theta)^2}{b'^2}=1$$

$$\text{or } x^2\left(\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}\right)+2xy\cos\theta\sin\theta\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)$$
$$+y^2\left(\frac{\sin^2\theta}{a'^2}+\frac{\cos^2\theta}{b'^2}\right)-1=0.$$

Since it touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, its equation is of the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}-1\right)\left(\frac{x\cos\alpha}{a}+\frac{y\sin\alpha}{b}+1\right)=0$$

$$\text{or } \frac{x^2}{a^2}(1+\lambda\cos^2\alpha)+\frac{y^2}{b^2}(1+\lambda\sin^2\alpha)+2\lambda xy\frac{\cos\alpha\sin\alpha}{ab}$$
$$-(1+\lambda)=0.$$

Hence $$\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}=\frac{1+\lambda\cos^2\alpha}{a^2(1+\lambda)},$$

$$\frac{\sin^2\theta}{a'^2}+\frac{\cos^2\theta}{b'^2}=\frac{1+\lambda\sin^2\alpha}{b^2(1+\lambda)}.$$

$$\cos\theta\sin\theta\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)=\lambda\frac{\cos\alpha\sin\alpha}{ab(1+\lambda)}.$$

$$\therefore\ a^2(1+\lambda)\left[\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}\right]$$

$$+b^2(1+\lambda)\left[\frac{\sin^2\theta}{a'^2}+\frac{\cos^2\theta}{b'^2}\right]=2+\lambda \quad \text{......(A)}$$

$$\left[\frac{\cos^2\theta}{a'^2}+\frac{\sin^2\theta}{b'^2}-\frac{1}{a^2(1+\lambda)}\right]\left[\frac{\sin^2\theta}{a'^2}+\frac{\cos^2\theta}{b'^2}-\frac{1}{b^2(1+\lambda)}\right]$$

$$=\cos^2\theta\sin^2\theta\left(\frac{1}{b'^2}-\frac{1}{a'^2}\right)^2 \quad \text{......(B)}$$

From (A), $$\frac{2+\lambda}{1+\lambda}=1+\frac{1}{1+\lambda}$$

$$=2k^2\cos^2\theta+\left(\frac{a^2}{b'^2}+\frac{b^2}{a'^2}\right)\sin^2\theta$$

Substituting in (B),

$$\left(\frac{\cos^2\theta}{a'^2}+\frac{b^2\sin^2\theta}{a^2a'^2}-\frac{1}{a^2}\right)\left(\frac{\cos^2\theta}{b'^2}+\frac{a^2\sin^2\theta}{b^2b'^2}-\frac{1}{b^2}\right)$$

$$=\frac{(a'^2-b'^2)^2}{a'^4b'^4}\cos^2\theta\sin^2\theta$$

or $$(a^2\cos^2\theta+b^2\sin^2\theta-a'^2)(b^2\cos^2\theta+a^2\sin^2\theta-b'^2)$$
$$=(a^2-b^2)^2\sin^2\theta\cos^2\theta$$

or $$a^2b^2(\cos^2\theta+\sin^2\theta)^2-(a^2b'^2+a'^2b^2)\cos^2\theta$$
$$-(b^2b'^2+a^2a'^2)\sin^2\theta+a'^2b'^2=0$$

or $$(a^2b'^2+a'^2b^2)\cos^2\theta+(b^2b'^2+a^2a'^2)\sin^2\theta$$
$$=(a'^2b'^2+a^2b^2)=(a'^2b'^2+a^2b^2)(\cos^2\theta+\sin^2\theta)$$

i. e. $(a'^2-b^2)(a^2-b'^2)\sin^2\theta=(a^2-a'^2)(b^2-b'^2)\cos^2\theta.$

$$\therefore\quad \tan^2\theta=\frac{(a^2-a'^2)(b^2-b'^2)}{(a'^2-b^2)(a^2-b'^2)}.$$

The result now follows from the fact $\frac{a}{a'}=\frac{b}{b'}$.

Exercises XLVII

1. If two conics $S=0$, $S'=0$ touch at a point P, all conics of the system $S+\lambda S'=0$ touch at P.

2. The polars of a point *w.r.* to all conics of the pencil $S+\lambda S'=0$ pass through a fixed point.

3. The locus of the poles of a fixed line $w.r.$ to the pencil of conics $S+\lambda S'=0$ is a conic.

4. Show that the locus of the poles of a given st. line $w. r.$ to conics which pass through the angular points of a given square is a rectangular hyperbola.

5. Find the three values of λ, for each of which the cartesian equation

$$(\lambda-1)(x+y+4)^2+(\lambda-2)(3x-4y+2)^2+(\lambda-3)(x-y-11)^2=0$$

represents a pair of straight lines. Hence or otherwise show that the three pairs of common chords of the two conics

$$(x+y+4)^2+(3x-4y+2)^2+(x-y-11)^2=0$$
$$(x+y+4)^2+2(3x-4y+2)^2+3(x-y-11)^2=0$$

have for their respective equations

$$2(x+y+4)^2+(3x-4y+2)^2=0\ ;\ (x+y+4)^2-(x-y-11)^2=0\ ;$$
$$(3x-4y+2)^2+2(x-y-11)^2=0.$$

[C. U., B.A. & B.Sc. Hons., 1927].

6. Show that the locus of the centres of all conics which pass through the points $(0, 0)$, $(0, 1)$, $(-\frac{1}{2}, \frac{3}{2})$, $(-2, 0)$ is the conic $2x^2-2y^2+4xy+5y-2=0$.

7. Prove that the locus of centres of all conics which pass through the centres of the inscribed and escribed circles of a triangle is the circumcircle of the triangle.

8. If the centre locus of a pencil of conics is a rectangular hyperbola, show that the system contains a circle.

9. The common self-conjugate triangle of the conics

$$ax^2+by^2=1, \qquad xy+gx+fy=0$$

has for its vertices the points (x_1, y_1), (x_2, y_2), (x_3, y_3), prove that

$$\frac{a}{f}\,x_1x_2x_3+\frac{b}{g}\,y_1y_2y_3=1.$$

[*Hint.*—The polars of (x_1, y_1) $w.r.$ to the two conics are identical, $\dfrac{ax_1}{y_1+g}=\dfrac{by_1}{x_1+f}=-\dfrac{1}{gx_1+fy_1}$. Thus the vertices of the triangle lie on the conics

$$agx^2+afxy+y+g=0, \quad bfy^2+bgxy+x+f=0.$$

These conics have common asymptotic directions, and therefore intersect in three finite points which are the vertices of the triangle.]

10. Prove that the locus of the centres of conics touching the four straight lines

$$y=mx,\ y=m'x,\ y=\lambda(x-a),\ y=\lambda'(x-a)$$

is the straight line

$$(2x-a)[mm'(\lambda+\lambda')-\lambda\lambda'(m+m')]=2y(mm'-\lambda\lambda').$$

11. Prove that the poles of the line p, whose equation is $lx+my+1=0$ w. r. to the conics of a confocal system $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$ lie on a line q. Prove further that, if p passes through a fixed point (α, β), q touches the parabola

$$(-\alpha x+\beta y+a^2-b^2)^2+4\alpha\beta xy=0.$$

Show that this parabola is also the envelope of the polars of (α, β) w.r. to the conics of the confocal system.

12. Show that the locus of the feet of the normals drawn from a fixed point (h, k) to each of the system of confocal conics given by the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$$

is the cubic curve $\frac{x}{y-k}+\frac{y}{x-h}=\frac{a^2-b^2}{hy-kx}$.

13. Show that the locus of the foci of parabolas, to which the triangle of reference is self-polar, is the nine-point circle.

14. Prove that the tangents at (f, g) to the conics that pass through it and are confocal with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, are given by the equation

$$fg\{(x-f)^2-(y-g)^2\}+(g^2-f^2+a^2-b^2)(x-f)(y-g)=0.$$

15. If a_1, b_1; a_2, b_2 be the semi-axes of the conics of the confocal system $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, which pass through a fixed point (x_0, y_0), show that

(*i*) $a_1^2a_2^2=(a^2-b^2)x_0^2$, $b_1^2b_2^2=-(a^2-b^2)y_0^2$

(*ii*) $x_0^2+y_0^2=a_1^2+b_2^2=a_2^2+b_1^2$.

[*Hint.*—$(a^2-b^2)(x_0^2+y_0^2)=a_1^2a_2^2-b_1^2b_2^2=a_1^2(a_2^2-b_2^2)+b_2^2(a_1^2-b_1^2)$.

Hence $x_0^2+y_0^2=a_1^2+b_2^2$, for $a_2^2-b_2^2=a_1^2-b_1^2=a^2-b^2$]

16. Show that the locus of the centres of all rectangular hyperbolas inscribed in the triangle of reference is the self-conjugate circle.

17. If ϕ denote the angle between the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$ from the point, the confocals through which have the primary semi-axes a_1, a_2, show that

$$\tan\phi=\frac{2\sqrt{(a_1^2-a^2)(a^2-a_2^2)}}{(a_1^2-a^2)-(a^2-a_2^2)} \text{ and } \tan\frac{\phi}{2}=\sqrt{\frac{a^2-a_2^2}{a_1^2-a^2}}.$$

Hence deduce that the locus of a point such that the tangents from it to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may contain an angle 2α is given by $a_1^2\cos^2\alpha+a_2^2\sin^2\alpha=a^2$, where a_1, a_2 are the primary semiaxes of the confocals through the point.

18. Show that the locus of the centres of the rectangular hyperbolas, with respect to which the triangle of reference is self-conjugate, is the circumcircle.

19. Show that if ψ be the angle which the tangents from P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ make with the tangent at P to the confocal through P, then

$$\sin\psi=\sqrt{\frac{\lambda_1}{\lambda_1-\lambda_2}}$$

where λ_1, λ_2 are the parameters of the confocal ellipse and hyperbola through P.

20. Show that the locus of the foci of all conics touching the four lines $l\alpha\pm m\beta\pm n\gamma=0$ is a cubic.

21. Prove that the two conics $ax^2+2hxy+by^2=1$ and $a'x^2+2h'xy+b'y^2=1$ can be placed so as to be confocal if

$$\frac{(a-b)^2+4h^2}{(ab-h^2)^2}=\frac{(a'-b')^2+4h'^2}{(a'b'-h'^2)^2}.$$

22. Show that the circumcircle of a triangle self-conjugate for a conic is orthogonal to the director circle of the conic.

23. The general equation of a conic confocal with

$$ax^2+2hxy+by^2=1$$

is $(x^2+y^2)(ab-h^2)+\lambda(ax^2+2hxy+by^2)=\dfrac{(a+\lambda)(b+\lambda)-h^2}{\lambda}$.

24. A line cuts two given conics in P, P′, and Q, Q′, so that the range $(\mathrm{PQP'Q'})=-1$; show that envelope of the line is a conic which touches the eight tangents to the given conics at their four points of intersection.

25. Prove that the general equation of a confocal with the conic S is $\Delta S+\lambda C\phi+\lambda^2=0$, where $\phi=0$ is the equation of the orthoptic circle in its normal form and Δ is the discriminant of ϕ.

26. Show that the general equation of conics whose foci are the given points $(a, b), (-a, -b)$ is

$$(x^2-a^2-\lambda)(y^2-b^2-\lambda)=(xy-ab)^2.$$

27. A pair of tangents to any confocal of the conic

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$$

pass respectively through the fixed points $(0, c_1)$, and $(0, c_2)$, show that the intersection of the tangents lies on the circle

$$(x^2+y^2-a^2+b^2)(c_1+c_2)=2y(c_1c_2-a^2+b^2).$$

28. If λ, μ are the values of the parameters for the parabolas of the confocal system $y^2=4\lambda(x+\lambda)$ which pass through P, prove that the angle ϕ between the tangents from P to the parabola $y^2=4k(x+k)$ is given by $\tan^2\phi=-(\lambda-k)(\mu-k)$.

29. Show that the tangent to the inner of the two concentric homothetic ellipses cuts the other in points whose eccentric angles have a constant difference.

30. Show that the tangent to the inner of the two concentric and homothetic ellipses and terminated by the outer is bisected at the point of contact.

31. Pairs of tangents are drawn to the conic $\alpha x^2+\beta y^2=1$, so as to be always parallel to conjugate diameters of the conic

$$ax^2+2hxy+by^2=1,$$

show that the locus of their intersection is a conic which is homothetic with the latter conic.

Msicellaneous Exercises XLVIII

1. A system of conics is drawn having four point contact with the conic

$$ax^2+2hxy+by^2+2fy=0$$

at the origin, prove that the orthoptic circle of these conics form a co-axal system whose limiting points are

$$(0,\ 0) \text{ and } \frac{-hf}{a^2+h^2},\ \frac{af}{a^2+h^2}.$$

2. When one of the base points of a pencil of conics is at infinity, so that there is one parabola in the system and the centre locus is a parabola, show that the former parabola is double the size of the latter.

3. A conic touches the sides of a parallelogram, show that the foci lie on a rectangular hyperbola through the corners.

4. Show that the general equation of conics having the points (a, b), (a', b') as their foci may be written as

$$\rho^2[x(b-b')-y(a-a')+ab'-a'b]^2+$$
$$2\rho[(x-a)(x-a')+(y-b)(y-b')]-1=0,$$

or $$\rho^2[(x-a)(b-b')-(y-b)(a-a')]^2$$
$$+2\rho[(x-a)(x-a')+(y-b)(y-b')]-1=0.$$

5. Prove that the two families of concentric conics which have their centre at the point (a, b) and touch the axes of x and y respectively at the origin are represented by the equations

$$(ay-bx)^2=A(y^2-2by),\quad (ay-bx)^2=B(x^2-2ax)$$

respectively, A and B having arbitrary values. Prove also that if $A+B=0$ the pairs of conics are confocal.

6. The conic

$$\frac{x^2}{a^2+\lambda a'^2}+\frac{y^2}{b^2+\lambda b'^2}=\frac{1}{1+\lambda}$$

is for all values of λ inscribed in the same quadrilateral.

7. Prove that the locus of the centres of all conics which have a double contact with a given conic, the chords of contact being in fixed directions, is the diameter of the given conic which is conjugate to the given direction.

8. The conic similar and similarly situated to the general conic $\Sigma ux^2+2\Sigma fyz=0$ and circumscribing the triangle of reference is

$$a(vc^2+wb^2-2bcf)yz+b(wa^2+uc^2-2acg)zx \\ +c(ub^2+va^2-2hab)xy=0,$$

the system of co-ordinates being trilinear.

Deduce that the general equation may represent a circle.

9. If $S=0$ be the equation of any conic in trilinear co-ordinates, prove that the equation of the similar and similarly situated conic through three given points (x_r, y_r, z_r) : $r=1, 2, 3$ is

$$\begin{vmatrix} S & x & y & z \\ S_1 & x_1 & y_1 & z_1 \\ S_2 & x_2 & y_2 & z_2 \\ S_3 & x_3 & y_3 & z_3 \end{vmatrix}=0$$

where S_r denotes the value of S for $x=x_r$, $y=y_r$, $z=z_r$.

10. If $S=0$, $S'=0$ be two conics and U, V, a pair of their chords of intersection, such that $S-S'=UV$, then

$$k^2U^2-2k(S+S')+V^2=0$$

represents a conic having double contact with S and S'.

[Hint. The equation can be written in either of the forms $(kU+V)^2-4k\,S=0$, $(kU-V)^2-4kS'=0$].

11. The general equation of a conic, having double contact with $S=0$, and $S+L^2+M^2=0$ is $S-(L\cos\alpha+M\sin\alpha)^2=0$.

12. If a conic has a double contact with two others which have the same focus and directrix, the chords of contact pass through the focus, and are perpendicular to each other.

13. The equation of a conic having double contact with

$$A^2x^2+B^2y^2=C, \quad A^2x^2+B^2y^2=D \text{ is}$$

$$Ax^2+By^2=\left(1-\frac{C}{D}\right)(Ax\cos\theta+By\sin\theta)^2.$$

14. Show that the equation of conic envelopes can be represented by the equation

$$\frac{x^2}{a+\lambda}+\frac{y^2}{b+\lambda}+\frac{z^2}{c+\lambda}=0,$$

where λ is the variable parameter. Find the centre locus of the system and prove that it is given by $\Sigma\ \frac{x}{p}\ (b-c)=0$, if $px+qy+rz=0$ is the line at infinity

15. If the conic $ux^2+vy^2+wz^2=0$ in areal co-ordinates touches at a finite point the conic similar and similarly situated. but which passes through the angular points of the triangle of reference, show that $u+v+w=0$, and that the conics are hyperbolas. (Pembroke, 1901)

16. If $\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}=0$, the three lines $\frac{x}{l_1}+\frac{y}{m_1}+\frac{z}{n_1}=0$,

etc., touch a conic inscribed in the triangle of reference. Prove further that the three lines form a triangle inscribed in a conic in which the triangle of reference is inscribed. (Pembroke, 1928)

17. Prove that, by a suitable choice of homogeneous co-ordinates, (x, y, z), the equation of any conic passing through four fixed points can be taken as

$$ax^2+by^2+cz^2=0\ ;\quad a+b+c=0.$$

Prove also that, if any point P of the conic is joined to the four fixed points, the cross-ratio of these four lines has one of the values $-\lambda/\mu$, where λ, μ are any two of the co-efficients.

18. If α, β are the given eccentric angles of the points P, Q on a variable ellipse of a given confocal family, and if the tangents at P and Q meet in T, prove that the locus of P is a confocal hyperbola, and that the locus of T is also a hyperbola. (Downing, 1929)

19. The foci of a confocal family divide the segment AB harmonically. Show that there is one member of the confocal family touching the four common tangents of the two circles with centres A and B, and radii a and b respectively, a and b being arbitrary. (Kings etc., 1930).

20. Show that the co-ordinates of the centre of the rectangular hyperbola passing through four fixed points, referred to

the common self-conjugate triangle of all the conics through the four points are given by

$$\frac{x}{a}(\eta^2-\zeta^2)=\frac{y}{b}(\zeta^2-\xi^2)=\frac{z}{c}(\zeta^2-\eta^2)$$

where (ξ, η, ζ) is any one of the four points and the system of co-ordinates is trilinear.

21. Show that the equation of the conic (in trilinear system of co-ordinates) which touches the sides of the triangle of reference and is confocal with the conic $\Sigma\, ayz=0$ is

$$\sqrt{x}\sin\frac{A}{2}+\sqrt{y}\sin\frac{B}{2}+\sqrt{z}\sin\frac{C}{2}=0.$$

22. Through four points there can in general be described two conics of given eccentricity, but if only one can be described, it is either a rectangular hyperbola, or its eccentricity e satisfies the equation

$$e^4\tan^2\alpha+4e^2-4=0,$$

where α is the angle between the axes of the two parabolas through the four points.

23. At a point (x', y') of a rectangular hyperbola $x^2-y^2=a^2$ is drawn the parabola of four-point contact. Prove that the equation of its directrix is $2xx'+2yy'=x'^2+y'^2$.

24. Show that the areal co-ordinates of the centre of the rectangular hyperbola of a system of four-point conics, referred to their self conjugate triangle are given by

$$(b^2z'^2-c^2y'^2)x=(c^2x'^2-a^2z'^2)y=(a^2y'^2-b^2x'^2)z$$

where (x', y', z') are the co-ordinates of the points in the areal system.

25. Show that $c(x^2+y^2)+2xy\sqrt{(a-c)(b-c)}-z^2=0$ has double contact with both of the conics

$$ax^2+by^2-z^2=0,\quad bx^2+ay^2-z^2=0.$$

Bibliography.

The following books were found useful in the preparation of this book. The books (1—9) are of advanced character and the student is advised to take them up at a later stage.

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2. *Darbox*—Principes Geometrie Analytique.
3. *Klein*—Volesungen Über Höbere Geometrie.
4. *Woods*—Higher Geometrie.
5. *Comessati*—Lezionic Di Geometria Analitica e Projectiva.
6. *H. Beck*—Co-ordinaten Geometrie.
7. *Hara and Ward*—Projective Geometry.
8. *Veblen and Young*—Projective Geometry.
9. *Serret*—Geometrie de Direction.
10. *Sommervile*—Analytical Conics.
11. *Jones*—Introduction to Algebraic Geometry.
12. *Askwith*—The Analytical Geometry of the Conic Section.
13. *Niewenglowski*—Geometric Analytique.
14. *Kowalewski*—Analytische Geometrie.
15. *Salmon*—Conic Section.
16. *Salmon, Fiedler, Dingeddey*—Analytische Geometrie der Kegel Sehnitte.
17. *Smith*—Conic Sections.
18. *Loney*—Conic Sections.
19. *Casey*—Analytical Geometry.
20. *Milne*—Homogeneous Co-ordinates.
21. *Ferrars*—Trilinear Co-ordinates.
22. *Scott*—Modern Geometry.
23. *Radford*—Problem Papers.
24. *Cozen*—Higher Mathematical Papers.

ANSWERS

Exercises I. (Page 8)

10. 2, 4.

Examples II. (Page 20)

1. (*i*) 5, (*ii*) 13, (*iii*) 55. **2.** (*i*) $\sqrt{13}$, (*ii*) $\sqrt{117}$.

8. -18. **9.** (*i*) 21, (*ii*) 25. **10.** $\left(6, \dfrac{-9}{2}\right)$, $\left(-6, \dfrac{23}{2}\right)$.

11. (*i*) $\left(\dfrac{11}{2}, \dfrac{11}{2}\right)$, $\left(\dfrac{3}{2}, \dfrac{5}{2}\right)$. (*ii*) $\left(\dfrac{11}{2}, \dfrac{3}{2}\right)$, $\left(\dfrac{1}{2}, \dfrac{27}{2}\right)$.

12. $(4, 1)$; $\sqrt{5}$. **13.** (*i*) $7:13$, $11:9$; (*ii*) $5:17$, $8:3$.

14. (*i*) $(3, 3)$; (*ii*) $(\frac{3}{2}, 0)$.

15. $\left\{\dfrac{2r_1r_2\cos\dfrac{\theta_1-\theta_2}{2}}{r_1+r_2}, \tfrac{1}{2}(\theta_1+\theta_2)\right\}$

16. (*i*) (*a*) $(x_1-x_2)^2+(y_1-y_2)^2=(x_2-x_3)^2+(y_2-y_3)^2$
$=(x_3-x_4)^2+(y_3-y_4)^2=(x_4-x_1)^2+(y_4-y_1)^2$,
and (*b*) $(x_1-x_3)^2+(y_1-y_3)^2=(x_2-x_4)^2+(y_2-y_4)^2$;
(*ii*) (*a*) $(x_1-x_2)^2+(y_1-y_2)^2=(x_3-x_4)^2+(y_3-y_4)^2$,
(*b*) $(x_1-x_4)^2+(y_1-y_4)^2=(x_2-x_3)^2+(y_2-y_3)^2$,
(*c*) $(x_1-x_3)^2+(y_1-y_3)^2=(x_2-x_4)^2+(y_2-y_4)^2$;
(*iii*) (*a*) of (*i*). (*iv*) (*a*), (*b*) of (*ii*).

Examples III. (Page 35)

1. $\dfrac{-3}{2}$. **2.** $\dfrac{1}{2}$. **3.** $\left(\dfrac{20}{13}, \dfrac{99}{26}\right)$, $\left(-\dfrac{20}{13}, \dfrac{291}{26}\right)$.

4. $x-3y=3$, $3x+y=9$. **5.** $x-y=1$, $x-y+3=0$, $x+y=3$, $x+y+1=0$. **8.** $y\sqrt{3}=x$, $x=0$. **9.** $\dfrac{\pi}{2}$.

10. Let $AB=a$, $BC=b$, $\angle BAD=\alpha$, AB, $\theta=0$; AD, $\theta=\alpha$; CD, $a\sin\alpha=r\sin\theta$; BC $a\sin\alpha=r\sin(\alpha-\theta)$; AC $\theta=\tan^{-1}\dfrac{b\sin\alpha}{a+b\cos\alpha}$; BD

$$ab\sin\alpha+br\sin(\theta-\alpha)=a\,r\sin\theta.$$

10. If $y=mx+c$ is the equation of the given line and θ the given angle, and (x', y') the given point, then the equations of the lines are

$$y-y'=\frac{m\pm\tan\theta}{1\mp m\tan\theta}(x-x'),$$

the upper signs going together and the lower signs going together.

Examples IV. (Page 45)

4. $x-y=\dfrac{17}{4}$, $x+4y=\frac{1}{2}$. **5.** $t_1+t_2+t_3=0$. **6.** (10, 4), (6, 0).

7. $bU-aV=0$ $\dfrac{U}{a}+\dfrac{V}{b}=1$.

Examples V. (Page 53)

1. (*i*) $3x+11y-49=0$ $11x-3y+1=0$.

(*ii*) $11x+23y-\dfrac{217}{19}$ $23x-11y+1=0$.

(*iii*) $-\sqrt{2}(x-2y+1)=x-3y+3$.
$\sqrt{2}\,(x-2y+1)=x-3y+3$.

In each case the second bisector bisects the angle in which the origin lies. The first bisector in (*i*) and second in (*ii*) and (*iii*) bisect the angles which contain (2, 2).

2. (*i*) $7x-9y-37=0$, (*ii*) $x-2y+1=0$.
3. (*i*) $9x+7y=12$. (*ii*) $4x+32y+25=0$, (*iii*) $x+3y+3=0$.
4. (*i*) acute. (*ii*) obtuse. **5.** (*i*) inside, (*ii*) outside opposite to C, (*iii*) outside opposite to B. (*iv*) in the angle vertically opposite to B. **6.** (*i*) $\left(\dfrac{35}{34}, \dfrac{-5}{2}\right)$, (*ii*) $\left(\dfrac{193}{112}, \dfrac{-7}{6}\right)$,

(*iii*) $\left(\dfrac{2}{11}, \dfrac{5}{11}\right)$, (*iv*) $(-11, -43)$. **7.** (2, 1).

Miscellaneous Examples VI. (Page 54)

3. $5x+12y=29$. **5.** $19x+3y+26=0$, $13x=23y-58$, $4x=y+3$. **6.** $119a^2$. **7.** (a, d). **13.** $\dfrac{-17}{9}, \dfrac{7}{41}$.

Examples VII. (Page 70)

1. (*i*) $\tan^{-1}\frac{1}{4}$, (*ii*) $\tan^{-1}(\sin\theta)$. **7.** (*i*) -2; (*ii*) $\dfrac{15}{2}$; $-\dfrac{5}{2}$;

(*iii*) $\lambda=6$. (*iv*) -5, $\dfrac{-35}{4}$; (*v*) $-1, -1, -1$. **8.** $y=0$.

$4x+3y=0$. **12.** (*i*) $(ab'-a'b)^2=4(bh'-b'h)(ah'-a'h)$;
(*ii*) $(bb'-aa')^2+4(bh'+a'h)(hb'+ah')=0$.

13. $\lambda=\dfrac{-\Delta}{C}$, $af^2-2fgh+bg^2=0$. **14.** $\dfrac{\pi}{6}$.

Examples VIII. (Page 80)

2. $X+\dfrac{c}{\sqrt{a^2+b^2}}=0$, $X^2-3Y^2=0$.

3. $X^2-3Y^2=0$ or $3X^2-Y^2=0$. **4.** $X^2-Y^2=0$.

5. $3x^2-8xy-3y^2-30y-27=0$.

7. (*i*) $xy=16$; (*ii*) $\frac{x^2}{9}+\frac{y^2}{4}=1$; (*iii*) $\frac{x^2}{9}-\frac{y^2}{4}=1$.

8. $2a$.

Miscellaneous Examples IX. (Page 81)

18. $\frac{2n\sqrt{(l^2+m^2)(h^2-ab)}}{am^2-2hlm+bl^2}, \frac{n^2\sqrt{h^2-ab}}{am^2-2hlm+bl^2}$.

Examples XI. (Page 100)

1. $3y+x=0$. 3. $(x-4)^2+(y-5)^2=100$.
9. $x^2+y^2-10x-10y+25=0$. $x^2+y^2+6x+6y+9=0$.
12. $3x-4y=20$. $4x+3y=35$. 13. $(-3, 2)$. 14. $(-1, -1)$.
18. $x^2+y^2-8x-2y+12=0$. 19. $x^2+y^2-2x+3y-3=0$.

Examples XII. (Page 111)

1. $(2, -1)$. 2. $(1, 2), (-2, -1)$.

Examples XIV. (Page 117)

1. $x^2+y^2-4x-6y-12=0$; $(-2, 0), (5, -1)$. 2. 1 and 5.
3. $y^2=4c(c-x)$.

Examples XV. (Page 146)

1. $(a^2+b^2)(x^2+y^2)+2c(ax+by)=0$.

7. $(-1, 1), (7, 7)$; $\left(\frac{-17}{25}, \frac{31}{25}\right), \left(\frac{-217}{25}, \frac{-119}{25}\right)$.

10. $3x-4y=9$, $15x+8y-81=0$. $y=\frac{9}{2}$, $72x+154y+9=0$.
11. (*i*) $-2x\pm y\sqrt{5}+3\mp\sqrt{5}=0$, $x\pm 2(y-1)\sqrt{2}=6$,
(*ii*) $x=1$, $3x+4y=5$, $y=2$. $4x-3y=5$.

14. $(-2, -1)$, $\left(\frac{8}{5}, \frac{-11}{5}\right)$.

15. $x^2+y^2+2\lambda(ax+by+c)=0$ where λ is a variable parameter. The line of centres is $bx-ay=0$, radical axis is $ax+by+c=0$. Limiting points are $(0, 0)$,

$$\left(\frac{-2ac}{a^2+b^2}, \frac{-2bc}{a^2+b^2}\right).$$

17. $x^2+y^2-2x(a+b)-y(a+b)+a^2+b^2+3ab=0$.

18. Limiting points are $\left\{-\left(\frac{g}{a}+\lambda l\right), -\left(\frac{f}{a}+\lambda m\right)\right\}$

where λ is a root of the equation

$$\lambda^2a^2(l^2+m^2)+2\lambda a(lg+mf-an)+g^2+f^2-ac=0.$$

27. $a+b+c=0$.

Examples XVII. (Page 174)

8. $20y^2-12xy-29x^2+4x+8y-4=0$.

Examples XVIII. (Page 175)

1. $y^2=4ax$. **2.** $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

Exercises XIX. (Page 178)

1. (*i*) real ellipse, (*ii*) hyperbola, (*iii*) hyperbola, (*iv*) real st. lines, (*v*) parabola, (*vi*) parabola.

2. (*i*) a pair of rt. lines if $\lambda=\pm1$, ellipse if $0<\lambda<1$, parabola if $\lambda=0$ and hyperbola if $\lambda<0$ or $\lambda>1$;

(*ii*) pair of lines if $\lambda=\pm\dfrac{1}{\sqrt{5}}$, parabola if $\lambda=\pm1$, ellipse if $-1<\lambda<1$, hyperbola if $\lambda<-1$ or $\lambda>1$;

(*iii*) pair of st. lines if $\lambda=0$, parabola for $\lambda=\pm1$, hyperbola for $-1<\lambda<1$; ellipse for $\lambda<-1$ or $\lambda>1$;

(*iv*) rt. lines if $\lambda=-2$, parabola if $\lambda=-1, 2$, ellipse for $-1<\lambda<2$ and hyperbola if $\lambda<-1$ or >2;

(*v*) st. lines if $\lambda=1$, parabola for $\lambda=-1$, ellipse if $-1<\lambda<1$, and hyperbola if $\lambda<-1$ or $\lambda>1$;

(*vi*) st. lines if $\lambda=-\frac{1}{2}, 1$; parabola if $\lambda=0$; ellipse if $0<\lambda<1$ and hyperbola if $\lambda<0$ or $\lambda>1$.

3. $\lambda=0$ imaginary lines, parallel lines $\lambda=-1$, real lines for $\lambda=2$.
If $-1<\lambda<0$, imaginary ellipse; $0<\lambda<1$ real ellipse.
If $\lambda=1$, parabola.
If $\lambda<-1$ or $\lambda>1$ hyperbola excepting for $\lambda=2$.

4. $4(x-2a)^3=27ay^2$. **6.** $a(y+f)^2=(x+g)(gx+fy+c)$.

12. $(1, 1)$.

Exercises XX. (Page 199)

13. $\dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=1$.

Miscellaneous Exercises XXI. (Page 221)

3. $By(y-k)+Ax(x-h)=[By(x-h)-Ax(y-k)]\times\tan\alpha$, where α is the given angle.

37. $\pm x\pm y=\dfrac{b^2-a^2}{\sqrt{a^2+b^2}}$.

Miscellaneous Exercises XXVIII. (Page 245)

2. $4y=x$. **3.** $y=\dfrac{9}{8}x$; $y=\dfrac{32}{9}x$. **4.** $5x=\pm4y$.

5. If (α, β) be the co-ordinates of either focus $\alpha=\beta=\pm a$, $\alpha=-\beta=\pm ia$, $x+y=\pm a$, $x-y=\pm ia$.

Exercises XXXVII. (Page 339)

3. If S denotes the left hand side of each equation, the equation of the asymptotes is $S+k=0$ where
(*i*) $k=-6$, (*ii*) $k=-4$, (*iii*) $k=-4$, (*iv*) $k=-7$.

4. With the notation of Ex. 3 above, the equations of the conjugate hyperbolas is $S+2k=0$.

5. The equations of the lines are
$x+y=0$, $7x-y+8=0$.
$\left(\frac{-3}{4}, \frac{3}{4}\right)$, $\left(\frac{-25}{24}, \frac{17}{24}\right)$.

6. $x+y=3$ $\left(\frac{16}{25}, \frac{59}{25}\right)$. **7.** $x-2y+5=0$, $\tan^{-1}\frac{1}{2}$, $2\sqrt{5}$.

8. $(-\sqrt{3}, \sqrt{3})$, $(\sqrt{3}, -\sqrt{3})$.

9. $x+y=2$, $x-y=0$ $r^2=\frac{5}{3}$, 5. $\frac{x^2}{5}+\frac{y^2}{\frac{5}{3}}=1$.

11. $3x+4y-5=0, \frac{17}{5}$. **12.** $\frac{(2x+y-3)^2}{80}+\frac{(x-2y+2)^2}{45}=1$

or $\frac{(2x+y-3)^2}{45}+\frac{(x-2y+2)^2}{80}=1$.

14. $2x^2+4xy+3y^2-3x-2y=0$.

15. $x^2+2xy-y^2+2x+4y+\frac{1}{2}=0$; $x^2+2xy-y^2+2x+4y+1=0$.

16. $(2x+3y+5)(5x+3y-8)+24=0$.

Exercises XXXVIII (Page 344)

1. If $2r$ is the length of either axis,

then $r^2=$ (*i*) $\frac{3}{2}, -6$; (*ii*) $1, \frac{-3}{2}$;

(*iii*) $\frac{3}{5}, \frac{2}{5}$; (*iv*) 9, 4; (*v*) $\frac{3}{5}, \frac{3}{10}$.

2. (*i*) $4\sqrt{2}$, (*ii*) 1.

Exercises XXXIX. (Page 352)

1. (*i*) (0,0), $\left(\frac{7}{3}, \frac{4}{3}\right)$, $21(x^2+y^2)+56x+36y+30=0$

(*ii*) $(0,-2)$, $\left(1, -\frac{3}{2}\right)$, $2(x^2+y^2)-2x+7y+7=0$

(*iii*) $(1,-2)$, $(-1, 1)$ $x^2+y^2+y+4=0$

(*iv*) $\left(\frac{-6}{5}, \frac{2}{5}\right)$, $\left(\frac{-8}{5}, \frac{-1}{5}\right)$, $x^2+y^2-2x+y+2=0$

(*v*) $(-1,-3)$, $(2, 2)$ $x^2+y^2-x+y-28=0$

(vi) $\left(\frac{8}{3}, \frac{4}{3}\right)$ $x+2y+\frac{25}{3}=0.$

2. $3(x^2+y^2)+2x+2y-2=0$

4. $\beta y^2(\alpha x^2+\beta y^2-1)=k^2(\beta y^2-1)^2$

Miscellaneous Exercises XL. (Page 355)

7. $(2, -3)$, $5x-3x+7=0$. **8.** $x-y+1=0$, $x-y-\frac{5}{3}=0$.

9. $36x-36y+77=0$, $\left(\frac{-23}{72}, \frac{-31}{72}\right)$.

10. $\left(\frac{-2}{5}, \frac{2}{5}\right)$, $\left(\frac{-31}{25}, \frac{13}{25}\right)$. **11.** Ellipse $\frac{1}{2}\sqrt{2}$.

12. $(1, -2)$, $2x+y=0$, $x-2y=5$, $\frac{1}{2}\sqrt{2}$.

13. $\frac{1}{2}\sqrt{2}$, $(x+2y)^2-(2x-y+3)^2=9$.

14. $10x+5y=31$, $5x-10y=3$.

15. Ellipse, $\left\{\frac{1}{5}(3\pm\sqrt{3}), \frac{2}{5}(2\pm\sqrt{3})\right\}$.

16. Directrix $3x+2y+4=0$, $\left(\frac{12}{13}, \frac{8}{13}\right)$, $3x+2y=8$, $\frac{1}{3}\sqrt{3}$.

17. $7y^2-24xy+20x=0$.

18. $(ab-h^2)(x^2+y^2)=(ax+hy)^2+(hx+by)^2$.

20. $3X^2-Y^2=\frac{1}{3}\Sigma(b-c)^2$. $\sqrt{\frac{2}{3}}$.

21. $2xy=1$, $16(x^2+y^2)-18xy-8(x+y)+1=0$.

22. Centre $\left(\frac{3}{2}, \frac{1}{2}\right)$, foci $\left\{\frac{3}{2}\pm\sqrt{2}, \frac{1}{2}\pm\frac{1}{3}\sqrt{2}\right\}$.

25. $(2ab-\lambda)(x^2+y^2)-2bx-2ay=0$.

31. $y^2=\frac{7}{9}(x-y)$. **32.** $2x^2-2xy+y^2=1$,
$337x^2-162xy+81y^2=144$.

36. $(x-y)^2=7x+5y$, $(x+2y)^2=7x+20y$,
$2x-2y=1$, $x+2y=\frac{47}{10}$. **39.** $8x^2+5y^2-4xy+24x-24y=0$.

43. $x+4=0$, $y+2=0$.

49. $(a't+b')x+y(at+b)+t^2(ba'-b'a)=0$.

Examples XLVII. (Page 458)

5. $\lambda=1, 2, 3$.

31. $ax^2+2hxy+by^2=\frac{a}{\alpha}+\frac{b}{\beta}$.

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