Probabilistic Classification

MLFA

Generative Model

- A story about how the observed data were “born”
- Story in the language of probability!

- Treat labels Y and features X as random variables
- Story outline:
- - Y created first
- - X created based on Y

Generative Model

- A story about how the observed data were "born"
- Story in the language of probability!

- Treat labels Y and features X as random variables
- Story outline:
- - Y created first $\quad$ $p(Y)$: prior distribution
- - X created based on Y $\quad$ $p(X|Y)$:class-conditional distribution
- Probabilistic Classification: $p(Y|X)$: posterior distribution

Prior Distribution

- Prior: information before observing X

	Y =1	Y = 2	Y = 3	Y = 4
	80	50	40	30

- P(Y = k) = ?
- Frequentist approach: just relative frequencies!

	Y = 1	Y = 2	Y = 3	Y = 4
	0.4	0.25	0.20	0.15

Posterior Distribution

- Posterior: information after observing X
- $P(Y = k \mid X) = ?$

- Bayes Theorem:
- $P(Y = k \mid X) = (p(X \mid Y = k) * p(Y = k)) / p(X)$

$$= K * p(X \mid Y = k) * p(Y = k)$$

Class-conditional Distribution

	Y = 1	Y = 2	Y = 3	Y = 4
X < 15	40	45	10	5
X > 15	40	5	30	25

- $P(X \mid Y = k) = ??$
- $P(X<15 \mid Y=1) = 40/(40+40) = 0.5$
- $P(X>15 \mid Y=3) = 30/(30+10) = 0.75$

- $P(Y = k \mid X) = ???$

Frequentist Approach: Direct estimation

	Y = 1	Y = 2	Y = 3	Y = 4	
X < 15	40	45	10	5	100
X > 15	40	5	30	25	100

- $P(Y = 1 \mid X<15) = 40 / 100 = 0.4$

	Y = 1	Y = 2	Y = 3	Y = 4	
$p(Y \mid X < 15)$	0.4	0.45	0.10	0.05	1.0
$p(Y \mid X > 15)$	0.4	0.05	0.30	0.25	1.0

- Similar to Decision Trees

Bayesian Approach: Posterior Distribution

	Y = 1	Y = 2	Y = 3	Y = 4
X < 15	40	45	10	5
X > 15	40	5	30	25

- $P(Y = 1 \mid X<15) = K * p(X<15 \mid Y=1) * p(Y=1) = K*(40/80)*(80/200)$
- $P(Y = 2 \mid X<15) = K * p(X<15 \mid Y=2) * p(Y=2) = K*(45/50)*(50/200)$
- $P(Y = 3 \mid X<15) = K * p(X<15 \mid Y=3) * p(Y=3) = K*(10/40)*(40/200)$
- $P(Y = 4 \mid X<15) = K * p(X<15 \mid Y=4) * p(Y=4) = K*\ (5/30)*(30/200)$
- K = ???

Posterior Distribution

	Y = 1	Y = 2	Y = 3	Y = 4
X < 15	40	45	10	5
X > 15	40	5	30	25

	Y = 1	Y = 2	Y = 3	Y = 4
Prior p(Y)	0.40	0.25	0.20	0.15
p(Y | X < 15)	0.40	0.45	0.10	0.05
P(Y | X > 15)	0.40	0.05	0.30	0.25

Bayesian or Frequentist?

- Frequentist approach: Estimate p(Y | X) from data
- Frequentist approach: More robust if some classes are rare
- Frequentist approach: More straightforward

- Bayesian approach: Estimate p(X | Y) and p(Y) from data
- Bayesian approach: More robust if some feature values are rare
- Bayesian approach: More interpretable in many real applications
- Huge advantage for high-dimensional features!!!

Probabilistic Classifier

- Predicted label : mode of the posterior distribution!
- $Y_{pred} = \text{argmax}_k \, p\,(Y = k \mid X)$
- Confidence of the prediction = $p(Y = Y_{pred} \mid X)$

- If Bayesian approach used for $p(Y \mid X)$: Bayesian Classifier!

Posterior Distribution

	Y = 1	Y = 2	Y = 3	Y = 4
X < 15	40	45	10	5
X > 15	40	5	30	25

	Y = 1	Y = 2	Y = 3	Y = 4	Ypred
Prior	0.40	0.25	0.20	0.15	1
X < 15	0.40	0.45	0.10	0.05	2
X > 15	0.40	0.05	0.30	0.25	1

Multi-dimensional Features

	Y = 1	Y = 2	Y = 3	Y = 4
X1<15	40	45	10	5
X1>15	40	5	30	25
X2 = a	40	30	15	30
X2 = b	40	20	25	0

- $P(Y = k \mid X1=12, X2=a) = ????$
- We need Joint Distribution of the features!!!

Multi-dimensional Features

	Y = 1	Y = 2	Y = 3	Y = 4
X1<15, X2=a	30	25	0	5
X1<15, X2=b	10	20	10	0
X1>15, X2=a	10	5	15	25
X1>15, X2=b	30	0	15	0

- $P(Y = 3 \mid X1=12, X2 =b) = K * P(X1<15, X2=b \mid Y = 3) *P(Y=3)$

  $= ?$

Multi-dimensional Features

	Y = 1	Y = 2	Y = 3	Y = 4
X1<15, X2=a	0.375	0.50	0	0.166
X1<15, X2=b	0.125	0.40	0.25	0
X1>15, X2=a	0.125	0.10	0.375	0.837
X1>15, X2=b	0.375	0	0.375	0

- $P(Y = 3 \mid X1=12, X2 =b) = K * P(X1<15, X2=b \mid Y = 3) * P(Y=3)$

$$= K * 0.25 * 0.20$$

Multi-dimensional Features

	Y = 1	Y = 2	Y = 3	Y = 4
X1<15, X2=a	0.375 * 0.4 * K1	0.50 * 0.25 * K1	0 * 0.2 * K1	0.166 * 0.15 * K1
X1<15, X2=b	0.125 * 0.4 * K2	0.40 * 0.25 * K2	0.25 * 0.2 * K2	0 * 0.15 * K2
X1>15, X2=a	0.125 * 0.4 * K3	0.10 * 0.25 * K3	0.375 * 0.2 * K3	0.837 * 0.15 * K3
X1>15, X2=b	0.375 * 0.4 * K4	0 * 0.25 * K4	0.375 * 0.2 * K4	0 * 0.15 * K4

- $P(Y = 3 \mid X1=12, X2 =b) = K2 * P(X1<15, X2=b \mid Y = 3) *P(Y=3)$
  $= K2 * 0.25 * 0.20$
  $= ??$

Naïve Bayes Classifier

- D-dimensional feature vector, M values each
- Rows of table = M**D

- Assumption: all features are independent (Naïve!)
- P(X1<15, X2 =b) = p(X1<15) * p(X2=b)

- D tables, rows of each table = M
- Naïve, but computationally efficient!

Naïve Bayes Classification

- $P(Y = k \mid X1<15, X2=b) = K * p(X1<15, X2=b \mid Y = k) * p(Y = k)$

$$= K * p(X1<15 \mid Y = k) * p(X2=b \mid Y = k) * p(Y = k)$$

Final prediction $= \text{argmax}_k\ p(X_1|Y=k)\ p(X_2|Y=k)*\ldots\ldots\ p(X_D|Y=k)*p(Y=k)$

Confidence $= \max_k\ p(X_1|Y=k)\ p(X_2|Y=k)*\ldots\ldots\ p(X_D|Y=k)*p(Y=k)$

Error in Bayes Classifier

- Bayes error probability = total probability of the non-mode classes!
- Risk of prediction = 1 – confidence of prediction
- Bayes error = expected risk (expectation over all X and Y)

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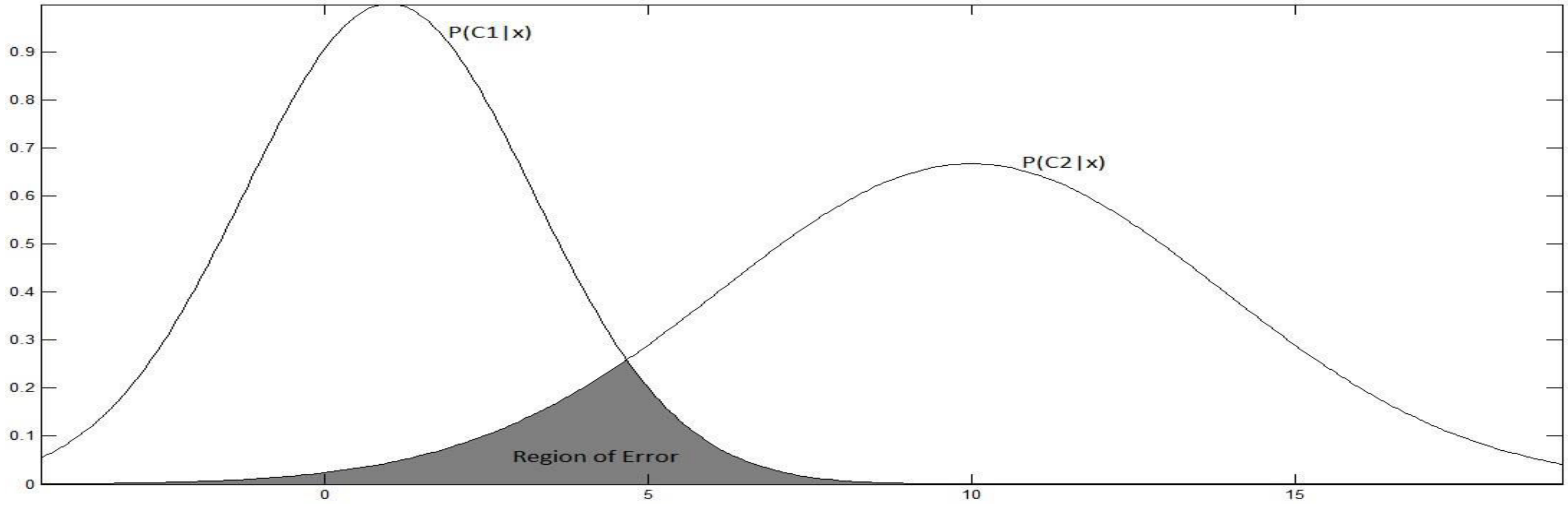
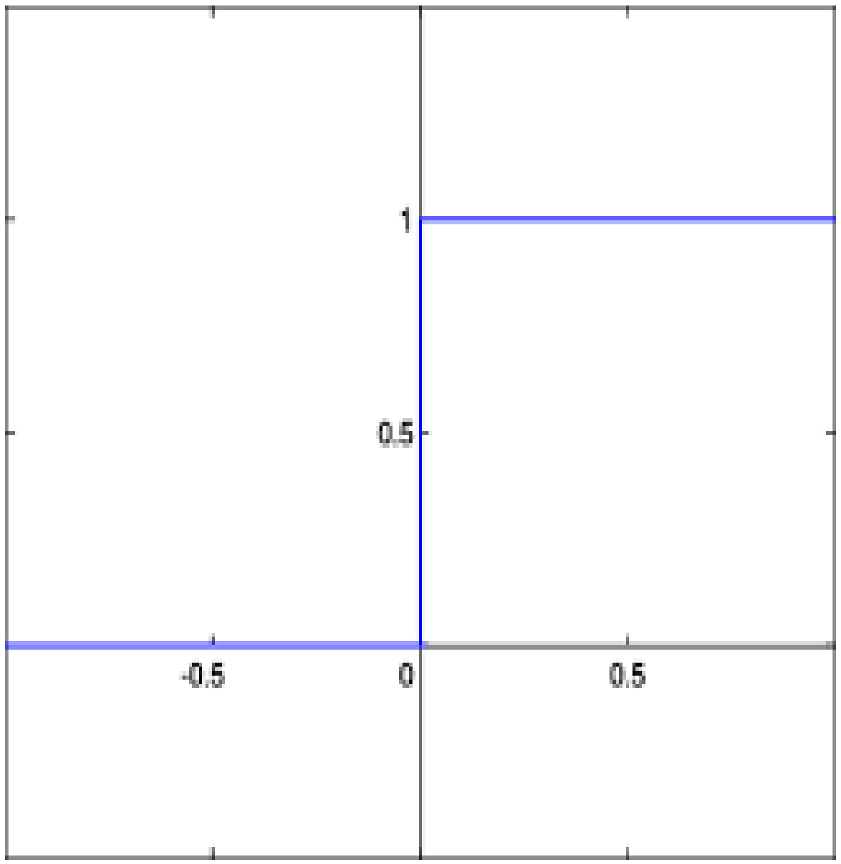


Logistic Regression for Classification



$P(Y = 1 \mid X) = 1$ if $X > 0$

$= 0$ if $X < 0$

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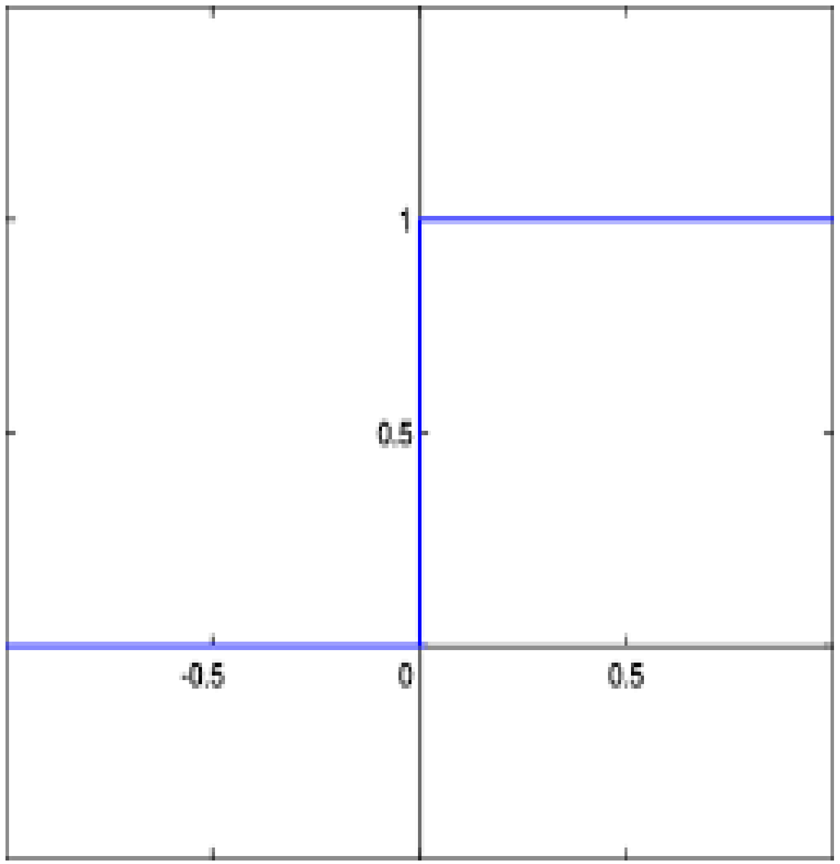

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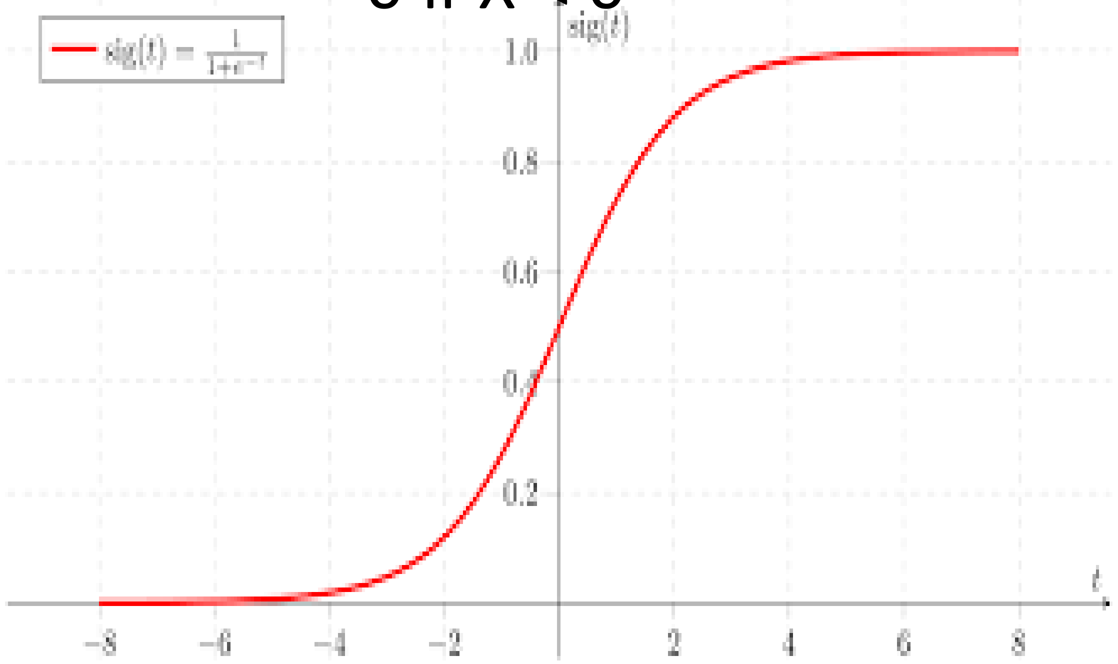

Logistic Regression

- $P(Y = 1 \mid X) = 1$ if $X > 0$

  $= 0$ if $X < 0$
- Approximation: $p(Y = 1 \mid X) = 1/(1 + \exp(-X))$

- Multi-dimensional features: consider weighted combination w.X

- $P(Y = 1 \mid X) = 1/(1 + \exp(-w.X))$ LOGISTIC

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- Multi-dimensional features: consider weighted combination $w.X$
- $P(Y = 1 \mid X) = 1/(1 + \exp(-w.X))$ LOGISTIC
- But how to find w? REGRESSION!