or $$\left[(n_1-1)+(n_2-1)+\ldots+(n_i-1)+\ldots+(n_k-1)\right]^2 = n^2-2nk+k^2$$

or $$\sum_{i=1}^{k}(n_i-1)^2 + \text{Non-negative crossterms} = n^2+k^2-2nk$$

or $$\sum_{i=1}^{k}(n_i^2-2n_i)+k+\text{ Non-negative crossterms } = n^2+k^2-2nk$$

Hence $$\sum_{i=1}^{k} n_i^2 \le n^2+k^2-2nk-k+2n = n^2+(k-1)(2n-1) \text{ proved}$$

10.12.1 Eulerian and Hamiltonian Graphs

We now consider two broad categories of problems where graph theory is used. In the first type of problems, we are interested to traverse a path using each edge of the graph exactly once. Such a problem arises in the delivery of goods through a network of roads where minimum distance is to be covered up in view of economy and time saving. The resulting graph is known as Euler Graph.

In the second type of problems, one has to visit each vertex exactly once. Such a path is useful to service engineers who must service machines on a regular basis. Each machine can be represented by a vertex. The resulting graph is known as Hamiltonian graph.

Euler Path: A path in a graph G is called an Euler path if it includes every edge exactly once. To find Euler path, vertices may be repeated.

Euler Circuit: If the initial and terminal vertices are same on an Euler path, then that path is called an Euler circuit. This is also called Euler line. Cleary, a graph may contain Euler path but not necessarily an Euler circuit.

Definition: Euler Graph

A graph G is called an Euler graph if it has at least one Euler circuit.

Example 1 Consider the graph shown in Fig. 10.45

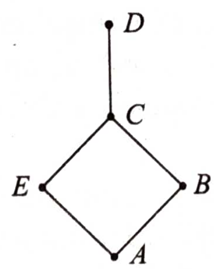

Figure 10.45

The path *DCEABC* contains all the edges exactly once, and hence is an *Euler path*.

However, it is *not* an Euler circuit as starting and ending at the same vertex is not possible without repeating an edge. Thus, the graph of Fig. 10.45 is not an Euler graph.

Example 2 Consider the graphs shown in Figs. 10.46(a) and (b)

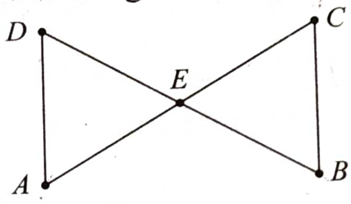

Figure 10.46(a)

The path *AECBEDA* is an Eulerian circuit, and hence the graph is an Euler graph.

Example 3 Consider the graph is Fig. 10.46(b)

The path *A B C D E F C G A* is an Eulerian circuit. Similarly, the path *A G C F E D C B A* is an Eulerian circuit, and hence the graph is an Eulerian graph. It can be observed that each vertex is of even degree.

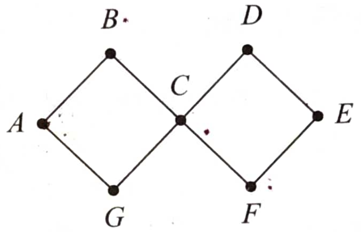

Figure 10.46(b)

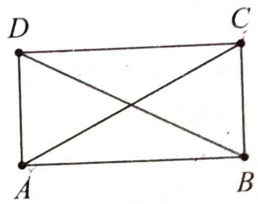

Figure 10.46(c)

A little experimentation will show that no Euler path or circuit is possible for the graph in Fig. 10.46(c). Hence, the graph is not an Euler graph. By its definition, path is always connected and hence Euler graph is always connected.

THEOREM 10.9 A connected graph G is an Euler graph if and only if each vertex of G is of even degree.

Proof: Let G be an Euler graph, and hence there will exist an Euler circuit, that is, a closed walk. In tracing this walk, every time the walk meets the vertex v_k, it will go through two new edges incident on v_k, the one we enter and the other to exist. This is true not only for all intermediate vertices of the walk but also for the starting vertex. Thus if G is an Euler graph, all vertices are of even degree. Conversely, if all vertices of G are of even degree, then G is an Euler graph can be proved.

10.12.2 Königsberg Bridge Problem

From the graph representing the problem of Königsberg Bridge shown in Fig. 10.1(b), we find that all its vertices are not of even degree, and hence it is not an Euler graph. Thus, it is not possible to walk over each of the seven bridges exactly once and return to the starting point. Thus this problem has no solution.

According to Euler, a connected graph is Eulerian if each vertex is of even degree. This theorem does not give any method to find the Eulerian circuit

We can use the following algorithm to find the Eulerian circuit.

10.12.3 Fleury's Algorithm

Let $G = (V, E)$ be a connected graph with each vertex of even degree.

Step 1: Select any starting vertex $v \in V$.

Example 2 Consider the graphs shown in Figs. 10.46(a) and (b)

Figure 10.46(a)

The path $AECBEDA$ is an Eulerian circuit, and hence the graph is an Euler graph.

Example 3 Consider the graph is Fig. 10.46(b)

The path $A\ B\ C\ D\ E\ F\ C\ G\ A$ is an Eulerian circuit. Similarly, the path $A\ G\ C\ F\ E\ D\ C\ B\ A$ is an Eulerian circuit, and hence the graph is an Eulerian graph. It can be observed that each vertex is of even degree.

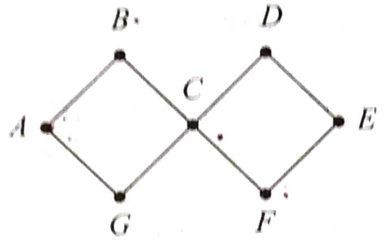

Figure 10.46(b)

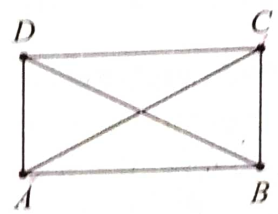

Figure 10.46(c)

A little experimentation will show that no Euler path or circuit is possible for the graph in Fig. 10.46(c). Hence, the graph is not an Euler graph. By its definition, path is always connected and hence Euler graph is always connected.

THEOREM 10.9 A connected graph G is an Euler graph if and only if each vertex of G is of even degree.

Proof: Let G be an Euler graph, and hence there will exist an Euler circuit, that is, a closed walk. In tracing this walk, every time the walk meets the vertex v_k, it will go through two new edges incident on v_k, the one we enter and the other to exist. This is true not only for all intermediate vertices of the walk but also for the starting vertex. Thus if G is an Euler graph, all vertices are of even degree. Conversely, if all vertices of G are of even degree, then G is an Euler graph can be proved.

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According to Euler, a connected graph is Eulerian if each vertex is of even degree. This theorem does not give any method to find the Eulerian circuit

We can use the following algorithm to find the Eulerian circuit.

10.12.3 Fleury's Algorithm

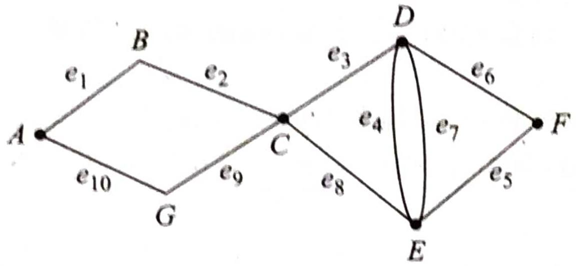
Let $G = (V, E)$ be a connected graph with each vertex of even degree.

Step 1: Select any starting vertex $v \in V$.

Step 2: Traverse any edge, choosing an edge that will disconnect the graph only if there is no other choice.

Step 3: Repeat Step 2 until no edge remains in E.
END

Example: Use Fleury's algorithm on the graph to find an Eulerian circuit.



Solution: Each vertex is of even degree, and hence an Euler circuit is possible.

Step 1: We can start from any vertex. Let us start at the vertex A. We can traverse either edge AB (i.e., e_1) or AG (i.e., e_{10})

We choose AB. At B, there is only one edge BC (i.e., e_2).

At C, we can choose either CD (i.e., e_3) or CE. Let us choose CD. Proceeding step by step, we obtain

$$A \xrightarrow{e_1} B \xrightarrow{e_2} C \xrightarrow{e_3} D \xrightarrow{e_4} E \xrightarrow{e_5} F \xrightarrow{e_6} D \xrightarrow{e_7} E \xrightarrow{e_8} C \xrightarrow{e_9} G \xrightarrow{e_{10}} A$$

10.12.4 Unicursal Graph

Consider the graph shown in Fig. 10.46(d)

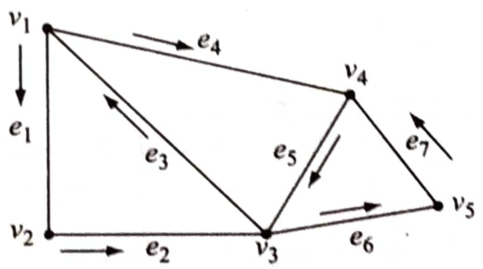


Figure 10.46(d)

The walk $v_1 \to v_2 \to v_3 \to v_1 \to v_4 \to v_3 \to v_5 \to v_4$ includes all edges without retracing any edge is not *closed*. The initial vertex is v_1 and the terminal vertex is v_4. Such an open walk is called an open Euler line or unicursal line. A graph having unicursal line is called *unicursal graph*.

Some Useful Results on Euler Graph

1. If every vertex of a connected graph G is of even degree, then the graph is an Euler graph.
2. If a connected graph has more than two vertices of odd degree, then there can be no Euler path. But if there are exactly two vertices of odd degree, then there exists an Euler path. Any Euler path in G must begin at one vertex of odd degree and end at the other.
3. If a graph G is connected and each vertex has even degree, then there is an Euler circuit in G.
4. If a graph G is connected and has one vertex of odd degree, then there can be no Euler circuit in G.