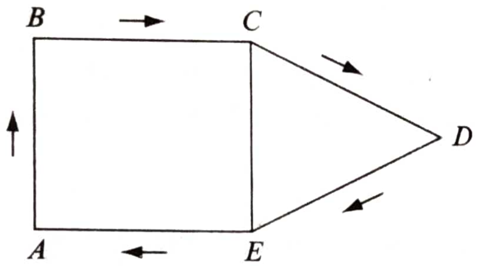
We now consider the second category of graph problems in which each vertex is to be visited exactly once. Such a path is useful to persons serving a set of vending machines on a regular basis. Each vending machine can be regarded as a vertex.

10.12.5 Hamiltonian Paths and Circuits

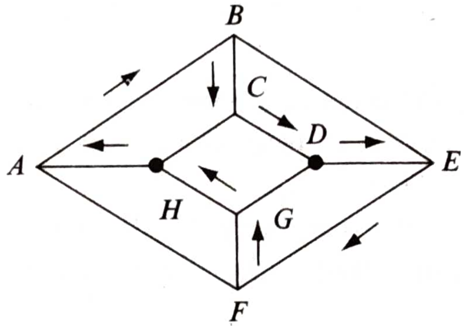
A *Hamiltonian path* in a connected graph G with $|V| \geq 3$ is a path that uses each vertex exactly once. A *Hamiltonian circuit* in a connected graph is a closed path that traverses each vertex exactly once except the starting vertex.

A graph with a closed path that includes every vertex exactly once is called a Hamiltonian graph named after the famous Irish mathematician Sir William Hamilton.



Graph G_1

Figure 10.47(a)



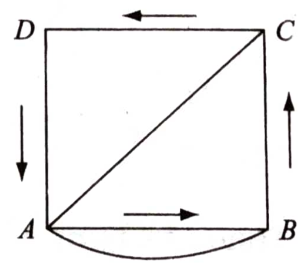
Graph G_2

Figure 10.47(b)

The graph G_1 shown in Fig. 10.47(a) is Hamiltonian as it contains a Hamiltonian circuit (i.e., closed path) $A \to B \to C \to D \to E \to A$

The graph G_2 shown is Fig. 10.47(b) $A \to B \to C \to D \to E \to F \to G \to H \to A$ is Hamiltonian

The graph G_3 shown in Fig. 10.48(a)

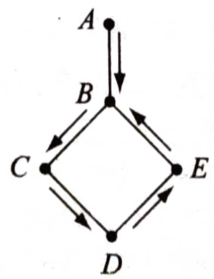


Graph G_3

Figure 10.48(a)

$A \to B \to C \to D \to A$ (choosing either edge from A to B) is Hamiltonian.

In the graph G_4 shown in Fig. 10.48(b)

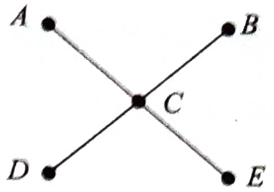


Graph G_4

Figure 10.48(b)

$A \to B \to C \to D \to E$ is a Hamiltonian path because it contains each vertex only once. It is easy to see that there is no Hamiltonian circuit for this graph. Hence, the graph G is not Hamiltonian.

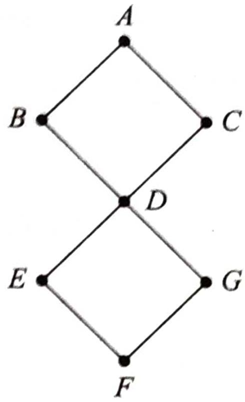
The graph G_5 shown in Fig. 10.48(c) has no Hamiltonian path. It has no Hamiltonian circuit, and hence the graph G_5 is not Hamiltonian.



Graph G_5

Figure 10.48(c)

The graph G_6 is not Hamiltonian because we have to pass through vertex D to return the home vertex choosing any vertex A, B, C to start with.



Graph G_6

Figure 10.48(d)

The following two theorems provide sufficient conditions for a simple connected graph to be Hamiltonian.

ORE'S THEOREM 10.10 Let G be a simple connected graph with $n \geq 3$ vertices, then G is Hamiltonian (i.e., G has a Hamiltonian circuit) if ?

$$\deg(u) + \deg(v) \geq n$$

for every pair of non-adjacent vertices u and v.

DIRAC'S THEOREM 10.11 A simple connected graph G with $n \geq 3$ is Hamiltonian is if $\deg(v) \geq \frac{n}{2}$ for every v in G.

Proof: The proof follows from Ore's theorem, that is, $\deg(u) + \deg(v) \geq n$ for every pair of non-adjacent vertices.

Corollary: The connected graph G with n vertices has a Hamiltonian circuit provided the number of edges in G, $e \geq \frac{1}{2}\left(n^2 - 3n + 6\right)$ *for* $n \geq 3$

Proof: Let the graph G be non-Hamiltonian. Then by Dirac's theorem, there will exist a pair of non-adjacent vertices u and v such that

$$\deg(u) + \deg(v) \leq n - 1$$

Let H be the sub-graph of G obtained by deleting the vertices u and v from G. The graph H will have $(n-2)$ vertices and e-deg(u) – deg(v) edges. Now the maximum number of edges of H can be ${}^{n-2}c_2$.

Hence,

$$e\text{-deg}(u) - \deg(v) \le {}^{n-2}c_2$$

$$\le \frac{(n-2)(n-3)}{2}$$

$$\le \frac{1}{2}\left(n^2 - 5n + 6\right)$$

$$\therefore e \le \frac{1}{2}\left(n^2 - 5n + 6\right) + (n-1)$$

$$\le \frac{1}{2}\left(n^2 - 5n + 6 + 2n - 2\right)$$

$$\le \frac{1}{2}\left(n^2 - 3n + 4\right)$$

$$< \frac{1}{2}\left(n^2 - 3n + 6\right)$$

which is a contradiction.

Note 10.12 Eulerian graph uses every edge exactly once but may repeat vertices while Hamiltonian graph uses each vertex exactly once (except for the first and last) but may skip edges.

The graphs given below illustrate comparison between Eulerian and Hamiltonian graphs. Eulerian as each vertex is of even degree and Hamiltonian as each vertex is of degree $\ge 4/2 = 2$

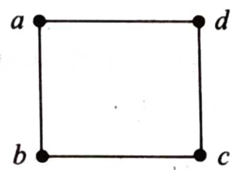


Figure 10.49(a)

Eulerian but non-Hamiltonian (Each vertex is of even degree).

$$a \to b \to c \to d \to e \to f \to d \to a$$

Since vertex d is repeated, and hence non-Hamiltonian.

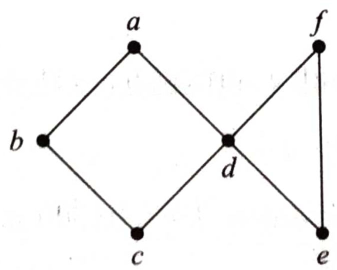


Figure 10.49(b)

Hamiltonian ($a \to b \to c \to d \to a$) but non-Eulerian (Four vertices of odd degree)

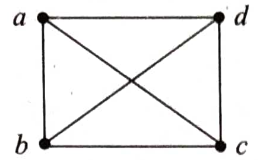


Figure 10.49(c)

Neither Eulerian (more than two vertices of odd degree) nor Hamiltonian (No circuit through all vertices is possible without repetition of vertices)

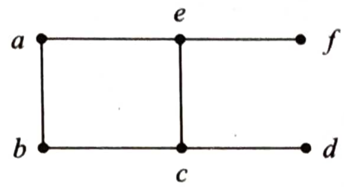


Figure 10.49(d)

Note 10.13

1. The graph in Fig. 10.49(a) is Hamiltonian as each vertex is of degree $\geq \frac{4}{2} = 2$ (there are four vertices).
2. The graph in Fig. 10.49(b) is non-Hamiltonian as there are 6 vertices and there is only one vertex of degree $\geq \frac{n}{2}$, that is, ≥ 3, and other are of degree < 3.
3. The graph in Fig. 10.49(c) is Hamiltonian. The numbers of vertices $n = 4$. Each vertex is of degree $\geq \frac{4}{2}$, that is, 2.
4. The graph in Fig. 10.49(d) is non-Hamiltonian. The number of vertices $n = 6$. There are only two vertices of degree $\geq \frac{n}{2}$, that is $\frac{6}{2} = 3$ and others are of degree <3.

Some Observations on Hamiltonian Circuit for the Graph $G = (V, E)$

1. If G has Hamiltonian Circuit of n vertices, it must consist of exactly n edges. If we remove any edge from the circuit, we shall have Hamiltonian path.
2. Hamiltonian path is a sub-graph of Hamiltonian circuit.
3. The length of a Hamiltonian path in a connected graph of n vertices is $(n - 1)$.
4. If G has Hamiltonian circuit, then each vertex of G must be of degree ≥ 2.

THEOREM 10.12 If a graph G has m edges and n vertices, then G has a Hamiltonian circuit if

$$m \geq \frac{1}{2}\left(n^2 - 3n + 6\right)$$

The converse of the theorem is not true, that is, the given condition is sufficient but not necessary as can be seen from the graph consisting of 8 vertices and 8 edges given in Fig. 10.49(e).

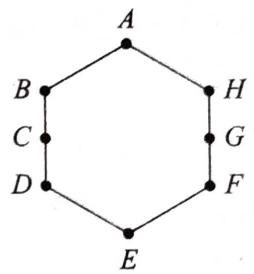


Figure 10.49(e)

The given condition is not satisfied, but there is Hamiltonian circuit for the graph given be $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow F \rightarrow G \rightarrow H \rightarrow A$

10.13 GRAPH COLOURING

Graph colouring problems have gained importance due to their wide practical applications in which 'colour' has different meaning. For example, if the graph represents a connected grid of cities, each city can be marked with the name of airline having most flights to and from the city. In this case, vertices are cities and colours are airline names.

The subject of graph colouring can be viewed to have its origin from the map colouring problem which was conjectured that four distinct colours are required to colour any map drawn on a plane such that no two regions (countries) sharing a common border have the same colour. The problem could not be solved for over a century. In 1976, two American mathematicians Kenneth Appel and Wolfan Haken found the solution with the aid of computer computations performed on almost 2000 configurations of graph that a minimum of four district colours are needed to colour the map. Graph colouring finds its application in scheduling conflict-free examinations for various courses minimizing time slots and also in the design of traffic light patterns at intersections in big cities.

Definition 10.2 Let $G=(V, E)$ be an undirected graph with no multiple edges and $C=\{c_1, c_2, ..., c_n\}$ a set of n colours.

A function $f: V \rightarrow C$ is called colouring of graph G if $f(v_i) \neq f(v_j)$, v_i and v_j being adjacent vertices of V.

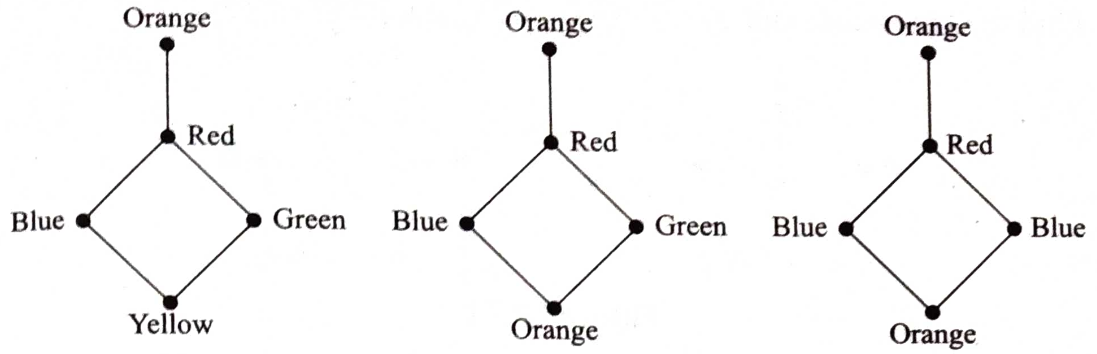
Thus colouring of a graph G means colouring of vertices with different colours.

A colouring is called proper colouring if any two adjacent vertices have different colours. The minimum number of colours needed for a proper colouring of graph G is called the chromatic number of G.

A graph that requires K different colours for its proper colouring is called K-chromatic. The number K is called chromatic number of G.

A graph with n vertices can be coloured by n colours by assigning a colour to each vertex. The proper colouring which is of interest in practical applications is the requirement of minimum number of colours.

Let us consider the graph with 5 vertices shown in Fig. 10.50(a).



(a) Five colours are required

(b) Four colours are required

(c) Three colours are required

Figure 10.50

Fig. 10.50(c) shows that a minimum of three colours are required for proper colouring, and hence its chromatic number is 3.

The definition of chromatic number leads to the following obvious results:

1. The chromatic number of a null graph is 1.
3. The chromatic number of a complete graph with n vertices is n.
3. The chromatic number of any graph with two or more vertices is ≥ 2.
4. The chromatic number of a graph having one or more edges is at least 2.
5. Every graph having a triangle is at least 3-chromatic.
6. A graph consisting of simply one circuit with the number of vertices $n \geq 3$ is 2-chromatic if n is even and 3-chromatic if n is odd. (If n is odd, nth and the first vertex will be adjacent and will have same colour and thus requiring a third colour.
7. Chromatic number of a bipartite graph $k_{m,\, n}$ is 2.

Note 10.14

1. The maximum number of colours required for colouring a connected graph G with n vertices is n.
2. If $d_{\max}$ is the maximum degree of vertices in a graph G, the chromatic number of $G \leq d_{\max} + 1$.

Example 1 Show that the chromatic number of the graph (Fig. 10.51) is 4.

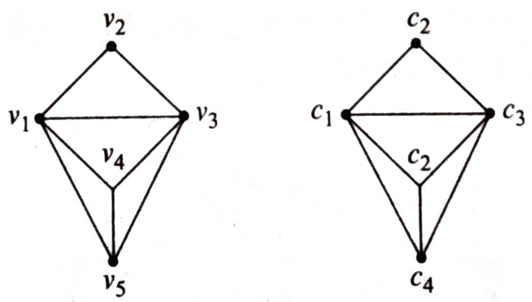


Figure 10.51

Solution: The triangle of vertices v_1, v_2, v_3 require three different colours say c_1, c_2, and c_3. The vertex v_4 is adjacent to v_1 and v_3 and hence must be assigned a colour different from that of v_1 and v_3. Hence, the colour of v_4 is c_2. Now v_5 must be assigned a colour different from c_1, c_2, and c_3 as v_5 is adjacent to v_1, v_4, and v_3. Let c_4 colour be assigned v_5. Hence chromatic number of the graph is 4.

Example 2 Find the chromatic number of the wheel graph (Fig. 10.52)

Figure 10.52

Solution: The triangle *abf* requires three colours c_1, c_2, and c_3. The vertex *c* being adjacent to vertices *b* and *f* cannot be assigned colours c_2 and c_3 and assign it c_1. The vertex *d* can be assigned c_2. The remaining vertex *e* cannot be assigned c_1, c_2, or c_3 as adjacent vertices *a*, *d*, and *f* have these colours. Hence, a colour c_4 will be assigned to vertex *e*. Hence, the wheel graph w_6 is 4 chromatic.

Example 3 Find the chromatic number of the graph K_5, that is, the number of colours necessary for proper colouring of the graph shown in Fig. 10.53.

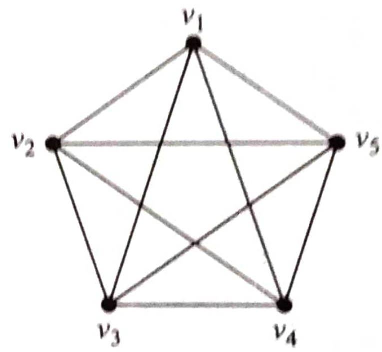


Figure 10.53

Solution: The graph contains triangle which will require 3 colours say for vertices v_1, v_2, v_3. The vertex v_4 cannot be assigned the colour assigned to v_1, v_2, and v_3 as this will amount to have adjacent vertices with the same colour. Hence v_4 will be assigned a fourth colour. Similarly, v_5 will also have to be assigned a new colour. Thus five colours will be needed. That is, chromatic number of K_5 is 5.

Example 4 Find the number of colours for proper colouring of $K_{2,3}$ (Fig. 10.54).

Solution: The chromatic number of any bipartite graph $K_{m,n}$ is 2. Since $K_{3,2}$ is a bipartite, two colours are required for its proper colouring.

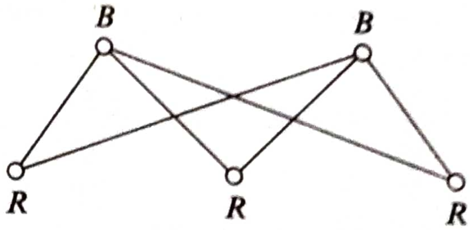
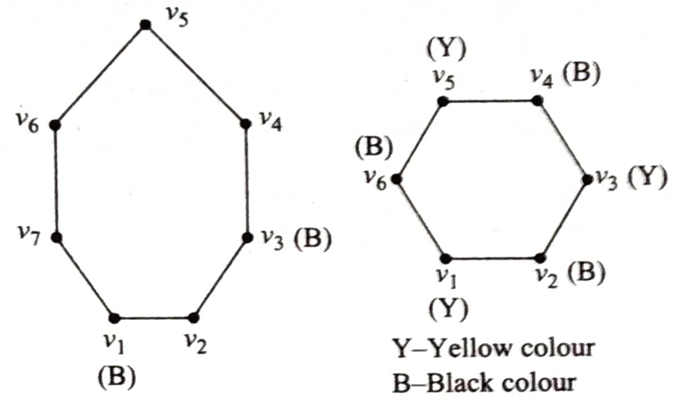


Figure 10.54

The vertex set $K_{2,3}$ can be partitioned into two disjoint subsets V_1 and V_2 such that every vertex in V_1 is adjacent to every vertex in V_2 and vice versa. In addition, no vertices in V_1 and in V_2 are adjacent. Let every vertex in V_1 be assigned one colour (black) and in V_2 another colour (red). Thus, only two colours are required for the graph $K_{2,3}$.

Example 5 Find the chromatic number of the cycle graph C_n.



Solution: Let n be even and vertices be $v_1, v_2, v_3, v_4, \ldots, v_{2n}$. Then odd vertices $v_1, v_3, v_5, \ldots, v_{2n-1}$ can be assigned yellow colour and the even numbered vertices $v_2, v_4, \ldots, v_{2n}$ black colour. If n is odd, let the vertices be $v_1, v_2, \ldots, v_{2n+1}$. If we assign black colour to odd vertices, then v_1 and v_{2n+1} (here v_1, v_7) are adjacent vertices which receive the same colour. Since this not acceptable, a third colour (say green) shall be assigned to v_{2n+1} (here v_7). Therefore, exactly three colours are needed. Hence, the chromatic number of cycle graph C_n is 2 if n is even and 3 otherwise.

10.13.1 Welch and Powell Algorithm

An algorithm developed by Welch and Powel can be used to find the number of colours for proper colouring of a graph.

Step 1: Arrange vertices in a sequence in the descending order of their degrees.

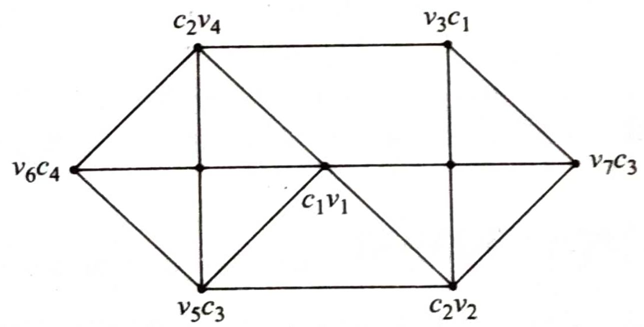
Step 2: Assign colour c_1 to the first vertex and then assign c_1 to each other vertex not adjacent to the previous vertex.

Step 3: Assign colour c_2 to the next non-coloured vertex in the sequence and repeat Step 2.

Step 4: Repeat the process until all the vertices have been painted.

It may be noted that the process may not always yield minimum number of colours.

Example 1 Use Welch and Powell algorithm to colour the vertices of graph G (Fig. 10.55) with minimum number of colours.



Graph G

Figure 10.55

Solution: The vertices of graph G are arranged in sequential order (descending order) of their degrees.

Vertex		v_1	v_2	v_4	v_5	v_3	v_6	v_7
Degree		5	4	4	4	3	3	3
Colour		c_1	c_2	c_2	c_3	c_1	c_4	c_3

Assign colour c_1 to the vertex v_1 of highest degree. The other vertex not adjacent to v_1 is v_3, and hence assign colour c_1 to v_3. Now choose next vertex v_2 and assign it colour c_2. The vertices not adjacent to v_2 are v_4 and v_6. Assign colour c_2 to v_4. Next choose vertex v_5 and assign it colour c_3. The vertex not adjacent to v_5 is v_7, and therefore assigns colour c_3 to v_7. The vertex v_6 can be assigned colour c_4. Hence, chromatic number of graph G is 4.

Example 2 Use Welch and Powell algorithm to colour the vertices of the graph (Fig. 10.56) having 12 vertices with minimum number of colours.

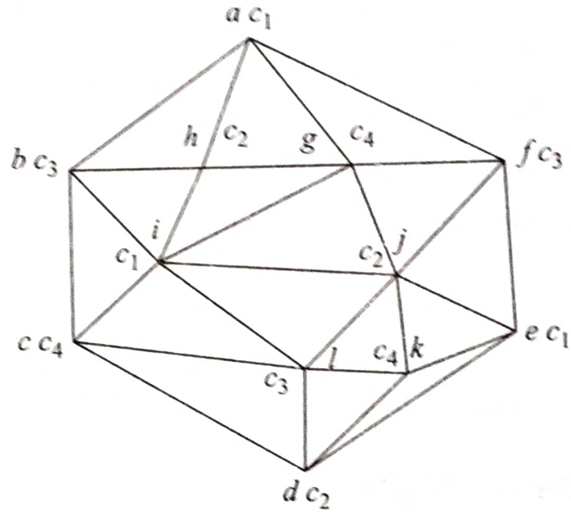


Figure 10.56

Solution:

Vertices	i	j	g	l	k	h	a	b	c	d	e	f
Degree	6	6	5	5	4	4	4	4	4	4	4	4
Colour	c_1	c_2	c_4	c_3	c_4	c_2	c_1	c_3	c_4	c_2	c_1	c_3

We can write the vertices in descending order of their degrees. Assign colour c_1 to the vertex of highest degree i or j. Let us assign to the vertex i and assign c_1 to vertices not adjacent to i, that is, to vertex a and e (f cannot be assigned c_1 otherwise a and c will have same colour). Now assign colour c_2 to vertex j(which also has highest degree) and assign colour c_2 to vertices d and h not adjacent to j. Assign colour c_3 to the next lower degree vertex l and assign colour c_3 to vertices non-adjacent vertices b and f. Now assign colour c_4 to the vertex g and assign c_4 to non-adjacent vertices c and k. Thus four colours are required for proper colouring of vertices.

10.14 CHROMATIC POLYNOMIAL

There are many different ways in which a graph G of n vertices can be properly coloured by using sufficiently large number of colours. This property of the graph can be expressed by a polynomial called chromatic polynomial of graph G.

Let $P_n(\lambda)$ denote the chromatic polynomial of a graph G with n vertices and its value given the number of ways of properly colouring using λ or lesser number of colours.

Let c_k be the different ways of properly colouring G using exactly k different colours. Now total number of ways of selecting k colours out of λ colours is