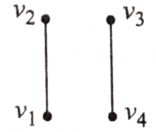
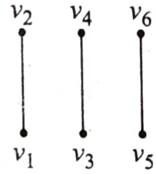
The number of edges is half the sum of all degree of vertices, that is, 2. Hence a possible graph is



which is a disconnected graph.

(3) In the given sequence (1, 1, 1, 1, 1, 1), the sum of degrees of all vertices is even, that is, 6 and also number of odd vertices is even, that is, 6. Hence, a graph is possible. For the graph, the number of edges will be half the sum of degrees of all vertices, that is, 3. Hence, a possible graph is a disconnected graph.



THEOREM 10.4 The maximum number of edges in a simple graph of n vertices is $n(n-1)/2$.

Proof: Let $G = (V, E)$ be a simple graph with n vertices $v_1, v_2, \ldots, v_n$. As the graph is simple, it has neither self-loop nor parallel edges, and hence at most each edge is incident to a two-element subset $\{v_i, v_j\}$ of V. The maximum number of two-element subsets of the set V is ${}^nc_2$.

Hence, the number of edges $\leq {}^nc_2$

$$\leq n(n-1)/2$$

10.2.8 Isolated and Pendant Vertex

In a graph, a vertex is said to be an isolated vertex if its degree is zero, that is, the vertex has no edge. In Fig. 10.10, vertices v_5 and v_6 are *isolated vertices*.

A vertex of degree 1 is called a pendant vertex. In Fig. 10.10, v_4 is a pendant vertex.

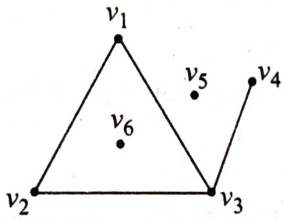


Figure 10.10

10.3 PLANAR AND NON-PLANAR GRAPHS

The concept of planarity finds its application in the design of printed circuit boards in computers.

A graph *G* is said to be planar if it can be drawn in a plane without crossings, that is, no two edges intersect geometrically except at a vertex to which both are incident. Any such drawing is a plane drawing.

A graph that cannot be drawn without a crossover between its edges is called a non-planar graph.

The following graphs shown in Fig. 10.11 are planar graphs:

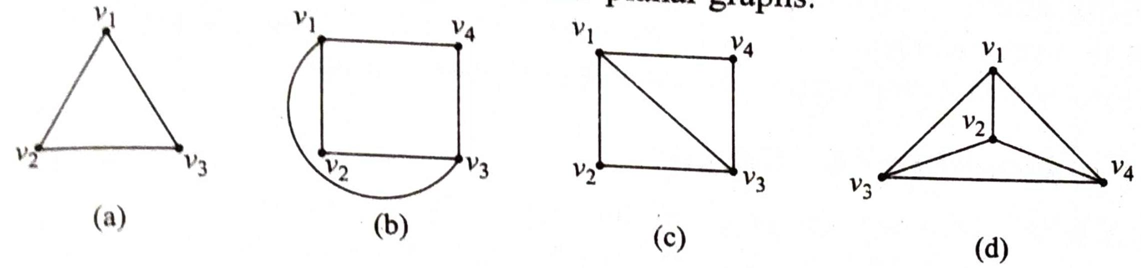
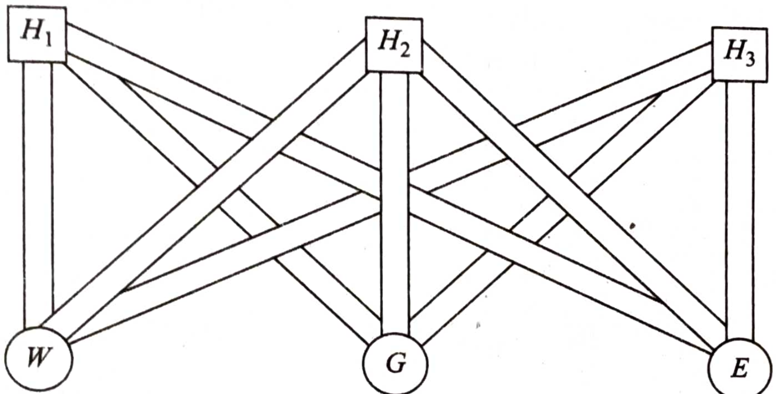


Figure 10.11

Consider a practical problem in which three houses H_1, H_2, and H_3 are to be connected to each of the three utilities water (W), gas (G), and electricity (E) by means of conduits.

The problem can be represented by a graph where conduits are shown as edges and houses and utility centres are vertices.



Non-planer graph

Figure 10.12

Figure 10.12 cannot be drawn without edges crossing over each other. Hence, a planar graph for the above problem is not possible. Figure 10.12 shows a non-planar graph as there is a crossover of the edges.

Note 10.3 A graph G with crossover does not mean that it is non-coplanar. There may be another way of drawing the graph in which edges do not cross over. Thus, before coming to a conclusion that a particular graph is non-planer, one must try redrawing it in different ways.

Consider the following graphs given in Figs. 10.13 and 10.14.

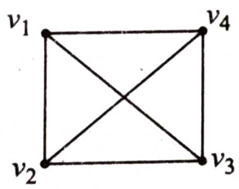


Figure 10.13 *Graph G (intersecting edges, and hence non-planar)*

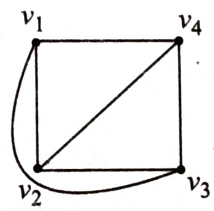


Figure 10.14 *Graph G (redrawn, and hence planar)*

The complete graph G consisting of four vertices v_1, v_2, v_3, and v_4 shown in Fig. 10.15 shows crossover of the edges while redrawn in Fig. 10.16 is planar. Thus, complete graph k_4 is planar.

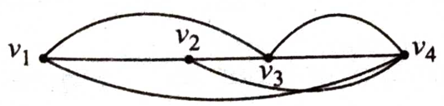


Figure 10.15 *Graph G (intersecting edges)*

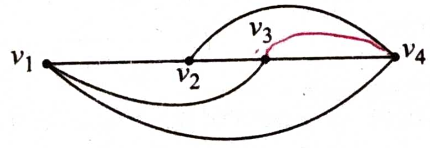


Figure 10.16 *Graph G (planar representation)*

The above discussion gives rise to a question as how to determine whether the graph $G=(V,E)$ is planar or non-planar. The answer was given by Polish mathematician Kasimir Kuratowski who gave specific non-planar graphs known as Kuratowski's graphs.

10.3.1 Kuratowski's Two Graphs

THEOREM 10.5(a) The complete graph of fine vertices is non-planar.

The graph shown in Fig. 10.17 is known as *first graph* of Kuratowski.

A complete graph is a simple graph in which every vertex is joined to every other vertex by an edge. When we join vertices (Fig. 10.17), it can be seen that the edge between v_1 and v_4 cannot be drawn without a crossover and hence the graph is non-planar.

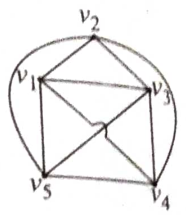
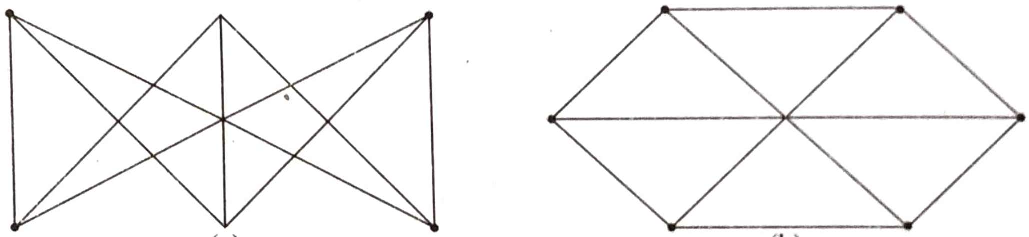


Figure 10.17

THEOREM 10.5(b) The second graph of Kuratowski, which is a complete regular graph with six vertices, is also non-planar.



(a) (b)

Figure 10.18

In literature, Kuratowski's first graph is denoted by K_5 and the second graph by $K_{3,3}$, the letter K for honouring Kuratowski.

Note 10.4 Important observations about Kuratowski's two graphs are the following:

(1) Both are regular and non-planar.
(2) Removal of one edge or a vertex makes each a planar graph.
(3) Kuratowski's first graph is the non-planar graph with the smallest number of vertices, and the second graph is non-planer with the smallest number of edges. Hence, both are simplest non-planar graphs.

10.4 REGION

A planar representation of a connected graph divides the plane into regions which are characterized by the set of edges or set of vertices forming its boundary. The unbounded region is also included in the representation.

Consider the planar graph shown in Fig. 10.19

The graph divides the plane in to five regions marked 1–5, the region 5 is infinite region. The graph has 3 vertices and 6 edges.

If r denotes the number of regions, e the edges, and v the vertices, then we have the relation:

$$r = e - v + 2$$

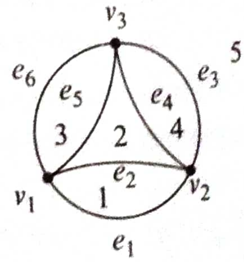


Figure 10.19

This relation is known as Euler's formula which gives the number of regions in a planar graph.

Example 1 Verify Euler's formula for the connected graph shown in Fig. 10.20

In the Fig. 10.20, we have

Number of vertices, $v = 5$

Number of edges, $e = 8$

Number of regions, $r = 5$

Now
$$r = e - v + 2$$
$$= 8 - 5 + 2$$
$$= 3 + 2 = 5$$

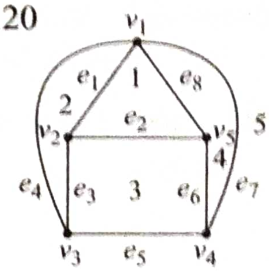


Figure 10.20

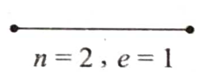
Hence, Euler's formula is verified.

THEOREM 10.6 Euler's Formula

In a connected planar graph G with n vertices and e edges, the number of regions r is $e - n + 2$, that is, $r = e - n + 2$.

Proof: The principle of mathematical induction is used for the value of r.

For $r = 1$ and 2, the statement is true as shown in Figs. 10.21(a) and (b).



$n = 2, e = 1$

Figure 10.21(a)

$n = 1 \quad e = 1$

Figure 10.21(b)

Let the result be true for every connected planar graph G with $r = k + 1$ regions, n vertices, and $e + 1$ edges.

If we select any edge (v_i, v_j) that belongs to regions r_1 and r_2 and delete it from G, then the resulting graph G' will have k regions, n vertices and e edges connected by the relation $r = e - n + 2$.

By re-introducing edge (v_i, v_j) into G', we shall have $k + 1 = (e + 1) - n + 2$ which shows that the result is valid for all values of r and e.

Some Important Results

(1) In a connected simple planar graph G of n (≥ 3) vertices, e edges, and no circuits of length 3, $e \leq 2n - 4$

(2) In a connected simple planar graph G of n (≥ 3) vertices, e (> 2) edges, and r regions,

(i) $3r \leq 2e$ (ii) $e \leq 3n - 6$.

Note 10.5 It may be noted that above conditions (1) and (2) are only necessary conditions for planarity but not sufficient. This means that mere satisfaction of the inequalities does not guarantee the planarity of the graph.

Example 1 Show that the K_5 is non-planar

Solution: The loop-free graph shown in Fig. 10.21(c) has 5 vertices and 10 edges.

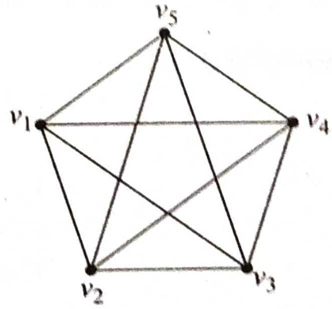


Figure 10.21(c)

For a planar graph
$$e \leq 3n - 6$$
i.e.,
$$3n - 6 \geq e \text{ or } 3n - e \geq 6 \qquad \text{(i)}$$

Here, $n = 5$, $e = 10$ and hence $3n - 6 = 3.5 - 6$
$$= 15 - 6 = 9$$

which does not satisfy the inequality (i). Hence, the graph K_5 is non-planar.

Example 2 A connected graph has 10 vertices. The degree of each vertex is 3. Find the number of regions into which the planner graph can split.

Solution: The number of vertices $n = 10$

Hence,

$$\sum_{i=1}^{3} d(v_i) = 2 \times \textit{number of edges}$$

$$30 = 2e \qquad \therefore\ e = 15$$

By Euler's formula, the number of regions $r = e - n + 2$

$$= 15 - 10 + 12$$

$$= 7$$

$\therefore$ The planar graph can split in 7 regions.

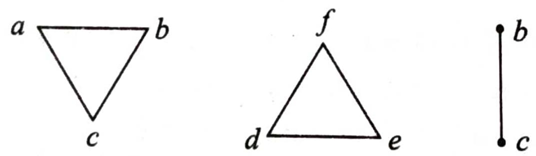
10.5 OPERATIONS ON GRAPHS

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs.

The union of G_1 and G_2 is the simple graph $G = (V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$.

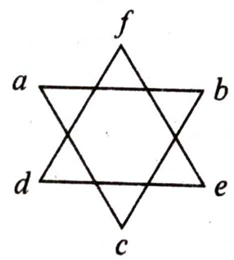
Note 10.6 If there is no common vertex between G_1 and G_2, then the graph $G_1 \cup G_2$ will be disconnected.

Example 1 Let G_1 and G_2 be the two graph



Graph G_1 Graph G_2

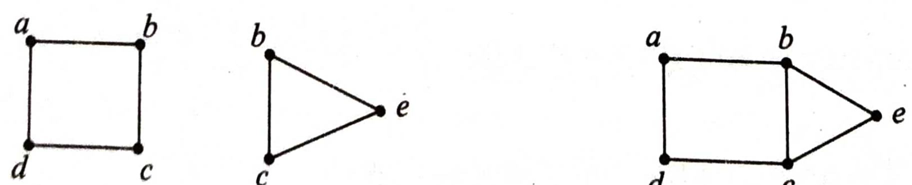
Then the graph $G = G_1 \cup G_2$ is



Graph G

The graph G is called Star of David

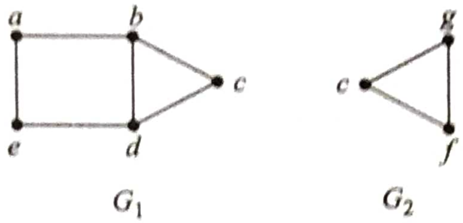
Example 2 Let



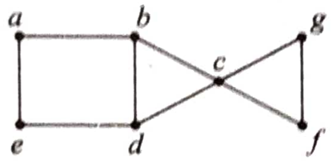
Graph G_1 Graph G_2 Graph G

Then the graph $G = G_1 \cup G_2$

Example 3 Let G_1 and G_2 be the two simple graphs



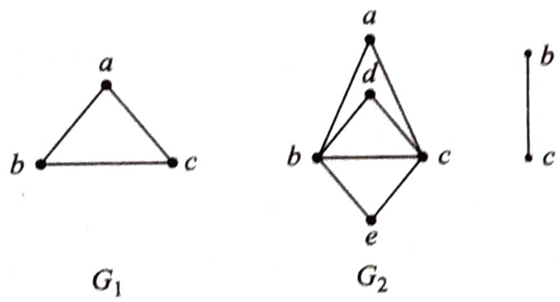
Then the union of the graphs G_1 and G_2, that is, $G = G_1 \cup G_2$ is



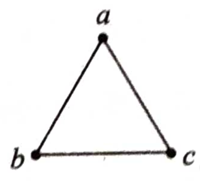
Intersection of Two Graphs: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. The intersection of G_1 and G_2 written as $G_1 \cap G_2$ is the graph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$

1. The intersection of graphs in Example 1 (10.5) is $G_1 \cap G_2 = \Phi$
2. The intersection of graphs in Example 2 (10.5) is the edge (b, c)

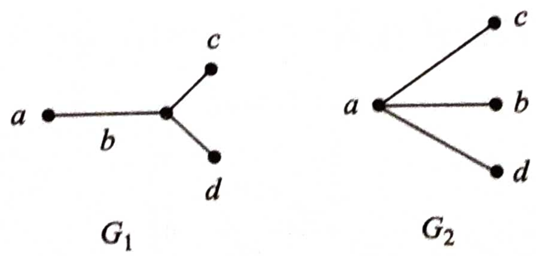
Example 1 Let



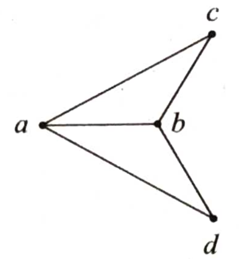
Then $G_1 \cap G_2$ is the graph.



Example 2 Find the union and intersection of the graphs

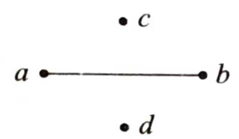


Solution: $G_1 \cup G_2$ is the graph



$(G_1 \cup G_2)$

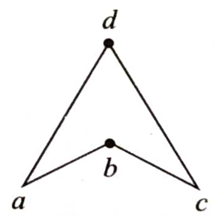
$G_1 \cap G_2$ is the graph consisting of edge (a, b) and isolated vertices c and d



$(G_1 \cap G_2)$

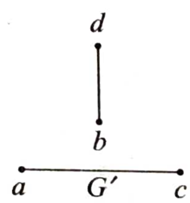
Complement: The complement of a simple graph G is a simple graph G' having the same vertex set of G and where two vertices adjacent in G are not adjacent in G'.

Example 1 Consider the graph G given by

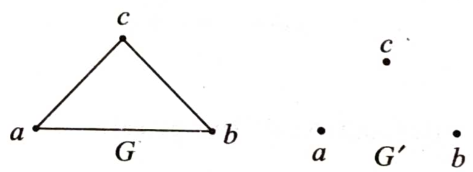


(Graph G)

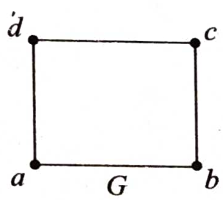
The complement graph G' is



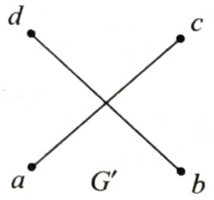
Example 2 The complement of the graph G is the graph G'. It consists of three isolated vertices.



Example 3 The complement of the graph G



is the graph G' given by

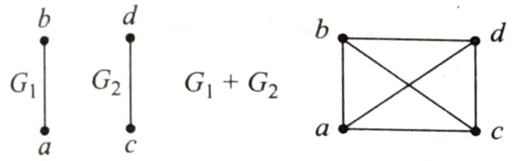


10.5.1 Sum of Two Graphs

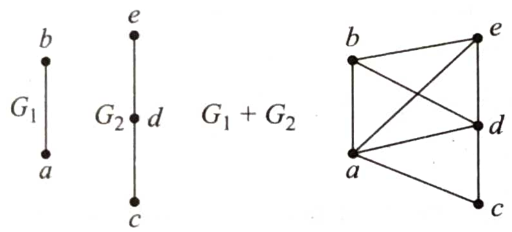
The sum of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denoted by $G_1 + G_2$ is defined as the graph $G = (V, E)$ if $V_1 \cap V_2 = \Phi$ and the vertex set V is the $V_1 \cup V_2$ and the edge set E consists of edges E_1 and E_2 and the edges joining each vertex of v_1 with each vertex of v_2.

Let

1.



2.



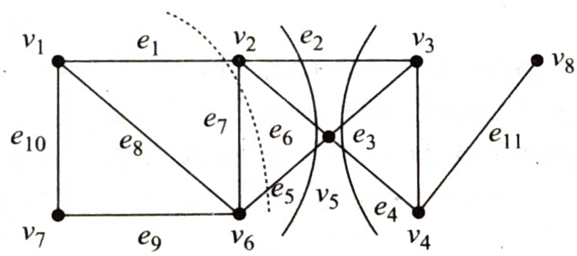
10.5.2 Cut-sets and Cut-vertices

Cut-sets play an important role in the design of communication systems of laying out telephone cables, transportation network of roads, bridges and water supply systems. The cut-sets are used to identify weak spots in a communication network.

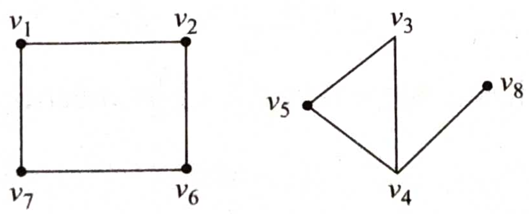
Cut-Set: If a graph G is connected and e is an edge such that $G–e$ is disconnected, then e is said to be a bridge or a cut edge.

If a set S of edges is such that its removal from G leaves $G–S$ disconnected, then S is called a *cut-set* provided the removal of no proper subset of S disconnects G.

Consider the graph



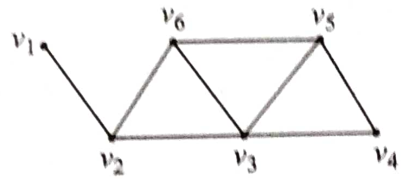
The removal of the set of edges $\{e_2, e_6, e_5\}$ disconnects the graph into two components and hence is a cut-set.



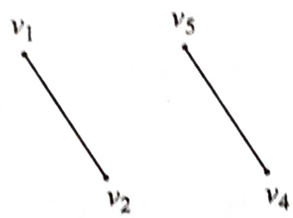
Note 10.7 It should be noted that removal of a vertex implies removal of all edges incident on that vertex. The reason for this is due to the fact that an edge does not exist without both and vertices. Further, every edge of a tree is a cut set since deletion of any edge breaks the tree into two parts.

Separating Set: A *separating set* S in a connected graph G is a set of vertices whose deletion disconnects the graph.

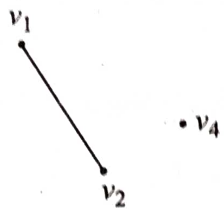
Consider the graph



The set of vertices $\{v_6, v_3\}$ separates the graph into two components:



Similarly, the set of vertices $\{v_6, v_3, v_5\}$ separates the graph into two components:



10.6 BIPARTITE GRAPH

A graph $G = (V, E)$ is a bipartite graph if its vertex set V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 with a vertex in V_2. No edge connects two vertices in V_1 or two vertices in V_2. Consider the following graphs shown in Fig. 10.22:

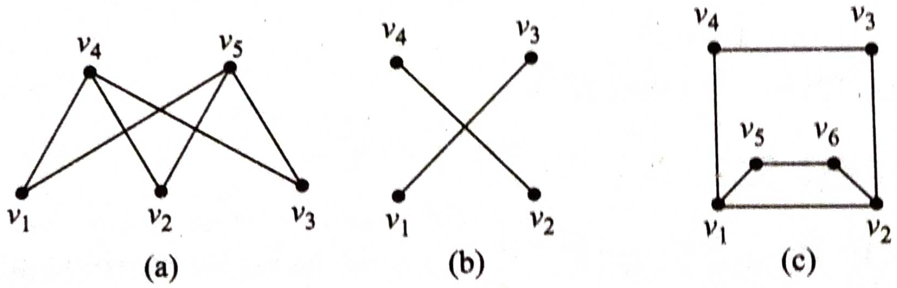


Figure 10.22

(a) $V_1 = \{v_1, v_2, v_3\}$
$V_2 = \{v_4, v_5\}$

(b) $V_1 = \{v_1, v_2\}$
$V_2 = \{v_3, v_4\}$

(c) $V_1 = \{v_1, v_3, v_6\}$
$V_2 = \{v_2, v_4, v_5\}$

Figs. 10.22(a)–(c) are all bipartite as the vertex set can be partitioned into two disjoint sets V_1 and V_2 such that there does not exist any edge connecting members within subsets V_1 or V_2.

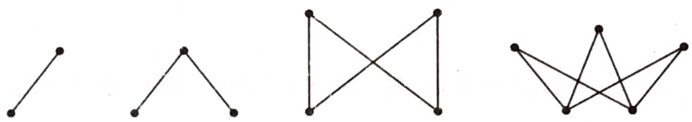
Note 10.8 A bipartite graph cannot have self-loops because self-loop connects the same vertex and it is not permitted in bipartite graphs. A bipartite graph may have parallel edges.

10.6.1 Complete Bipartite Graph

A bipartite graph $G = (V, E)$ is said to be a complete bipartite if each vertex of V_1 is connected to each vertex of V_2 where V_1 and V_2 are the partitions of the vertex set V. Complete bipartite graph G is denoted by $K_{m,n}$, where m and n are the number of vertices in vertex sets V_1 and V_2.

Note 10.9 A complete bipartite graph $K_{m,n}$ has $m + n$ vertices and mn edges. If $m = n$, then $K_{m,n}$ is regular as all vertices are of the same degree.

Some Complete Bipartite Graphs



$K_{1,1}$ $K_{1,2}$ $K_{2,2}$ $K_{2,3}$

Figure 10.23

The graphs $K_{1,1}$ and $K_{2,2}$ are regular.

Example 1 A graph that contains a triangle cannot be bipartite.

Solution: A triangle consists of 3 vertices, and it is not possible to divide these vertices into two disjoint sets such that the vertices of each set are not joined among themselves. Hence, the graph cannot be bipartite.

Example 2 Show that the graph $K_{3,3}$ is non-planar.

Solution: A complete bipartite graph has $(3 + 3) = 6$ vertices and $3.3 = 9$ edges.

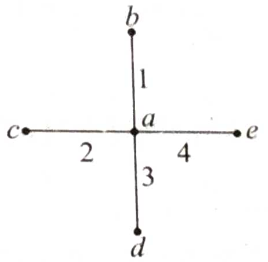
Hence,
$$n = 6, e = 9$$
$$\therefore 3n - 6 = 3.6 - 6 = 18 - 6 = 12.$$

Hence, the condition $e \leq 3n - 6$ is satisfied.
As the graph has no circuit of length 3,
$$e \leq 2n - 4$$
But $2n - 4 = 2.6 - 4 = 12 - 4 = 8$
Hence, $e = 9 \nleq 8$ is not satisfied.
Therefore, the graph $K_{3,3}$ is non-planar.

10.7 ISOMORPHISM

Two graphs $G = (V, E)$ and $G' = (V', E')$ are said to be isomorphic to each other if there exists a function $f: V \to V'$ which is one-to-one and onto, that is, there exists one-to-one correspondence between their vertices and also the edges such that incidence relationships are preserved. If the edge e_K is incident on vertices v_i and v_j in G, then the corresponding edge e'_K in G' must be incident on the vertices v_i' and v_j' in G', which corresponds to vertices v_i and v_j. In other words, if $f(v_i) = v_i'$, $f(v_j) = v_j'$ and $f(e_K) = e_K'$, that is, $(v_i, v_j) \in E$ if and only if $[f(v_i), f(v_j)] \in E'$.

Example 1



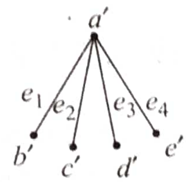


Figure 10.24 (a) **Figure 10.24 (b)**

The graphs (a) and (b) in Fig. 10.24 are isomorphic as the vertices a, b, c, d, and e correspond to the vertices a', b', c', d', and e' and the edges 1, 2, 3, 4 correspond to e_1, e_2, e_3, and e_4.

Example 2

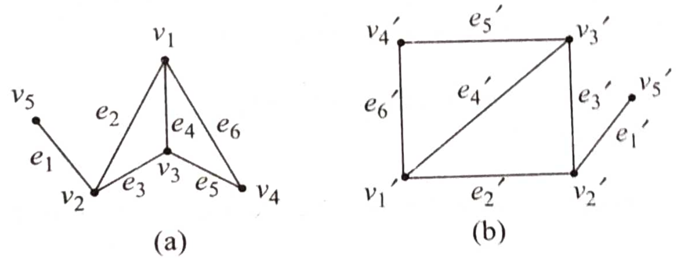


Figure 10.25

The graphs (a) and (b) in Fig. 10.25 are isomorphic. The vertices v_1', v_2', v_3', v_4' and v_5' correspond to v_1, v_2, v_3, v_4, and v_5. The edges e_1', e_2', e_3', e_4', e_5', and e_6' correspond to e_1, e_2, e_3, e_4, e_5, and e_6, respectively, and hence graph (a) is isomorphic to graph (b) given in Fig. 10.25.

From the definition of isomorphism of two graphs, it follows that the graphs must have

(1) The same number of vertices.
(2) The same number of edges.
(3) An equal number of vertices with given degree.

However, these conditions are not sufficient. The graph given in Fig. 10.26(a) and (b) are *not isomorphic*.

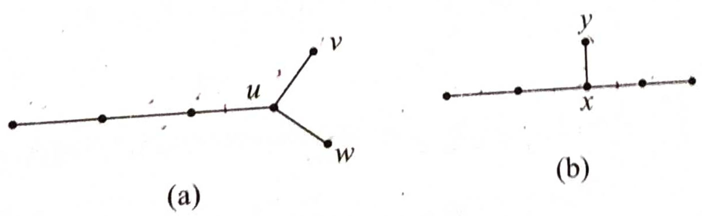


Figure 10.26

If the graphs are isomorphic, vertex u must correspond to the vertex x as there are no other vertex of degree 3. In graph (b), there is only one pendant vertex y adjacent to x, whereas in graph (a), there are two pendant vertices v and w adjacent to u.