cables in cities. The concept of minimum spanning tree is discussed, which helps in all network design problems. Graph colouring is also introduced due to its widespread applications.

10.1 INTRODUCTION

Graph theory with the passage of time has emerged as a special branch of mathematics which has applications in various branches of engineering such as computer science, electrical network, mechanical engineering, rail roads, highways, bridges, and several other areas of chemistry, genetics, operations research, linguistics, and management science.

A graph can be used to represent almost any physical situation involving discrete objects and relationship between them. Like many other discoveries, graph theory originated from the celebrated königsberg seven bridges problem which was solved in 1736 by Swiss mathematician Leonhard Euler. The problem is as follows: Two islands A and B formed by the Pregel River in Königsberg were connected to each other and to the banks C and D with seven bridges as shown in Fig. 10.1(a)

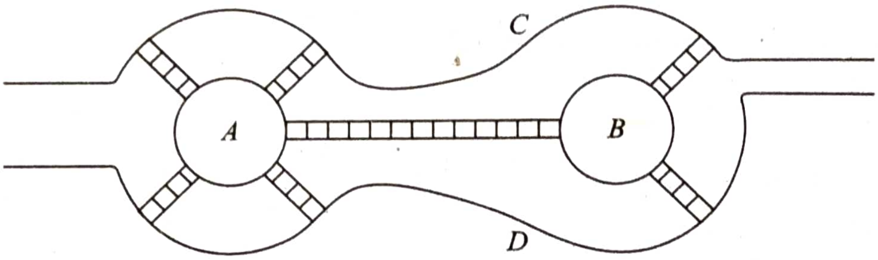


Figure 10.1(a)

The problem was to start at any of the four land areas A, B, C, or D, walk over each of the seven bridges exactly once and return to the starting point (without swimming across the river. Mathematically, the maximum number of possible paths for moving across the seven bridges is $7! = 5040$.

In 1736, Euler constructed a mathematical model (Fig. 10.1(b)) of the problem in which the points A and B represent the islands, and C and D represent the river banks.

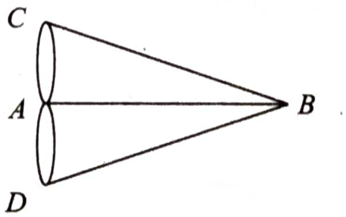


Figure 10.1(b)

The lines/arcs represent seven bridges. Euler concluded that no such walk is possible. Fig. 10.1(b) consisting of points A, B, C, and D and the line segments/arcs joining them is called a graph. The points are called vertices and the lines are called edges.

10.2 GRAPH DEFINITION

A graph $G = (V, E)$ is a system consisting of a set V of vertices $\{v_1, v_2, \ldots\}$ and a set E of edges $\{e_1, e_2, \ldots\}$ such that each edge e_k is identified with an unordered pair of vertices (v_i, v_j). The vertices v_i and v_j are called end vertices of the edge e_k.

Consider the graph (Fig. 10.2) $G = (V, E)$ where

$$V = \{v_1, v_2, v_3, v_4, v_5\} \text{ and } E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$$

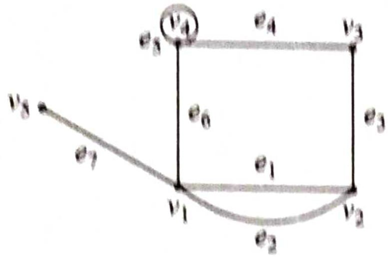
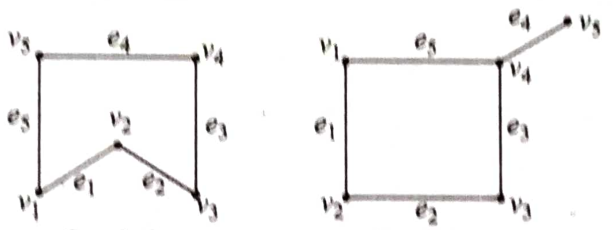


Figure 10.2

Self-Loop: An edge associated with a vertex pair (v_i, v_i), starting and ending at the same vertex is called a self-loop or simply a loop. In Fig. 10.2, edge e_5 is a loop.

Parallel Edges: The edges associated with the same pair of vertices are called parallel edges. In Fig. 10.2, e_1 and e_2 are parallel edges.

Simple Graph: A graph that has neither self-loops nor parallel edges is called a simple graph.



Graph G_1
(a)

Graph G_2
(b)

Figure 10.3

The graphs G_1 and G_2 in Figs. 10.3(a) and (b) are simple graphs.

10.2.1 Finite and Infinite Graphs

The definition of graph does not require the vertex set V and the edge set E to be finite; but in almost all the applications, these are finite.

A graph with a finite number of vertices and also a finite number of edges is called a finite graph, else it is called an infinite graph. The study here is confined to finite graphs.

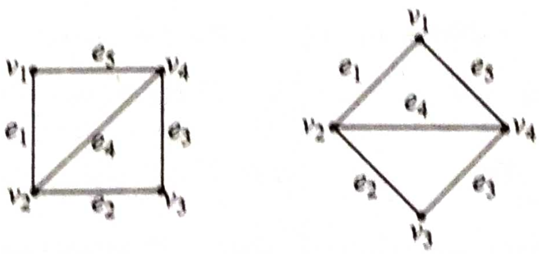


Figure 10.4(a) *Same graph drawn differently*

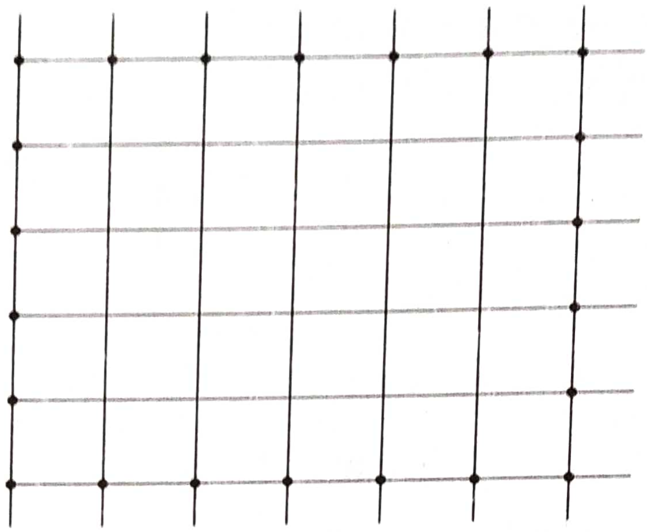


Figure 10.4(b) *Portion of an infinite graph*

Note 10.1 In drawing a graph, it is immaterial whether the lines joining vertices are drawn straight or curved, long or short. The importance lies in the incidence between the edges and vertices [Fig. 10.4(a)].

Multigraph: A graph that contains self-loops or parallel edges or both is called a multigraph. The graph shown is Fig. 10.2 is one such graph.

10.2.2 Directed and Undirected Graph

A graph in which direction is associated with each edge is called a directed graph. A directed graph is a one-way road where traffic flow is allowed only in one direction. A directed edge (v_j, v_k) allows flow from the vertex v_j to v_k In case of an undirected graph, the edge between the vertices v_j and v_k are considered as an ordered pair and is taken as a two-way road with traffic flows in both directions.

An example of a directed graph is the transportation network as given in Fig. 10.5(a) in which A is a factory and M is the market connected to A through cities B, C, and D via road and the manufactured product will be transported through these roads which are edges in the graph.

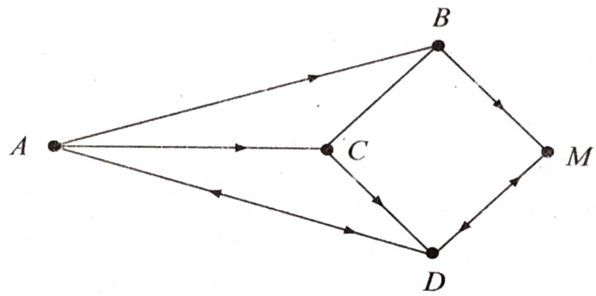


Figure 10.5(a) *Directed graph*

The directions on the graph show the pathway that can be used to approach the market.

10.2.3 Weighted Graph

In many practical problems such as travelling salesman's problem, Chinese postman problem and railway network connecting cities, a non-negative number can be assigned to each edge which is called weight of the edge. Such a graph is called a weighted graph. The weight assigned to an edge e is denoted by $w(e)$. The weights associated with the edges may either be distances or time needed to cover the distances. The problem is to find the minimum path or minimum time or minimum cost to reach the destination.

The graph shown in Fig. 10.5(b) is an example of a weighted graph connecting manufacturing unit P to the market M and the weights represent the distances.

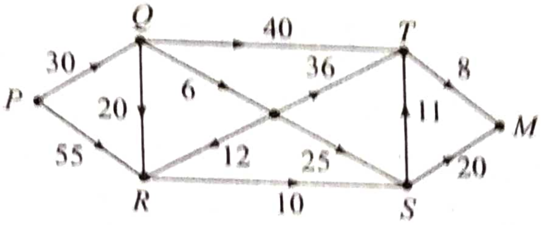


Figure 10.5(b)

The shortest path is

$$P \to Q \to U \to R \to S \to T \to M = 77 \text{ km}$$

10.2.4 Incidence and Degree

When a vertex v_k is an end vertex of some edge e_j, then v_k and e_j are said to be incident with (on or to) each other. In Fig. 10.2, the edges e_1, e_2, e_6, and e_7 are incident with vertex v_1.

Two vertices are said to be adjacent if they are the end vertices of the same edge. Two edges are said to be adjacent if they have a common vertex.

Degree of a Vertex: The number of edges incident on a vertex v_k with self-loops counted twice is called the degree of the vertex v_k and is written as $d(v_k)$

In Fig. 10.2, $d(v_1) = 4$, $d(v_2) = 3$, $d(v_3) = 2$, $d(v_4) = 4$, $d(v_5) = 1$

The sum of degrees of all vertices in any graph is twice the number of edges.

Taking Fig. 10.2 as an example, it can be seen that

$$\sum_{k=1}^{5} d(v_k) = d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5)$$

$$= 4 + 3 + 2 + 4 + 1 = 14$$

$$= \text{twice the number of edges.}$$

THEOREM 10.1 Handshake Theorem

In any graph $G = (V, E)$ with e number of edges and n vertices $v_1, v_2, \ldots, v_n$

$$\sum_{i=1}^{n} d(v_i) = 2e \qquad \text{(i)}$$

Proof: In the case of an undirected graph, the degree of a vertex v is the number of edges meeting at v. Now every edge has exactly two vertices, and hence each edge is counted twice, once at each end. Thus the sum of degrees of all vertices is equal to twice the number of edges.

Note 10.2 The name of the theorem is handshaking theorem because the number of hands involved in handshake of persons will always be even as two hands are involved in every handshake.

Example 1 Consider the graph (Fig. 10.6) with four vertices a, b, c, and d.

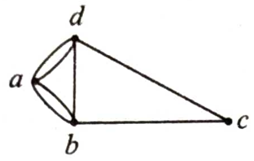


Figure 10.6

$$d(a) = 4,\ d(b) = 4,\ d(c) = 2,\ d(d) = 4$$

The sum of degrees all vertices is

$$= 4 + 4 + 2 + 4 = 14$$

Since the sum is even, there might be a graph with $14/2 = 7$ edges.
Fig. 10.6 demonstrates such a graph.

Even and Odd Vertex: A vertex is said to be an even vertex if its degree is even. In case its degree is odd, the vertex is called an odd vertex.

THEOREM 10.2 The number of vertices of odd degree in a graph is always even.

Proof: Considering even and odd vertices separately for a graph, we have

$$\sum_{i=1}^{n} d(v_i) = \sum_{\text{even vertices}} d(v_j) + \sum_{\text{odd vertices}} d(v_k) \qquad \text{(i)}$$

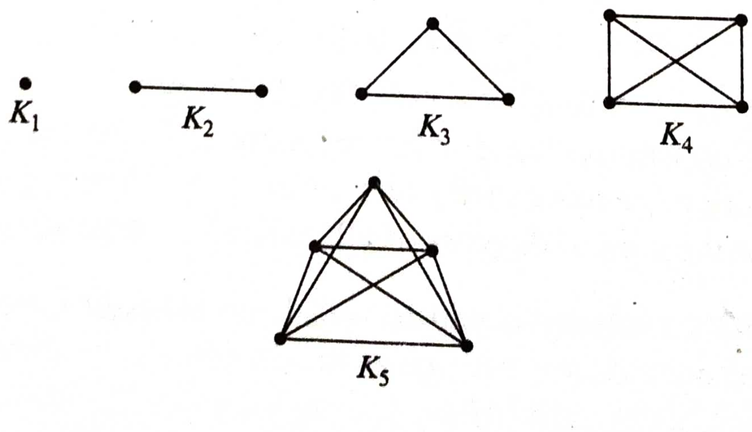
The term on the L.H.S is even. The first term on the R.H.S. is also even being the sum of even terms. Hence, $\sum_{\text{odd vertices}} d(v_k)$ is even which is possible when the number of odd vertices is even.

Indegree and Outdegree: In a directed graph, the number of edges ending at a vertex v is called indegree of v and is denoted by indegree (v). Similarly, the number of edges beginning from the vertex v is called its outdegree and is denoted by outdegree (v). A self-loop contributes two degrees (one outdegree and one indegree).

10.2.5 Complete Graph

In a graph G of n vertices $v_1, v_2, \ldots, v_n$ if every vertex is connected to every other vertex, then G is called a complete graph and is denoted by K_n.

The complete graphs K_1 to K_5 are the following:



Complete graphs are useful in modelling round-robin tournaments where every team plays every other team exactly once. With n teams participating in the tournament, K_n provides the solution.

THEOREM 10.3 If $G = (V, E)$ is a directed graph, then the sum of outdegree vertices is equal to the sum of indegree vertices which equals to the number of edges.

This is

$$\sum_i in \deg (v_i) = \sum_j out \deg (v_j) = e \quad \text{(the number of edges).}$$

This is illustrated with the help of an example given in Fig. 10.7.

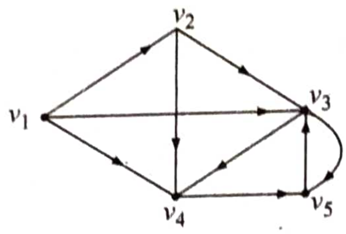


Figure 10.7

Vertex	v_1	v_2	v_3	v_4	v_5
Indegree	0	1	3	3	2
Outdegree	3	2	2	1	1

Sum of indegree vertices = 0 + 1 + 3 + 3 + 2 = 9

Sum of outdegree vertices = 3 + 2 + 2 + 1 + 1 = 9

Number of edges = 9

10.2.6 Regular Graph

A graph in which all vertices are of equal degree is called a regular graph.

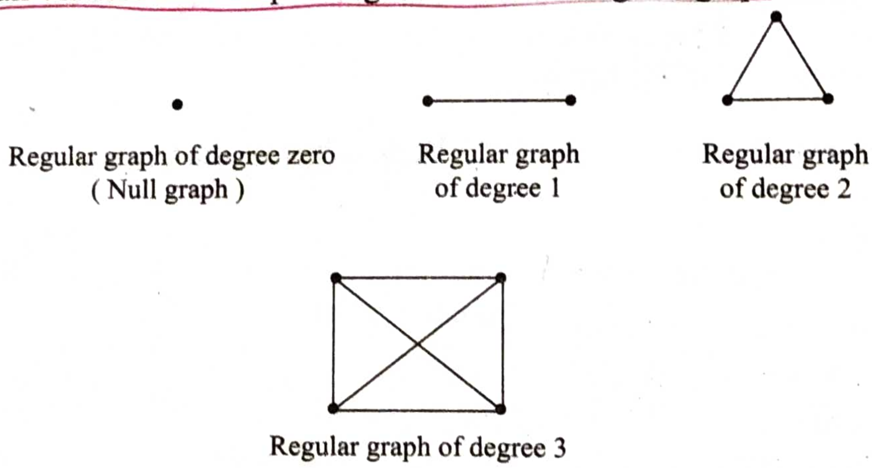


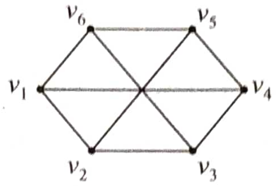
Figure 10.8

Example 1 Is it possible to draw a 3-regular graph with 5 vertices?

Solution: There are 5 vertices and each vertex must be of degree 3 as the graph is 3-regular. Thus, the sum of all degrees is $3 \times 5 = 15$ which is odd. As the sum of degrees of all vertices must be even, such a graph is *not* possible.

Example 2 Draw a 3-regular graph with 6 vertices. Find the number of edges.

Solution:



The number of edges

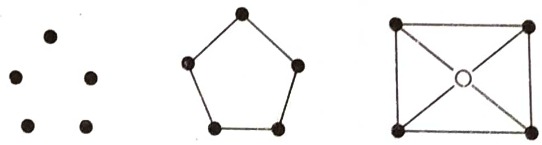
$$= \frac{1}{2}\text{ (sum of degrees of all vertices)}$$

$$= \frac{1}{2}.6.3 = 9 \text{ (as each vertex is of degree 3)}$$

Cycle Graph, Wheel Graph, and Null Graph: A *cycle graph* of order n is a connected graph whose edges form a cycle of length n. Cycle graphs are denoted by C_n.

A wheel of order n is a graph obtained by joining a single new vertex (the hub) to each vertex of a cycle graph of order $n - 1$. Wheels of order n are denoted by W_n.

A null graph of order n is a graph with n vertices and no edges. Null graphs of order n are denoted by N_n. Each vertex in a null graph is an isolated vertex.



N_5 $\qquad$ C_5 $\qquad$ W_5

10.2.7 Degree Sequence of Graph

Let $v_1, v_2, \ldots, v_n$ be the vertices of a graph G and let $d_1, d_2, \ldots, d_n$ be their degrees, respectively. If the sequence $(d_1, d_2, \ldots, d_n)$ is monotonically increasing, it is called degree sequence of graph G. Consider the graph shown in Fig. [10.9(a)]

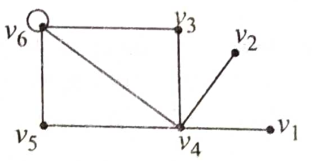


Figure 10.9(a)

Now $\qquad \deg(v_1) = 1, \deg(v_2) = 1, \deg(v_3) = 2,$

$\qquad \deg(v_4) = 5, \deg(v_5) = 2, \deg(v_6) = 5.$

Hence, the degree sequence of the graph is (1, 1, 2, 2, 5, 5).

Example 1 Is there a simple graph corresponding to degree sequence (1, 1, 2, 2)?

Solution: The number of odd-degree vertices is even, and therefore a simple graph is possible. Further, the sum of degrees of all vertices is even, and hence a simple graph exists. The possible graph can be a graph with four vertices and is shown in Fig. 10.9(b).

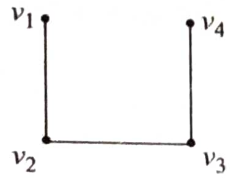


Figure 10.9(b)

Example 2 Is there a simple graph corresponding to degree sequence (1, 1, 4, 4)?

Solution: The number of odd vertices is even and also the sum of degrees of all vertices, that is, $1 + 1 + 4 + 4 = 10$ is even, and hence a simple graph is possible. But in the case of simple graph G of n vertices, the degree of any vertex cannot exceed $(n - 1)$. Here, there are 4 vertices and two of them are of degree 4, which is not possible and hence the graph is not possible.

Example 3 For the graph G, shown in Fig. [10.9(c)], write the degree sequence of G. Also find the number of odd-degree vertices and the number of edges. Verify handshake theorem.

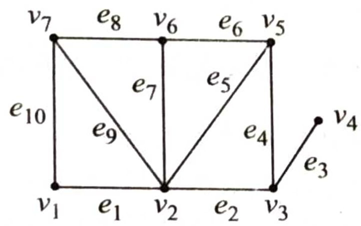


Figure 10.9(c)

Solution:

$\deg(v_1) = 2$,	$\deg(v_2) = 5$,	$\deg(v_3) = 3$
$\deg(v_4) = 1$,	$\deg(v_5) = 3$,	$\deg(v_6) = 3$
$\deg(v_7) = 3$		

Hence, the degree sequence of G is (1, 2, 3, 3, 3, 3, 5)

The number of odd-degree vertices is 6

Now, the sum of degrees of all vertices is $1 + 2 + 3 + 3 + 3 + 3 + 5 = 20$

The number of edges is 10 (from e_1 to e_{10})

Hence, the sum of degrees of all vertices is equal to twice the number of edges (i.e., handshaking theorem).

Example 4 Draw the graph if it exists for the following degree sequences:

(1) (1, 1, 1) (2) (1, 1, 1, 1) (3) (1, 1, 1, 1, 1, 1)

Solution:

(1) In the given sequence (1, 1, 1), the sum of degrees of all vertices is odd, and hence no simple graph can be drawn.
(2) In the given sequence (1, 1, 1, 1), the sum of degrees of all vertices is even, that is, 4. The number of odd-degree vertices is even, and hence a graph is possible which has 4 vertices of degree 1 each.