Note 10.10 The necessary condition for the graph G and G' to be isomorphic are as follows:

1. Both G and G' have same number of vertices.
2. Both G and G' have same number of edges.
3. Both G and G' have same degree sequences.

The graphs are not isomorphic even if one of the above conditions is not satisfied. However, these conditions are *not* sufficient.

10.8 REPRESENTATION OF GRAPHS IN COMPUTER MEMORY

Although graphs are geometric figures, they can be represented by matrix and hence can be stored in computer memory. The following representations are commonly used:

1. Adjacency matrix (vertex–vertex adjacency matrix)
2. Incidence matrix (vertex–edge incidence matrix)

10.8.1 Adjacency Matrix of Undirected Graph

Consider an undirected graph G with n vertices $v_1, v_2, v_3, \ldots, v_n$ and no parallel edges and loops. The adjacency matrix $A = [a_{ij}]$ of order $n \times n$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if vertex } v_i \text{ is adjacent to } v_j \\ 0 & \text{otherwise} \end{cases}$$

The adjacency matrix of the graph consisting of five vertices given in Fig. 10.27 is

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

(adjacency matrix)

Some Important Results of Adjacency Matrix

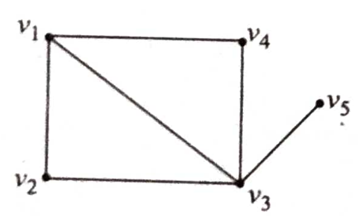


Figure 10.27

1. Adjacency matrix A is symmetric. That is, $a_{ij} = a_{ji}$ for all i and j.
2. All the diagonal elements are zero as there are no loops.
3. Adjacency matrix for a graph with n vertices has n^2 elements.

10.8.2 Incidence Matrix

Let G be a graph with n vertices and m edges. The incidence matrix $M = [m_{ij}]$ of order $n \times m$ is defined by

$$m_{ij} = 1 \quad \text{if vertex } V_i \text{ is incident on edge } e_j$$
$$= 0 \quad \text{otherwise}$$

The matrix has a row for every vertex and a column for every edge.

Consider the graph shown in Fig. 10.28 consisting of 5 vertices and 7 edges which can be represented by the incidence matrix of order 5×7, that is,

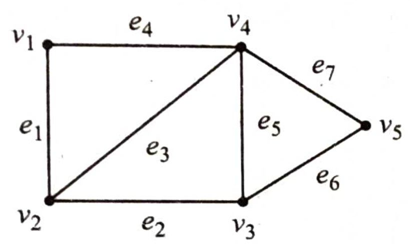


Figure 10.28

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

(incidence matrix)

Some Important Results for Incidence Matrix

1. Each column contains exactly *two* unit elements as each edge is connected in the two vertices.
2. A row with all its elements zero will correspond to an isolated vertex.
3. A row with single-unit element will correspond to a pendant vertex.
4. The number of elements in row k represents the degree of the vertex v_K.
5. If the graph is connected with n vertices, then the rank of incidence matrix M is $(n - 1)$.

10.8.3 Matrix Representation of Directed Graph: The Adjacency Matrix

The adjacency matrix $A = [a_{ij}]$ for a directed graph G with n vertices is an $n \times n$ matrix defined by

$$a_{ij} = 1 \quad \text{if edge beginning at vertex } v_i \text{ ends at } v_j$$
$$= 0 \quad \text{otherwise}$$

The non-zero elements in the matrix are equal to the number of edges in the graph.

The adjacency matrix of order 5×5 for the directed graph given in Fig. 10.29 is the following:

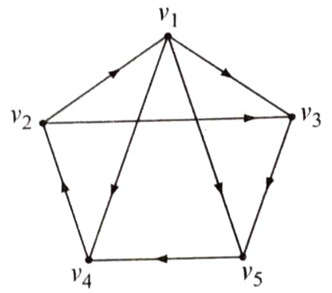


Figure 10.29

$$\begin{array}{c c} & \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(adjacency matrix)

Incidence Matrix: The incidence matrix $M = [m_{ij}]$ for a directed graph G of n vertices and m edges is of order $n \times m$ and is defined by

$$m_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of edge } e_j \\ -1 & \text{if } v_i \text{ is the final vertex of edge } e_j \\ 0 & \text{if } v_i \text{ is not incident on edge } e_j \end{cases}$$

The incidence matrix for the graph given in Fig. 10.30 is the following:

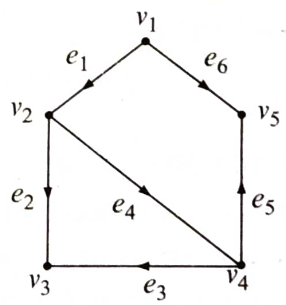


Figure 10.30

$$M = \begin{array}{c c} & \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \end{array}.$$

10.9 REPRESENTATION OF MULTI GRAPH

When graph G is a multigraph, then a_{ij} is the number of edges joining the vertex v_i to the vertex v_j.

Let G be a multigraph consisting of n vertices, then the $n \times n$ adjacency matrix $A = [a_{ij}]$ is defined by,

$$a_{ij} = k$$
$$= 0$$

where k is the number of edges joining the vertex v_i to the vertex v_j otherwise.

The adjacency matrix of order 4×4 for the multigraph shown in Fig. 10.31 is the following:

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 3 & 2 \\ 1 & 1 & 1 & 0 \\ 3 & 1 & 0 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

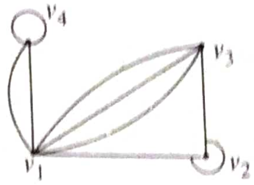


Figure 10.31

Example 1 Find the adjacency matrix and the incident matrix of the multigraph shown in Fig. 10.32

$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \end{array}$$

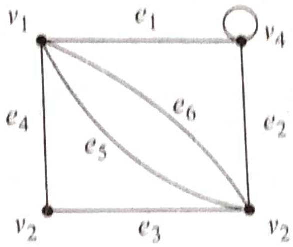


Figure 10.32

The incidence matrix M is the $n \times m$ matrix whose ijth entry is 1 if vertex i is incident to edge e_j and 0 otherwise.

$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

10.10 WALK IN A GRAPH

A walk is a finite alternating sequence of vertices and edges starting and ending with vertices such that each edge is incident with the vertices preceding and following. In a walk, no edge traversed more than once although a vertex can appear more than once.

Figs. 10.33(a) and (b) illustrate two walks in the graph:

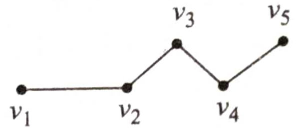


Figure 10.33(a)

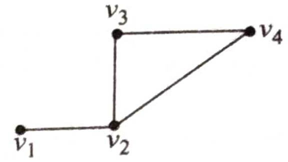


Figure 10.33(b)

1. $v_1 \to v_2 \to v_3 \to v_4 \to v_5$ is a walk
2. $v_1 \to v_2 \to v_3 \to v_4 \to v_2$ is a walk (here vertex v_2 is repeated)

Path: A path is a walk through a sequence of vertices:

$$v_1 \to v_2 \to v_3 \to \ldots v_{n-1} \to v_n$$

without any repetition of vertices such a path is also known as simple path or an elementary path. In a path, terminal vertices are of degree 1, whereas all the intermediate vertices are of degree 2. The walk shown in Fig. 10.33(b), that is, $v_1 \to v_2 \to v_3 \to v_4 \to v_2$ is *not* a path as vertex v_2 is repeated.

Trail: A walk in which all edges are distinct is called a Trail. Thus a trail is a path in which no edge is repeated. The initial and final vertices of a walk are called terminal vertices.

Closed Walk: A walk can begin and end at the same vertex and such a walk is called a closed walk. If terminal vertices are distinct, the walk is called an open walk.

Example 1 $v_1 \to v_2 \to v_5 \to v_2 \to v_4 \to v_4 \to v_5$ is a walk in Fig. 10.34(a)

Example 2 $v_1 \to v_2 \to v_3 \to v_4 \to v_4 \to v_5$ is a walk in Fig. 10.34(a)

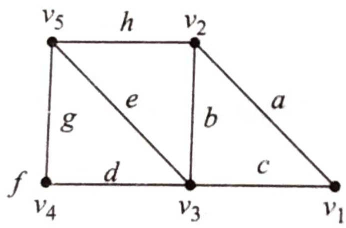


Figure 10.34(a)

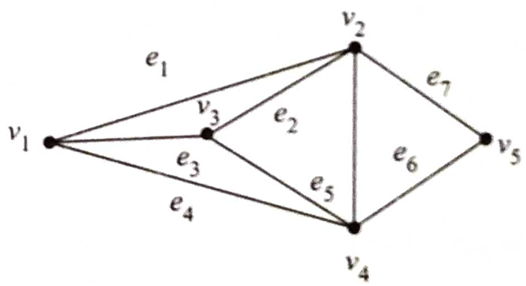


Figure 10.34(b)

Example 3 In Fig. 10.34(a) $v_1 \to v_2 \to v_5 \to v_3$ is a path.

Length of a Path: The number of edges in a path is called the length of the path.

Example 1 In Fig. 10.34(b), $v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_7} v_5 \xrightarrow{e_6} v_4 \xrightarrow{e_5} v_3$ is a path of length 4. This is an open path.

Example 2 In Fig. 10.34(b), the path $v_1 \to v_2 \to v_5 \to v_4 \to v_3 \to v_1$ is a closed path.

Note 10.11 A loop can be included in a walk but not in a path.

10.10.1 Circuit

A closed walk in which no vertex is repeated is called a *circuit*. Thus, a circuit is a *path* that begins and ends at the same vertex.

In other words, it is a closed non-intersecting walk.

In Fig. 10.34(a)

1. $v_1 \xrightarrow{a} v_2 \xrightarrow{b} v_3 \xrightarrow{c} v_1$ is a circuit.
2. $v_1 \xrightarrow{c} v_3 \xrightarrow{e} v_5 \xrightarrow{h} v_2 \xrightarrow{a} v_1$ is a circuit.

Example 1 The following graphs are circuits:

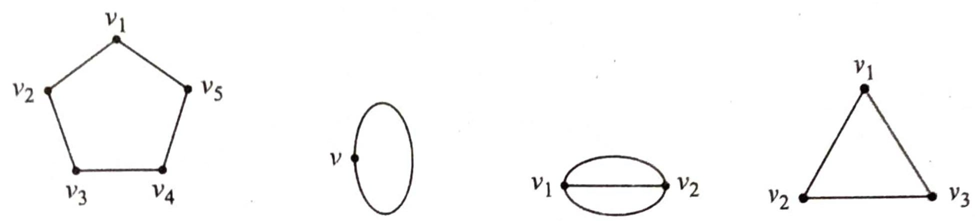


Figure 10.35(a) Figure 10.35(b) Figure 10.35(c) Figure 10.35(d)

Example 2 Find all the circuits of the graph shown in Fig. 10.35(e)

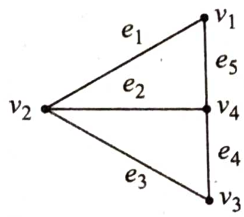


Figure 10.35(e)

(1) $v_1 \to v_2 \to v_4 \to v_1$ is a circuit.
(2) $v_2 \to v_3 \to v_4 \to v_2$ is a circuit.
(3) $v_2 \to v_4 \to v_1 \to v_2$ is a circuit.
(4) $v_3 \to v_4 \to v_1 \to v_2 \to v_3$ is a circuit.

Example 3 Draw a circuit from the graph which includes all the vertices given in Fig. 10.35(f).

Solution:

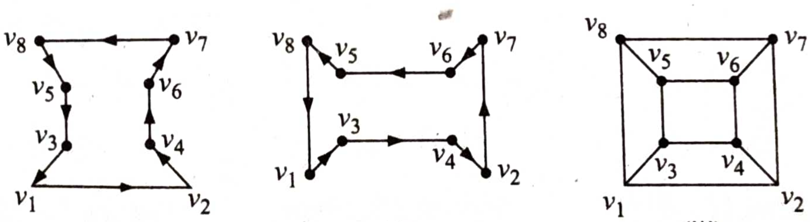


Figure 10.35(f)

The circuits are (i) $v_1\ v_2\ v_4\ v_6\ v_7\ v_8\ v_5\ v_3\ v_1$

and (ii) $v_1\ v_3\ v_4\ v_2\ v_7\ v_6\ v_5\ v_8\ v_1$ (except the initial and final)

10.11 SUB-GRAPH

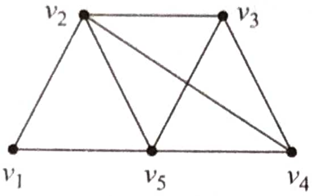
In some problems, only a part of the entire graph need be considered. Such a graph is called the sub-graph of the given graph. Sub-graphs play an important role in analysing the properties of graphs. The concept of sub-graph is similar to that of a subset, and hence the following results are true:

1. Every graph is a sub-graph of itself.
2. Every vertex in a graph G is a sub-graph of G.
3. Every edge together with its end vertices is also a sub-graph of G.

Definition 10.1 A graph H is said to be a **sub-graph** of a graph G if all its vertices and edges are in G and each edge of H has the same end vertices in H as in G.

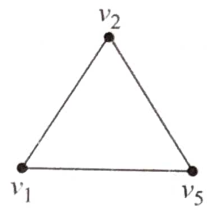
In other words, if $G = (V, E)$ and $H = (V_1, E_1)$ are two graphs such that $V_1 \subseteq V$ and $E_1 \subseteq E$, then H is called a **sub-graph** of G.

Example 1 Let G be the original graph shown in Fig. 10.36



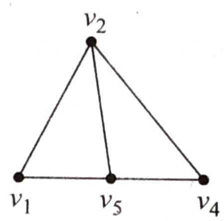
Graph G

Figure 10.36



Graph G_1

Figure 10.37(a)

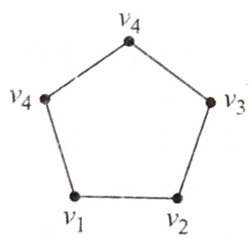


Graph G_2

Figure 10.37(b)

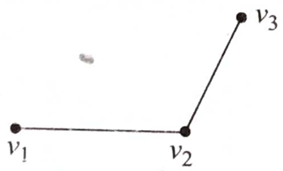
The graph G_1 and G_2 shown in Figs. 10.37(a) and (b) are sub-graphs of graph G given in Fig. 10.36.

Example 2



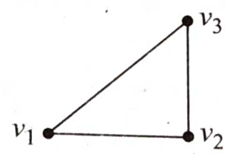
Graph G

Figure 10.38(a)



Graph G_1

Figure 10.38(b)

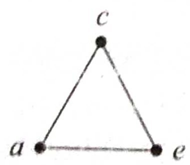


Graph G_2

Figure 10.38(c)

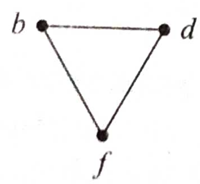
The graph G_1 is a sub-graph of G but the graph G_2 is not a sub-graph of G as no edge between the vertices v_1 and v_3 is present in the original graph G. [Fig. 10.38(a)]

Example 3 The graphs G_1 and G_2 (Fig. 10.39) are sub-graphs of graph G (Fig. 10.40).



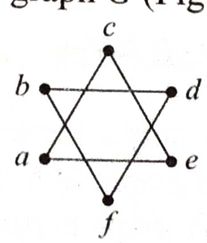
Graph G_1

Figure 10.39(a)



Graph G_2

Figure 10.39(a)



Graph G Star of David

Figure 10.40

10.11.1 Spanning Sub-graph

The graph H is called a spanning sub-graph of G if H contains all vertices of G but not necessarily all edges of G.

Consider the graph G shown in Fig. 10.41.

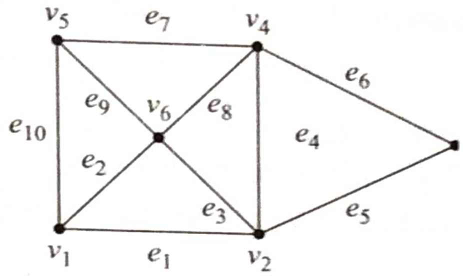


Figure 10.41

The following are spanning sub-graphs G:

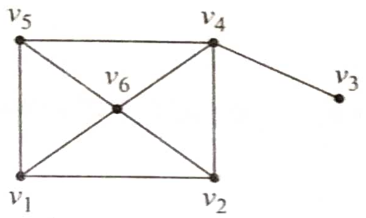


Figure 10.42(a) *Deleting edge e_5*

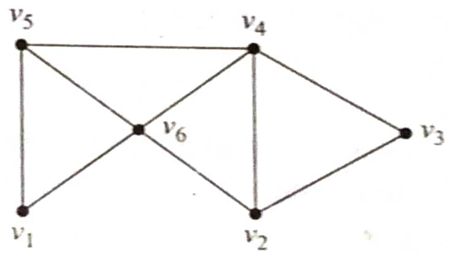


Figure 10.42(b) *Deleting edge e_1*

1. $G–e_5$
2. $G–e_1$

10.12 CONNECTED AND DISCONNECTED GRAPHS

A graph G is said to be connected if there exists at least one path between every pair of vertices of G, otherwise it is disconnected.

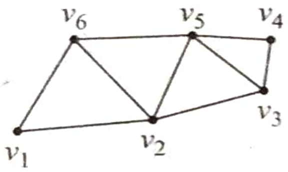


Figure 10.43(a) *connected graph*

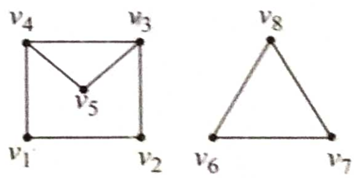


Figure 10.43(b) *Disconnected graph*

The graph in Fig. 10.43(a) is connected, while the graph in Fig. 10.43(b) is disconnected. A null graph of more than one vertex is disconnected. Further, any disconnected graph will have two or more connected sub-graphs which are called components of the original graph.

Let us consider a graph shown in Fig. 10.44 which has three connected components:

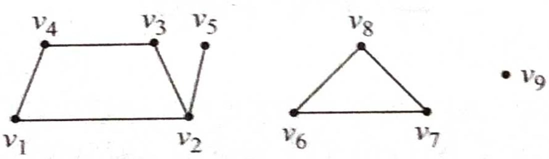


Figure 10.44