Acoustic Harmonics and the Quantum Analog

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Introduction

Classical systems allow one to make assumptions about properties of their state such as a position measurement. This can then allow for a consecutive velocity measurement, which coupled with a measured mass then yields momentum. Quantum systems do not allow for this application of deterministic logic. Instead, they are dictated by probabilistic interpretations. This is best represented by the images on the right where the physical sounds waves are represented by wave equations (1) and (2) and the energies of the quantum waves are wave functions (3) and (4). Note, an ansatz was used for (2).*

* $p(\vec{r},t) = p(\vec{r})\cos(\omega t)$

Classical Interpretation

(1) $$\frac{\partial^2 p}{\partial t^2} = \frac{1}{\rho\kappa}\Delta p$$

Time-Dependent Helmholtz Equation

(2) $$-\frac{\omega^2}{c^2}p(\vec{r}) = \Delta p(\vec{r})$$

Time-Independent Helmholtz Equation

Analogous Quantum System

(3) $$E\psi(x) = -\frac{\hbar^2}{2m}\Delta\psi(x)$$

1-Dimensional Form of a Particle in a Box

(4) $$E\psi(\vec{r}) = -\frac{\hbar^2}{2m}\Delta\psi(\vec{r}) - \frac{e^2}{r}\psi(\vec{r})$$

3-Dimensional Form of an Electron in a Hydrogen Atom

Sound Waves in a Tube

To analyze the quantum system, we must use the same mathematical method of taking an ansatz to see how it operates. An ansatz is simply an approximate function by which we can analyze a solution for. We can then perform an ansatz for both the classical and quantum system by assuming a sine-cosine waveform that varies in position and time. Both (5) and (6) must also satisfy their respective boundary conditions, which we will talk about later. The time-dependent solution is what is useful and will allow us to make further interpretations later about their eigenstates and eigenfrequencies.

(5) Ansatz of Time-Dependent Sound Wave Form

$$p(x) = p_0 \cos(kx + \alpha)\cos(\omega t)$$

(6) Ansatz of Time-Dependent Quantum Wave Form

$$\psi(x,t) = A\sin(kx + \alpha)\, e^{-i\omega t}$$

Sounds Waves in a Spherical Resonator

The difference between the classical system of sound waves and the electron wave field remains only in its radial components. As shown in (7) and (8), the spherical function remains unchanged with the same magnetic quantum number m and azimuthal quantum number l. So, when we observe the radial function, we can use the coefficients to later make a statement on the eigenvalues of the radial function.

(7) $$p(r,\theta,\varphi) = Y_l^m(\theta,\varphi)\, f(r)$$ Sound Wave in Spherical Coordinates

(8) $$\psi(r,\theta,\varphi) = Y_l^m(\theta,\varphi)\, \chi_l(r)$$ Quantum Wave in Spherical Coordinates

(9) $$-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]Y_l^m(\theta,\varphi) = l(l+1)Y_l^m(\theta,\varphi)$$ Spherical Function

(10) $$-\frac{\hbar^2}{2mr}\frac{\partial^2}{\partial r^2}r\chi(r)-\frac{l(l+1)\hbar^2}{2mr^2}\chi(r)-\frac{e^2}{r}\chi(r) = E\chi(r)$$ Quantum Radial Function

(11) $$-\frac{\partial^2 f}{\partial r^2}-\frac{2}{r}\frac{\partial f}{\partial r}+\frac{l(l+1)}{r^2}f(r) = \frac{\omega^2}{c^2}f(r)$$ Classical Radial Function

Posed Question

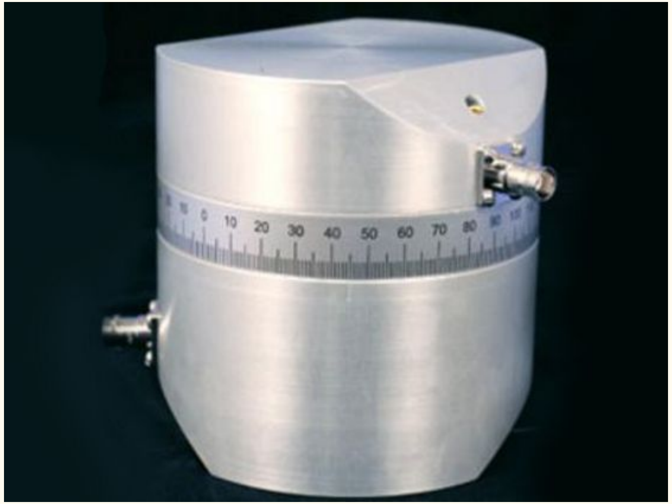
What properties of a classical system contain analogous properties in a quantum system?

Experimental Description - Overview

<u>Two parts to this lab:</u>

Soundwaves in a Tube

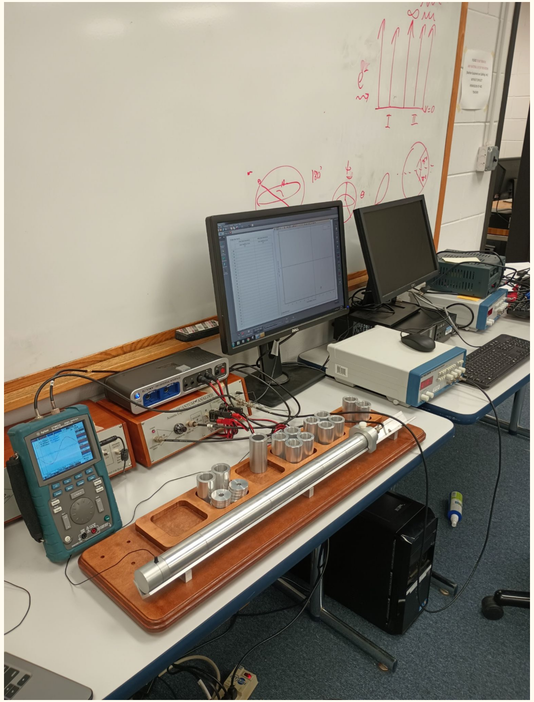
Spherical Resonator



Experimental Description - Materials

All Materials used in lab are as follows:

- Complete Tube stand
- Spherical Resonator
- Quantum Analog Controller
- Pasco Interface
- Function Generator
- Oscilloscope

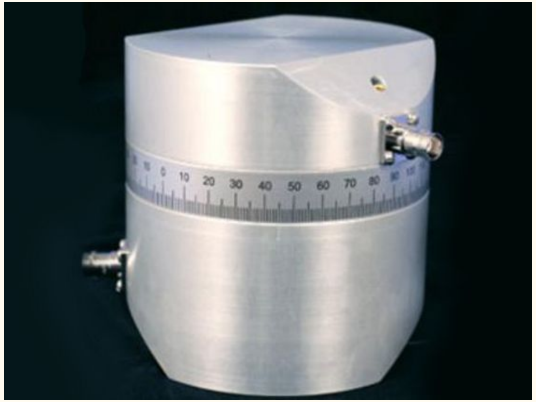


Experimental Description - Part 1

- Interface and Apparatus wired
- Speaker & Mic Situated
- Aluminum Tubes put on track
    - Differing lengths
    - Iris Barriers
- Frequency Swept from 100 Hz - 10 kHz
- Data recorded on PASCO software

Experimental Description - Part 2

- Spherical Resonator wired
  - Upper Hemisphere: Microphone
  - Lower Hemisphere: Speaker
- Frequency Swept to find Resonances
  - Frequency then kept constant at specific Resonance pt.
- Upper Hemisphere rotated by 360 deg
  - Observing how resonances change w.r.t angle
- Frequency Sweeps conducted at 10 deg increments
  - Observing change in resonance amplitudes



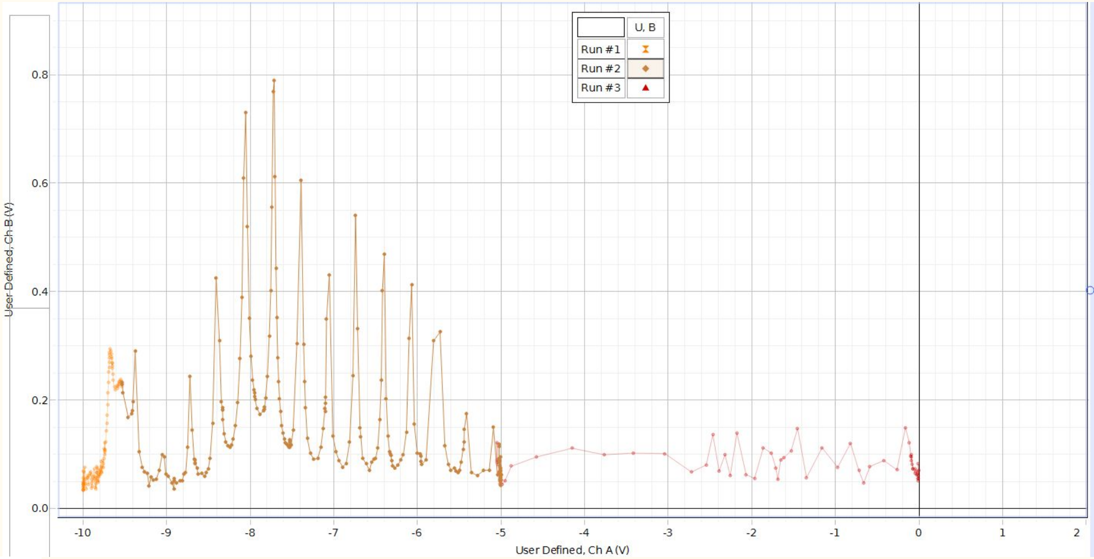
Hypothetical Results

For the sound waves within a tube, the eigenfunctions is the energy of the quantum wave and the frequency of the sound wave. These can be both implemented with the eigenvalue k, which will be explained later in detail.. For the sound waves within a spherical resonator, the radial equation is satisfied by two eigenvalues, the quantum energy level and the the spherical eigenfrequencies shown by (10) and (11). Therefore, we expect to see a relationship between these factors on the graphical interpretations.

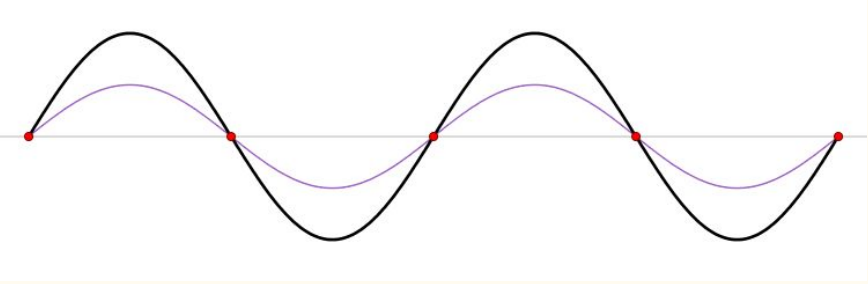
$$Y_l^m(\theta, \phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} P_l^m(\cos\theta) e^{im\phi} \tag{12}$$

Results & Analysis - Resonant Frequencies 1

- Appear as peaks in the graph
- X-axis: Frequency (recorded as voltage by software and then converted with a linear expression)
- Y-axis: Amplitude of air wave (Sound)



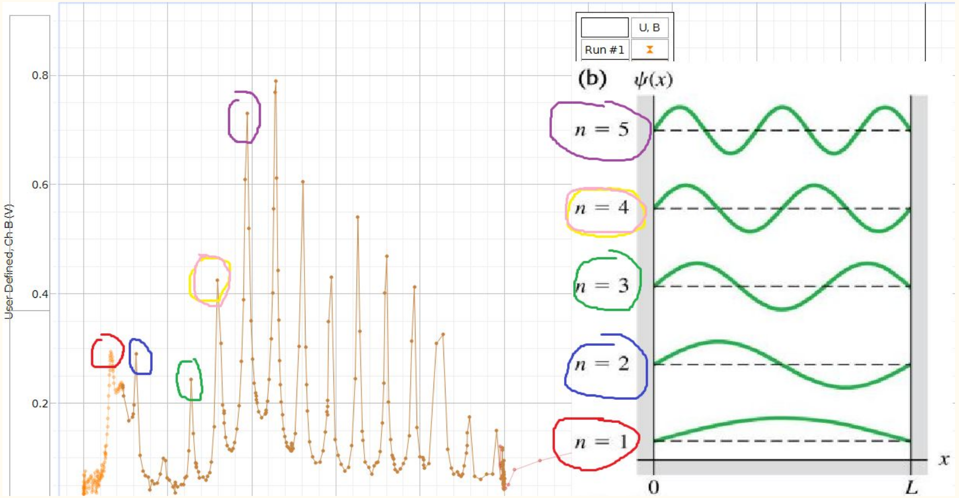
Amplitude vs. Frequency graph of a 52.5 cm tube



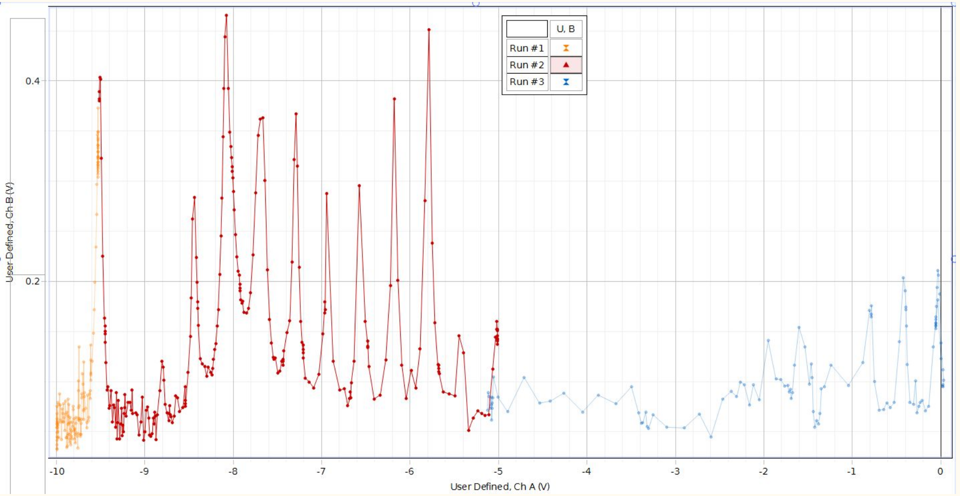
Results & Analysis - Resonant Frequencies 2

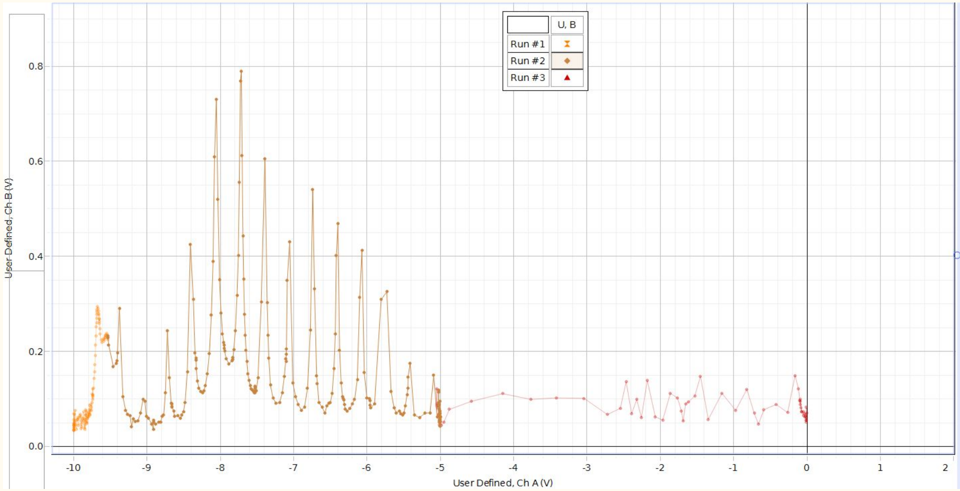
QM Interpretation:

- Amplitude Peaks within the tube at different resonant frequencies correspond to the different energies of the quantum waveforms inside of a box.



Results & Analysis - Tube Length Change



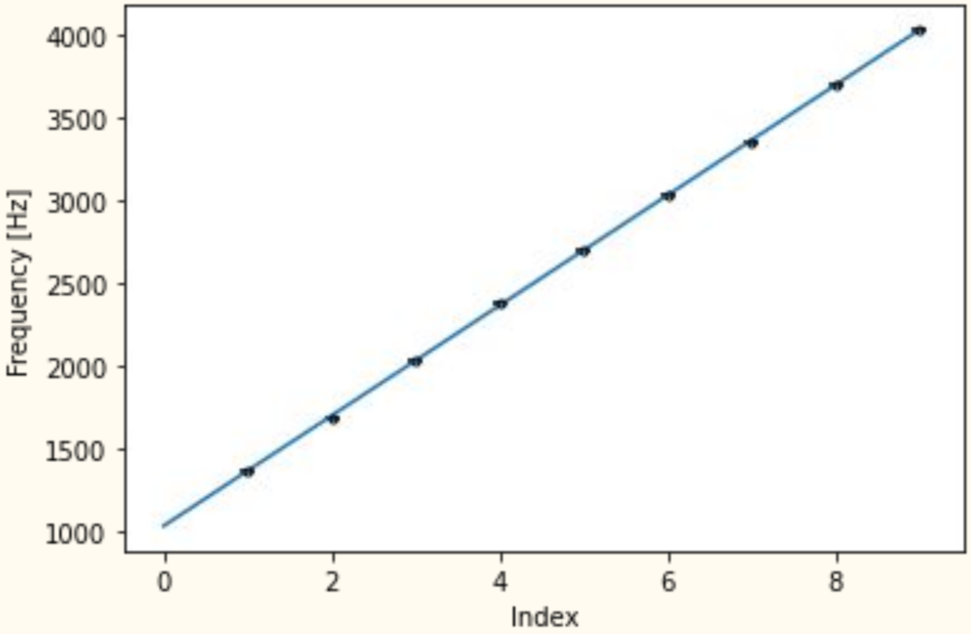


a) Tube Length = 45 cm

b) Tube Length = 52.5 cm

- Long tubes = higher frequencies for resonance and larger amplitudes

Results & Analysis - Speed of Sound



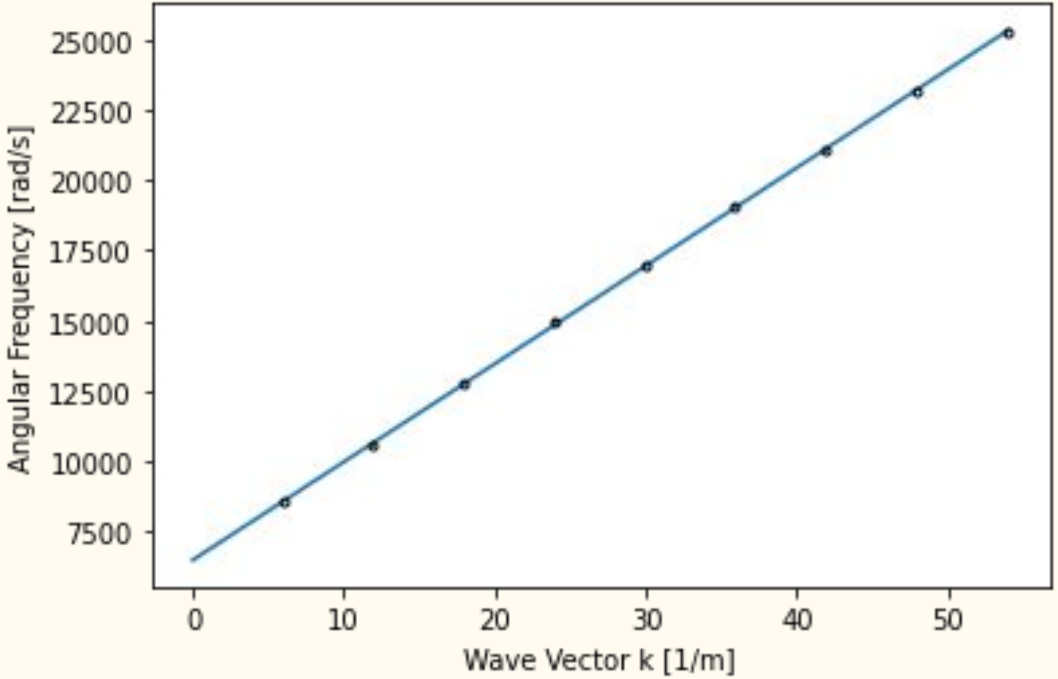
$$2L = n\frac{c}{f} \qquad f(n) = \frac{c}{2L} * n$$

- For function: f = n * (c / 1.05) + B
  - c = 348.84493 +/- 4.05343 m/s
  - B = 1034.61142 +/- 21.72353 1/s

(Speed of Sound in dry air @ 20 C = 343 m/s)

Resonance Frequency vs Index (L = 52.5cm)

Results & Analysis - Dispersion Relation of Sound



Angular Frequency vs Wave Vector (L = 52.5cm)

$$2\pi * f = \frac{2\pi * nc}{2L}$$

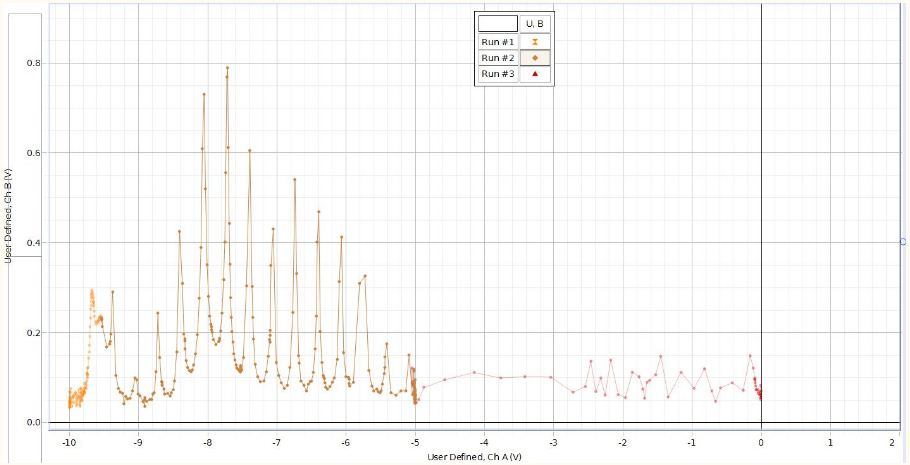
$$\omega = \frac{n\pi c}{L}$$

$$\omega(k) = kc$$

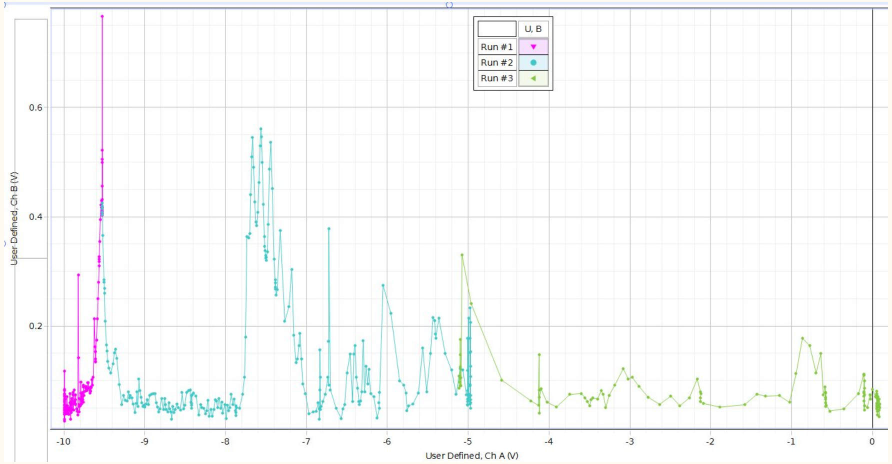
c = 348.84493 +/- 0.75264 m/s

B = 6500.65530 +/- 25.34411 rad/s

Results & Analysis - Tube Barrier Effects (Irises) 1

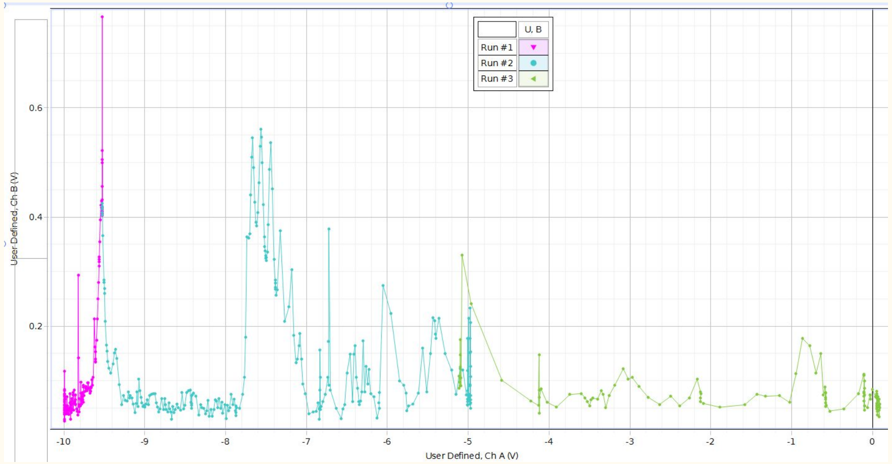


a) Amplitude vs Frequency w/ no Iris

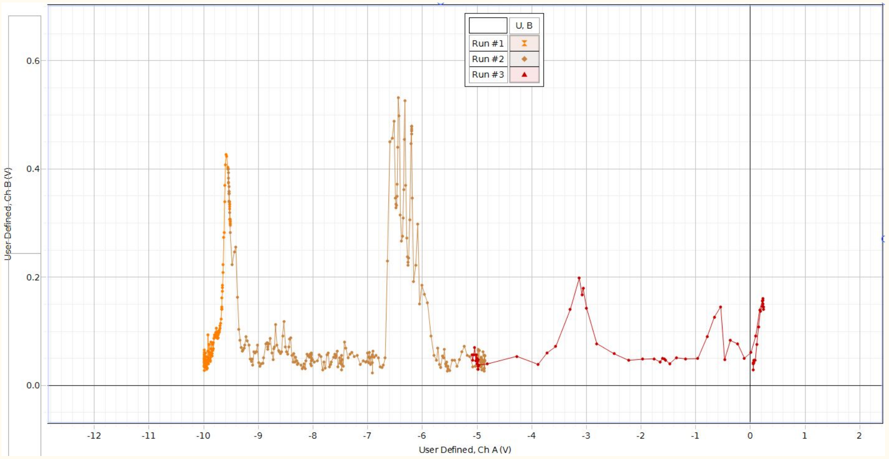


b) Amplitude vs Frequency w/ 7 irises

Results & Analysis - Tube Barrier Effects (Irises) 2



a) 7 Irises



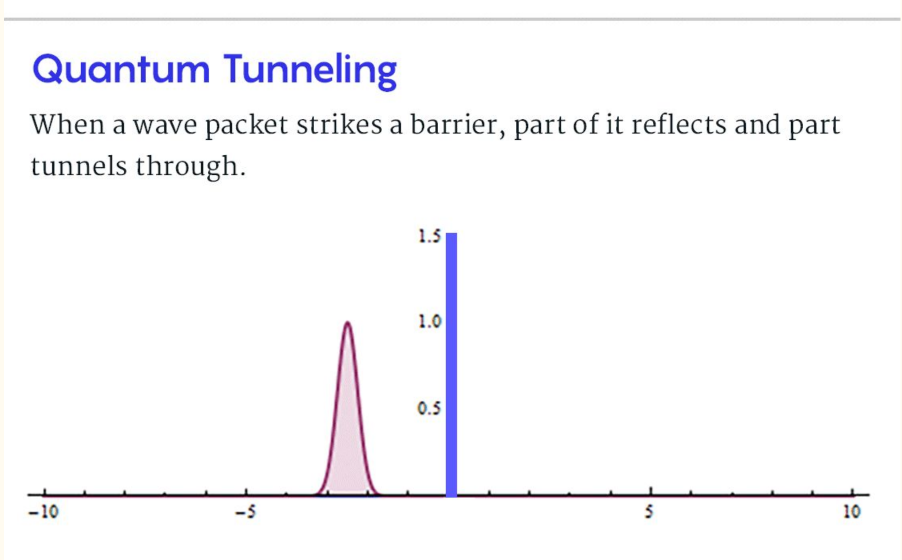
b) 10 Irises

- Resonance Frequencies have:
  - Steep peaks in Amplitude
  - Resonant Frequencies are much more condensed
  - Appears like peaks decrease in an exponential fashion

Results & Analysis - Tube Barrier Effects (Irises) 3

QM Interpretation::

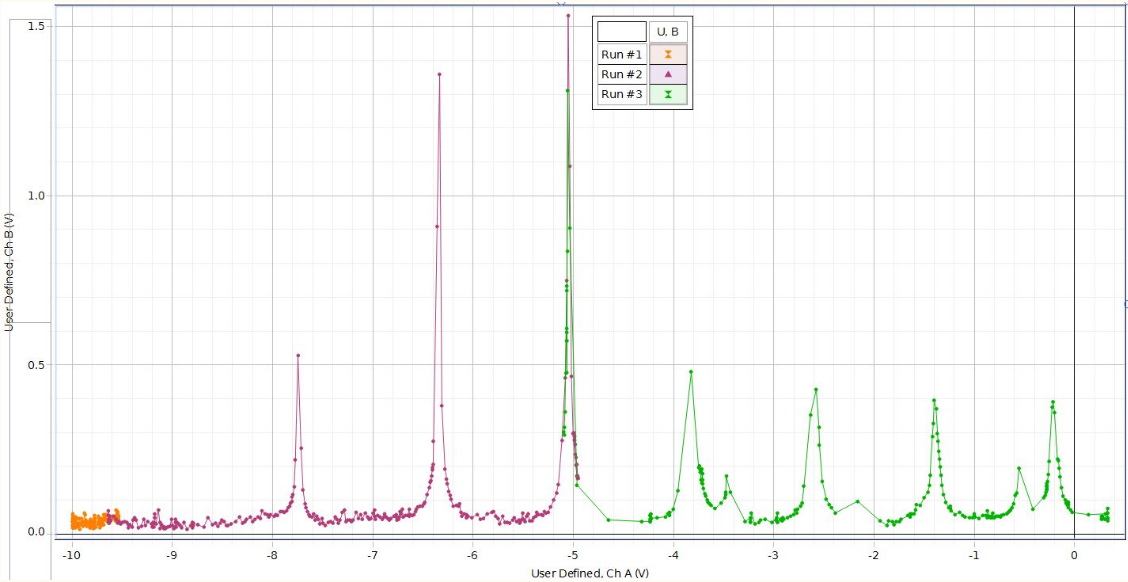
- Irises between the tubes seems to replicate the effects of Quantum Tunneling
    - When Sound hits barrier, there is a large spike in frequency
    - Causes multiple peaks in amplitude to form which dissipate quickly
    - Is the wave through the Iris an exponential decay as in QM?



https://www.quantamagazine.org/quantum-tunnel-shows-particles-can-break-the-speed-of-light-20201020/

Results & Analysis - Spherical Resonator 1

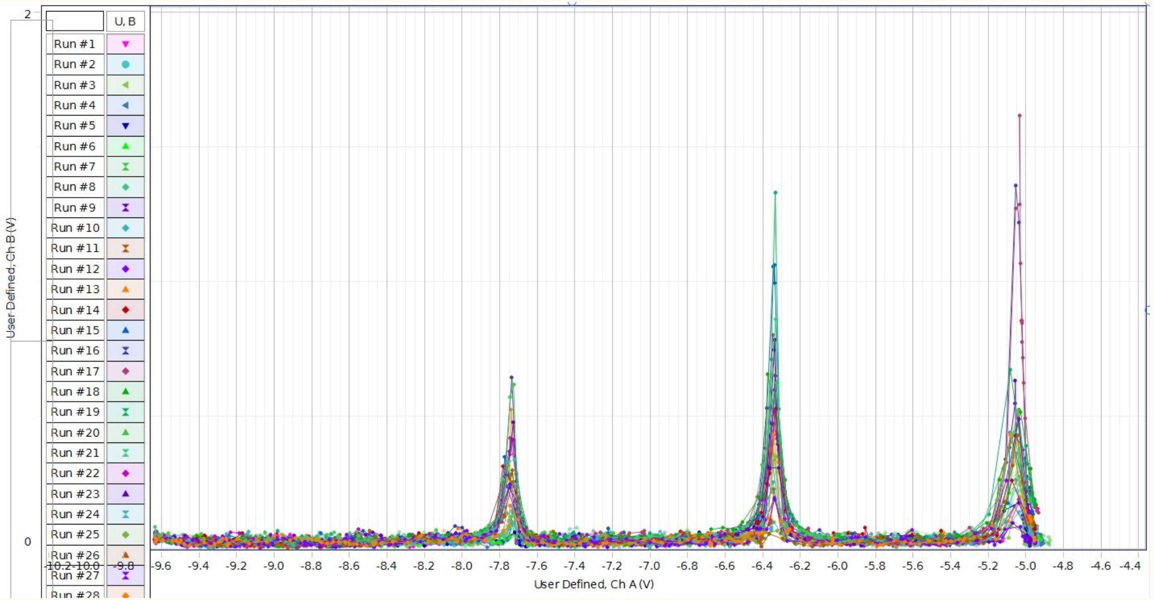
- Frequency Sweep (100Hz-10kHz) of the spherical resonator at the zero degree marking (α=0)



- Resonant Peaks appear on graphs similarly to the peaks in the previous part of this research

Results & Analysis - Spherical Resonator 2

- When Resonator is spun and frequency is swept, the resonant frequency remains constant
- The Amplitude, however, changes in magnitude



Resonant Frequencies of Spherical Resonator captured at every 10 degree increment

Aside:: Legendre Polynomials

Useful mathematical formula in which to analyze the waveforms of hydrogen with different values if l::

Spherical Harmonics and Legendre Polynomials:

The spherical harmonics $Y_l^m(\theta,\varphi)$ can be written as

$$Y_l^m(\theta,\varphi) \propto P_l^m(\cos\theta)\, e^{im\varphi}$$

to apply to curve fitting they are as follows::

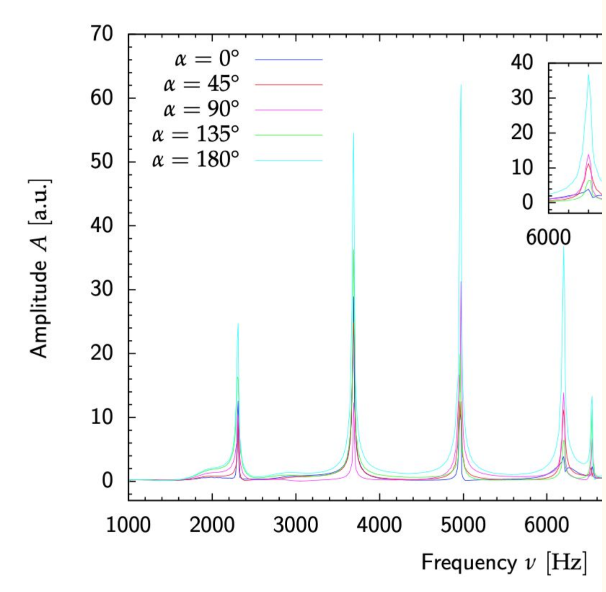
$$P_0(\cos\theta) = 1$$

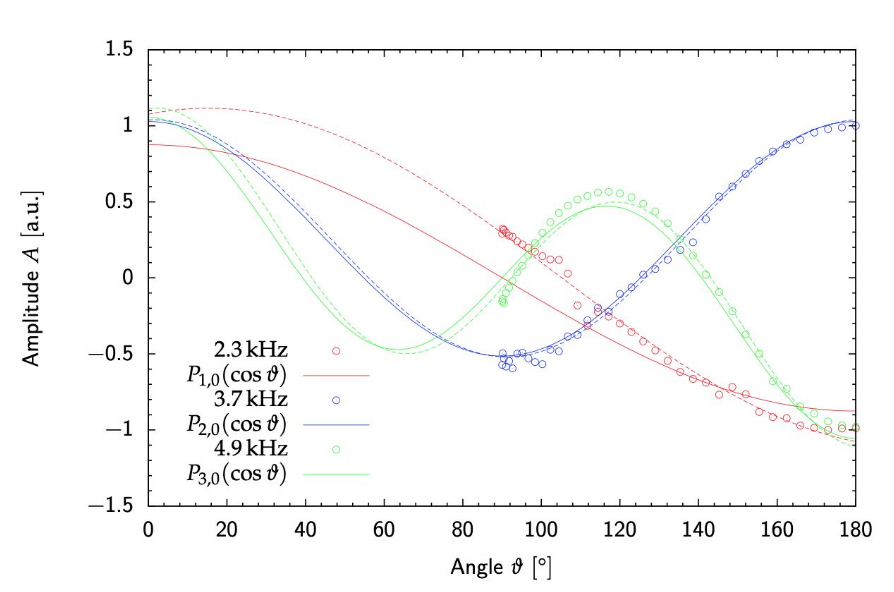
$$P_1(\cos\theta) = \cos\theta$$

$$P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$$

$$P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$$

Results & Analysis - Spherical Resonator L values

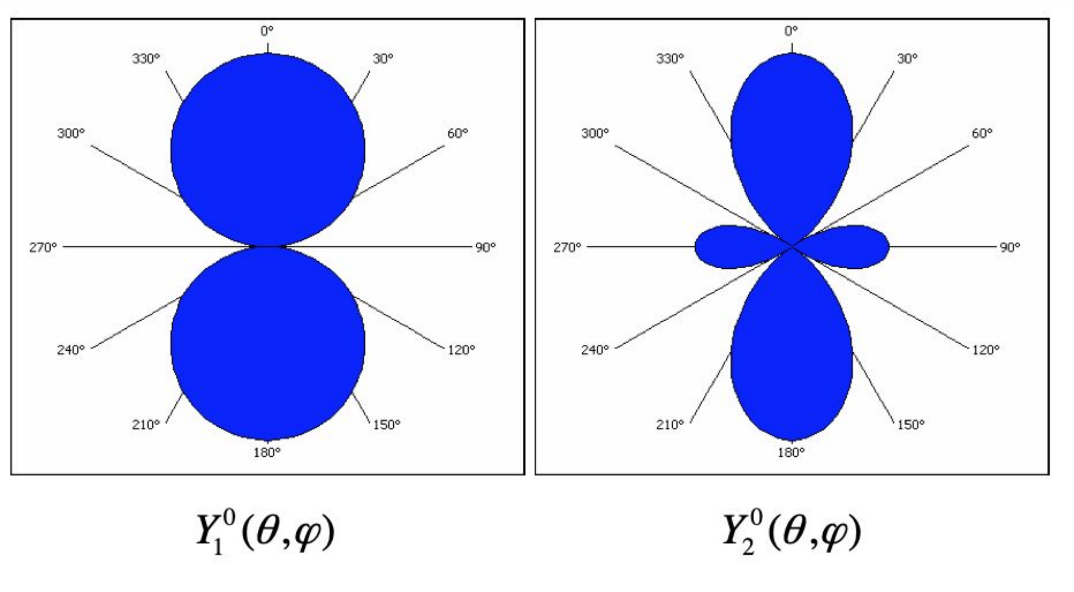




Researchers given the same experimental setup at Stuttgart University fit the values of amplitude at the first three resonant frequencies and found that the value for azimuthal quantum number l is: 1, 2, and 3 for the first three peaks

Results & Analysis - Spherical Resonator in QM

- Quantum number l is useful in showing the solutions to Schrodinger's equation for a hydrogen atom
- This equation can also be curve fit using the data of amplitudes at each resonant frequency recorded



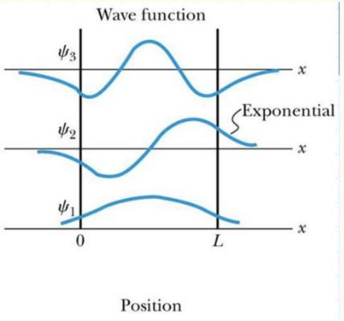
Polar representations of Quantum Waveform for Hydrogen Atom

Conclusion: Sound Waves in a Tube

Analogous Properties

1. Energy and Frequency $$E(k) = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 n^2 \pi^2}{2mL^2}$$

2. Potential Barriers and Irises



3. Phase Factors

Non-Analogous Properties

1. Position Measurement $$|\Psi(x)|^2$$

2. The Schrödinger Potential $$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = E\psi$$

3. Boundary Conditions

Conclusion: Sound Waves in a Spherical Resonator

Analogous Properties

1. Energy and Frequency

2. The Spherical Function

3. Magnetic and Azimuthal Quantum Number

Non-Analogous Properties

1. The Radial Function

2. Radial Quantum Number

3. Energy Level Decomposition

Orbital Energy levels in the Hydrogen atom

References

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