Sathyabama Institute of Science and Technology

SMTA1402 - Probability and Statistics

Unit-1 Probability Concepts and Random Variable

Random Experiment

An experiment whose outcome or result can be predicted with certainty is called a Deterministic experiment.

Although all possible outcomes of an experiment may be known in advance the outcome of a particular performance of the experiment cannot be predicted owing to a number of unknown causes. Such an experiment is called a Random experiment.

(e.g.) Whenever a fair dice is thrown, it is known that any of the 6 possible outcomes will occur, but it cannot be predicted what exactly the outcome will be.

Sample Space

The set of all possible outcomes which are assumed equally likely.

Event

A sub-set of S consisting of possible outcomes.

Mathematical definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Let $n(S)$ and $n(A)$ be the number of elements of S and A. then the probability of event A occurring is denoted as P(A), is denoted by

$$P(A) = \frac{n(A)}{n(S)}$$

Note: 1. It is obvious that $0 \le P(A) \le 1$.
2. If A is an impossible event, $P(A) = 0$.
3. If A is a certain event , $P(A) = 1$.

A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$ A set of events is said to be mutually exclusive if the occurrence of any one them excludes the occurrence of the others. That is, set of the events does not occur simultaneously,

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n, \ldots) = 0$$

Axiomatic definition of Probability

Let S be the sample space and A be an event associated with a random experiment. Then the probability of the event A, P(A) is defined as a real number satisfying the following axioms.

1. $0 \le P(A) \le 1$
2. $P(S) = 1$
3. If A and B are mutually exclusive events, $P(A \cup B) = P(A) + P(B)$ and
4. If $A_1, A_2\ A_3, \ldots, A_n, \ldots$ are mutually exclusive events,
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n, \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots + P(A_n) \ldots$$

Important Theorems

Theorem 1: Probability of impossible event is zero.

Proof: Let S be sample space (certain events) and ϕ be the impossible event.

Certain events and impossible events are mutually exclusive.

$P(S \cup \phi) = P(S) + P(\phi)$ (Axiom 3)

$S \cup \phi = S$

$P(S) = P(S) + P(\phi)$

$P(\phi) = 0$, hence the result.

Theorem 2: If $\overline{A}$ is the complementary event of A, $P(\overline{A}) = 1 - P(A) \leq 1$.

Proof: Let A be the occurrence of the event

$\overline{A}$ be the non-occurrence of the event .

Occurrence and non-occurrence of the event are mutually exclusive.

$$P(A \cup \overline{A}) = P(A) + P(\overline{A})$$

$$A \cup \overline{A} = S \quad \Rightarrow \quad P(A \cup \overline{A}) = P(S) = 1$$

$$\therefore \quad 1 = P(A) + P(\overline{A})$$

$$P(\overline{A}) = 1 - P(A) \leq 1.$$

Theorem 3: (Addition theorem)

If A and B are any 2 events,

$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B)$.

Proof: We know, $A = A\overline{B} \cup AB$ and $B = \overline{A}B \cup AB$

$$\therefore \quad P(A) = P(A\overline{B}) + P(AB) \text{ and } P(B) = P(\overline{A}B) + P(AB) \quad \text{(Axiom 3)}$$

$$P(A) + P(B) = P(A\overline{B}) + P(AB) + P(\overline{A}B) + P(AB)$$

$$= P(A \cup B) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \leq P(A) + P(B).$$

Note: The theorem can be extended to any 3 events, A,B and C

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$

Theorem 4: If $B \subset A$, $P(B) \leq P(A)$.

Proof: A and $A\overline{B}$ are mutually exclusive events such that $B \cup A\overline{B} = A$

$$\therefore \quad P(B \cup A\overline{B}) = P(A)$$

$$P(B) + P(A\overline{B}) = P(A) \quad \text{(Axiom 3)}$$

$$P(B) \leq P(A)$$

Conditional Probability

The conditional probability of an event B, assuming that the event A has happened, is denoted by P(B/A) and defined as

$$P(B/A) = \frac{P(A \cap B)}{P(A)}, \text{ provided } P(A) \neq 0$$

Product theorem of probability

Rewriting the definition of conditional probability, We get

$$P(A \cap B) = P(A)P(A/B)$$

The product theorem can be extended to 3 events, A, B and C as follows:

$$P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$$

Note: 1. If $A \subset B$, $P(B/A) = 1$, since $A \cap B = A$.

2. If $B \subset A$, $P(B/A) \geq P(B)$, since $A \cap B = B$, and $\frac{P(B)}{P(A)} \geq P(B)$,

As $P(A) \leq P(S) = 1$.

3. If A and B are mutually exclusive events, $P(B/A) = 0$, since $P(A \cap B) = 0$.
4. If $P(A) > P(B)$, $P(A/B) > P(B/A)$.
5. If $A_1 \subset A_2$, $P(A_1/B) \leq P(A_2/B)$.

Independent Events

A set of events is said to be independent if the occurrence of any one of them does not depend on the occurrence or non-occurrence of the others.

If the two events A and B are independent, the product theorem takes the form $P(A \cap B) = P(A) \times P(B)$, Conversely, if $P(A \cap B) = P(A) \times P(B)$, the events are said to be independent (pair wise independent).

The product theorem can be extended to any number of independent events, If $A_1\ A_2\ A_3 \ldots\ A_n$ are n independent events, then

$$P(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n) = P(A_1) \times P(A_2) \times P(A_3) \times \ldots \times P(A_n)$$

Theorem 4:

If the events A and B are independent, the events $\bar{A}$ and B are also independent.

Proof:

The events $A \cap B$ and $\bar{A} \cap B$ are mutually exclusive such that $(A \cap B) \cup (\bar{A} \cap B) = B$

$\therefore \quad P(A \cap B) + P(\bar{A} \cap B) = P(B)$

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$= P(B) - P(A)\,P(B)$ (∴ A and B are independent)

$= P(B)\,[1 - P(A)]$

$= P(\bar{A})\,P(B).$

Theorem 5:

If the events A and B are independent, the events $\bar{A}$ and $\bar{B}$ are also independent.

Proof:

$$P(\bar{A} \cap \bar{B}) = P\left(\overline{A \cup B}\right) = 1 - P(A \cup B)$$

$= 1 - [\,P(A) + P(B) - P(A \cap B)]$ (Addition theorem)

$= [1 - P(A)] - P(B)\,[1 - P(A)]$

$= P(\bar{A})P(\bar{B}).$

Problem 1:

From a bag containing 3 red and 2 balck balls, 2 ball are drawn at random. Find the probability that they are of the same colour.

Solution :

Let A be the event of drawing 2 red balls

B be the event of drawing 2 black balls.

$\therefore \quad P(A \cup B) = P(A) + P(B)$

$$= \frac{3C_2}{5C_2} + \frac{2C_2}{5C_2} = \frac{3}{10} + \frac{1}{10} = \frac{2}{5}$$

Problem 2:

When 2 card are drawn from a well-shuffled pack of playing cards, what is the probability that they are of the same suit?

Solution :

Let A be the event of drawing 2 spade cards

B be the event of drawing 2 claver cards

C be the event of drawing 2 hearts cards

D be the event of drawing 2 diamond cards.

$$\therefore P(A \cup B \cup C \cup D) = 4\frac{13C_2}{52C_2} = \frac{4}{17}.$$

Problem 3:

When A and B are mutually exclusive events such that P(A) = 1/2 and P(B) = 1/3, find $P(A \cup B)$ and $P(A \cap B)$.

Solution :

$P(A \cup B) = P(A) + P(B) = 5/6$; $P(A \cap B) = 0$.

Problem 4:

If P(A) = 0.29, P(B) = 0.43, find $P(A \cap \overline{B})$, if A and B are mutually exclusive.

Solution :

We know $A \cap \overline{B} = A$

$P(A \cap \overline{B}) = P(A) = 0.29$

Problem 5:

A card is drawn from a well-shuffled pack of playing cards. What is the probability that it is either a spade or an ace?

Solution :

Let A be the event of drawing a spade

B be the event of drawing a ace

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{4}{13}.$$

Problem 6:

If P(A) = 0.4, P(B) = 0.7 and P(A $\cap$ B) = 0.3, find P($\bar{A} \cap \bar{B}$).

Solution :

$$P(\bar{A} \cap \bar{B}) = 1 - P(A \cup B)$$
$$= 1 - [P(A) + P(B) - P(A \cap B)]$$
$$= 0.2$$

Problem 7:

If P(A) = 0.35, P(B) = 0.75 and P(A $\cup$ B) = 0.95, find P($\bar{A} \cup \bar{B}$).

Solution :

$$P(\bar{A} \cup \bar{B}) = 1 - P(A \cap B) = 1 - [P(A) + P(B) - P(A \cup B)] = 0.85$$

Problem 8:

A lot consists of 10 good articles, 4 with minor defects and 2 with major defects. Two articles are chosen from the lot at random(with out replacement). Find the probability that (i) both are good, (ii) both have major defects, (iii) at least 1 is good, (iv) at most 1 is good, (v) exactly 1 is good, (vi) neither has major defects and (vii) neither is good.

Solution :

(i) P(both are good) = $\dfrac{10C_2}{16C_2} = \dfrac{3}{8}$

(ii) P(both have major defects) = $\dfrac{2C_2}{16C_2} = \dfrac{1}{120}$

(iii) P(at least 1 is good) = $\dfrac{10C_1 6C_1 + 10C_2}{16C_2} = \dfrac{7}{8}$

(iv) P(at most 1 is good) = $\dfrac{10C_0 6C_2 + 10C_1 6C_1}{16C_2} = \dfrac{5}{8}$

(v) P(exactly 1 is good) = $\dfrac{10C_1 6C_1}{16C_2} = \dfrac{1}{2}$

(vi) P(neither has major defects) = $\dfrac{14C_2}{16C_2} = \dfrac{91}{120}$

(vii) P(neither is good) = $\dfrac{6C_2}{16C_2} = \dfrac{1}{8}$.

Problem 9:

If A, B and C are any 3 events such that P(A) = P(B) = P(C) = 1/4, P(A $\cap$ B) = P(B $\cap$ C) = 0; P(C $\cap$ A) = 1/8. Find the probability that at least 1 of the events A, B and C occurs.

Solution :

Since P(A $\cap$ B) = P(B $\cap$ C) = 0; P(A $\cap$ B $\cap$ C) = 0

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

$$= \frac{3}{4} - 0 - 0 - \frac{1}{8} = \frac{5}{8}.$$

Problem 10:

A box contains 4 bad and 6 good tubes. Two are drawn out from the box at a time. One of them is tested and found to be good. What is the probability that the other one is also good?

Solution :

Let A be a good tube drawn and B be an other good tube drawn.

P(both tubes drawn are good) = P(A ∩ B) = $\frac{6C_2}{10C_2} = \frac{1}{3}$

P(B/A) = $\frac{P(A \cap B)}{P(A)} = \frac{1/3}{6/10} = \frac{5}{9}$ (By conditional probability)

Problem 11:

In shooting test, the probability of hitting the target is 1/2, for a, 2/3 for B and ¾ for C. If all of them fire at the target, find the probability that (i) none of them hits the target and (ii) at least one of them hits the target.

Solution :

Let A, B and C be the event of hitting the target .
P(A) = 1/2, P(B) = 2/3, P(C) = 3/4
P($\overline{A}$) = 1/2, P($\overline{B}$) = 1/3, P($\overline{C}$) = 1/4

P(none of them hits) = P($\overline{A} \cap \overline{B} \cap \overline{C}$) = P($\overline{A}$) × P($\overline{B}$) × P($\overline{C}$) = 1/24

P(at least one hits) = 1 – P(none of them hits)
= 1 – (1/24) = 23/24.

Problem 12:

A and B alternatively throw a pair of dice. A wins if he throws 6 before B throws 7 and B wins if he throws 7 before A throws 6. If A begins, show that his chance of winning is 30/61.

Solution :

Let A be the event of throwing 6
B be the event of throwing 7.

P(throwing 6 with 2 dice) = 5/36 P(throwing 7 with 2 dice) = 1/6
P(not throwing 6) = 31/36 P(not throwing 7) = 5/6

A plays in I, III, V,……trials.
A wins if he throws 6 before Be throws 7.

$$P(A \text{ wins}) = P(A \cup \overline{A}\,\overline{B}A \cup \overline{A}\,\overline{B}\,\overline{A}\,\overline{B}A \cup \ldots\ldots)$$
$$= P(A) + P(\overline{A}\,\overline{B}A) + P(\overline{A}\,\overline{B}\,\overline{A}\,\overline{B}A) + \ldots\ldots$$
$$= \frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)\frac{5}{36} + \left(\frac{31}{36} \times \frac{5}{6}\right)^2 \frac{5}{36} + \ldots$$
$$= \frac{30}{61}$$

Problem 13:

A and B toss a fair coin alternatively with the understanding that the first who obtain the head wins. If A starts, what is his chance of winning?

Solution :

P(getting head) = 1/2 , P(not getting head) = 1/2

A plays in I, III, V,……trials.
A wins if he gets head before B.

$$P(\text{A wins}) = P(A \cup \bar{A}\,\bar{B}\,A \cup \bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A \cup \ldots\ldots)$$
$$= P(A) + P(\bar{A}\,\bar{B}\,A) + P(\bar{A}\,\bar{B}\,\bar{A}\,\bar{B}\,A) + \ldots\ldots$$
$$= \frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)\frac{1}{2} + \left(\frac{1}{2}\times\frac{1}{2}\right)^2\frac{1}{2} + \cdots$$
$$= \frac{2}{3}$$

Problem 14:

A problem is given to 3 students whose chances of solving it are 1/2 , 1/3 and 1/4 . What is the probability that (i) only one of them solves the problem and (ii) the problem is solved.

Solution :

P(A solves) = 1/2 P(B) = 1/3 P(C) = 1/4
$P(\bar{A}) = 1/2$, $P(\bar{B}) = 2/3$, $P(\bar{C}) = 3/4$

P(none of them solves) = $P(\bar{A} \cap \bar{B} \cap \bar{C}) = P(\bar{A}) \times P(\bar{B}) \times P(\bar{C}) =$ 1/4

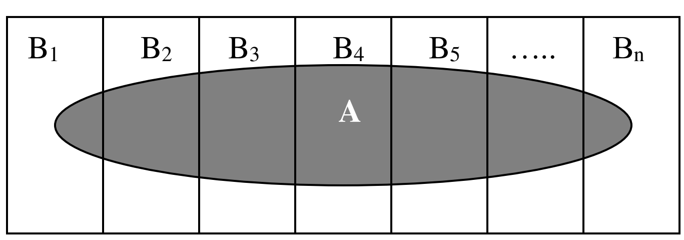
P(at least one solves) = 1 – P(none of them solves)
= 1 – (1/4) = 3/4 .

Baye's Theorem

Statement: If $B_1, B_2, B_3, \ldots B_n$ be a set of exhaustive and mutually exclusive events associated with a random experiment and A is another event associated with B_i, then

$$P(B_i / A) = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}$$

Proof :



The shaded region represents the event A, A can occur along with $B_1, B_2, B_3, \ldots B_n$ that are mutually exclusive.

$\therefore$ $AB_1, AB_2, AB_3, \ldots, AB_n$ are also mutually exclusive.

Also $A = AB_1 \cup AB_2 \cup AB_3 \cup \ldots \cup AB_n$

$$P(A) = P(AB_1) + P(AB_2) + P(AB_3) + \ldots + P(AB_n)$$
$$= \sum_{i=1}^{n} P(AB_i)$$
$$= \sum_{i=1}^{n} P(B_i) \times P(A/B_i) \quad \text{(By conditional probability)}$$

P(B_i/A) = $\frac{P(B_i) \times P(A/B_i)}{P(A)} = \frac{P(B_i) \times P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \times P(A/B_i)}$.

Problem 15:

Ina bolt factory machines A, B, C manufacture respectively 25%, 35% and 40% of the total. Of their output 5%, 4% and 2% are defective bolts. A bolt is drawn at random from the produce and is found to be defective. What are the probabilities that it was manufactured by machines A, B and C.

Solution :

Let B_1 be bolt produced by machine A
B_2 be bolt produced by machine B
B_3 be bolt produced by machine C
Let A/B_1 be the defective bolts drawn from machine A
A/B_2 be the defective bolts drawn from machine B
A/B_3 be the defective bolts drawn from machine C.

P(B_1) = 0.25 P(A/B_1) = 0.05
P(B_2) = 0.35 P(A/B_2) = 0.04
P(B_3) = 0.40 P(A/B_3) = 0.02

Let B_1/A be defective bolts manufactured by machine A
B_2/A be defective bolts manufactured by machine B
B_3/A be defective bolts manufactured by machine C

$P(A) = \sum_{i=1}^{3} P(B_i) \times P(A/B_i)$ = (0.25) × (0.05) + (0.35) × (0.04) + (0.4) × (0.02)

= 0.0345

P(B_1/A) = $\frac{P(B_1) \times P(A/B_1)}{P(A)}$ = 0.3623

P(B_2/A) = $\frac{P(B_2) \times P(A/B_2)}{P(A)}$ = 0.405

P(B_3/A) = $\frac{P(B_3) \times P(A/B_3)}{P(A)}$ = 0.231 .

Problem 16 :

The first bag contains 3 white balls, 2 red balls and 4 black balls. Second bag contains 2 white, 3 red and 5 black balls and third bag contains 3 white, 4 red and 2 black balls. One bag is chosen at random and from it 3 balls are drawn. Out of three balls two balls are white and one is red. What are the probabilities that they were taken from first bag, second bag and third bag.

Solution :

Let P(selecting the bag) = P(A_i) = 1/3 , i = 1, 2, 3.

P(A/B_1) = $\frac{3C_2 2C_1}{9C_3} = \frac{6}{84}$ $P(A) = \sum_{i=1}^{3} P(B_i) \times P(A/B_i)$ = 0.0746

P(A/B_2) = $\frac{2C_2 3C_1}{10C_3} = \frac{3}{120}$

$$P(A/B_3) = \frac{3C_2\,4C_1}{9C_3} = \frac{12}{84}$$

$$P(B_1/A) = \frac{P(B_1)\times P(A/B_1)}{P(A)} = 0.319$$

$$P(B_2/A) = \frac{P(B_2)\times P(A/B_2)}{P(A)} = 0.4285$$

$$P(B_3/A) = \frac{P(B_3)\times P(A/B_3)}{P(A)} = 0.638$$

Random Variable

Random Variable:

A random variable is a real valued function whose domain is the sample space of a random experiment taking values on the real line $\mathbb{R}$.

Discrete Random Variable:

A discrete random variable is one which can take only finite or countable number of values with definite probabilities associated with each one of them.

Probability mass function:

Let X be discrete random variable which assuming values $x_1, x_2, \ldots, x_n$ with each of the values, we associate a number called the probability $P(X = x_i) = p(x_i), (i = 1, 2, \ldots, n)$ this is called the probability of x_i satisfying the following conditions

i. $p_i \geq 0 \ \forall i$ i.e., $p_i\,'s$ are all non-negative

ii. $\sum_{i=1}^{n} p_i = p_1 + p_2 + \ldots + p_n = 1$ i.e., the total probability is one.

Continuous random variable:

A continuous random variable is one which can assume every value between two specified values with a definite probability associated with each.

Probability Density Function:

A function f is said to be the probability density function of a continuous random variable X if it satisfies the following properties.

i. $f(x) \geq 0; \ -\infty < x < \infty$

ii. $\int_{-\infty}^{\infty} f(x)dx = 1.$

Distribution Function or Cumulative Distribution Function

i. Discrete Variable:

A distribution function of a discrete random variable X is defined as $P(X \leq x) = \sum_{x_i \leq x} P(x_i)$.

ii. Continuous Variable:

A distribution function of a continuous random variable X is defined as $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)\,dx$.

Mathematical Expectation

The expected value of the random variable X is defined as

i. If X is discrete random variable $E(X) = \sum_{i=1}^{\infty} x_i p(x_i)$ where $p(x)$ is the probability function of x.

ii. If X is continuous random variable $E(X)=\int_{-\infty}^{\infty} xf(x)dx$ where $f(x)$ is the probability density function of x.

Properties of Expectation:

1. If C is constant then $E(C)=C$

Proof:

Let X be a discrete random variable then $E(x)=\sum xp(x)$

Now $E(C)=\sum Cp(x)$

$$=C\sum p(x) \qquad \text{since } \sum_{i=1}^{n} p_i = p_1+p_2+...+p_n=1$$

$$=C$$

2. If a,b are constants then $E(ax+b)=aE(x)+b$

Proof:

Let X be a discrete random variable then $E(x)=\sum xp(x)$

Now $E(ax+b)=\sum (ax+b)\,p(x)$

$$=\sum axp(x)+\sum bp(x)$$

$$=a\sum xp(x)+b\sum p(x) \qquad \text{since } \sum_{i=1}^{n} p_i = p_1+p_2+...+p_n=1$$

$$=aE(x)+b$$

3. If a and b are constants then $Var(ax+b)=a^2Var(x)$

Proof:

$$Var(ax+b)=E\left[\left(ax+b-E(ax+b)\right)^2\right]$$

$$=E\left[\left(ax+b-aE(x)-b\right)^2\right]$$

$$=E\left[a^2\left(x-E(x)\right)^2\right]$$

$$=a^2E\left[\left(x-E(x)\right)^2\right]$$

$$=a^2Var(x).$$

4. If a is constant then $Var(ax)=a^2Var(x)$

Proof:

$$Var(ax)=E\left[\left(ax-E(ax)\right)^2\right]$$

$$=E\left[\left(ax-aE(x)\right)^2\right]$$

$$=E\left[a^2\left(x-E(x)\right)^2\right]$$

$$=a^2E\left[\left(x-E(x)\right)^2\right]$$

$$=a^2Var(x).$$

5. Prove that $Var(x) = E(x^2) - [E(x)]^2$

Proof:

$$Var(x) = E\left[\left(x - E(x)\right)^2\right]$$
$$= E\left[x^2 + \left(E(x)\right)^2 - 2xE(x)\right]$$
$$= E\left[x^2 + \mu^2 - 2x\mu\right]$$
$$= E(x^2) + E(\mu^2) - E(2x\mu)$$
$$= E(x^2) + \mu^2 - 2\mu E(x)$$
$$= E(x^2) + \mu^2 - 2\mu^2$$
$$= E(x^2) - \mu^2$$
$$Var(x) = E(x^2) - [E(x)]^2$$

Problem.1

If the probability distribution of X is given as

X	:	1	2	3	4
$P\ X$	:	0.4	0.3	0.2	0.1

Find $P(1/2 < X < 7/2 / X > 1)$

Solution:

$$P\{1/2 < X < 7/2 / X > 1\} = \frac{P\{(1/2 < X < 7/2) \cap X > 1\}}{P(X > 1)}$$

$$= \frac{P(X = 2 \, or \, 3)}{P(X = 2, 3 \, or \, 4)}$$

$$= \frac{P(X=2) + P(X=3)}{P(X=2) + P(X=3) + P(X=4)}$$

$$= \frac{0.3 + 0.2}{0.3 + 0.2 + 0.1} = \frac{0.5}{0.6} = \frac{5}{6}.$$

Problem.2

A random variable X has the following probability distribution

X	:	-2	-1	0	1	2	3
$P\ X$	:	0.1	K	0.2	$2K$	0.3	$3K$

a) Find K, b) Evaluate $P(X < 2)$ and $P(-2 < X < 2)$

b) Find the cdf of X and d) Evaluate the mean of X.

Solution:

a) Since $\sum P(X) = 1$

$$0.1 + K + 0.2 + 2K + 0.3 + 3K = 1$$

$$6K + 0.6 = 1$$

$$6K = 0.4$$

$$K = \frac{0.4}{6} = \frac{1}{15}$$

b) $P(X < 2) = P(X = -2, -1, 0 \, or \, 1)$

$$= P(X = -2) + P(X = -1) + P(X = 0) + P(X = 1)$$

$$= \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{2}{15}$$

$$= \frac{3 + 2 + 6 + 4}{30} = \frac{15}{30} = \frac{1}{2}$$

$P(-2 < X < 2) = P(X = -1, 0 \, or \, 1)$

$$= P(X = -1) + P(X = 0) + P(X = 1)$$

$$=\frac{1}{15}+\frac{1}{5}+\frac{2}{15}$$
$$=\frac{1+3+2}{15}=\frac{6}{15}=\frac{2}{5}$$

c) The distribution function of X is given by $F(x)$ defined by

$X=x$	$P(X=x)$	$F(x)=P(X\le x)$
-2	$\frac{1}{10}$	$F(x)=P(X\le -2)=\frac{1}{10}$
-1	$\frac{1}{15}$	$F(x)=P(X\le -1)=\frac{1}{6}$
0	$\frac{2}{10}$	$F(x)=P(X\le 0)=\frac{11}{30}$
1	$\frac{2}{15}$	$F(x)=P(X\le 1)=\frac{1}{2}$
2	$\frac{3}{10}$	$F(x)=P(X\le 2)=\frac{4}{5}$
3	$\frac{3}{15}$	$F(x)=P(X\le 3)=1$

d) Mean of X is defined by $E(X)=\sum xP(x)$

$$E(X)=\left(-2\times\frac{1}{10}\right)+\left(-1\times\frac{1}{15}\right)+\left(0\times\frac{1}{5}\right)+\left(1\times\frac{2}{15}\right)+\left(2\times\frac{3}{10}\right)+\left(3\times\frac{1}{5}\right)$$
$$=-\frac{1}{5}-\frac{1}{15}+\frac{2}{15}+\frac{3}{5}+\frac{3}{5}=\frac{16}{15}.$$

Problem.3

A random variable X has the following probability function:

X	: 0	1	2	3	4	5	6	7
$P\ X$	: 0	K	$2K$	$2K$	$3K$	K^2	$2K^2$	$7K^2+K$

Find (i) K , (ii) Evaluate $P(X<6), P(X\ge 6)$ and $P(0<X<5)$

(iii). Determine the distribution function of X .

(iv). $P(1.5<X<4.5/X>2)$

(v). $E(3x-4), Var(3x-4)$

(vi). The smallest value of n for which $P(X\le n)>\frac{1}{2}$.

Solution:

(i) Since $\sum_{x=0}^{7}P(X)=1$,

$$K+2K+2K+3K+K^2+2K^2+7K^2+K=1$$
$$10K^2+9K-1=0$$
$$K=\frac{1}{10}\quad or\quad K=-1$$

As $P(X)$ cannot be negative $K=\frac{1}{10}$

(ii) $P(X<6)=P(X=0)+P(X=1)+...+P(X=5)$

$$=\frac{1}{10}+\frac{2}{10}+\frac{2}{10}+\frac{3}{10}+\frac{1}{100}+...=\frac{81}{100}$$

Now $P(X\geq 6)=1-P(X<6)$

$$=1-\frac{81}{100}=\frac{19}{100}$$

Now $P(0<X<5)=P(X=1)+P(X=2)+P(X=3)=P(X=4)$

$$=K+2K+2K+3K$$

$$=8K=\frac{8}{10}=\frac{4}{5}.$$

(iii) The distribution of X is given by $F(x)=P(X\leq x)$

$X=x$	$P(X=x)$	$F(x)=P(X\leq x)$
0	0	$F(x)=P(X\leq 0)=0$
1	$\frac{1}{10}$	$F(x)=P(X\leq 1)=\frac{1}{10}$
2	$\frac{2}{10}$	$F(x)=P(X\leq 2)=\frac{3}{10}$
3	$\frac{2}{10}$	$F(x)=P(X\leq 3)=\frac{5}{10}$
4	$\frac{3}{10}$	$F(x)=P(X\leq 4)=\frac{8}{10}$
5	$\frac{1}{100}$	$F(x)=P(X\leq 5)=\frac{81}{100}$
6	$\frac{2}{100}$	$F(x)=P(X\leq 6)=\frac{83}{100}$
7	$\frac{17}{100}$	$F(x)=P(X\leq 7)=1$

(iv) $P(1.5<X<4.5/X>2)=\dfrac{P(x=3)+P(x=4)}{1-\left[P(x=0)+P(x=1)+P(x=2)\right]}$

$$=\frac{\frac{5}{10}}{1-\left[\frac{3}{10}\right]}=\frac{5}{7}$$

(v) $E(x)=\sum xp(x)$

$$=1\times\frac{1}{10}+2\times\frac{2}{10}+3\times\frac{2}{10}+4\times\frac{3}{10}+5\times\frac{1}{100}+6\times\frac{2}{100}+7\times\frac{17}{100}$$

$E(x)=3.66$

$E(x^2)=\sum x^2p(x)$

$$=1^2\times\frac{1}{10}+2^2\times\frac{2}{10}+3^2\times\frac{2}{10}+4^2\times\frac{3}{10}+5^2\times\frac{1}{100}+6^2\times\frac{2}{100}+7^2\times\frac{17}{100}$$

$$E\left(x^2\right)=16.8$$

Mean $=E(x)=3.66$

Variance $=E\left(x^2\right)-\left[E(x)\right]^2$

$$=16.8-(3.66)^2$$
$$=3.404$$

(vi) The smallest value of n for which $P(X\le n)>\frac{1}{2}$ is 4

Problem.4

The probability mass function of random variable X is defined as $P(X=0)=3C^2$, $P(X=1)=4C-10C^2$, $P(X=2)=5C-1$, where $C>0$, and $P(X=r)=0$ if $r\ne 0,1,2$.

Find (i). The value of C.

(ii). $P(0<X<2/x>0)$.

(iii). The distribution function of X.

(iv). The largest value of x for which $F(x)<\frac{1}{2}$.

Solution:

(i) Since $\sum_{x=0}^{x=2}p(x)=1$

$p(0)+p(1)+p(2)=1$

$3C^2+4C-10C^2+5C-1=1$

$7C^2-9C+2=0$

$C=1,\frac{2}{7}$

$C=1$ is not applicable

$\therefore C=\frac{2}{7}$

The Probability distribution is

X	:	0	1	2
$P(X)$	:	$\frac{12}{49}$	$\frac{16}{49}$	$\frac{21}{49}$

(ii) $P\left[0<x<2/x>0\right]=\frac{P\left[(0<x<2)\cap x>0\right]}{P[x>0]}$

$$=\frac{P[0<x<2]}{P[x>0]}=\frac{P[x=1]}{P[x=1]+P[X=2]}$$

$$P\left[0<x<2/x>0\right]=\frac{\frac{16}{49}}{\frac{16}{49}+\frac{21}{49}}=\frac{16}{37}$$

(iii). The distribution function of X is

X	$F(X=x)=P(X\le x)$
0	$F(0)=P(X\le 0)=\frac{12}{49}=0.24$
1	$F(1)=P(X\le 1)=P(X=0)+P(X=1)=\frac{12}{49}+\frac{16}{49}=0.57$
2	$F(2)=P(X\le 2)=P(X=0)+P(X=1)+P(X=2)=\frac{12}{49}+\frac{16}{49}+\frac{21}{49}=1$

(iv) The Largest value of x for which $F(x)=P(X\le x)<\frac{1}{2}$ is 0.

Problem.5

If $P(x)=\begin{cases}\frac{x}{15}; x=1,2,3,4,5\\ 0 \quad ; elsewhere\end{cases}$

Find (i) $P\{X=1 or 2\}$ and (ii) $P\{1/2<X<5/2/x>1\}$

Solution:

i) $P(X=1\, or\, 2)=P(X=1)+P(X=2)$

$$=\frac{1}{15}+\frac{2}{15}=\frac{3}{15}=\frac{1}{5}$$

ii) $P\left(\frac{1}{2}<X<\frac{5}{2}/x>1\right)=\frac{P\left\{\left(\frac{1}{2}<X<\frac{5}{2}\right)\cap(X>1)\right\}}{P(X>1)}$

$$=\frac{P\{(X=1or2)\cap(X>1)\}}{P(X>1)}$$

$$=\frac{P(X=2)}{1-P(X=1)}$$

$$=\frac{2/15}{1-(1/15)}=\frac{2/15}{14/15}=\frac{2}{14}=\frac{1}{7}.$$

Problem.6

A continuous random variable X has a probability density function $f(x)=3x^2$, $0\le x\le 1$. Find $'a'$ such that $P(X\le a)=P(X>a)$.

Solution:

Since $P(X\le a)=P(X>a)$, each must be equal to $\frac{1}{2}$ because the probability is always 1.

$$\therefore P(X\le a)=\frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)dx=\frac{1}{2}$$

$$\int_0^a 3x^2dx=\frac{1}{2}\Rightarrow 3\left[\frac{x^3}{3}\right]_0^a=a^3=\frac{1}{2}.$$

$$\therefore a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

Problem.7

A random variable X has the p.d.f $f(x)$ given by $f(x) = \begin{cases} Cxe^{-x}; \ if \ x > 0 \\ 0 \quad ; \ if \ x \le 0 \end{cases}$ Find the value of C and cumulative density function of X.

Solution:

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^{\infty} Cxe^{-x}dx = 1$$

$$C\left[x\left(-e^{-x}\right) - \left(e^{-x}\right)\right]_0^{\infty} = 1$$

$$C = 1$$

$$\therefore f(x) = \begin{cases} xe^{-x}; x > 0 \\ 0 \quad ; x \le 0 \end{cases}$$

Cumulative Distribution of x is

$$F(x) = \int_0^x f(x)dt = \int_0^x xe^{-x}dx = \left[-xe^{-x} - e^{-x}\right]_0^x = -xe^{-x} - e^{-x} + 1$$

$$= 1 - (1+x)e^{-x},\ x > 0.$$

Problem.8

If a random variable X has the p.d.f $f(x) = \begin{cases} \frac{1}{2}(x+1); -1 < x < 1 \\ \quad 0 \quad ; otherwise \end{cases}$. Find the mean and variance of X.

Solution:

$$\text{Mean} = \mu_1' = \int_{-1}^{1} xf(x)\,dx = \frac{1}{2}\int_{-1}^{1} x(x+1)\,dx = \frac{1}{2}\int_{-1}^{1}\left(x^2 + x\right)dx$$

$$= \frac{1}{2}\left(\frac{x^3}{3} + \frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{3}$$

$$\mu_2' = \int_{-1}^{1} x^2 f(x)dx = \frac{1}{2}\int_{-1}^{1}\left(x^3 + x^2\right)dx = \frac{1}{2}\left[\frac{x^4}{4} + \frac{x^3}{3}\right]_{-1}^{1}$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{3} - \frac{1}{4} + \frac{1}{3}\right]$$

$$= \frac{1}{2}.\frac{2}{3} = \frac{1}{3}$$

$$Variance = \mu_2' - \left(\mu_1'\right)^2$$

$$= \frac{1}{3} - \frac{1}{9} = \frac{3-1}{9} = \frac{2}{9}.$$

Problem.9

A continuous random variable X that can assume any value between $X = 2$ and $X = 5$ has a probability density function given by $f(x) = k(1+x)$. Find $P(X < 4)$.

Solution:

Given X is a continuous random variable whose pdf is $f(x) = \begin{cases} k(1+x), 2 < x < 5 \\ 0 \ \ , Otherwise \end{cases}$.

Since $\int_{-\infty}^{\infty} f(x)\,dx = 1 \Rightarrow \int_{2}^{5} k(1+x)dx = 1$

$$k\left[\frac{(1+x)^2}{2}\right]_2^5 = 1$$

$$k\left[\frac{(1+5)^2}{2} - \frac{(1+2)^2}{2}\right] = 1$$

$$k\left[18 - \frac{9}{2}\right] = 1$$

$$k\left[\frac{27}{2}\right] = 1 \Rightarrow k = \frac{2}{27}$$

$$\therefore f(x) = \begin{cases} \frac{2(1+x)}{27}, 2 < x < 5 \\ 0 \ \ , Otherwise \end{cases}$$

$$P(X < 4) = \frac{2}{27}\int_{2}^{4} (1+x)\,dx$$

$$= \frac{2}{27}\left[\frac{(1+x)^2}{2}\right]_2^4 = \frac{2}{27}\left[\frac{(1+4)^2}{2} - \frac{(1+2)^2}{2}\right] = \frac{2}{27}\left[\frac{25}{2} - \frac{9}{2}\right] = \frac{2}{27}\frac{16}{2} = \frac{16}{27}.$$

Problem.10

A random variable X has density function given by $f(x) = \begin{cases} 2e^{-2x}; x \ge 0 \\ 0 \quad ; x < 0 \end{cases}$. Find m.g.f

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_0^{\infty} e^{tx} f(x)\,dx = \int_0^{\infty} e^{tx} 2e^{-2x} dx$$

$$= 2\int_0^{\infty} e^{(t-2)x} dx$$

$$= 2\left[\frac{e^{(t-2)x}}{t-2}\right]_0^{\infty} = \frac{2}{2-t}, t < 2.$$

Problem.11

The pdf of a random variable X is given by $f(x) = \begin{cases} 2x, \ 0 \le x \le b \\ 0, \ \ otherwise \end{cases}$. For what value of b is $f(x)$ a valid pdf? Also find the cdf of the random variable X with the above pdf.

Solution:

Given $f(x)=\begin{cases}2x, & 0\le x\le b\\ 0, & otherwise\end{cases}$

Since $\int_{-\infty}^{\infty} f(x)\,dx=1 \Rightarrow \int_{0}^{b} 2x\,dx=1$

$$\left[2\frac{x^2}{2}\right]_0^b=1$$

$$\left[b^2-0\right]=1 \;\Rightarrow\; \text{b}=1$$

$$\therefore f(x)=\begin{cases}2x, & 0\le x\le 1\\ 0, & otherwise\end{cases}$$

$$F(x)=P(X\le x)=\int_0^x f(x)\,dx=\int_0^x 2xdx=\left[2\frac{x^2}{2}\right]_0^x=x^2 \;,\; 0\le x\le 1$$

$$F(x)=P(X\le x)=\int_{-\infty}^{x} f(x)\,dx=\int_{-\infty}^{x} 0\,dx=0 \;,\; x<0$$

$$F(x)=P(X\le x)=\int_{-\infty}^{0} f(x)\,dx+\int_0^1 f(x)\,dx+\int_1^x f(x)\,dx$$

$$=\int_{-\infty}^{0} 0\;dx+\int_0^1 2x\,dx+\int_1^x 0\,dx \;=\left[2\frac{x^2}{2}\right]_0^1=1 \;,\; x>1$$

$$F(x)=\begin{cases}0, & x<0\\ x^2, & 0\le x\le 1\\ 1, & x>1\end{cases}$$

Problem.12

A random variable X has density function $f(x)=\begin{cases}\dfrac{K}{1+x^2}, & -\infty<x<\infty\\ 0 \;, & Otherwise\end{cases}$. Determine K and the distribution functions. Evaluate the probability $P(x\ge 0)$.

Solution:

Since $\int_{-\infty}^{\infty} f(x)dx=1$

$$\int_{-\infty}^{\infty}\frac{K}{1+x^2}dx=1$$

$$K\int_{\infty}^{\infty}\frac{dx}{1+x^2}=1$$

$$K\left(\tan^{-1}x\right)_{-\infty}^{\infty}=1$$

$$K\left(\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right)=1$$

$$K\pi=1$$

$$K = \frac{1}{\pi}$$

$$F(x) = \int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{K}{1+x^2}dx$$

$$= \frac{1}{\pi}\left[\tan^{-1}x - \left(-\frac{\pi}{2}\right)\right]$$

$$F(x) = \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}x\right], -\infty < x < \infty$$

$$P(X \geq 0) = \frac{1}{\pi}\int_{0}^{\infty}\frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1}x\right)_0^{\infty}$$

$$= \frac{1}{\pi}\left(\frac{\pi}{2} - \tan^{-1}0\right) = \frac{1}{2}.$$

Problem.13

If X has the probability density function $f(x) = \begin{cases} Ke^{-3x}, x > 0 \\ 0 \quad ,otherwise \end{cases}$ find K, $P[0.5 \leq X \leq 1]$ and the mean of X .

Solution:

Since $\int_{-\infty}^{\infty} f(x)dx = 1$

$$\int_{0}^{\infty} Ke^{-3x}dx = 1$$

$$K\left[\frac{e^{-3x}}{-3}\right]_0^{\infty} = 1$$

$$\frac{K}{3} = 1$$

$$K = 3$$

$$P(0.5 \leq X \leq 1) = \int_{0.5}^{1} f(x)dx = 3\int_{0.5}^{1} e^{-3x}dx = \cancel{3}\left(\frac{e^{-3} - e^{-1.5}}{-\cancel{3}}\right) = \left[e^{-1.5} - e^{-3}\right]$$

Mean of $X = E(x) = \int_{0}^{\infty} xf(x)dx = 3\int_{0}^{\infty} xe^{-3x}dx$

$$= 3\left[x\left(\frac{-e^{-3x}}{3}\right) - 1\left(\frac{e^{-3x}}{9}\right)\right]_0^{\infty} = \frac{3\times 1}{9} = \frac{1}{3}$$

Hence the mean of $X = E(X) = \frac{1}{3}$.

Problem.14

If X is a continuous random variable with pdf given by

$$f(x)=\begin{cases} Kx & in \quad 0\le x\le 2 \\ 2K & in \quad 2\le x\le 4 \\ 6K-Kx & in \quad 4\le x\le 6 \\ 0 & elsewhere \end{cases}$$. Find the value of K and also the cdf $F(x)$.

Solution:

Since $\int_{\infty}^{\infty} F(x)dx=1$

$$\int_0^2 Kxdx+\int_2^4 2Kdx+\int_4^6 (6k-kx)dx=1$$

$$K\left[\left(\frac{x^2}{2}\right)_0^2+(2x)_2^4+\int_4^6\left(6x-\frac{x^2}{2}\right)_4^6\right]=1$$

$$K\left[\cancel{2}+\cancel{8}-4+36-18-24+8\right]=1$$

$$8K=1$$

$$K=\frac{1}{8}$$

We know that $F(x)=\int_{-\infty}^{x} f(x)dx$

If $x<0$, then $F(x)=\int_{-\infty}^{x} f(x)dx=0$

If $x\in(0,2)$, then $F(x)=\int_{-\infty}^{x} f(x)dx$

$$F(x)=\int_{-\infty}^{0} f(x)dx+\int_0^x f(x)dx$$

$$=\int_{-\infty}^{0} 0dx+\int_0^x Kxdx=\int_{-\infty}^{0} 0dx+\frac{1}{8}\int_0^x xdx$$

$$F(x)=\left(\frac{x^2}{16}\right)_0^x=\frac{x^2}{16},0\le x\le 2$$

If $x\in(2,4)$, then $F(x)=\int_{-\infty}^{0} f(x)dx+\int_0^2 f(x)dx+\int_2^x f(x)dx$

$$=\int_{-\infty}^{0} 0dx+\int_0^2 Kxdx+\int_2^x 2Kdx$$

$$=\int_0^2\frac{x}{8}dx+\int_2^x\frac{1}{4}dx=\left(\frac{x^2}{16}\right)_0^2+\left(\frac{x}{4}\right)_2^x$$

$$=\frac{1}{4}+\frac{x}{4}-\frac{1}{2}$$

$$F(x)=\frac{x}{4}-\frac{4}{16}=\frac{x-1}{4},2\le x<4$$

If $x \in (4,6)$, then $F(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{x} K(6-x)dx$

$$= \int_{0}^{2} \frac{x}{8}dx + \int_{2}^{4} \frac{1}{4}dx + \int_{4}^{x} \frac{1}{8}(6-x)dx$$

$$= \left(\frac{x^2}{16}\right)_0^2 + \left(\frac{x}{4}\right)_2^4 + \left(\frac{6x}{8} - \frac{x^2}{16}\right)_4^x$$

$$= \frac{1}{4} + 1 - \frac{1}{2} + \frac{6x}{8} - \frac{x^2}{16} - 3 + 1$$

$$= \frac{4+16-8+12x-x^2-48+16}{16}$$

$$F(x) = \frac{-x^2+12x-20}{16}, 4 \le x \le 6$$

If $x > 6$, then $F(x) = \int_{-\infty}^{0} 0dx + \int_{0}^{2} Kxdx + \int_{2}^{4} 2Kdx + \int_{4}^{6} K(6-x)dx + \int_{6}^{\infty} 0dx$

$$F(x) = 1, x \ge 6$$

$$\therefore F(x) = \begin{cases} 0 & ; x \le 0 \\ \frac{x^2}{16} & ; 0 \le x \le 2 \\ \frac{1}{4}(x-1) & ; 2 \le x \le 4 \\ \frac{-1}{16}(20-12x+x^2) & ; 4 \le x \le 6 \\ 1 & ; x \ge 6 \end{cases}$$

Problem.15

A random variable X has the P.d.f $f(x) = \begin{cases} 2x, 0 < x < 1 \\ 0 \ \ , Otherwise \end{cases}$

Find (i) $P\left(X < \frac{1}{2}\right)$ (ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right)$ (iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right)$

Solution:

(i) $P\left(x < \frac{1}{2}\right) = \int_{0}^{1/2} f(x)dx = \int_{0}^{1/2} 2xdx = 2\left(\frac{x^2}{2}\right)_0^{1/2} = \frac{2 \times 1}{8} = \frac{1}{4}$

(ii) $P\left(\frac{1}{4} < x < \frac{1}{2}\right) = \int_{1/4}^{1/2} f(x)dx = \int_{1/4}^{1/2} 2xdx = 2\left(\frac{x^2}{2}\right)_{1/4}^{1/2}$

$$= 2\left(\frac{1}{8} - \frac{1}{32}\right) = \left(\frac{1}{4} - \frac{1}{16}\right) = \frac{3}{16}.$$

(iii) $P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \frac{P\left(X > \frac{3}{4} \cap X > \frac{1}{2}\right)}{P\left(X > \frac{1}{2}\right)} = \frac{P\left(X > \frac{3}{4}\right)}{P\left(X > \frac{1}{2}\right)}$

$$P\left(X > \frac{3}{4}\right) = \int_{3/4}^{1} f(x)\,dx = \int_{3/4}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{3/4}^{1} = 1 - \frac{9}{16} = \frac{7}{16}$$

$$P\left(X > \frac{1}{2}\right) = \int_{1/2}^{1} f(x)\,dx = \int_{1/2}^{1} 2x\,dx = 2\left(\frac{x^2}{2}\right)_{1/2}^{1} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$P\left(X > \frac{3}{4} / X > \frac{1}{2}\right) = \frac{\frac{7}{16}}{\frac{3}{4}} = \frac{7}{16} \times \frac{4}{3} = \frac{7}{12}.$$

Problem.16

Let the random variable X have the p.d.f $f(x) = \begin{cases} \frac{1}{2} e^{-\frac{x}{2}} & , x > 0 \\ 0 & , otherwise. \end{cases}$ .Find the moment generating function, mean & variance of X .

Solution:

$$M_X(t) = E\left(e^{tx}\right) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{0}^{\infty} e^{tx} \frac{1}{2} e^{-x/2}\,dx$$

$$= \frac{1}{2}\int_{0}^{\infty} e^{-\left(\frac{1}{2}-t\right)x}\,dx = \frac{1}{2}\left[\frac{e^{-\left(\frac{1}{2}-t\right)x}}{-\left(\frac{1}{2}-t\right)}\right]_0^{\infty} = \frac{1}{1-2t},\ if\ t < \frac{1}{2}.$$

$$E(X) = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{2}{(1-2t)^2}\right]_{t=0} = 2$$

$$E\left(X^2\right) = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{8}{(1-2t)^3}\right]_{t=0} = 8$$

$$Var(X) = E\left(X^2\right) - \left[E(X)\right]^2 = 8 - 4 = 4.$$