25.01.2022

1) If x and y are independent random variables uniformly distributed in $(0,1)$. Find the distribution of xy.

Sol: Given that x and y are uniformly distributed in $(0,1)$.

WKT, In uniform distribution,

the p.d.f is,

$$f(x) = \begin{cases} \frac{1}{b-a} ; & a < x < b \\ 0 ; & \text{otherwise} \end{cases}$$

$$\therefore f(x) = \begin{cases} \frac{1}{1-0} ; & 0 < x < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$f(x) = 1 ; \quad 0 < x < 1$$

$|||^{ly}$ y is also uniformly distributed in $(0,1)$

$$f(y) = 1 ; \quad 0 < y < 1.$$

Since x and y are indep. random variables.

$$\therefore f(x,y) = f(x) \cdot f(y) \quad (\because P(A \cap B) = P(A) \cdot P(B))$$

$$= 1 \times 1$$

$$\Rightarrow f(x,y) = 1 ; \quad 0 < x < 1, \; 0 < y < 1$$

Let the transformation be $U = xy$ and $V = y$

$u = xy ; \quad v = y$

$\hookrightarrow u = xv \quad \Rightarrow \quad x = u$

$\hookrightarrow u = x \cdot v \Rightarrow x = \frac{u}{v}$

$\therefore \; x = \frac{u}{v}$ and $y = v$

$$\frac{\partial x}{\partial u} = \frac{1}{v} \; ; \; \frac{\partial y}{\partial u} = 0$$

$$\frac{\partial x}{\partial v} = \frac{-u}{v^2} \; ; \; \frac{\partial y}{\partial v} = 1$$

WKT, Jacobian $(J) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

$$J = \begin{vmatrix} \frac{1}{v} & \frac{-u}{v^2} \\ 0 & 1 \end{vmatrix}$$

$$= \frac{1}{v} \cdot 1 - \left(\frac{-u}{v^2}\right) \cdot 0$$

$$J = \frac{1}{v}$$

$$|J| = \left|\frac{1}{v}\right| = \frac{1}{v}$$

$\therefore$ The joint pdf of u and v is,

$$f(u,v) = |J| \, f(x,y)$$

$$= \frac{1}{v} \cdot 1$$

$$\boxed{f(u,v) = \frac{1}{v}}$$

To find the limits of u and v :-

Since $x = \frac{u}{v}$; $y = v$

$0 < x < 1$; $0 < y < 1$

$0 < \frac{u}{v} < 1$; $0 < v < 1$

$\times v \Rightarrow 0 < u < v$; $0 < v < 1$

$0 < u < v < 1$

hence the limits of u and v is,

$0 < u < v$ and $u < v < 1$

WKT,

The Marginal pdf of $u = xy$ is,

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\, dv$$

$$= \int_{v=u}^{1} \frac{1}{v} \cdot dv$$

$$= (\log v)_{v=u}^{1} \qquad \left(\int \frac{dx}{x} = \log x\right)$$

$$= \log 1 - \log u$$

$$= 0 - \log u$$

$$= -\log u$$

$$= \log u^{-1} \qquad (\because n \log m = \log m^n)$$

$$f(u) = \log\left(\frac{1}{u}\right)$$

$\therefore$ The marginal p.d.f of $u = xy$ is,

$$f(u) = \log\left(\frac{1}{u}\right).$$

05) Let (X, Y) be a two dimensional non-negative continuous random variable having the joint density $f(x,y) = 4xy\,e^{-(x^2+y^2)}$, $x>0, y>0$ find the density function of $U = \sqrt{X^2+Y^2}$.

Soln: Given $f(x,y) = 4xy\,e^{-(x^2+y^2)}$, $x>0, y>0$

$U = \sqrt{X^2+Y^2}$, $\boxed{Y = V}$

Now $X^2+Y^2 = U^2 \Rightarrow X^2 = U^2 - Y^2 = U^2 - V^2$

$\therefore \boxed{X = \sqrt{u^2-v^2}}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{2\sqrt{u^2-v^2}}(2u) & \frac{1}{2\sqrt{u^2-v^2}}(-2v) \\ 0 & 1 \end{vmatrix}$$

$$= \frac{u}{\sqrt{u^2-v^2}}.$$

$$\therefore f(u,v) = |J|\, f(x,y) = \frac{u}{\sqrt{u^2-v^2}}\, 4xy\, e^{-(x^2+y^2)}$$

$$= \frac{u}{\cancel{\sqrt{u^2-v^2}}}\, 4\cdot\cancel{\sqrt{u^2-v^2}}\,(v)\, e^{-u^2}$$

$$\boxed{f(u,v) = 4uv\,e^{-u^2}}$$

Limits for U & V

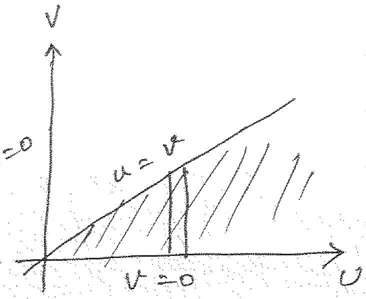
$x > 0 \Rightarrow \sqrt{u^2-v^2} > 0 \Rightarrow u^2 - v^2 > 0 \Rightarrow u^2 > v^2$

$\Rightarrow \boxed{u > v}$

$y > 0 \Rightarrow \boxed{v > 0}$ Also $u > 0$, $\because x>0, y>0, u = \sqrt{x^2+y^2}$

To find $f(u)$

$$f(u) = \int_0^u f(u,v)\,dv$$



$$= \int_0^u 4uv\,e^{-u^2}\,dv$$

$$= 4ue^{-u^2}\left(\frac{v^2}{2}\right)_0^u = 4ue^{-u^2}\left(\frac{u^2}{2}\right)$$

$$= 2u^3e^{-u^2}$$

$$\therefore \boxed{f(u) = 2u^3e^{-u^2}, \quad u > 0}$$