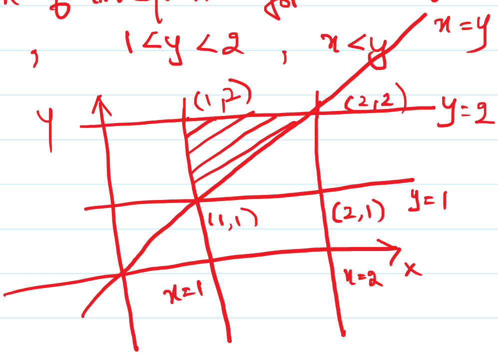
Problem:

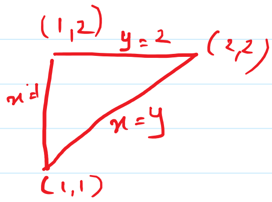
The joint probability density function of the two dimensional r.v is $f(x,y) = \frac{8xy}{9}$, $1 < x < y < 2$. Find the conditional density function of $X$ given $Y$ and $Y$ given $X$. Also verify the conditional density function are valid. Find $P(X < 3/2)$.

Soln.

Given region of integration for $f(x,y)$

$1 < x < 2$, $1 < y < 2$, $x < y$




Marginal densities of $X$ and $Y$.

$$f(x) = \int_x^2 f(x,y)\,dy = \int_x^2 \frac{8xy}{9}\,dy = \frac{\cancel{8}^{4} x}{9}\left[\frac{4-x^2}{\cancel{2}}\right]$$

$$f(x) = \frac{4x}{9}\left[4-x^2\right] \qquad 1 \le x \le 2$$

$$f(y) = \int_1^y f(x,y)\,dx = \int_1^y \frac{8xy}{9}\,dx = \frac{\cancel{8}^{4} y}{9}\left[\frac{x^2}{\cancel{2}}\right]_1^y = \frac{4y}{9}\left[y^2-1\right]$$

$$f(y) = \frac{4y}{9}\left[y^2-1\right], \qquad 1 \le y \le 2$$

Conditional density function of $X$ given $Y$.

$$f(x/y) = \frac{f(x,y)}{f(y)} = \frac{8xy/9}{\frac{4y}{9}\left[y^2-1\right]} = \frac{\cancel{8}^{2} x \cancel{y}}{\cancel{9}} \times \frac{\cancel{9}}{\cancel{4}\cancel{y}\left[y^2-1\right]} = \frac{2x}{y^2-1}$$

$$f(x|y) = \frac{2x}{y^2-1}, \qquad 1 < x < y$$

Conditional density function of $Y$ given $X$

$$f(y|x) = \frac{f(x,y)}{f(x)} = \frac{8xy/9}{\frac{4x}{9}\left[4-x^2\right]} = \frac{\cancel{8}^{2}\cancel{x}y}{\cancel{9}} \times \frac{\cancel{9}}{\cancel{4x}\left[4-x^2\right]} = \frac{2y}{4-x^2}$$

$$f(y|x) = \frac{2y}{4-x^2} \qquad x<y<2$$

Verification:

$$\int_1^y f(x/y)\,dx = \int_1^y \frac{2x}{y^2-1}\,dx = \frac{2}{y^2-1}\int_1^y x\,dx$$

$$= \frac{\cancel{2}}{y^2-1}\left[\frac{x^2}{\cancel{2}}\right]_1^y = \frac{1}{\cancel{y^2-1}}\left[\cancel{y^2-1}\right]$$

$$=1$$

$f(x|y)$ is Valid.

Find $P(X<3/2)$

$$= \int_1^{3/2} f(x)\,dx$$

$$= \int_1^{3/2} \frac{4x}{9}\left[4-x^2\right] dx$$

$$= \frac{4}{9}\int_1^{3/2} (4x-x^3)\,dx$$

Also

$$\int_x^2 f(y|x)\,dy = \int_x^2 \frac{2y}{4-x^2}\,dy = \frac{\cancel{2}}{4-x^2}\left[\frac{y^2}{\cancel{2}}\right]_x^2$$

$$= \frac{1}{\cancel{4-x^2}}\left[\cancel{4-x^2}\right] = 1$$

$f(y|x)$ is Valid.

Transformation of Random Variables:

$$(X,Y) \rightarrow (U,V)$$

$U$ & $V$ are defined by $U=f_1(x,y)$, $V=f_2(x,y)$

Procedure:

1. Find the joint p.d.f of $(X,Y)$ if it is not given.

2. $U=f_1(x,y)$ & $V=f_2(x,y)$ will be given.
   From this we have to express the relation in the form
   $x=g_1(u,v)$ $y=g_2(u,v)$.

3. Find
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

4. Find the joint p.d.f of $(U,V)$ using the formula
$$f(u,v) = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \cdot f(x,y)$$
[Here $f(x,y) \rightarrow$ can be written in terms of $u$ & $v$].

5. Find the marginal densities $f(u)$ & $f(v)$ from $f(u,v)$

6. Find the limits for $f(u,v)$, $f(u)$ & $f(v)$ using the relation in step 2.

x —— x

Problem:

If the joint density of $x_1$ & $x_2$ is given by

$f(x_1, x_2) = e^{-(x_1+x_2)}$ , $x_1, x_2 > 0$ find the p.d.f of

$y = \frac{x_1}{x_1+x_2}$.

Soln. For Convenience $x_1 = x$ $x_2 = y$.

$\therefore f(x,y) = e^{-(x+y)}$ , $x,y > 0$

Given $U = \frac{x}{x+y}$. Let $V = x+y$

$U = \frac{x}{V} \Rightarrow \boxed{x = UV}$

Also $V = x+y$

$V = UV + y$

$\Rightarrow \boxed{y = V - UV}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} = v - vu + vu = v$$

$\therefore f(u,v) = \frac{\partial(x,y)}{\partial(u,v)} \cdot f(x,y)$

$= v\, e^{-(uv - v - uv)}$

$= v e^{-v}$

Limits for $u$ & $v$

$x > 0$ $y > 0$

$x > 0 \Rightarrow x = uv$

$uv > 0 \Rightarrow u > 0$ & $v > 0$

$y > 0 \Rightarrow y = v - uv \Rightarrow v - uv > 0$

$v > uv$

$\Rightarrow 1 > u$

$\Rightarrow u < 1$

$\therefore f(u,v) = ve^{-v}$ $0 < u < 1$ , $v > 0$

To find $f(U)$

$\int u\,dv = uv - \int v\,du$

$f(u) = \int_0^{\infty} f(u,v)\,dv = \int_0^{\infty} v e^{-v}\,dv$

To find $f(v)$

$$f(u) = \int_0^\infty f(u,v)\,dv = \int_0^\infty v e^{-v}\,dv$$

$$\int u\,dv = uv - \int v\,du$$

$u = v \quad dv = e^{-v}$

Integrating by parts

$du = dv \quad v = -e^{-v}$

$$= \left[ v(-e^{-v}) \right]_0^\infty - \int_0^\infty -e^{-v}\,dv$$

$$= 0 - \left[ e^{-v} \right]_0^\infty$$

$$= -(0-1) = 1$$

x — x