Sathyabama Institute of Science and Technology

SMTA1402 - Probability and Statistics

UNIT-3 TWO DIMENSIONAL RANDOM VARIABLE

1. Let X and Y have joint density function $f(x,y)=2,\ 0<x<y<1$. Find the marginal density function. Find the conditional density function Y given $X=x$.

Solution:

Marginal density function of X is given by

$$f_X(x)=f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy$$

$$=\int_{x}^{1} f(x,y)\,dy=\int_{x}^{1} 2dy=2(y)_x^1$$

$$=2(1-x),0<x<1.$$

Marginal density function of Y is given by

$$f_Y(y)=f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx$$

$$=\int_{0}^{y} 2dx=2y,0<y<1.$$

Conditional distribution function of Y given $X=x$ is $f\left(y/x\right)=\dfrac{f(x,y)}{f(x)}=\dfrac{2}{2(1-x)}=\dfrac{1}{1-x}$.

2. Verify that the following is a distribution function. $F(x)=\begin{cases}0 & ,x<-a\\ \frac{1}{2}\left(\frac{x}{a}+1\right) & ,-a<x<a\\ 1 & ,x>a\end{cases}$.

Solution:

$F(x)$ is a distribution function only if $f(x)$ is a density function.

$$f(x)=\frac{d}{dx}\left[F(x)\right]=\frac{1}{2a},\ -a<x<a$$

$$\int_{-\infty}^{\infty} f(x)=1$$

$$\therefore \int_{-a}^{a}\frac{1}{2a}dx=\frac{1}{2a}\left[x\right]_{-a}^{a}=\frac{1}{2a}\left[a-(-a)\right]$$

$$=\frac{1}{2a}.2a=1.$$

Therefore, it is a distribution function.

3. Prove that $\int_{x_1}^{x_2} f_X(x)\,dx = p(x_1 < x < x_2)$

Solution:

$$\int_{x_1}^{x_2} f_X(x)\,dx = \left[F_X(x)\right]_{x_1}^{x_2}$$

$$= F_X(x_2) - F_X(x_1)$$

$$= P[X \le x_2] - P[X \le x_1]$$

$$= P[x_1 \le X \le x_2]$$

4. A continuous random variable X has a probability density function $f(x) = 3x^2,\ \ 0 \le x \le 1$. Find '$a$' such that $P(X \le a) = P(X > a)$.

Solution:

Since $P(X \le a) = P(X > a)$, each must be equal to $\frac{1}{2}$ because the probability is always 1.

$$\therefore P(X \le a) = \frac{1}{2}$$

$$\Rightarrow \int_0^a f(x)\,dx = \frac{1}{2}$$

$$\int_0^a 3x^2\,dx = \frac{1}{2} \Rightarrow 3\left[\frac{x^3}{3}\right]_0^a = a^3 = \frac{1}{2}.$$

$$\therefore a = \left(\frac{1}{2}\right)^{\frac{1}{3}}$$

5. Suppose that the joint density function $f(x,y) = \begin{cases} Ae^{-x-y}, & 0 \le x \le y,\ 0 \le y \le \infty \\ 0 & , otherwise \end{cases}$ Determine A.

Solution:

Since $f(x,y)$ is a joint density function

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dxdy = 1.$$

$$\Rightarrow \int_0^{\infty}\int_0^{y} Ae^{-x}e^{-y}\,dxdy = 1$$

$$\Rightarrow A\int_0^{\infty} e^{-y}\left(\frac{e^{-x}}{-1}\right)_0^{y} dy = 1$$

$$\Rightarrow A\int_0^{\infty}\left[e^{-y} - e^{-2y}\right]dy = 1$$

$$\Rightarrow A\left[\frac{e^{-y}}{-1}-\frac{e^{-2y}}{-2}\right]_0^{\infty}=1$$

$$\Rightarrow A\left[\frac{1}{2}\right]=1 \Rightarrow A=2$$

6. Examine whether the variables X and Y are independent, whose joint density function is $f(x,y)=xe^{-x(y+1)}, 0<x,y<\infty$.

Solution:

The marginal probability function of X is

$$f_X(x)=f(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\int_0^{\infty} xe^{-x(y+1)}dy$$

$$=x\left[\frac{e^{-x(y+1)}}{-x}\right]_0^{\infty}=-\left[0-e^{-x}\right]=e^{-x},$$

The marginal probability function of Y is

$$f_Y(y)=f(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\int_0^{\infty} x\,e^{-x(y+1)}dx$$

$$=x\left\{\left[\frac{e^{-x(y+1)}}{-(y+1)}\right]_0^{\infty}-\left[\frac{e^{-x(y+1)}}{(y+1)^2}\right]\right\}_0^{\infty}$$

$$=\frac{1}{(y+1)^2}$$

Here $f(x).f(y)=e^{-x}\times\frac{1}{(1+y)^2}\neq f(x,y)$

$\therefore$ X and Y are not independent.

7. If X has an exponential distribution with parameter 1. Find the pdf of $y=\sqrt{x}$

Solution:

Since $y=\sqrt{x}$, $x=y^2$

Since X has an exponential distribution with parameter 1, the pdf of X is given by

$$f_X(x)=e^{-x},\ x>0 \qquad \left[\ f(x)=\lambda e^{-\lambda x},\lambda=1\right]$$

$$\therefore f_Y(y)=f_X(x)\left|\frac{dx}{dy}\right|$$

$$=e^{-x}2y=2ye^{-y^2}$$

$$f_Y(y)=2ye^{-y^2},\ \ y>0$$

8. If X is uniformly distributed random variable in $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, Find the probability density function of $Y=tanX$.

Solution:

Given $Y = tanX \Rightarrow x = tan^{-1}y$

$$\therefore \frac{dx}{dy} = \frac{1}{1+y^2}$$

Since X is uniformly distribution in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

$$f_X(x) = \frac{1}{b-a} = \frac{1}{\frac{\pi}{2}+\frac{\pi}{2}}$$

$$f_X(x) = \frac{1}{\pi}, \ -\frac{\pi}{2} < x < \frac{\pi}{2}$$

Now $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right| = \frac{1}{\pi}\left(\frac{1}{1+y^2}\right), \ -\infty < y < \infty$

$$\therefore f_Y(y) = \frac{1}{\pi\left(1+y^2\right)}, \ -\infty < y < \infty$$

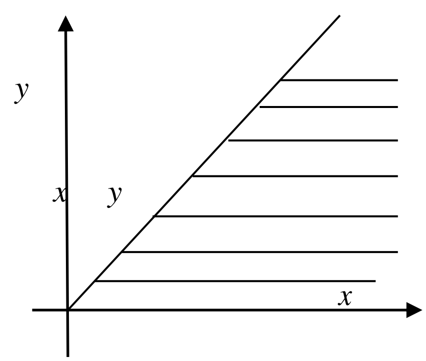
9. If the Joint probability density function of (x, y) is given by $f(x, y) = 24y(1-x)$, $0 \le y \le x \le 1$ Find $E(XY)$.

Solution:

$$E(xy) = \int_0^1 \int_y^1 xyf(x, y)\,dxdy$$

$$= 24\int_0^1 \int_y^1 xy^2(1-x)\,dxdy$$

$$= 24\int_0^1 y^2\left[\frac{1}{6} - \frac{y^2}{2} + \frac{y^3}{3}\right]dy = \frac{4}{15}.$$



10. If X and Y are random Variables, Prove that $Cov(X,Y) = E(XY) - E(X)E(Y)$

Solution:

$$cov(X,Y) = E\left[\left(X - E(X)\right)\left(Y - E(Y)\right)\right]$$

$$= E\left(XY - \bar{X}Y - \bar{Y}X + \bar{X}\bar{Y}\right)$$

$$= E(XY) - \bar{X}E(Y) - \bar{Y}E(X) + \bar{X}\bar{Y}$$

$$= E(XY) - \bar{X}\bar{Y} - \bar{X}\bar{Y} + \bar{X}\bar{Y}$$

$$= E(XY) - E(X)E(Y) \qquad \left[\ E(X) = \bar{X}, E(Y) = \bar{Y}\right]$$

11. If X and Y are independent random variables prove that $cov(x, y) = 0$

Proof:

$cov(x,y) = E(xy) - E(x)E(y)$

But if X and Y are independent then $E(xy) = E(x)E(y)$

$cov(x,y) = E(x)E(y) - E(x)E(y)$

$cov(x,y) = 0.$

12. Write any two properties of regression coefficients.

Solution:

1. Correction coefficients is the geometric mean of regression coefficients
2. If one of the regression coefficients is greater than unity then the other should be less than 1.

$$b_{xy} = r\frac{\sigma_y}{\sigma_x} \text{ and } b_{yx} = r\frac{\sigma_x}{\sigma_y}$$

If $b_{xy} > 1$ then $b_{yx} < 1$.

13. Write the angle between the regression lines.

The slopes of the regression lines are

$$m_1 = r\frac{\sigma_y}{\sigma_x}, m_2 = \frac{1}{r}\frac{\sigma_y}{\sigma_x}$$

If θ is the angle between the lines, Then

$$\tan\theta = \frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\left[\frac{1-r^2}{r}\right]$$

When $r = 0$, that is when there is no correlation between x and y, $\tan\theta = \infty$ (or) $\theta = \frac{\pi}{2}$ and so the regression lines are perpendicular

When $r = 1$ or $r = -1$, that is when there is a perfect correlation $+ve$ or $-ve$, $\theta = 0$ and so the lines coincide.

15. i). Two random variables are said to be orthogonal if correction is zero.

ii). If $X = Y$ then correlation coefficients between them is 1.

16.a). The joint probability density function of a bivariate random variable X,Y is $f_{XY}(x,y)=\begin{cases}k(x+y), & 0<x<2,\ 0<y<2\\ 0 & , otherwise\end{cases}$ where $'k'$ is a constant.

i. Find k.
ii. Find the marginal density function of X and Y.
iii. Are X and Y independent?
iv. Find $f_{Y/X}\left(y/x\right)$ and $f_{X/Y}\left(x/y\right)$.

Solution:

(i). Given the joint probability density function of a brivate random variable (X,Y) is

$$f_{XY}(x,y)=\begin{cases}K(x+y), & 0<x<2,\ 0<y<2\\ 0 & , otherwise\end{cases}$$

Here $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dxdy=1 \Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(x+y)\,dxdy=1$

$$\int_0^2\int_0^2 K(x+y)\,dxdy=1 \Rightarrow K\int_0^2\left[\frac{x^2}{2}+xy\right]_0^2 dy=1$$

$$\Rightarrow K\int_0^2 (2+2y)\,dy=1$$

$$\Rightarrow K\left[2y+y^2\right]_0^2=1$$

$$\Rightarrow K[8-0]=1$$

$$\Rightarrow K=\frac{1}{8}$$

(ii). The marginal p.d.f of X is given by

$$f_X(x)=\int_{-\infty}^{\infty} f(x,y)\,dy=\frac{1}{8}\int_0^2 (x+y)\,dy$$

$$=\frac{1}{8}\left[xy+\frac{y^2}{2}\right]_0^2=\frac{1+x}{4}$$

$\therefore$ The marginal p.d.f of X is

$$f_X(x)=\begin{cases}\frac{x+1}{4}, & 0<x<2\\ 0 & , otherwise\end{cases}$$

The marginal p.d.f of Y is

$$f_Y(y)=\int_{-\infty}^{\infty} f(x,y)\,dx=\frac{1}{8}\int_0^2 (x+y)\,dx$$

$$=\frac{1}{8}\left[\frac{x^2}{2}+yx\right]_0^2$$

$$=\frac{1}{8}\left[2+2y\right]=\frac{y+1}{4}$$

$\therefore$ The marginal p.d.f of Y is

$$f_Y\left(y\right)=\begin{cases}\dfrac{y+1}{4}, & 0<y<2\\ 0 & , otherwise\end{cases}$$

(iii). To check whether X and Y are independent or not.

$$f_X\left(x\right)f_Y\left(y\right)=\frac{\left(x+1\right)}{4}\frac{\left(y+1\right)}{4}\neq f_{XY}\left(x,y\right)$$

Hence X and Y are not independent.

(iv). Conditional p.d.f $f_{Y/X}\left(y/x\right)$ is given by

$$f_{Y/X}\left(y/x\right)=\frac{f\left(x,y\right)}{f_X\left(x\right)}=\frac{\frac{1}{8}\left(x+y\right)}{\frac{1}{4}\left(x+1\right)}=\frac{1}{2}\frac{\left(x+y\right)}{\left(x+1\right)}$$

$$f_{Y/X}\left(y/x\right)=\frac{1}{2}\left(\frac{x+y}{x+1}\right),\ 0<x<2,\ 0<y<2$$

(v) $P\left(0<y<\frac{1}{2}\Big/x=1\right)=\int_0^2 f_{Y/X}\left(y/x=1\right)dy$

$$=\frac{1}{2}\int_0^{\frac{1}{2}}\frac{1+y}{2}dy=\frac{5}{32}.$$

17.a). If X and Y are two random variables having joint probability density function $f\left(x,y\right)=\begin{cases}\frac{1}{8}\left(6-x-y\right), & 0<x<2,\ 2<y<4\\ 0 & , otherwise\end{cases}$ Find (i) $P\left(X<1\cap Y<3\right)$

(ii) $P\left(X+Y<3\right)$ (iii) $P\left(X<1\big/Y<3\right)$.

b). Three balls are drawn at random without replacement from a box containing 2 white, 3 red and 4 black balls. If X denotes the number of white balls drawn and Y denotes the number of red balls drawn find the joint probability distribution of X,Y .

Solution:

a).

$$P\left(X<1\cap Y<3\right)=\int_{y=-\infty}^{y=3}\int_{x=-\infty}^{x=1}f\left(x,y\right)dxdy$$

$$=\int_{y=2}^{y=3}\int_{x=0}^{x=1}\frac{1}{8}\left(6-x-y\right)dxdy$$

$$= \frac{1}{8}\int_2^3\int_0^1 (6-x-y)\,dxdy$$

$$= \frac{1}{8}\int_2^3 \left[6x - \frac{x^2}{2} - xy\right]_0^1 dy$$

$$= \frac{1}{8}\int_2^3 \left[\frac{11}{2} - y\right]dy = \frac{1}{8}\left[\frac{11y}{2} - \frac{y^2}{2}\right]_2^3$$

$$P(X<1 \cap Y<3) = \frac{3}{8}$$

(ii). $P(X+Y<3) = \int_0^1 \int_2^{3-x} \frac{1}{8}(6-x-y)\,dydx$

$$= \frac{1}{8}\int_0^1 \left[6y - xy - \frac{y^2}{2}\right]_2^{3-x} dx$$

$$= \frac{1}{8}\int_0^1 \left[6(3-x) - x(3-x) - \frac{(3-x)^2}{2} - [12-2x-2]\right]dx$$

$$= \frac{1}{8}\int_0^1 \left[18 - 6x - 3x + x^2 - \frac{(9+x^2-6x)}{2} - (10-2x)\right]dx$$

$$= \frac{1}{8}\int_0^1 \left[18 - 9x + x^2 - \frac{9}{2} - \frac{x^2}{2} + \frac{6x}{2} - 10 + 2x\right]dx$$

$$= \frac{1}{8}\int_0^1 \left[\frac{7}{2} - 4x + \frac{x^2}{2}\right]dx$$

$$= \frac{1}{8}\left[\frac{7x}{2} - \frac{4x^2}{2} + \frac{x^3}{6}\right]_0^1 = \frac{1}{8}\left[\frac{7}{2} - 2 + \frac{1}{6}\right]$$

$$= \frac{1}{8}\left[\frac{21-12+1}{6}\right] = \frac{1}{8}\left(\frac{10}{6}\right) = \frac{5}{24}.$$

(iii). $P\left(X<1 \big/ Y<3\right) = \frac{P(x<1 \cap y<3)}{P(y<3)}$

The Marginal density function of Y is $f_Y(y) = \int_0^2 f(x,y)\,dx$

$$= \int_0^2 \frac{1}{8}(6-x-y)\,dx$$

$$= \frac{1}{8}\left[6x - \frac{x^2}{2} - yx\right]_0^2$$

$$= \frac{1}{8}[12-2-2y]$$

$$= \frac{5-y}{4},\ 2 < y < 4.$$

$$P\left(X < 1 \middle/ Y < 3\right) = \frac{\int_{x=0}^{x=1}\int_{y=2}^{y=3} \frac{1}{8}(6-x-y)\,dxdy}{\int_{y=2}^{y=3} f_Y(y)\,dy}$$

$$= \frac{\frac{3}{8}}{\int_2^3 \left(\frac{5-y}{4}\right)dy} = \frac{\frac{3}{8}}{\frac{1}{4}\left[5y - \frac{y^2}{2}\right]_2^3}$$

$$= \frac{3}{8} \times \frac{8}{5} = \frac{3}{5}.$$

b). Let X takes 0, 1, 2 and Y takes 0, 1, 2 and 3.

$P\left(X = 0, Y = 0\right) = P($drawing 3 balls none of which is white or red)

$P($ all the 3 balls drawn are black)

$$= \frac{4C_3}{9C_3} = \frac{4\times 3\times 2\times 1}{9\times 8\times 7} = \frac{1}{21}.$$

$P\left(X = 0, Y = 1\right) = P($drawing 1 red ball and 2 black balls)

$$= \frac{3C_1 \times 4C_2}{9C_3} = \frac{3}{14}$$

$P\left(X = 0, Y = 2\right) = P($ drawing 2 red balls and 1 black ball)

$$= \frac{3C_2 \times 4C_1}{9C_3} = \frac{3\times 2\times 4\times 3}{9\times 8\times 7} = \frac{1}{7}.$$

$P\left(X = 0, Y = 3\right) = P($ all the three balls drawn are red and no white ball)

$$= \frac{3C_3}{9C_3} = \frac{1}{84}$$

$P\left(X = 1, Y = 0\right) = P($drawing 1White and no red ball)

$$= \frac{2C_1 \times 4C_2}{9C_3} = \frac{\frac{2\times 4\times 3}{1\times 2}}{\frac{9\times 8\times 7}{1\times 2\times 3}}$$

$$= \frac{12\times 1\times 2\times 3}{9\times 8\times 7} = \frac{1}{7}.$$

$P\left(X = 1, Y = 1\right) = P($drawing 1White and 1 red ball)

$$= \frac{2C_1 \times 3C_1}{9C_3} = \frac{2\times 3}{\frac{9\times 8\times 7}{1\times 2\times 3}} = \frac{2}{7}$$

$P(X=1,Y=2)=P($ drawing 1White and 2 red ball)

$$=\frac{2C_1\times 3C_2}{9C_3}=\frac{2\times 3\times 2}{\frac{9\times 8\times 7}{1\times 2\times 3}}=\frac{1}{14}$$

$P(X=1,Y=3)=0$ (Since only three balls are drawn)

$P(X=2,Y=0)=P($ drawing 2 white balls and no red balls)

$$=\frac{2C_2\times 4C_1}{9C_3}=\frac{1}{21}$$

$P(X=2,Y=1)=P($ drawing 2 white balls and no red balls)

$$=\frac{2C_2\times 3C_1}{9C_3}=\frac{1}{28}$$

$P(X=2,Y=2)=0$

$P(X=2,Y=3)=0$

The joint probability distribution of X,Y may be represented as

X \ Y	0	1	2	3
0	$\frac{1}{21}$	$\frac{3}{14}$	$\frac{1}{7}$	$\frac{1}{84}$
1	$\frac{1}{7}$	$\frac{2}{7}$	$\frac{1}{14}$	0
2	$\frac{1}{21}$	$\frac{1}{28}$	0	0

18.a). Two fair dice are tossed simultaneously. Let X denotes the number on the first die and Y denotes the number on the second die. Find the following probabilities.

(i) $P(X+Y)=8$, (ii) $P(X+Y\geq 8)$, (iii) $P(X=Y)$ and (iv) $P\left(X+Y=6/_{Y=4}\right)$.

b) The joint probability mass function of a bivariate discrete random variable (X,Y) in given by the table.

Y \ X	1	2	3
1	0.1	0.1	0.2
2	0.2	0.3	0.1

Find

i. The marginal probability mass function of X and Y.
ii. The conditional distribution of X given $Y=1$.
iii. $P(X+Y<4)$

Solution:

a). Two fair dice are thrown simultaneously

$$S=\begin{Bmatrix}(1,1)(1,2)\ldots(1,6)\\(2,1)(2,2)\ldots(2,6)\\ . \quad . \quad \ldots \quad .\\(6,1)(6,2)\ldots(6,6)\end{Bmatrix},\ n(S)=36$$

Let X denotes the number on the first die and Y denotes the number on the second die.

Joint probability density function of (X,Y) is $P(X=x,Y=y)=\frac{1}{36}$ for

$x=1,2,3,4,5,6$ and $y=1,2,3,4,5,6$

(i) $X+Y=\{$ the events that the no is equal to 8 $\}$

$$=\{(2,6),(3,5),(4,4),(5,3),(6,2)\}$$

$$P(X+Y=8)=P(X=2,Y=6)+P(X=3,Y=5)+P(X=4,Y=4)$$
$$+P(X=5,Y=3)+P(X=6,Y=2)$$
$$=\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}+\frac{1}{36}=\frac{5}{36}$$

(ii) $P(X+Y\geq 8)$

$$X+Y=\begin{Bmatrix}(2,6)\\(3,5),(3,6)\\(4,4),(4,5),(4,6)\\(5,3),(5,4)(5,5),(5,6)\\(6,2),(6,3),(6,4),(6,5)(6,6)\end{Bmatrix}$$

$$\therefore P(X+Y\geq 8)=P(X+Y=8)+P(X+Y=9)+P(X+Y=10)$$
$$+P(X+Y=11)+P(X+Y=12)$$
$$=\frac{5}{36}+\frac{4}{36}+\frac{3}{36}+\frac{2}{36}+\frac{1}{36}=\frac{15}{36}=\frac{5}{12}$$

(iii) $P(X=Y)$

$$P(X=Y)=P(X=1,Y=1)+P(X=2,Y=2)+\ldots\ldots+P(X=6,Y=6)$$
$$=\frac{1}{36}+\frac{1}{36}+\ldots\ldots\ldots\ldots+\frac{1}{36}=\frac{6}{36}=\frac{1}{6}$$

(iv) $P\left(X+Y=6\middle/Y=4\right)=\dfrac{P(X+Y=6\cap Y=4)}{P(Y=4)}$

Now $P(X+Y=6\cap Y=4)=\frac{1}{36}$

$$P(Y=4)=\frac{6}{36}$$

$$\therefore P\left(X+Y=6 \middle/ Y=4\right)=\frac{\frac{1}{36}}{\frac{6}{36}}=\frac{1}{6}.$$

b). The joint probability mass function of (X,Y) is

$Y \backslash X$	1	2	3	Total
1	0.1	0.1	0.2	0.4
2	0.2	0.3	0.1	0.6
Total	0.3	0.4	0.3	1

From the definition of marginal probability function

$$P_X(x_i)=\sum_{y_j} P_{XY}(x_i,y_j)$$

When $X=1$,

$$P_X(x_i)=P_{XY}(1,1)+P_{XY}(1,2)$$
$$=0.1+0.2=0.3$$

When $X=2$,

$$P_X(x=2)=P_{XY}(2,1)+P_{XY}(2,2)$$
$$=0.2+0.3=0.4$$

When $X=3$,

$$P_X(x=3)=P_{XY}(3,1)+P_{XY}(3,2)$$
$$=0.2+0.1=0.3$$

$\therefore$ The marginal probability mass function of X is

$$P_X(x)=\begin{cases}0.3 & when\ x=1\\ 0.4 & when\ x=2\\ 0.3 & when\ x=3\end{cases}$$

The marginal probability mass function of Y is given by $P_Y(y_j)=\sum_{x_i} P_{XY}(x_i,y_j)$

When $Y \quad 1$, $P_Y(y=1)=\sum_{x_i=1}^{3} P_{XY}(x_i,1)$

$$=P_{XY}(1,1)+P_{XY}(2,1)+P_{XY}(3,1)$$
$$=0.1+0.1+0.2=0.4$$

When $Y \quad 2$, $P_Y(y=2)=\sum_{x_i=1}^{3} P_{XY}(x_i,2)$

$$=P_{XY}(1,2)+P_{XY}(2,2)+P_{XY}(3,2)$$
$$=0.2+0.3+0.1=0.6$$

$\therefore$ Marginal probability mass function of Y is

$$P_Y(y)=\begin{cases}0.4 & when\ y=1\\ 0.6 & when\ y=2\end{cases}$$

(ii) The conditional distribution of X given $Y \quad 1$ is given by

$$P\left(X = x/_{Y=1}\right) = \frac{P(X = x \cap Y = 1)}{P(Y=1)}$$

From the probability mass function of Y, $P \quad y \quad 1 \quad P_y \; 1 \quad 0.4$

When $X \quad 1$, $P\left(X = 1/_{Y=1}\right) = \dfrac{P(X = 1 \cap Y = 1)}{P(Y=1)}$

$$= \frac{P_{XY}(1,1)}{P_Y(1)} = \frac{0.1}{0.4} = 0.25$$

When $X \quad 2$, $P\left(X = 2/_{Y=1}\right) = \dfrac{P_{XY}(2,1)}{P_Y(1)} = \dfrac{0.1}{0.4} = 0.25$

When $X \quad 3$, $P\left(X = 3/_{Y=1}\right) = \dfrac{P_{XY}(3,1)}{P_Y(1)} = \dfrac{0.2}{0.4} = 0.5$

(iii). $P(X+Y<4) = P\{(x,y)/x+y<4 \;\; Where \;\; x = 1,2,3; \;\; y = 1,2\}$

$$= P\{(1,1),(1,2),(2,1)\}$$

$$= P_{XY}(1,1) + P_{XY}(1,2) + P_{XY}(2,1)$$

$$= 0.1 + 0.1 + 0.2 = 0.4$$

19.a). If X and Y are two random variables having the joint density function $f(x,y) = \frac{1}{27}(x+2y)$ where x and y can assume only integer values 0, 1 and 2, find the conditional distribution of Y for $X \quad x$.

b). The joint probability density function of X, Y is given by $f_{XY}(x,y) = \begin{cases} xy^2 + \dfrac{x^2}{8}, & 0 \le x \le 2, \quad 0 \le y \le 1 \\ 0 & , \; otherwise \end{cases}$. Find (i) $P \;\; X \quad 1$, (ii) $P \;\; X$ [redacted] and (iii) $P \;\; X \quad Y \quad 1$

Solution:

a). Given X and Y are two random variables having the joint density function

$$f(x,y) = \frac{1}{27}(x+2y) ----(1)$$

Where $x \quad 0,1,2$ and $y \quad 0,1,2$

Then the joint probability distribution X and Y becomes as follows

$X \backslash Y$	0	1	2	$f_1(x)$
0	0	$\frac{1}{27}$	$\frac{2}{27}$	$\frac{3}{27}$

1	$\frac{2}{27}$	$\frac{3}{27}$	$\frac{4}{27}$	$\frac{9}{27}$
2	$\frac{4}{27}$	$\frac{5}{27}$	$\frac{6}{27}$	$\frac{15}{27}$

The marginal probability distribution of X is given by $f_1(X)=\sum_j P(x,y)$ and is calculated in the above column of above table.

The conditional distribution of Y for X is given by $f_1\left(Y=y/X=x\right)=\frac{f(x,y)}{f_1(x)}$ and is obtained in the following table.

Y \ X	0	1	2
0	0	$\frac{1}{3}$	$\frac{2}{3}$
1	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$
2	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{2}$

$$P\left(Y=0/X=0\right)=\frac{P(X=0,Y=0)}{P(X=0)}=\frac{0}{\frac{6}{27}}=0$$

$$P\left(Y=1/X=0\right)=\frac{P(X=0,Y=1)}{P(X=0)}=\frac{\frac{2}{27}}{\frac{6}{27}}=\frac{1}{3}$$

$$P\left(Y=2/X=0\right)=\frac{P(X=0,Y=2)}{P(X=0)}=\frac{\frac{4}{27}}{\frac{6}{27}}=\frac{2}{3}$$

$$P\left(Y=0/X=1\right)=\frac{P(X=1,Y=0)}{P(X=1)}=\frac{\frac{1}{27}}{\frac{9}{27}}=\frac{1}{9}$$

$$P\left(Y=1/X=1\right)=\frac{P(X=1,Y=1)}{P(X=1)}=\frac{\frac{3}{27}}{\frac{9}{27}}=\frac{3}{9}=\frac{1}{3}$$

$$P\left(Y=2/X=1\right)=\frac{P(X=1,Y=2)}{P(X=1)}=\frac{\frac{5}{27}}{\frac{9}{27}}=\frac{5}{9}$$

$$P\left(Y=0/X=2\right)=\frac{P\left(X=2,Y=0\right)}{P\left(X=2\right)}=\frac{\frac{2}{27}}{\frac{12}{27}}=\frac{1}{6}$$

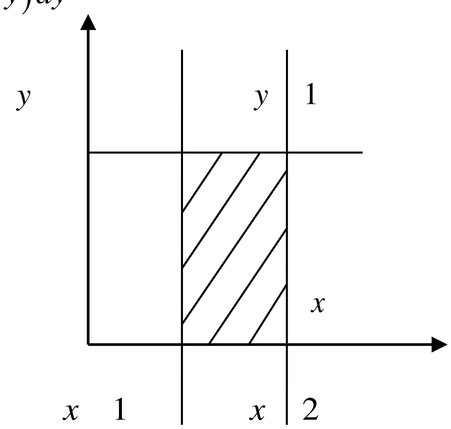
$$P\left(Y=1/X=2\right)=\frac{P\left(X=2,Y=1\right)}{P\left(X=2\right)}=\frac{\frac{4}{27}}{\frac{12}{27}}=\frac{1}{3}$$

$$P\left(Y=2/X=2\right)=\frac{P\left(X=2,Y=2\right)}{P\left(X=2\right)}=\frac{\frac{6}{27}}{\frac{12}{27}}=\frac{1}{2}$$

b). Given the joint probability density function of X,Y is $f_{XY}\left(x+y\right)=xy^2+\frac{x^2}{8}$, $0\le x\le 2,\ 0\le y\le 1$

(i). $P\left(X>1\right)=\int_1^\infty f_X\left(x\right)dx$

The Marginal density function of X is $f_X\left(x\right)=\int_0^1 f\left(x,y\right)dy$



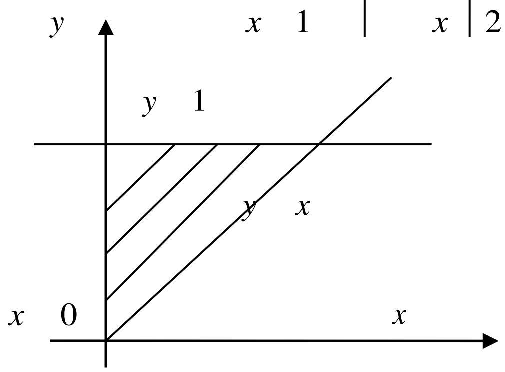
$$f_X\left(x\right)=\int_0^1\left(xy^2+\frac{x^2}{8}\right)dy$$

$$=\left[\frac{xy^2}{3}+\frac{x^2y}{8}\right]_0^1=\frac{x}{3}+\frac{x^2}{8},\ 1<x<2$$

$$P\left(X>1\right)=\int_1^2\left(\frac{x}{3}+\frac{x^2}{8}\right)dx$$

$$=\left[\frac{x^2}{6}+\frac{x^3}{24}\right]_1^2=\frac{19}{24}.$$

(ii) $P\left(X<Y\right)=\int\int_{R_2} f_{XY}\left(x,y\right)dxdy$



$$P\left(X<Y\right)=\int_{y=0}^{1}\int_{x=0}^{y}\left(xy^2+\frac{x^2}{8}\right)dxdy$$

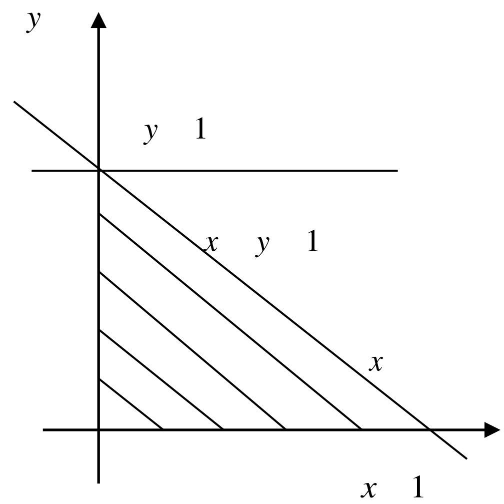
$$=\int_0^1\left[\frac{x^2y^2}{2}+\frac{x^3}{24}\right]_0^y dy$$

$$=\int_0^1\left(\frac{y^4}{2}+\frac{y^3}{24}\right)dy=\left[\frac{y^5}{10}+\frac{y^4}{96}\right]_0^1$$

$$=\frac{1}{10}+\frac{1}{96}=\frac{96+10}{960}=\frac{53}{480}$$

(iii) $P\left(X+Y \le 1\right) = \iint\limits_{R_3} f_{XY}\left(x, y\right) dxdy$

Where R_3 is the region



$$P\left(X+Y \le 1\right) = \int_{y=0}^{1} \int_{x=0}^{1-y} \left(xy^2 + \frac{x^2}{8} \right) dxdy$$

$$= \int_{y=0}^{1} \left[\left(\frac{x^2 y^2}{2} + \frac{x^3}{24} \right) \right]_0^{1-y} dy$$

$$= \int_{y=0}^{1} \left(\frac{\left(1-y\right)^2 y^2}{2} + \frac{\left(1-y\right)^3}{24} \right) dy$$

$$= \int_0^1 \left(\frac{\left(1+y^2-2y\right) y^2}{2} + \frac{\left(1-y\right)^3}{24} \right) dy$$

$$= \left[\left[\frac{y^3}{3} + \frac{y^5}{5} - \frac{2y^2}{4} \right] \frac{1}{2} + \frac{\left(1-y\right)^4}{96} \right]_0^1$$

$$= \frac{1}{6} + \frac{1}{10} - \frac{1}{4} + \frac{1}{96} = \frac{13}{480}.$$

20).a). If the joint distribution functions of X and Y is given by

$$F\left(x, y\right) = \begin{cases} \left(1-e^{x}\right)\left(1-e^{-y}\right), & x > 0,\ y > 0 \\ 0 & , otherwise \end{cases}$$

i. Find the marginal density of X and Y.

ii. Are X and Y independent.

iii. $P\ 1 \quad X \quad 3,\ 1 \quad Y \quad 2$.

b). The joint probability distribution of X and Y is given by

$$f\left(x, y\right) = \begin{cases} \frac{6-x-y}{8}, & 0 < x < 2,\ 2 < y < 4 \\ 0 & , otherwise \end{cases}$$. Find $P\ 1 \quad Y \quad 3/X \quad 2$.

Solution:

a). Given $F\left(x, y\right) = \left(1-e^{-x}\right)\left(1-e^{-y}\right)$

$$= 1 - e^{-x} - e^{-y} + e^{-(x+y)}$$

The joint probability density function is given by

$$f\left(x, y\right) = \frac{\partial^2 F\left(x, y\right)}{\partial x \partial y}$$

$$= \frac{\partial^2}{\partial x \partial y}\left[1 - e^{-x} - e^{-y} + e^{-(x+y)}\right]$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right]$$

$$\therefore f(x,y) = \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0 \\ 0 & , \ otherwise \end{cases}$$

(ii) The marginal probability function of X is given by

$$f(x) = f_X(x)$$

$$= \int_{-\infty}^{\infty} f(x,y)\,dy = \int_{0}^{\infty} e^{-(x+y)}dy$$

$$= \left[\frac{e^{-(x+y)}}{-1}\right]_0^{\infty}$$

$$= \left[-e^{-(x+y)}\right]_0^{\infty}$$

$$= e^{-x},\ x > 0$$

The marginal probability function of Y is

$$f(y) = f_Y(y)$$

$$= \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \int_{0}^{\infty} e^{-(x+y)}dx = \left[-e^{-(x+y)}\right]_0^{\infty}$$

$$= e^{-y},\ y > 0$$

$$\therefore f(x)f(y) = e^{-x}e^{-y} = e^{-(x+y)} = f(x,y)$$

$\therefore$ X and Y are independent.

(iii) $P(1 < X < 3, 1 < Y < 2) = P(1 < X < 3) \times P(1 < Y < 2)$ [Since X and Y are independent]

$$= \int_1^3 f(x)dx \times \int_1^2 f(y)\,dy.$$

$$= \int_1^3 e^{-x}dx \times \int_1^2 e^{-y}dy$$

$$= \left[\frac{e^{-x}}{-1}\right]_1^3 \times \left[\frac{e^{-y}}{-1}\right]_1^2$$

$$= \left(-e^{-3} + e^{-1}\right)\left(-e^{-2} + e^{-1}\right)$$

$$= e^{-5} - e^{-4} - e^{-3} + e^{-2}$$

b). $P \; 1 \quad Y \quad 3/X \quad 2 \qquad \int_{1}^{3} f \; y/x \quad 2 \; dy$

$$f_X(x) = \int_2^4 f(x,y)\,dy$$

$$= \int_2^4 \left(\frac{6-x-y}{8} \right) dy$$

$$= \frac{1}{8} \left(6y - xy - \frac{y^2}{2} \right)_2^4$$

$$= \frac{1}{8} (16 - 4x - 10 + 2x)$$

$$f\left({}^{y}\!/_{x} \right) = \frac{f(x,y)}{f(x)} = \frac{\frac{6-x-y}{8}}{\frac{6-2x}{8}} = \frac{6-x-y}{6-2x}$$

$$P \; 1 \;\; Y \;\; {}^{3}\!/_{X} \;\; 2 \qquad \overset{3}{\underset{1}{}} \; f \; {}^{y}\!/_{x} \;\; 2 \; dy$$

$$= \int_2^3 \left(\frac{4-y}{2} \right) dy$$

$$= \frac{1}{2} \left[4y - \frac{y^2}{2} \right]_2^3$$

$$= \frac{1}{2} \left[4y - \frac{y^2}{2} \right]_2^3 = \frac{1}{2} \left[14 - \frac{17}{2} \right] = \frac{11}{4}.$$