Probability (Defn):

$$P(\text{Event}) = \frac{\text{No. of favourable to the event}}{\text{Total No. of Exhaustive events}}$$

Ex:- 1) In tossing a coin,

$S = \{H, T\}$

$P(H) = \frac{1}{2}$, $P(T) = \frac{1}{2}$

2) In throwing a die,

$S = \{1, 2, 3, 4, 5, 6\}$

$P(1) = \frac{1}{6}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{6}$ .... $P(6) = \frac{1}{6}$

~~Axioms of Probab~~

Axioms of Probability:-

1) $0 \leq P(E) \leq 1$

2) $\Sigma P(E) = 1$

3) If $A_1, A_2 \ldots A_n$ are mutually Exclusive Events (M.E.E) then

$P(A_1 \cup A_2 \cup \ldots A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$

Problems:

~~In~~ A bag contains 5 red, 4 Black and 3 white balls. What is the probability that if 3 balls are drawn

i) all are red balls.

ii) 2 Black & 1 white balls.

iii) 1 white, 1 black & 1 red ball.

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Soln:-

Total no.of. balls = 12

i) $P(3 \text{ Red balls}) = \frac{5C_3}{12C_3} = \frac{\frac{5\times4\times3}{1\times2\times3}}{\frac{12\times11\times10}{1\times2\times3}} = \frac{1}{22}$

ii) $P(2B, 1W) = \frac{4C_2 \times 3C_1}{12C_3} = \frac{6\times3}{220} = \frac{18}{220} = \frac{9}{110}$

iii) $P(1R, 1B, 1W) = \frac{5\times4\times3}{220} = \frac{20\times3}{220} = \frac{3}{11}$

2) What is the Probability that 53 Sundays in a leap year?

Soln:-

In a leap year we have 52 Sundays and the remaining 2 days are Sunday & monday, M&T, T&W, W&Th, Th.F, F&S, S&S

$\therefore P(53 \text{ Sundays}) = \frac{2}{7}$.

Defn:-

Independent events:-

A set of events is said to independent if the occurrence of any one does not depend on the occurrence of other events.

If two events A&B are independent, Then

$$\boxed{P(A\cap B) = P(A) \times P(B).}$$

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Problems:-

1) If A & B are independent events then Prove the following:

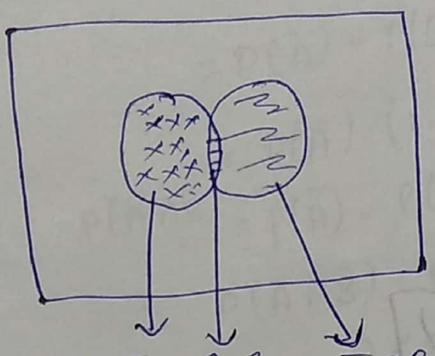
i) $\overline{A}$ & B are independent.

ii) A & $\overline{B}$ are independent

iii) $\overline{A}$ & $\overline{B}$ are independent.

Soln:-

Venn diagram:-



$A \cap \overline{B}$ $\quad$ $A \cap B$ $\quad$ $\overline{A} \cap B$

i) TP: $\overline{A}$ & B are independent events,

ETP: $P(\overline{A} \cap B) = P(\overline{A}) \cdot P(B)$

WKT,

$B = (A \cap B) \cup (\overline{A} \cap B)$

$P(B) = P[(A \cap B) \cup (\overline{A} \cap B)]$

$= \cancel{P[(A \cap B)}$

$= \cancel{P[} P(A \cap B) + P(\overline{A} \cap B)$

$P(B) = P(A) \cdot P(B) + P(\overline{A} \cap B)$ $\quad$ ($\because A \cap B$ & $\overline{A} \cap B$ are M.E.E's)

$P(B) - P(A) \cdot P(B) = P(\overline{A} \cap B)$

$P(B)(1 - P(A)) = P(\overline{A} \cap B)$

$P(B) \cdot P(\overline{A}) = P(\overline{A} \cap B)$ $\quad$ [$\because \Sigma P(E) = 1$

$P(A) + P(\overline{A}) = 1$

$\Rightarrow P(\overline{A}) = 1 - P(A)$]

$\Rightarrow P(\overline{A} \cap B) = P(\overline{A}) \cdot P(B)$

i.e., $\overline{A}$ & B are independent

$A \times B = C$
$\bar{A} = \bar{B} = C$

ii) T.P: A & $\bar{B}$ are independent

E.T.P: $P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})$

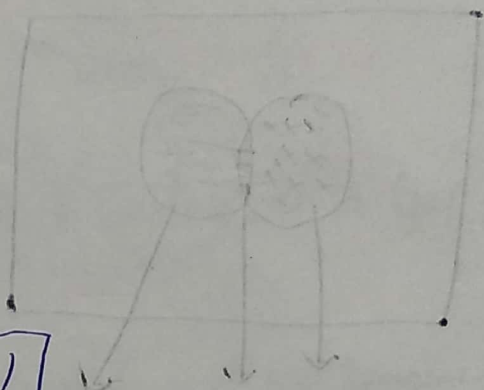
WKT,

$$A = (A \cap B) \cup (A \cap \bar{B})$$

$$P(A) = P\left[(A \cap B) \cup (A \cap \bar{B})\right]$$

$$P(A) = P(A \cap B) + P(A \cap \bar{B}) \quad (\because A \cap B \text{ \& } A \cap \bar{B} \text{ are M.E.E})$$

$$P(A) = P(A) \cdot P(B) + P(A \cap \bar{B}) \quad (\because \cancel{P(A} \; A \cap B$$



$$P(A) - P(A) \cdot P(B) = P(A \cap \bar{B})$$

$$P(A)(1 - P(B)) = P(A \cap \bar{B})$$

$$P(A)\, P(\bar{B}) = P(A \cap \bar{B})$$

$$\boxed{\therefore P(A \cap \bar{B}) = P(A) \cdot P(\bar{B})}$$

iii) T.P: $\bar{A}$ & $\bar{B}$ are independent

E.T.P: $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$.

$$\bar{A} = \overline{(A \cap B) \cup (A \cap \bar{B})}$$

$$\bar{A} = (\overline{A \cap B}) \cap (\overline{A \cap \bar{B}})$$

$$\bar{A} = (\bar{A} \cup \bar{B}) \cap (\bar{A} \cup B)$$

$$P(\bar{A}) = [P(\bar{A}) + P(\bar{B})] \cap [P(\bar{A}) + P(B)]$$

$$P(\bar{A}) = P$$

iii) T.P : $\bar{A}$ & $\bar{B}$ are independent,

I.T.P: $P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$ (Demorgan's law)

$= 1 - P(A \cup B)$

$= 1 - (P(A) + P(B) - P(A \cap B))$

$= 1 - P(A) - P(B) + P(A \cap B)$

$= P(\bar{A}) - P(B) + P(A) \cdot P(B)$

$= P(\bar{A}) - P(B)(1 - P(A))$

$= P(\bar{A}) - P(B) P(\bar{A})$

$= P(\bar{A})(1 - P(B))$

$P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

$\therefore P(\bar{A} \cap \bar{B}) = P(\bar{A}) \cdot P(\bar{B})$

$\therefore$ $\bar{A}$ & $\bar{B}$ are independent events.

8/1/19

Conditional Probability:-

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

/ → Conditional along

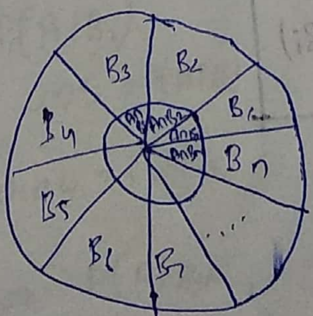
IIIly,

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

$\Rightarrow P(A \cap B) = P(B) P(A/B)$

$= P(A) P(B/A)$

Theorem of Probability:-



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The inner Circle represents the event 'A'. It can occur also along with $B_1, B_2, B_3 \ldots B_n$ that are exhaustive and mutually exclusive

Therefore $A = (A\cap B_1) \cup (A\cap B_2) \cup (A\cap B_3) \cup \ldots \cup (A\cap B_n)$

$$P(A) = P(A\cap B_1) + P(A\cap B_2) + P(A\cap B_3) + \ldots + P(A\cap B_n)$$

(∵ All are M.E. E's)

$$= \sum_{i=1}^{n} P(A\cap B_i) \longrightarrow ①$$

Now, by Conditional Probability:-

$$P(A|B_i) = \frac{P(A\cap B_i)}{P(B_i)}$$

$$P(B_i)\,P(A|B_i) = P(A\cap B_i) \longrightarrow ②$$

Sub ② in ① =>

$$\boxed{P(A) = \sum_{i=1}^{n} P(B_i)\,P(A/B_i)} \longrightarrow \text{Total Probability.}$$

Baye's Theorem:-

Stmt:- If $B_1, B_2 \ldots B_n$ are a set of exhaustive and mutually Exclusive events associated with a random experiment and A is another event associated with B_i then BAYE'S Theorem states that

$$\boxed{P(B_i/A) = \frac{P(B_i)\,P(A/B_i)}{\sum_{i=1}^{n} P(B_i)\,P(A/B_i)}}$$

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Proof:-

To First Prove Theorem of total Probability,

$$\boxed{P(A) = \sum_{i=1}^{n} P(B_i)\, P(A/B_i)}$$

Now,

$$P(B_i/A) = \cancel{P(B}\ \frac{P(A \cap B_i)}{P(A)}$$

$$= \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}$$

$$\boxed{\therefore\ P(B_i/A) = \frac{P(B_i)P(A/B_i)}{\sum_{i=1}^{n} P(B_i) \cdot P(A/B_i)}}$$

Hence Proved//

Problems:-

1) In a Bolt factory, machines B_1, B_2 & B_3 Produce 25%, 35% & 40% of the total output respectively of their outputs 5%, 4% & 2% are defective bolts respectively. If a bolt is chosen at random from the combined output & it is a defective, what is the Probability that it was Produced by machine B_1 or B_2 or B_3?

Soln:-

Let A be the defective bolts Produced from the combined output.

Given that,

	$P(B_i)$	$P(A/B_i)$	$P(B_i)\,P(A/B_i)$
$P(B_1)$	0.25	$0.05 = P(A/B_1)$	$0.0125 = P(B_1)\,P(A/B_1)$
$P(B_2)$	0.35	$0.04 = P(A/B_2)$	$0.014 = P(B_2)\,P(A/B_2)$
$P(B_3)$	0.40	$0.02 = P(A/B_3)$	$0.008 = P(B_3)\,P(A/B_3)$

$$\sum_{i=1}^{3} P(B_i)\ P(A/B_i) = 0.0345$$

By Baye's Theorem;

$$P(B_1/A) = \frac{P(B_1)\, P(A/B_1)}{\sum_{i=1}^{3} P(B_i)\, P(A/B_i)}$$

$$= \frac{0.0125}{0.0345} = 0.3623$$

$$P(B_2/A) = \frac{P(B_2)\, P(A/B_2)}{\sum_{i=1}^{3} P(B_i)\, P(A/B_i)} = \frac{0.014}{0.0345} = 0.4058$$

$$P(B_3/A) = \frac{P(B_3)\, P(A/B_3)}{\sum_{i=1}^{3} P(B_i)\, P(A/B_i)} = \frac{0.008}{0.0345} = 0.2319 \,//$$

2) A bag contains 5 balls and it is not known how many of them are white. 2 balls are drawn at random from the bag and they are to be white. What is the probability that all the balls in the bag are white.

Soln:-

Since 2 white balls have been drawn out the ~~bak~~ bag must have contained 2, 3, 4, 5 white balls.

Let B_1 = Event of the bag containing 2 white balls,

B_2 = Event of the ~~bag~~ bag " 3 " ".

B_3 = " " " " 4 white balls

B_4 = Event of the bag " 5 white Balls

A = Event of drawing 2 white balls.

$$\therefore P(A/B_1) = \frac{2C_2}{5C_2} = \frac{1}{10} = 0.1$$

$$P(A/B_2) = \frac{3C_2}{5C_2} = \frac{3}{10} = 0.3$$

$$P(A/B_3) = \frac{4C_2}{5C_2} = \frac{6}{10} = 0.6$$

$$P(A/B_4) = \frac{5C_2}{5C_2} = 1.$$

$\because$ The no of white balls in the bag is not known

$\therefore$ All B_i's are Equally Likely.

i.e., $P(B_1) = 1/4$, $P(B_2) = 1/4$, $P(B_3) = 1/4$, $P(B_4) = 1/4$.

By Baye's Theorem:-

$$P(B_4/A) = \frac{P(A/B_4) \cdot P(B_4)}{\sum_{i=1}^{4} P(B_i) \cdot P(A/B_i)}$$

$$= \frac{1(1/4)}{\frac{1}{4}(0.1) + \frac{1}{4}(0.3) + \frac{1}{4}(0.6) + \frac{1}{4}(1)}$$

$$= \frac{\frac{1}{4}}{0.025 + 0.075 + 0.15 + 0.25}$$

$$\frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{4} \times 2 = \frac{1}{2}$$

$$= \frac{\frac{1}{4}}{0.5} = \frac{0.25}{0.5} = 0.5 = 1/2$$

3) A Bag Contains 10 white & 3 Black Balls. Another Bag Contains 3 white & 5 Black Balls. Two Balls are drawn at random from the first bag & placed in the 2nd bag, and then 1 ball is taken at random from the ~~later~~ latter. what is the Probability that it is a white ball.

Soln:- The two balls are transferred from 1st bag to 2nd bag may be both white / both black / one white and one black.

Let B_1 = Event of drawing 2 white balls from Bag 1.

B_2 = Event of drawing 2 Black Balls from Bag 1.

B_3 = Event of drawing 1W & 1B Balls from Bag 1

$P(B_1) = \cancel{\frac{{}^{10}C_2}{{}^{13}C_2}} =$

Clearly, all B_i's are mutually exclusive events.

Let A = Event of drawing a white ball from the 2nd bag after transfer.

$$P(B_1) = \frac{{}^{10}C_2}{{}^{13}C_2} = \frac{45}{78} = 0.5769$$

$$P(B_2) = \frac{{}^{3}C_2}{{}^{13}C_2} = \frac{3}{78} = 0.0384$$

$$P(B_3) = \frac{{}^{10}C_1 \times {}^{3}C_1}{{}^{13}C_2} = \frac{30}{78} = 0.3846$$

$$P(A|B_1) = \frac{5}{10} = 0.5$$

$$P(A|B_2) = \frac{3}{10} = 0.3$$

$$P(A|B_3) = \frac{4}{10} = \frac{2}{5} = 0.4$$

By ~~Baye's Theorem~~:

By Theorem of Total Probability:-

$$P(A) = P(B_1) \cdot P(A|B_1) + P(B_2) \cdot P(A|B_2) + P(B_3) \cdot P(A|B_3)$$

$$= 0.5769(0.5) + 0.0384(0.3) + 0.3846(0.4)$$

$$= 0.4538$$

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RANDOM VARIABLE

Discrete (Finite (or) Countable)

Continuous (infinite (or) uncountable)

Discrete Random Variable:-

If values are finite and countable then the random variables are called discrete random variable. Its probability mass function is denoted by $P(x)$ (or) $P(x=x)$. It's total probability is $\boxed{\Sigma P(x)=1}$

Continuous Random Variable:-

If the values are infinite and uncountable then the random variable is called as continuous random variable. Its probability density function (PDF) is denoted by $f(x)$'s. Its total probability is $\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$

Cumulative ~~Discrete~~ Distribution function (CDF):-

It is denoted by $F(x)$. If x is a discrete random variable then CDF is

$$\boxed{F(x) = \sum_{1}^{x} p(x)}$$

If x is continuous random variable then CDF is

$$\boxed{F(x) = \int_{-\infty}^{\infty} f(x)\,dx}$$

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If distribution function is given then the density function is $\boxed{f(x) = \frac{d}{dx} F(x)}$

1) A random variable (x) has the following probability distribution

x:	0	1	2	3	4	5	6	7
P(x)	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

i) Find the value of k

ii) $P(2 \le x \le 5)$

iii) $P(x \ge 6)$

iv) $P(1.5 < x < 4.5 / x > 2)$

v) The smallest value of λ for which $P(x \le \lambda) > 1/2$

Soln:-

⟹ i) It is a discrete R.V

Its total probability is $\boxed{\sum_{x=0}^{7} p(x) = 1}$

i.e, $\sum_{x=0}^{7} p(x) = 1$

$0 + k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$10k^2 + 9k = 1$

$10k^2 + 9k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

$(k+1)(10k-1) = 0$

$k = -1, 1/10$

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k=-1 will not be Possible because there won't be any negative value in Probability.

So the answer is $\boxed{k = 1/10}$

ii) $P(2 \le X \le 5)$

$$\therefore \sum_{x=2}^{5} P(x)$$

$$= 2k + 2k + 3k + k^2$$

$$= 7k + k^2$$

$$= \frac{1}{100} + \frac{7}{10} \quad (\text{Sub } k = 1/10)$$

$$= \frac{71}{100} \;//$$

iii) $P(x \ge 6)$

$$\therefore \sum_{x=6}^{7} P(x)$$

$$= 2k^2 + 7k^2 + k$$

$$= 9k^2 + k$$

$$= 9\left(\frac{1}{100}\right) + \frac{1}{10}$$

$$= \frac{19}{100} \;//$$

iv) $P(1.5 < x < 4.5 / x > 2) = \dfrac{P(1.5 < x < 4.5 \cap x > 2)}{P(x > 2)}$

$P(1.5 < 4.5 \cap x > 2) = P(x=3) + P(x=4)$

$$= 2k + 3k$$

$$= 5k$$

$P(1.5 < x < 4.5 \cap x > 2) = 5/10 \quad (\text{Sub } k = 1/10)$

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Similarly, $P(X>2) = P(X=3) + P(X=4) + P(X=5) + P(X=6) + P(X=7)$

$= 2k + 3k + k^2 + 2k^2 + 7k^2 + k$

$= 10k^2 + 6k$

$= \frac{10}{100} + \frac{6}{10}$

$$\boxed{P(X>2) = 7/10}$$

$\therefore P(1.5 < X < 4.5 / X>2) = \frac{5/10}{7/10} = 5/7$

v) To find λ if $P(X \le \lambda) > 1/2$

x	$P(x)$	Cumulative D.F $(F(x))$
0	0	0
1	1/10	$0 + 1/10 = 1/10$
2	2/10	$1/10 + 2/10 = 3/10$
3	2/10	$3/10 + 2/10 = 5/10$
4	3/10	$5/10 + 3/10 = 8/10$
5	1/100	$8/10 + 1/100 = 81/100$
6	2/100	$81/100 + 2/100 = 83/100$
7	17/100	$83/100 + 17/100 = \frac{100}{100} = 1$

To find λ. $P(X \le \lambda) > 1/2$ from the C.D.F

2) A random variable has the following Probability Distribution

x	-2	-1	0	1	2	3
$p(x)$	0.1	k	0.2	$2k$	0.3	$3k$

i) To find value of k

ii) Evalute $P(x<2)$ & $P(-2<X<2)$

iii) Find C.D.F of x

iv) Find mean of X

i)

$\rightarrow$ It is discrete R.V

Total Probability is

$$\sum_{-2}^{3} p(x) = 1$$

$$0.1 + k + 0.2 + 2k + 0.3 + 3k = 1$$

$$0.6 + 6k = 1$$

$$6k = 1 - 0.6$$

$$6k = 0.4$$

$$\boxed{k = 0.066}$$

ii) $P(x<2)$

$$\sum_{-2}^{1} p(x) = 1 \quad (x=-2) + (x=-1) + (x=0) + (x=1)$$

$$= 0.1 + 0.066 + 0.2 + 2(0.066)$$

$$= 0.498$$

$P(-2<X<2)$

$$= 0.066 + 0.2 + 2(0.066)$$

$$= 0.398$$

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iii)

X	P(X)	C.D.F (F(x))
-2	0.1	0.1
-1	0.066	0.1+0.066 = 0.166
0	0.2	0.166+0.2 = 0.366
1	0.132	0.366+0.132 = 0.498
2	0.3	0.498+0.3 = 0.798
3	0.198	0.996 = 1

iv) mean of x

Formula of mean = $E(x)$ where E = Expectation ~~Excep~~

$= \sum x P(x)$

$= -2(0.1) + -1(0.066) + 0(0.2) + 1(0.132) + 2(0.3) + 3(0.198)$

$= 1.061$

3) If the density function of a Continuous Random Variable X is given by

$$f(x) = \begin{cases} ax & ; 0 \le x \le 1 \\ a & ; 1 \le x \le 2 \\ 3a - ax & ; 2 \le x \le 3 \\ 0 & ; \text{otherwise} \end{cases}$$

i) Find the values of a

ii) Find the CDF of X

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Soln:-

$\Rightarrow$ Since X is a Continuous Random variable,

Its total Probability is

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = 1}$$

In this Problem the interval is 0 to 1 & 1 to 2 & 2 to 3 & 3 to ∞.

$$= \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a-ax)\,dx + \int_3^{\infty} 0\,dx = 1$$

$$= a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left(3ax - \frac{ax^2}{2}\right)_2^3 = 1$$

$$a\left(\frac{1}{2}\right) + a(2-1) + \left(3a(3) - \frac{9a}{2}\right) - \left(3a(2) - \frac{a4}{2}\right) = 1$$

$$\frac{a}{2} + a + 9a - \frac{9a}{2} - 6a - \frac{4a}{2} = 1$$

$$\frac{a}{2} + a + 9a - \frac{9a}{2} - 6a - 2a = 1$$

$$6a - 4a = 1$$

$$2a = 1$$

$$\boxed{a = 1/2}$$

WkT C.D.F is

$$F(x) = \int_{-\infty}^{\infty} f(x)\,dx$$

$$\int_0^x f(x)\,dx = \int_0^x ax\,dx$$

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$F(x) = a\left(\frac{x^2}{2}\right) = \frac{x^2}{4}$

$\therefore 0 \le x \le 1$

$F(x) = \int_0^1 ax\,dx + \int_1^x a\,dx$

$= a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^x$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(x^2-1\right) = \frac{1}{4} + \frac{1}{2}\left(x^2-1\right)$

$\therefore 1 \le x \le 2$ //

$F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^x (3a-ax)\,dx$

$= a\left(\frac{x^2}{2}\right)_0^1 + a(x)_1^2 + \left(3a - \frac{ax^2}{2}\right)_2^x$

$= \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}(2-1) + \left[\frac{3}{2}x - \frac{1}{2}\cdot\frac{x^2}{2} - \frac{6}{2} + \frac{1}{2}(2)\right]$

$= \frac{1}{4} + \frac{1}{2} + \frac{5x}{2} - \frac{x^2}{4} - 2$

$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4}$

$\therefore 2 \le x \le 3$

$F(x) = \int_0^1 ax\,dx + \int_1^2 a\,dx + \int_2^3 (3a-ax)\,dx + \int_3^\infty 0\cdot dx$

$= 1$

$\therefore x > 3$ otherwise

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1) A Continuous random variable x has a P.D.F $f(x) = kx^2 e^{-x}$, $0 < x < \infty$

i) Find the value of k

ii) Find the mean

iii) Find the variance

Soln:

Since it is a continuous random variable

Total Probability $= 1$

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

In this problem we have 0 to ∞

So

$$\int_{0}^{\infty} kx^2 e^{-x}\, dx = 1$$

Bernoulli Formula:

$$k\left[x^2\left(\frac{e^{-x}}{-1}\right) - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty} = 1$$

$$k(2) = 1$$

$$\boxed{k = 1/2}$$

To find mean:-

$$\text{mean} = E(x)$$

$$= \int_{-\infty}^{\infty} x.f(x)\, dx = \int_0^{\infty} x.k\, x^2 e^{-x}\, dx$$

$$= k\int_0^{\infty} x^3 e^{-x}\, dx$$

$$= \frac{1}{2}\left[x^3\left(\frac{e^{-x}}{-1}\right) - 3x^2\left(\frac{e^{-x}}{-1}\right) + 6x\left(\frac{e^{-x}}{-1}\right) - 6\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$$

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$= \frac{1}{2} \times 6 = 3$

$\boxed{\text{mean} = E(x) = 3}$

iii) Variance $= E(x^2) - (E(x))^2$

To find $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

$= \int_0^{\infty} x^2 k x^2 e^{-x} dx$

$= k \int_0^{\infty} x^4 e^{-x} dx$

$= \frac{1}{2} \Big(x^4 \left(\frac{e^{-x}}{-1}\right) - 4x^3 \left(\frac{e^{-x}}{1}\right) + 12x^2 \left(\frac{e^{-x}}{-1}\right) - 24x \left(\frac{e^{-x}}{1}\right) + 24 \left(\frac{e^{-x}}{-1}\right) \Big]_0^{\infty}$

$= \frac{1}{2}(24) = 12$

$\boxed{E(x^2) = 12}$

$\therefore \cancel{Var(E)}$

$\therefore Var(x) = E(x^2) - (E(x))^2$

$= 12 - 3^2$

$\boxed{Var(x) = 3}$ //

---

5) The distribution function of a Random Variable X is given by F(x)

$F(x) = 1 - (1+x)e^{-x}, \; x \geq 0.$ P.N 01.2.25

i) Find density function

ii) Find Mean & Variance.

Soln:-

i) $f(x) = \frac{d}{dx} f(x)$

$= \frac{d}{dx}\left[1-(1+x)e^{-x}\right]$

$= 0 - \left((1+x)e^{-x}.(-1) + e^{-x}.1\right)$

$f(x) = -e^{-x}.(-1-x+1)$

$f(x) = xe^{-x}$

$\boxed{\therefore x \geq 0}$

ii) mean $E(x) = \int_{-\infty}^{\infty} x f(x)\, dx$

$= \int_{0}^{\infty} x . xe^{-x} dx$

$= \int_{0}^{\infty} x^2 e^{-x} dx = \left[x^2\left[\frac{e^{-x}}{-1}\right] - 2x\left(\frac{e^{-x}}{1}\right) + 2\left(\frac{e^{-x}}{-1}\right)\right]_0^{\infty}$

$\boxed{\text{mean} : E(x) = 2}$

Variance $= E(x^2) - E(x)$

$E(x^2) = \int_0^{\infty} x^2 f(x) dx$

$= \int_0^{\infty} x^2(1-(1+x)e^{-x}$

$= \int_0^{\infty} x^2(1-e^{-x}-xe^{-x})$

$= \int_0^{\infty} x^2 - x^2e^{-x} - x^3e^{-x}$ $\cancel{\int_0^{\infty} xe^{x}}$

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~~one function of one~~

---

one function of one random variable:

If x is a continuous random variable $f_x(x)$ and $g(x)$ is a strictly monotomic function of x such that $y = g(x)$ then density function of $f_Y(y) = \dfrac{f_x(x)}{|g'(x)|}$.

1) If the PDF of random variable

$$X = f_X(x) = 2x, \; 0 < x < 1.$$

Find the PDF and CDF of $Y = 3X+1$..

$$\Rightarrow Y = 3X + 1$$

$$\frac{dY}{dX} = 3 \therefore Y = g(x)$$

$\therefore$ It is greater than 0 thus if it is positive then we say that they are strictly monotonic function. To find the limit substitute $0 < x < 1$ in X $\therefore$ limit are 1 to 4.

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$$f_X(y) = \frac{f_X(x)}{|g'(x)|} \rightarrow \text{Absolute value of first derivative.}$$

$$= \frac{2x}{3}, \quad 1<y<4$$

$\therefore$ From ques we get Y as $Y = 3X+1$

$$\therefore X = \left(\frac{y-1}{3}\right)$$

$$= \frac{2}{3}\left(\frac{y-1}{3}\right), \quad 1<y<4$$

$$= \frac{2(y-1)}{9}, \quad 1<y<4$$

$\therefore$ CDF of Y :

$$F(y) = \int_1^y f_X(y)\,dy$$

$$= \int_1^y \frac{2}{9}(y-1)\,dy = \frac{2}{9}\int_1^y y\,dy - dy$$

$$= \frac{2}{9}\int_1^y \frac{y^2}{2} - y = \frac{2}{9}\left[\frac{y^2}{2} - y\right]_1^2$$

$$= \frac{1}{9}\left[(y-1)^2 - 0\right]$$

$$F(y) = \frac{(y-1)^2}{9}, \quad 1<y<4. //$$

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2) $f_X(x) = x^{-x}, x > 0$

$Y = 2X + 1$

Find $P(Y \geq 5)$

Soln:

$Y = g(x)$ is an monotonically increasing function.

$$f_Y(Y) = \frac{f_X(x)}{|g'(x)|}$$

when $X > 0, y > 1$

$$f_Y(y) = \frac{e^{-x}}{2}, \; y > 1$$

$$= \frac{e^{-\left(\frac{y-1}{2}\right)}}{2}, \; y > 1$$

$$P(y \geq 5) = \int_5^{\infty} f(x)\,dx$$

$$= \frac{1}{2}\int_5^{\infty} e^{-\left(\frac{y-1}{2}\right)} dy = \frac{1}{2}\left[\frac{e^{-\left(\frac{y-1}{2}\right)}}{-\frac{1}{2}}\right]_5^{\infty}$$

$$= -\left[e^{-\infty} - e^{5}\right]_{4}$$

3) If PDF of $f_X(x) = 2x, \; 0 < x < 1$ and $Y = \sqrt{x}$

Soln:

$Y = \sqrt{x}$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

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$\therefore$ It is >0 which means it is monotonically increasing. Thus we get limit as 0 to 1.

$$f_Y(y) = \frac{f_X(x)}{|g'(x)|}, \quad 0<y<1$$

$$= \frac{2x}{\frac{1}{2\sqrt{x}}}, \quad 0<x<1$$

$$= 4x\sqrt{x}, \quad 0<x<1$$

$$= 4(y^2)(y), \quad 0<y<1.$$

$$= 4y^3, \quad 0<y<1 \,//$$

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22/1/19

Transformation of one Random variable :-

1) Given the Random Variable X with density function $f(x) = \{$

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 0, & \text{elsewhere.} \end{cases}$$ Find the PDF of $Y = 8x^3$.

Soln:-

$Y = 8x^3$.

WKT,

$$\boxed{f(y) = f(x) \cdot \left|\frac{dx}{dy}\right|} \to ①$$

$$y = 8x^3$$

$$\frac{y}{8} = x^3$$

$$\left(\frac{y}{8}\right)^{1/3} = (x^3)^{1/3}$$

$$\boxed{\frac{1}{2} y^{1/3} = x}$$

$$\frac{dx}{dy} = \frac{1}{2} \cdot \frac{1}{3} y^{-2/3}$$

$$\left|\frac{dx}{dy}\right| = \frac{1}{6} y^{-2/3}$$

$$① \Rightarrow f(y) = 2x \cdot \frac{1}{6} y^{-2/3}$$

$$= 2 \cdot \frac{1}{2} y^{1/3} \cdot \frac{1}{6} y^{-2/3}$$

$$f(y) = \frac{1}{6} y^{-1/3} ; \quad 0 < y < 8 \, //$$

(when $x = 0$ then $y = 8 \cdot 0^3 = 0$
$x = 1$ then $y = 8 \cdot 1^3 = 8$)

2) If the Continuous Random Variable X has the PDF

$$f(x) = \begin{cases} \frac{2}{9}(x+1) ; & -1 < x < 2 \\ 0 ; & \text{elsewhere} \end{cases}$$

$x = \sqrt{y}$
$x = y^{1/2}$ $-1/2$
$= 1/2 \; y$
$= \frac{1}{2\sqrt{y}}$

Then, find the PDF of $Y = X^2$.

Soln:- $Y = X^2 \Rightarrow X = \sqrt{Y}$

~~wkt~~
The transformation of $Y = x^2$ is not monotonic in $(-1, 2)$.
So we divide the interval in to two parts $(-1,1)$ & $(1,2)$. Since $(-1,1)$ is a symmetrical interval $\therefore$ Y lies b/w ~~abd~~ $(0,1)$.

Therefore, when $-1 < x < 1$ i.e., $0 < y < 1$

Taking average.

→ Taking average Since the given interval is b/w $(-)$ve & $(+)$ve.

$$f(y) = \frac{1}{2\sqrt{y}}\left(f(\sqrt{y}) + f(-\sqrt{y})\right)$$

$$f(y) = \frac{1}{2\sqrt{y}}\left(\frac{2}{9}(\sqrt{y}+1) + \frac{2}{9}(-\sqrt{y}+1)\right)$$

$$= \frac{1}{2\sqrt{y}}\left(\frac{2}{9}(\sqrt{y}+1-\sqrt{y}+1)\right)$$

$$= \frac{2}{9\sqrt{y}} ; \quad 0 < y < 1.$$

and $1 < x < 2$ i.e., $1 < y < 4$

$$f(y) = f(x)\left|\frac{dx}{dy}\right|$$

$$= \frac{2}{9}(x+1) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{2}{9}(\sqrt{y}+1) \cdot \frac{1}{2\sqrt{y}}$$

$$= \frac{\sqrt{y}+1}{9\sqrt{y}} = \frac{1}{9}\left(\frac{\sqrt{y}}{\sqrt{y}} + \frac{1}{\sqrt{y}}\right)$$

$$\boxed{f(y) = \frac{1}{9}\left(1 + \frac{1}{\sqrt{y}}\right)}$$

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Problems:-

1) Find the value of k for the PDF $f(x) = \begin{cases} kx(2-x) & ; 0<x<2 \\ 0 & ; \text{otherwise} \end{cases}$

and hence find its mean & variance.

Soln:-

WKT

The total Prob is,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_0^2 kx(2-x)\,dx = 1$$

$$k\int_0^2 (2x-x^2)\,dx = 1$$

$$k\left(\frac{2x^2}{2} - \frac{x^3}{3}\right)_0^2 = 1$$

$$k\left(4-\frac{8}{3}\right) = 1$$

$$k\left(\frac{4}{3}\right) = 1$$

$$\boxed{\therefore k = \frac{3}{4}}$$

$$\boxed{\text{Mean} = E(x)} = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_0^2 x\,kx(2-x)\,dx$$

$$E(x) = \frac{3}{4}\int_0^2 (2x^2 - x^3)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^3}{3} - \frac{x^4}{4}\right]_0^2 = \frac{3}{4}\left[\frac{2(8)}{3} - \frac{16}{4}\right]$$

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$$= \frac{3}{4}\left[\frac{16}{3} - \frac{16}{4}\right] = \frac{3}{4}\left[\frac{64-48}{12}\right]$$

$$= \frac{\cancel{3}}{\cancel{4}}\left[\frac{\cancel{16}\,4}{\cancel{12}\,4}\right] = 1$$

$\boxed{\therefore \text{Mean} = 1}$

16
3
48
64
48
16

wkt,

Variance : ~~E(x) - E(x^2)~~ ~~E(x^2) - (1~~ wkt,

$= 1 \; \cancel{E(x^2)}$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx = \int_0^2 x^2 k x(2-x)\,dx$$

$$= k\int_0^2 x^3(2-x)\,dx = \frac{3}{4}\int_0^2 (2x^3 - x^4)\,dx$$

$$= \frac{3}{4}\left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 = \frac{3}{4}\left[\frac{2\times 16}{4} - \frac{32}{5}\right]$$

$$= \frac{3}{4}\left[8 - \frac{32}{5}\right] = \frac{3}{4}\left[\frac{40-32}{5}\right] = \frac{3}{\cancel{4}}\left[\frac{\cancel{8}\,2}{5}\right]$$

$$= \frac{6}{5}$$

Variance $= \cancel{1 - \frac{6}{5} = \frac{5-6}{5} = -\frac{1}{5}}$

wkt

$\boxed{Var(x) = E(x^2) - (E(x))^2}$

$$= \frac{6}{5} - 1$$

$\boxed{Var(x) = \frac{1}{5}}$

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2) A Random Variable X has a PDF $f(x) = \begin{cases} \frac{k}{1+x^2}, & -\infty < x < \infty. \end{cases}$

Find i) k value, ii) Distribution function of x.

Soln:-

WKT,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} \frac{k}{1+x^2}\,dx = 1$$

$$k\left(\tan^{-1} x\right)_{-\infty}^{\infty} = 1$$

$$k\left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right) = 1 \qquad \left(\because \tan^{-1}(-x) = -\tan^{-1}x\right)$$

$$k\left(2\tan^{-1}\infty\right) = 1$$

$$k\left(\cancel{2} \times \pi/\cancel{2}\right) = 1$$

$$k\pi = 1$$

$$\boxed{k = 1/\pi}$$

ii) Distributive function is,

$$F(x) = \int_{-\infty}^{x} f(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{k}{(1+x^2)}\,dx = \frac{1}{\pi}\int_{-\infty}^{x} \frac{dx}{1+x^2} = \frac{1}{\pi}\left(\tan^{-1}x\right)_{-\infty}^{x}$$

$$= \frac{1}{\pi}\left(\tan^{-1}x - \tan^{-1}(-\infty)\right)$$

$$= \frac{1}{\pi}\left(\tan^{-1}x + \tan^{-1}\infty\right)$$

$$\boxed{F(x) = \frac{1}{\pi}\left(\tan^{-1}x + \frac{\pi}{2}\right)}$$

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3) A Random Variable X has the PDF $f(x) = \begin{cases} \lambda x e^{-x}, & x \geq 0 \\ 0 ; & x < 0. \end{cases}$

Find i) λ value, ii) Distribution function, iii) mean & variance.

Soln:-

WKT

i) $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$\int_{0}^{\infty} \lambda x e^{-x} = 1$

$\lambda \int_{0}^{\infty} x e^{-x} = 1$

$\lambda \left[x e^{-x} - 1 \times \frac{e^{-x}}{-1} \right]_0^{\infty} = 1$

$\lambda \left[-x e^{-x} + e^{-x} \right]_0^{\infty} = 1$

$\lambda \left(e^{-x}(1-x) \right]_0^{\infty} = 1$

$\lambda \left[e^{-\infty}(1-\infty) - e^{-0}(1-0) \right] = 1$

$\lambda (0 - 1 \times 1) = 1$

$\lambda(-1) = 1$

$\boxed{\lambda = -1}$

Bernoulli's formula,

$\int \cancel{u\,dv = uv - u'v_1 + u''v_2}$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

u - diff, v - integration.

Bernoulli's formula:

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 + \ldots$

ii) $F(x) = \lambda \int_{-\infty}^{x} x e^{-x}\,dx = \lambda \left[e^{-x}(1-x) \right]_{\infty}^{x} = 1 \left[e^{-x}(1-x) - (0) \right]$

$= 1\, e^{-x}(1-x)$

$\boxed{F(x) = e^{-x}(1-x)}$

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iii) mean = $E(x)$. $= \int_{\infty}^{0} x f(x)$

$= \cancel{\lambda \int_{\infty}^{0} x\, (xe^{-x}}$

$= \lambda \int_{\infty}^{0} x(xe^{-x})\,dx$

$= \lambda \int_{\infty}^{0} x^2 e^{-x}\,dx$

$= -1\left[x^2 e^{-x} - 2x\frac{e^{-x}}{-1} + 2\frac{e^{-x}}{-1}\right]_{\infty}^{0}$

$= \cancel{-1[0-0+2(0)]}$

$= 0 \; -1[0-0-2] - (0-0-0]$

$= 2$ //

$x^3e^{-x} = x^3e^{-x} - \left(3x^2\frac{e^{-x}}{-1}\right) + \left(6x\frac{e^{-x}}{-1}\right) - \left(6\frac{e^{-x}}{-1}\right)$

$= x^3e^{-x} + 3x^2e^{-x} - 6xe^{-x} + 6e^{-x}$

Variance = $E(x^2) = \int_{\infty}^{0} x^2 f(x) = \lambda \int_{\infty}^{0} x^2(xe^{-x})\,dx$

$= -1\int_{\infty}^{0} x^2(xe^{-x})\,dx$

$= -1\int_{\infty}^{0} x^3 e^{-x}\,dx$

$= -1\left[x^3e^{-x} + 3x^2e^{-x} - 6xe^{-x} + 6e^{-x}\right]_{\infty}^{0}$

$= -1[6] \quad = -6$ //

Variance = 6 //

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4) A Discrete Random Variable X has a Probability function

x: 0 1 2 3 4 5 6 7 8

$p(x)$: a 3a 5a 7a 9a 11a 13a 15a 17a.

Find i) value of a, ii) $P(x<3)$, $P(x\geq 3)$, iii) Distribution function.

Soln:- $\sum P(x) = 1$

$a+3a+5a+7a+9a+11a+13a+15a+17a = 1$

$81a = 1$

$\boxed{a = 1/81}$

ii) $P(x<3) = \cancel{a}\, P(x=0) + P(x=1) + P(x=2)$

$= a+3a+5a$

$= 9a$

$P(x\geq 3) = P(x=3)+P(x=4)+P(x=5)+P(x=6)+P(x=7)+P(x=8)$

$= 7a+9a+11a+13a+15a+17a$

$= \cancel{\frac{72}{8}}\ 72a$

$= \frac{72}{81} = \frac{8}{9}$ //.

~~iii) $\sum_{1}^{x} xp(x)$~~

iii)

x	p(x)	C.D.F (F(x))
0	a = 1/81	1/81
1	3a = 3/81	1/81+3/81 = 4/81
2	5a = 5/81	4/81+5/81 = 9/81
3	7a = 7/81	9/81+7/81 = 16/81
4	9a = 9/81	16/81+9/81 = 25/81
5	11a = 11/81	25/81+11/81 = 36/81
6	13a = 13/81	13/81 + 36/81 = 49/81
7	15a = 15/81	49/81 + 15/81 = 64/81
8	17a = 17/81	$\frac{64}{81}+\frac{17}{81} = \frac{81}{81} = 1$

$$\begin{array}{r} 36 \\ +13 \\ \hline 49 \\ +15 \\ \hline 64 \\ 17 \\ \hline 81 \end{array}$$

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