OPTION PRICING USING MONTE CARLO SIMULATIONS

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Abstract:

This project is about valuing the stock options using the Black-Scholes method, Monte Carlo simulation method and its variance reduction techniques. The focus of this project would be to compare the Numerical values obtained with the actual values in the market for the stock indices. Options are financial instruments that play an important role in the finance industry, they are mainly used in the hedging, speculation and arbitraging. There has been always a need to value these options as they are widely used in investing. Options were considered as an obscure financial instrument due to the lack of valuation techniques in the past. This obscurity was broken when Black – Scholes devised a model in 1973 which could value these options and this escalated the options trading. In this project the sample market prices of the market will be considered for the valuation. The method that will be used widely in the project would be the Monte Carlo Simulation techniques. So, the valuation of the option would be done using the Black-Scholes formula with the help of the Monte Carlo simulation. Therefore, the main goal of this study is how can Monte Carlo and the variance reduction technique be applied to finance?

Table of Contents

1. Introduction:

Derivatives in general can be called as the risk management instruments which derives its value from its underlying assets. Here the underlying assets include stocks, currencies, commodities, bonds etc... They use these instruments to prevent themselves from the uprising uncertainties that can happen in the markets due to fluctuations in the asset prices. Most used examples of derivatives are Futures, Options, Forwards, Swaps and Exotics. Derivatives are traded in different ways in different countries, one such exchange is the Chicago Board Options Exchange. The trading of options also happens on OTC which is known as Over The Counter market, where trading of contracts happen between just two parties, without an intermediate party. There is always a risk involved in trading, something which helps the traders to minimize the risk is called hedging. Hedging helps in reducing the uncertainty risk that happens during the fluctuations of price of an asset. This strategy of reducing the risk is not only used by the individual investors but also by large corporations and portfolio managers. In simple words we could say hedging as securing the investments A's loss by offsetting it with investment B's profit. Options trading is one of the key derivatives that are used in hedging.

1.1 Options:

An option can be defined as one of the most common form of derivative, as an option always derive its value from its underlying asset. In fact, most exchange-traded options have stocks as their underlying asset, but they could also have any other type of security or commodity such as indexes, interest rates, bonds, currencies, swaps, baskets of assets. Technically, it is possible to place an option on anything that can be purchased: coffee, cocoa, gold, silver, oil, gas, a plane, watch and so on. People who buy options are called holders and people who sell options are called writers. Holding the option gives the buyer the right to do something with it, but it does not obligate the buyer (holder) to exercise the option. The buyer of the option has the right, but not the obligation to exercise the option at the expiration date.

Options are mainly carried out in two forms, one is the currency option and the other one is index options. An index option is a financial derivative that gives the holder the right, but not the obligation, to buy or sell the value of an underlying index, such as the Standard and Poor's (S&P) 500, at the stated exercise price on or before the expiration date of the option. No actual stocks are bought or sold; index options are always cash-settled. A key difference between the equity options and index options is that the index options exposes the investors to the market, to take one trading decision. Some examples of the index options market are S&P 500, Dow Jones Industrial Average, NSE. Unlike the stocks that investors can hold forever, the options have an expiration cycle defined in the binding contract. Below is a graph which shows us the volume of trading that is happening in the options of S&P 500. The daily average volume of trading on the index options seems to only increases every year and it looks tremendous.

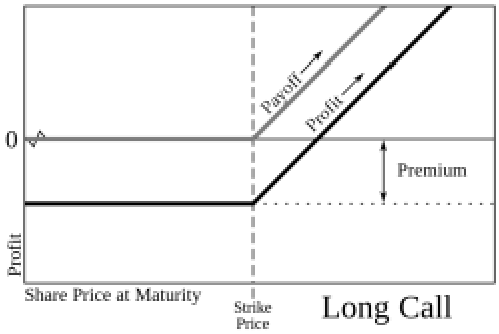
It should be emphasized that an option gives the holder the right to buy or sell, but not the obligation. The holder may opt not to exercise the right. This is what forms the main difference between options and forwards and futures, where the holder is obligated to buy or sell the underlying asset. It costs nothing to enter into a forward or futures contract, whereas there is a cost to acquiring an option. Index options are mostly European style options which means we can settle it only with cash on the expiration date.

Long and Short in Trading:

In trading terms there is quite a difference between the terms used, like the difference between “sell” and “short”, when a trader says he is going to short a stock means he is going to sell the stock even before owning the share. When a stock is short, the profit is limited, whereas when it is a long the profit is unlimited. It is important to notice that there should not be any confusions between these terms when it comes to the trading.

1.2 Call Options:

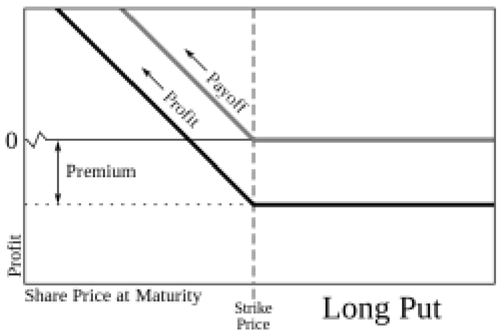
A call option gives the holder (buyer/ one who is long call), the right to buy specified quantity of the underlying asset at the strike price on or before expiration date in case of American option. The seller (one who is short call) however, has the obligation to sell the underlying asset if the buyer of the call option decides to exercise his option to buy.



The above diagram represents the payoff and the profit of the two types of the European option relative to the price of the underlying asset at maturity. A call option can be exercised if the asset price is greater than the strike price of the option. Otherwise, the asset can be acquired in the market at a lower price. Even in the case where the asset price is greater than the strike price, the option may not yield positive profit because of the premium paid to buy the option.

1.3 Put Options:

A Put option gives the holder (buyer/ one who is long Put), the right to sell specified quantity of the underlying asset at the strike price on or before an expiry date in case of American option. The seller of the put option (one who is short Put) however, has the obligation to buy the underlying asset at the strike price if the buyer decides to exercise his option to sell.



The above diagram shows an opposite behaviour from a call option. Giving the owner the right to sell the asset, this option yields higher profit while the market price is less than the strike price.

When it comes to the options trading we need to understand about three main concepts to make money, and they are In the money, At the money and Out of the money. These three concepts can be explained in the form of a simple table shown below.

	Call Options	Put Options
In the money	Strike Price< Spot Price of underlying asset	Strike Price > Spot Price of underlying asset
At the money	Strike Price = Spot Price of underlying asset	Strike Price = Spot Price of underlying asset
Out of the money	Strike Price > Spot Price of underlying asset	Strike Price< Spot Price of underlying asset

1.4 European Options:

A European option may be exercised only at the expiration date of the option, i.e. at a single pre-defined point in time. The payoff is same as the American options like max{(S-K),0} for a call option and max {(K- S,0} for the put option where K is the strike price and S is the spot price. Indexes are generally represented by the European options. However, European options are generally easier to analyse than American options, and some of the properties of an American option are frequently deduced from those of its European counterpart.

1.4.1 Example:

For example, an investor gets a July call option on stock XYZ with a $50 strike price. At expiration, the spot price of stock XYZ is $75. In this case, the owner of the call option has the right to purchase the stock at $50 and exercises the option, making $25, or ($75 - $50), per share. However, in this scenario, if the spot price of stock XYZ is $30 at expiration, it does not make sense to exercise the option to purchase the stock at $50 when the same stock could be purchased in the spot market for $30. In this case, the payoff is $0. Note the payoff and profit are different. To calculate the profit from the option, the cost of the contract must be subtracted from the payoff. In this sense, the most an investor in the option can lose is the premium price paid for the option.

1.5 American Options:

The American options are the ones which can be exercised anytime during its lifetime. Because of this feature, it makes the options superior to the European options. The value of the option if it is exercised is given by max(0, S – K) if it is a call and max(0, K-S), if it is a put. However, in most cases time premium associated with the remaining life of an option makes early exercise sub-optimal.

1.5.1 Example:

The same example as the European options can be used for the American options as well, except that there is no time limit for this.
There are also other less used options such as Bermuda options, Canary options, Verde options, Swing options etc.

1.6 Currency Options:

Currency options are widely available, they are traded in various ways. A currency option is a contract that grants the buyer the right, but not the obligation, to buy or sell a specified currency at a specified exchange rate on or before a specified date. For this right, a premium is paid to the seller, the amount of which varies depending on the number of contracts if the option is bought on an exchange, or on the nominal amount of the option if it is done on the over-the-counter market. Currency options are one of the most common ways for corporations, individuals or financial institutions to hedge against adverse movements in exchange rates. The risk factors one should consider while trading in the currency options are the prevailing spot rate, interbank deposit rates for each of the currencies, and the current implied volatility level for the expiration date.

In contrast to forwards, (futures) contracts, an option is the right, but not the obligation, to buy (or sell) an asset under specified terms. In the currency market (the Forex market for example) it means the right to buy (or sell) some amount of foreign currency at a fixed moment of time in future at a previously stipulated exchange rate. A foreign exchange option, (commonly shortened to just FX option), is a derivative where the owner has the right but not the obligation to exchange money denominated in one currency into another currency at a pre-agreed exchange rate on a specified date.

1.6.1 Example:

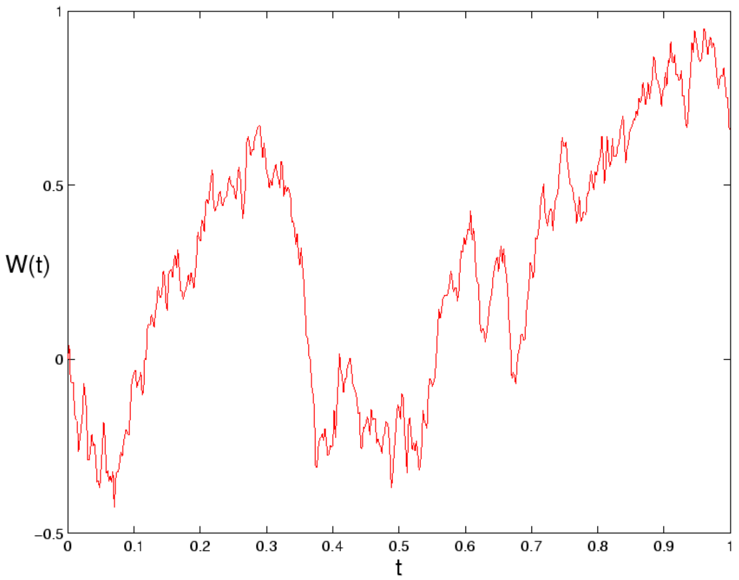
The call and put options in the currency trading can be explained as below. The investor who buys a call option anticipates that asset price will increase, for example, in the Forex market that the EURO/USD exchange rate will go up, so that at the expiration date he/she can buy the asset at the strike price K and sell it at the spot price S. His/her gain is then max(0,S-K). If the spot price decreases to less than the strike price K, the option is not exercised. Whereas, the person who buys a put option expects that asset price will decrease (or for example in Forex market EURO/USD exchange rate goes down), so that at the expiration date he/she can sell the asset at the strike price K and buy it at the spot price S. His/her gain is then max(0,K-S). If the spot price increases to more than the strike price K, the option is not exercised.

1.7 Weiner Process or Geometric Brownian Motion:

One of the most common models in finance is the geometric Brownian motion (GBM), which is also called as the Weiner process. A Weiner process can be called a stochastic process $\{W_t\}_{t\geq 0+}$ indexed by non-negative real numbers t with the following properties.

I. $W_0 = 0$
II. With the probability 1, the function $t \rightarrow W_t$ is continuous on t
III. The process $\{W_t\}_{t\geq 0}$ has stationary, independent increments
IV. The increment $W_{t+s} - W_s$ has the normal (0, t) distribution

To put it in simple words, a Weiner process can be said as a continuous stochastic process which can be further understood clearly by the image shown below.



A stochastic process is St is said to follow a Geometric Brownian Motion if it satisfies the following stochastic differential equation.

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

Where μ is the measure of the expected growth rate of the stock price and σdW describes the stochastic change in the stock price. Here dW_t stands for the below equation

$\Delta W = W(t + \Delta t) - W(t)$ as $\Delta t \to 0$.

W_t is the Weiner process and σ is the volatility.

2. Main Components

2.1 Valuation of Options:

The concept of valuing the options have always been an interesting concept in the field of finance. One of the most important question that arises in the minds of traders is that at what price one should buy or sell their option contracts. Both the call and put options are valuable only before they expire on their maturity date, after their maturity date, their value will become zero. However, the options will fluctuate a lot before their maturity date and this fluctuation will depend on the underlying asset price which changes dynamically according to the market sentiments. This is where the importance of valuing the option comes to play for the traders, and in 1973 Black and Scholes devised a model to evaluate the price of an option. However, the model generated inaccuracies over time to time. Though it has its own flaws it is one of the methods used by the traders for option pricing.

2.2 Determinants of Option value

Value of underlying asset – We can say that as this value increases, the right to buy at a fixed price (calls) will also become more valuable and the right to sell at a fixed price (puts) will become less valuable.

Variance in that value – As the variance increases, both calls and puts will become more valuable.

Expected Dividends – the dividends can reduce the value of the calls but it can increase the value of puts.

Life of the option – The longer the life of an option, more the benefits the trader would get.

Risk free interest rate - The total number of options contracts outstanding in the market at any given point of time.

2.3 Put - Call parity:

This is one of the most important concepts when it comes to the European put and call options that have the same strike price and the time to mature. In mathematical terms we could say the below relationship as the put-call parity.

$$c + Ke^{-rT} == \mathrm{p} + \mathrm{S}_0$$

Where,

c → Call premium

K → Strike price of call and put

P → Put premium

r → Annual Interest Rate

T → Time in years

S_0 → Initial pricing of the underlying asset

The above equation shows us that the value of a European call with a certain exercise price and exercise date can be deduced from the value of a European put with the same exercise price and exercise date, and vice versa. The put-call parity establishes the relationship between put and call prices of a share, with the same strike price and same maturity. From the put prices, the call prices can be deduced, and vice versa. It can be further explained by the below two equations,

$$c = p + S_0 - Xe^{-rT} \qquad (1)$$

and

$$p = c - S_0 + Xe^{-rT} \qquad (2)$$

Also, to be noted that this put-call parity does not apply to the American options as they can be exercised anytime during the contract. If the options and stock positions do not have the same value, then the opportunities for arbitrage arise, in which the investors could make profitable and risk-free trades until put-call parity returns.

2.3.1 Example:

Let’s consider the following information

S_0 = 100; K = 110, Risk free rate = 5%, Prevailing price of the put option = 6,

Expiry time = 1 year

Applying the values of the information in our equation (1), the price of the call option should not be more than or less than 1.36, this value was obtained by the below calculation.

$$\left(6 + 100 - \frac{110}{e^{0.05}}\right) = 1.36$$

So, this says us that at exactly 1.36 we should buy a call option and short the put option and short the share as at 1.36 it creates a situation of no profit/ no loss. In a similar way we could arrive at the upfront payment that must be paid as

$$1.36 - 100 + 6 = 104.64$$

If this amount is invested at a risk-free rate of 5%, it will exactly come to the value of 110 which is our K and the price of the share that is locked. So, if the value of the call is less than or equal to 1.36 we can make a risk-free profit, else we can short the call and long the put and long the share to earn a risk-free profit.

2.4 Upper and Lower bounds of an Option:

One of the most important concepts in option valuation is the upper and the lower bounds of the options. They can be said as the maximum and the minimum value that the options can attain during its time period. In the language of options, the maximum limit to which the options can go is called the Upper bound and the minimum limit to which the options can go can be said as the Lower bounds.

The put-call parity shows us that the European call option can be found from the value of put option with the same strike price and maturity.

$$c = p + S_0 - Xe^{-rT}$$

If $p \geq 0$, then $c \geq S_0 - Xe^{-rT}$ where we can say that $S_0 - Xe^{-rT}$ is the lower bound of the call option. The same can be written as below

$$S_0 - Xe^{-rT} \leq c \leq S_0$$

If there is no put call parity between the call and put options then there will arise an opportunity of arbitrage.

2.4.1 Example:

Let's consider a lower bound for 6 months European call option with strike price as $35 and the initial stock price as $40 and the risk-free interest rate as 5%/annum. Find the lower bound of the call option.

Now according to the problem, we have the following data $S_0 = 40$, $X = 35$, $T = 0.5$, and $r = 0.05$

The lower bound of the call option can be calculated as follows

$$S_0 - Xe^{-rT} \leq c$$

$$40 - 35\, e^{-0.05*0.5} = 5.864$$

So, the lower bound of the European call option will be 5.864.

2.5 Arbitrage free pricing model:

No arbitrage principle states that there are no opportunities to make a risk-free profit. This is one of the key concepts in financial mathematics when it comes to derivative pricing methods. During the modelling we always assume that there is no arbitrage possible in the market. The no-arbitrage assumption implies no risk-free profit, which in turn, implies that a risk-less portfolio has no more than the risk-free rate of return. It also implies that two portfolios have the same present value if they are exposed to the same sources of risk; this is also known as the law of one price.

An important breakthrough in this regard was made by Black & Scholes (1973), where using the no-arbitrage principle they derive a partial differential equation (PDE) that helps price certain generic options. Since then, this principle has spurred enormous research in determining the prices of financial securities. Roughly speaking, the no arbitrage pricing principle states that two portfolios of securities that have the same payoffs in every possible scenario, should have the same price. Otherwise, by buying low and selling high, sure profit, or 'arbitrage', can be created from zero investment.

No-arbitrage pricing requires that the market for the instruments in the replicating portfolio be complete. Also, the assumption is that these instruments can be traded continuously and without frictions (such as transaction costs or taxes). No-arbitrage pricing can be used only where the markets for the underlying assets are complete. Nevertheless, it is still possible to use no-arbitrage pricing for markets that are "nearly" complete, either by assuming away the incompleteness (useful for developed markets with very small transaction costs, for example). No-arbitrage pricing has the benefit of not involving the investor's attitude towards risk.

2.5.1 Example:

Consider a three months European call and put options with the exercise price of $12 are trading at $3 and $6 respectively. Stock price is $8 and interest rate 5%. Check if there is an arbitrage opportunity.

Here we can use the concept of the put-call parity to check if there exists an arbitrage opportunity. So, according to the equation of put-call parity the value of p should be equal to 6 as given in the question. The put-call parity can be checked as below

$$p = c - S_0 + Xe^{-rT}$$

According to the values in the question we can rewrite the above equation as below

$$p = 3 - 8 + 12e^{-(0.05*0.25)}$$

$$p = 6.851 < 6$$

Here 6 is the value of put given in the question, so we could conclude that it violates the put-call parity. The traders could use this as an arbitrage opportunity to gain their profit, so they will

- Buy a put option for $6
- Sell a call option for $3
- Buy a share for $8

2.6 Black -Scholes Theory:

In the real-world scenario, the stock prices fluctuate at every moment. As the number of continuous occurrences become huge, and the price changes become small, the distribution of price changes should be normal. So, we could say that it's better to apply a normal distribution to the stock prices. But, In the case of stock prices, however, normal distribution is not applicable, as a standard normal distribution assumes positive and negative values scattered along a mean of 0. Stock prices can never be negative – hence, stock prices do not follow normal distribution. But, we can say that the changes in stock prices do – hence, stock prices are taken to be normal on the log scale – that is, logs of stock prices are normally distributed.

Considering the above theory and applying the properties of GBM and solving the partial differential equations Fisher Black and Myron Scholes (1973) along with Merton presented a theory for option pricing. The influence of option pricing theory on finance practice has not been limited to plain options, however, the "Black-Scholes" methodology has played a fundamental role in supporting the development of new financial instruments around the globe. The derivation of the Black-Scholes-Merton pricing formula is based on the following assumptions:

- The market is perfect - there are no transaction costs or taxes. Trading takes place continuously. Borrowing and short-selling are with no restriction.
- The underlying asset follows a stochastic differential equation in the form of a geometric Brownian motion,

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t$$

Where α is the expected rate of return on the asset which is constant, σ is the volatility which is a constant as well, and dW_t is the increments of a standard Brownian motion.

- The risk-free rate of interest is constant over time and the options are European.

So, Black and Scholes arrived at a partial differential equation as below

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial S} rS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}\sigma^2 S^2 - rV = 0,$$

Solving all these equations they finally arrived at a formula which can find the value of put and call options as given below. However, the formula is applicable to European call and put options on non-dividend paying stocks.

$$c = S_0 N(d_1) - Xe^{-rT} N(d_2)$$

$$p = Xe^{-rT} N(-d_2) - S_0 N(-d_1) \qquad \text{Where}$$

$$d_1 = \frac{\ln\left(\frac{S_0}{X}\right) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{(T)}$$

Here σ is the standard deviation in the natural logs of the asset values.

Assumptions of Black Scholes Method:

- Prices are Log-normally distributed.
- No dividend till expiration of call option.
- Applies to European style options only.

The two parts of the Black Scholes equation relates to the hedge ratio, and the discounted value of the expected pay-off from the option where the prevailing price is less than the strike price. The first part of the equation which has the $N(d_1)$ represents the number of shares that needs to be held long to create the riskless portfolio. The second part $N(d_2)$ is the amount of money that needs to be in hand today to create a riskless position. The difference between the two values is the value of the call option and with different signs for the put option.

We can get a better estimate of valuing the options if we use some kind of a simulation technique, one of the most famous technique is the Monte Carlo simulation method, which works based on repeated computation and random sampling.

Hence, the idea of using the Monte Carlo simulation technique seemed to be better to do the option pricing.

2.7 Numerical Techniques:

Numerical techniques are practical methods that are used by both academic researchers and market professionals. Depending on the problem at hand, one method may be more convenient or computationally cheaper to use than another. Some well-known numerical procedures are *finite-difference methods, binomial and trinomial trees, quadrature methods, and Monte Carlo simulations.*

2.8 Monte Carlo Simulation:

Monte Carlo Option Price is a method often used in Mathematical finance to calculate the value of an option with multiple sources of uncertain-ties and random features, such as changing interest rates, stock prices or exchange rates, etc. This method is called Monte Carlo simulation, naming after the city of Monte Carlo, which is noted for its casinos. Monte Carlo (MC) simulation is considered as one of the robust and widely used method to price derivatives and estimate the risk of a portfolio

The Monte Carlo method is one of the statistical simulation method which uses random numbers to perform simulation calculations. It is commonly used in estimating multi-dimensional integrations because of its advantage when dealing with high dimensional problems, including options on multiple assets, asset processes with jumps, stochastic interest rates or stochastic volatilities.

Options can be priced by Monte Carlo simulation. First, the price of the underlying asset is simulated by random number generation for several paths. Then, the value of the option is found by calculating the average of discounted returns over all paths. Since the option is priced under risk-neutral measure, the discount rate is the risk-free interest rate.

However, the desired result is taken as an average over many observations. This highlights a weakness of the Monte Carlo method, namely the low convergence rate, though different variance reduction techniques can help mitigate the problem, although considerably increases computational cost.

The simulation consists of three main steps. At first, we use the Geometric Brownian Motion defined by equation, and we will calculate the stock price at maturity. Second, we calculate the payoff of the option based on the stock price and finally we discount the payoff at the risk-free rate to today's price. Repeating the above procedure for a reasonable number of times, gives a good estimate of the average payoff and the price of the option.

The advantage of the Monte Carlo simulation method is to deal with path dependent options. The superiority of the Monte Carlo simulation method is that it can simulate the underlying asset price path by path, calculate the payoff associated with the information for each simulated path and utilize the average discounted payoff to approximate the expected discounted payoff, which is the value of path-dependent options.

Though there is this advantage of the Monte Carlo simulation method, it also causes the difficulty to apply this method to price American options. This is because it is difficult to derive the holding value (or the continuation value) at any time point t based on one single subsequent path. This is one of the reasons why Monte Carlo method cannot easily handle situations where there are early exercise opportunities, as in the case of American options. Though it has its own drawbacks, it can be applied to model the European put, call options and currencies with the right computational power at hand.

2.8.1 Standard Monte Carlo method:

This model approximates the option price as following

$$c = E[e^{-rT}(S(T) - K)^+]$$

$$p = E\left[e^{-rT}\big(K - S(T)\big)^+\right]$$

Where c and p represents European call, and put options. The factor e^{-rT} is for discounting the price for time 0 instead of T. Since the S evolves according the Geometric Brownian Motion, the methodology will be as follows.

Methodology for Monte Carlo Simulations (call option):

Step 1: Generate m independent standard normal random variables like $z_1, z_2, \dots, z_m \sim N(0,1)$

Step 2: Let S be defined as $S(T) = S(0)e^{\left(\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\sqrt{T}\,Z_i\right)}$, for each Z_i

Step 3: Let $c_i = e^{-rT}(S(T) - K)^+$, for each Z_i

Step 4: Set $c \approx (x_m) = \ 1/m \ \sum_{i=1}^{m} X_i$

2.8.1.1 Example:

Let's calculate the value of an European call option using the Monte Carlo method. We have two important formulae, the first formula helps us to calculate the asset price at the maturity, and the second one will help us find the value of the option at the maturity. The formula was already mentioned, repeating it for further clear reference.

$$S(T) = S(0)\exp[\left(r - \frac{\sigma^2}{2}\right)T + \sigma W_T]$$

$$= S(0)\exp[\left(r - \frac{\sigma^2}{2}\right)T + \sigma\epsilon_i\sqrt{T}] \text{ where } \epsilon_i \sim N(0,1) \qquad \text{(i)}$$

$$\text{Value of option at maturity} = \max(S(T) - K, 0) \qquad \text{(ii)}$$

Now the algorithm of Monte Carlo to calculate the value of the option is as below

$$
\begin{aligned}
&\mathit{for}\ i = 1\ \mathit{to}\ M \\
&\qquad \mathit{compute\ an}\ N(0,1)\ \mathit{sample}\ \epsilon_i \\
&\qquad \mathit{set}\ S_i = S_0 e^{[(r-\frac{\sigma^2}{2})T + \sigma\epsilon_i\sqrt{T}]} \\
&\qquad \mathit{set}\ X_i = \max(S_i - K, 0) \\
&\mathit{end} \\
&\mathit{set\ Mean} = \frac{1}{M}\sum_{i=1}^{M} X_i \\
&\mathit{set\ Estimate\ Of\ Value\ Of\ Option} = e^{-rT} \times \mathit{Mean}
\end{aligned}
$$

By looking at the above algorithm we could say that there is no need to apply Monte Carlo method as the Black-Scholes formula already gives us the exact solution. But there will arise lots of complicated situations where this cannot be applied.

According to the J.Hull the estimated error of Monte Carlo during the value of the option is

$$\frac{\omega}{\sqrt{M}}$$

where ω is the standard deviation of the sample payoffs. This tells us that the error in the value of the option is inversely proportional to the square root of the number of trials done. So, to increase the accuracy of our estimate, we must increase the number of trials by a factor of 4. Thus, we will need a very large value of M to estimate the value of the option to a reasonable accuracy. This is where the time consumption of Monte Carlo method comes, where the computers can take lots of time to find the estimate. There are several variance reduction methods which can considerably reduce the computational costs. To name a few variance reduction techniques, we have the *Antithetic Variate Technique*, *Stratified Sampling*, *Quasi Monte Carlo method*. In this project we will be mainly using the Antithetic Variate Technique.

2.8.2 Antithetic Variate Technique:

The algorithm of this technique looks very similar to the Monte Carlo algorithm except that we estimated the price of an European-style option by generating M independent samples, $X_1, X_2, .., X_M$, of the payoff from the option by using M random $N(0,1)$ samples, $\epsilon_1, \epsilon_2, .., \epsilon_M$, whereas in Antithetic Variable Technique, we will be generating $2M$ samples $X_1, X_2, .., X_M$ and $\overline{X_1}, \overline{X_2}, .., \overline{X_M}$. If X_i is calculated using ϵ_i, then $\overline{X_\iota}$ will be calculated using $-\epsilon_i$. In the normal MC method, we calculated the mean of X_i's but here we will be calculating the mean of $\widehat{X_\iota}$'s, where $\widehat{X_\iota} = \frac{1}{2}(X_i + \overline{X_\iota})$.

2.8.2.1 Algorithm for Antithetic Variance Method:

The below is the algorithm for the variance reduction technique.

$$
\begin{aligned}
&for\ i = 1\ to\ M \\
&\qquad compute\ an\ N(0,1)\ sample\ \epsilon_i \\
&\qquad set\ S_i = S_0 e^{[(r-\frac{\sigma^2}{2})T + \sigma\epsilon_i\sqrt{T}]} \\
&\qquad set\ \overline{S}_i = S_0 e^{[(r-\frac{\sigma^2}{2})T - \sigma\epsilon_i\sqrt{T}]} \\
&\qquad set\ X_i = \max(S_i - K, 0) \\
&\qquad set\ \overline{X}_i = \max(\overline{S}_i - K, 0) \\
&\qquad set\ \widehat{X}_i = \tfrac{1}{2}(X_i + \overline{X}_i) \\
&end \\
&set\ Mean = \tfrac{1}{M}\textstyle\sum_{i=1}^{M} \widehat{X}_i \\
&set\ Estimate\ Value\ Of\ Option = e^{-rT} \times Mean
\end{aligned}
$$

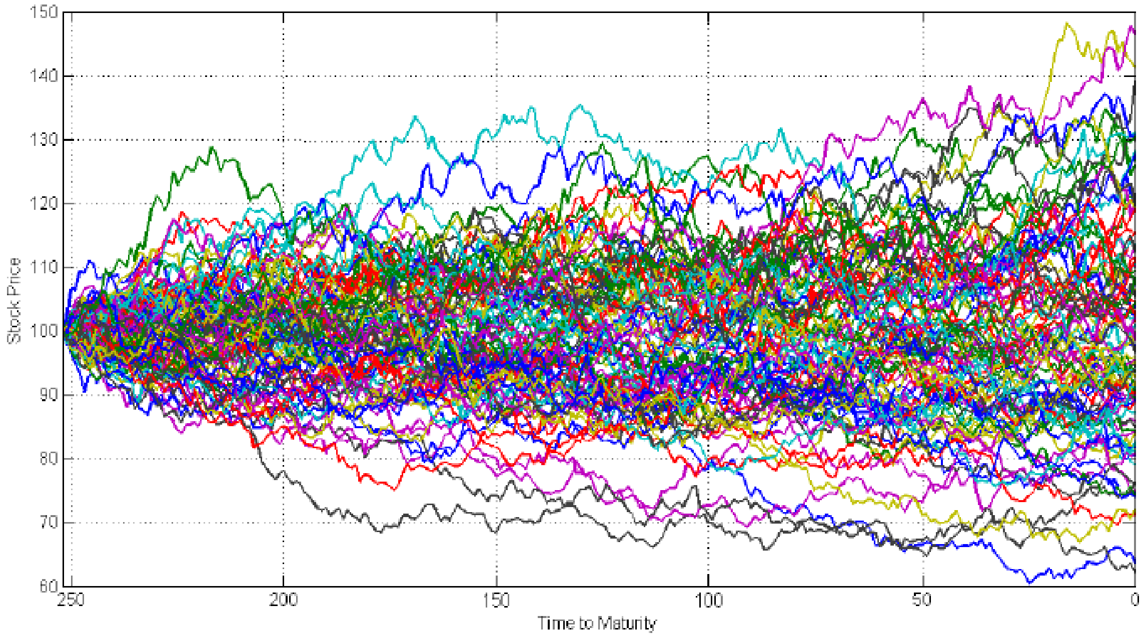
So, to get the value of the option price we should take the mean that is mentioned in the algorithm above. To put the complicated math expressions in simpler terms we could say that the Antithetic Variable Technique use two Monte Carlo simulations and takes the average. It doubles the sample size; it uses the original MC simulation along with its negatively correlated result.

3.Results and Empirical Analysis

3.1 Aim

The aim of this project is to verify the values between the analytical method and the numerical method. Here the analytical method used is the Black Scholes method and the numerical method that is considered is the Monte Carlo simulation. The algorithm and code was done using Python which is nowadays widely used in the finance field. Libraries such as pandas, scipy, numpy helps us during such mathematical computing. The calculations are performed on the European style options where there is no limit for the expiry time. Here, the code has been designed is such a way that the values of the options can be plugged in and the code will generate the value for both the Black Scholes and Monte Carlo simulation model.

A typical Geometric Brownian Motion that comes when a Monte Carlo simulation is used and it looks like the image given below. The image says us that the price of an option can go up till 149 or can go as lower as 60 where the time frame given for the option is taken as 250 months approximately.

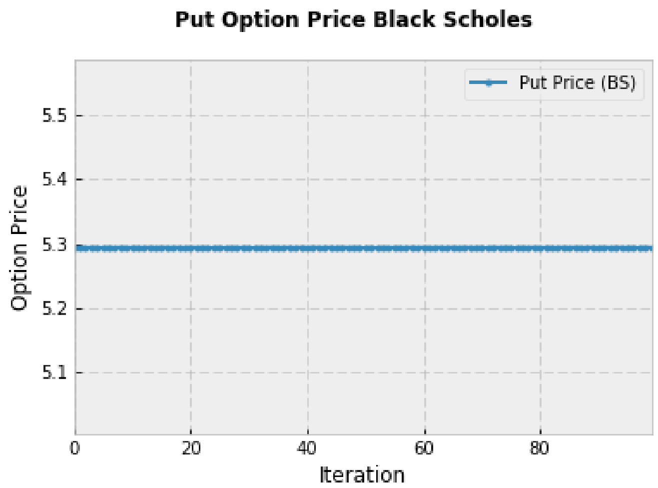


3.2 Valuation of Option using Black Scholes

3.2.1 Put option

Using the python code, we get the following value by inputting the below parameters. The output can be seen in the below image.

```
S0      stock price at time 0: 95
E       strike price: 95
rf      continuously compounded risk-free rate: 0.05
sigma   volatility of the stock price per year: 0.2
T       time to maturity in trading years: 1
```

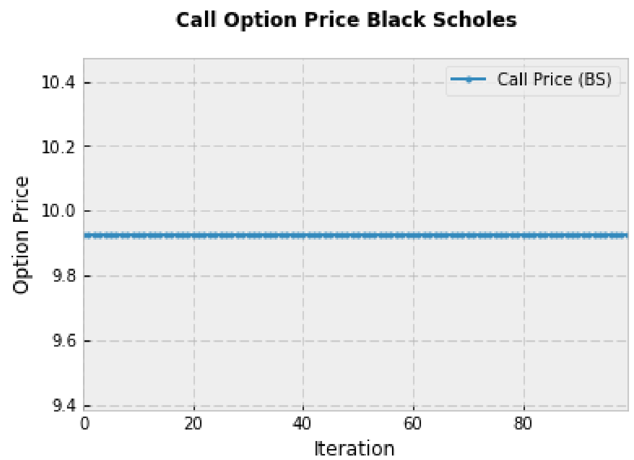


So, by using the Black Scholes formula we get a value of 5.3 for the put option.

3.2.2 Call option

Now, for the same parameters we can calculate the call option price.

```
S0      stock price at time 0: 95
E       strike price: 95
rf      continuously compounded risk-free rate: 0.05
sigma   volatility of the stock price per year: 0.2
T       time to maturity in trading years: 1
```



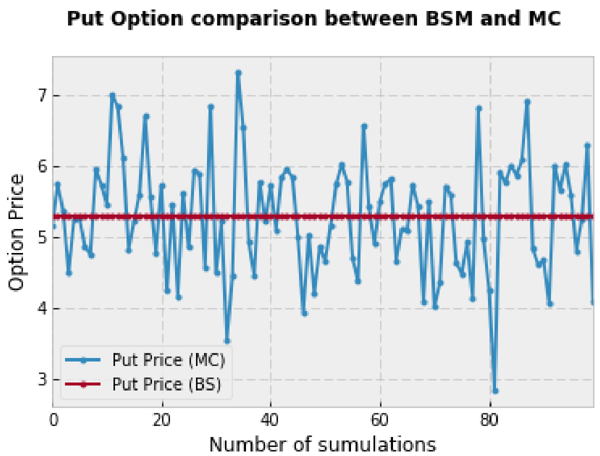
By inputting the above parameters, we get a value of 9.93 using the Black Scholes formula.

3.3 Comparison of Option prices using Monte Carlo Simulation and Black-Scholes

3.3.1 Put Options

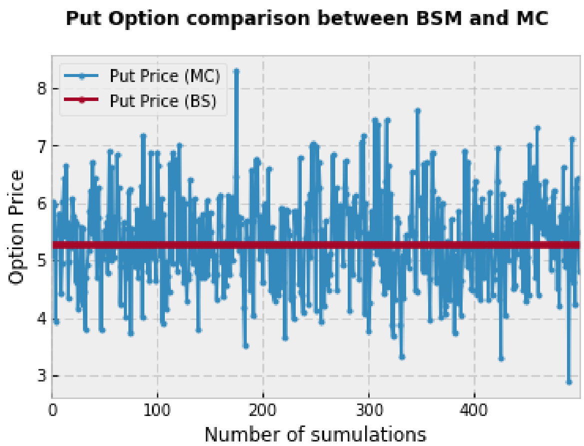
Below is the calculation of the same option price using the Monte Carlo simulation, we get a closer value of the option by increasing the number of simulations in the code. The images below will give better comparison on the analytical and numerical calculation.

```
Sim      number of simulations: 100
```



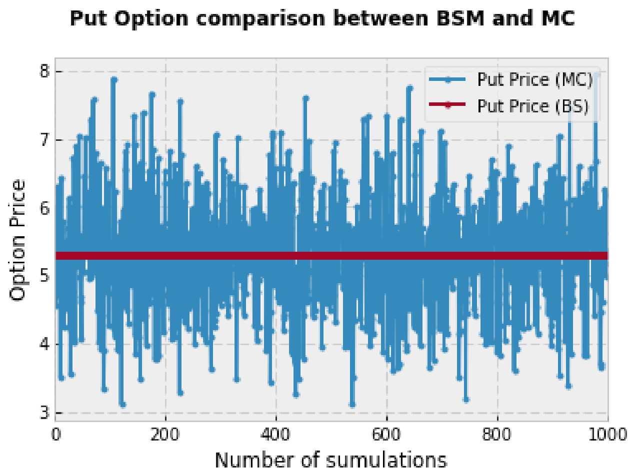

```
Time taken to compute the code in seconds:
 0.5931167808987539
Standard Error Between MC Sim and BS Model:
  0.0579424406141
```

```
Sim      number of simulations: 500
```



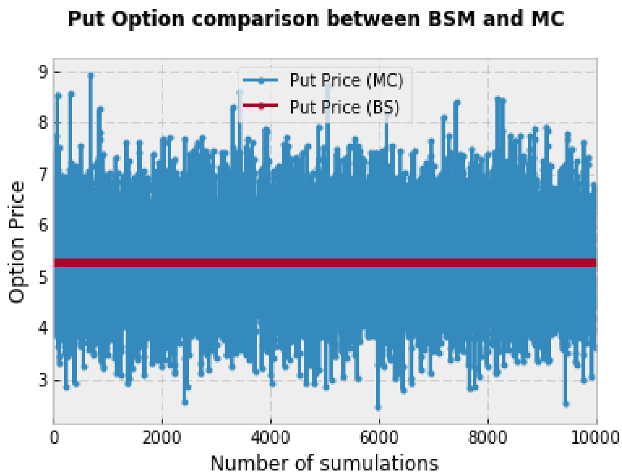

```
Time taken to compute the code in seconds:
 1.2264124153516605
Standard Error Between MC Sim and BS Model:
  0.0258669559337
```

```
Sim      number of simulations: 1000
```




```
Time taken to compute the code in seconds:
 1.8638396063124674
Standard Error Between MC Sim and BS Model:
  0.0187284693736
```

```
Sim      number of simulations: 10000
```

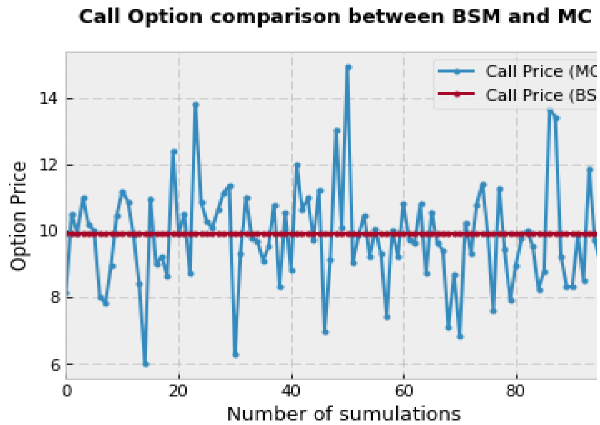



```
Time taken to compute the code in seconds:
 14.3293500554405
Standard Error Between MC Sim and BS Model:
  0.00586039496312
```

3.3.2 Call Options

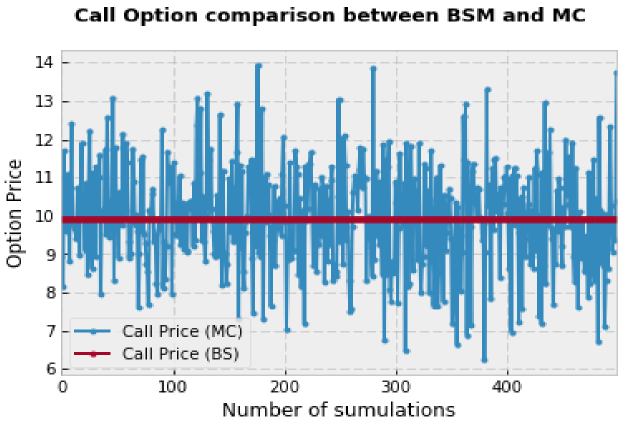
Following the same procedure for the call options using the code, we get the following results.

Sim number of simulations: 100



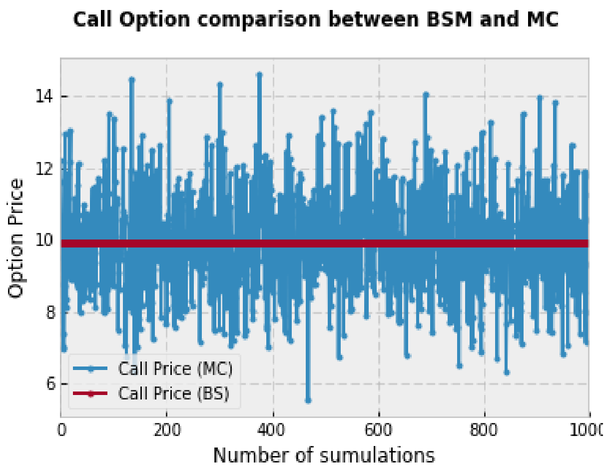
Time taken to compute the code in seconds:
0.8965892088735927
Standard Error Between MC Sim and BS Model:
0.109179495057

Sim number of simulations: 500



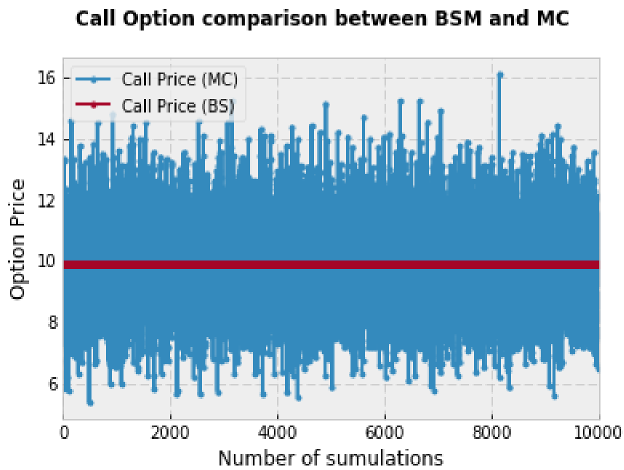
Time taken to compute the code in seconds:
1.3786443504613999
Standard Error Between MC Sim and BS Model:
0.0425003226102

Sim number of simulations: 1000



Time taken to compute the code in seconds:
2.064532795152445
Standard Error Between MC Sim and BS Model:
0.0318721374327

Sim number of simulations: 10000



Time taken to compute the code in seconds:
15.352363334036454
Standard Error Between MC Sim and BS Model:
0.00992302408433

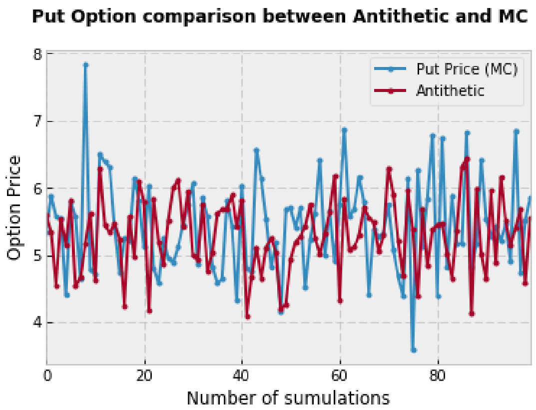
The simulation images show us the computational time for the number of simulations that it takes into account. We can also see that more the number of simulations the lesser is the error between the BSM and MC model. But taking into the computational cost, the time increases drastically when the number of simulations increase.

3.4 Comparison between Monte Carlo and Antithetic Variance

3.4.1 Put options

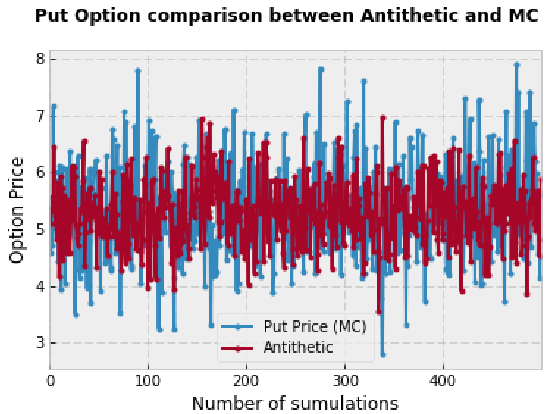
So, since Monte Carlo takes a lot of computing effort and time, there are methods to reduce its variance to improve its efficiency and accuracy. One such variance reduction method is the Antithetic Variate Technique, where the sample size of the data is double and run through the same simulations. It can be further seen by the results give below.

```
Sim     number of simulations: 100
```



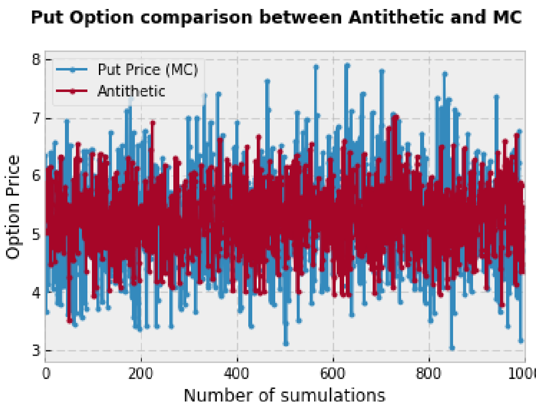

```
Time taken to compute the code in seconds:
 0.4355304010823602
Standard Error Between MC Sim and Antithetic Model:
  0.0113625103094
```

```
Sim     number of simulations: 500
```



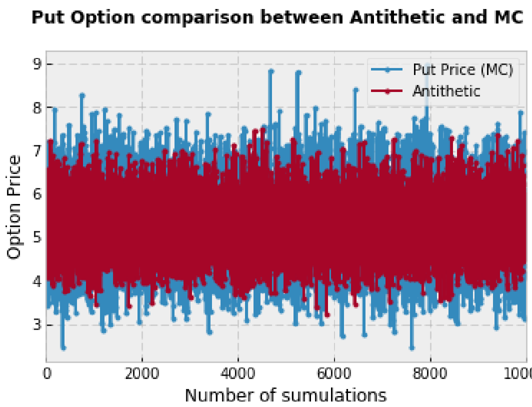

```
Time taken to compute the code in seconds:
 1.188487287297903
Standard Error Between MC Sim and Antithetic Model:
  0.00836171957885
```

```
Sim     number of simulations: 1000
```




```
Time taken to compute the code in seconds:
 1.4408626860677032
Standard Error Between MC Sim and Antithetic Model:
  0.00598687111312
```

```
Sim     number of simulations: 10000
```

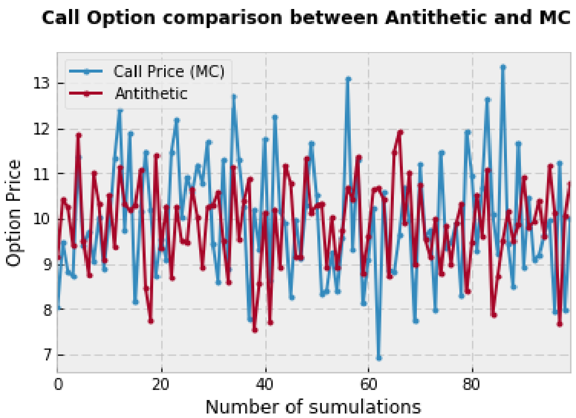



```
Time taken to compute the code in seconds:
 16.514736213168362
Standard Error Between MC Sim and Antithetic Model:
  0.0017494839216
```

3.4.2 Call Options

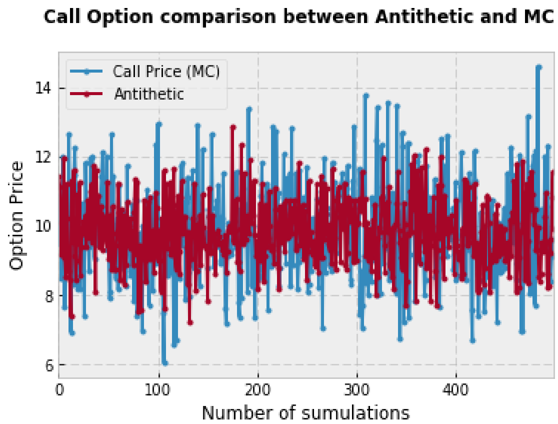
Now, comparing the methods for the call options, the following results are obtained.

```
Sim      number of simulations: 100
```



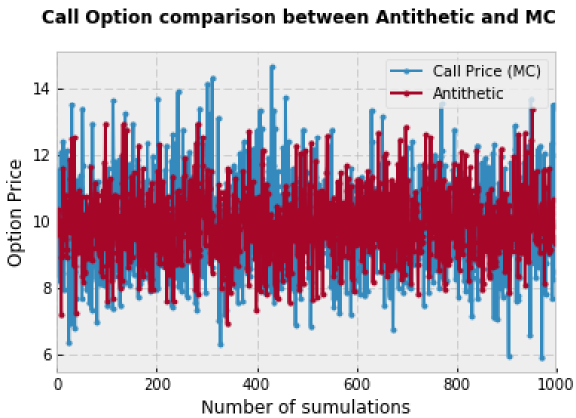

```
Time taken to compute the code in seconds:
 0.725733671364651
Standard Error Between MC Sim and Antithetic Model:
  0.0270314381735
```

```
Sim      number of simulations: 500
```



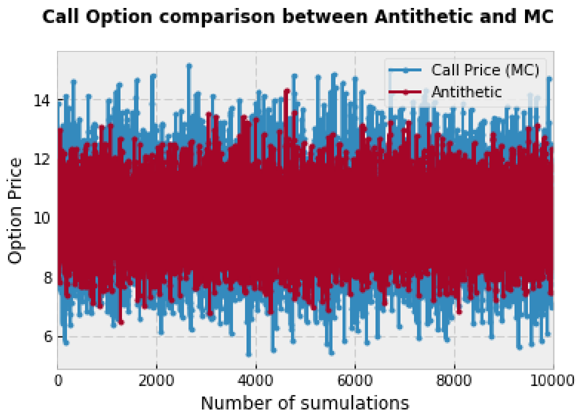

```
Time taken to compute the code in seconds:
 1.1611097751447232
Standard Error Between MC Sim and Antithetic Model:
  0.0138232954147
```

```
Sim      number of simulations: 1000
```




```
Time taken to compute the code in seconds:
 2.0525562877373886
Standard Error Between MC Sim and Antithetic Model:
  0.00815489978794
```

```
Sim      number of simulations: 10000
```




```
Time taken to compute the code in seconds:
 11.183813228039071
Standard Error Between MC Sim and Antithetic Model:
  0.00297616125393
```

Here we can see that by using the Antithetic Variance Method, the time taken for the calculations of the option prices have come down well and the standard error between the black scholes method and the simulation method has reduced drastically. This shows us that the

Antithetic variance technique can be used to find a better and faster results. A summary of all the calculations with time and standard error rate are given below in the table.

3.5 Summary of the Results:

For Put Options				
	Black Scholes Model & Monte Carlo Simulation		Black Scholes & Antithetic Variate Technique	
Simulations	Computational Time	Standard Error Rate	Computational Time	Standard Error Rate
100	0.5931	0.0579	0.4355	0.0113
500	1.2264	0.0258	1.1884	0.0083
1000	1.8638	0.0187	1.4408	0.0059
10000	14.3293	0.0058	16.5147	0.0017
For Call Options				
	Black Scholes Model & Monte Carlo Simulation		Black Scholes & Antithetic Variate Technique	
Simulations	Computational Time	Standard Error Rate	Computational Time	Standard Error Rate
100	0.8965	0.1091	0.7257	0.027
500	1.3786	0.0425	1.1611	0.0138
1000	2.0645	0.0318	2.0525	0.0081
10000	15.3523	0.0099	11.1838	0.0029

4. Conclusion

Through all these empirical analysis, it could be said that both the Monte Carlo methods and its variance reduction technique are equally good in valuing the options. It can also be seen that more the simulations, better the results of both the techniques are, the values of both the numerical techniques converge to the value of the analytical method. However, though both the methods are almost strong, the values show that the Antithetic Variate Technique has a better edge over the general Monte Carlo simulation. By the comparison of the values it could be strongly said that these techniques can be applied to the field of finance, considering it as a risk-free market. There are also other techniques which can be applied to value the same, such as finite difference methods, PDE methods with high accuracy rates. One of the greatest hindrance to such valuation projects is obtaining the real data, which would cost a great cost. All the valuation was done considering the underlying stock follows a log normal distribution. So, to conclude a better valuation study can be done if there is more computational power and a large-scale environment.

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14. Python for finance, analyse big financial data by Yves Hilpisch

Appendix:

```
import numpy as np
import scipy.stats as ss
import pandas as pd
import matplotlib.pyplot as plt
import time
#import resource
from matplotlib import style
style.use('bmh')
time_start = time.clock()

class OptionPricing:
    #Normal Black Scholes Model where...
    #S0 = Underlying Price
    #E = Strike Price
    #T = Expiration
    #rf = Risk Free
    #srigma = volatility
    #iterations = number of prices to simulate

    #This is a constructor which will assign initial variables for the option calculation
    def __init__(self, S0, E, T, rf, sigma, iterations, num_simulations, Otype):
        self.S0 = S0
        self.E = E
        self.T = T
        self.rf = rf
        self.sigma = sigma
        self.iterations = iterations
        self.num_simulations = num_simulations
        self.Otype = Otype

    def option_sim_gb(self):
        #Columns For DataFrame
        if self.Otype == 'C':
            cols = ["Call Price (MC)"]
```

```
        else:
            cols = ["Put Price (MC)"]

        #rows for data frame appended to this list
        data_row = []

        #Enter loop monte carlo simulation
        for x in range(1,self.num_simulations+1):

            #Array of options data
            data= np.zeros([self.iterations, 2])
            rand = np.random.normal(0,1,[1, self.iterations])

            #Calculate Stock Price applying geometric brownian motion to bs model
            stock_price = self.S0*np.exp(self.T*(self.rf - 0.5*self.sigma**2)+ \
                                         self.sigma*np.sqrt(self.T)*rand)

            if self.Otype == 'C':
                #Calculate max
                data[:,1] = stock_price - self.E

                #Average for MC Simulation
                average = np.sum(np.amax(data, axis=1))/float(self.iterations)

                #Calculate call price
                call_price = np.exp(-1.0*self.rf*self.T)*average

                #Append price as row to df
                data_row.append([call_price])
            else:

                #Note the calculation for Put Options differs here with the max
                data[:,1] = self.E - stock_price
```

```
                #Average for MC Simulation
                average = np.sum(np.amax(data, axis=1))/float(self.iterations)

                #Calculate put price
                put_price = np.exp(-1.0*self.rf*self.T)*average



                #Append price as row to df
                data_row.append([put_price])

        #DataFrame for simulation
        df = pd.DataFrame(data_row, columns=cols)
        return df

    #Functions Reserved For Black Scholes Model
    def d1(self, S0, E, rf, sigma, T):
        return (np.log(self.S0/E) + (self.rf + self.sigma**2 / 2) * \
                self.T)/(self.sigma * np.sqrt(self.T))

    def d2(self, S0, E, r, sigma, T):
        return (np.log(self.S0 / self.E) + (self.rf - self.sigma**2 / 2) * \
                self.T) / (self.sigma * np.sqrt(self.T))

    #Black Scholes Model
    def BlackScholes(self):
        data_row = []

        #We enter a loop here to graph our static call/put price along with MC results
        for x in range(1,self.num_simulations+1):
            if self.Otype=="C":
```

```
                call_price =  self.S0 * ss.norm.cdf(self.d1(self.S0, self.E, self.rf, \
                            sigma, self.T)) - self.E * np.exp(-self.rf * self.T) * \
                            ss.norm.cdf(self.d2(self.S0, self.E, self.rf, self.sigma, self.T))
                data_row.append([call_price])
                cols = ["Call Price (BS)"]

            else:
                put_price =  self.E * np.exp(-self.rf * self.T) * ss.norm.cdf(-self.d2(self.S0, \
                            self.E, self.rf, self.sigma, self.T)) - self.S0 * ss.norm.cdf(-self.d1(self.S0, \
                            self.E, self.rf, self.sigma, self.T))

                cols = ["Put Price (BS)"]

                data_row.append([put_price])

        df = pd.DataFrame(data_row, columns=cols)
        return df

    #Antithetic Model
    def antithetic(self):
        if self.Otype=="C":
            #Set df
            cols = ["Call Price (MC)", "Call Price (BS)"]
            df = pd.DataFrame(columns=cols)

            #Call gb sim to get greater sample size
            op1 = np.round(m.option_sim_gb(), 2)
            op2 = np.round(m.option_sim_gb(), 2)

        else:
            #Set df
            cols = ["Put Price (MC)", "Put Price (BS)"]
            df = pd.DataFrame(columns=cols)
```

```python
            #Call gb sim to get greater sample size

            op1 = np.round(m.option_sim_gb(), 2)
            op2 = np.round(m.option_sim_gb(), 2)

        #Combine df and get mean of rows
        df = pd.concat([op1, op2], axis=1)
        df['Antithetic'] = df.mean(axis=1)

        return df

    #Print Option Specs
    def specs(self):
        print("S0\tstock price at time 0:", self.S0)
        print("E\tstrike price:", self.E)
        print("rf\tcontinuously compounded risk-free rate:", self.rf)
        print("sigma\tvolatility of the stock price per year:", self.sigma)
        print("T\ttime to maturity in trading years:", self.T)
        print("Sim\tnumber of simulations:", self.num_simulations)

    #Plot Options DF
    def plotdf(self, title, df):
        plot_df = df
        plot_df.plot(style='.-')
        plt.suptitle(title, fontweight="bold")
        plt.xlabel('Number of sumulations')
        plt.ylabel('Option Price')
        plt.show()
```

```python
if __name__=="__main__":
    #Here we define the different option parameters
    #**********MODIFY PARAMS HERE***************#
    S0 = 95
    E = 95
    T = 1
    rf = 0.05
    sigma = 0.2
    iterations = 100
    num_simulations = 10000
    Otype = 'P'
    #**********MODIFY PARAMS HERE***************#

    #Initialize constructor
    m = OptionPricing(S0, E, T, rf, sigma, iterations, num_simulations, Otype)

    #Option Specs
    m.specs()

    #Here we plot the results of our simulations compared with BS formula and other MC approaches

    #Get df for regular mc sim
    gb_df = m.option_sim_gb()

    #Get Black Scholes output
    bs = m.BlackScholes()


    #Get Antithetic df
    anti_df = m.antithetic()
    anti_df = anti_df['Antithetic']
```

```python

    #Combine and plot
    df = pd.concat([gb_df, anti_df], axis=1)
    #df = pd.concat([bs], axis=1)
    if Otype == 'C':
        m.plotdf("Call Option comparison between Antithetic and MC", df)
        #m.plotdf("Call Option Pricing Comparison", df)
    else:
        m.plotdf("Put Option comparison between Antithetic and MC", df)
        #m.plotdf("Put Option Pricing Comparison", df)
    time_elapsed = (time.clock() - time_start)
    print('Time taken to compute the code in seconds:\n',time_elapsed)

    #Calculate Standard Error of Each Method
    df.sem()
    #Calculate for GB and BS
    std_err = df.iloc[:,0:2].std().std()/np.sqrt(len(df))
    print('Standard Error Between MC Sim and Antithetic Model:\n ', std_err)

```