Generalised Stokes Theorem

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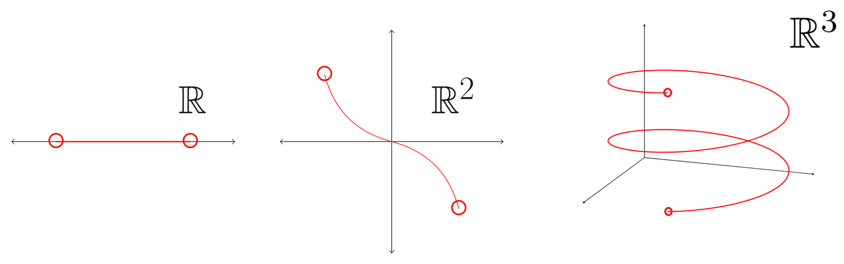
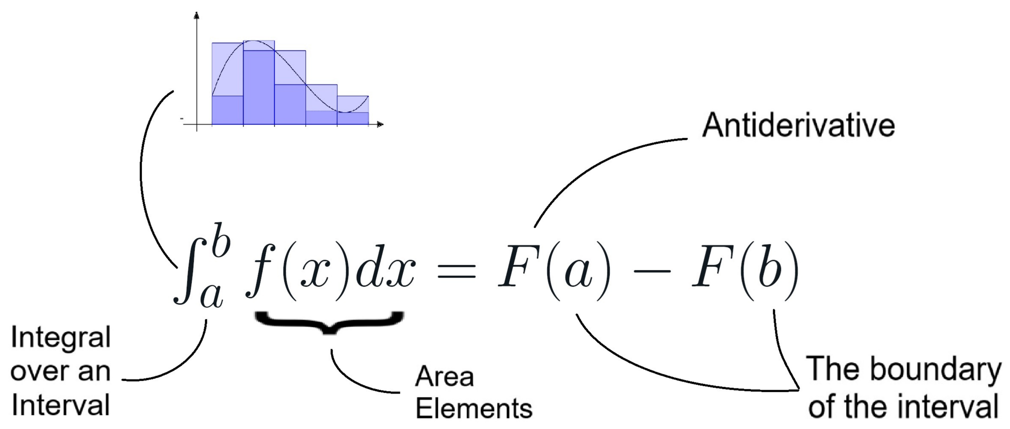
Context

- Gauss was studying surfaces, and wanted to define curvature in a way that was independent of the $3-$dimensional space that the surface is embedded in. The vector calculus view of surfaces is an extrinsic view. But there is also an intrinsic geometry of these surfaces.



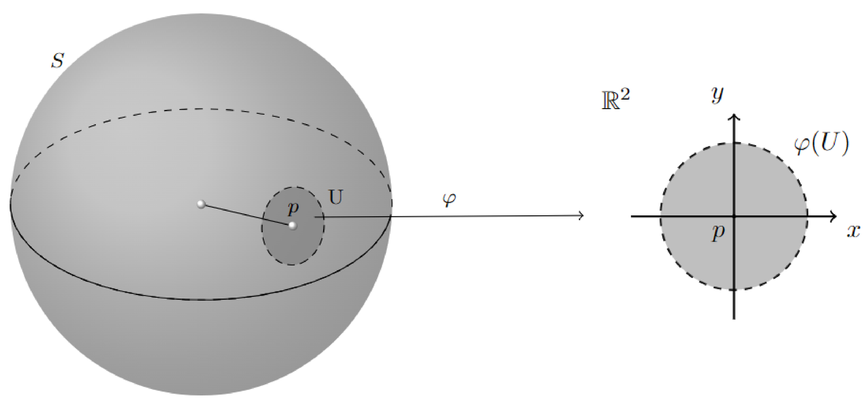
  This intrinsic view of surfaces, is developed in the theory of manifolds.

- The generalised stokes theorem is an analog to the fundamental theorem of calculus, in the setting of manifolds. The fundamental theorem of calculus states:



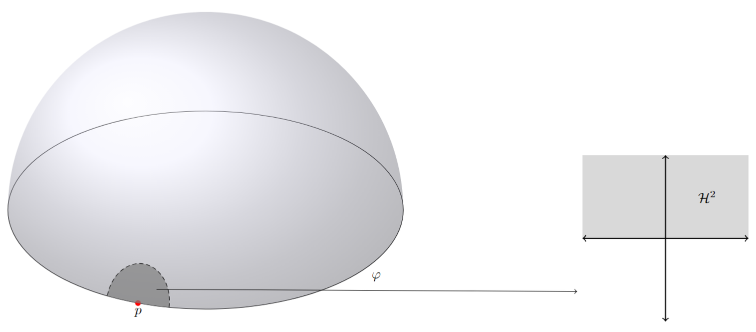
Manifolds

- Manifolds are a topological space such that each point is locally homeomorphic to an open set in a Euclidean space. For example, circles, spheres, torus, möbius strip etc.

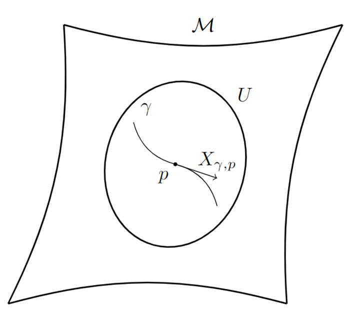


  In the above, the neighborhood paired with the homeomorphism (U, φ) is called a *chart*. Because at all points this manifold resembles a Euclidean space, a collection of these charts for neighborhoods of each point on the manifold, is given the term *Atlas*.

- smooth manifolds.
- There are surfaces that dont fully fit into this definition like the hemisphere. The neighborhood of a point on its boundary is not homeomorphic to a Euclidean space, because it is not an open set. For this rather than the entire euclidean space, we restrict the domain of the homeomorphism to the Euclidean Half space. Such structures are called *Manifolds with boundary*



- Tangent Space: Recall the view of tangents as the direction of motion on a path on the surface. This is the idea behind the tangent space. We consider paths on the manifold and the directional derivative operator along these paths on functions on the manifold as the tangent vectors.



$\gamma : (-\epsilon, \epsilon) \subset \mathbb{R} \to U \subset M$ (A path on U, $\gamma(0) = p$)
$f : U \to \mathbb{R}$ (A function on U)
$X_{\gamma,p}(f) := (f \circ \gamma)'(0)$ (The directional derivative of f at p in the direction of γ)

Differential Forms

- Tensors: $k-$linear, real valued functions, $V^k \to \mathbb{R}$. Examples: the dot product, a two tensor, the determinant, a $k-$tensor.
  Tensor product: combines a k and an l tensor to give a $k+l$ tensor. Defined as
  $$(K \otimes L)(v_1, v_2, \ldots, v_{k+l}) = K(v_1 \ldots v_k) \cdot L(v_k + 1, \ldots, v_{k+l})$$
- The dual of a vector space V is the space of linear, real valued functions on V. This is denoted V^*. So it is also the space of 1-tensors on V. The dual space is isomorphic to the original vector space.
- Forms on $\mathbb{R}^n$: A $k-$form on $\mathbb{R}^n$ is an alternating $k-$tensor
- forms on manifold, 1 forms as cotangent vetors
- exterior derivative
- pull backs

Integration

- integral of forms on Rn
- partitions of unity
- integral of forms on M
- change of variables
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statement, visual importance