## 1. Chains

A singular 0-cube in $\mathbb{R}^n$ is a function $c : \{0\} \to \mathbb{R}^n$ and a (piecewise smooth) singular $k$-cube in $\mathbb{R}^n$ is a function

$$c : [0,1]^k \to \mathbb{R}^n$$

that can be extended to a smooth function $\phi : V \to \mathbb{R}^n$ defined on an open subset $V$ of $\mathbb{R}^k$ containing $[0,1]^k$. (We will use $c$ for $\phi$ for simplicity). A singular 1-cube in $\mathbb{R}^n$ is called a singular curve in $\mathbb{R}^n$ and a singular two cube in $\mathbb{R}^n$ is called a singular surface in $\mathbb{R}^n$. The restriction of the identity function $I : \mathbb{R}^n \to \mathbb{R}^n$ to $[0,1]^n$ is a singular $n$-cube called the standard $n$-cube in $\mathbb{R}^n$ and denoted by $I^n$.

**Definition 1.1.** A (piecewise smooth) singular $k$-chain in $\mathbb{R}^n$ is a (formal) $\mathbb{Z}$-linear combination of (piecewise smooth) singular $k$-cubes in $\mathbb{R}^n$, i.e. a (piecewise smooth) singular $k$-chain is of the form

$$n_1\gamma_1 + \cdots + n_s\gamma_s$$

where $n_1, \cdots, n_s \in \mathbb{Z}$ and $\gamma_1, \cdots, \gamma_s : [0,1]^k \to \mathbb{R}^n$.

A (piecewise smooth) singular $k$-chain in $\mathbb{R}^n$ can be rewritten as

$$c = \sum_\gamma n_\gamma \gamma$$

where $\gamma$ runs through the set of all (piecewise smooth) singular $k$-cubes and $n_\gamma = 0$ for all but finitely many (piecewise smooth) singular $k$-cubes $\gamma$ in $\mathbb{R}^n$. The set of all (piecewise smooth) singular $k$-chains in $\mathbb{R}^n$ is denoted by $C_k(\mathbb{R}^n)$.

We define the sum of two (piecewise smooth) singular $k$-chains $c = \sum_\gamma n_\gamma\gamma$ and $c' = \sum_\gamma n'_\gamma \gamma$ by

$$c + c' = \sum_\gamma (n_\gamma + n'_\gamma)\gamma.$$

Then $C_k(\mathbb{R}^n)$ forms an abelian group.

Let $I^k : [0,1]^k \to \mathbb{R}^k$ be the standard $k$-cubes in $\mathbb{R}^k$. For each $1 \le i \le k$ and $0 \le a \le 1$, we defines functions $I^k_{(i,a)} : [0,1]^{k-1} \to \mathbb{R}^k$ by

$$I^k_{(i,0)}(t_1, \cdots, t_{k-1}) = (t_1, \cdots, t_{i-1}, 0, t_i, \cdots, t_k)$$
$$I^k_{(i,1)}(t_1, \cdots, t_{k-1}) = (t_1, \cdots, t_{i-1}, 1, t_i, \cdots, t_k).$$

We call $I^k_{(i,a)}$ the $(i,a)$-th face of $I^k$. The (algebraic) boundary of $I^k$ is defined to be

$$\partial_k I^k = \sum_{i=1}^k \sum_{a=0}^1 (-1)^{i+a} I^k_{(i,a)}.$$

Then $\partial_k I^k$ is a (piecewise smooth) singular $k-1$-chain in $\mathbb{R}^n$.

If $\gamma : [0,1]^k \to \mathbb{R}^n$ is a (piecewise smooth) singular $k$-cube in $\mathbb{R}^n$, we define the $(i,a)$-th face of $\gamma$ to be

$$\gamma_{(i,a)} = \gamma \circ I^k_{(i,a)}$$

for any $1 \le i \le k$ and for $0 \le a \le 1$. The algebraic boundary of $\gamma$ is defined to be

$$\partial_k \gamma = \sum_{i=1}^k \sum_{a=0}^1 (-1)^{i+a} \gamma_{(i,a)}.$$

In general, we define the (algebraic) boundary of a (piecewise smooth) singular $k$-chain $c$ to be the following singular $k-1$-chain

$$\partial_k c = \sum_{\gamma} n_\gamma (\partial_k \gamma).$$

**Lemma 1.1.** The function $\partial_k : C_k(\mathbb{R}^n) \to C_{k-1}(\mathbb{R}^n)$ is an abelian group homomorphism such that

$$\partial_{k-1} \circ \partial_k = 0$$

for any $k \geq 1$.

We denote $\ker \partial_k$ by $Z_k(\mathbb{R}^n)$ and $\operatorname{Im} \partial_{k+1}$ by $B_k(\mathbb{R}^n)$. Elements of $Z_k(\mathbb{R}^n)$ are called (piecewise smooth) singular $k$-cycles in $\mathbb{R}^n$ and elements of $B_k(\mathbb{R}^n)$ are called (piecewise smooth) singular $k$-boundaries in $\mathbb{R}^n$. Since $\partial_{k-1} \circ \partial_k = 0$, $B_k(\mathbb{R}^n)$ is an abelian subgroup of $Z_k(\mathbb{R}^n)$. We define the $k$-th (piecewise smooth) singular homology group of $\mathbb{R}^n$ to be the quotient group

$$H_k(\mathbb{R}^n) = Z_k(\mathbb{R}^n)/B_k(\mathbb{R}^n).$$

**Remark.** One can prove that

$$H_k(\mathbb{R}^n) = \begin{cases} 0 & \text{if } k > 0 \\ \mathbb{Z} & \text{if } k = 0. \end{cases}$$

Now let us state the Stoke's Theorem. For convenience, all the chains and cubes mentioned below are assumed to be piecewise smooth.

Let $\omega = f(x) dx_1 \wedge \cdots \wedge dx_n$ be any $n$-form on an open subset $U$ of $\mathbb{R}^n$. We define the integral of $\omega$ over a Jordan measurable subset $S$ of $\mathbb{R}^n$ contained in $U$ to be

$$\int_S \omega = \int_S f(x) d\mu$$

where $\int_S f(x) d\mu$ is the Riemann integral of the function $f$ over $S$.

**Remark.** It you are not familiar with the notion of Jordan measurable sets in $\mathbb{R}^n$, you take $S$ to be any $n$-dimensional compact interval $S = \prod_{i=1}^n [a_i, b_i]$.

If $\omega$ is a $k$-form on $\mathbb{R}^n$ and $\gamma : [0,1]^k \to \mathbb{R}^n$ is a singular $k$-cube, we define

$$\int_\gamma \omega = \int_{[0,1]^k} \gamma^* \omega.$$

In general, if $c = \sum_\gamma n_\gamma \gamma$ is a singular $k$-chain in $\mathbb{R}^n$, we define the integral of $\omega$ over $c$ by

$$\int_c \omega = \sum_\gamma n_\gamma \int_\gamma \omega.$$

**Theorem 1.1.** (Stoke's Theorem) Let $\omega$ be any $k-1$ form on $\mathbb{R}^n$ and $c$ be any $k$-chain in $\mathbb{R}^n$. Then

$$\int_c d\omega = \int_{\partial c} \omega.$$

Let us prove the case when $\omega = Q(x,y) dy$ is a one form on $\mathbb{R}^2$ and $c$ is any singular two chain in $\mathbb{R}^2$. At first, we prove that

$$\int_{\partial I^2} \omega = \int_{I^2} d\omega$$

where $I^2 : [0,1] \times [0,1] \to \mathbb{R}^2$ is the standard 2-cube. Since $I^2$ is the restriction of the identity function $I : \mathbb{R}^2 \to \mathbb{R}^2$,

$$(I^2)^* d\omega = d\omega = Q_x dx \wedge dy.$$

By definition.

$$\begin{aligned}\int_{I^2} d\omega &= \int_{[0,1]^2} (I^2)^* d\omega = \int_{[0,1]^2} Q_x(x,y) dx \wedge dy \\ &= \iint_{[0,1]\times[0,1]} Q_x(x,y) dA = \int_0^1 \left( \int_0^1 Q_x(x,y) dx \right) dy \\ &= \int_0^1 (Q(1,y) - Q(0,y)) dy.\end{aligned}$$

Since $\partial I^2 = I^2_{(2,0)} + I^2_{(1,1)} - I^2_{(2,1)} - I^2_{(1,0)}$,

$$\int_{\partial I^2} \omega = \int_{I^2_{(2,0)}} \omega + \int_{I^2_{(1,1)}} \omega - \int_{I^2_{(2,1)}} \omega - \int_{I^2_{(1,0)}} \omega.$$

One can show that

$$(I^2_{(2,0)})^* \omega = (I^2_{(2,1)})^* \omega = 0, \quad (I^2_{(1,1)})^* \omega = Q(1,t) dt, \quad (I^2_{(1,0)})^* \omega = Q(0,t) dt.$$

As a consequence,

$$\int_{I^2} d\omega = \int_0^1 (I^2_{(1,1)})^* \omega - \int_0^1 (I^2_{(1,0)})^* \omega = \int_0^1 (Q(1,t) - Q(0,t)) dt$$

which coincides with $\int_{I^2} d\omega$. Now let us prove that the statement is true for any singular 2-cube $\gamma$ in $\mathbb{R}^2$. To do this, we need the following Lemma and its Corollary.

**Lemma 1.2.** Let $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ be any smooth functions. For any $r$ form $\omega$ on $\mathbb{R}^p$,

$$(g \circ f)^* \omega = f^*(g^* \omega).$$

*Proof.* When $\omega = h$ is a zero form,

$$(g \circ f)^* h = h \circ g \circ f = f^*(h \circ g) = f^*(g^* h).$$

Assume that $\omega$ is a $r$ form on $\mathbb{R}^k$. For each $p \in \mathbb{R}^n$, and each $(v_1)_p, \cdots, (v_r)_p$ in $T_p(\mathbb{R}^n)$,

$$\begin{aligned}(g \circ f)^* \omega(p)((v_1)_p, \cdots, (v_r)_p) &= \omega((g \circ f)(p))(d(g \circ f)_p(v_1)_p, \cdots, d(g \circ f)_p(v_r)_p) \\ &= \omega((g \circ f)(p))(dg_{f(p)}(df_p(v_1)_p), \cdots, dg_{f(p)}(df_p(v_r)_p)) \\ &= (g^* \omega)(f(p))(df_p((v_1)_p), \cdots, df_p((v_1)_p)) \\ &= f^*(g^* \omega)(p)((v_1)_p, \cdots, (v_r)_p).\end{aligned}$$

Thus $(g \circ f)^* \omega(p) = f^*(g^* \omega)(p)$ for any $p \in \mathbb{R}^n$. Hence the statement is true. $\square$

**Corollary 1.1.** Let $\gamma$ be any singular $k$-cube on $\mathbb{R}^n$ and $\omega$ be any $k$-form on $\mathbb{R}^n$. Then

$$\int_\gamma f^* \omega = \int_{f \circ \gamma} \omega$$

for any smooth function $f : \mathbb{R}^n \to \mathbb{R}^n$.

*Proof.* Using the previous lemma, we find

$$(f \circ \gamma)^* \omega = \gamma^*(f^* \omega).$$

By definition,

$$\int_{f \circ \gamma} \omega = \int_{[0,1]^k} (f \circ \gamma)^* \omega = \int_{[0,1]^k} \gamma^*(f^* \omega) = \int_\gamma f^* \omega.$$

□

Let $\gamma$ be any 2-cube in $\mathbb{R}^2$. We consider $\gamma$ as a smooth function from an open subset of $\mathbb{R}^2$ containing $[0,1] \times [0,1]$ to $\mathbb{R}^2$. By definition, the boundary of $\gamma$ is

$$\partial \gamma = \gamma \circ I^2_{(2,0)} + \gamma \circ I^2_{(1,1)} - \gamma \circ I^2_{(2,1)} - \gamma \circ I^2_{(1,0)}.$$

By Corollary 1.1,

$$\int_{\partial \gamma} \omega = \int_{I^2_{(2,0)}} \gamma^* \omega + \int_{I^2_{(1,1)}} \gamma^* \omega - \int_{I^2_{(2,1)}} \gamma^* \omega - \int_{I^2_{(1,0)}} \gamma^* \omega = \int_{\partial I^2} \gamma^* \omega.$$

By Stoke's Theorem for standard 2-cube,

$$\int_{\partial I^2} \gamma^* \omega = \int_{I^2} d(\gamma^* \omega).$$

Since $d(\gamma^* \omega) = \gamma^*(d\omega)$, we find

$$\int_{I^2} d(\gamma^* \omega) = \int_{I^2} \gamma^*(d\omega) = \int_{\gamma \circ I^2} d\omega = \int_\gamma d\omega.$$

We find that the statement is true for any singular 2-cube $\gamma$ in $\mathbb{R}^2$. In general, if $c = \sum_\gamma n_\gamma \gamma$ is a 2-chain, then

$$\int_{\partial c} \omega = \sum_\gamma n_\gamma \int_{\partial \gamma} \omega = \sum_\gamma n_\gamma \int_\gamma d\omega = \int_c d\omega.$$

We prove that the statement is true for any 2-cubes in $\mathbb{R}^2$ for $\omega = Q(x,y)dy$. When $\omega = P(x,y)dx$, the proof is similar. When $\omega = P(x,y)dx + Q(x,y)dy$, we let $\omega_1 = P(x,y)dx$ and $\omega_2 = Q(x,y)dy$. Using Stoke's theorem for $\omega_1$ and for $\omega_2$ respectively, we obtain

$$\int_{\partial c} \omega = \int_{\partial c} \omega_1 + \int_{\partial c} \omega_2 = \int_c d\omega_1 + \int_c d\omega_2 = \int_c d\omega.$$

Here we use the fact that $d\omega = d\omega_1 + d\omega_2$. The idea of the above proof can be applied to the proof of Stoke's Theorem for general cases. Let us prove that

$$\int_{\partial I^k} \omega = \int_{I^k} d\omega$$

holds for $k-1$ form of the form $\omega = f(x) dx_2 \wedge \cdots \wedge dx_k$. Then

$$d\omega = f_{x_1} dx_1 \wedge \cdots \wedge dx_k.$$

By definition and the Fubini's Theorem,

$$\begin{aligned}
\int_{I^k} d\omega &= \int_{[0,1]^k} f_{x_1} dx_1 \wedge \cdots \wedge dx_k \\
&= \int_{[0,1]^{k-1}} (f(1, x_2, \cdots, x_n) - f(0, x_2, \cdots, x_n)) d\mu_{k-1}.
\end{aligned}$$

Here $d\mu_{k-1}$ is the Jordan measure on $\mathbb{R}^{k-1}$. On the other hand,
$$(I^k_{(i,a)})^*\omega = 0 \text{ for } 2 \leq i \leq k$$
and
$$(I^k_{(1,0)})^*\omega = f(0,t_1,\cdots,t_{k-1})dt_1 \wedge \cdots \wedge dt_{k-1}$$
$$(I^k_{(1,1)})^*\omega = f(1,t_1,\cdots,t_{k-1})dt_1 \wedge \cdots \wedge dt_{k-1}.$$
By definition, $I^k = \sum_{i=1}^k \sum_{a=0}^1 (-1)^{i+a} I^k_{(i,a)}$, and hence
$$\int_{\partial I^k} \omega = \sum_{i=1}^k \sum_{a=0}^1 (-1)^{i+a} \int_{I^k_{(i,a)}} \omega.$$
By the previous observation,
$$\int_{I^k_{(i,a)}} \omega = \int_{[0,1]^{k-1}} (I^k_{(i,a)})^*\omega = 0 \text{ for } 2 \leq i \leq k$$
and
$$\int_{I^k_{(1,0)}} \omega = \int_{[0,1]^{k-1}} (I^k_{(1,0)})^*\omega = \int_{[0,1]^{k-1}} f(0,t_1,\cdots,t_{k-1})d\mu_{n-1}$$
$$\int_{I^k_{(1,1)}} \omega = \int_{[0,1]^{k-1}} (I^k_{(1,1)})^*\omega = \int_{[0,1]^{k-1}} f(1,t_1,\cdots,t_{k-1})d\mu_{n-1}.$$
We see that
$$\int_{\partial I^k} \omega = \int_{[0,1]^{k-1}} (f(1,t_1,\cdots,t_{k-1}) - f(0,t_1,\cdots,t_{k-1}))d\mu_{n-1}$$
which coincides with $\int_{I^k} d\omega$. For the case when $\omega$ is a $k-1$ form of the form
$$\omega = f(x)dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge \cdots dx_k,$$
the proof is similar. If $\omega$ is a $k-1$ form of the form
$$\omega = \sum_{i=1}^k f_i(x)dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge \cdots dx_k,$$
we write $\omega_i = f_i(x)dx_1 \wedge \cdots dx_{i-1} \wedge dx_{i+1} \wedge \cdots \wedge \cdots dx_k$. Then
$$\int_{\partial I^k} \omega_i = \int_{I^k} d\omega_i.$$
Since $d\omega = \sum_{i=1}^k d\omega_i$, we find
$$\int_{\partial I^k} \omega = \sum_{i=1}^k \int_{\partial I^k} \omega_i = \sum_{i=1}^k \int_{I^k} d\omega_i = \int_{I^k} \left( \sum_{i=1}^k d\omega_i \right) = \int_{I^k} d\omega.$$

If $\gamma : [0,1]^k \to \mathbb{R}^n$ is a $k$-cube on $\mathbb{R}^n$ and $\omega$ is a $k$-form on $\mathbb{R}^n$, then
$$\int_{\partial\gamma} \omega = \int_{\partial I^k} \gamma^*\omega = \int_{I^k} d(\gamma^*\omega) = \int_{I^k} \gamma^*(d\omega) = \int_\gamma d\omega.$$
We prove that the theorem holds for any $k$-cubes and any $k$-forms. One can show that the Stoke's theorem holds for any $k$-form and for any $k$-chains on $\mathbb{R}^n$.