GRADUATE STUDIES IN MATHEMATICS 227

Geometric Structures on Manifolds

William M. Goldman

AMS
American Mathematical Society

10.1090/gsm/227

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Figure 15.1 on page 339 is courtesy of François Guéritaud.

2020 *Mathematics Subject Classification.* Primary 22-XX, 51-XX, 53-XX, 57-XX.

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Library of Congress Cataloging-in-Publication Data

Names: Goldman, William Mark, author.
Title: Geometric structures on manifolds / William M. Goldman.
Description: Providence, Rhode Island : American Mathematical Society, [2022] | Series: Graduate studies in mathematics, 1065-7339 ; 227 | Includes bibliographical references and index.
Identifiers: LCCN 2022030398 | ISBN 9781470471033 (hardcover) | ISBN 9781470471989 (paperback) | ISBN 9781470471972 (ebook)
Subjects: LCSH: Manifolds (Mathematics) | AMS: Topological groups, Lie groups. | Geometry. | Differential geometry. | Manifolds and cell complexes.
Classification: LCC QA613 .G65 2022 | DDC 516.3/6–dc23
LC record available at https://lccn.loc.gov/2022030398

10 9 8 7 6 5 4 3 2 1 27 26 25 24 23 22

Dedicated to Carolyn, Emily, Evan, Lizzie, Michael, Amelia, Liana, Leonardo, Jonah and the memory of Morris Cohen and Stanley Goldman

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Preface

This book explores geometric structures on manifolds locally modeled on a classical geometry.

This subject mediates between *topology* and *geometry,* where a fixed topology is given local coordinate systems in the geometry of a homogeneous space of a Lie group. A familiar example puts Euclidean geometry on a manifold; such a *Euclidean structure* is nothing more than a Riemannian metric of zero curvature. In this sense, the topology of the 2-dimensional sphere $\mathbb{S}^2$ is incompatible with the geometry of Euclidean space: *There is no metrically accurate atlas of the world.* In contrast, however, the topology of the 2-dimensional torus $\mathbb{T}^2$ *does* support Euclidean geometry. Indeed, the classification of Euclidean structures on the torus is part of a rich and central area of mathematics (elliptic curves, modular forms). Indeed, Euclidean structures on $\mathbb{T}^2$ are classified by the action of the modular group on the Poincaré upper halfplane.

Topology and geometry communicate via *group theory. Topology* contributes its group, — the fundamental group — and *Geometry* contributes the group of symmetries of the given geometry. Thus our approach starts from the Klein–Lie algebraicization of geometry via Lie groups and homogeneous spaces, and quickly evolves into studying representations of discrete groups in Lie groups.

This book surveys the theory, with a special emphasis on affine and projective geometry. Many important geometries (for example, hyperbolic geometry) have projective models, and these projective models unify the diverse geometries.

This work is based on examples. I have tried to present examples as a way to suggest the general theory. Because of the dramatic growth of this subject in the last decades, I tried to collect many facets of this subject and present them from a single viewpoint. Since Ehresmann's 1936 initiation of this subject, there have been many "success stories" in the classification of geometric structures on a given topology. I have to tried to present some of these in this book.

Despite the profound interrelations between different geometries, each geometry enjoys special features. A developing map only goes so far, and heavier machinery is often required, drawing on techniques special to the particular geometry. Learning new techniques and adapting to different areas of mathematics has been an exciting part of this journey.

Furthermore some of the material — which I feel should be better known — is unpublished, untranslated, or aimed at a different readership. The literature suffers from many errors (including some of my own) which I have tried to correct and clarify. However, I am certain many errors still persist, and I take full responsibility.

This book is suitable for a graduate textbook and contains many exercises. Some exercises are routine and others are more difficult. Many are used in other parts of the text. Others are meant to introduce ideas and examples before a subsequent detailed discussion.

To preserve the expository flow, several developments have been put in appendices. I have tried to illustrate geometric ideas with pictures and algebraic ideas with tables.

I have tried to keep the prerequisites fairly minimal. Material from beginning graduate courses in topology, differential geometry, and algebra are assumed, although some of the material which is crucial or less standard is summarized. The relationship between Lie groups and Lie algebras is heavily used, but little of the general structure theory/representation theory is assumed.

I began this area of research working with Dennis Sullivan and Bill Thurston at Princeton University in 1976. Their influence is evident throughout this work. Thurston formulated his *geometrization* of 3-manifolds in the context of geometric structures modeled on 3-dimensional Riemannian homogeneous spaces. Since then the study of more general (but not necessarily Riemannian) locally homogeneous geometric structures has become a very active field with interactions to other areas of mathematics and physics.

Several important topics have been omitted or only briefly mentioned. Flat structures on Riemann surfaces — namely, singular Euclidean structures modeled on translations — are barely mentioned despite their fundamental role in modern Teichmüller theory. Their strata are fascinating and mysterious examples of incomplete complex affine manifolds. Nor are holomorphic affine and projective structures on complex manifolds. The algebraic theory of character varieties and representations of fundamental groups, is not really developed thoroughly. In particular the theory of surface group representations into Lie groups of higher rank, sometimes called *higher Teichmüller theory*, is not extensively discussed, despite its remarkable recent activity. Integral affine structures (important in mirror symmetry) are not discussed. Other very natural topics in this subject have not been discussed in detail, for reasons of space: These include the convex decomposition theorem of Suhyoung Choi, completeness results of Carrière and Klingler for constant curvature Lorentzian manifolds, and affine structures with diagonal holonomy as developed by Smilie and Benoist.

I welcome suggestions, comments, and feedback of (almost) all sorts. Through the AMS bookstore, I plan to maintain a website of errata, comments, graphics, and interactive software in connection with this book.

I hope you enjoy this journey as much as I have!

Bill Goldman

College Park, Maryland

`wmg@umd.edu`
`http://www.math.umd.edu/~wmg`

Acknowledgments

This book grew out of lecture notes *Projective Geometry on Manifolds,* from a course I gave at the University of Maryland in 1988. Since then I have given minicourses at international conferences, graduate courses and summer schools on various aspects of this material.

Comments and suggestions by Suhyoung Choi, Daniele Alessandrini, Jean-Philippe Burelle, Francisco Arana Herrera, Daryl Cooper, Charles Daly, Jeff Danciger, François Guéritaud, Misha Kapovich, Fanny Kassel, Qiongling Li, John Loftin, Darren Long, Ludovic Marquis, Richard Schwartz, Stephan Tillmann, Tengren Zhang and various anonymous referees have been extremely helpful. The final version of the manuscript greatly benefitted from an extremely careful reading by Olivier Guichard.

I am grateful to David Dumas for organizing a working seminar at University of Illinois at Chicago in Fall 2021 using this book, and the participants Lydia Holley, Stephen Mackes, Amelia Pampilio, and Jennifer Vaccaro for comments and suggestions. I especially wish to thank Lydia Holley for helping with the illustrations.

This work has benefitted tremendously from a large community of research mathematicians at various career stages, who have contributed in many ways to its development. I sincerely apologize for any omissions.

My interest in this subject began with my 1977 undergraduate thesis [**145**] from Princeton University. There Dennis Sullivan and Bill Thurston suggested looking at affine and projective structures, which I continued to pursue in graduate school in Berkeley with my doctoral adviser Moe Hirsch. David Fried and John Smillie spent summers in Berkeley and the four of us discussed affine structures extensively. Their influence is clearly evident from this book's content. This led to correspondence with Jacques Vey and Jacques Helmstetter in Grenoble. Conversations with my teachers at that time, especially Dan Burns, Shiing-Shen Chern, Bob Gunning, Shoshichi Kobayashi, Joe Wolf, and S.T. Yau were particularly valuable for specific parts of this work.

While a postdoc, I benefitted from many discussions with Scott Wolpert, John Millson, Walter Neumann, Ravi Kulkarni, and Domingo Toledo on specific parts of this work. My collaboration with Millson was greatly influenced by the generosity of Pierre Deligne, whose letter to Millson taught us, among other things, the power of schemes.

I also want to express my gratitude to François Labourie for many discussions at an early stage, as well as Yves Benoist for many interesting conversations and his support of this project, beginning with my original 1988 notes. I am also grateful fo Grisha Margulis for many discussions about proper affine actions, and my collaboration with Todd Drumm and Virginie Charette on *Margulis spacetimes.*

Master's theses by Anton Lukyanenko and Ryan Hoban and lecture notes by Greg Laun and Son Lam Ho from graduate courses were especially helpful in writing this manuscript.

The series of conferences *Crystallographic Groups and their Generalizations* (1996, 1999, 2002, 2005, 2008, and 2011), in Kontrijtk, Belgium (and later in Oostende) were also very formative, and I am extremely grateful to Paul Igodt and Karel Dekimpe of the Katholieke University of Louvain in Kortrijt for organizing these highly stimulating events.

I am grateful to the hospitality of the University of Maryland, Mathematical Sciences Research Institute, the Institute for Advanced Study, Brown University, International Center for Theoretical Physics, Institute for Computation and Experimentation in Research in Mathematics, and Institut Henri Poincaré. This work was completed while on sabbatical at the Institute for Advanced Study in 2021–2022, and I benefitted from the superb working conditions there.

I would like to thank Ed Dunne at the American Mathematical Society for suggesting, at the turn of the last century, expanding the original 1988 notes into a book in this series. His advice on the process of publishing over the years was extremely useful. I also want to thank Eko Hironaka for more informative suggestions and encouragement of this project. Without her gentle but firm support—especially near the end of this project—this work would have taken even longer to complete.

This work was partially supported by grants from the National Science Foundation, DMS1065965, DMS1406281, and most recently DMS1709701, and *Research Networks in the Mathematical Sciences* grant DMS1406281, supporting the network *GEometric structures And Representations* (GEAR). Directed by Steven Bradlow, myself, Steve Kerckhoff, Richard Wentworth, and Anna Wienhard. It has facilitated much activity in this subject, and this is reflected in this text. Semester Research Awards (Fall 2005 and Fall 2010) from the General Research Board of the University of Maryland; the Oswald Veblen Fund at IAS (2007); memberships at IHP, MSRI, IAS, and ICERM; travel funding from the Clay Foundation and a sabbatical grant for 2014–2015 from the Simons Foundation have also supported this work.

Finally I thank my family, without whom none of this would have been remotely possible, for their enormous and deep support. I am sincerely grateful to my wife Emily; offspring Evan, Lizzie, and Michael; grandchildren Amelia, Liana, Leo, and Jonah; my mother Carolyn; and the memory of my father Stanley Goldman and grandfather Morris Cohen, all of to whom this book is dedicated.

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Introduction

Symmetry powerfully unifies the various notions of geometry. Based on ideas of Sophus Lie, Felix Klein's 1872 Erlangen program proposed that geometry is the study of properties of a space X invariant under a group G of transformations of X. For example Euclidean geometry is the geometry of n-dimensional Euclidean space $\mathbb{R}^n$ invariant under its group of rigid motions. This is the group of transformations which transforms an object ξ into an object congruent to ξ. In Euclidean geometry one can speak of points, lines, parallelism of lines, angles between lines, distance between points, area, volume, and many other geometric concepts. All these concepts can be derived from the notion of distance, that is, from the metric structure of Euclidean geometry. Thus any distance-preserving transformation or *isometry* preserves all of these geometric entities.

Notions more primitive than that of distance are the *length* and *speed* of a smooth curve. Namely, the distance between points a, b is the infimum of the length of curves γ joining a and b. The length of γ is the integral of its speed $\|\gamma'(t)\|$. Thus Euclidean geometry admits an infinitesimal description in terms of the *Riemannian metric tensor,* which allows a measurement of the size of the velocity vector $\gamma'(t)$. In this way standard Riemannian geometry generalizes Euclidean geometry by imparting Euclidean geometry to each tangent space.

Other geometries "more general" than Euclidean geometry are obtained by removing the metric concepts, but retaining other geometric notions. *Similarity geometry* is the geometry of Euclidean space where the equivalence relation of congruence is replaced by the broader equivalence relation of similarity. It is the geometry invariant under similarity transformations. Similarity geometry does not involve distance, but rather involves angles, lines, and parallelism. *Affine geometry* arises when one speaks only of points, lines and the relation of parallelism. And when one removes the notion of parallelism and only studies lines, points and the relation of incidence between them (for example, three points being *collinear* or three lines being *concurrent*) one arrives at *projective geometry.* However in projective geometry, one must enlarge the space to *projective space,* which is the space upon which all the projective transformations are defined.

Here is a basic example illustrating the differences among the various geometries. A particle moving along a smooth path has a well-defined velocity vector field, representing its *infinitesimal displacement* at any time. This uses only the differentiable structure of $\mathbb{R}^n$. The magnitude of the velocity is the *speed,* which makes sense in Euclidean geometry. Thus "motion at unit speed" (that is, "arc-length-parametrized geodesic") is a meaningful concept there. But in affine geometry, the concept of "speed" or "arc-length" must be abandoned: yet "motion at constant speed" remains meaningful since the property of moving at constant speed along a straight line can be characterized as motion with zero acceleration. This is equivalent to the parallelism of the velocity vector field. In projective geometry this notion of "constant speed along a straight line" (or "parallel velocity") must be further weakened to the concept of "projective parameter" introduced by J. H. C. Whitehead [**346**].

Synthetic projective geometry was developed by the architect Desargues in 1636–1639 out of attempts to understand the geometry of perspective. Two hundred years later non-Euclidean (hyperbolic) geometry was developed independently — and practically simultaneously — by Bolyai in 1833 and Lobachevsky in 1826–1829. These geometries were unified in 1871 by Klein who noticed that Euclidean, affine, hyperbolic, and elliptic geometry were all "present" in projective geometry.

Later in the nineteenth century, mathematical crystallography developed, leading to the theory of *Euclidean crystallographic groups.* Answering Hilbert's eighteenth problem on the finiteness of the number of space groups in any given dimension n, Bieberbach developed a structure theory in 1911–1912. For torsion free groups, the quotient spaces identified with *flat Riemannian manifolds* of dimension n, that is, Riemannian n-manifolds having zero sectional curvature. Such Riemannian structures are locally isometric to Euclidean space E^n. In particular, every point has an open neighborhood isometric to an open subset of E^n. These local isometries define a local Euclidean geometry on the neighborhood. Furthermore on overlapping neighborhoods, the local Euclidean geometries "agree," that is, they are related by restrictions of global isometries $\mathsf{E}^n \to \mathsf{E}^n$. The neighborhoods form *coordinate patches,* the local isometries from the patches to

E^n are the *coordinate charts,* and the restrictions of isometries of E^n are the corresponding *coordinate changes.* In this way a flat Riemannian manifold is defined by a coordinate atlas for a *Euclidean structure.*

More generally, for any geometry one can define geometric structures on a manifold M modeled on the homogeneous space (G, X). A geometric atlas consists of an open covering of M by patches $U \hookrightarrow M$, together with a system of charts $U \xrightarrow{\psi} X$ such that the coordinate changes are locally restrictions of transformations of X which lie in G.

The plethora of different geometries suggests that, at least at a superficial level, no general inclusive theory of locally homogeneous geometric structures exists. Each geometry has its own features and idiosyncrasies, and special techniques particular to each geometry are used in each case. For example, a surface modeled on $\mathbb{C}\mathsf{P}^1$ has the underlying structure of a Riemann surface, and viewing a $\mathbb{C}\mathsf{P}^1$-structure as a projective structure on a Riemann surface provides a satisfying classification of $\mathbb{C}\mathsf{P}^1$-structures. Namely, as was presumably understood by Poincaré, *the deformation space of* $\mathbb{C}\mathsf{P}^1$*-structures on a closed surface* Σ *with* $\chi(\Sigma) < 0$ *identifies with a holomorphic affine bundle over the Teichmüller space of* Σ. When X is a complex manifold upon which G acts biholomorphically, holomorphic mappings provide a powerful tool in the study, a class of local mappings more flexible than "constant" maps (maps which are "locally in G") but more rigid than general smooth maps. Another example occurs when X admits a G-invariant connection, such as an invariant (pseudo-)Riemannian structure. Then the geodesic flow provides a powerful tool for the study of (G, X)-manifolds.

We emphasize the interplay between different mathematical techniques as an attractive aspect of this general subject. See [**160**] for a recent historical account of this material.

Organization of the text

The book divides into three parts. Part One describes affine and projective geometry and provides some of the main background on these extensions of Euclidean geometry. As noted by Lie and Klein, most classical geometries can be modeled in projective geometry. We introduce projective geometry as an extension of affine geometry, so we begin with a detailed discussion of affine geometry as an extension of Euclidean geometry and projective geometry as an extension of affine geometry. Part Two describes how to put the geometry of a Klein geometry (G, X) on a manifold M, and gives the basic examples and constructions. One goal is to *classify* the (G, X)-structures on a fixed topology in terms of a *deformation space* whose points correspond to equivalence classes of *marked structures,* whereby a marking is an extra piece of information which fixes the topology as the geometry of

M varies. Part Three describes recent developments in this general theory of locally homogeneous geometric structures.

Part One: Affine and projective geometry

Chapter 1 introduces affine geometry as the geometry of parallelism. Two objects are *parallel* if they are related by a *translation.* Translations form a vector space V, and act *simply transitively* on affine space. That is, for two points $p, q \in \mathsf{A}$ there is a unique translation taking p to q. In this way, points in A identify with the vector space V, but this identification depends on the (arbitrary) choice of a basepoint, or *origin* which identifies with the zero vector in V. One might say that an affine space is a vector space, where the origin is forgotten. More accurately, the special role of the zero vector is suppressed, so that all points are regarded equally.

The action by translations now allows the definition of *acceleration* of a smooth curve. A curve is a *geodesic* if its acceleration is zero, that is, if its velocity is parallel. In affine space itself, unparametrized geodesics are straight lines; a parametrized geodesic is a curve following a straight line at "constant speed." Of course, the "'speed" itself is undefined, but the notion of "constant speed" just means that the acceleration is zero.

This notion of parallelism is a special case of the notion of an *affine connection,* except the existence of *globally defined* translations effecting the notion of parallelism is a special feature to our setting — the setting of *flat connections.* Just as Euclidean geometry is affine geometry with a parallel Riemannian metric, other linear-algebraic notions enhance affine geometry with parallel tensor fields. The most notable (and best understood) are flat Lorentzian (and pseudo-Riemannian) structures.

Chapter 2 develops the geometry of projective space, viewed as the compactification of affine space. *Ideal points* arise as "where parallel lines meet." A more formal definition of an ideal point is an equivalence class of lines, where the equivalence relation is parallelism of lines. Linear families (or *pencils*) of lines form planes, and indeed the set of ideal points in a projective space form a *projective hyperplane,* that is, a projective space of one lower dimension. Projective geometry appears when the ideal points lose their special significance, just as affine geometry appears when the zero vector $\mathbf{0}$ in a vector space loses its special significance.

However, we prefer a more efficient (if less synthetic) approach to projective geometry in terms of linear algebra. Namely, the *projective space associated to a vector space* V is the space $\mathsf{P}(\mathsf{V})$ of 1-dimensional linear subspaces of V (that is, lines in V passing through $\mathbf{0}$). *Homogeneous coordinates* are introduced on projective space as follows. Since a 1-dimensional linear subspace is determined by any nonzero element, its coordinates determine a point in projective space. Furthermore the homogeneous coordinates are uniquely defined up to *projective equivalence,* that is, the equivalence relation defined by multiplication by nonzero scalars. Projectivizing linear subspaces of V produces projective subspaces of $\mathsf{P}(\mathsf{V})$, and projectivizing linear automorphisms of V yields *projective automorphisms,* or *collineations* of $\mathsf{P}(\mathsf{V})$.

The equivalence of the geometry of incidence in $\mathsf{P}(\mathsf{V})$ with the algebra of V is remarkable. Homogeneous coordinates provide the "dictionary" between projective geometry and and linear algebra. The collineation group is compactified as a projective space of "projective endomorphisms;" this will be useful for studying limits of sequences of projective transformations. These "singular projective transformations" are important in controlling developing maps of geometric structures, as developed in the second part.

Chapter 3 discusses, first from the classical viewpoint of polarities, the Cayley–Beltrami–Klein model for hyperbolic geometry. Polarities are the geometric version of nondegenerate symmetric or skew-symmetric bilinear forms on vector spaces. They provide a natural context for hyperbolic geometry, which is one of the principal examples of geometry in this study.

The Hilbert metric on a properly convex domain in projective space is introduced and is shown to be equivalent to the categorically defined Kobayashi metric [**220**, **222**]. Later this notion is extended to manifolds with projective structure.

Chapter 3 develops notions of convexity. The Cayley–Beltrami–Klein metric on hyperbolic space is a special case of the Hilbert metric on properly convex domains. These define natural metric structures on certain well-studied projective structures. An application of the Hilbert metric is Vey's semisimplicity theorem [**339**], which is later used to characterize closed hyperbolic projective manifolds as quotients of sharp convex cones. Then another metric (due to Vinberg [**340**]) is introduced, and is used to give a new proof of *Benzécri's Compactness theorem* [**46**] that the collineation group acts properly and cocompactly on the space of convex bodies in projective space — in particular the quotient is a compact (Hausdorff) manifold. This is used to characterize the boundary of convex domains which cover convex projective manifolds. Recently Benzécri's theorem has been used by Cooper, Long, and Tillmann [**100**] in their study of cusps of $\mathbb{R}\mathsf{P}^n$-manifolds.

Part Two: Geometric manifolds

The second part globalizes these geometric notions to manifolds, introducing *locally homogeneous geometric structures* in the sense of Whitehead [**345**] and Ehresmann [**122**] in Chapter 5. We associate to every transformation group (G, X) a category of geometric structures on manifolds locally modeled on the geometry of X invariant under the group G. Because of the "rigidity" of the local coordinate changes of open sets in X which arise from transformations in G, these structures on M intimately relate to the fundamental group $\pi_1(M)$.

Chapter 5 presents three different viewpoints to study these structures. First are coordinate atlases for the pseudogroup arising from (G, X). Using the aforementioned rigidity, these are globalized in terms of a *developing map*

$$\widetilde{M} \xrightarrow{\mathsf{dev}} X,$$

defined on the universal covering space $\widetilde{M}$ of the geometric manifold M. The developing map is equivariant with respect to the holonomy homomorphism

$$\pi_1(M) \xrightarrow{h} G$$

which represents the group $\pi_1(M)$ of deck transformations of $\widetilde{M} \to M$ in G. Each of these two viewpoints represents M as a quotient: in the coordinate atlas description, M is the quotient of the disjoint union

$$\mathcal{U} := \coprod_{\alpha \in A} U_\alpha$$

of the coordinate patches U_α; in the second description, M is represented as the quotient of $\widetilde{M}$ by the action of the group $\pi_1(M)$. While a map defined on a connected space $\widetilde{M}$ may seem more tractable than a map defined on the disjoint union $\mathcal{U}$, the space $\widetilde{M}$ can still be quite large.

The third viewpoint replaces $\widetilde{M}$ with M and replaces the developing map by a section of a bundle defined over M. The bundle is a *flat bundle,* (that is, has *discrete structure group* in the sense of Steenrod [**317**]). The corresponding *developing section* is characterized by transversality with respect to the foliation arising from the flat structure. This replaces the coordinate charts (respectively the developing map) being local diffeomorphisms into X.

Chapter 6 discusses examples of geometric structures from these three points of view. Although the main interest in these notes is structures modeled on affine and projective geometry, we describe other interesting structures.

These structures interrelate: Geometries may "contain" or "refine" other geometries. For example, affine geometry *contains* Euclidean geometry — abandon the metric notions but retain the notion of *parallelism.* This corresponds to the inclusion of the Euclidean isometry group (consisting of transformations $x \longmapsto Ax + b$, where A is orthogonal) as a subgroup of the affine automorphism group (consisting of transformations $x \longmapsto Ax + b$

where A is only assumed to be linear). Other examples include the projective and conformal models for non-Euclidean geometry. In these examples, the model space of the refined geometry is an open subset of the larger model space, and the transformations in the refined geometry are restrictions of transformations in the larger geometry.

This *hierarchy of geometries* plays a crucial role in the theory. This is simply the geometric interpretation of the inclusion relations between closed subgroups of Lie groups. This *algebraicization* of geometries in the 19^{th} century by Lie and Klein satisfactorily organized the proliferation of classical geometries. This viewpoint is the cornerstone in our construction and classification of geometric structures. The classification of geometric manifolds often shows that a manifold modeled on one geometry may actually have a *stronger* geometry. For example, Fried's theorem [**135**] (see 11.4) asserts a closed manifold M with a similarity structure is either Euclidean or a manifold modeled on $\mathbb{R}^n \backslash \{\mathbf{0}\} \cong \mathbb{S}^{n-1} \times \mathbb{R}$ with its invariant (product) Riemannian metric. In particular M admits a finite covering space which is either a flat torus (the Euclidean case) or a *Hopf manifold,* a cyclic quotient of $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

Chapter 7 deals with the general classification of (G, X)-structures from the point of view of developing sections. The main result is an important observation due to Thurston [**323**] that the *deformation space* of marked (G, X)-structures on a fixed topology Σ is itself "locally modeled" on the quotient of the space $\mathsf{Hom}\big(\pi_1(\Sigma), G)\big)$ by the group $\mathsf{Inn}(G)$ of inner automorphisms of G. The description of $\mathbb{R}\mathsf{P}^1$-manifolds is described in this framework. The deformation space, however, is a non-Hausdorff 1-manifold, while the subspace consisting of closed affine 1-manifolds identifies with $[0, \infty)$. For affine structures on $\mathbb{T}^2$, the deformation space is not even a (non-Hausdorff) manifold.

Chapter 8 deals with the important notion of *completeness,* for taming the developing map. In general, the developing map may be quite pathological — even for closed (G, X)-manifolds — but under various hypotheses, can be proved to be a covering space onto its image. However, the main techniques borrow from Riemannian geometry, and involves *geodesic completeness* of the Levi–Civita connection (the Hopf–Rinow theorem). A complete affine manifold M is a quotient $\Gamma\backslash\mathsf{A}$, where A is an affine space and $\Gamma < \mathsf{Aff}(\mathsf{A})$ is a discrete subgroup acting properly on A. Equivalently, a developing map $\widetilde{M} \to \mathsf{A}$ is a homeomorphism (an affine isomorphism) of the universal covering space $\widetilde{M}$ onto A.

This requires relating geometric structures to *connections,* since all of the locally homogeneous geometric structures discussed in this book can be approached through this general concept. However, we do *not* discuss the general notion of *Cartan connections,* but rather refer to the excellent introduction to this subject by R. Sharpe [**305**]. Some aspects of the general theory of affine connections have been relegated to Appendix B.

Chapter 8 introduces some of the basic examples in our theory. Bieberbach's theorems [**53, 54**] successfully describe the structure and classification of Euclidean structures on closed manifolds:[1] *Every closed Euclidean manifold* M *is a biquotient* $\Lambda\backslash\mathbb{R}^/\Phi$ *where* $\Lambda < \mathbb{R}^n$ *is a lattice and* Φ *is a finite group of automorphisms of* Λ. In other words M is finitely covered by *flat torus,* such as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

One wonders if a similar picture holds when M is only assumed to be *affine,* and this is the *Auslander–Milnor question,* (sometimes called the "Auslander conjecture"): whether the fundamental group $\pi_1(M)$ is virtually polycyclic. In that case, M is finitely covered by a *solvmanifold* $\Gamma\backslash G$ where G is a solvable Lie group and $\Gamma < G$ is a lattice. Here G has a left-invariant complete affine structure, meaning that it acts simply transitively and affinely on affine space. It plays the role of the group of translations for Euclidean manifolds.

This question is open for closed manifolds, but Margulis [**253**] found proper affine actions of the 2-generator free group $\mathbb{F}_2$ on A^3, and their quotients, called *Margulis spacetimes,* are discussed in §15.4.

The first 3-dimensional examples are described, including $\mathsf{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, the Heisenberg nilmanifold $\mathsf{Heis}_{\mathbb{Z}}\backslash\mathsf{Heis}_{\mathbb{R}}$, and hyperbolic torus bundles $\mathsf{Sol}_{\mathbb{Z}}\backslash\mathsf{Sol}_{\mathbb{R}}$. They represent three of the eight *Thurston geometries* in dimension 3.

We classify complete affine structures on the 2-torus $\mathbb{T}^2$ (originally due to Kuiper [**234**]). The Hopf manifolds introduced in §6.2 are fundamental examples of incomplete structures. That affine structures on compact manifolds are generally incomplete is one dramatic difference between affine geometry and traditional Riemannian geometry.

The successful classification of affine (and projective) strucutres on $\mathbb{T}^2$ began with Kuiper [**234**] in the convex case. It was completed by Nagano–Yagi [**279**] and Arrowsmith–Furness [**141**]; Baues [**33**] provides an excellent exposition. They provide many basic examples, some of which generalize to higher dimensions. The classification of affine (and projective) 2-manifolds is somewhat messy but provides a paradigm for the problems discussed in this book. The classification is revisited several times to motivate some of the general theory, including deformation spaces and affine Lie groups.

[1]See Charlap [**84**] for a good exposition of this theory.

Part Three: Affine and projective structures

Chapter 9 begins the classification of affine structures on surfaces. We prove Benzécri's theorem [**45**] that a closed surface Σ admits an affine structure if and only if its Euler characteristic vanishes. We discuss the famous conjecture of Chern that the Euler characteristic of a closed affine manifold vanishes, giving the proof of Kostant–Sullivan [**225**] in the complete case.

Chapter 10 offers a detailed study of left-invariant affine structures on Lie groups. We will call a Lie group with a left-invariant affine structure an *affine Lie group.* These provide many examples; in particular all the non-radiant affine structures on $\mathbb{T}^2$ are *invariant* affine structures on the *Lie group* $\mathbb{T}^2$. For these structures the holonomy homomorphism and the developing map blend together in an intriguing way.[2] Covariant differentiation of left-invariant vector fields lead to well-studied nonassociative algebras called *algèbres symétriques à gauche* or (*left-symmetric* algebras). Such algebras have the property that their *associators* are s c in the left two variables. Commutator defines the structure of an underlying Lie algebra. Associative algebras correspond to *bi-invariant affine structures,* so the "group objects" in the category of affine manifolds correspond naturally to associative algebras. These structures were introduced by Ernest Vinberg [**340**] in his study of homogeneous convex cones in affine space, and further developed by Jean-Louis Koszul and his school. We take a decidedly geometric approach to these ubiquitous mathematical structures. For example, many closed affine surfaces are affine Lie groups.

Chapter 11 describes the question (apparently first raised by L. Markus [**254**]) of whether, for a closed orientable affine manifold, completeness is equivalent to *parallel volume.* The existence of a parallel volume form is equivalent to unimodularity of the linear holonomy group, that is, whether the holonomy preserves volume. An "infinitesimal analog" of this question for left-invariant affine structures on Lie groups is the conceptual and suggestive result that completeness is equivalent to parallelism of *right-invariant* vector fields, (Exercise 10.3.9 in §10.3.5.)

This tantalizing question has led to much research, subsuming various questions which we discuss. Carrière's proof that compact flat Lorentzian manifolds are complete [**78**] is a special case, and Smillie's nonexistence theorem is another special case, discussed in §11.3. Section 11.2 treats the case when the affine holonomy group Γ is nilpotent. Another example is Fried's sharp classification of closed similarity manifolds [**135**] (proved independently by a much different argument by Vaisman–Reischer [**331**]).

[2] Perhaps this provides a conceptual basis for the unexpected relation between the 1-dimensional property of geodesic completeness and the top-dimensional property of volume-preserving holonomy.

Chapter 12 expounds the notions of "hyperbolicity" of Vey [**337**] and Kobayashi [**222**]. *Hyperbolic affine manifolds* are quotients of properly convex cones. A closed hyperbolic manifold is a radiant suspension of an $\mathbb{RP}^n$-manifold, which itself is a quotient of a divisible domain. In particular we describe how a *completely incomplete* closed affine manifold must be affine hyperbolic in this sense. (That is, we tame the developing map of an affine structure with *no* two-ended complete geodesics.) This striking result is similar to the tameness where *all* geodesics are complete — complete manifolds are also quotients. The key ingredient is the *infinitesimal Kobayashi pseudo-metric,* which measures the (in)completeness of a geodoesic with given velocity.

Chapter 13 summarizes some aspects of the now blossoming subject of $\mathbb{RP}^2$-structures on surfaces, in terms of the explicit coordinates and deformations which extend some of the classic geometric constructions on the deformation space of hyperbolic structures on closed surfaces. We describe the analog of Fenchel–Nielsen coordinates and other coordinate systems, briefly mentioning a more analytic approach due independently to Loftin and Labourie. Then we describe the grafting construction, and the first examples, due to Smillie and Sullivan–Thurston, of a projective structure on $\mathbb{T}^2$ with pathological developing map.

Chapter 14 describes the classic subject of $\mathbb{CP}^1$-manifolds, which traditionally identify with *projective structures on Riemann surfaces.* Using the Schwarzian derivative, these structures are classified by the points of a holomorphic affine bundle over the Teichmüller space of Σ. This parametrization (presumably known to Poincaré), is remarkable in that is completely *formal,* using standard facts from the theory of Riemann surfaces. One knows precisely the deformation space without any knowledge of the developing map (besides it being a local biholomorphism). This is notable because the developing maps can be pathological; indeed the first examples of pathological developing maps were $\mathbb{CP}^1$-manifolds on hyperbolic surfaces. The theory of projective structures on Riemann surfaces is a suggestive paradigm for a successful classifaction of highly nontrivial geometric structures.

Chapter 15 surveys known results, and the many open questions, in dimension three. This complements Thurston's book [**324**] and expository articles of Scott [**302**] and Bonahon [**56**], which deal with geometrization and the relations to 3-manifold topology. In particular we describe the classification, due to Serge Dupont [**119, 120**], of projective structures on hyperbolic torus bundles

Prerequisites

This book is aimed roughly at first-year graduate students and advanced undergraduate students, although some knowledge of advanced material will be useful.

For general treatments of geometry, we refer to the two-volume text of Berger [**49**, **50**] (see also Berry–Pansu–Berry–Saint Raymond [**51**]) and Coxeter [**102**].

We also assume basic familiarity with elementary topology, smooth manifolds, and the rudiments of Lie groups and Lie algebras. Much of this can be found in Lee's book "Introduction to Smooth Manifolds" [**244**], including its appendices. For topology, we require basic familiarity with the notion of metric spaces, covering spaces, and fundamental groups.

Fiber bundes, as discussed in the still excellent treatise of Steenrod [**317**], or the more modern treatment of principal bundles given in Sontz [**315**], will be used.

Some familiarity with the properties of proper maps and proper group actions will also be useful.

Some familiarity with the theory of connections in fiber bundles and vector bundles is useful, for example, Kobayashi–Nomizu [**224**], or Milnor [**269**], do Carmo [**113**] Lee [**244**], O'Neill [**283**].

We put the discussion of Fenchel–Nielsen coordinates on Fricke space in the context of Darboux's theorem in symplectic geometry; we recommend §22 of Lee [**244**] for a good general treatment consistent with our notation.

Notation, terminology, and general background

Vectors and matrices

We work over a field k, usually the field $\mathbb{R}$ of real numbers, but sometimes the field $\mathbb{C}$ of complex numbers. We shall denote vectors and matrices in bold font. Let V be a vector space over k of dimension n. A basis determines an isomorphism $\mathsf{V} \cong \mathsf{k}^n$. Thus a vector in V corresponds to a column vector:

$$\mathbf{v} \quad \longleftrightarrow \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix}$$

A *covector* is defined as a linear functional $\mathsf{V} \xrightarrow{\omega} \mathsf{k}$, corresponding to a row vector:

$$\omega \quad \longleftrightarrow \quad \begin{bmatrix} \omega_1 & \ldots & \omega_n \end{bmatrix}$$

and the duality pairing between V and V^* is:

$$\begin{aligned} \mathsf{V} \times \mathsf{V}^* &\longrightarrow \mathsf{k} \\ (\mathbf{v}, \omega) &\longmapsto v^i \omega_i \end{aligned}$$

(summation over paired indices). A linear transformation $\mathsf{k}^m \longrightarrow \mathsf{k}^n$ is defined by an $m \times n$ matrix

$$\mathbf{A} = \begin{bmatrix} A^i{}_j \end{bmatrix}$$

mapping

$$\mathsf{k}^m \xrightarrow{\mathbf{A}} \mathsf{k}^n$$

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^m \end{bmatrix} \longmapsto \begin{bmatrix} A^1{}_j v^j \\ \vdots \\ A^n{}_j v^j \end{bmatrix}$$

where $j = 1, \dots, m$.

Affine vector fields on A correspond to affine maps $\mathsf{A} \to \mathsf{A}$:

$$(A^i{}_j x^j + b^i)\partial_i \quad \longleftrightarrow \quad \hat{\mathbf{A}} := \left[\mathbf{A} \;\middle|\; \mathbf{b}\right]$$

where

$$\mathbf{A} = \begin{bmatrix} A^1{}_1 & \dots & A^1{}_i & \dots & A^1{}_n \\ \vdots & & \vdots & & \vdots \\ A^i{}_1 & \dots & A^i{}_j & \dots & A^i{}_n \\ \vdots & & \vdots & & \vdots \\ A^n{}_1 & \dots & A^n{}_j & \dots & A^n{}_n \end{bmatrix}$$

is the linear part and

$$\mathbf{b} = \begin{bmatrix} b^1 \\ \vdots \\ b^i \\ \vdots \\ b^n \end{bmatrix}$$

is the translational part. In this notation,

$$\left[\mathbf{A} \;\middle|\; \mathbf{b}\right] = \left[\begin{array}{ccc|c} A^1_1 & \dots & A^1_n & b^1 \\ \vdots & & \vdots & \vdots \\ \dots & A^i{}_j & \dots & b^i \\ \vdots & & \vdots & \vdots \\ A^n_1 & \dots & A^n_n & b^n \end{array}\right]$$

Projective equivalence of vectors. Denote the multiplicative group of nonzero scalars in k by $\mathsf{k}^\times$, and let V be a vector space over k. Then $\mathsf{k}^\times$ acts by scalar multiplication on V. Define nonzero vectors $\mathbf{w}, \mathbf{u} \in \mathsf{V}$ to be *projectively equivalent* if and only if $\exists \lambda \in \mathsf{k}^\times$ such that $\mathbf{w} = \lambda \mathbf{u}$. Projective equivalence classes $[\mathbf{v}]$ of nonzero vectors $\mathbf{v}$ form the *projective space* $\mathsf{P}(\mathsf{V})$ associated to V. Denote the projective equivalence class of a vector

$$\mathbf{v} = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \in \mathsf{V}$$

by

$$[\mathbf{v}] := \left[\!\!\left[\begin{matrix} v^1 \\ \vdots \\ v^n \end{matrix} \right]\!\!\right]$$

and the projective equivalence class of a covector $\omega = \left[\, \omega^1 \ \dots \ \omega^n \,\right] \in \mathsf{V}^*$ by

$$[\omega] := \left[\omega^1 \quad \dots \quad \omega^n\right].$$

Projective equivalence classes of nonzero *covectors* comprise the projective space $\mathsf{P}(\mathsf{V}^*)$ *dual* to $\mathsf{P}(\mathsf{V})$. Just as points in the projective space $\mathsf{P}(\mathsf{V})$ correspond to projective equivalence classes of vectors, projective hyperplanes in $\mathsf{P}(\mathsf{V})$ correspond to projective equivalence classes of covectors (see §3.1).

General topology

For background in topology see Lee [**244**], Willard [**349**], Hatcher [**186**], and Greenberg–Harper [**172**].

If A and B are topological spaces, and $A \xrightarrow{f} B$ is a continuous map, then f is a *local homeomorphism* if and only if $\forall a \in A$, the restriction $f|_U$ is a homeomorphism $U \to f(U)$ for some open neighborhood $U \ni a$.

If A is a topological space, and $B \subset A$ is a subspace, then we write $B \subset\subset A$ if B is compact (in the subspace topology).

We denote the space of mappings $A \longrightarrow B$ by $\mathsf{Map}(A, B)$, given the compact-open topology and the group of homeomorphisms $X \to X$ by $\mathsf{Homeo}(X)$.

If f_n (for $n = 1, 2, \dots, \infty$) are mappings on a space X, write $f_n \rightrightarrows f_\infty$ if f_n converges *uniformly* to f_∞ on X, with respect to a given *uniform structure* (for exampe a metric) on X.

Suppose (X, d) is a metric space. If $x \in X, r > 0$, define the *(open) ball* with center x and radius r as:

$$\mathsf{B}_r(x) \ := \ \big\{ y \in X \mid \mathsf{d}(x, y) < r \big\}.$$

The open balls in a metric space are partially ordered by inclusion. More generally, if $A \subset X$, define

$$\mathsf{B}_r(A) \;:=\; \big\{ y \in X \mid \exists a \in A \text{ such that } \mathsf{d}(y,a) < r \big\}.$$

If (X, d) is a metric space, and $S, T \subset\subset X$, then define their *Hausdorff distance*

$$\mathsf{d}(S,T) \;:=\; \mathsf{inf}\big\{ r \in \mathbb{R} \mid S \subset \mathsf{B}_r(T) \text{ and } T \subset \mathsf{B}_r(S) \big\}.$$

If X is compact, then the set of closed subsets of X with Hausdorff distance d is a metric space.

Denote the group of isometries of a metric space (X, d) by $\mathsf{Isom}(X, \mathsf{d})$, or just $\mathsf{Isom}(X)$ if the context is clear.

Fundamental group and covering spaces. For this material, we recommend the first chapter of Hatcher [**186**].

If $[a,b] \xrightarrow{\gamma} X$ is a continuous path, write

$$\gamma(a) \overset{\gamma}{\rightsquigarrow} \gamma(b)$$

to indicate that γ runs between its two endpoints $\gamma(a), \gamma(b)$. Two such paths are *relatively homotopic* if they are homotopic by a homotopy fixing their endpoints. In that case we write $\gamma_1 \simeq \gamma_2$.

Fix an (arbitrary) basepoint $p_0 \in X$. A *loop based at* p_0 is a path $p_0 \overset{\gamma}{\rightsquigarrow} p_0$, that is, a continuous map $[0,1] \xrightarrow{\gamma} X$ with

$$\gamma(0) = p_0 = \gamma(1).$$

The *fundamental group* $\pi_1(M; p_0)$ corresponding to p_0 consists of relative homotopy classes $[\gamma]$ of based loops γ.

The group operation is defined by *concatenation* of paths: Suppose

$$[a_i, b_i] \xrightarrow{\gamma_i} X, \text{ for } i = 1, 2$$

are paths, with $b_1 = a_2$ and $\gamma_1(b_1) = \gamma_2(a_2)$. Define $\gamma_1 \star \gamma_2$ to be the continuous path

$$\gamma_1(a_1) \rightsquigarrow \gamma_2(b_2),$$

given by:

$$\begin{aligned} [a_1, b_2] &\xrightarrow{\gamma_1 \star \gamma_2} X \\ t &\longmapsto \begin{cases} \gamma_1(t) & \text{if } a_1 \le t \le b_1 \\ \gamma_2(t) & \text{if } a_2 \le t \le b_2 \end{cases} \end{aligned}$$

If γ_1, γ_2 are loops based at p_0, so is $\gamma_1 \star \gamma_2$, and concatenation defines a binary operation on $\pi_1(X, p_0)$.

The *constant path* p_0 defines an identity element on $\pi_1(X, p_0)$ since

$$p_0 \star \gamma \simeq \gamma \star p_0 \simeq \gamma.$$

Define the *inverse* of a path $[a,b] \xrightarrow{\gamma} M$

$$\begin{aligned}[a,b] &\xrightarrow{\gamma^{-1}} M \\ t &\longmapsto \gamma(a+b-t).\end{aligned}$$

If γ is a loop based at p_0, then

$$\gamma \star \gamma^{-1} \simeq \gamma^{-1} \star \gamma \simeq p_0,$$

obtaining *inversion* in $\pi_1(M; p_0)$. If $[a_3, b_3] \xrightarrow{\gamma_3} X$ with $\gamma_2(b_2) = \gamma_3(a_2)$, then

$$(\gamma_1 \star \gamma_2) \star \gamma_3 \simeq \gamma_1 \star (\gamma_2 \star \gamma_3),$$

implying associativity. Thus $\pi_1(X, p_0)$ is indeed a group.

Under rather general conditions on X (such as being a topological manifold) define the *universal covering space* (corresponding to p_0)

$$\widetilde{X^{(p_0)}} \xrightarrow{\Pi} X$$

as the collection of relative homotopy classes of paths γ starting at p_0, and ending at another point which we will call $\Pi(\gamma)$. Give $\widetilde{X^{(p_0)}}$ the quotient topology, which is the coarsest topology such that Π is continuous.

Then Π is a local homeomorphism, and indeed a *Galois covering space* (or regular covering space) with covering group $\pi_1(X, p_0)$.

The (left) action on $\widetilde{X^{(p_0)}}$ by deck transformations from $\pi_1(X, p_0)$ is defined as follows. Choose a point $p \in X$, a path $p_0 \overset{\eta}{\rightsquigarrow} p$ and a loop γ based at p_0. The action of $[\gamma]$ on $[\eta]$ is defined by:

$$[\eta] \xmapsto{[\gamma]} [\gamma \star \eta].$$

The action is free and proper, preserves $p = \Pi([\eta])$. The quotient map naturally identifies with Π and the quotient space $\widetilde{X^{(p_0)}}/\pi_1(X, p_0)$ naturally identifies with X.

Smooth manifolds

We shall work in the context of smooth manifolds, for which a good general reference is Lee [**244**]. This will enable the use of differential calculus locally, and notions of smooth mappings between manifolds. A *smooth manifold* is a Hausdorff space built from open subsets of $\mathbb{R}^n$, which we call *coordinate patches.* The *coordinate changes* are general smooth locally invertible maps. If M and N are given such structures, a continuous map $M \longrightarrow N$ is *smooth* if in the local coordinate charts it is given by a smooth map.

Smooth functions $M \longrightarrow \mathbb{R}$ form a commutative associative $\mathbb{R}$-algebra which we denote $\mathsf{C}^\infty(M)$.

This structure enables the *tangent bundle* $\mathsf{T}M$, whose points are the *infinitesimal displacements* of points in M. That is, to every smooth curve $(a,b) \xrightarrow{\gamma} M$, and parameter t with $a \le t \le b$, is a *velocity vector*

$$\gamma'(t) \in \mathsf{T}_{\gamma(t)}M$$

representing the infinitesimal effect of displacing $\gamma(t)$ along γ. Since the local coordinates change by general smooth locally invertible maps, there is no natural way of identifying these infinitesimal displacements at *different* points. Therefore we attach to each point $p \in M$, a "copy" T_pM of the model space $\mathbb{R}^n$, which represents the vector space of *infinitesimal displacements of* p. It is important to note that although the *fibers* T_pM are disjoint, that the union

$$\mathsf{T}M := \bigcup_{p\in M} \mathsf{T}_pM$$

is topologized as a smooth manifold (indeed, a smooth *vector bundle),* and not as the disjoint union (see below).

The velocity vector of a smooth curve is a *tangent vector at p,* which can be defined in two equivalent ways:

- Equivalence classes of smooth curves $\gamma(t)$ with $\gamma(0) = p$, where curves $\gamma_1 \sim \gamma_2$ if and only if
$$\frac{d}{dt}\bigg|_{t=0} f\circ\gamma_1(t) = \frac{d}{dt}\bigg|_{t=0} f\circ\gamma_2(t)$$
for all smooth functions $U \xrightarrow{f} \mathbb{R}$, where $U \subset M$ is an open neighborhood of p.
- Linear operators $\mathsf{C}^\infty(M) \xrightarrow{D} \mathbb{R}$ satisfying
$$D(fg) = D(f)g(p) + f(p)D(g).$$

The tangent space T_pM is a vector space *linearizing* the smooth manifold M at the point $p \in M$.

The space of tangent vectors forms a smooth vector bundle $\mathsf{T}M \xrightarrow{\Pi} M$, with fiber $\Pi^{-1}(p) := \mathsf{T}_pM$. If $U \ni p$ is a coordinate patch, then $\Pi^{-1}(U)$ identifies with $U \times \mathbb{R}^n$, and this defines a smooth coordinate atlas on $\mathsf{T}M$.

Let M, N be smooth manifolds, and $p \in M$. A mapping

$$M \xrightarrow{f} N$$

is *differentiable at* p if every infinitesimal displacement $\mathbf{v} \in \mathsf{T}_pM$ maps to an infinitesimal displacement $\mathsf{D}_pf(\mathbf{v}) \in \mathsf{T}_qN$, where $q = f(p)$. That is, if γ is a smooth curve with $\gamma(0) = p$ and $\gamma'(0) = \mathbf{v}$, then we require that

$f \circ \gamma$ is a smooth curve through q at $t = 0$; then we call the new velocity $(f \circ \gamma)'(0) \in \mathsf{T}_q N$ the value of the *differential* or *derivative*

$$\begin{aligned} \mathsf{T}_p M &\xrightarrow{(\mathsf{D}f)_p} \mathsf{T}_q N \\ \mathbf{v} &\longmapsto (f \circ \gamma)'(0). \end{aligned}$$

Clearly a smooth mapping is differentiable in the above sense.

If P is a third smoooth manifold, and $N \xrightarrow{g} P$ is a smooth map, the composition $M \xrightarrow{g \circ f} P$ is defined, and is a smooth map. The *Chain Rule* expresses the derivative of the composition as the composition of the derivatives of f and g: and $M \xrightarrow{f} N \xrightarrow{g} P$ are smooth maps, then the differential of a composition

$$\begin{tikzcd} M \arrow[r, "f"'] \arrow[rr, bend left, "g \circ f"] & N \arrow[r, "g"'] & P \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \mathsf{T}_x M \arrow[r, "(\mathsf{D}f)_x"'] \arrow[rr, bend left, "\big(\mathsf{D}(g \circ f)\big)_x"] & \mathsf{T}_{f(x)} N \arrow[r, "(\mathsf{D}g)_{f(x)}"'] & \mathsf{T}_{(g \circ f)(x)} P \end{tikzcd}$$

that is, $\mathsf{D}(g \circ f)_x = (\mathsf{D}g)_{f(x)} \circ (\mathsf{D}f)_x$.

If M, N are smooth manifolds, a *diffeomorphism* $M \longrightarrow N$ is an invertible smooth mapping whose inverse is also smooth. In particular a diffeomorphism is a smooth homeomorphism.

Now suppose $M \xrightarrow{f} N$ is a smooth map and $p \in M$ such that the differential

$$\mathsf{T}_p M \xrightarrow{(\mathsf{D}f)_p} \mathsf{T}_{f(p)} N$$

is an isomorphism of vector spaces. The *Inverse Function Theorem* guarantees the existence of an open neighborhood $U \ni p$ such that the restriction $f|_U$ is a diffeomorphism $U \to f(U)$. In particular $f(U) \subset N$ is open. Furthermore U can be chosen so that $(\mathsf{D}f)_q$ is an isomorphism for every $q \in U$. Such a map is called a *local diffeomorphism* (at p). Proposition 4.30 Lee [**244**] characterizes when a local diffeomrorphism is a smooth covering space.

Under the C^∞ topology, diffeomorphisms $M \to M$ form a topological group, denoted by $\mathsf{Diff}(M)$. Indeed $\mathsf{Diff}(M)$ has more structure as a *Fréchet Lie group.* If N is a smooth manifold, then a map $N \to \mathsf{Diff}(M)$ is *smooth* if the natural composition $N \times M \to M$ is smooth. A smooth homomorphism $\mathbb{R} \xrightarrow{\Phi} \mathsf{Diff}(M)$ is called a *smooth flow* on M.

Denote the group of diffeomorphisms of a smooth manifold X by $\mathsf{Diff}(X)$, with the C^∞ topology (uniform convergence to all orders, on all $K \subset\subset X$). If f, g are smooth maps between smooth manifolds $X \longrightarrow Y$, then we say that f and g are *isotopic* if and only if there is a smooth path

$$\phi_t \in \mathsf{Diff}(X),\ 0 \le t \le 1,$$

with $\phi_0 = \mathbb{I}_X$ such that $g = \phi_1 \circ f$. Denote this relation by $f \simeq g$.

Vector fields. A *vector field* on M is a section of the tangent bundle $\mathsf{T}M \xrightarrow{\Pi} M$, that is a mapping $M \xrightarrow{\xi} \mathsf{T}M$ such that

$$\Pi \circ \xi = \mathbb{I}_M,$$

or, equivalently, $\xi(p) \in \mathsf{T}_pM$ for all $p \in M$. Denote the space of all vector fields on M by $\mathsf{Vec}(M)$. Just as individual tangent vectors at $p \in M$ define derivations $\mathsf{C}^\infty(M) \longrightarrow \mathbb{R}$ over the evaluation map

$$\begin{aligned} \mathsf{C}^\infty(M) &\longrightarrow \mathbb{R} \\ f &\longmapsto f(p) \end{aligned}$$

vector fields in $\mathsf{Vec}(M)$ define derivations of the algebra $\mathsf{C}^\infty(M)$.

Let $a < b \in \mathbb{R}$. A smooth curve $(a, b) \xrightarrow{\gamma} M$ is an *integral curve* for $\xi \in \mathsf{Vec}(M)$ if and only if

$$\gamma'(t) = \xi\big(\gamma(t)\big) \in \mathsf{T}_{\gamma(t)}M$$

for all $a < t < b$. If Φ is a smooth flow as above, then for each $p \in M$,

$$\begin{aligned} \mathbb{R} &\xrightarrow{\Phi_p} M \\ t &\longmapsto \Phi(t)(p) \end{aligned}$$

is a smooth curve in M with velocity vector field $(\Phi_p)'(t) \in \mathsf{T}_{\Phi_p(t)}M$. In particular

$$\xi(p) := (\Phi_p)'(0) \in \mathsf{T}_pM \text{ (since } \Phi_p(0) = p)$$

and Φ_p being an integral curve of ξ defines a smooth vector field $\xi \in \mathsf{Vec}(M)$.

The *Fundamental Theorem on Flows* is a statement in the converse direction: every vector field $\xi \in \mathsf{Vec}(M)$ is tangent to a *local flow.* That is, through every point there exists a unique maximal integral curve, defined for some open interval (a, b) containing 0. When M is a closed manifold, then the integral curves are defined on all of $\mathbb{R}$ and corresponds to a flow Φ on M. Such a vector field is called *complete.* More generally (Lee [**244**], Theorem 9.16), if ξ is compactly supported, it is complete. See Lee [**244**], §9, for full details; a precise statement of the Fundamental Theorem on Flows is given in Theorem 9.12.

If f is a *local diffeomorphism,* and $\xi \in \mathsf{Vec}(N)$, then define the *pullback* $f^*\xi \in \mathsf{Vec}(M)$ by:

$$(f^*\xi)_p := \big((\mathsf{D}f)_p\big)^{-1}(\xi_{f(p)}). \tag{0.1}$$

In particular, in the terminology of Lee [**244**], the vector fields ξ and $f^*\xi$ are f-related.

Suppose that $M \xrightarrow{f} N$ is a smooth map and $\xi \in \mathsf{Vec}(M)$ and $\eta \in \mathsf{Vec}(N)$ are *f-related* vector fields, that is,

$$(\mathsf{D}f)_p\big(\xi(p)\big) = \eta\big(f(p)\big), \quad \forall p \in M.$$

The *Naturality of Flows* (Lee [**244**], Theorem 9.13) implies that if $\Phi(t)$ is the local flow defined by $\xi \in \mathsf{Vec}(M)$ and $\Psi(t)$ the local flow on N defined by $\eta \in \mathsf{Vec}(N)$, then

$$f\big(\Phi_t(p)\big) = \Psi_t\big(f(p)\big)$$

whenever these objects are defined.

Tensor fields and differential forms. Given any smooth vector bundle $\mathsf{W} \longrightarrow M$ over a smooth manifold M, its sections $M \to E$ comprise a module $\Gamma(\mathsf{W})$ over the ring $\mathsf{C}^\infty(M)$. For example

$$\mathsf{Vec}(M) = \Gamma(\mathsf{T}M)$$

is a $\mathsf{C}^\infty(M)$-module; $\mathsf{C}^\infty(M)$ acts on $\mathsf{Vec}(M)$ by scalar multiplication of vector fields by functions. Furthermore $\mathsf{Vec}(M)$ is a Lie algebra under Lie bracket. Although Lie multiplication is not $\mathsf{C}^\infty(M)$-blinear, these two structures relate via:

$$[f\,\xi,\, g\,\eta] \;=\; fg[\xi,\eta] + f\,(\xi g)\,\eta - g\,(\eta f)\,\xi.$$

Suppose V, W are vector bundles over M. A bundle map $\mathsf{V} \to \mathsf{W}$ determines a homomorphism of $\mathsf{C}^\infty(M)$-modules $\Gamma(\mathsf{V}) \longrightarrow \Gamma(\mathsf{W})$. Conversely an $\mathbb{R}$-linear mapping $\Gamma(V) \longrightarrow \Gamma(\mathsf{W})$ corresponds to a bundle map if and only if it is linear over $\mathsf{C}^\infty(M)$, that is, a homomorphism of $\mathsf{C}^\infty(M)$-modules. Such bundle maps identify with sections of the vector bundle $\mathsf{Hom}(\mathsf{V},\mathsf{W})$. Compare Lee [**244**], §5.16.

For example, suppose that W is a vector bundle over M and

$$\underbrace{\mathsf{Vec}(M) \times \cdots \times \mathsf{Vec}(M)}_{k} \longrightarrow \Gamma(\mathsf{W})$$

is multilinear over $\mathsf{C}^\infty(M)$. Then F is induced by a vector bundle homomorphism

$$\otimes^k \mathsf{T}M \;:=\; \underbrace{\mathsf{T}M \otimes \cdots \otimes \mathsf{T}M}_{k} \longrightarrow \mathsf{W},$$

or, equivalently, a section of $\mathsf{Hom}\big(\otimes^k(\mathsf{T}M), \mathsf{W}\big)$. Denote the space of such W-*valued covariant tensor fields* by

$$\mathcal{T}^k(M;\mathsf{E}) \longleftrightarrow \Gamma\Big(\mathsf{Hom}\big(\otimes^k \mathsf{T}M, \mathsf{W}\big)\Big).$$

The case when $k = 1$ is particularly important. Then

$$\mathsf{Hom}\big(\otimes^k \mathsf{T}M, \mathsf{W}\big) \cong \mathsf{T}^*M \otimes \mathsf{W},$$

whose sections are W-*valued* 1-*forms* on M. When $\mathsf{W} = \mathsf{T}M$, these are sections of

$$\mathsf{End}(\mathsf{T}M) := \mathsf{Hom}(\mathsf{T}M, \mathsf{T}M) \longleftrightarrow \mathsf{T}^*M \otimes \mathsf{T}M,$$

which we call *endomorphism fields.* An example of an endomorphism field is the identity map on $\mathsf{T}M$, which we can also think of as a $\mathsf{T}M$-valued 1-form. It is sometimes called the *solder form.*

Exterior differential calculus

Sections of the associated exterior algebra bundle $\Lambda^k(\mathsf{T}M)$ are *exterior differential forms* of degree $k \geq 0$; they comprise the $\mathsf{C}^\infty(M)$-module $\mathcal{A}^k(M)$. The direct sum

$$\mathcal{A}^*(M) := \bigoplus_{k=0}^{n} \mathcal{A}^k(M)$$

is a graded algebra over $\mathsf{C}^\infty(M)$; if $\alpha \in \mathcal{A}^p(M)$ then p is its *degree* and we write $p = |\alpha|$. Explicitly, If $\alpha \in \mathcal{A}^p(M)$, $\beta \in \mathcal{A}^q(M)$, their exterior product $\alpha \wedge \beta \in \mathcal{A}^{p+q}(M)$ is defined by:

$$(\alpha\wedge\beta)(\xi_1,\dots,\xi_{p+q}) \;=\; \frac{1}{p!q!}\sum_{\sigma\in\mathfrak{S}_{p+q}} (-1)^{|\sigma|}\alpha\big(\xi_{\sigma(1)},\dots,\xi_{\sigma(p)}\big)\,\beta\big(\xi_{\sigma(p+1)},\dots,\xi_{\sigma(p+q)}\big)$$

where $\xi_1,\dots,\xi_{p+q} \in \mathsf{Vec}(M)$.

This graded algebra is associative and graded-commutative under exterior (wedge) product, where *graded-commutativity* means

$$\beta\wedge\alpha \;=\; (-1)^{pq}\alpha\wedge\beta.$$

A collections of maps $\mathcal{A}^p(M) \xrightarrow{D_p} \mathcal{A}^{p+k}(M)$ is a *derivation of degree* k if and only if

$$D_{p+q}(\alpha\wedge\beta) \;=\; (D_p\alpha)\wedge\beta) + (-1)^{pk}\alpha\wedge D_q(\beta),$$

and if D, D' are derivations, their *commutator* $[D, D']$ defined by:

$$[D, D'] \;:=\; D\circ D' - (-1)^{|D||D'|}D'\circ D$$

is a derivation of degree $|D| + |D'|$. We describe three derivations: *exterior differentiation* of degree $+1$, *interior multiplication* of degree -1 and depending on a vector field, and *Lie differentiation* of degree 0 and depending on a vector field.

- *Exterior differentiation* $\mathcal{A}^k(M) \xrightarrow{d} \mathcal{A}^{k+1}(M)$ is a derivation of degree 1:
$$d(\alpha \wedge \beta) \;=\; d\alpha \wedge \beta + (-1)^{|\alpha|}\alpha \wedge d\beta$$
and satisfies:
$$df(\xi) \;=\; \xi f$$
for $\xi \in \mathsf{Vec}(M)$. Furthermore $d \circ d = 0$, and these properties uniquely characterize d.
- For any vector field $\xi \in \mathsf{Vec}(M)$, *interior multiplication* (or contraction) ι_ξ defined by
$$\iota_\xi(\omega)(\eta_1, \dots, \eta_{k-1}) \;:=\; \omega(\xi, \eta_1, \dots, \eta_{k-1}),$$
for $\omega \in \mathcal{A}^k(M), \eta_1, \dots, \eta_{k-1} \in \mathsf{Vec}(M)$.
- If $\xi \in \mathsf{Vec}(M)$ generates a local flow Φ_t, and ω is a tensor field on M, then the *Lie derivative* $\mathfrak{L}_\xi(\omega) \in \mathsf{Vec}(M)$ is defined as:
$$\mathfrak{L}_\xi(\omega) \;:=\; \frac{\partial}{\partial t}\Big|_{t=0} (\Phi_t)_*(\omega)$$
and defines a derivation of degree 0 on $\mathcal{A}^*(M)$.

Cartan's magic formula (Lee [**244**], Proposition 18.13) relates these these derviations through the graded commutator operation:

$$[d, \iota_\xi] \;:=\; d\iota_\xi + \iota_\xi d \;=\; \mathfrak{L}_\xi \tag{0.2}$$

Furthermore the graded commutator

$$[\mathfrak{L}_\xi,\, \iota_\eta] \;:=\; \mathfrak{L}_\xi\, \iota_\eta - \iota_\eta\, \mathfrak{L}_\xi \;=\; \iota_{[\xi,\eta]} \tag{0.3}$$

(Lee [**244**], Proposition 18.9(e)). In particular, if $\alpha \in \mathcal{A}^1(M), \omega \in \mathcal{A}^2(M)$ and $\xi, \eta, \zeta \in \mathsf{Vec}(M)$,[3] then

$$d\alpha(\xi, \eta) \;=\; \xi\alpha(\eta) - \eta\alpha(\xi) - \alpha([\xi, \eta]) \tag{0.4}$$

$$\begin{aligned} d\omega(\xi, \eta, \zeta) \;=\;& \xi\omega(\eta, \zeta) + \eta\omega(\zeta, \xi) + \zeta\omega(\xi, \eta) - \\ & \omega([\xi, \eta], \zeta) - \omega([\eta, \zeta], \xi) - \omega([\zeta, \xi], \eta) \end{aligned} \tag{0.5}$$

[3] Use the formula for $d\omega$ as found in Lee [**244**], or Kobayashi–Nomizu [**224**] — but note that [**224**] uses the "Alt-convention" for differential forms, Lee [**244**], §12, p.302.

For any subbundle $E \subset \mathsf{T}M$, the annihilators

$$\mathsf{Ann}^p(E) \ := \ \Big\{ \alpha \in \mathcal{A}^p(M) \ \Big| \ \alpha(v_1, \dots, v_p) = 0, \ \forall v_1, \dots, v_p \in E \Big\}$$

define an ideal in the graded algebra $\mathcal{A}^*(M)$. Integrability is equivalent to this ideal being stable under d, that is, $\mathsf{Ann}^*(E)$ is a *differential ideal* in $\mathcal{A}^*(M)$ (Lee [**244**], Proposition 19.9).

Connections on vector bundles

We briefly summarize some general facts about Koszul connections which we use later. Compare Kobayashi–Nomizu [**224**] for more details.

If W is a vector bundle, then a *connection* on W is an $\mathbb{R}$-bilinear mapping

$$\begin{aligned} \mathsf{Vec}(M) \times \Gamma(\mathsf{W}) &\longrightarrow \Gamma(W) \\ (\xi, w) &\longmapsto \nabla_\xi(w) \end{aligned}$$

satisfying:

$$\begin{aligned} \nabla_{f\xi}(w) &= f\, \nabla_\xi w \\ \nabla_\xi(fw) &= f\, \nabla_\xi w \, + \, (\xi f)\, w \end{aligned}$$

for all $f \in \mathsf{C}^\infty(M)$. We call $\nabla_\xi w$ the *covariant derivative* of w with respect to ξ. Alternatively, tensoriality implies that this bilinear mapping is equivalent to an $\mathbb{R}$-linear mapping

$$\Gamma(\mathsf{W}) \xrightarrow{\nabla} \Gamma\big(\mathsf{T}^*M \otimes \mathsf{W}\big)$$

(called the *covariant differential*) satisfying

$$\nabla(fw) \ = \ f\, \nabla w + df \otimes w$$

for all $f \in \mathsf{C}^\infty(M)$. (Compare [**224**],Propositions 2.10,2.11,2.12.) The difference between two connections on W, as linear maps

$$\Gamma(\mathsf{W}) \to \Gamma(\mathsf{T}^*M \otimes \mathsf{W}) \ = \ \mathcal{A}^1(M, \mathsf{W}),$$

is an $\mathsf{End}(\mathsf{W})$-valued 1-form. Indeed, the vector space $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$ of $\mathsf{End}(\mathsf{W})$-valued 1-forms acts simply transitively on the space $\mathfrak{A}(\mathsf{W})$ of connections on E. That is, $\mathfrak{A}(\mathsf{W})$ is an affine space with underlying vector space $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$, in other words, an $\mathcal{A}^1\big(M, \mathsf{End}(\mathsf{W})\big)$*-torsor.*

The *Riemann curvature tensor* (or simply the *curvature*) of ∇ is the $\mathsf{End}(\mathsf{W})$-valued exterior 2-form

$$\mathsf{Riem}_\nabla \ \in \ \mathcal{A}^2\big(M; \mathsf{End}(\mathsf{W})\big) \ = \ \Gamma\Big(\mathsf{Hom}\big(\Lambda^2 \mathsf{T}M, \mathsf{End}(\mathsf{W})\big)\Big)$$

defined by

$$\begin{aligned} \mathsf{Vec}(M) \times \mathsf{Vec}(M) \times \Gamma(\mathsf{W}) &\xrightarrow{\mathsf{Riem}_\nabla} \qquad \Gamma(\mathsf{W}) \\ (\xi, \eta; w) \qquad &\longmapsto \Big(\nabla_\xi \nabla_\eta - \nabla_\eta \nabla_\xi - \nabla_{[\xi,\eta]}\Big) w. \end{aligned}$$

A pleasant exercise is to show that this expression for Riem_∇ is $\mathsf{C}^\infty(M)$-trilinear, and thus corresponds to an $\mathsf{End}(\mathsf{W})$-valued exterior 2-form

$$\mathsf{Riem}_\nabla \in \mathsf{A}^2\big(M; \mathsf{End}(\mathsf{W})\big).$$

(Compare Kobayashi–Nomizu [**224**], §5, Chapter 6 of Burago–Burago–Ivanov [**68**] for Riemannian connections, and Appendix C of Milnor–Stasheff [**273**].)

The covariant differential operator

$$\mathcal{A}^0(M) \;=\; \Gamma(\mathsf{W}) \xrightarrow{\nabla} \Gamma(\mathsf{T}^*M \otimes \mathsf{W}) \;=\; \mathcal{A}^0(M, \mathsf{W})$$

extends to a mapping, the *covariant exterior differential,*

$$\mathcal{A}^k(M; \mathsf{W}) \xrightarrow{D_\nabla} \mathcal{A}^{k+1}(M; \mathsf{W})$$

such that if $\alpha \in \mathcal{A}^k(M), \beta \in \mathcal{A}^l(M; \mathsf{W})$, (so that $\alpha \wedge \beta \in \mathcal{A}^{l+1}(M; \mathsf{W})$), then

$$D_\nabla(\alpha \wedge \beta) \;=\; d\alpha \wedge \beta + (-1)^k \alpha \wedge D_\nabla \beta.$$

Then the square

$$\mathcal{A}^k(M; \mathsf{W}) \xrightarrow{D_\nabla \,\circ\, D_\nabla} \mathcal{A}^{k+2}(M; \mathsf{W})$$

equals exterior multiplication/composition with Riem_∇. In particular, if ∇ is *flat,* that is, $\mathsf{Riem}_\nabla = 0$, then $(D_\nabla)^2 = 0$ and W-valued de Rham cohomlogy $H^k(M; \mathsf{W})$ is defined.

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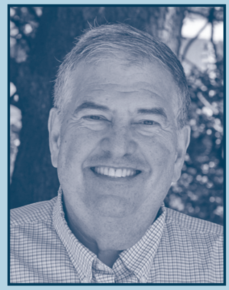

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The theory of geometric structures on manifolds which are locally modeled on a homogeneous space of a Lie group traces back to Charles Ehresmann in the 1930s, although many examples had been studied previously. Such locally homogeneous geometric structures are special cases of Cartan connections where the associated curvature vanishes. This theory received a big boost in the 1970s when W. Thurston put his geometrization program for 3-manifolds in this context. The subject of this book is more ambitious in scope. Unlike Thurston's eight 3-dimensional geometries, it covers structures which are not metric structures, such as affine and projective structures.

This book describes the known examples in dimensions one, two and three. Each geometry has its own special features, which provide special tools in its study. Emphasis is given to the interrelationships between different geometries and how one kind of geometric structure induces structures modeled on a different geometry. Up to now, much of the literature has been somewhat inaccessible and the book collects many of the pieces into one unified work. This book focuses on several successful classification problems. Namely, fix a geometry in the sense of Klein and a topological manifold. Then the different ways of locally putting the geometry on the manifold lead to a "moduli space". Often the moduli space carries a rich geometry of its own reflecting the model geometry.

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ISBN 978-1-4704-7103-3
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