VISUALIZING EXTERIOR CALCULUS

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Abstract. Exterior calculus, broadly, is the structure of differential forms. These are usually presented algebraically, or as computational tools to simplify Stokes' Theorem. But geometric objects like forms should be visualized! Here, I present visualizations of forms that attempt to capture their geometry.

Contents

Introduction

What's the natural way to integrate over a smooth manifold? Well, a k-dimensional manifold is (locally) parametrized by k coordinate functions. At each point we get k tangent vectors, which together form an infinitesimal k-dimensional volume element. And how should we measure this element? It seems reasonable to ask for a function f taking k vectors, so that

- f is linear in each vector
- f is zero for degenerate elements (i.e. when vectors are linearly dependent)
- f is smooth

These are the properties of a *differential form.*

Introducing structure to a manifold usually follows a certain pattern:

- Create it in $\mathbb{R}^n$. We define the exterior algebra in section 1.
- Introduce it locally in coordinate neighborhoods. We devote sections 2 and 3 to this.
- Understand it globally by patching together the coordinate neighborhoods. This is beyond the scope of this paper, except for a few remarks at the end of section 3.

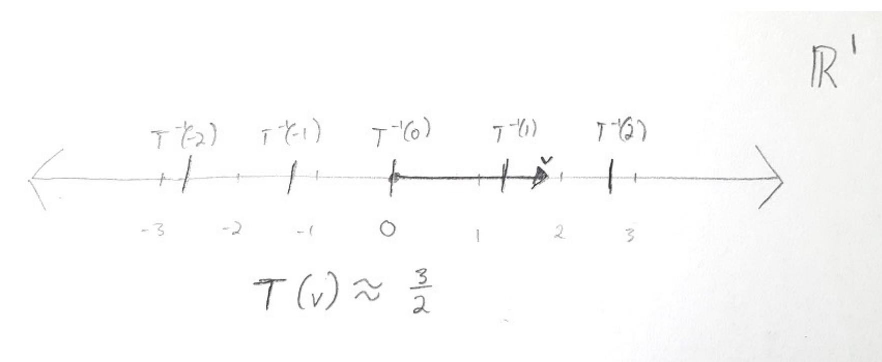
1. Exterior algebra

Given a vector space[1] V, we can construct its **exterior algebra**[2] $\Lambda(V)$, a sequence of vector spaces $\Lambda^k(V)$ which represent k-dimensional volume elements.We measure these volume elements via the dual space $\Lambda^k(V)^\vee$. This generalizes the line integrals seen in a typical introductory multivariable calculus class: $\int \mathbf{F} \cdot \mathrm{d}\mathbf{r}$. Here, "$\mathbf{F}\ \cdot$" is a dual vector measuring the line element $\mathrm{d}\mathbf{r}$.

We face a double challenge: to see the *exterior algebra* of *dual vectors.* We'll tackle this by first visualizing dual vectors, and then visualizing the exterior algebra of (regular) vectors and dual vectors in tandem.

1.1. **Vectors and dual vectors.** Visually, a vector in $\mathbb{R}^n$ is an arrow whose tail is at the origin. It's less clear how to visualize dual vectors. Since there is an isomorphism $V^\vee \cong V$, why not simply visualize $V^\vee$ as V itself? The main problem with this approach is that it transforms the wrong way under a change of basis. For example, suppose we're initially measuring in feet and switch to inches: if our old basis is $\{e_1, \ldots, e_n\}$, then our new basis is $\{e_1/12, \ldots, e_n/12\}$. Vectors appear to get 12 times bigger. But for any $T \in V^\vee$ and $v \in V$, the value of $T(v)$ is independent of basis, so T must *shrink* by a factor of 12, not *grow.*

Instead, in $\mathbb{R}^1$, view a dual vector a collection of "tick marks" on the axis:



In $\mathbb{R}^n$, we can do the same, with a caveat. We must choose an axis with tick marks, and project the vector into this axis.

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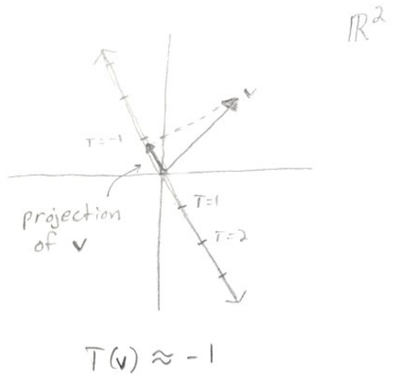
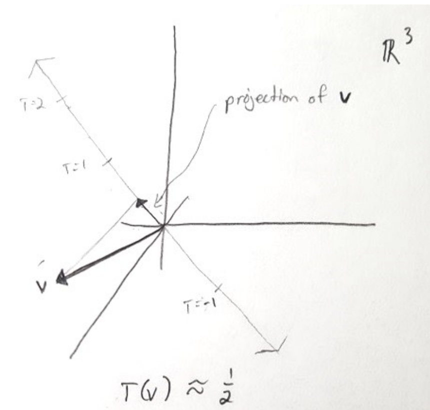
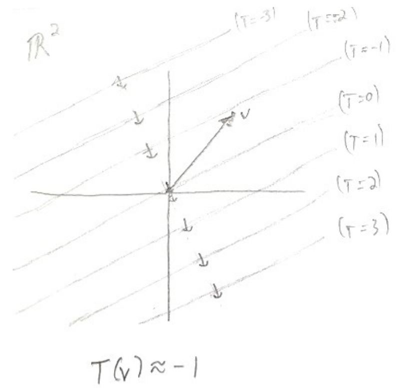
[1] In this paper, all vector spaces are finite-dimensional.

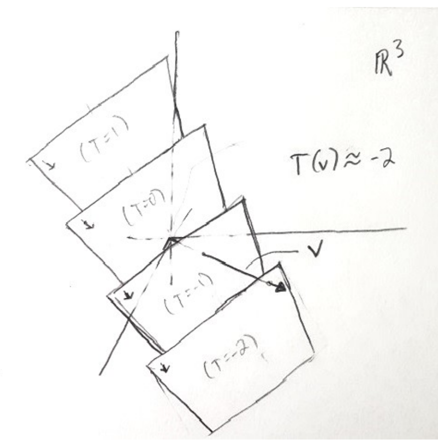
[2] The notation "Λ" seems to be a corruption of the symbol $\wedge$, the exterior/wedge product. Some authors write $\bigwedge(V)$ instead.





This doesn't really measure the vector: it measures the *projection.* We can measure vectors more directly by orthogonally extending the tick marks. This yields a series of lines in $\mathbb{R}^2$, planes in $\mathbb{R}^3$, and hyperplanes in general.





Formally, we can think of this geometric representation of a dual vector T as the set of pre-images $T^{-1}(n)$ of integers n.

1.2. **The wedge of two vectors.** Allow me to start with a historical note. In $\mathbb{R}^3$, in addition to the vector space operations of addition and scaling, we have the dot product and cross product. We compute the dot product $u \cdot v$ by projecting u along v and then multiplying their signed lengths. We compute the cross product $u \times v$ by first forming the parallelogram P spanned by u and v and then setting the length of $u \times v$ to be the area of P, in the direction of the (oriented) normal to P.

These operations were not always called "dot product" and "cross product". Hermann Grassman essentially invented linear algebra in his 1844 book *Die lineale Ausdehnungslehre* ([3]), including the exterior algebra. In the preface, he calls these the "interior product"[3] and "exterior product," respectively. He reasons that the dot product takes its maximum value when one vector is *inside* the span of the other, while the cross product is only nonzero if each vector is *outside* the span of the other.

[3] This is not the same as the interior multiplication we define later! Also, this is probably where the term "inner product" comes from.

Anyways, back to describing multi-dimensional volume. The cross product is essentially the right idea—it's no accident that it shows up whenever we compute a surface integral. To generalize it, we need to fix a quirk of the cross product: it describes a 2-dimensional *area* as a 1-dimensional *vector*. The fix is to describe the result as a "2-dimensional vector"![4]

We implement this fix by defining a new operation, the **wedge product** (or **exterior product**) $u \wedge v$ of vectors u and v. We want it to satisfy many of the same properties as the cross product:

- $(u_1 + u_2) \wedge v = u_1 \wedge v + u_2 \wedge v$
- $c(u \wedge v) = (cu) \wedge v = u \wedge (cv)$
- $u \wedge v = -(v \wedge u)$, so in particular, $v \wedge v = 0$

1.3. **Defining the exterior algebra.** The wedge product looks essentially like the tensor product, but with some extra relations (in the third bullet above). So, it should seem natural to define the space of 2-vectors, $\Lambda^2(V)$, as a quotient of the tensor power $V \otimes V$. Before we do this, recall a little trick, which shows that the relations $v \wedge v = 0$ imply $u \wedge v = -v \wedge u$.

$$0 = (u + v) \wedge (u + v) = u \wedge u + u \wedge v + v \wedge u + v \wedge v = u \wedge v + v \wedge u.$$

We now define

$$\Lambda^2(V) = \frac{V \otimes V}{I},$$

where I is the subspace of $V \otimes V$ generated by the tensors $v \otimes v$. We can generalize this definition to higher powers:

Definition 1.1. Let V be a vector space, and k a nonnegative integer. Then

$$\Lambda^k(V) = \frac{V^{\otimes k}}{I_k},$$

where I_k is the subspace of $V^{\otimes k}$ generated by tensors $v_1 \otimes \cdots \otimes v_k$, where at least one of the v_i is repeated. (Or equivalently, where the v_i are linearly dependent.)

Recall that, if $\{e_1, \ldots, e_n\}$ is a basis for V, then the elements

$$e_I = e_{i_1} \wedge \cdots \wedge e_{i_k}$$

form a basis for $\Lambda^k(V)$. Here, I ranges over all strictly increasing k-tuples in $\{1, \ldots, n\}$. The dimension of $\Lambda^k(V)$ is the number of such tuples, which is $\binom{n}{k}$.

Finally, treating the exterior algebra as a whole makes the definition even simpler: Let $T(V)$ be the tensor algebra for V. Then

$$\Lambda(V) = T(V)/I,$$

where I is the ideal generated by the tensors $v \otimes v$.[5]

[4] Perhaps I'm being unfair to the cross product here. It really is remarkable that in $\mathbb{R}^3$, we can describe area using vectors, and the cross product is particularly useful since it spits out an object of the same type as its inputs.

[5] The ideal I is different from the subspace I from the definition of $\Lambda^2(V)$, since now we also use $\otimes$ when generating I.

1.4. **The wedge of two dual vectors.** We've seen that the wedge of two vectors looks like a stretchy, oriented parallelogram. How about the wedge of two dual vectors? First, recall the usual definition of 2-forms.

Proposition 1.2. *Let V be a finite-dimensional vector space. Then*

$$\Lambda^2(V^\vee) = W,$$

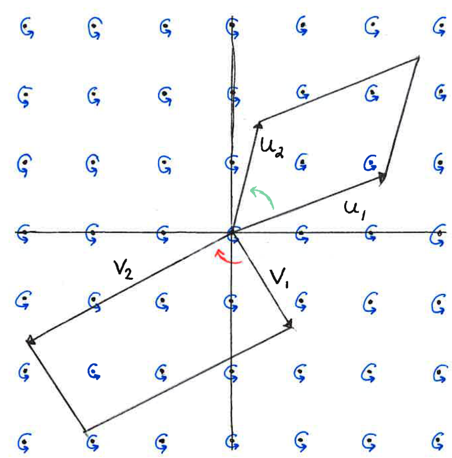
where W is the space of bilinear, alternating[6] *functions $V \times V \to \mathbb{R}$.*

Let's start simply, in $\mathbb{R}^2$. Here, we have a way to measure oriented parallelograms: the determinant. Indeed, the function

$$(u, v) \mapsto \det\left(\begin{bmatrix} u & v \end{bmatrix}\right)$$

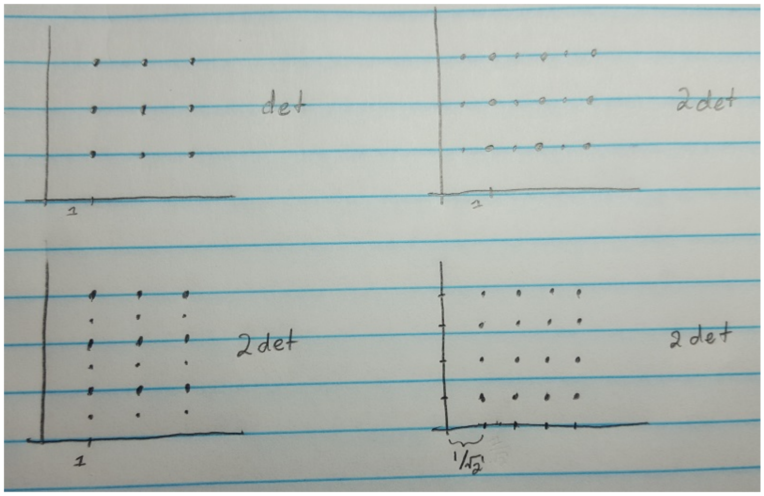
is bilinear and alternating. The determinant in $\mathbb{R}^2$ is a particular 2-form; recall that in general, the determinant $\det_n$ in $\mathbb{R}^n$ is equal to $e_1^\vee \wedge \cdots \wedge e_n^\vee$, where $\left\{e_j^\vee\right\}$ is the usual basis for $(\mathbb{R}^n)^\vee$.

Here's how we can visualize the determinant: Place a dot at each lattice point in the plane. To approximate $\det(u \wedge v)$, draw the parallelogram $u \wedge v$ and count how many dots are inside. Keep track of orientations with a small circle around each dot. For example, in the below figure, $\det(u_1 \wedge u_2) \approx 3$ and $\det(v_1 \wedge v_2) \approx -4$.

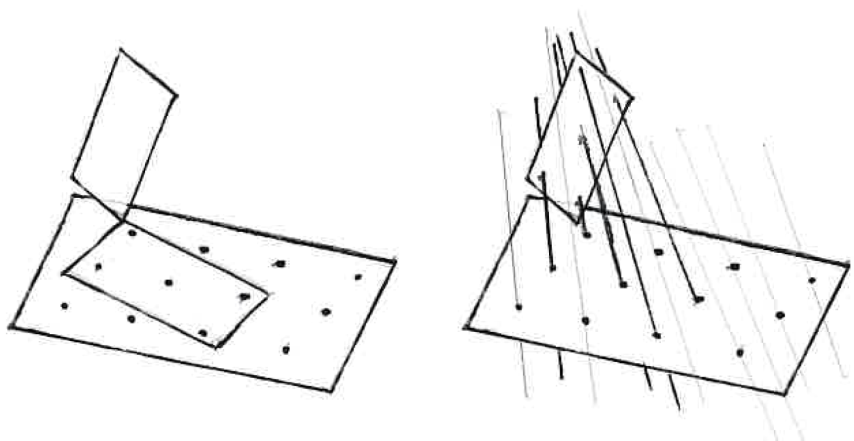


From now on, I'll mostly ignore orientations. To rescale the determinant, move the dots closer together or farther apart, reversing the circles if the scalar is negative. Due to the bilinearity of the determinant, it doesn't matter how you choose to move the dots.

[6] A function is **alternating** if its sign flips when swapping any two of its arguments.



In $\mathbb{R}^3$, we can get a 2-form by first choosing a plane and placing dots, projecting the 2-vector onto the plane, and counting the dots in the projection. Like with dual vectors, we can un-project the dots, which gives us a collection of *evenly spaced lines.*



To summarize: we can geometrically represent the wedge of two dual vectors $\alpha \wedge \beta$ as $\alpha^{-1}(\mathbb{Z}) \cap \beta^{-1}(\mathbb{Z})$. In general, we represent a (pure) k-form $\omega_1 \wedge \cdots \wedge \omega_k$ as

$$\omega_1^{-1}(\mathbb{Z}) \cap \cdots \cap \omega_k^{-1}(\mathbb{Z}).$$

1.5. $\mathbb{R}^3$ **has no mixed wedges.** There's an important distinction between *pure* and *mixed* tensors. Pure tensors are those that can be written as a single term $v_1 \otimes \cdots \otimes v_k$, whereas mixed tensors cannot. For example, while you can write

$$u_1 \otimes v_1 + u_1 \otimes v_2 + u_2 \otimes v_1 + u_2 \otimes v_2 = (u_1 + u_2) \otimes (v_1 + v_2),$$

you typically cannot find vectors a and b so that

$$u_1 \otimes v_1 + u_2 \otimes v_2 = a \otimes b.$$

As long as the dimensions of V and W are at least 2, we can pick linearly independent vectors $u_1, u_2 \in U$ and $v_1, v_2 \in V$. Then, $U \otimes V$ will have mixed tensors, such as $u_1 \otimes v_1 + u_2 \otimes v_2$. The same, however, is *not always* true for $\Lambda^k(V)$.

Proposition 1.3. *Let V have dimension n. Then $\Lambda^k(V)$ has mixed wedges if and only if* $1 < k < n-1$.

This may explain why differential forms developed much later than classical multivariable calculus. In three or fewer dimensions, we never need to worry about mixed wedges, so we can integrate over lines and surfaces using regular vectors. Even in higher dimensions, we can integrate over curves and hypersurfaces with impunity. We only needed forms when we wanted to be able to integrate 2-dimensional surfaces in $\mathbb{R}^4$, for example.

Proof sketch of ($\Leftarrow$). Consider the function[7] $J : V^k \to \Lambda^k(V)$ given by

$$J(v_1, \dots, v_k) = v_1 \wedge \cdots \wedge v_k.$$

Then $\Lambda^k(V)$ has mixed tensors if and only if J is not surjective.

The function J is not linear, but it is smooth. ($\Lambda^k(V)$ is a smooth manifold since it is a finite-dimensional real vector space.) The derivative of J is a $\binom{n}{k} \times nk$ matrix[8], which has the form

$$DJ = \begin{bmatrix} A & | & -A & | & \cdots & | & (-1)^{k+1}A \end{bmatrix}$$

because the wedge product is alternating. Since A is a $\binom{n}{k} \times n$ matrix,

$$\operatorname{rank}(DJ) = \operatorname{rank}(A) \leq n.$$

When $1 < k < n-1$, we have $\operatorname{rank}(DJ) \leq n < \binom{n}{k}$, so every point of V^k is a critical point. Hence, by Sard's theorem, J cannot be surjective. □

Proof sketch of ($\Rightarrow$). The space $\Lambda^n(V)$ has dimension 1, so all wedges are parallel to some pure wedge, and are hence pure. When $k = 1$, the space $\Lambda^k(V)$ is just V itself and J is the identity map.

Let P be the set of pure $(n-1)$-wedges. We show that P is closed under addition. There is an isomorphism $\perp\colon P \to V$, given by taking the normal vector, scaled by volume. (In $\mathbb{R}^3$, $\perp$ maps $u \wedge v$ to $u \times v$.) Though many parallelepipeds can have the same normal vector, they're all equivalent when considered as multivectors, so $\perp$ is injective. Add pure wedges by adding their normal vectors and applying $\perp^{-1}$. □

1.6. **Interior multiplication.** Instead of thinking of a k-form as a linear function $\Lambda^k(V) \to \mathbb{R}$, we may think of it as an alternating k-linear function $V^k \to \mathbb{R}$. (In other words, $\Lambda^k(V^\vee) \cong \Lambda^k(V)^\vee$.) Hence, we can define an operation $\iota\colon V \times \Lambda^k(V^\vee) \to \Lambda^{k-1}(V^\vee)$ by plugging a specified vector v into the first argument of a form:

$$\iota_v \omega = \omega(v, -).$$

Here, I'm omitting the free variables: the above definition means

$$(\iota_v(\omega))(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1}).$$

We call ι **interior multiplication**.[9]

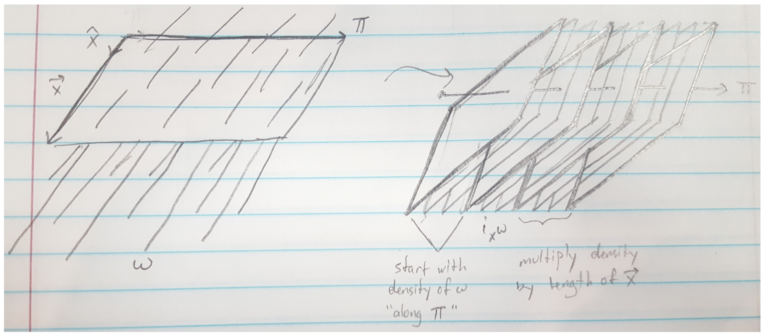
[7] I chose J for **j**oin, since J combines vectors. The more obvious choice, W for **w**edge, could be confused for another vector space.

[8] Here, I'm picking a basis for V and using the corresponding basis for $\Lambda^k(V)$.

[9] The operation ι has many names: interior product, interior derivative, interior multiplication, contraction, insertion, and probably several more. It is "interior" because it generalizes the inner product: a 1-form ω can be represented via $\omega = a\cdot$ for some vector a, so

$$\iota_v \omega = \omega(v) = a \cdot v.$$

We visualize interior multiplication as follows: From a k-form ω, a density of $(n-k)$-dimensional spaces, we obtain a $(k-1)$ form $\iota_x\omega$ of $(n-k+1)$-dimensional spaces by extending the spaces along direction x and rescaling properly.



In the above illustration, "Π" is the **p**arallelepiped that we're evaluating ω on.

Here's the reasoning behind this visualization. Consider a pure k-form ω in $\mathbb{R}^n$ and a vector v. (We restrict to pure forms since they are easier to visualize.) There are two cases:

- Suppose $\iota_v\omega$ is not identically zero. Then we can find $\omega_2, \ldots, \omega_k$ so that

$$\omega = \omega_1 \wedge \cdots \wedge \omega_k,$$

  where ω_1 is the dual vector to v (here, I mean $\omega_1(w) = \frac{1}{\|v\|^2} v \cdot w$). Now, we have

$$\iota_v\omega = \omega_2 \wedge \cdots \wedge \omega_k.$$

  Then ω is represented visually by

$$\omega_1^{-1}(\mathbb{Z}) \cap \underbrace{\left(\omega_2^{-1}(\mathbb{Z}) \cap \cdots \cap \omega_k^{-1}(\mathbb{Z})\right)}_{\text{visual representation of } \iota_v\omega},$$

  so we visually get from ω to $\iota_v\omega$ by "un-intersecting" with $\omega_1^{-1}(\mathbb{Z})$. In other words, take each subspace making up the density of ω and span it with v.
- If $\iota_v\omega$ is identically zero, we cheat a little: v is already "in" ω, so $\iota_v\omega$ looks the same as ω. Now, $\iota_v\omega$ is being considered as a $(k-1)$-form, but has $(n-k)$-planes instead of $(n-k+1)$-planes, so is degenerate.

2. Integration

We now understand multivectors and exterior forms in Euclidean space, so we can let them vary and do calculus to them. In this section, we describe how to put exterior algebra on manifolds, and how to integrate over manifolds.

The reason ι is sometimes called a "derivative" is that it is an **anti-derivation**; in other words, if ω is a k-form, then

$$\iota_v(\omega \wedge \eta) = \iota_v(\omega) \wedge \eta + (-1)^k \omega \wedge (\iota_v \eta).$$

2.1. **Differential forms.** A **differential k-form** ω associates to each point p in a smooth manifold M an exterior k-form $\omega_p \in \Lambda^k(T_pM)$, so that ω "varies smoothly". What does it mean to vary smoothly?

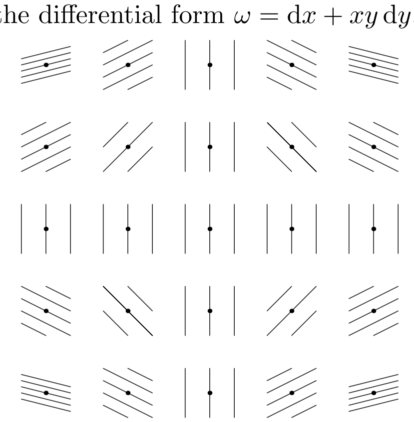
In open submanifolds of $\mathbb{R}^n$, we can write ω_p uniquely as

$$\omega_p = \sum_{I \text{ inc}} a_I(p)\, \mathrm{d}x_I\ ,$$

Then, "varying smoothly" means the functions $a_I(p)$ are smooth. The key here is that open submanifolds of $\mathbb{R}^n$ have a global coordinate chart, so the $\mathrm{d}x_I$ make sense everywhere. To make sense of varying smoothly on other smooth manifolds, we need the notion of a vector bundle, specifically the k-th exterior bundle. For details about this construction, see [4, p. 359].

The space of all differential k-forms on a manifold is often denoted $\Omega^k(M)$. One mnemonic is that the shape "Ω" is a smoothed-out version of the shape "Λ".

2.2. **Visualizing differential forms.** We visualize differential forms like how we visualize vector fields. To visualize vector fields, we pick several points in M. At each point p, draw the vector corresponding to p, using p as the origin. To visualize differential forms, just replace "draw the vector" with "draw the exterior form". For example, below is the differential form $\omega = \mathrm{d}x + xy\,\mathrm{d}y$.



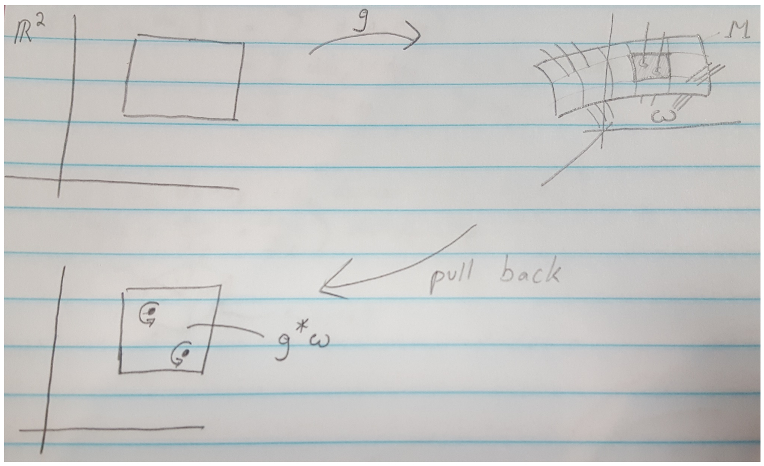
2.3. **Integrating differential forms over manifolds.** We rigorously define the integral of a differential form as follows:

(1) If ω is a differential n-form in $\mathbb{R}^n$, we know $\omega = a(x)\det$ for some smooth function $a : \mathbb{R}^n \to \mathbb{R}$. We then define
$$\int_K \omega = \int_K a,$$
where $K \subset \mathbb{R}^n$ is compact, and the RHS is the usual Riemann integral.

(2) If ω is a differential k-form in a n-dimensional manifold M, and K is a compact set contained inside a coordinate neighborhood $g(U)$ with parametrization $g\colon U \to M$, then we **pull back** ω to a differential form $g^*\omega$ over U and define
$$\int_K \omega = \int_{g^{-1}(K)} g^*\omega,$$
where the RHS integral was defined in the first step.

(3) Integrating over a noncompact set A is fairly technical, and involves partitions of unity. Essentially, this involves breaking A into many compact sets K_i, possibly with overlap, and then integrating over each K_i while accounting for the overlap.

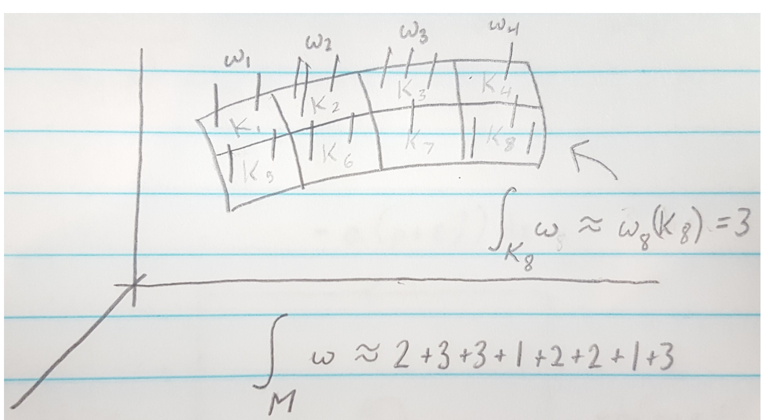
Pullback looks like projecting the linear spaces in ω into the tangent space to M, and then sending it backwards to U via g.



While pullback is crucial to *defining* these integrals, we can ignore it when *visualizing* them. Visually, this is how we integrate a differential n-form ω over a n-dimensional manifold M:

(1) Break M into tiny n-dimensional pieces K_i
(2) Approximate each K_i with a tangent n-vector v_i with proper orientation
(3) In each K_i, approximate ω with an exterior n-form ω_i
(4) Evaluate $\omega_i(v_i)$, and add them all up

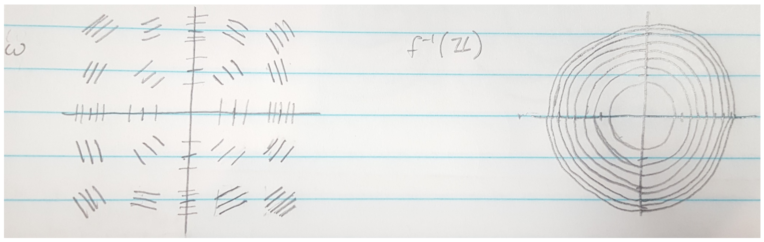
We can take a shortcut. Since the K_i are tiny, they are almost identical to the v_i. So, we just need to give an orientation to each K_i and evaluate "$\omega_i(K_i)$", in the same way we would visually evaluate $\omega_i(v_i)$: by counting how many pieces of ω_i pass through K_i. The image below shows the process with this shortcut.



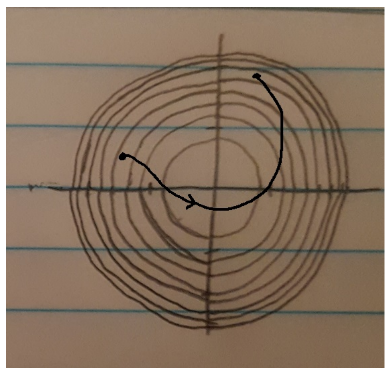
In some cases, we can streamline the visualization even more. For example, let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be smooth, and consider its gradient form $\omega \in \Omega^1(\mathbb{R}^2)$, defined as[10]

$$\omega_p(v) = v \cdot (\mathrm{grad} f)(p),$$

where $\mathrm{grad} f$ is the gradient of f. In this case, we can (imprecisely) describe ω as a linear approximation to the level sets $f^{-1}(\mathbb{Z})$.



Now, to integrate ω over a curve c, we can simply see how many times c passes through the level sets of f. In the below figure, $\int_c \omega \approx (-3) + 5 = 2$.



3. Differentiation

In single-variable calculus, there is essentially one derivative you can define—you can interpret the derivative of a function in many ways, but each yields the same object. However, in exterior calculus, some of those interpretations diverge, giving different mathematical objects. In this section, we describe the main one: the **exterior derivative**.

3.1. Exterior Derivative.

We can define the derivative of a function $f : \mathbb{R} \to \mathbb{R}$ via the fundamental theorem of calculus

$$f(b) - f(a) = \int_a^b f'(x)\,\mathrm{d}x\,.$$

Pick some $x \in \mathbb{R}$. When the interval $[a, b]$ is small, we can approximate

$$\int_a^b f'(x)\,\mathrm{d}x \approx (b - a) f'(x),$$

[10] The form ω is actually the exterior derivative of f, which we'll define later. This explains why you need a Riemannian metric to define a gradient vector field: the gradient is essentially a 1-form, and you need an inner product for the isomorphism $V \cong V^\vee$.

so
$$f'(x) \approx \frac{f(b)-f(a)}{b-a}.$$
This is essentially the same as the difference quotient. However, the similarity ends when we step up in dimension.

Now, we can think of flux and divergence. Let $\mathbf{F}$ be a smooth vector field in $\mathbb{R}^3$, and B be a closed unit ball centered at p with boundary S. The divergence theorem states
$$\int_S \text{flux of } \mathbf{F} \text{ over } S = \int_B \text{div}\mathbf{F}.$$
When the ball B is small, we can approximate
$$\text{div}\mathbf{F} \approx \frac{1}{\text{vol}B}\int_S \text{flux of } \mathbf{F} \text{ over } S$$
and, in fact, could define divergence as
$$\text{div}\mathbf{F} = \lim_{\epsilon\to 0}\frac{1}{\text{vol}(\epsilon B)}\int_{\epsilon S}(\text{flux of } \mathbf{F} \text{ over } \epsilon S).$$
We can interpret the fundamental theorem of calculus as a "divergence theorem" by thinking of $f(b)$ as the amount of f leaving $[a,b]$ at b, and $-f(a)$ as the amount of f leaving $[a,b]$ at a. The negative sign indicates that at a, f leaves to the *left*.

Using this flux-and-divergence interpretation of the derivative for inspiration, we can define the exterior derivative. (This discussion and definition was adapted from [1, p. 188].)

Definition 3.1. Let ω be a k-form, and let $v_1,\dots,v_{k+1}$ be vectors. For all $\epsilon > 0$, let $\epsilon\Pi$ be the (oriented) parallelepiped spanned by the vectors $\epsilon v_1,\dots,\epsilon v_{k+1}$. The **exterior derivative** of ω, denoted $\mathrm{d}\omega$, is defined by
$$\mathrm{d}\omega\,(v_1,\dots,v_{k+1}) = \lim_{\epsilon\to 0}\frac{1}{\text{vol}(\epsilon\Pi)}\int_{\partial(\epsilon\Pi)}\omega.$$

In other words, $\mathrm{d}\omega$ is defined to make the equation
$$\int_\Pi \mathrm{d}\omega = \int_{\partial\Pi}\omega$$
true for tiny parallelepipeds Π. But we use tiny parallelepipeds to approximate manifolds, so the Generalized Stokes' Theorem shouldn't be a surprise.

Theorem 3.2 (Generalized Stokes' Theorem)**.** *Let M be an oriented manifold with boundary ∂M, and ω be a differential k-form on ∂M. Then*
$$\int_M \mathrm{d}\omega = \int_{\partial M}\omega.$$

With our definition of the exterior derivative, the proof of Stokes' Theorem is conceptually straightforward:

(1) Show it holds when M is a parallelepiped
(2) By collapsing parts of the boundary of a parallelepiped, show it holds when M is a simplex, and therefore a polyhedron
(3) Approximate M by polyhedra. (If M has no global coordinate chart, then use a partition of unity.) Show the formula holds in the limit.

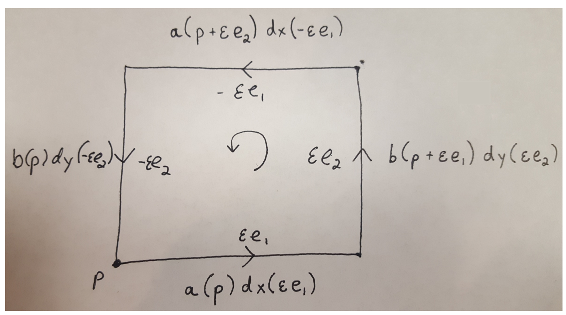
The trickiest part of proving Stokes' Theorem is showing that our definition is well-defined (i.e. the limit exists and produces a k-form), and is equivalent to the usual definition. I won't prove this, but here's an example illustrating a particular case. In the figure below, we're evaluating $\mathrm{d}\omega_p\,(\varepsilon e_1, \varepsilon e_2)$, where $\omega = a\,\mathrm{d}x + b\,\mathrm{d}y$ is a 1-form in $\mathbb{R}^2$ and $\{e_1, e_2\}$ is the standard basis for $\mathbb{R}^2$. Based on the picture,

$$\mathrm{d}\omega_p\,(\varepsilon e_1, \varepsilon e_2) = \Big(-\varepsilon a(p+\varepsilon e_2) + \varepsilon a(p)\Big) + \Big(\varepsilon b(p+\varepsilon e_1) - \varepsilon b(p)\Big) \approx \varepsilon\left(\varepsilon\frac{\partial b}{\partial x} - \varepsilon\frac{\partial a}{\partial y}\right).$$

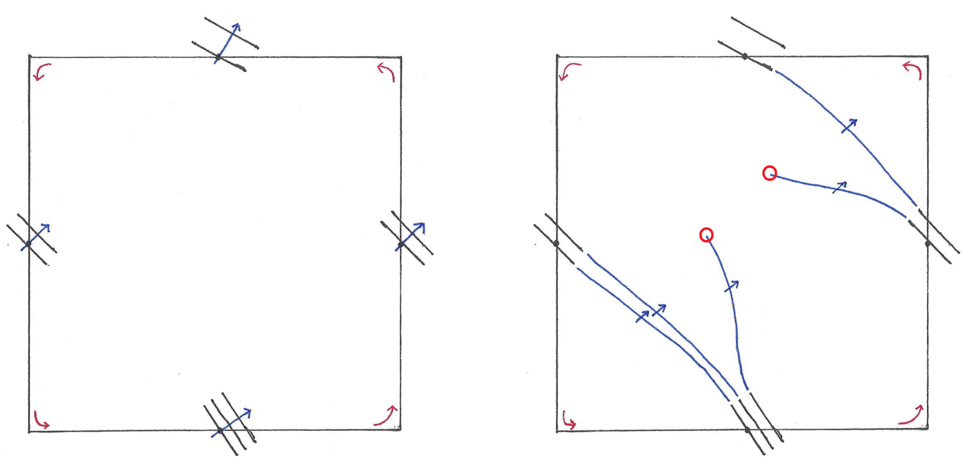
This agrees with the usual definition, since

$$\mathrm{d}\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right)\mathrm{d}x \wedge \mathrm{d}y\,.$$

The same sort of pattern happens in higher dimensions: opposite faces have opposite orientations, so we're essentially computing partial derivatives. The orientation of the parallelepiped gives us the correct signs on the partial derivatives.

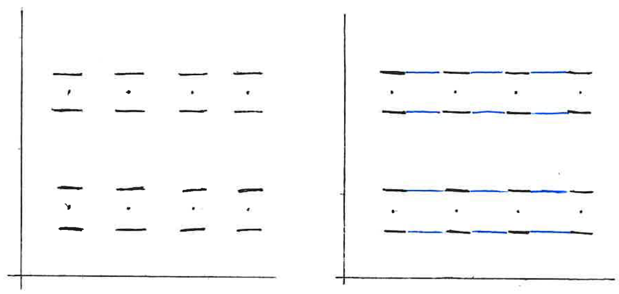


3.2. **Visualizing the exterior derivative.** We can use the ideas of flux and divergence to visualize the exterior derivative. This is clearest with 1-forms in $\mathbb{R}^2$. Consider the below 1-form ω, which we've drawn along a small square loop centered about a point p. The flux of ω about this loop is roughly $3 + 2 - 1 - 2 = 2$, so ω has divergence 2 near p. To visualize this, consider trying to "stitch up" the lines of ω like in the below right. There's no way to do this perfectly; there will always be two "loose threads". Hence, we can visualize $\mathrm{d}\omega$ at p with two dots.

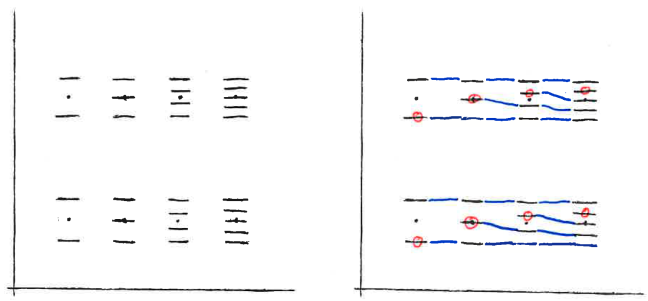


3.3. **Closed and exact forms.** Recall that a form ω is *exact* if $\omega = \mathrm{d}\eta$ for some η, and *closed* if $\mathrm{d}\omega = 0$. Since $\mathrm{d}^2 = 0$, all exact forms are closed. The converse is only partially true: by the Poincaré Lemma, all closed forms are *locally exact*. That is, all points have a neighborhood on which the form is exact.[11]

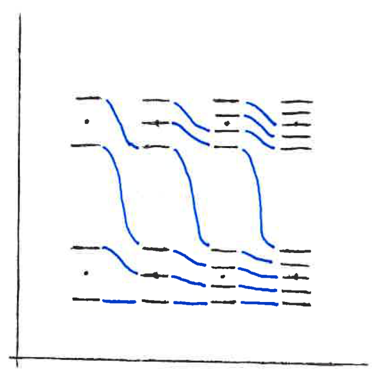
Hence, we can interpret the exterior derivative as a *local barrier to exactness*. To do this, select several points, and then try to stitch up ω as before. There is a difference: previously, we selected several points around a single central point, whereas now we just scatter some points. For example, consider the form $\mathrm{d}y$. Since $\mathrm{d}(\mathrm{d}y) = 0$, we expect that $\mathrm{d}y$ could be stitched up. Indeed, we can simply extend the horizontal lines to meet:



If we try to stitch up $x\,\mathrm{d}y$ in the same way, we end up with loose threads popping up at each lattice point of the plane. The set of lattice points is precisely our visual representation of $\mathrm{d}x \wedge \mathrm{d}y$, which is $\mathrm{d}(x\,\mathrm{d}y)$.



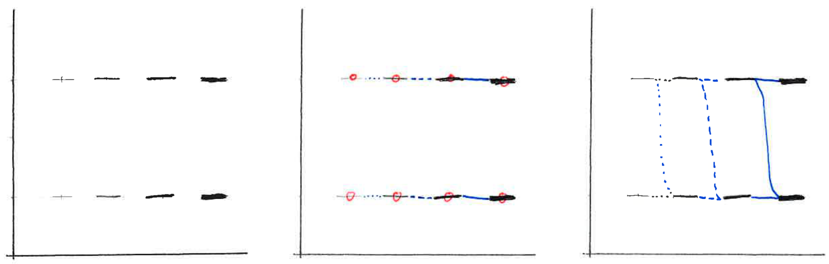
One might object: why not stitch up $x\,\mathrm{d}y$ differently, like the following?



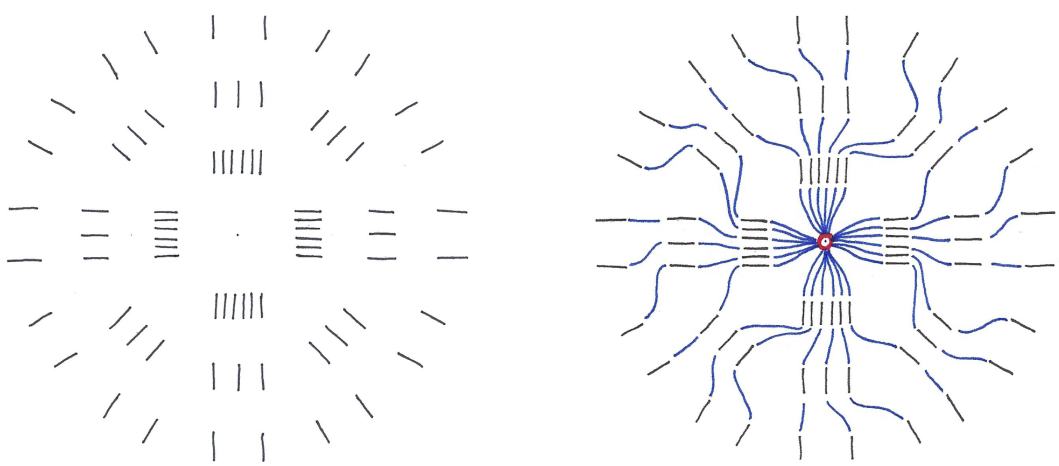
[11]We could continue down this path: by checking which locally exact forms are not globally exact, we can probe the topology of our manifold. This leads to the *de Rham cohomology*.

The problem is that in the middle, the stitching doesn't agree with the form—it's curved, but the form is horizontal. *But wasn't the correct stitching also curved?* It looks like it was, but it wasn't. The below image shows, from left to right,

(1) what the form $x\,\mathrm{d}y$ actually looks like
(2) what the correct patching looks like
(3) what the incorrect patching looks like



So, a form *isn't* closed if stitching it together requires loose threads. What about a form that is closed, but not exact? The typical example is the form $\mathrm{d}\theta = (y\,\mathrm{d}x - x\,\mathrm{d}y)/(x^2+y^2)$ defined on $\mathbb{R}^2 \setminus 0$. As you can see below, $\mathrm{d}\theta$ has many loose threads starting from the origin, which is outside the domain of $\mathrm{d}\theta$. (The stitching looks unnatural because I used a small number of points. With more points, $\mathrm{d}\theta$ should look like many spokes sticking out of the origin.)



3.4. **Future work: Lie derivative.** The exterior derivative generalizes flux and divergence. If we instead generalize the "difference quotient", we end up with the directional derivative, and then the **Lie derivative** $L_v\omega$ of a k-form ω along a path given by the vector field v.

Though the Lie derivative and exterior derivative are quite different ways of generalizing the derivative to exterior calculus, they are related by a simple formula, called **Cartan's magic formula**, which also bring in interior multiplication.

$$L_v = \mathrm{d}\iota_v + \iota_v\mathrm{d}.$$

To prove this, one usually shows the LHS and RHS are a special type of operator called a *derivation*, which implies it need only be shown for 0-forms and 1-forms. This proof gives an interesting view of the algebraic structure of these operators.

However, it certainly isn't visual. I spent much of the summer hunting for compatible visualizations of interior multiplication and the Lie and exterior derivatives, hoping that all would fit in the same picture and shine a new light on this formula. Such a picture remains elusive. Maybe you can find it! Arnold hints at a visual proof with a different style of visualization in [1, p. 198].

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