Department of Mathematical Sciences
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Discrete Mathematics

Induction

22.4. Prove the following equations by induction. In each case, n is a positive integer.

a. $\displaystyle\sum_{i=1}^{n}(3i-2)=\frac{n(3n-1)}{2}$.

Using lecture notations we have following statements which are related to n:

$\boxed{1} \rightsquigarrow 3\cdot 1-2=\dfrac{1(3\cdot 1-1)}{2}$,

$\boxed{2} \rightsquigarrow (3\cdot 1-2)+(3\cdot 2-2)=\dfrac{2(3\cdot 2-1)}{2}$,

$\boxed{3} \rightsquigarrow (3\cdot 1-2)+(3\cdot 2-2)+(3\cdot 3-2)=\dfrac{3(3\cdot 3-1)}{2}$,

$\boxed{4} \rightsquigarrow (3\cdot 1-2)+(3\cdot 2-2)+(3\cdot 3-2)+(3\cdot 4-2)=\dfrac{4(3\cdot 4-1)}{2}$,

$$\vdots$$

$\boxed{\text{n}} \rightsquigarrow (3\cdot 1-2)+(3\cdot 2-2)+\cdots+(3\cdot n-2)=\dfrac{n(3n-1)}{2}$,

etc.

Proof. **Basis step:** The case $n=1$ is true is true because both sides of the equation

$$\sum_{i=1}^{1}(3i-2)=3\cdot 1-2=1 \quad \text{and} \quad \frac{1(3\cdot 1-1)}{2}=1$$

evaluate to 1.

Induction hypothesis: Suppose the result is true for $k\in\mathbb{N}$; that is, we assume we have

$$\sum_{i=1}^{k}(3i-2)=(3\cdot 1-2)+(3\cdot 2-2)+\cdots+(3\cdot k-2)=\frac{k(3k-1)}{2}$$

(We must prove that the result is true for $k+1$; that is using the equation $\displaystyle\sum_{i=1}^{k}(3i-2)=\frac{k(3k-1)}{2}$ we must prove $\displaystyle\sum_{i=1}^{k+1}(3i-2)=\frac{(k+1)(3(k+1)-1)}{2}=\frac{(k+1)(3k+2)}{2}$)

Since

$$\begin{aligned}
\sum_{i=1}^{k+1}(3i-2) &= \underbrace{(3\cdot 1-2)+(3\cdot 2-2)+\cdots+(3\cdot k-2)}_{\sum_{i=1}^{k}(3i-2)=\frac{k(3k-1)}{2}}+(3(k+1)-2) \\
&= \frac{k(3k-1)}{2}+(3(k+1)-2) \\
&= \frac{k(3k-1)+2(3(k+1)-2)}{2} \\
&= \frac{3k^2+5k+2}{2}
\end{aligned}$$

and $\dfrac{(k+1)(3k+2)}{2} = \dfrac{3k^2+5k+2}{2}$, we have

$$\sum_{i=1}^{k+1}(3i-2) = \frac{(k+1)(3(k+1)-1)}{2}.$$

In other words we have shown:

- **Basis step:** The statement $\boxed{1}$ is true.
- **Induction hypothesis:**
  "If statement $\boxed{1}$ is true, then statement $\boxed{2}$ is true"
  "If statement $\boxed{2}$ is true, then statement $\boxed{3}$ is true"
  "If statement $\boxed{3}$ is true, then statement $\boxed{4}$ is true"
  "If statement $\boxed{4}$ is true, then statement $\boxed{5}$ is true"

  $\vdots$

  "If statement $\boxed{\text{k}}$ is true, then statement $\boxed{\text{k+1}}$ is true"

Therefore the statement $\boxed{\text{n}}$ is true for every n, or in other words the equation $\displaystyle\sum_{i=1}^{n}(3i-2) = \frac{n(3n-1)}{2}$ is true for every positive integer n.

b. $\displaystyle\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}.$

Using lecture notations we have following statements which are related to n:

$\boxed{1} \rightsquigarrow 1^3 = \dfrac{1^2(1+1)^2}{4},$

$\boxed{2} \rightsquigarrow 1^3 + 2^3 = \dfrac{2^2(2+1)^2}{4},$

$\boxed{3} \rightsquigarrow 1^3 + 2^3 + 3^3 = \dfrac{3^2(3+1)^2}{4},$

$\boxed{4} \rightsquigarrow 1^3 + 2^3 + 3^3 + 4^3 = \dfrac{4^2(4+1)^2}{4},$

$$\vdots$$

$\boxed{\text{n}} \rightsquigarrow 1^3 + 2^3 + 3^3 + 4^3 + \cdots n^3 = \dfrac{n^2(n+1)^2}{4}.$

etc.

Proof. **Basis step:** The case $n = 1$ is true is true because both sides of the equation

$$\sum_{i=1}^{1} i^3 = 1^3 = 1 \quad \text{and} \quad \tfrac{1^2(1+1)^2}{4} = 1$$

evaluate to 1.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\sum_{i=1}^{k} i^3 = 1^3 + 2^3 + \cdots + k^3 = \frac{k^2(k+1)^2}{4}$$

(We must prove that the result is true for $k+1$; that is using the equation $\displaystyle\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}$ we must prove $\displaystyle\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4} = \frac{(k+1)^2(k+2)^2}{4}$)

Since

$$\sum_{i=1}^{k+1} i^3 = \underbrace{1^3+2^3+\cdots+k^3}_{\sum_{i=1}^{k} i^3 = \frac{k^2(k+1)^2}{4}} + (k+1)^3$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3 = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2(k^2+4(k+1))}{4} = \frac{(k+1)^2(k^2+4k+4)}{4}$$

and $\dfrac{(k+1)^2(k+2)^2}{4} = \dfrac{(k+1)^2(k^2+4k+4)}{4}$, we have

$$\sum_{i=1}^{k+1} i^3 = \frac{(k+1)^2((k+1)+1)^2}{4}.$$

In other words we have shown:

- **Basis step:** The statement $\boxed{1}$ is true.
- **Induction hypothesis:**
  "If statement $\boxed{1}$ is true, then statement $\boxed{2}$ is true"
  "If statement $\boxed{2}$ is true, then statement $\boxed{3}$ is true"
  "If statement $\boxed{3}$ is true, then statement $\boxed{4}$ is true"
  "If statement $\boxed{4}$ is true, then statement $\boxed{5}$ is true"

  $\vdots$

  "If statement $\boxed{\text{k}}$ is true, then statement $\boxed{\text{k+1}}$ is true"

Therefore the statement $\boxed{\text{n}}$ is true for every n, or in other words the equation $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$ is true for every positive integer n.

c. $\sum_{i=1}^{n} 9 \cdot 10^{i-1} = 10^n - 1.$

Proof. **Basis step:** The case $n = 1$ is true is true because both sides of the equation

$$\sum_{i=1}^{1} 9 \cdot 10^{i-1} = 9 \cdot 10^0 = 9 \quad \text{and} \quad 10^1 - 1 = 9$$

evaluate to 9.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\sum_{i=1}^{k} 9 \cdot 10^{i-1} = 10^k - 1$$

(We must prove that the result is true for $k+1$; that is using the equation $\sum_{i=1}^{k} 9 \cdot 10^{i-1} = 9 + 9 \cdot 10 + 9 \cdot 10^2 + \cdots + 9 \cdot 10^{k-1} = 10^k - 1$ we must prove $\sum_{i=1}^{k+1} 9 \cdot 10^{i-1} = 10^{k+1} - 1.$)

Since

$$\begin{aligned}
\sum_{i=1}^{k+1} 9 \cdot 10^{i-1} &= \underbrace{9 + 9 \cdot 10 + 9 \cdot 10^2 + \cdots + 9 \cdot 10^{k-1}}_{\sum_{i=1}^{k} 9 \cdot 10^{i-1} = 10^k - 1} + 9 \cdot 10^{(k+1)-1} \\
&= 10^k - 1 + 9 \cdot 10^k \\
&= 10^k + 9 \cdot 10^k - 1 \\
&= 10^k(1+9) - 1 = 10^k \cdot 10 - 1 \\
&= 10^{k+1} - 1
\end{aligned}$$

the equation $\sum_{i=1}^{n} 9 \cdot 10^{i-1} = 10^n - 1$ is true for for every positive integer n.

d. $\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$

Proof. **Basis step:** The case $n = 1$ is true is true because both sides of the equation

$$\textstyle\sum_{i=1}^{1} \frac{1}{i(i+1)} = \frac{1}{1(1+1)} = \frac{1}{2} \quad \text{and} \quad 1 - \frac{1}{1+1} = 1 - \frac{1}{2} = \frac{1}{2}$$

evaluate to $\frac{1}{2}$.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\sum_{i=1}^{k} \frac{1}{i(i+1)} = 1 - \frac{1}{k+1}$$

We must prove that the result is true for $k+1$; that is using the equation $\sum_{i=1}^{k} \frac{1}{i(i+1)} = \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2} + \cdots + \frac{1}{k\cdot(k+1)} = 1 - \frac{1}{k+1}$ we must prove $\sum_{i=1}^{k+1} \frac{1}{i(i+1)} = 1 - \frac{1}{(k+1)+1} = 1 - \frac{1}{k+2}$

Using Induction hypothesis we have

$$\begin{aligned}
\sum_{i=1}^{k+1} \frac{1}{i(i+1)} &= \underbrace{\frac{1}{1\cdot 2} + \frac{1}{1\cdot 2} + \cdots + \frac{1}{k\cdot(k+1)}}_{\sum_{i=1}^{k} \frac{1}{i(i+1)} = 1 - \frac{1}{k+1}} + \frac{1}{(k+1)((k+1)+1)} \\
&= 1 - \frac{1}{k+1} + \frac{1}{(k+1)(k+2)} = 1 - \left(\frac{1}{k+1} - \frac{1}{(k+1)(k+2)}\right) \\
&= 1 - \left(\frac{k+2}{(k+1)(k+2)} - \frac{1}{(k+1)(k+2)}\right) \\
&= 1 - \frac{k+2-1}{(k+1)(k+2)} = 1 - \frac{k+1}{(k+1)(k+2)} \\
&= 1 - \frac{1}{(k+2)}
\end{aligned}$$

Therefore $\sum_{i=1}^{n} \frac{1}{i(i+1)} = 1 - \frac{1}{n+1}$ is true for for every positive integer n.

e. $\sum_{i=0}^{n} x^i = \frac{1 - x^{n+1}}{1 - x}$, here $x \neq 1$

Proof. **Basis step:** The case $n = 1$ is true is true because the left-hand side is $\sum_{i=0}^{1} x^i = x^0 = x^1 = 1 + x$. and the right is

$\dfrac{1-x^{1+1}}{1-x} = \dfrac{1-x^2}{1-x}$. Since $1-x^2 = (1+x)(1-x)$, this reduces to $1+x$ as required.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$$

(We must prove that the result is true for $k+1$; that is using the equation $\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}$ we must prove $\sum_{i=0}^{k+1} x^i = \frac{1-x^{(k+1)+1}}{1-x} = \frac{1-x^{k+2}}{1-x}$)

Using Induction hypothesis we have

$$\begin{aligned}
\sum_{i=0}^{k+1} x^i &= \underbrace{1+x+x^2+\cdots+x^k}_{\sum_{i=0}^{k} x^i = \frac{1-x^{k+1}}{1-x}} + x^{k+1} \\
&= \frac{1-x^{k+1}}{1-x} + x^{k+1} = \frac{1-x^{k+1}}{1-x} + \frac{(1-x)x^{k+1}}{1-x} \\
&= \frac{1-x^{k+1}}{1-x} + \frac{x^{k+1}-x^{k+2}}{1-x} \\
&= \frac{1-x^{k+1}+x^{k+1}-x^{k+2}}{1-x} = \frac{1-x^{k+2}}{1-x}
\end{aligned}$$

Therefore $\sum_{i=0}^{n} x^i = \dfrac{1-x^{n+1}}{1-x}$ is true for for every integer $n > 0$.

22.5. Prove the following inequalities bu induction. In each case, n is a positive integer.

a. $2^n \leq 2^{n+1} - 2^{n-1} - 1$.

Proof. **Basis step:** The left-hand side evaluates to $2^1 = 2$ and the right hand side evaluates to $2^{1+1} - 2^{1-1} - 1 = 4 - 1 - 1 = 2$, so the basic case is true, because $2 \leq 2$.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$2^k \leq 2^{k+1} - 2^{k-1} - 1.$$

(We must prove that the result is true for $k+1$; that is using $2^k \leq 2^{k+1} - 2^{k-1} - 1$ we must prove $2^{k+1} \leq 2^{k+1+1} - 2^{k+1-1} - 1 = 2^{k+2} - 2^k - 1$)

To his end, we multiply both sides of $2^k \leq 2^{k+1} - 2^{k-1} - 1$ by 2:

$$2 \cdot 2^k \leq 2(2^{k+1} - 2^{k-1} - 1)$$

So we have $2^{k+1} \leq 2 \cdot 2^{k+2} - 2 \cdot 2^{k-1} - 2 = 2^{k+2} - 2^k - 1 - 1 \leq 2^{k+2} - 2^k - 1$.

(For any integer a we have $a - 1 \leq a$)

b. $\displaystyle\prod_{i=1}^{n} \left(1 - \frac{1}{2^i}\right) \geq \frac{1}{4} + \frac{1}{2^{n+1}}$.

Proof. **Basis step:** The left-hand side evaluates to $\prod_{i=1}^{1} \left(1 - \frac{1}{2^i}\right) = \frac{1}{2}$ and the right hand side evaluates to $\frac{1}{4} + \frac{1}{2^{1+1}} = \frac{1}{4} + \frac{1}{4}$, so the basic case is true, because $\frac{1}{2} \geq \frac{1}{4}$.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\prod_{i=1}^{k} \left(1 - \frac{1}{2^i}\right) = \left(1 - \frac{1}{2^1}\right)\left(1 - \frac{1}{2^2}\right) \cdots \left(1 - \frac{1}{2^k}\right) \geq \frac{1}{4} + \frac{1}{2^{k+1}}.$$

(We must prove that the result is true for $k+1$; that is using $\prod_{i=1}^{k} \left(1 - \frac{1}{2^i}\right) \geq \frac{1}{4} + \frac{1}{2^{k+1}}$ we must prove $\displaystyle\prod_{i=1}^{k+1} \left(1 - \frac{1}{2^i}\right) \geq \frac{1}{4} + \frac{1}{2^{k+2}}$.)

To his end, we multiply both sides of $\prod_{i=1}^{k}\left(1-\frac{1}{2^i}\right) \geq \frac{1}{4}+\frac{1}{2^{k+1}}$ by $1-\frac{1}{2^{k+1}}$ to get:

$$\begin{aligned}
\prod_{i=1}^{k+1}\left(1-\frac{1}{2^i}\right) &= \left(1-\frac{1}{2^1}\right)\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{2^k}\right)\left(1-\frac{1}{2^{k+1}}\right) \\
&\geq \left(\frac{1}{4}+\frac{1}{2^{k+1}}\right)\left(1-\frac{1}{2^{k+1}}\right) \\
&= \frac{1}{4}+\frac{1}{2^{k+1}}-\frac{1}{2^{k+3}}-\frac{1}{2^{2k+2}} \\
&= \frac{1}{4}+\frac{2^{k+1}-2^{k-1}-1}{2^{2k+2}} \\
&\geq \frac{1}{4}+\frac{2^k}{2^{2k+2}} \qquad \text{by part } \mathbf{a.} \\
&= \frac{1}{4}+\frac{1}{2^{k+1}}.
\end{aligned}$$

c. $\displaystyle\sum_{i=1}^{2^n}\frac{1}{i} \geq 1+\frac{n}{2}.$

Proof. **Basis step:** The left-hand side evaluates to $\sum_{i=1}^{2\cdot 1}\frac{1}{i} = \frac{1}{1}+\frac{1}{2}=\frac{3}{2}$ and the right hand side evaluates to $1+\frac{1}{2}=\frac{3}{2}$, so the basic case is true, because $\frac{3}{2}\geq\frac{3}{2}$.

Induction hypothesis: Suppose the result is true for $k \in \mathbb{N}$; that is, we assume we have

$$\sum_{i=1}^{2^k}\frac{1}{i} \geq 1+\frac{k}{2}.$$

(We must prove that the result is true for $k+1$; that is using $\displaystyle\sum_{i=1}^{2^k}\frac{1}{i} \geq 1+\frac{k}{2}$ we must prove $\displaystyle\sum_{i=1}^{2^{k+1}}\frac{1}{i} \geq 1+\frac{k+1}{2}$.)

$$
\begin{aligned}
\sum_{i=1}^{2^{k+1}} \frac{1}{i} &= \underbrace{1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2^k}}_{\sum_{i=1}^{2^k} \frac{1}{i} \geq 1 + \frac{k}{2}} + \underbrace{\frac{1}{2^k+1} + \frac{1}{2^k+2} + \cdots + \frac{1}{2^{k+1}}}_{2^k-\text{terms}} \\
&\quad \text{(there are } 2^k \text{ additional terms and they are all greater than of equal to } \tfrac{1}{2^{k+1}}\text{)} \\
&\geq 1 + \frac{k}{2} + \underbrace{\frac{1}{2^{k+1}} + \frac{1}{2^{k+1}} + \cdots + \frac{1}{2^{k+1}}}_{2^k-\text{terms}} \qquad . \\
&= 1 + \frac{k}{2} + \frac{1}{2} \\
&= 1 + \frac{k+1}{2}
\end{aligned}
$$