Title of the Article

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SAME AS TITLE OF THE ARTICLE

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Abstract. Here you will write the abstract of the article. No references to inside the article or bibliography is allowed here. Any mathematical notation that is not universally well-defined can not be used here without its definition. For example, $\mathbb{N}, \mathbb{R}, \mathbb{C}$ are universally known to denote the set of natural numbers, set of real numbers, and set of complex numbers, respectively, while any notation used for Legendre symbol or multiplier of a fixed point or a congruence residue class are not universally known.

1. Introduction

2. Prerequisite

2.1. **Subsection 1.** An example of citing an article from bibliography, here is [1].

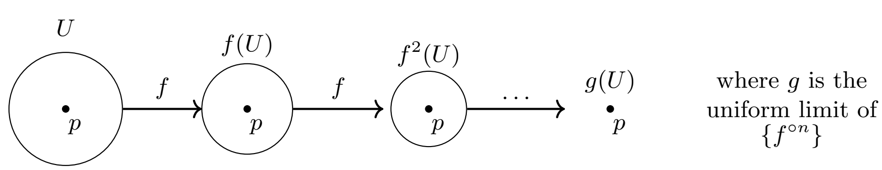
2.2. **Subsection 2.**

3. Notations

Lemma 3.1. *Schwartz Lemma*

4. Attracting Fixed points

Definition 4.1. Given a map f and a fixed point p, we call the point *topologically attracting*, if there exists a neighborhood U of p, such that the family of functions $\{f^{\circ n}\}_{n\in\mathbb{N}}$ converges uniformly to the constant function $g(z) = p$ on U.



One can think of this as a point, which has a neighborhood which is shrunk down further and further under repeated action by f. After infinite iterations of f, this neighborhood will be shrunk down to just the point. To characterize this, we use the multiplier.

Lemma 4.2. *A fixed point p for a holomorphic function f is topologically attracting if and only if its multiplier λ satisfies $|\lambda| < 1$*

Proof. ($|\lambda| < 1 \implies$ *topologically attracting*)
First, recall that through möbius conjugations, we can send p to 0, argue there, and apply the inverse transformation to bring the arguments back to p. We aim to find a neighborhood that f contracts geometrically. using that, we can show that the family of iterates of f will converge geometrically to the constant 0 function. So, we show that $\exists r$ such that $\forall z \in \mathbb{D}_r$

$$f(z) \leq c \cdot |z|$$

where $c < 1$. Choose any c such that $|\lambda| < c < 1$.

WLOG, let $f(0) = 0$. Taylors thorem tells us that for some small enough r_0, $f(z) = \lambda z + O(z^2)$ on $\mathbb{D}_{r_0}$. That is,

$$\begin{aligned} &|f(z) - \lambda z| \leq c_0 \cdot |z^2| && \forall z \in \mathbb{D}_{r_0} \\ &|f(z)| \leq |\lambda z| + c_0 \cdot |z^2| = (|\lambda| + c_0|z|) \cdot |z| && (4.1) \end{aligned}$$

Since this holds on $\mathbb{D}_{r_0}$, it also holds on $\mathbb{D}_r$ for all $r \leq r_0$. By the archimedian principle, we know that there exists a small enough r_1 such that $0 < c_0 \cdot r < c - |\lambda| \implies |\lambda| + c_0 \cdot r < c$. Setting $r = \min\{r_0, r_1\}$ and continuing from (4.1) we can say that,

$$|f(z)| \leq (|\lambda| + c_0 \cdot r) \cdot |z| < c \cdot |z| \qquad \forall z \in \mathbb{D}_r$$

Now, the claim is that on this neighborhood $\mathbb{D}_r$ of 0, the family of functions $\{f^{\circ n}\}$ converges uniformly. So, $\forall z \in \mathbb{D}_r$,

$$|f(z)| < c \cdot |z| < c \cdot r$$

since $c < 1$, $f(z)$ is in $\mathbb{D}_r$. So we can also say $\forall z \in \mathbb{D}_r$

$$|f \circ f(z)| < c \cdot |f(z)| < c^2 \cdot r$$

Continuing this argument,

$$|f^{\circ n}(z)| < c^n |z| < c^n r$$

This converges uniformly to 0 as $n \to \infty$. So 0 is topologically attracting.

(*topologically attracting* $\implies |\lambda| < 1$) If 0 is a topologically attracting fixed point, then there exists a neighborhood U of 0 such that $\{f^{\circ n}\}$ converges uniformly to the constant 0 function on U. So, for all $\epsilon > 0$ there exists an $N > 0$ such that $\forall n > N$,

$$|f^{\circ n}(z)| < \epsilon \qquad \forall z \in U$$

Consider the disk $\mathbb{D}_\epsilon \subset U$. The above statement implies that $\mathbb{D}_\epsilon$ is mapped onto a proper subset of itself by some iterate $f^{\circ n}$. Using Schwartz lemma (3.1) on this iterate and $\mathbb{D}_\epsilon$, we get that the multiplier of $f^{\circ n}$, which is λ^n satisfies $|\lambda^n| < 1$. But this implies that $|\lambda| < 1$.

□

Theorem 4.3. *Koenigs Linearisation theorem*
Given a geometrically attracting fixed point or a repelling fixed point of f, there exists a local holomorphic change of co-ordinates ϕ, such that $\phi \circ f \circ \phi^{-1}(w) = \lambda w$ for all w in some neighborhood of 0. Further, ϕ is unique upto multiplication by a constant.

A restatement of the above, is that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d, "\phi"'] & f(U) \arrow[d, "\phi"] \\ \mathbb{D}_\epsilon \arrow[r, "w \mapsto \lambda w"] & \mathbb{D}_\epsilon \end{tikzcd}$$

Proof. **Existence:** First we show that such a function exists. Define the following family of functions, $\{\phi_n\}$ as

$$\phi_n(z) = \frac{f^{\circ n}(z)}{\lambda^n}$$

□

4.1. **Subsection 1.**

4.2. **Subsection 2.**

5. Main Section 2

5.1. **Subsection 1.** You can also label and summon sections and subsections as in Section 5, Subsection 5.1.

5.2. **Subsection 2.**

6. Applications and connections

6.1. Subsection 1.

6.2. Subsection 2.

Acknowledgement

Here you can write your acknowledgement.

First Line, Thank me, hehe.

Second Line, Thank LGP programme.

Third Line onwards, Thank whoever you want.

References

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[4] Benedetto, Robert L. & Vefa Goksel. Misiurewicz polynomials and dynamical units, part I. International Journal of Number Theory: Vol. 19, No. 06, pp. 1249-1267 (2023). https://doi.org/10.1142/S1793042123500616

[5] Benedetto, Robert L. & Vefa Goksel. Misiurewicz polynomials and dynamical units, Part II. (2022) ArXiv. https://doi.org/10.48550/arXiv.2203.14431

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